SIMPLE method for Navier-Stokes equations with implicit time stepping

Mech587 Project #3

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Project 3 Cheat sheet

- This is a summary for the equations you need to solve:

1. Predictor for velocity (Take u velocity as an example)

$$\boxed{\left(\frac{\boldsymbol{I}}{\Delta t}-\frac{0.5}{Re}\right)u_k^* = \frac{\boldsymbol{I}}{\Delta t}u^n - \left(1.5\boldsymbol{A}(u^n) - 0.5\boldsymbol{A}(u^{n-1})\right) + \frac{0.5}{Re}\boldsymbol{D}(u^n) - \boldsymbol{\delta_x}(p_k^n)}$$

2. Solve for pressure correction (with u_k^*, p_k^n)

$$\boxed{\begin{aligned}&-\frac{1}{a}\left(\frac{p'_{i+1}-2p'_{i}+p'_{i-1}}{\Delta x^2}+\frac{p'_{j+1}-2p'_{j}+p'_{j-1}}{\Delta y^2}\right)\\&=-\left(\frac{u_{i+1}^*-u_{i-1}^*}{2\Delta x}+\frac{p_{i+2}^n-4p_{i+1}^n+6p_i^n-4p_{i-1}^n+p_{i-2}^n}{4a(\Delta x)^2}\right)\\&-\left(\frac{v_{j+1}^*-v_{j-1}^*}{2\Delta y}+\frac{p_{j+2}^n-4p_{j+1}^n+6p_j^n-4p_{j-1}^n+p_{j-2}^n}{4a(\Delta y)^2}\right)\end{aligned}}$$

3. Update pressure, solve for new u^*

$$\boxed{p_{k+1}^n = p_k^n + \omega_p p' \qquad u_{k+1}^* = f(u^n, p_{k+1}^n)\ \text{(step 1)}}$$

4. After u_{k+1}^*, p_{k+1}^n converge, which gives value u_c^*, p_c^n, correct velocity

$$\boxed{u^{n+1} = u_c^* - \frac{1}{a}\frac{p'_{c,i+1}-p'_{c,i-1}}{2\Delta x},\ v^{n+1} = v_c^* - \frac{1}{a}\frac{p'_{c,j+1}-p'_{c,j-1}}{2\Delta y},}$$

Where $a = \left(\frac{1}{\Delta t}+\frac{1}{Re\Delta x^2}+\frac{1}{Re\Delta y^2}\right)$, $\omega_p = 0.7$ is an under-relaxation factor used in this project

Non-dimensionalized Navier-Stokes Equations

- Consider the Navier Stokes equations:
  - Momentum equation

X-momentum conservation
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

Y-momentum conservation
$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

Mass conservation
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

Where $Re=\frac{\rho UL}{\mu}$ is the Reynolds number. U is a characteristic velocity which is a constant

- Note that while we have three unknowns and three equations, there is no equation which can be used to evolve the pressure!
- To resolve this issue, we employ a pressure correction method. The method is composed by the following steps:
  1) Start with initial guess u^n, v^n and p^n
  2) Evolution u and v according to the momentum conservation equation to get intermedia variable u^* and v^*
  3) Come up with corrections $u^{n+1}=u^*+u'$, $v^{n+1}=v^*+v'$, $p^{n+1}=p^n+p'$ such that the corrected velocity satisfy the mass conservation. In this step, we get a equation for p', which update p from time step n to n+1

Step 1: Prediction

- With initial guess u^n, v^n and p^n, we want to evolve u and v according to the momentum conservation equation to get intermedia variable u^* and v^*

X-momentum conservation
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

Y-momentum conservation
$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

- We want to solve the equation in conservative form:

X-momentum conservation
$$\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

Y-momentum conservation
$$\frac{\partial v}{\partial t}+\frac{\partial uv}{\partial x}+\frac{\partial v^2}{\partial y}=-\frac{\partial p}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

- The equivalence of these two forms can be derived by:

$$\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}=u\frac{\partial u}{\partial x}+\frac{\partial u}{\partial x}u+u\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}v$$

Due to mass conservation, $u\frac{\partial u}{\partial x}+u\frac{\partial v}{\partial y}=u\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$

Conservative form

- We want to solve the equation in conservative form:

X-momentum conservation

$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

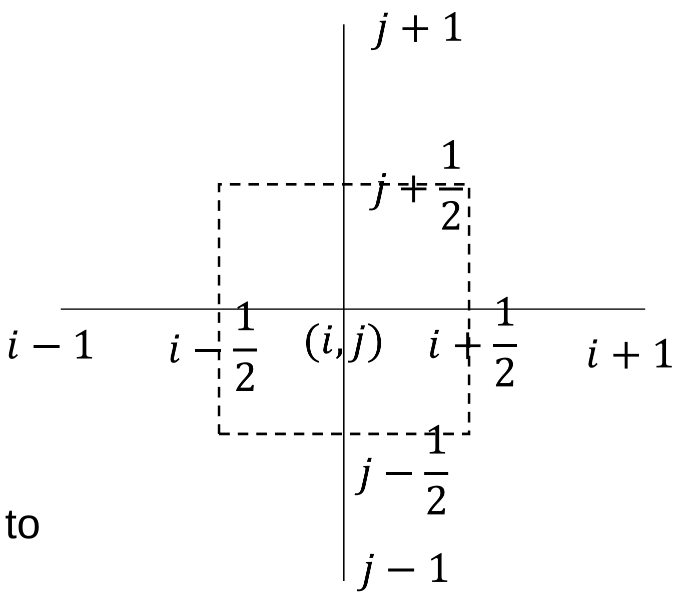
Y-momentum conservation

$$\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

- The conservation at the numerical level can be proved by rearranging the equation as:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(u^2 + p - \frac{1}{Re}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(uv - \frac{1}{Re}\frac{\partial u}{\partial y}\right) = 0$$

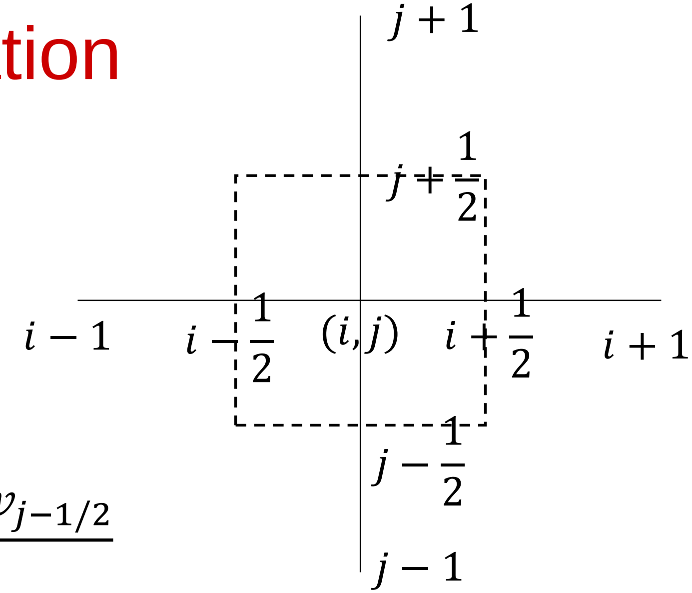
$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\left(uv - \frac{1}{Re}\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(v^2 + p - \frac{1}{Re}\frac{\partial u}{\partial y}\right) = 0$$



- And consider the integration in the control volume enclosed by the dashed line:

- 1D demonstration is provided in handout 5 with regard to why it is conservative.

