MECH 479/587

Computational Fluid Dynamics

Module 2b: PDE Classification

Rajeev K. Jaiman

Winter Term 1, 2022

What is PDE?

- A differential equation that contains unknown multivariable functions and their partial derivatives
  - PDEs are distinguished from ODEs by the fact that they contain derivatives more than one independent variable

Consider a scalar partial differential equation (PDE) of the general form

A general implicit form of PDE is:

$$F\left(x, y, \ldots, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \ldots, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial xy}, \frac{\partial^2 u}{\partial y^2}, \ldots\right) = 0$$

where $x, y, \ldots$ are the indepdenent variables, and $u = u(x, y, \ldots)$ is the dependent variable.

Classification of PDEs

- PDEs are classified based on the mathematical concept of characteristics that are lines (2D) or surfaces (3D) along which certain properties remain constant
  - Certain derivatives may be discontinuous
  - Characteristics are related in which “information” can be transmitted in physical system

- PDEs can be classified into hyperbolic, parabolic and elliptic ones
  - Each class of PDEs models a different kind of physical processes
  - Number of initial/boundary conditions depends on the PDE type
  - Different solution methods are required for PDEs of different type

Classification of PDEs

- <u>Hyperbolic equations</u>
  - Information propagates in certain directions at finite speeds; the solution is a superposition of multiple simple waves
    - *Wave-like* solutions
- <u>Parabolic equations</u>
  - Information travels downstream/forward in time; the solution can be constructed using a marching/time-stepping method
    - *Damped wave-like*
- <u>Elliptic equations</u>
  - Information propagates in all directions at infinite speed; describe equilibrium phenomena (unsteady problems are never elliptic)
    - *Not wave-like*

Some Examples

- 1D scalar equations

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0 \qquad \text{Advection}$$

$$\frac{\partial \phi}{\partial t} - \nu\frac{\partial^2 \phi}{\partial x^2} = 0 \qquad \text{Diffusion}$$

$$\frac{\partial^2 \phi}{\partial t^2} - \nu\frac{\partial^2 \phi}{\partial x^2} = 0 \qquad \text{Wave propagation}$$

Evolution in time

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \text{Laplace equation}$$

Link with the 2D Navier-Stokes Equation

- X-momentum

Diffusion

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

Advection part

- Continuity

$$\boxed{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0}$$

Laplace-type

Quasi-Linear 1st Order PDEs

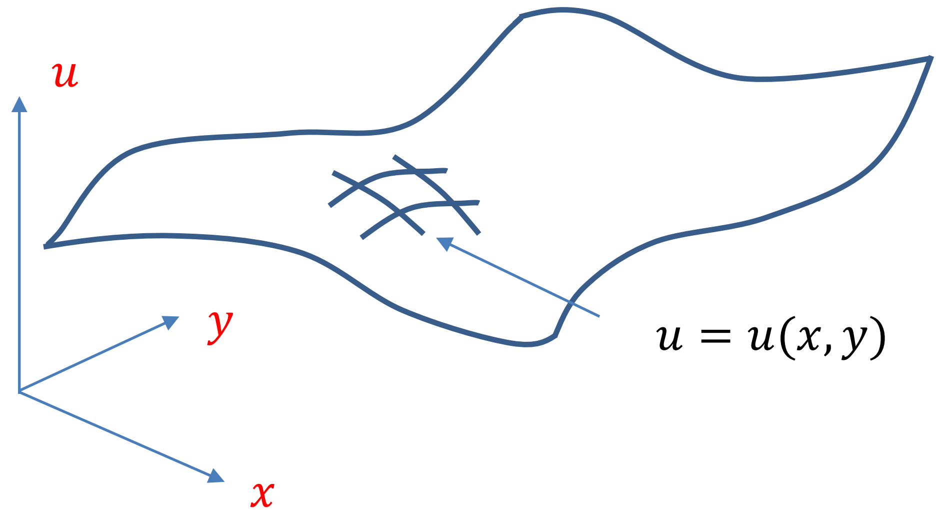
- Quasi-linear first order equation

$$a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}=c$$

$$a=a\left(x,y,u\right)$$

$$b=b\left(x,y,u\right)$$

$$c=c\left(x,y,u\right)$$



Quasi-Linear PDE

- Arbitrary change in u is given by

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

- The original equation and the condition for an infinitesimal change

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c \quad \Rightarrow \boxed{\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, -1\right)} \cdot (a, b, c) = 0$$

