The University of British Columbia
MECH 479

Module 3 Example Problems

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Example 1

Consider the below discretization

$$\left(\frac{\partial T}{\partial y}\right)_{i,j} = \frac{1}{6h}\left(-11T_{i,j} + 18T_{i,j+1} - 9T_{i,j+2} + 2T_{i,j+3}\right) + O\left(h^3\right)$$

Find the truncation error for the above approximation.

Hint: Express the variables on the RHS by Taylor series expansions about the point (i, j):

$$\begin{aligned}
T_{i,j+1} &= T_{i,j} + \left(\tfrac{\partial T}{\partial y}\right)_{i,j}\Delta y + \left(\tfrac{\partial^2 T}{\partial y^2}\right)_{i,j}\tfrac{(\Delta y)^2}{2!} + \left(\tfrac{\partial^3 T}{\partial y^3}\right)_{i,j}\tfrac{(\Delta y)^3}{3!} + \left(\tfrac{\partial^4 T}{\partial y^4}\right)_{i,j}\tfrac{(\Delta y)^4}{4!} + \cdots \\
T_{i,j+2} &= T_{i,j} + \left(\tfrac{\partial T}{\partial y}\right)_{i,j}(2\Delta y) + \left(\tfrac{\partial^2 T}{\partial y^2}\right)_{i,j}\tfrac{(2\Delta y)^2}{2!} + \left(\tfrac{\partial^3 T}{\partial y^3}\right)_{i,j}\tfrac{(2\Delta y)^3}{3!} + \left(\tfrac{\partial^4 T}{\partial y^4}\right)_{i,j}\tfrac{(2\Delta y)^4}{4!} + \cdots \\
T_{i,j+3} &= T_{i,j} + \left(\tfrac{\partial T}{\partial y}\right)_{i,j}(3\Delta y) + \left(\tfrac{\partial^2 T}{\partial y^2}\right)_{i,j}\tfrac{(3\Delta y)^2}{2!} + \left(\tfrac{\partial^3 T}{\partial y^3}\right)_{i,j}\tfrac{(3\Delta y)^3}{3!} + \left(\tfrac{\partial^4 T}{\partial y^4}\right)_{i,j}\tfrac{(3\Delta y)^4}{4!} + \cdots
\end{aligned}$$

Now substitute the RHS expressions above for the terms that appear in the RHS and simplify. We find that

$$\frac{1}{6h}\left(-11T_{i,j} + 18T_{i,j+1} - 9T_{i,j+2} + 2T_{i,j+3}\right) = \left(\frac{\partial T}{\partial y}\right)_{i,j} + \frac{1}{4}\left(\frac{\partial^4 T}{\partial y^4}\right)_{i,j}(\Delta y)^3 + \cdots$$

so, the truncation error has been identified and is

$$\frac{1}{4}\left(\frac{\partial^4 T}{\partial y^4}\right)_{i,j}(\Delta y)^3 + \cdots = O(\Delta y)^3$$

Example 2

Develop a finite difference approximation with truncation error of $O(\Delta y)^2$ for $\partial T/\partial y$ at point (i, j) using $T_{i,j}$, $T_{i,j+1}$, and $T_{i,j+2}$, when the grid spacing is not uniform.
Hint: To find a second-order approximation for $\frac{\partial T}{\partial y}$, let

$$\frac{\partial T}{\partial y} = \frac{cT_{i,j} + aT_{i,j+1} + bT_{i,j+2}}{\Delta y_1} + O\left(\Delta y_1^2\right)$$