Spatial discretization



- For X-Momentum equation:

$$\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial u}{\partial t}+\frac{u_{i+1/2}^2-u_{i-1/2}^2}{\Delta x}+\frac{u_{j+1/2}v_{j+1/2}-u_{j-1/2}v_{j-1/2}}{\Delta y}$$
$$=-\frac{p_{i+\frac{1}{2}}-p_{i-\frac{1}{2}}}{\Delta x}+\frac{1}{Re}\left(\frac{u_{i+1}-2u_i+u_{i-1}}{\Delta x^2}+\frac{u_{j+1}-2u_j+u_{j-1}}{\Delta y^2}\right)$$

where

$$\begin{aligned}
u_{i+1/2}&=0.5(u_{i+1}+u_i) & u_{i-1/2}&=0.5(u_i+u_{i-1})\\
u_{j+1/2}&=0.5(u_{j+1}+u_j) & u_{j-1/2}&=0.5(u_j+u_{j-1})\\
v_{j+1/2}&=0.5(v_{j+1}+v_j) & v_{j-1/2}&=0.5(v_j+v_{j-1})\\
p_{i+1/2}&=0.5(p_{i+1}+p_i) & p_{i-1/2}&=0.5(p_i+p_{i-1})
\end{aligned} \qquad (1)$$

- Note that all these data at half is intermediate. We only store u, v, p at positions with integer index
- Similar equation can be derived for Y-momentum conservation:

Temporal discretization

- Following the Project 2, we use a implicit/explicit scheme
  - Adam-Bathforth 2 for convection
  - Trapezoidal for diffusion
- For X-Momentum equation:

$$\frac{\partial u}{\partial t}+\frac{u_{i+1/2}^2-u_{i-1/2}^2}{\Delta x}+\frac{u_{j+1/2}v_{j+1/2}-u_{j-1/2}v_{j-1/2}}{\Delta y}=-\frac{p_{i+\frac{1}{2}}-p_{i-\frac{1}{2}}}{\Delta x}+\frac{1}{Re}\left(\frac{u_{i+1}-2u_i+u_{i-1}}{\Delta x^2}+\frac{u_{j+1}-2u_j+u_{j-1}}{\Delta y^2}\right)$$

Denote $$A(u^n)=\frac{u_{i+1/2}^2-u_{i-1/2}^2}{\Delta x}+\frac{u_{j+1/2}v_{j+1/2}-u_{j-1/2}v_{j-1/2}}{\Delta y}$$

with all the velocities from time step of n

$$D(u^n)=\frac{u_{i+1}-2u_i+u_{i-1}}{\Delta x^2}+\frac{u_{j+1}-2u_j+u_{j-1}}{\Delta y^2}$$

with all the velocities from time step of n

$$\delta_x(p^n)=\frac{p_{i+1/2}^n-p_{i-1/2}^n}{\Delta x}$$ with all the pressure from time step of n

- We have: $$\frac{1}{\Delta t}\left(u_{i,j}^*-u_{i,j}^n\right)=-\left(1.5A(u^n)-0.5A(u^{n-1})\right)+\frac{0.5}{Re}D(u^n)+\frac{0.5}{Re}D(u^*)-\delta_x(p^n)$$

- Matrix form:

$$\boxed{\left(\frac{\boldsymbol{I}}{\Delta t}-\frac{0.5}{Re}\right)u^*=\frac{\boldsymbol{I}}{\Delta t}u^n-\left(1.5\boldsymbol{A}(u^n)-0.5\boldsymbol{A}(u^{n-1})\right)+\frac{0.5}{Re}\boldsymbol{D}(u^n)-\boldsymbol{\delta_x}(p^n)} \qquad (2)$$

- Y-momentum equation can be solved in a similar manner.
- You can solve it in exactly the same way with solving the project 2 to get u^*, v^*.

Step 2: Correction

- From the prediction step :

$$\frac{1}{\Delta t}\left(u_{i,j}^{*}-u_{i,j}^{n}\right)=-\left(1.5A(u^{n})-0.5A(u^{n-1})\right)+\frac{0.5}{Re}D(u^{n})+\frac{0.5}{Re}D(u^{*})-\frac{p_{i+\frac{1}{2}}^{n}-p_{i-\frac{1}{2}}^{n}}{\Delta x} \quad (3)$$

- We are hoping that after correction :

$$\frac{1}{\Delta t}\left(u_{i,j}^{n+1}-u_{i,j}^{n}\right)=-\left(1.5A(u^{n})-0.5A(u^{n-1})\right)+\frac{0.5}{Re}D(u^{n})+\frac{0.5}{Re}D(u^{n+1})-\frac{p_{i+\frac{1}{2}}^{n+1}-p_{i-\frac{1}{2}}^{n+1}}{\Delta x} \quad (4)$$

- (4) – (3), we have the equation for the correction:

$$\frac{1}{\Delta t}u_{i,j}^{\prime}-\frac{0.5}{Re}D(u^{\prime})=-\frac{p_{i+\frac{1}{2}}^{\prime}-p_{i-\frac{1}{2}}^{\prime}}{\Delta x} \qquad \frac{1}{\Delta t}v_{i,j}^{\prime}-\frac{0.5}{Re}D(v^{\prime})=-\frac{p_{j+\frac{1}{2}}^{\prime}-p_{j-\frac{1}{2}}^{\prime}}{\Delta y} \quad (5)$$

where

$$u^{*}+u^{\prime}=u^{n+1} \qquad v^{*}+v^{\prime}=v^{n+1} \qquad p^{n}+p^{\prime}=p^{n+1}$$

- The corrected velocities should satisfy

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\Rightarrow\frac{u_{i+1/2}^{n+1}-u_{i-1/2}^{n+1}}{\Delta x}+\frac{v_{j+1/2}^{n+1}-v_{j-1/2}^{n+1}}{\Delta y}=0 \quad (6)$$

