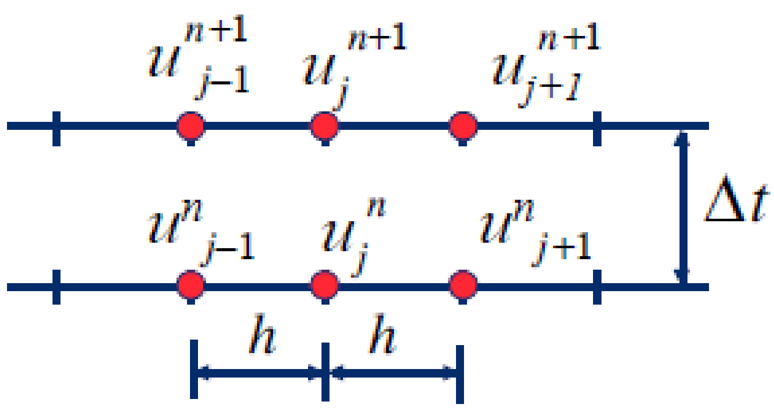
THE UNIVERSITY OF BRITISH COLUMBIA
Department of Mechanical Engineering
MECH 479/587 Computational Fluid Dynamics
Winter Term 1, 2022

Problem Set #3. Due Nov 4, 2022

Question 1: Consider the following finite difference approximation

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -\frac{1}{2}\left(\frac{c}{2h}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{c}{2h}\left(u_{j+1}^{n+1} - u_{j-1}^{n+1}\right)\right)$$

where c denotes the constant speed and h is the grid size.



a) Write down the modified (equivalent differential) equation.
b) What differential equation is being approximated?
c) Determine the accuracy of the scheme.
d) Use the von Neumann's procedure to derive an equation for the stability condition.

(20 pts)

Question 2: Let us discretize the 1D diffusion equation ($\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$) using the following spatial-temporal finite difference approximation:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\alpha}{2(\Delta x)^2}\left(\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right) + \left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)\right)$$

a) Check if the above discretization is consistent.
b) Use von Neumann analysis and determine if (and under what conditions) the finite difference scheme is stable.
c) Discuss the convergence of the finite difference scheme using the Lax Equivalence Theorem.

(10 pts)

Question 3: Consider a simplified 2D species transport (mass diffusion) equation of coronavirus-laden droplets represented by scalar field ϕ

$$\frac{\partial \phi}{\partial t} = \alpha\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right)$$

where α denotes the diffusion parameter. Perform a discretization via explicit forward difference in time and the central difference in space. Evaluate the stability condition through the von Neumann analysis. For simplicity, you can assume the mesh resolution $\Delta x = \Delta y = h$.

(20 pts)

1) $$\frac{u_j^{n+1}-u_j^n}{\Delta t} = -\frac{1}{2}\left[\frac{c}{2h}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{c}{2h}\left(u_{j+1}^{n+1} - u_{j-1}^{n+1}\right)\right]$$

a) Modified

$$\frac{1}{\Delta t}\left[\cancel{u_j^n} + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} + \frac{\Delta t^3}{3!} u_{ttt} + \dots - \cancel{u_j^n}\right]$$

$$= \frac{-c}{4h}\left[\cancel{u_j^n} + h u_x + \cancel{\frac{h^2}{2} u_{xx}} + \frac{h^3}{3!} u_{xxx} + \dots - \cancel{u_j} + h u_x - \cancel{\frac{h^2}{2} u_{xx}}\right.$$

$$+ \frac{h^3}{3!} u_{xxx} - \dots + \cancel{u_j^{n+1}} + h u_x^{n+1} + \cancel{\frac{h^2}{2} u_{xx}^{n+1}} + \frac{h^3}{3!} u_{xxx}^{n+1} + \dots$$

$$\left. - \cancel{u_j^{n+1}} + u_x^{n+1} h - \cancel{u_{xx}^{n+1} \frac{h^2}{2}} + u_{xxx}^{n+1} \frac{h^3}{3!} \dots \right]$$

$$\begin{cases} u_x^{n+1} = u_x^n + \Delta t u_{xt} + \frac{\Delta t^2}{2} u_{xtt} + \frac{\Delta t^3}{3!} u_{xttt} + \dots \\ u_{xxx}^{n+1} = u_{xxx}^n + \Delta t u_{xxxt} + \frac{\Delta t^2}{2} u_{xxxtt} + \frac{\Delta t^3}{3!} u_{xxxttt} + \dots \end{cases}$$

$$u_t + c u_x = -\frac{\Delta t}{2} u_{tt} - \frac{c}{4h} 2h \Delta t u_{xt} - \frac{\Delta t^2}{3!} u_{ttt} - \frac{c}{4h} 2h \frac{\Delta t^2}{2} u_{xtt} -$$