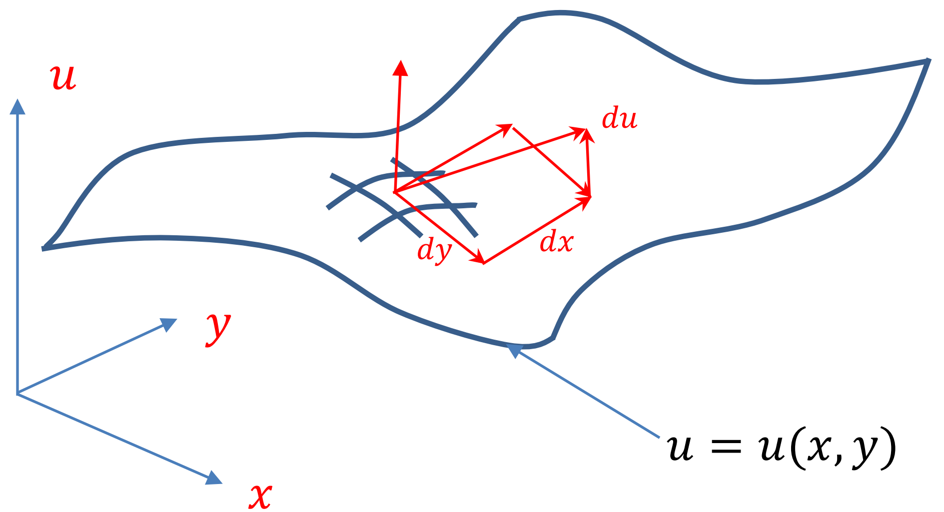
$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \quad \Rightarrow \boxed{\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, -1\right)} \cdot (dx, dy, du) = 0$$

Normal to the surface

- Both (a, b, c) and (dx, dy, du) lie on the surface
  - The vector field (a, b, c) is tangent to the surface u

Visual Inspection

- The solution of this equation defines a single valued surface *u(x,y)* in three-dimensional space:



- Total derivative

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

Characteristics

- Considering a perturbation in the direction of (a, b, c) with arbitrary scaling *ds*

$$(dx, dy, du) = ds(a, b, c)$$

- Separating the components will give the characteristics

$$\Rightarrow \underbrace{\frac{dx}{ds} = a; \quad \frac{dy}{ds} = b}_{\frac{dx}{dy} = \frac{a}{b}}; \quad \frac{du}{ds} = c$$

- Given the initial conditions, the equations can be integrated in time
  - ▶ These integral curves are the characteristics

Characteristics Cont'd

- Three equations specify in the *x-y* plane

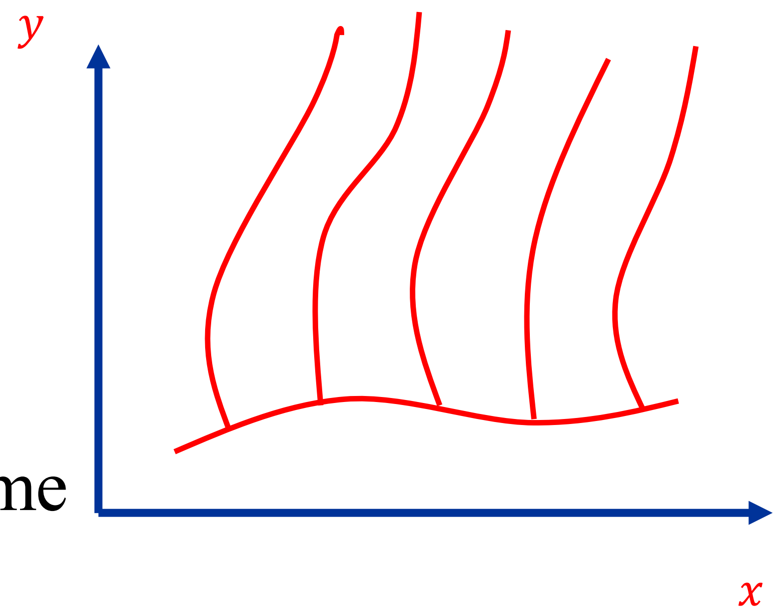
$$\frac{dx}{ds} = a; \quad \frac{dy}{ds} = b; \quad \frac{du}{ds} = c$$