Step 2: Correction

- Expand (6), we have:

$$\frac{u^*_{i+1/2} - u^*_{i-1/2}}{\Delta x} + \frac{v^*_{j+1/2} - v^*_{j-1/2}}{\Delta y} + \frac{u'_{i+1/2} - u'_{i-1/2}}{\Delta x} + \frac{v'_{j+1/2} - v'_{j-1/2}}{\Delta y} = 0 \qquad (7)$$

Where u^*, v^* is known. With equation (5):

$$\frac{1}{\Delta t} u'_{i,j} - \frac{0.5}{Re} D(u') = -\frac{p'_{i+\frac{1}{2}} - p'_{i-\frac{1}{2}}}{\Delta x} \qquad \frac{1}{\Delta t} v'_{i,j} - \frac{0.5}{Re} D(v') - \frac{p'_{j+\frac{1}{2}} - p'_{j-\frac{1}{2}}}{\Delta y}$$

- If we can write equation (5) as $u' = f(p'), v' = g(p')$, substitute into equation (7), we have:

$$\frac{u^*_{i+1/2} - u^*_{i-1/2}}{\Delta x} + \frac{v^*_{j+1/2} - v^*_{j-1/2}}{\Delta y} + \frac{f(p')_{i+1/2} - f(p')_{i-1/2}}{\Delta x} + \frac{g(p')_{j+1/2} - g(p')_{j-1/2}}{\Delta x} = 0 \qquad (8)$$

Where u^*, v^* known, we can solve equation (8) for p'.
Then, with p' solved, we can solve equation (5) for u' and v'.
Then, we can update the solution with

$$u^* + u' = u^{n+1} \qquad v^* + v' = v^{n+1} \qquad p^n + p' = p^{n+1}$$

Step 2: Correction

- Now let's focus on equation (7). We have several technique details to discuss.

$$\frac{u'_{i+1/2} - u'_{i-1/2}}{\Delta x} + \frac{v'_{j+1/2} - v'_{j-1/2}}{\Delta y} = -\left(\frac{u^*_{i+1/2} - u^*_{i-1/2}}{\Delta x} + \frac{v^*_{j+1/2} - v^*_{j-1/2}}{\Delta y}\right) \tag{9}$$

  - How to get $u' = f(p')$ and $v' = g(p')$?
  - With u^* and v^* known at the positions with integer index, how to interpolate for values at half location?

Elimination for implicit temporal discretization

- For $u' = f(p')$ and $v' = g(p')$, we need to look into the linear system (5):

$$\frac{1}{\Delta t}u'_{i,j} - \frac{0.5}{Re}D(u') = -\frac{p'_{i+\frac{1}{2}} - p'_{i-\frac{1}{2}}}{\Delta x} \qquad \frac{1}{\Delta t}v'_{i,j} - \frac{0.5}{Re}D(v') = -\frac{p'_{j+\frac{1}{2}} - p'_{j-\frac{1}{2}}}{\Delta y}$$

- which gives us:

$$-\frac{1}{2Re\Delta x^2}u'_{i-1} - \frac{1}{2\Delta Re y^2}u'_{j-1} + \left(\frac{1}{\Delta t} + \frac{1}{Re\Delta x^2} + \frac{1}{Re\Delta y^2}\right)u'_{i,j} - \frac{1}{2Re\Delta y^2}u'_{j+1} - \frac{1}{2Re\Delta x^2}u'_{i+1} = -\frac{p'_{i+\frac{1}{2}} - p'_{i-\frac{1}{2}}}{\Delta x}$$

- Here, we use a very brutal simplification for $u' = f(p')$, which is throwing away all the off-diagonal terms! This gives us

$$u'_{i,j} = -\frac{1}{\left(\frac{1}{\Delta t} + \frac{1}{Re\Delta x^2} + \frac{1}{Re\Delta y^2}\right)}\frac{p'_{i+\frac{1}{2}} - p'_{i-\frac{1}{2}}}{\Delta x} \qquad (10)$$

Discussions:

- This difficulty arise from the implicit temporal discretization. If we use explicit schemes for convection and diffusion, then the LHS matrix will be $\boldsymbol{I}$ instead of $\left(\frac{\boldsymbol{I}}{\Delta t} - \frac{0.5}{Re}\boldsymbol{D}\right)$
- Yes, it is not perfect in terms of derivation. But it allows us to use implicit method, which allows large Δt. We get a huge gain in code performance!
- This simplification will cause convergence issue. Because all u' are at the same order of magnitude. This can be more or less compensated by under-relaxation etc. For more discussion, please see from equation 7.71 onwards,: *Ferziger, Joel H., Milovan Perić, and Robert L. Street. Computational methods for fluid dynamics. Vol. 3. Berlin: springer, 2002.*

Elimination for implicit temporal discretization

- Now we have equation (10):

$$u'_{i,j} = -\frac{1}{\left(\frac{1}{\Delta t} + \frac{1}{Re\Delta x^2} + \frac{1}{Re\Delta y^2}\right)} \frac{p'_{i+\frac{1}{2}} - p'_{i-\frac{1}{2}}}{\Delta x} \qquad v'_{i,j} = -\frac{1}{\left(\frac{1}{\Delta t} + \frac{1}{Re\Delta x^2} + \frac{1}{Re\Delta y^2}\right)} \frac{p'_{j+\frac{1}{2}} - p'_{j-\frac{1}{2}}}{\Delta x}$$

- However, the LHS of equation (9), we need $u'_{i\pm 1/2}$ and $v'_{j\pm 1/2}$. We use:

$$u'_{i+1/2} = -\frac{1}{\left(\frac{1}{\Delta t} + \frac{1}{Re\Delta x^2} + \frac{1}{Re\Delta y^2}\right)} \frac{p'_{i+1} - p'_{i}}{\Delta x} \qquad u'_{i-1/2} = -\frac{1}{\left(\frac{1}{\Delta t} + \frac{1}{Re\Delta x^2} + \frac{1}{Re\Delta y^2}\right)} \frac{p'_{i} - p'_{i-1}}{\Delta x}$$

$$v'_{j+1/2} = -\frac{1}{\left(\frac{1}{\Delta t} + \frac{1}{Re\Delta x^2} + \frac{1}{Re\Delta y^2}\right)} \frac{p'_{j+1} - p'_{j}}{\Delta x} \qquad v'_{j-1/2} = -\frac{1}{\left(\frac{1}{\Delta t} + \frac{1}{Re\Delta x^2} + \frac{1}{Re\Delta y^2}\right)} \frac{p'_{j} - p'_{j-1}}{\Delta x}$$

