MECH 479: Module 2 - Characteristic Form of First-Order PDE System

Winter Term 1, Year 2021/22

Aim: Understand the characteristic form of first-order PDE system and its connection with eigenvalues and eigenvectors.

Example 1: Consider system of first-order PDE system for the wave equation $u_{tt} = c^2 u_{xx}$.
Let us introduce variables v and w as follows: $v = \frac{\partial u}{\partial t}, \quad w = c\frac{\partial u}{\partial x}$, then we may write

$$\begin{aligned} \frac{\partial v}{\partial t} &= c\frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial t} &= c\frac{\partial v}{\partial x} \end{aligned} \tag{1}$$

The above equation can be written as system of equations:

$$\frac{\partial \mathbf{u}}{\partial t} + [A]\frac{\partial \mathbf{u}}{\partial x} = 0$$

where

$$\mathbf{u} = \begin{bmatrix} v \\ w \end{bmatrix}$$

$$[A] = \begin{bmatrix} 0 & -c \\ -c & 0 \end{bmatrix}$$

The eigenvalues λ of $[A]$ matrix are found from

$$\det \| [A] - \lambda [I] \mid = 0$$

Thus

$$\begin{vmatrix} -\lambda & -c \\ -c & -\lambda \end{vmatrix} = 0$$

Or

$$\lambda^2 - c^2 = 0$$

The roots of this characteristic equation are

$$\begin{aligned} \lambda_1 &= +c \\ \lambda_2 &= -c \end{aligned}$$

These are the characteristic differential equations for the wave equation, i.e.,

$$\left(\frac{dx}{dt}\right)_1 = +c$$

$$\left(\frac{dx}{dt}\right)_2 = -c$$

The system of equations in this example is hyperbolic, and we see that the eigenvalues of the $[A]$ matrix represent the characteristic differential equations of the wave equation.
Example 2: General characteristic form of compressible flow system (*This is somewhat advance and we will discuss more in Module 5.*)

Consider a system of first-order partial differential equations:

$$\frac{\partial \mathbf{u}}{\partial t} + A\frac{\partial \mathbf{u}}{\partial x} = 0 \tag{2}$$

where $\mathbf{u} = \mathbf{u}(x,t)$ and A is a square matrix. This equation is more general than it first appears - if the coefficient of $\partial \mathbf{u}/\partial t$ is an invertible matrix, simply multiply both sides of the equation by the inverse of the matrix to put the equation in form Eq. (2). This equation is linear if $A = const.$ and $quasi$ -linear if $A = A(\mathbf{u}, x, t)$. An equation or system of equations with a complete wave description is sometimes called hyperbolic. On the other hand, the system of Equations (1) is hyperbolic if and only if A is diagonalizable. In other words, the system of Equations is hyperbolic if and only if

$$Q^{-1}AQ = \Lambda$$

for some matrix Q, where Λ is a diagonal matrix. More specifically, Λ is a diagonal matrix whose diagonal elements λ_i are characteristic values or eigenvalues of A, Q is a matrix whose columns $\mathbf{r}_i$ are right characteristic vectors or right eigenvectors of A, and Q^{-1} is a matrix whose rows l_i are left characteristic vectors or left eigenvectors of A. As most students will recall from elementary linear algebra, right characteristic vectors are defined as follows:

$$A\mathbf{r}_i = \lambda_i \mathbf{r}_i$$

Multiply both sides of Equation (2) by Q^{-1} to obtain

$$Q^{-1}\frac{\partial \mathbf{u}}{\partial t} + Q^{-1}A\frac{\partial \mathbf{u}}{\partial x} = 0 \tag{3}$$

The above equation is called "characteristic form". We define the characteristic variable $d\mathbf{v} = Q^{-1}d\mathbf{u}$ and then above equation becomes:

$$\frac{\partial \mathbf{v}}{\partial t} + Q^{-1}AQ\frac{\partial \mathbf{v}}{\partial x} = 0 \tag{4}$$

which can be expressed as

$$\frac{\partial \mathbf{v}}{\partial t} + \Lambda\frac{\partial \mathbf{v}}{\partial x} = 0 \tag{5}$$

The characteristic form is a wave form. To see this, consider the i -th equation in Eq. (5):

$$\frac{\partial v_i}{\partial t} + \lambda_i\frac{\partial v_i}{\partial x} = 0$$

This is like first-order advection equation except that, for quasi-linear systems of equations, λ_i depends on all of the characteristic variables and not just on the single characteristic variable v_i. Despite this difference, the same analysis applies

$$v_i = \text{ const.} \quad \text{ for } \quad \frac{dx}{dt} = \lambda_i$$

The curves $dx = \lambda_i dt$ are called wavefronts or characteristics; the variables v_i are called signals or information carried by the waves or characteristic variables; and the characteristic values λ_i are called wave speeds or characteristic speeds or signal speeds. The term "characteristic" is used here because the analysis depends heavily on the characteristic values and characteristic vectors of matrix A.