Given initial conditions:

$$x = x(s, t_0); \; y = y(s, t_0); \quad u = u(s, t_0)$$

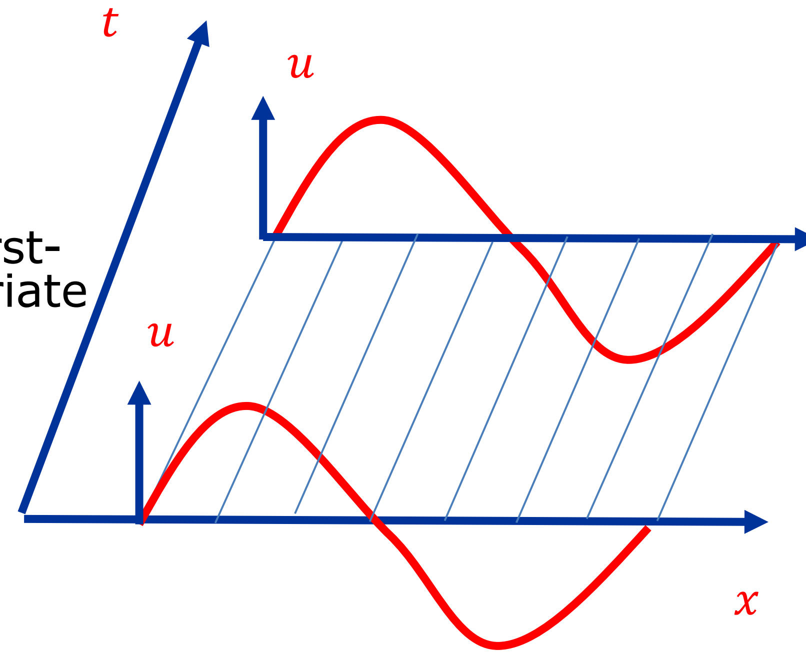
the equations can be integrated in time

slope $\dfrac{dx}{dy} = \dfrac{a}{b}$



- Along the characteristics, the original PDE becomes the ODE

Example: Linear Advection Equation



- Goal: We want to transform this linear first-order PDE into an ODE along the appropriate curve

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

- Characteristics are given by

$$\frac{dt}{ds} = 1; \quad \frac{dx}{ds} = c; \quad \frac{du}{ds} = 0;$$

$$\frac{dx}{dt} = c; \quad du{=}0;$$

Likewise previous example:

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx$$

$$\Rightarrow (dt, dx, du) = ds(1, c, 0)$$

- Remarks:
  - ▶ Solution moves along straight characteristics
  - ▶ Solution remains constant

Example: Linear Advection Equation

- Along the characteristics, the solution remains constant
  - Along the the characteristics $(x(s), t(s))$, the original PDE becomes the ODE
- To determine the general solution, it is enough to find the characteristics by solving the characteristic system of ODEs

- $\dfrac{dt}{ds} = 1$ letting $t(0) = 0$ we know $t = s$
- $\dfrac{dx}{ds} = c$ letting $x(0) = x_0$ we know $x = cs + x_0$
- $\dfrac{du}{ds} = 0$ letting $\mathrm{u}(0) = g(x_0)$ we know $u(x(t), t) = g(x_0) = g(x - ct)$

Linear Advection Equation

- ❑ Graphically

$$u(x,t) = g(x - ct) \quad \text{where } g(x) = u(x, t = 0)$$

Verify:

Set $\alpha(x,t) = x - ct$

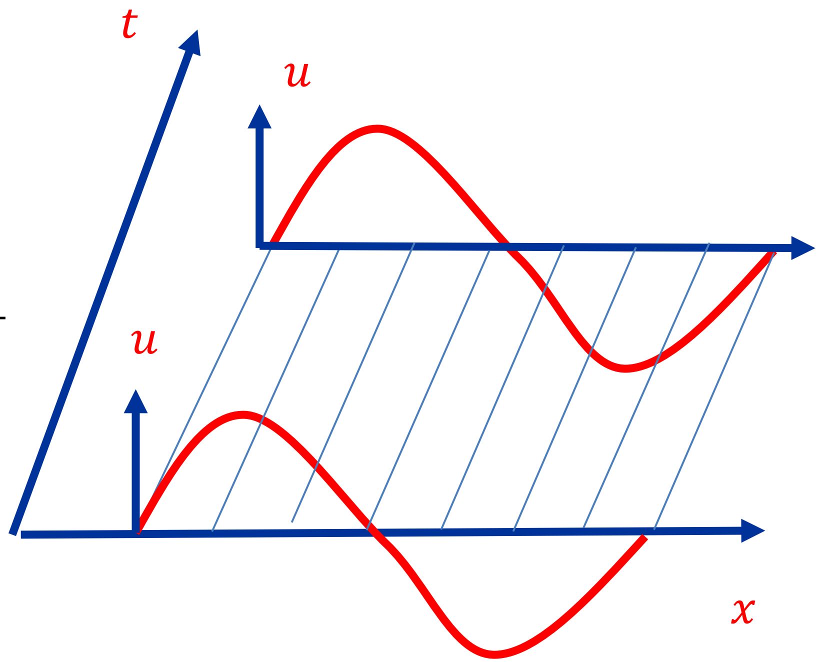
Then

$$\frac{\partial u}{\partial t} = \frac{\partial g}{\partial \alpha}\frac{\partial \alpha}{\partial t} = \frac{\partial g}{\partial \alpha}(-c) = -c\frac{\partial g}{\partial \alpha}$$

$$\frac{\partial u}{\partial x} = \frac{\partial g}{\partial \alpha}\frac{\partial \alpha}{\partial x} = \frac{\partial g}{\partial \alpha}\frac{\partial (x - ct)}{\partial x} = \frac{\partial g}{\partial \alpha}\left[\underbrace{\frac{\partial x}{\partial x}}_{1} - \underbrace{\frac{\partial (ct)}{\partial x}}_{0}\right] = \frac{\partial g}{\partial \alpha}$$

Substitute into the original equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$



- ❑ Solution:
  - ► Waves travel with constant speed
  - ► Preserve the initial waveform

Some Examples

- Add a source term

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = u$$

- Nonlinear (quasi-linear) advection equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

- Advection-diffusion (Burger) equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \epsilon\frac{\partial^2 u}{\partial x^2} = 0$$

Classification of PDE using eigenvalue method

We shall now discuss how we can classify a given PDE. Let us begin with a general system of PDE in 2-D given as

$$a_1 \frac{\partial u}{\partial x} + b_1 \frac{\partial u}{\partial y} + c_1 \frac{\partial v}{\partial x} + d_1 \frac{\partial v}{\partial y} = 0$$

$$a_2 \frac{\partial u}{\partial x} + b_2 \frac{\partial u}{\partial y} + c_2 \frac{\partial v}{\partial x} + d_2 \frac{\partial v}{\partial y} = 0$$

We can rewrite this as

$$\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} \frac{\partial \mathbf{W}}{\partial x} + \begin{bmatrix} b_1 & d_1 \\ b_2 & d_2 \end{bmatrix} \frac{\partial \mathbf{W}}{\partial y} = 0 \quad \text{where} \quad \mathbf{W} = \begin{pmatrix} u \\ v \end{pmatrix}$$

We simplify this equation

$$\frac{\partial \mathbf{W}}{\partial x} + \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}^{-1} \begin{bmatrix} b_1 & d_1 \\ b_2 & d_2 \end{bmatrix} \frac{\partial \mathbf{W}}{\partial y} = 0$$

Eigenvalue method

We simplify this equation

$$\frac{\partial \mathbf{W}}{\partial x}+\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}^{-1}\begin{bmatrix} b_1 & d_1 \\ b_2 & d_2 \end{bmatrix}\frac{\partial \mathbf{W}}{\partial y}=0$$

We define the matrix $\mathbf{N}=\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}^{-1}\begin{bmatrix} b_1 & d_1 \\ b_2 & d_2 \end{bmatrix}$ we determine the eigenvalues of matrix

and the classification is

If all eigenvalues are **real**, then the system is **hyperbolic;**
If all eigenvalues are **complex**, then the system is **elliptic;**
If the eigenvalues are mixed real and complex values,
then the system is mixed **elliptic-hyperbolic**.