- Substitute into equation (9), we have:

$$\boxed{\text{LHS of (9)} = -\frac{1}{\left(\frac{1}{\Delta t} + \frac{1}{Re\Delta x^2} + \frac{1}{Re\Delta y^2}\right)} D(p')} \qquad (11)$$

where

$$D(p') = \frac{p'_{i+1} - 2p'_{i} + p'_{i-1}}{\Delta x^2} + \frac{p'_{j+1} - 2p'_{j} + p'_{j-1}}{\Delta y^2}$$

Elimination for implicit temporal discretization

Discussion on the previous slide

- we have equation (5):

$$\left(\frac{1}{\Delta t}-\frac{0.5}{Re}D\right)u' = -\frac{p'_{i+\frac{1}{2}}-p'_{i-\frac{1}{2}}}{\Delta x}$$

- And after simplification, we have equation (10):

$$u'_{i,j} = -\frac{1}{\left(\frac{1}{\Delta t}+\frac{1}{Re\Delta x^2}+\frac{1}{Re\Delta y^2}\right)}\frac{p'_{i+\frac{1}{2}}-p'_{i-\frac{1}{2}}}{\Delta x}$$

- In case where you discretize the convective term in an implicit manner, you will have

$$\left(\frac{1}{\Delta t}-\frac{0.5}{Re}D + C(u^k)\right)u' = -\frac{p'_{i+\frac{1}{2}}-p'_{i-\frac{1}{2}}}{\Delta x}$$

- Still, we can write it as:

$$a_{i,j}u'_{i,j} + \sum_{nb} a_{nb}u'_{nb} = -\frac{p'_{i+\frac{1}{2}}-p'_{i-\frac{1}{2}}}{\Delta x}$$

Where nb denotes the neighbor points

- And after simplification (throw away nb terms)

$$u'_{i,j} = -\frac{1}{a_{i,j}}\frac{p'_{i+\frac{1}{2}}-p'_{i-\frac{1}{2}}}{\Delta x}$$

- Note that $a_{i,j}$ now involves u^k, which is no longer a constant. Now we can have:

$$u'_{i+1/2} = -\frac{1}{a_{i+1/2}}\frac{p'_{i+1}-p'_i}{\Delta x}$$

- Where $\frac{1}{a_{i+1/2}}$ can be approximated through interpolation:

$$\frac{1}{a_{i+1/2}} = \frac{1}{2}\left(\frac{1}{a_i}+\frac{1}{a_{i+1}}\right)$$

Checkerboard problem

- Now let's focus on the RHS of the equation (9)

$$\text{RHS of (8)} = -\left(\frac{u^*_{i+1/2} - u^*_{i-1/2}}{\Delta x} + \frac{v^*_{j+1/2} - v^*_{j-1/2}}{\Delta y}\right)$$

- Note that u^*, v^* are only stored at positions with integer index. We need to interpolate at half positions
- The immediate idea is to do $u^*_{i+1/2} = 0.5(u^*_i + u^*_{i+1})$ Now, we will discuss why this don't work well
- let's look back at the governing equation for u^*, which is equation (2) and do some algebra (to keep focus, you can directly jump to equation (12) two page later)

$$\frac{1}{\Delta t}\left(u^*_{i,j} - u^n_{i,j}\right) = -\left(1.5A(u^n) - 0.5A(u^{n-1})\right) + \frac{0.5}{Re}D(u^n) + \frac{0.5}{Re}D(u^*) - \frac{p^n_{i+\frac{1}{2}} - p^n_{i-\frac{1}{2}}}{\Delta x}$$

- Rearrange as

$$\underbrace{\frac{I}{\Delta t}u^*_{i,j} - \frac{0.5}{Re}D(u^*)}_{a_{i,j}u^*_{i,j} + \sum_{nb} a_{nb}u^*_{nb}} = \underbrace{\frac{I}{\Delta t}u^n - \left(1.5A(u^n) - 0.5A(u^{n-1})\right) + \frac{0.5}{Re}D(u^n)}_{b^n} - \frac{p^n_{i+\frac{1}{2}} - p^n_{i-\frac{1}{2}}}{\Delta x}$$

Checkerboard problem

➢ Rearrange as

$$\underbrace{\frac{1}{\Delta t}u^*_{i,j} - \frac{0.5}{Re}D(u^*)}_{a_{i,j}u^*_{i,j} + \sum_{nb} a_{nb}u^*_{nb}} = \underbrace{\frac{1}{\Delta t}u^n - \left(1.5A(u^n) - 0.5A(u^{n-1})\right) + \frac{0.5}{Re}D(u^n)}_{b^n} - \frac{p^n_{i+\frac{1}{2}} - p^n_{i-\frac{1}{2}}}{\Delta x}$$

$$a_{i,j}u^*_{i,j} + \sum_{nb} a_{nb}u^*_{nb} = b^n - \frac{p^n_{i+\frac{1}{2}} - p^n_{i-\frac{1}{2}}}{\Delta x}$$

$$u^*_{i,j} = \underbrace{\frac{b^n - \sum_{nb} a_{nb}u^*_{nb}}{a_{i,j}}}_{u^{nb,*}_{i,j}} - \frac{1}{a_{i,j}}\frac{p^n_{i+\frac{1}{2}} - p^n_{i-\frac{1}{2}}}{\Delta x}$$

$$u^*_{i,j} = u^{nb,*}_{i,j} - \frac{1}{a_{i,j}}\frac{p^n_{i+\frac{1}{2}} - p^n_{i-\frac{1}{2}}}{\Delta x}$$

Checkerboard problem

- With

$$u^*_{i,j} = u^{nb,*}_{i,j} - \frac{1}{a_{i,j}} \frac{p^n_{i+\frac{1}{2}} - p^n_{i-\frac{1}{2}}}{\Delta x} \tag{12}$$

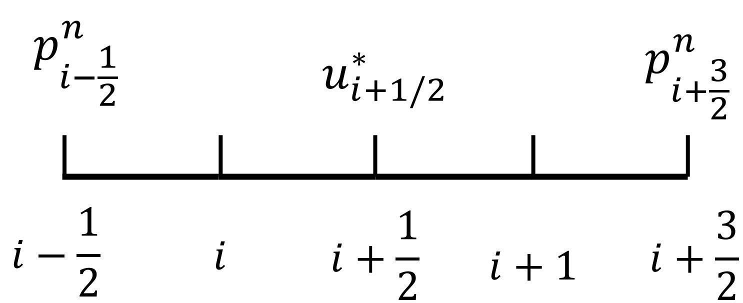
- If we do interpolation as $u^*_{i+1/2} = 0.5(u^*_i + u^*_{i+1})$, we reach

$$u^*_{i+1/2} = 0.5(u^{nb,*}_{i,j} + u^{nb,*}_{i+1}) - 0.5\left(\frac{1}{a_{i,j}} \frac{p^n_{i+\frac{1}{2}} - p^n_{i-\frac{1}{2}}}{\Delta x} + \frac{1}{a_{i+1,j}} \frac{p^n_{i+\frac{3}{2}} - p^n_{i+\frac{1}{2}}}{\Delta x}\right)$$

