1.2 a) $\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} = 0$

$$\frac{\partial \phi}{\partial t} = \underbrace{-u\frac{\partial \phi}{\partial x} - v\frac{\partial \phi}{\partial y}}_{\text{convective term}}$$

BS - $\frac{u^n_{i,j} - u^n_{i-1,j}}{\Delta x} + \frac{u^n_{i,j} - u^n_{i,j-1}}{\Delta y}$

FS - $\frac{u^n_{i+1,j} - u^n_{i,j}}{\Delta x} + \frac{u^n_{i,j} - u^n_{i,j-1}}{\Delta y}$

Upwind

$$-\frac{u\Delta t}{\Delta x}(u^n_{i,j} - u^n_{i-1,j}) - \frac{v\Delta t}{\Delta y}(u^n_{i,j} - u^n_{i,j-1}), \quad u>0, v>0 \quad (1)$$

$$-\frac{u\Delta t}{\Delta x}(u^n_{i+1,j} - u^n_{i,j}) - \frac{v\Delta t}{\Delta y}(u^n_{i,j+1} - u^n_{i,j}), \quad u<0, v<0 \quad (2)$$

$$-\frac{u\Delta t}{\Delta x}(u^n_{i,j} - u^n_{i-1,j}) - \frac{v\Delta t}{\Delta y}(u^n_{i,j+1} - u^n_{i,j}), \quad u>0, v<0 \quad (3)$$

$$-\frac{u\Delta t}{\Delta x}(u^n_{i+1,j} - u^n_{i,j}) - \frac{v\Delta t}{\Delta y}(u^n_{i,j} - u^n_{i,j-1}), \quad u<0, v>0 \quad (4)$$

Define

$u^+ = \frac{1}{2}(u + |u|)$, $u^+ = u$ if $u>0$, 0 if $u<0$ (5)

$u^- = \frac{1}{2}(u - |u|)$, $u^- = u$ if $u<0$, 0 if $u>0$ (6)

$v^+ = \frac{1}{2}(v + |v|)$, $v^+ = v$ if $v>0$, 0 if $v<0$ (7)

$v^- = \frac{1}{2}(v - |v|)$, $v^- = v$ if $v<0$, 0 if $v>0$ (8)

Combine the previous to get a general 2D upwind discretization

$$= -\frac{\Delta t}{\Delta x}\left[u^{+}\left(u^{n}_{i,j} - u^{n}_{i-1,j}\right) + u^{-}\left(u^{n}_{i+1,j} - u^{n}_{i,j}\right)\right]$$

$$- \frac{\Delta t}{\Delta y}\left[v^{+}\left(u^{n}_{i,j} - u^{n}_{i,j-1}\right) + v^{-}\left(u^{n}_{i,j+1} - u^{n}_{i,j}\right)\right]$$

$$= -\frac{\Delta t}{\Delta x}\left[\frac{1}{2}\left(u+|u|\right)\left(u^{n}_{i,j} - u^{n}_{i-1,j}\right) + \frac{1}{2}\left(u-|u|\right)\left(u^{n}_{i+1,j} - u^{n}_{i,j}\right)\right]$$

$$- \frac{\Delta t}{\Delta y}\left[\frac{1}{2}\left(v+|v|\right)\left(u^{n}_{i,j} - u^{n}_{i,j-1}\right) + \frac{1}{2}\left(v-|v|\right)\left(u^{n}_{i,j+1} - u^{n}_{i,j}\right)\right]$$

$$= -\frac{\Delta t}{\Delta x}\Big[\frac{1}{2}\left(u\,u^{n}_{i,j} - u\,u^{n}_{i-1,j} + |u|\,u^{n}_{i,j} - |u|\,u^{n}_{i-1,j}\right)$$
$$+ \frac{1}{2}\left(u\,u^{n}_{i+1,j} - u\,u^{n}_{i,j} - |u|\,u^{n}_{i+1,j} + |u|\,u^{n}_{i,j}\right)\Big]$$