Quasi-linear 2nd Order PDEs

- Consider

$$a\frac{\partial^2 u}{\partial x^2}+b\frac{\partial^2 u}{\partial x\partial y}+c\frac{\partial^2 u}{\partial y^2}=d$$

where

$$a=a\left(x,y,u,u_x,u_y\right)$$

$$b=b\left(x,y,u,u_x,u_y\right)$$

$$c=c\left(x,y,u,u_x,u_y\right)$$

$$d=d\left(x,y,u,u_x,u_y\right)$$

Second-Order PDE

- First write the 2nd order PDE as a system of first order equations
- Define

$$\phi = \frac{\partial u}{\partial x} \quad \text{and} \quad \psi = \frac{\partial u}{\partial y}$$

- Then

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = d \quad \rightarrow \quad a\frac{\partial \phi}{\partial x} + b\frac{\partial \phi}{\partial y} + c\frac{\partial \psi}{\partial y} = d$$

- Other equation can be:

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \rightarrow \frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} = 0$$

Transformed Equations

- We can transform 2nd order PDE

$$a\frac{\partial^2 u}{\partial x^2}+b\frac{\partial^2 u}{\partial x\partial y}+c\frac{\partial^2 u}{\partial y^2}=d$$

into first order PDE

$$a\frac{\partial \phi}{\partial x}+b\frac{\partial \phi}{\partial y}+c\frac{\partial \psi}{\partial y}=d$$

$$\frac{\partial \psi}{\partial x}-\frac{\partial \phi}{\partial y}=0$$

In the matrix form

$$\begin{pmatrix}\dfrac{\partial \phi}{\partial x}\\[2ex] \dfrac{\partial \psi}{\partial x}\end{pmatrix}+\begin{pmatrix} b/a & c/a \\ -1 & 0\end{pmatrix}\begin{pmatrix}\dfrac{\partial \phi}{\partial y}\\[2ex] \dfrac{\partial \psi}{\partial y}\end{pmatrix}=\begin{pmatrix} d/a \\ 0\end{pmatrix}$$

$$\Rightarrow \mathbf{W}_x+\mathbf{A}\mathbf{W}_y=\mathbf{S} \quad \text{where } \mathbf{W}=\begin{pmatrix}\phi\\ \psi\end{pmatrix}$$

Characteristic Lines

- The total derivative

$$\frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial y}{\partial x}\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial x} + \alpha\frac{\partial \phi}{\partial y}$$

- Are there lines along which the solution can be determined by ODE's?

$$\mathrm{m}_1\left(\frac{\partial \phi}{\partial x} + \frac{b}{a}\frac{\partial \phi}{\partial y} + \frac{c}{a}\frac{\partial \psi}{\partial y}\right) + m_2\left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) = m_1\frac{d}{a}$$

- Will the above equation will be equal to?

$$\mathrm{m}_1\left(\frac{\partial \phi}{\partial x} + \alpha\frac{\partial \phi}{\partial y}\right) + m_2\left(\frac{\partial \psi}{\partial x} + \alpha\frac{\partial \psi}{\partial y}\right) = m_1\frac{d}{a}$$

Determinant Condition (1)

- Characteristic lines exist if

$$m_1 \frac{b}{a} - m_2 = m_1 \alpha$$

$$m_1 \frac{c}{a} = m_2 \alpha$$

- In the matrix form:

$$\begin{pmatrix} \frac{b}{a} - \alpha & -1 \\ \frac{c}{a} & -\alpha \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left(A^T - \alpha I \right) m = 0$$

- The determinant condition is:

$$\det\left(A^T - \alpha I \right) = 0$$

Determinant Condition (2)

- Determinant

$$\det\left(A^T - \alpha I\right) = 0$$

$$-\alpha\left(\frac{b}{a} - \alpha\right) + \frac{c}{a} = 0$$

- Solving:

$$\alpha = \frac{1}{2a}\left(b \pm \sqrt{b^2 - 4ac}\right)$$

- PDE types:

$\sqrt{b^2 - 4ac} > 0$ Two real characteristics

$\sqrt{b^2 - 4ac} = 0$ One real characteristic

$\sqrt{b^2 - 4ac} < 0$ No real characteristics

Example 1:

$$\frac{\partial^2 u}{\partial x^2} - c^2 \frac{\partial^2 u}{\partial y^2} = 0$$