Substituting the following Taylor series expansions

$$\begin{aligned}
T_{i,j} =& T\left(x_0, y_0\right) \\
T_{i,j+1} =& T\left(x_0, y_0 + \Delta y_1\right) = T\left(x_0, y_0\right) + \frac{\partial T}{\partial y}\Delta y_1 + \frac{1}{2}\frac{\partial^2 T}{\partial y^2}\Delta y_1^2 + \cdots \\
T_{i,j+2} =& T\left(x_0, y_0 + \Delta y_1 + \Delta y_2\right) = T\left(x_0, y_0\right) \\
&+ \frac{\partial T}{\partial y}\left(\Delta y_1 + \Delta y_2\right) + \frac{1}{2}\frac{\partial^2 T}{\partial y^2}\left(\Delta y_1 + \Delta y_2\right)^2 + \cdots
\end{aligned}$$

we find that

$$a + b + c = 0$$
$$a + (1+\alpha)b = 1$$
$$a + (1+\alpha)^2 b = 0$$

where $\alpha = \frac{\Delta y_2}{\Delta y_1}$. Solving these equations for a, b, c gives

$$a = \frac{1+\alpha}{\alpha}$$
$$b = -\frac{1}{\alpha(1+\alpha)}$$
$$c = -\frac{2+\alpha}{1+\alpha}$$

The T.E. is $(1+\alpha)\frac{(\Delta y_1)^2}{6}\frac{\partial^3 T}{\partial y^3}$

Example 3

Consider the following FDE:

$$(u_x)_{i+1} + 4(u_x)_i + (u_x)_{i-1} = \frac{3}{h}(u_{i+1} - u_{i-1})$$

Find the truncation error. Hint: Using the following Taylor series expresssions:

$$(u_x)_{i+1} = (u_x)_i + (u_{xx})_i h + (u_{xxx})_i \frac{h^2}{2} + (u_{xxxx})_i \frac{h^3}{6} + (u_{xxxxx})_i \frac{h^4}{24} + \cdots$$
$$(u_x)_{i-1} = (u_x)_i - (u_{xx})_i h + (u_{xxx})_i \frac{h^2}{2} - (u_{xxxx})_i \frac{h^3}{6} + (u_{xxxxx})_i \frac{h^4}{24} - \cdots$$
$$u_{i+1} = u_i + (u_x)_i h + (u_{xx})_i \frac{h^2}{2} + (u_{xxx})_i \frac{h^3}{6} + (u_{xxxx})_i \frac{h^4}{24} + (u_{xxxxx})_i \frac{h^5}{120} + \cdots$$
$$u_{i-1} = u_i - (u_x)_i h + (u_{xx})_i \frac{h^2}{2} - (u_{xxx})_i \frac{h^3}{6} + (u_{xxxx})_i \frac{h^4}{24} - (u_{xxxxx})_i \frac{h^5}{120} + \cdots$$

we find

$$(u_x)_{i+1} + 4(u_x)_i + (u_x)_{i-1} = 6(u_x)_i + (u_{xxx})_i h^2 + (u_{xxxxx})\frac{h^4}{12} + \cdots$$
$$\frac{3}{h}(u_{i+1} - u_{i-1}) = \frac{3}{h}\left[2(u_x)_i h + (u_{xxx})_i \frac{h^3}{3} + (u_{\mathrm{xxxxx}})_i \frac{h^5}{60}\right] + \cdots$$

which simplifies to

$$(u_x)_{i+1} + 4(u_x)_i + (u_x)_{i-1} = \frac{3}{h}(u_{i+1} - u_{i-1}) + \frac{1}{30}(u_{\mathrm{xxxxx}})_i h^4$$

Hence the truncation error is $O\left(h^4\right)$

Example 4

Application of Taylor Table: Consider 3-point backward difference operator for $\frac{\partial u}{\partial x}$ up to order $\mathrm{O}(\Delta x)^p$. The idea is to determine values of $\alpha_{-2}, \alpha_0, \alpha_{-1}, \cdots$ such that

$$U_j' + \alpha_{-2}U_{j-2} + \alpha_{-1}U_{j-1} + \alpha_0 U_j + \cdots = O(\Delta x^p)$$

$U_j'-$ Differential operator of interest to have minimum truncation error $O(\Delta x^p)^*$

Similar to the example in the handout, let's say that we have 3 unknowns $\alpha_{-2}, \alpha_{-1}, \alpha_0$