$$-\frac{\Delta t}{\Delta y}\Big[\frac{1}{2}\left(v\,u^{n}_{i,j} - v\,u^{n}_{i,j-1} + |v|\,u^{n}_{i,j} - |v|\,u^{n}_{i,j-1}\right) +$$
$$\frac{1}{2}\left(v\,u^{n}_{i,j+1} - v\,u^{n}_{i,j} - |v|\,u^{n}_{i,j+1} + |v|\,u^{n}_{i,j}\right)\Big]$$

$$= -\frac{\Delta t}{2\Delta x}\Big[u\,\cancel{u^{n}_{i,j}} - u\,u^{n}_{i-1,j} + \overbrace{|u|\,u^{n}_{i,j}}^{2} - |u|\,u^{n}_{i-1,j} +$$
$$u\,u_{i+1,j} - \cancel{u\,u^{n}_{i,j}} - |u|\,u^{n}_{i+1,j} + \cancel{|u|\,u^{n}_{i,j}}\Big]$$

$$- \frac{\Delta t}{2\Delta y}\Big[v\,\cancel{u^{n}_{i,j}} - v\,u^{n}_{i,j-1} + \overbrace{|v|\,u^{n}_{i,j}}^{3} - |v|\,u^{n}_{i,j-1} +$$
$$v\,u^{n}_{i,j+1} - v\,\cancel{u^{n}_{i,j}} - |v|\,u^{n}_{i,j+1} + \cancel{|v|\,u^{n}_{i,j}}\Big]$$

$$\approx -\frac{\Delta t}{2\Delta x}\left[u\left(u^n_{i+1,j}-u^n_{i-1,j}\right)+|u|\left(2u^n_{i,j}-u^n_{i-1,j}-u^n_{i+1,j}\right)\right]$$

$$-\frac{\Delta t}{2\Delta y}\left[v\left(u^n_{i,j+1}-u^n_{i,j-1}\right)+|v|\left(2u^n_{i,j}-u^n_{i,j-1}-u^n_{i,j+1}\right)\right]$$

$$\approx -\frac{u\Delta t}{2\Delta x}\left(u^n_{i+1,j}-u^n_{i-1,j}\right)+\frac{|u|\Delta t}{2\Delta x}\left(u^n_{i+1,j}-2u^n_{i,j}+u^n_{i-1,j}\right)$$

$$-\frac{v\Delta t}{2\Delta y}\left(u^n_{i,j+1}-u^n_{i,j-1}\right)+\frac{|v|\Delta t}{2\Delta y}\left(u^n_{i,j+1}-2u^n_{i,j}+u^n_{i,j-1}\right)$$

1.2b) $f\,T - \dfrac{u^{n+1}_{i,j}-u^n_{i,j}}{\Delta t}$

Complete upwind

$$u^{n+1}_{i,j}=u^n_{i,j}-\frac{u\Delta t}{2\Delta x}\left(u^n_{i+1,j}-u^n_{i-1,j}\right)+\frac{|u|\Delta t}{2\Delta \alpha}\left(u^n_{i+1,j}-2u^n_{i,j}+u^n_{i-1,j}\right)$$

$$-\frac{v\Delta t}{2\Delta y}\left(u^n_{i,j+1}-u^n_{i,j-1}\right)+\frac{|v|\Delta t}{2\Delta y}\left(u^n_{i,j+1}-2u^n_{i,j}+u^n_{i,j-1}\right)$$

1.3a)