$$\frac{c}{4h} 2h \frac{\Delta t^3}{3!} u_{xttt} - \frac{\Delta t^3}{4!} u_{tttt} - \frac{c}{4h}\frac{4h^3}{3!} u_{xxx} - \frac{c}{4h}\frac{2h^3}{3!} \Delta t u_{xxxt}$$

$$u_t + c u_x = -\frac{\Delta t}{2} c^2 u_{xx} - \frac{c}{4h} 2h \Delta t \cdot \frac{1}{-c} c^2 u_{xx} - \frac{\Delta t^2 (-c^3)}{3!} u_{xxx}$$

$$- \frac{c}{4h} 2h \frac{\Delta t^2}{2} \left(\frac{1}{-c}\right)(-c^3) u_{xxx} - \frac{c}{4h} 2h \frac{\Delta t^3}{3!} \left(\frac{1}{-c}\right) c^4 u_{xxxx}$$

$$- \frac{\Delta t^3}{4!} c^4 u_{xxxx} - \frac{c}{4h}\frac{4h^3}{3!} u_{xxx} - \frac{c}{4h}\frac{2h^3}{3!} \Delta t \frac{1}{-c^3} c^4 u_{xxxx}$$

$$U_t + C U_x = U_{xxx}\left[\Delta t^2 C^3\left(-\frac{1}{12}\right) - \frac{Ch^2}{6}\right\} + \ldots$$

b) The differential equation found in a)

c) Scheme accuracy $O(\Delta t^2, \Delta x^2)$

d) $$U_j^{n+1} - U_j^n = -\frac{\Delta t}{2}\frac{C}{2h}\left[U_{j+1}^n - U_{j-1}^n + U_{j+1}^{n+1} - U_{j-1}^{n+1}\right]$$

Substitute $U_j^n = V^n e^{ikx_j}$

$$V^{n+1}e^{ikx_j} - V^n e^{ikx_j} = S\left[V^n e^{ikx_j} e^{ik\Delta x} - V^n e^{ikx_j} e^{-ik\Delta x} + V^{n+1} e^{ikx_j} e^{ik\Delta x} - V^{n+1} e^{ikx_j} e^{-ik\Delta x}\right]$$

$\theta = k\Delta x$, $G = \frac{V^{n+1}}{V^n}$

$$G - 1 = S\left[e^{i\theta} - e^{-i\theta} + Ge^{i\theta} - Ge^{-i\theta}\right]$$

$$G - 1 = S\left[2i\sin\theta(1+G)\right]$$

$$G\left[1 - 2i\sin\theta\right] = 1 + 2S \cdot i\sin\theta$$

$$G = \frac{1+2S\cdot i\sin\theta}{1-2S\cdot i\sin\theta} = \frac{(2S\cdot i\sin\theta + 1)^2}{1-4S^2\sin^2\theta} = \frac{1+4S\cdot i\sin\theta + 4S^2\sin^2\theta}{1-4S^2\sin^2\theta}$$

$$= \frac{1+4S^2\sin^2\theta}{1-4S^2\sin^2\theta} + \frac{4S\cdot i\sin\theta}{1-4S^2\cdot\sin^2\theta}$$

$$|G| = \sqrt{\left(\frac{1+4S^2\sin^2\theta}{1-4S^2\sin^2\theta}\right)^2 - \left(\frac{4S\cdot\sin\theta}{1-4S^2\sin^2\theta}\right)^2} \qquad S^2\sin^2\theta = A$$

$$|G| = \sqrt{\frac{1+8A^2+16A^4-16A^4}{(1-4A^2)^2}} = \sqrt{\frac{(1-4A^2)^2}{(1-4A^2)^2}} = \pm 1$$

This discretization is stable

2a) $u^n_{i\pm1} = u^n_i \pm \Delta x (u_x)^n_i + \frac{\Delta x^2}{2}(u_{xx})^n_i \pm \frac{\Delta x^3}{3!}(u_{xxx})^n_i + \frac{\Delta x^4}{4!}(u_{xxxx})^n_i + \dots$

$u^n_i = u^n_i$

$u_j^{n+1} = u_j^n + \Delta t (u_t)^n_i + \frac{\Delta t^2}{2}(u_{tt})^n_i + \frac{\Delta t^3}{3!}(u_{ttt})^n_i + \frac{\Delta t^4}{4!}(u_{tttt})^n_i + \dots$

LHS

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} = \frac{1}{\cancel{\Delta t}}\Big[\cancel{u^n_i} + \cancel{\Delta t}(u_t)^n_i + \frac{\Delta t^2}{2}(u_{tt})^n_i + \frac{\Delta t^3}{3!}(u_{ttt})^n_i + \frac{\Delta t^4}{4!}(u_{tttt})^n_i - \cancel{u^n_i}$$

$$= (u_t)^n_i \quad \frac{\Delta t}{2}(u_{tt})^n_i + \frac{\Delta t^2}{3!}(u_{ttt})^n_i + \frac{\Delta t^3}{4!}(u_{tttt})^n_i$$