- With current discretization, $a_{i,j} = a_{i+1,j} = \left(\frac{1}{\Delta t} + \frac{1}{Re\Delta x^2} + \frac{1}{Re\Delta y^2}\right)$ is a constant.

$$u^*_{i+1/2} = 0.5(u^{nb,*}_{i,j} + u^{nb,*}_{i+1}) - \frac{1}{\left(\frac{1}{\Delta t} + \frac{1}{Re\Delta x^2} + \frac{1}{Re\Delta y^2}\right)}\left(\frac{p^n_{i+\frac{3}{2}} - p^n_{i-\frac{1}{2}}}{2\Delta x}\right)$$

- See 1D stencil,

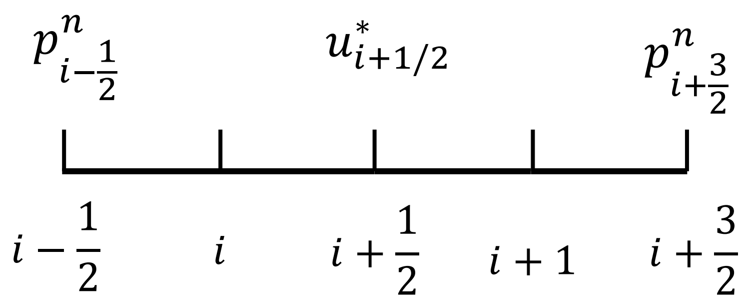


Checkerboard problem

- With current discretization, $a_{i,j} = a_{i+1,j} = \left(\frac{1}{\Delta t} + \frac{1}{Re\Delta x^2} + \frac{1}{Re\Delta y^2}\right)$ is a constant.

$$u^*_{i+1/2} = 0.5(u^{nb,*}_{i,j} + u^{nb,*}_{i+1}) - \frac{1}{\left(\frac{1}{\Delta t} + \frac{1}{Re\Delta x^2} + \frac{1}{Re\Delta y^2}\right)}\left(\frac{p^n_{i+\frac{3}{2}} - p^n_{i-\frac{1}{2}}}{2\Delta x}\right)$$

- See 1D stencil,



- It is equivalent to solve pressure in a mesh which is one level coarser than the current grid! Which leads to bad coupling between the pressure and velocity.
- This is referred to as the checkerboard problem

Checkerboard problem

- For the current equation

$$u^{*}_{i+1/2} = 0.5(u^{nb,*}_{i,j} + u^{nb,*}_{i+1}) - \frac{1}{\left(\frac{1}{\Delta t} + \frac{1}{Re\Delta x^2} + \frac{1}{Re\Delta y^2}\right)}\left(\frac{p^{n}_{i+\frac{3}{2}} - p^{n}_{i-\frac{1}{2}}}{2\Delta x}\right)$$

- To further demonstrate the problem, suppose we have the following pressure distribution:

$p =$	80	40	80	40
	$i-3/2$	$i-1/2$	$i+1/2$	$i+3/2$

- If you check this pressure field, while it is oscillator and should produce flow motion due to pressure gradient, with the lose coupling nothing is going to happen!

- Observation: the checkerboard problem is introduced immediately when the pressure is considered and we are taking average for half locations

Rhie-Chow correction

- To resolve this, we use Rhie and Chow method. Instead of taking the average $u^*_{i+1/2} = 0.5(u^*_i + u^*_{i+1})$ (which cause the issue), we assume that the $u^*_{i+1/2}$ exist, which essentially uses the spirit of the staggered grid:

$$u^*_{i+1/2} = u^{nb,*}_{i+1/2} - \frac{1}{a_{i+1/2}} \frac{p^n_{i+1} - p^n_i}{\Delta x}$$

- And approximate non-existing data $u^{nb,*}_{i+1/2}$ as:

$$u^{nb,*}_{i+1/2} = 0.5(u^{nb,*}_{i,j} + u^{nb,*}_{i+1})$$

- Compare with the averaging algorithm $u^*_{i+1/2} = 0.5(u^*_i + u^*_{i+1})$,

$$u^*_{i+1/2} = 0.5(u^{nb,*}_{i,j} + u^{nb,*}_{i+1}) - \frac{1}{\left(\frac{1}{\Delta t} + \frac{1}{Re\Delta x^2} + \frac{1}{Re\Delta y^2}\right)} \left(\frac{p^n_{i+\frac{3}{2}} - p^n_{i-\frac{1}{2}}}{2\Delta x}\right)$$

  the checkerboard problem is gone

- Essentially, we are offsetting the steps of involving pressure and taking average

Rhie-Chow correction

- Now let's do some algebra. According to the explanation provided just now, what we end up with is:

$$u^*_{i+1/2} = 0.5(u^{nb,*}_{i,j} + u^{nb,*}_{i+1}) - \frac{1}{a_{i+1/2}} \frac{p^n_{i+1} - p^n_i}{\Delta x}$$

- For $u^{nb,*}_{i+1}$ and $u^{nb,*}_{i,j}$, from equation (12) we have:

$$u^{nb,*}_{i,j} = u^*_{i,j} + \frac{1}{a_{i,j}} \frac{p^n_{i+\frac{1}{2}} - p^n_{i-\frac{1}{2}}}{\Delta x} \qquad u^{nb,*}_{i+1} = u^*_{i+1} + \frac{1}{a_{i+1}} \frac{p^n_{i+\frac{3}{2}} - p^n_{i+\frac{1}{2}}}{\Delta x}$$

- Substitute and eliminate, we have:

$$u^*_{i+1/2} = 0.5(u^*_{i,j} + u^*_{i+1}) + \frac{0.5}{a_{i,j}} \frac{p^n_{i+\frac{1}{2}} - p^n_{i-\frac{1}{2}}}{\Delta x} + \frac{0.5}{a_{i+1}} \frac{p^n_{i+\frac{3}{2}} - p^n_{i+\frac{1}{2}}}{\Delta x} - \frac{1}{a_{i+1/2}} \frac{p^n_{i+1} - p^n_i}{\Delta x}$$