2nd Order Backward

$$\frac{3u_i^n - 4u_{i-1}^n + u_{i-2}^n}{2\Delta x}$$

2nd Order forward

$$\frac{-u_{i+2} + 4u_{i+1} - 3u_{i}}{2\Delta x}$$

Define

$u^+ = \frac{1}{2}(u + |u|)$, $u^+ = u$ if $u > 0$, 0 if $u < 0$

$u^- = \frac{1}{2}(u - |u|)$, $u^- = u$ if $u < 0$, 0 if $u > 0$

$v^+ = \frac{1}{2}(v + |v|)$, $v^+ = v$ if $v > 0$, 0 if $v < 0$

$v^- = \frac{1}{2}(v - |v|)$, $v = v$ if $v < 0$, 0 if $v > 0$

2nd order Upwind

$$\approx -\frac{\Delta t}{2\Delta x}\left[u^+(3u_{i,j}^n - 4u_{i-1,j}^n + u_{i-2,j}^n) + u^-(-3u_{i,j}^n + 4u_{i+1,j}^n - u_{i+2,j}^n)\right]$$
$$-\frac{\Delta t}{2\Delta y}\left[v^+(3u_{i,j}^n - 4u_{i,j-1}^n + u_{i,j-2}^n) + v^-(-3u_{i,j}^n + 4u_{i,j+1}^n - u_{i,j+2}^n)\right]$$

$$= -\frac{\Delta t}{2\Delta x}\left[\frac{1}{2}(u + |u|)(3u_{i,j}^n - 4u_{i-1,j}^n + u_{i-2,j}^n) + \frac{1}{2}(u - |u|)(-3u_{i,j}^n + 4u_{i+1,j}^n - u_{i+2,j}^n)\right]$$

$$-\frac{\Delta t}{2\Delta y}\left[\frac{1}{2}(v + |v|)(3u_{i,j}^n - 4u_{i,j-1}^n + u_{i,j-2}^n) + \frac{1}{2}(v - |v|)(-3u_{i,j}^n + 4u_{i,j+1}^n - u_{i,j+2}^n)\right]$$

$$= -\frac{\Delta t}{4\Delta x}\Big[u(\cancel{3u_{i,j}^n} - 4u_{i-1,j}^n + u_{i-2,j}^n - \cancel{3u_{i,j}^n} + 4u_{i+1,j}^n - u_{i+2,j}^n)$$
$$+ |u|(\underbrace{\overset{6}{\cancel{3}}u_{i,j}}_{} - 4u_{i-1,j}^n + u_{i-2,j}^n + \cancel{3u_{i,j}^n} - 4u_{i+1,j}^n + u_{i+2,j}^n)\Big]$$

$$-\frac{\Delta t}{4\Delta y}\Big[v(\cancel{3u_{i,j}^n} - 4u_{i,j-1}^n + u_{i,j-2}^n - \cancel{3u_{i,j}^n} + 4u_{i,j+1}^n - 4u_{i,j+2}^n)$$

$$+ |v|(\overset{6}{\cancel{3}}u_{i,j}^n - 4u_{i,j-1}^n + u_{i,j-2}^n + \cancel{3u_{i,j}^n} - 4u_{i,j+1}^n + u_{i,j+2}^n)\Big]$$

$$= -\frac{u\Delta t}{4\Delta x}\left[u_{i-2,j}^{n} - 4u_{i-1,j}^{n} + 4u_{i+1,j}^{n} - u_{i+2,j}^{n}\right]$$

$$-\frac{|u|\Delta t}{4\Delta x}\left[u_{i-2,j}^{n} - 4u_{i-1,j}^{n} + 6u_{i,j}^{n} - 4u_{i+1,j}^{n} + u_{i+2,j}^{n}\right]$$

$$-\frac{v\Delta t}{4\Delta y}\left[u_{i,j-2}^{n} - 4u_{i,j-1}^{n} + 4u_{i,j+1}^{n} - u_{i,j+2}^{n}\right]$$

$$-\frac{|v|\Delta t}{4\Delta y}\left[u_{i,j-2}^{n} - 4u_{i,j-1}^{n} + 6u_{i,j}^{n} - 4u_{i,j+1}^{n} + 4_{i,j+2}^{n}\right]$$

1.3b) 2^{nd} Order Adams Bashforth method

$$u^{n+1} = u^{n} + \Delta t\left(\frac{3}{2}g(t^{n}, u^{n}) - \frac{1}{2}g(t^{n-1}, u^{n-1})\right)$$