- Comparing

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = d$$

- Gives: $a = 1;\ \ b = 0;\ \ c = \ -c^2;\ \ d = 0;$

$$b^2 - 4ac = 0^2 + 4.1.c^2 = 4c^2 > 0$$

Hyperbolic

Example 2:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

- Comparing

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = d$$

- Gives: $a = 1; \quad b = 0; \quad c = 1; \quad d = 0;$

$$b^2 - 4ac = 0^2 - 4.1.1 = -4 < 0$$

Elliptic

Example 3:

$$\frac{\partial u}{\partial x} - k\frac{\partial^2 u}{\partial y^2} = 0$$

- Comparing

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = d$$

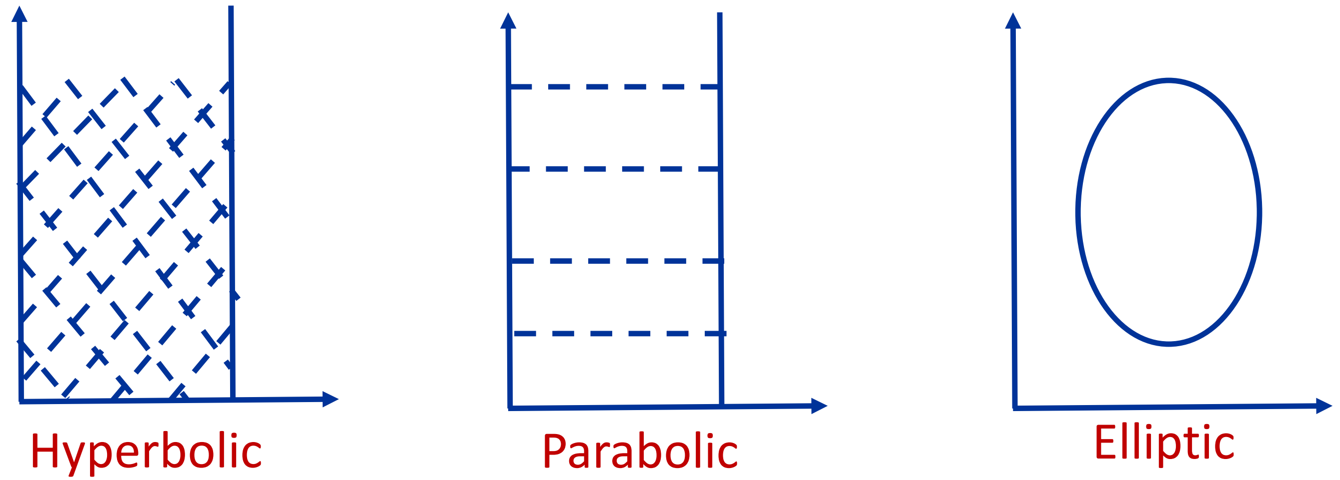
- Gives: $a = 0;\quad b = 0;\quad c = -k;\quad d = 0;$

$$b^2 - 4ac = 0^2 + 4.0.1 = 0$$

Parabolic

Relevance of PDE Classification

- ❑ Proper specifications of initial and boundary conditions
- ❑ Exploring and understanding different physical behaviors
- ❑ Development of appropriate numerical techniques



Link with the 2D Navier-Stokes Equation

- X-momentum

Parabolic

$$\underbrace{\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}}_{\text{Hyperbolic part}}+\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

- Continuity

$$\boxed{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0}$$

Elliptic

Note on the Navier-Stokes Equations

- The Navier-Stokes equations are <u>mixed</u> and they contain the parabolic, hyperbolic and elliptic characteristics behavior

- The different types of the equation require specialized techniques.

- Based on the underlying physical parameters, one behavior can dominant
  - For example, for inviscid compressible flow, only the hyperbolic part sustains

Example: The Navier-Stokes Equations

- Model problem for incompressible flow equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \bullet \nabla \mathbf{u} + \frac{\nabla p}{\rho} = \nu \Delta \mathbf{u}$$

$$\nabla \bullet \mathbf{u} = \mathbf{0}$$

These equations can be split into three pieces

$$Hyperbolic: \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \bullet \nabla \mathbf{u} = \mathbf{0}$$

$$Parabolic: \frac{\partial \mathbf{u}}{\partial t} = \nu \Delta \mathbf{u}$$

$$Elliptic: \ \Delta p = \nabla \bullet \left(-\mathbf{u} \bullet \nabla \mathbf{u} + \nu \Delta \mathbf{u} \right)$$

Boundary Conditions (B.C.):