- In the current discretization, we have $a = a_{i,j} = a_{i+1} = a_{i+1/2} = \left(\frac{1}{\Delta t} + \frac{1}{Re\Delta x^2} + \frac{1}{Re\Delta y^2}\right)$
- As a result:

$$u^*_{i+1/2} = 0.5(u^*_{i,j} + u^*_{i+1}) + \frac{1}{a} \frac{0.5p^n_{i+\frac{3}{2}} - p^n_{i+1} + p^n_i - 0.5p^n_{i-\frac{1}{2}}}{\Delta x}$$

Rhie-Chow correction

- Again, data at half location are just intermedia values which doesn't exist. Recall in equation system (1)

$$p_{i+3/2} = 0.5(p_{i+2} + p_{i+1}) \quad p_{i-1/2} = 0.5(p_i + p_{i-1})$$

- Substitute into the equation:

$$u^*_{i+1/2} = 0.5(u^*_{i,j} + u^*_{i+1}) + \frac{1}{a}\frac{0.5p^n_{i+\frac{3}{2}} - p^n_{i+1} + p^n_i - 0.5p^n_{i-\frac{1}{2}}}{\Delta x}$$

- We end up with:

$$u^*_{i+1/2} = 0.5(u^*_{i,j} + u^*_{i+1}) + \frac{1}{a}\frac{p^n_{i+2} - 3p^n_{i+1} + 3p^n_i - p^n_{i-1}}{4\Delta x}$$

- Similarly, we can derive $u^*_{i-1/2}, v^*_{i+1/2}, v^*_{i-1/2}$. Substate them into equation (9), we get:

$$\text{RHS of (9)} = -\left(\frac{u^*_{i+1} - u^*_{i-1}}{2\Delta x} + \frac{p^n_{i+2} - 4p^n_{i+1} + 6p^n_i - 4p^n_{i-1} + p^n_{i-2}}{4a(\Delta x)^2}\right)$$
$$-\left(\frac{v^*_{j+1} - v^*_{j-1}}{2\Delta y} + \frac{p^n_{j+2} - 4p^n_{j+1} + 6p^n_j - 4p^n_{j-1} + p^n_{j-2}}{4a(\Delta y)^2}\right)$$

Rhie-Chow correction

- At the end, we reach:

$$\text{RHS of }(9) = -\left(\frac{u^*_{i+1} - u^*_{i-1}}{2\Delta x} + \frac{p^n_{i+2} - 4p^n_{i+1} + 6p^n_i - 4p^n_{i-1} + p^n_{i-2}}{4a(\Delta x)^2}\right) - \left(\frac{v^*_{j+1} - v^*_{j-1}}{2\Delta y} + \frac{p^n_{j+2} - 4p^n_{j+1} + 6p^n_j - 4p^n_{j-1} + p^n_{j-2}}{4a(\Delta y)^2}\right) \quad (13)$$

Where $a = \left(\frac{1}{\Delta t} + \frac{1}{Re\Delta x^2} + \frac{1}{Re\Delta y^2}\right)$

- Discussion
  - $p^n_{i+2} - 4p^n_{i+1} + 6p^n_i - 4p^n_{i-1} + p^n_{i-2}$ is proportional to $\partial^4 p/\partial x^4$
  - If we compare with the averaging scheme from $u^*_{i+1/2} = 0.5(u^*_i + u^*_{i+1})$, which results in:

$$\text{RHS of }(9) = -\left(\frac{u^*_{i+1} - u^*_{i-1}}{2\Delta x} + \frac{v^*_{j+1} - v^*_{j-1}}{2\Delta y}\right)$$

  - We can see that the Rhie-Chow correction is actually adding a forth order damping in pressure for stabilization
  - You can find more information in section 8.2.1, *Ferziger, Joel H., Milovan Perić, and Robert L. Street. Computational methods for fluid dynamics. Vol. 3. Berlin: springer, 2002.*

Step 2: Correction

- Put equation (11) and (13) together, we have the equation for the corrected pressure:

$$\boxed{\begin{aligned}&-\frac{1}{a}\left(\frac{p'_{i+1}-2p'_{i}+p'_{i-1}}{\Delta x^2}+\frac{p'_{j+1}-2p'_{j}+p'_{j-1}}{\Delta y^2}\right)\\&=-\left(\frac{u^*_{i+1}-u^*_{i-1}}{2\Delta x}+\frac{p^n_{i+2}-4p^n_{i+1}+6p^n_{i}-4p^n_{i-1}+p^n_{i-2}}{4a(\Delta x)^2}\right)\\&-\left(\frac{v^*_{j+1}-v^*_{j-1}}{2\Delta y}+\frac{p^n_{j+2}-4p^n_{j+1}+6p^n_{j}-4p^n_{j-1}+p^n_{j-2}}{4a(\Delta y)^2}\right)\end{aligned}} \tag{14}$$

Where $a=\left(\frac{1}{\Delta t}+\frac{1}{Re\Delta x^2}+\frac{1}{Re\Delta y^2}\right)$

- You can solve it in exactly the same way with solving the project 1.

Step 3: Update

- We update the pressure as:

$$p_{k+1}^{n} = p_k^n + \omega_p p'$$

- Where $\omega_p = 0.7$ is an under-relaxation factor
- Then we go back to the prediction step (equation 2) to predict u_{k+1}^{*}, v_{k+1}^{*} with p_{k+1}^{n} until convergence, which means the mass is conserved.

- Lastly, we proceed in time according to equation (10):

$$u'_{i,j} = -\frac{1}{a}\frac{p'_{i+\frac{1}{2}} - p'_{i-\frac{1}{2}}}{\Delta x} \qquad v'_{i,j} = -\frac{1}{a}\frac{p'_{j+\frac{1}{2}} - p'_{j-\frac{1}{2}}}{\Delta y}$$

- where

$$p'_{i+1/2} = 0.5(p'_{i+1} + p'_i), p'_{i-1/2} = 0.5(p'_i + p'_{i-1}), p'_{j+1/2} = 0.5\left(p'_{j+1} + p'_j\right), p'_{j-1/2} = 0.5(p'_j + p'_{j-1})$$

- We have explicit equations for u' and v':

$$\boxed{\begin{gathered} u'_{i,j} = -\frac{1}{a}\frac{p'_{i+1} - p'_{i-1}}{2\Delta x} \qquad v'_{i,j} = -\frac{1}{a}\frac{p'_{j+1} - p'_{j-1}}{2\Delta y} \\ u^{n+1} = u_c^* + u',\ v^{n+1} = v_c^* + v', p^{n+1} = p_c^n + \omega_p p' \end{gathered}}$$

Where $a = \left(\frac{1}{\Delta t} + \frac{1}{Re\Delta x^2} + \frac{1}{Re\Delta y^2}\right)$, p_c^n, u_c^*, v_c^* are converged pressure and predicted velocity field accordingly.