Taking

$$-\frac{u\Delta t}{4\Delta x}\left[u_{i-2,j}^{n} - 4u_{i-1,j}^{n} + 4u_{i+1,j}^{n} - u_{i+2,j}^{n}\right]$$

$$-\frac{|u|\Delta t}{4\Delta x}\left[u_{i-2,j}^{n} - 4u_{i-1,j}^{n} + 6u_{i,j}^{n} - 4u_{i+1,j}^{n} + u_{i+2,j}^{n}\right]$$

$$-\frac{v\Delta t}{4\Delta y}\left[u_{i,j-2}^{n} - 4u_{i,j-1}^{n} + 4u_{i,j+1}^{n} - u_{i,j+2}^{n}\right]$$

$$-\frac{|v|\Delta t}{4\Delta y}\left[u_{i,j-2}^{n} - 4u_{i,j-1}^{n} + 6u_{i,j}^{n} - 4u_{i+1,j}^{n} + 4_{i+2,j}^{n}\right]$$

as $g(t^n, u^n)$ (spatial derivatives) we obtain the following

$$
\begin{aligned}
u_{i,j}^{n+1} = u_{i,j}^{n} + \Delta t \Bigg[\frac{3}{2} \Bigg(& -\frac{u\Delta t}{4\Delta x}\left[u_{i-2,j}^{n} - 4u_{i-1,j}^{n} + 4u_{i+1,j}^{n} - u_{i+2,j}^{n}\right] \\
& - \frac{|u|\Delta t}{4\Delta x}\left[u_{i-2,j}^{n} - 4u_{i-1,j}^{n} + 6u_{i,j}^{n} - 4u_{i+1,j}^{n} + u_{i+2,j}^{n}\right] \\
& - \frac{v\Delta t}{4\Delta y}\left[u_{i,j-2}^{n} - 4u_{i,j-1}^{n} + 4u_{i+1,j}^{n} - u_{i+2,j}^{n}\right] \\
& - \frac{|v|\Delta t}{4\Delta y}\left[u_{i,j-2}^{n} - 4u_{i,j-1}^{n} + 6u_{i,j}^{n} - 4u_{i+1,j}^{n} + u_{i+2,j}^{n}\right] \Bigg) \\
- \frac{1}{2} \Bigg(& -\frac{u\Delta t}{4\Delta x}\left[u_{i-2,j}^{n-1} - 4u_{i-1,j}^{n-1} + 4u_{i+1,j}^{n-1} - u_{i+2,j}^{n-1}\right] \\
& - \frac{|u|\Delta t}{4\Delta x}\left[u_{i-2,j}^{n-1} - 4u_{i-1,j}^{n-1} + 6u_{i,j}^{n-1} - 4u_{i+1,j}^{n-1} + u_{i+2,j}^{n-1}\right] \\
& - \frac{v\Delta t}{4\Delta y}\left[u_{i,j-2}^{n-1} - 4u_{i,j-1}^{n-1} + 4u_{i+1,j}^{n-1} - u_{i+2,j}^{n-1}\right] \\
& - \frac{|v|\Delta t}{4\Delta y}\left[u_{i,j-2}^{n-1} - 4u_{i,j-1}^{n-1} + 6u_{i,j}^{n-1} - 4u_{i+1,j}^{n-1} + u_{i+2,j}^{n-1}\right] \Bigg)
\end{aligned}
$$

Central difference - Advection

$$\left(u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y}\right) = u\left(\frac{\phi^n_{i+1,j} - \phi^n_{i-1,j}}{2\Delta x}\right) + v\left(\frac{\phi^n_{i,j+1} - \phi^n_{i,j-1}}{2\Delta y}\right)$$

Central difference - Diffusion

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\phi^n_{i+1,j} - 2\phi^n_{i,j} + \phi^n_{i-1,j}}{\Delta x^2} + \frac{\phi^n_{i,j+1} - 2\phi^n_{i,j} + \phi^n_{i,j-1}}{\Delta y^2}$$

2.1) $$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} = \alpha\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right)$$

