The University of British Columbia
MECH 479/587

Module 2 Example Problems

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Review

1. First-order PDE system

$$\underset{\sim}{W} = \begin{pmatrix} u \\ v \end{pmatrix} \qquad \frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial y} \quad \ldots$$

$$\frac{\partial \underset{\sim}{W}}{\partial x} + \underset{\sim}{[A]} \frac{\partial \underset{\sim}{W}}{\partial y} = 0$$

2. Second-order PDE : u

$$\frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial^2 u}{\partial x \partial y} \quad \ldots$$

$$\Rightarrow a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial x \partial y} + c \frac{\partial^2 u}{\partial y^2} = \underbrace{d(x, y, \ldots)}_{\Downarrow}$$

$$\Downarrow$$

$$D = \underbrace{b^2 - 4ac}$$

$D > 0$ → Hyperbolic

$D = 0$ → Parabolic

$D < 0$ → Elliptic

Example 1

Consider the following PDE describing the advection behavior (first-order wave equation):

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0.$$

with the wave speed $c = 1$. The initial condition is $u = u_0(x)$ on $t = 0$

Solution: Loosely speaking, a characteristic is a curve or line along which a PDE reduces to an ODE. Suppose that one follows some particular path $x(t)$. Then, along that line, $u = u(x(t), t)$ is a function of t only and its total derivative with respect to t is (using chain-rule)

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}$$

Compare this with the original differential equation. It is exactly the same as the LHS of that equation provided that the chosen path satisfies

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1$$

or

$$x = t + \text{ const}$$

Along these lines, $\frac{\mathrm{d}u}{\mathrm{d}t} = 0$ (an ordinary differential equation). Hence, u propagates unchanged along the lines.

Example 2

The Lapalce equation can be written as first-order Cauchy-Riemann equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$\frac{\partial u}{\partial x} = +\frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

and

$$\frac{\partial \mathbf{w}}{\partial x} + [A]\frac{\partial \mathbf{w}}{\partial y} = 0$$

where

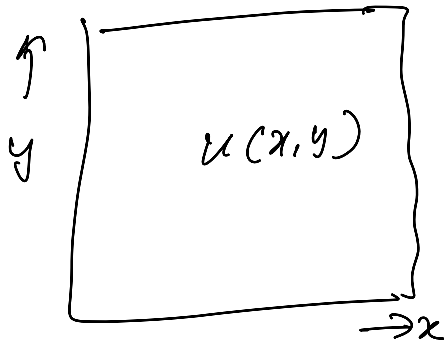
$$\mathbf{w} = \begin{bmatrix} u \\ v \end{bmatrix}$$

and

$$[A] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$b^2 - 4ac$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^{\boxed{2}} v}{\partial y^2} = 0$$



$\rightarrow$ $\dfrac{\partial u}{\partial x} = +\dfrac{\partial v}{\partial y}$ $\rightarrow$ Diff. w.r.t. x

$\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ $\rightarrow$ Diff. w.r.t y

Write in vector-Matrix form:

$$\underset{\sim}{w} = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\frac{\partial \underset{\sim}{w}}{\partial x} + [A]\frac{\partial \underset{\sim}{w}}{\partial y} = 0$$

$$[A] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$\det(A - \lambda I) \Rightarrow +\lambda^2 + 1 = 0$

$= \pm i$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \alpha\frac{\partial^2 u}{\partial x^2} \qquad b^2 - 4ac = 0$$

Example 3

Show that the one-dimensional Navier-Stokes equation without pressure gradient (known as the 'viscous' Burger's equation) $\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \alpha\frac{\partial^2 u}{\partial x^2}}$ is parabolic in x, t.

Compare with the form $a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x y} + c\frac{\partial^2 u}{\partial y^2} = d$, we have $a = -\alpha, b = 0, c = 0$. Because $b^2 - 4ac = 0$, the given PDE is parabolic in x, t.

Compare

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \alpha\frac{\partial^2 u}{\partial x^2}$$

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = d$$

$$a = -\alpha, \quad b = 0, \quad c = 0$$

Discriminant : $b^2 - 4ac$

$$0 - 4(-\alpha)0 = 0$$

$$\boxed{b^2 - 4ac = 0} \Rightarrow \text{Parabolic}$$

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{2}\dfrac{\partial u}{\partial x} + 1\dfrac{\partial v}{\partial x} = 0 & \text{--- } ① \\[2ex] \dfrac{\partial v}{\partial t} + \dfrac{\partial u}{\partial x} + \dfrac{1}{2}\dfrac{\partial v}{\partial x} = 0 & \text{--- } ② \end{cases}$$

$$\underset{\sim}{W} = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad (t, x)$$

$$\frac{\partial}{\partial t}\begin{bmatrix} u \\ v \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{bmatrix}}_{[A]} \frac{\partial}{\partial x}\begin{Bmatrix} u \\ v \end{Bmatrix} = 0$$

$$\frac{\partial \underset{\sim}{W}}{\partial t} + [A]\frac{\partial W}{\partial x} = 0$$

(a) Find eigenvalues :

$$\det(A - \lambda I) = 0$$

$$\left| \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} \frac{1}{2} - \lambda & 1 \\ 1 & \frac{1}{2} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \left(\frac{1}{2} - \lambda\right)^2 - 1 = 0$$

$$\underline{\lambda_1} = \frac{3}{2}, \quad \lambda_2 = -\frac{1}{2}$$

<u>Hyperbolic</u> !