RHS

$$\frac{\alpha}{2\Delta x^2}\left(u^{n+1}_{i+1} - 2u^{n+1}_i + u^{n+1}_{i-1} + u^n_{i+1} - 2u^n_i + u^n_{i-1}\right) =$$

$$\Big[u_i^{n+1} + \cancel{\Delta x (u_x)^{n+1}_i} + \frac{\Delta x^2}{2!}(u_{xx})^{n+1}_i + \cancel{\frac{\Delta x^3}{3!}(u_{xxx})^{n+1}_i} + \frac{\Delta x^4}{4!}(u^{(4)})^{n+1}_i\Big]$$

$$-2\Big[u^n_i + \Delta t (u_t)^n_i + \frac{\Delta t^2}{2!}(u_{tt})^n_i + \frac{\Delta t^3}{3!}(u_{ttt})^n_i + \frac{\Delta t^4}{4!}(u^{(4)})^n_i$$

$$+\Big[u^{n+1}_i - \cancel{\Delta x(u_x)^{n+1}_i} + \frac{\Delta x^2}{2!}(u_{xx})^{n+1}_i - \cancel{\frac{\Delta x^3}{3!}(u_{xxx})^{n+1}_i} + \frac{\Delta x^4}{4!}(u^{(4)})^{n+1}_i\Big]$$

$$+\Big[u^n_i + \cancel{\Delta x(u_x)^n_i} + \frac{\Delta x^2}{2!}(u_{xx})^n_i + \cancel{\frac{\Delta x^3}{3!}(u_{xxx})^{n+1}_i} + \frac{\Delta x^4}{4!}(u^{(4)})^n_i\Big]$$

$$-2u^n_i + \Big[u^n_i - \cancel{\Delta x(u_x)^n_i} + \frac{\Delta x^2}{2!}(u_{xx})^n_i - \cancel{\frac{\Delta x^3}{3!}(u_{xxx})^{n+1}_i} + \frac{\Delta x^4}{4!}(u^{(4)})^n_i\Big]\Big)$$

$$= \Big[\Big(2U_i^{n+1} + \Delta x^2 (U_{xx})_i^{n+1} + \frac{2\Delta x^4}{4!} (U_{xxxx})_i^{n+1} \Big) -$$

$$2\Big[U_i^n + \Delta t (U_t)_i^n + \frac{\Delta t^2}{2!} (U_{tt})_i^n + \frac{\Delta t^3}{3!} (U_{ttt})_i^n + \frac{\Delta t^4}{4!} (U_{tttt})_i^n \Big]$$

$$+ \Big(2U_i^n + \Delta x^2 (U_{xx})_i^{n+1} + \frac{2\Delta x^4}{4!} (U_{xxxx})_i^n \Big) \Big]$$

Let LHS = RHS

$$(U_t)_i^n + \frac{\Delta t}{2} (U_{tt})_i^n + \frac{\Delta t^2}{3!} (U_{ttt})_i^n + \frac{\Delta t^3}{4!} (U_{tttt})_i^n = \frac{\alpha}{2\Delta x^2}$$

$$\Big[\Big(2\Big[U_i^n + \Delta t (U_t)_i^n + \frac{\Delta t^2}{2!} (U_{tt})_i^n + \frac{\Delta t^3}{3!} (U_{ttt})_i^n + \frac{\Delta t^4}{4!} (U_{tttt})_i^n \Big]$$

$$+ \Delta x^2 \Big[(U_{xx})_i^n + \Delta t (U_{xxt})_i^n + \frac{\Delta t^2}{2!} (U_{xxtt})_i^n + \frac{\Delta t^3}{3!} (U_{xxttt})_i^n$$

$$+ \frac{\Delta t^4}{4!} (U_{xxtttt})_i^n \Big]$$

$$+ \frac{2\Delta x^4}{4!} \Big[U_{xxxx}^n + \Delta t (U_{xxxxt})_i^n + \frac{\Delta t^2}{2!} (U_{xxxxtt})_i^n + \frac{\Delta t^3}{3!} (U_{xxxxttt})_i^n$$

$$+ \frac{\Delta t^4}{4!} (U_{xxxxtttt})_i^n \Big] -$$

$$2\Big[U_i^n + \Delta t (U_t)_i^n + \frac{\Delta t^2}{2!} (U_{tt})_i^n + \frac{\Delta t^3}{3!} (U_{ttt})_i^n + \frac{\Delta t^4}{4!} (U_{tttt})_i^n \Big]$$