We can let $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \nabla^2 \phi$

Newton Raphson

- Start with guess ϕ_0
- update $\phi_{k+1} = \phi_k + \delta\phi_k$
- Hope $\nabla^2 \phi_{k+1} = 0$

If $\nabla^2 \phi_{k+1} = 0 \rightarrow \nabla^2(\phi_k + \delta\phi_k) = 0$

Only doing for diffusion $\rightarrow$ Linear

$\nabla^2(\phi_k + \delta\phi_k) = \nabla^2 \phi_k + \nabla^2(\delta\phi_k) \Rightarrow \nabla^2(\delta\phi_k) = -\nabla^2\phi_k$

Update the solution $\phi_{k+1} = \phi_k + \delta\phi_k$ until $\|\delta\phi_k\|_2 < tol$

We have $\nabla^2(\delta\phi_k) = -\nabla^2\phi_k$

True solution is $\nabla^2\phi = 0$ so we have $-\nabla\phi_k$ as our residual

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\delta\phi_k = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi_k$$

We discretize ϕ using Central difference,

$$A\,\delta\phi_k = -R$$

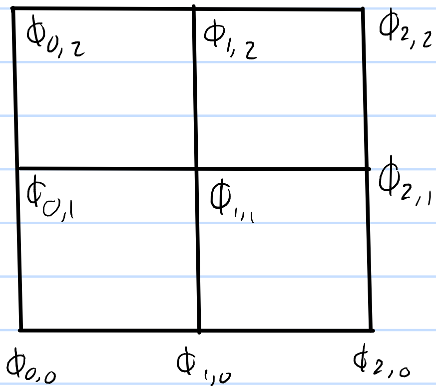
Discretizing

$$\left(\frac{\partial^2}{\partial x^2}\delta\phi\right)_{i,j} = \frac{\delta\phi_{i-1,j} - 2\delta\phi_{i,j} + \delta\phi_{i+1,j}}{\Delta x^2}$$

$$\left(\frac{\partial^2}{\partial y^2}\delta\phi\right)_{i,j} = \frac{\delta\phi_{i,j-1} - 2\delta\phi_{i,j} + \delta\phi_{i,j+1}}{\Delta y^2}$$

so $$\left(\frac{\partial^2 \delta\phi}{\partial x^2} + \frac{\partial^2 \delta\phi}{\partial y^2}\right)_{i,j} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\Delta x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\Delta y^2}$$

Take 3x3 matrix



We can discretize $\phi_{1,1}$ to be

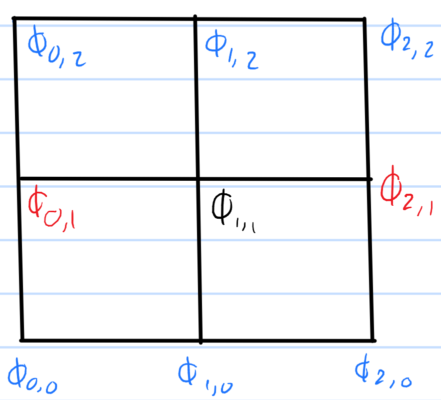
$$\frac{\phi_{0,1}-2\phi_{1,1}+\phi_{2,1}}{\Delta x^2}+\frac{\phi_{1,0}-2\phi_{1,1}+\phi_{1,2}}{\Delta y^2}$$

Isolate all ϕ

$$\delta\phi_{1,1}=\frac{1}{\Delta x^2}\delta\phi_{0,1}+\frac{1}{\Delta y^2}\delta\phi_{1,0}+\left(-\frac{2}{\Delta x^2}-\frac{2}{\Delta y^2}\right)\delta\phi_{1,1}+\frac{1}{\Delta y^2}\delta\phi_{1,2}+\frac{1}{\Delta x^2}\delta\phi_{2,1}$$