(ii) Eigenvectors

$$A X_1 = \underline{\lambda_1} X_1$$

$$\Rightarrow X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A X_2 = \lambda_2 X_2$$

$$\Rightarrow X_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\underset{\sim}{Q}, \quad \underset{\sim}{S}, \quad \underset{\sim}{L}$

$$\underset{\sim}{L} = \begin{bmatrix} \vdots & \vdots \\ X_1 & X_2 \\ \vdots & \vdots \end{bmatrix}$$

$$= \begin{bmatrix} \frac{X_1}{\|X_1\|} & \frac{X_2}{\|X_2\|} \end{bmatrix}$$

$$\underset{\sim}{L} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

This is eigenvector matrix !

$$L^{-1}\left(\frac{\partial \underset{\sim}{w}}{\partial t} + \underline{[A]} \frac{\partial w}{\partial x} = 0 \right)$$

$$\frac{\partial (L^{-1} w)}{\partial t} + L^{-1} A \frac{\partial w}{\partial x} = 0$$

$I = L L^{-1}$

$$\Rightarrow \frac{\partial}{\partial t}(L^{-1} w) + L^{-1} A L L^{-1} \frac{\partial w}{\partial x} = 0$$

$$\frac{\partial}{\partial t}\underbrace{(L^{-1}W)} + \underbrace{L^{-1}AL}\frac{\partial(L^{-1}W)}{\partial x} = 0$$

$$\Downarrow$$

$$L^{-1}AL = \Lambda$$

$$W = \begin{pmatrix} u \\ v \end{pmatrix} \qquad = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$\boxed{\underbrace{L^{-1}W}}$ $\Rightarrow$ Characteristic variables

$\underline{\Phi_1}$ $\quad$ $\underline{\Phi_2}$

$$\rightarrow \left.\begin{aligned} \frac{\partial \Phi_1}{\partial t} + \underline{\lambda}_1 \frac{\partial \Phi_1}{\partial x} &= 0 \\ \frac{\partial \Phi_2}{\partial t} + \lambda_2 \frac{\partial \Phi_2}{\partial x} &= 0 \end{aligned}\right\} \begin{aligned} &\to \Phi_1 \\ &\to \Phi_2 \end{aligned}$$

$$\left.\begin{aligned}\phi_1 &= \frac{1}{\sqrt{2}}(u-v)\\ \phi_2 &= \frac{1}{\sqrt{2}}(u+v)\end{aligned}\right\}$$

These are two characteristic variables!

Example 3

Consider the system

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 0 \qquad — \quad (1)$$

$$\frac{\partial v}{\partial t} + \frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial v}{\partial x} = 0 \qquad — \quad (2)$$

(a) Write the system in matrix form and obtain the matrix A:

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = 0$$

$$\underbrace{U = \begin{bmatrix} u \\ v \end{bmatrix}}$$

$$\frac{\partial U}{\partial t} + \underbrace{\begin{bmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{bmatrix}}_{A} \frac{\partial}{\partial x}\begin{bmatrix} u \\ v \end{bmatrix} = 0$$

(b) Find the eigenvalues of A and show that the system is hyperbolic.

$$\det(A - \lambda I) = 0 \qquad \boxed{\underbrace{\lambda_1 = \frac{3}{2}, \quad \lambda_2 = -\frac{1}{2}}_{\text{Real Eigenvalues}}}$$

Hyperbolic: Directions / Characteristic can be found!