Boundary condition

- In this project, we use wall boundary condition for all boundaries:

$$u = v = u' = v' = 0 \qquad \text{on all the boundaries}$$

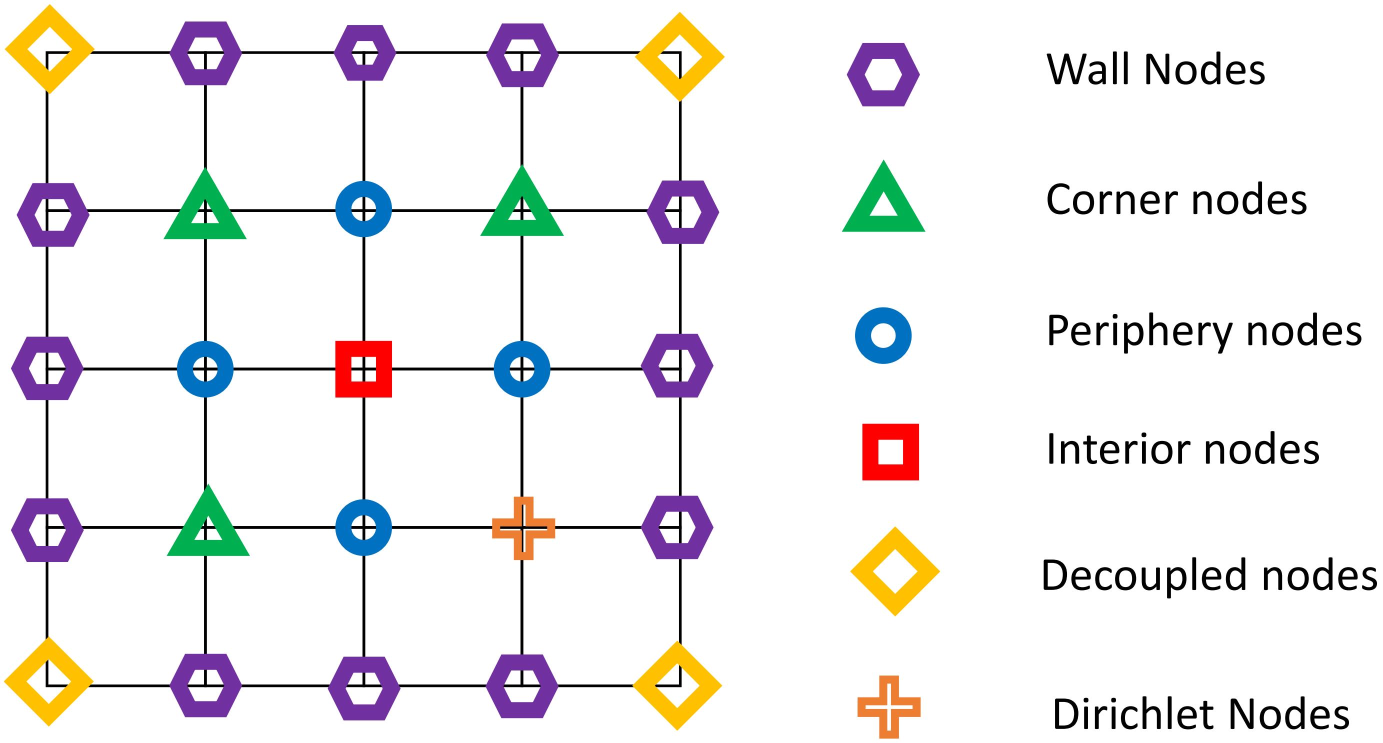
- The complexity lies on the boundary condition for equation (14)

$$-\frac{1}{a}\left(\frac{p'_{i+1} - 2p'_{i} + p'_{i-1}}{\Delta x^2} + \frac{p'_{j+1} - 2p'_{j} + p'_{j-1}}{\Delta y^2}\right)$$
$$= -\left(\frac{u^*_{i+1} - u^*_{i-1}}{2\Delta x} + \frac{p^n_{i+2} - 4p^n_{i+1} + 6p^n_{i} - 4p^n_{i-1} + p^n_{i-2}}{4a(\Delta x)^2}\right)$$
$$-\left(\frac{v^*_{j+1} - v^*_{j-1}}{2\Delta y} + \frac{p^n_{j+2} - 4p^n_{j+1} + 6p^n_{j} - 4p^n_{j-1} + p^n_{j-2}}{4a(\Delta y)^2}\right)$$

- The equation can be used for all interior nodes far from the boundary

$$i = 2,3, \cdots Nx - 3, j = 2,3, \cdots Ny - 3,$$

Boundary condition



Boundary condition for wall nodes

- On the boundary point
  $i = 0, i = nx - 1\ j = 0, j = ny - 1$
- For wall boundary condition, we have $u^n = v^n = u^{n+1} = v^{n+1} = 0$
- Therefore $u' = v' = 0$
- According to equation (10), we have:

$$u'_{1,j} = -\frac{1}{\left(\frac{1}{\Delta t} + \frac{1}{Re\Delta x^2} + \frac{1}{Re\Delta y^2}\right)} \frac{p'_{0.5,j} - p'_{-0.5,j}}{\Delta x} = 0$$

- We approximate

$$p'_{0.5,j} - p'_{-0.5,j} = 0$$

- As a crude approximation, we shift the grid by 0.5:

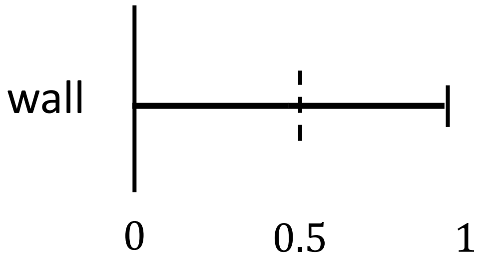
$$p'_{1,j} - p'_{0,j} = 0$$

- Which is the Neumann boundary condition
- Similarly, we have

$$p'_{i,1} - p'_{i,0} = 0\ p'_{nx-1,j} - p'_{nx-2,j} = 0, p'_{i,ny-1} - p'_{i,ny-2} = 0$$