At boundary nodes - If Neumann we need to update $\delta\phi$

- If Dirichlet, value is known so $\delta\phi=0$



- In this case we have Neumann boundary on top & bottom

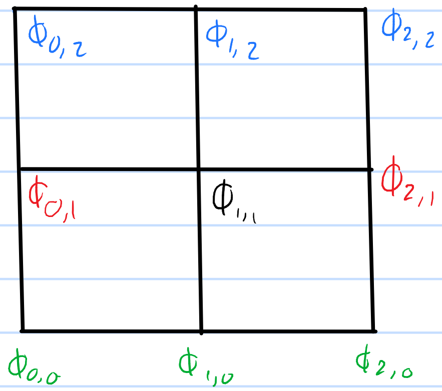
- Dirichlet Boundary
- Neumann Boundary

At Neumann boundaries we obtain $\frac{\partial\phi}{\partial n}=h$, where h is the Prescribed Neumann value

We will apply forward difference in y on bottom boundary & top boundary on top. We get the following

$$\frac{\delta\phi_{0,1}^{k+1}-\delta\phi_{0,0}^{k+1}}{\Delta y}=h\rightarrow\frac{(\phi_{1,1}^{k}+\delta\phi_{1,1}^{k})-(\phi_{1,0}^{k}+\delta\phi_{1,0}^{k})}{\Delta y}=h$$

$$\frac{1}{\Delta y}\delta\phi_{0,1}-\frac{1}{\Delta y}\delta\phi_{0,0}=h-\frac{\phi_{0,1}-\phi_{0,0}}{\Delta y}\leftarrow \text{Point } 0,0$$

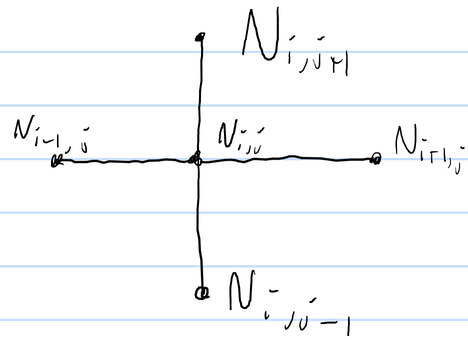


$\delta\phi_{0,1} = 0$ $\quad \delta\phi_{0,0} =$

$\delta\phi_{2,1} = 0$

r, dPhi, phi, V, G,

α



for bottom Boundary

$$\underbrace{\frac{1}{\Delta y}\delta\phi^k_{i,1} - \frac{1}{\Delta y}\delta\phi^k_{i,0}} = \underbrace{h - \frac{\phi^k_{i,1} - \phi^k_{i,0}}{\Delta y}}$$

for Top boundary

$$\frac{1}{\Delta y}\delta\phi_{i,N_y-1} - \frac{1}{\Delta y}\delta\phi_{i,N_y-2} = h - \frac{\phi_{i,N_y-2} - \phi_{i,N_y-1}}{\Delta y}$$

Matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\Delta x^2} & 0 & \frac{1}{\Delta y^2} & \left(-\frac{2}{\Delta x^2} - \frac{2}{\Delta y^2}\right) & \frac{1}{\Delta y^2} & 0 & \frac{1}{\Delta x^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta\phi_{0,0} \\ \delta\phi_{0,1} \\ \delta\phi_{0,2} \\ \delta\phi_{1,0} \\ \delta\phi_{1,1} \\ \delta\phi_{1,2} \\ \delta\phi_{2,0} \\ \delta\phi_{2,1} \\ \delta\phi_{2,2} \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

$$-\frac{1}{\Delta y}\delta\phi_{i,0} = h - \frac{\phi_{i,1}-\phi_{i,0}}{\Delta y} - \frac{1}{\Delta y}\delta\phi_{i,1}$$

$$\delta\phi_{i,0} = (\phi_{i,1}-\phi_{i,0}) + \delta\phi_{i,1} - h\Delta y$$

for Top boundary

$$\frac{1}{\Delta y}\delta\phi_{i,N_y-2} - \frac{1}{\Delta y}\delta\phi_{i,N_y-1} = h - \frac{\phi_{i,N_y-2}-\phi_{i,N_y-1}}{\Delta y}$$