(c) Derive the left and right eigenvectors and obtain the matrix L which diagonalizes A. Let's obtain the characteristic variables as follows:

$$\begin{cases} A X_1 = \lambda_1 X_1 \Rightarrow X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ A X_2 = \lambda_2 X_2 \Rightarrow X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{cases}$$

$$\boxed{L} = \left[\frac{X_1}{\|X_1\|}, \frac{X_2}{\|X_2\|} \right] = \boxed{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}$$

$$\rightarrow \quad \underline{\frac{\partial (L^{-1} U)}{\partial t} + L^{-1} A L \frac{\partial (L^{-1} U)}{\partial x} = 0}$$

Characteristic variables are:

$\boxed{L^{-1} U}$ will give two sets of characteristic variables

$$\left.\begin{array}{l} \phi_1 \rightarrow \frac{1}{\sqrt{2}} (u - v) \\ \phi_2 \rightarrow \frac{1}{\sqrt{2}} (u + v) \end{array}\right\}$$

characteristic Eqns

$$\left\{\boxed{\begin{array}{l} \frac{\partial \phi_1}{\partial t} + \lambda_1 \frac{\partial \phi_1}{\partial x} = 0 \\ \frac{\partial \phi_2}{\partial t} + \lambda_2 \frac{\partial \phi_2}{\partial x} = 0 \end{array}}\right.$$

$AX = \lambda X$

$$A \Rightarrow L^{-1} A L = \Lambda$$

Eigenvalue Decomposition (Review)

$$\underset{\sim}{A}\underset{\sim}{x} = \lambda \underset{\sim}{x}$$

Consider square matrix & positive definite

$$\begin{bmatrix} & & \\ & \underline{A} & \\ & & \end{bmatrix}_{m \times m} \underset{\sim}{x} = \lambda \underset{\sim}{x} \qquad \underset{\sim}{x} = \text{eigenvector}$$

Eigenvector matrix: $\underset{\sim}{Q} = \begin{bmatrix} \vdots & \vdots & \vdots \\ \underset{\sim}{x}_1 & \underset{\sim}{x}_2 & \underset{\sim}{x}_3 \\ | & | & \vdots \end{bmatrix}$

Multiply with $\underset{\sim}{Q}^{-1}$:

$$\underset{\sim}{Q}^{-1}\left(\underset{\sim}{A}\,\underset{\sim}{Q} = \Lambda \underset{\sim}{Q}\right)$$

$$\Downarrow$$

$$\underset{\sim}{A} = Q^{-1} \Lambda Q \qquad \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \end{bmatrix}$$

Spectral Theorem

(Eigenvalue Decomposition)

Example 6

Classify the system of equations:

$$\frac{\partial}{\partial t}\begin{bmatrix} u \\ v \end{bmatrix} + \left(\frac{\partial}{\partial x}\right)\underbrace{\begin{bmatrix} 0 & 8 \\ 2 & 0 \end{bmatrix}}_{A}\begin{bmatrix} u \\ v \end{bmatrix} = \boxed{\begin{aligned} \frac{\partial u}{\partial t} + 8\frac{\partial v}{\partial x} &= 0 \\ \frac{\partial v}{\partial t} + 2\frac{\partial u}{\partial x} &= 0 \end{aligned}}$$

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 8 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$$

$$\Rightarrow \det(A - \lambda I) = 0$$

$$\lambda_1 = 4, \quad \lambda_2 = -4$$

Two real eigenvalues

$\Rightarrow$ Hyperbolic

$$\left.\begin{aligned} \frac{\partial \phi_1}{\partial t} + \lambda_1 \frac{\partial \phi_1}{\partial x} &= 0 \\ \frac{\partial \phi_2}{\partial t} + \lambda_2 \frac{\partial \phi_2}{\partial x} &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} &\phi_1 \\ &\phi_2 \end{aligned}\right\}$$

$$a_1 = 1, \quad b_1 = 0, \quad c_1 = 0, \quad d_1 = 8$$
$$a_2 = 0, \quad b_2 = 2, \quad c_2 = 1, \quad d_2 = 0$$

$$N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 8 \\ 2 & 0 \end{bmatrix} \tag{1}$$

Solve $\det(N - \lambda I) = 0$, we have $\lambda_1 = 4$, $\lambda_2 = -4$. Two real eigenvalues, the system of equations is hyperbolic.

Example 7

Determine the mathematical character of the equations given by below:

$$\beta^2 \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$
$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

$$A = \begin{bmatrix} 0 & -\frac{1}{\beta^2} \\ -1 & 0 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

if $\beta = 0$, $v = v(x)$. Along the characteristics $x = const$ the value of v keeps the same. $-\frac{\partial u}{\partial y} = \frac{\partial v(x)}{\partial x}$ becomes an ODE along characteristics $y = const$. So the system is hyperbolic

if $\beta \neq 0$, we have

$$A = \begin{bmatrix} 0 & -\frac{1}{\beta^2} \\ -1 & 0 \end{bmatrix} \tag{2}$$

$\det(A - \lambda I) = \lambda^2 - \frac{1}{\beta^2} = 0$. Because $\frac{1}{\beta^2} > 0$, we have two real eigenvalues. So the system is hyperbolic.

Example 8

Given the following second order partial differential equation

$d(\cdot\cdot\cdot)$

$$1\,\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial x \partial y} - \frac{1}{2}\frac{\partial^2 u}{\partial y^2} + \boxed{\left(\frac{\partial u}{\partial y}\right)^2 - 2\frac{\partial u}{\partial x} + 7} = 0$$

(a) Classify the equation (hyperbolic, parabolic, or elliptic).