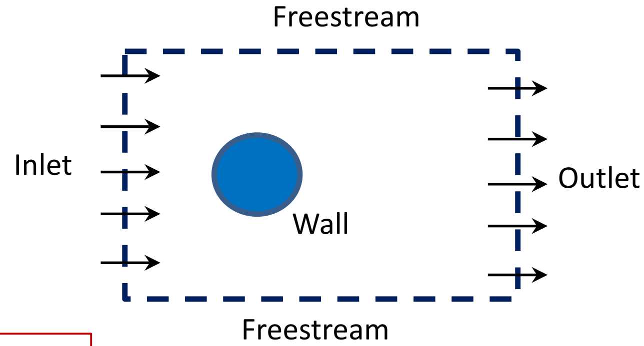
For **viscous** flow, we have $\mathbf{u}_{\text{fluid}} = \mathbf{u}_{\text{wall}}$ for the fluid on the solid wall.

This condition is known as no-slip condition.

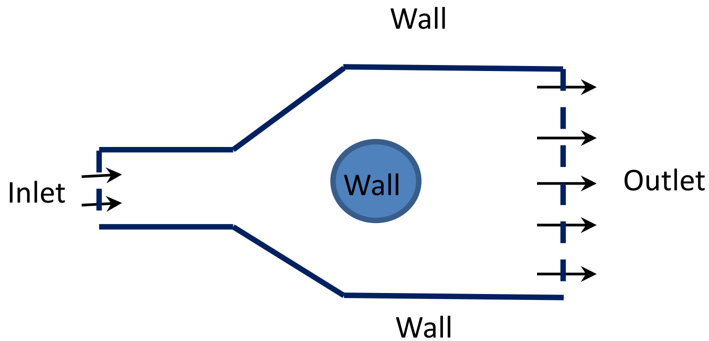
For **inviscid** flow, $(\mathbf{u}_{\text{fluid}}) \cdot (\mathbf{n}_{\text{wall}}) = 0$. This is called the no-penetration condition, or slip condition.

Initial Conditions (I.C.):

We need to provide $\mathbf{u}$ at time $t = 0$.



External flow



Internal flow

Well-posed Problem

Definition: A mathematical problem is well-posed when there exists one solution to the problem (Existence), which must also be the only solution (Uniqueness) and depends continuously on all the given data (Continuity).

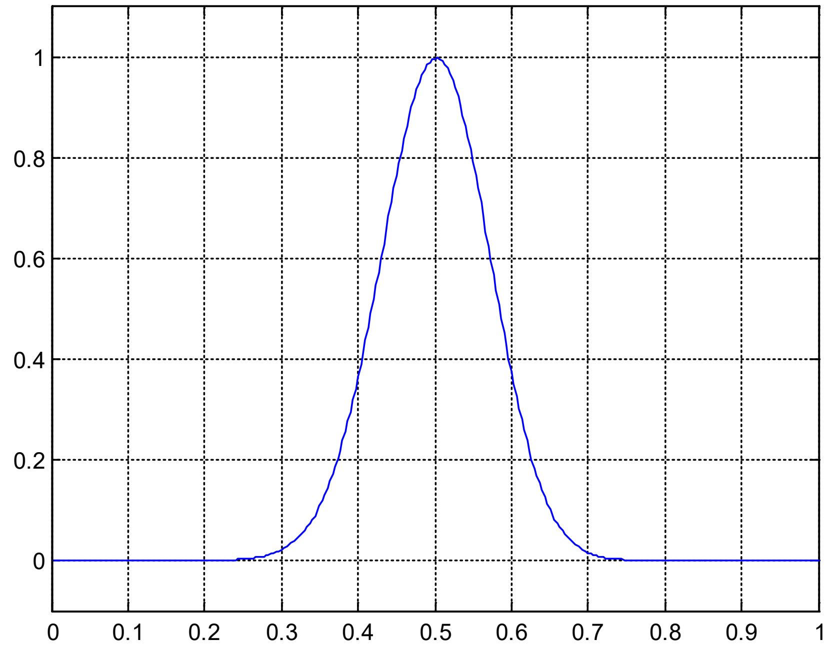
To determine whether or not a problem is well-posed, we must consider the governing equations, boundary conditions and initial conditions.

Demo: PDE Classification

- Matlab code

```
% PDE demo - advection, diffusion, dissipation

% Discretization
m = 300;
h = 1 / m;
x = h * (1:m)';
…
…
```



Rajeev K. Jaiman
Email: rjaiman@mech.ubc.ca