	U_j	U'_j	$U_j^{(2)}$	$U_j^{(3)}$
U'_j	0	1	0	0
$\alpha_0 U_j$	α_0	0	0	0
$\alpha_{j-1} U_{j-1}$	α_{j-1}	$-\alpha_{j-1}\Delta x$	$\frac{1}{2}\alpha_{j-1}\Delta x^2$	$-\frac{1}{6}\alpha_{j-1}\Delta x^3$
$\alpha_{j-2} U_{j-2}$	α_{j-2}	$-\alpha_{j-2}2\Delta x$	$\frac{1}{2}\alpha_{j-2}(2\Delta x)^2$	$-\frac{1}{6}\alpha_{j-2}(2\Delta x)^3$
LHS	RHS1	RHS2	RHS3	RHS4

hence we can set three equations of RHS equal to zero (starting from 1^{st} column), i.e.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_{-2} \\ \alpha_{-1} \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\Delta x \\ 0 \end{bmatrix}$$

which gives $[\alpha_{-2}, \alpha_{-1}, \alpha_0] = -\frac{1}{2\Delta x}[1, -4, 3]$.

$$\left(\frac{\partial u}{\partial x}\right)_j \equiv U'_j = \frac{1}{2\Delta x}\left(u_{j-2} - 4u_{j-1} + 3u_j\right) + O\left(\Delta x^2\right)$$

Example 5

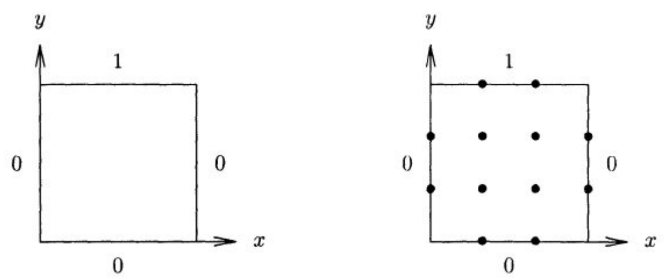

Figure 1: Laplace example (from Heath book)

Consider the Laplace equation on the unit square

$$u_{xx} + u_{yy} = 0, \quad 0 \le x \le 1, \quad 0 \le y \le 1$$

with boundary conditions as shown on the left in Fig. 2. Including boundaries, we define a uniform mesh in the domain, as shown on the right in Fig. 2. We wish compute the approximate solution at the interior points

$$\left(x_i, y_j\right) = (ih, jh), \quad i, j = 1, \ldots, n$$

where in our example $n = 2$ and $h = 1/(n+1) = 1/3$. We replace the second derivatives in the equation with the second-order centered difference approximation at each interior mesh point to obtain the finite difference equations

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} = 0, \quad i, j = 1, \ldots, n$$

where $u_{i,j}$ is an approximation to the true solution $u\left(x_i, y_j\right)$ and represents one of the given boundary values if i or j is 0 or $n+1$. Simplifying and writing out the resulting four equations explicitly, we obtain

$$\begin{aligned}
4u_{1,1} - u_{0,1} - u_{2,1} - u_{1,0} - u_{1,2} &= 0 \\
4u_{2,1} - u_{1,1} - u_{3,1} - u_{2,0} - u_{2,2} &= 0 \\
4u_{1,2} - u_{0,2} - u_{2,2} - u_{1,1} - u_{1,3} &= 0 \\
4u_{2,2} - u_{1,2} - u_{3,2} - u_{2,1} - u_{2,3} &= 0
\end{aligned}$$

Writing these four equations in matrix form, we obtain

$$\boldsymbol{Ax} = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{1,2} \\ u_{2,2} \end{bmatrix} = \begin{bmatrix} u_{0,1} + u_{1,0} \\ u_{3,1} + u_{2,0} \\ u_{0,2} + u_{1,3} \\ u_{3,2} + u_{2,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \boldsymbol{b}$$

The above system can be solved via Matlab yielding:

$$\boldsymbol{x} = \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{1,2} \\ u_{2,2} \end{bmatrix} = \begin{bmatrix} 0.125 \\ 0.125 \\ 0.375 \\ 0.375 \end{bmatrix}$$