Boundary condition for Periphery/corner nodes

- Following the same procedure, now we have:



$$u'_{1.5} = -\frac{1}{\left(\frac{1}{\Delta t} + \frac{1}{Re\Delta x^2} + \frac{1}{Re\Delta y^2}\right)} \frac{p'_2 - p'_1}{\Delta x} \qquad u'_{0.5} = -\frac{1}{2\left(\frac{1}{\Delta t} + \frac{1}{Re\Delta x^2} + \frac{1}{Re\Delta y^2}\right)} \frac{p'_1 - p'_0}{\Delta x} = 0$$

$$v'_{1,j+1/2} = -\frac{1}{\left(\frac{1}{\Delta t} + \frac{1}{Re\Delta x^2} + \frac{1}{Re\Delta y^2}\right)} \frac{p'_{1,j+1} - p'_{1,j}}{\Delta y} \qquad v'_{1,j-1/2} = -\frac{1}{\left(\frac{1}{\Delta t} + \frac{1}{Re\Delta x^2} + \frac{1}{Re\Delta y^2}\right)} \frac{p'_{1,j} - p'_{1,j-1}}{\Delta y}$$

- Which gives:

$$\text{LHS of (9) at } (1,j) = \frac{u'_{1.5} - u'_{0.5}}{\Delta x} + \frac{v'_{1,j+1/2} - v'_{1,j-1/2}}{\Delta y}$$

$$= -\frac{1}{a(\Delta y)^2} p'_{1,j-1} + \left(\frac{1}{a(\Delta x)^2} + \frac{2}{a(\Delta y)^2}\right) p'_{1,j} - \frac{1}{a(\Delta y)^2} p'_{1,j+1} - \frac{1}{a(\Delta x)^2} p'_2$$

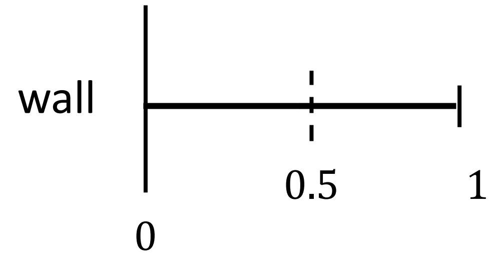
- Similar condition can be derived for the rest periphery nodes and corner nodes
- Another way is to do interpolation as written in page 13 (not used for this project)

Boundary condition for Dirichlet/ decoupled nodes

- Only the gradient is used in the momentum equation. Therefore, the decoupled nodes is not used in the calculation. You can specify arbitrary value for them
- Without the reference value, suppose p' is a solution of equation (14), then $p' + C$ will be a viable solution as well. The solution is not unique, therefore the problem is not solvable. At the matrix level, the LHS matrix is ill-conditioned.
- In this case, we need to give a reference value for p'. The value should be direction involved in the calculation (can not be wall nodes or corner nodes). Therefore, we select the point adjacent to the right bottom corner as the reference point:

$$p'_{\mathrm{nx}-2,1} = const$$

Boundary condition



- For boundary immediately adjacent to the wall
  $i = 1, i = nx - 2\ j = 1, j = ny - 2$
- Take the left boundary as an example
  RHS of (9) at $(1, j)$

$$= -\left(\frac{u_2^* - u_0^*}{2\Delta x} + \frac{p_3^n - 4p_2^n + 6p_1^n - 4p_0^n + p_{-1}^n}{4a(\Delta x)^2}\right)$$

$$-\left(\frac{v_{1,j+1}^* - v_{1,j-1}^*}{2\Delta x} + \frac{p_{1,j+2}^n - 4p_{1,j+1}^n + 6p_{1,j}^n - 4p_{1,j-1}^n + p_{1,j-2}^n}{4a(\Delta y)^2}\right)$$

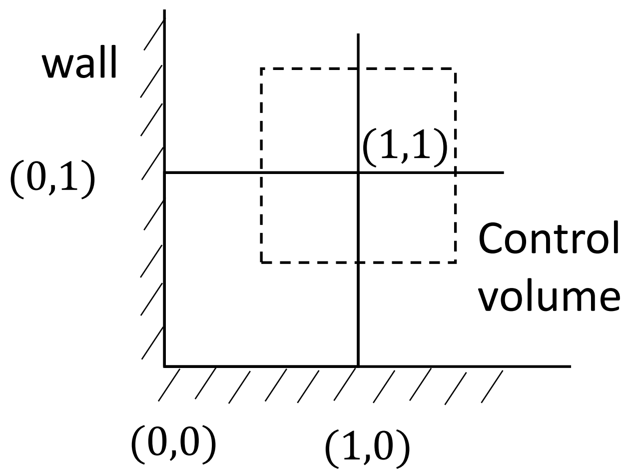
- The difficulty we are facing is that p_{-1}^n is unknown. To close the equation, we can use extrapolation:

$$p_{-1}^n = p_0^n - \Delta x \frac{\partial p}{\partial x} + (\Delta x)^2 \frac{\partial^2 p}{\partial x^2} - \cdots$$

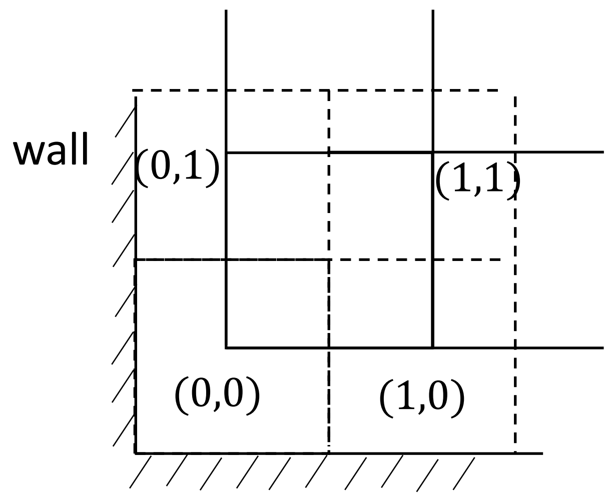
- In this project, we use second order extrapolation

Boundary condition

- Discussion
  - As we can see, we made a lot of crude approximation for boundary conditions for p'. Note that p' is used for velocity correction to ensure mass conservation. Therefore, the error introduced at the boundary for p' will be reflected as errors in terms of mass conservation.
  - If we change the stencil such that the control volumes (show as dashed line) are perfectly aligned with the boundaries (as shown by stencil (2) in figure below), the boundary condition for p' naturally comes. However, we will find ourselves in another difficulty: the wall boundary condition for velocity can not be satisfied exactly because we don't have data point on the wall!
  - In the incompressible flow regime, flow penetrating wall is much more unphysical than loss in mass conservation. Besides, upon convergence, $p' \rightarrow 0$. Therefore we chose the current method.



Current stencil



Stencil (2)