Compare with the form $a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial xy} + c\frac{\partial^2 u}{\partial y^2} = \boxed{d}$, $a = 1, b = 3, c = -\frac{1}{2}$

$b^2 - 4ac > 0$. The equation is hyperbolic.

$b^2-4ac > 0$

(b) Write it as a system of first order equations

Define $\phi = \frac{\partial u}{\partial x}$ and $\psi = \frac{\partial u}{\partial y}$:

$$\frac{\partial \phi}{\partial x} + 3\frac{\partial \phi}{\partial y} - \frac{1}{2}\frac{\partial \psi}{\partial y} + \psi^2 - 2\phi + 7 = 0$$
$$\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} = 0$$

Example 9

Consider the following quasi-linear partial differential equation:

$$x\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=0$$

Suppose the domain is $x \geq 0, t \geq 0$.. Write down the characteristic equation.

Solution:

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0$$

$$\frac{dx}{dt}=c$$

$$\frac{\partial u}{\partial t}+\frac{1}{x}\frac{\partial u}{\partial x}=0$$

$$\rightarrow \frac{\partial u}{\partial t}+\frac{dx}{dt}\frac{\partial u}{\partial x}=0$$

Characteristic Eqn:

$$\frac{dx}{dt}=\frac{1}{x}$$

$$\Rightarrow \int x\,dx=\int dt$$

$$\boxed{\frac{x^2}{2}=t+c}$$

$u = \text{Const}$

Consider:

$$e^x \frac{\partial u}{\partial t} + t \frac{\partial u}{\partial x} = 0$$

Write down characteristic eq^n:

$$\frac{dx}{dt} = \frac{t}{e^x}$$

$$\rightarrow \boxed{\frac{t^2}{2} = e^x + c}$$

Example 10

Consider the ODE: $\boxed{\frac{du}{dt} = -u}$ with initial condition $u(0) = 1$. The exact solution is

$$u(t) = e^{-t}$$

Forward Euler:
$u^{n+1} = u^n - u^n\Delta t \to \boxed{u^{n+1} = (1-\Delta t)u^n}$
$\left|\frac{u^{n+1}}{u^n}\right| \leq 1$ only if $\Delta t \leq 2$.

We can write a numerical sequence for the Forward Euler method:

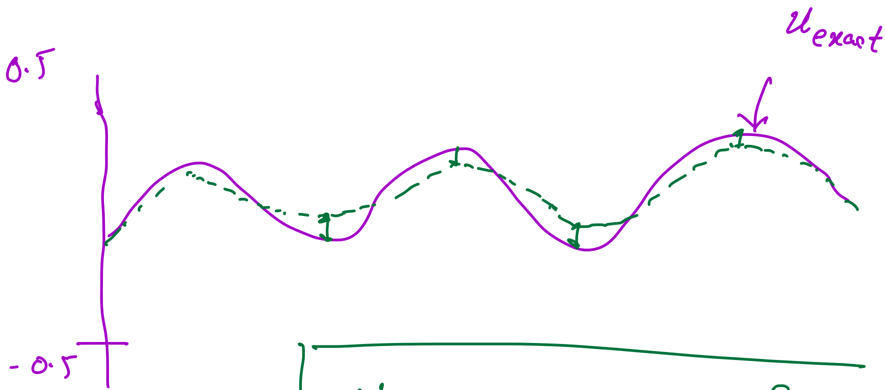
$$\begin{aligned} u^{n+1} &= (1-\Delta t)u^n = (1-\Delta t)^2 u^{n-1} \\ &= \ldots\ldots\ldots\ldots = (1-\Delta t)^n u^1 \end{aligned}$$

Obviously, u oscillates unless $\Delta t \leq 1$

Now let's use Backward Euler:

$$u^{n+1} = u^n - u^{n+1}\Delta t \quad \to \quad \boxed{\frac{u^{n+1}}{u^n} = \frac{1}{(1+\Delta t)}}$$

$$\left|\frac{u^{n+1}}{u^n}\right| \leq 1 \quad \text{for all } \Delta t$$



$$\text{Error} = \frac{\sqrt{\sum_{j=1}^{N}\left(u_j - u_{exact}\right)^2}}{N}$$

$$N = \frac{T}{\Delta t} \qquad\qquad p = 2$$

$$E = C(\Delta t)^{p} = C\left(\frac{1}{\Delta t}\right)^{-p}$$

Taking the log:

$$\ln E = \ln\left(C\left(\frac{1}{\Delta t}\right)^{-p}\right)$$

$$= \ln C - \underline{p}\,\ln\left(\frac{1}{\Delta t}\right)$$