$$+ \Big(2U_i^n + \Delta x^2 (U_{xx})_i^n + \frac{2\Delta x^4}{4!} (U_{xxxx})_i^n \Big) \Big]$$

$$(u_t)_i^n = \frac{\alpha}{2\Delta x^2}\Big[\cancel{2(u)_i^n} + \cancel{2\Delta t(u_t)_i^n} + \cancel{\Delta t^2(u_{tt})_i^n} + \cancel{\frac{2\Delta t^3}{3!}(u_{ttt})_i^n}$$

$$+ \cancel{\frac{2\Delta t^4}{4!}(u_{tttt})_i^n} + \Delta x^2(u_{xx})_i^n + \Delta x^2\Delta t(u_{xxt})_i^n +$$

$$\frac{\Delta x^2\Delta t^2}{2!}(u_{xxtt})_i^n + \frac{\Delta x^2\Delta t^3}{3!}(u_{xxttt})_i^n + \frac{\Delta x^2\Delta t^4}{4!}(u_{xxtttt})_i^n +$$

$$\frac{2\Delta x^4}{4!}(u_{xxxx})_i^n + \frac{2\Delta x^4\Delta t}{4!}(u_{xxxxt})_i^n + \frac{\Delta x^4\Delta t^2}{4!}(u_{xxxxtt})_i^n +$$

$$\frac{2\Delta x^4\Delta t^3}{3!\cdot 4!}(u_{xxxxttt})_i^n + \frac{2\Delta x^4\Delta t^4}{4!\cdot 4!}(u_{xxxxtttt})_i^n - \cancel{2u_i^n} - \cancel{2\Delta t(u_t)_i^n}$$

$$- \cancel{\Delta t^2(u_{tt})_i^n} - \cancel{\frac{2\Delta t^3}{3!}(u_{ttt})_i^n} - \cancel{\frac{2\Delta t^4}{4!}(u_{tttt})_i^n} + 2u_i^n +$$

$$\Delta x^2(u_{xx})_i^n + \frac{2\Delta x^4}{4!}(u_{xxxx})_i^n\Big] - \frac{\Delta t}{2}(u_{tt})_i^n - \frac{\Delta t^2}{3!}(u_{ttt})_i^n - \frac{\Delta t^3}{4!}(u_{tttt})_i^n$$

Sum all higher order terms into value Θ

$(u_t)_i^n = \alpha u_{xx} + \Theta$ If we subtract the original PDE we are just left with Θ. Taking

$\lim_{\Delta t, \Delta x \to 0} \Theta = 0$. The discretization is consistent

b) Plug $U^n e^{-ikx_j}$ into discretization

$$\frac{1}{\Delta t}\left[U^{n+1}e^{-ikx_j} - U^n e^{-ikx_j}\right] = \frac{\alpha}{2\Delta x^2}\Big[U^{n+1}e^{-ikx_j}e^{ik\Delta x} - 2U^{n+1}e^{ikx_j}$$

$$+ U^{n+1}e^{ikx_j}e^{-ik\Delta x} + U^n e^{ikx_j}e^{ik\Delta x} - 2U^n e^{ikx_j} +$$

$$U^n e^{ikx_j}e^{-ik\Delta x}\Big]$$

$$U^{n+1}e^{ikx_j} - U^n e^{ikx_j} = \frac{\alpha\Delta t}{2\Delta x^2}\Big[U^{n+1}e^{ikx_j}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)$$

$$+ U^n e^{ikx_j}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)\Big]$$

$$\cancel{e^{ikx_j}}\left(U^{n+1} - U^n\right) = \frac{\alpha\Delta t}{2\Delta x^2}\cancel{e^{ikx_j}}\left(U^{n+1} + U^n\right)\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)$$

$$U^{n+1} - U^n = \frac{\alpha\Delta t}{2\Delta x^2}\left(U^{n+1} + U^n\right)\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)$$

$$U^{n+1} - U^n = \frac{\alpha\Delta t}{2\Delta x^2}U^{n+1}\left(e^{-ik\Delta x} - 2 + e^{-ik\Delta x}\right) + \frac{\alpha\Delta t}{2\Delta x^2}U^n\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)$$

$$U^{n+1} - \frac{\alpha\Delta t}{2\Delta x^2}U^{n+1}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right) = \frac{\alpha\Delta t}{2\Delta x^2}U^n\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right) + U^n$$

$$U^{n+1}\left(1 - \frac{\alpha\Delta t}{2\Delta x^2}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)\right) = U^n\left(1 + \frac{\alpha\Delta t}{2\Delta x^2}\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)\right)$$

Let $S = \frac{\alpha\Delta t}{2\alpha^2}$, $\frac{U^{n+1}}{U^n} = G$, $k\Delta x = \theta$

$$G = \frac{1 + S(\cos\theta - 1)}{1 - S(\cos\theta - 1)}$$

$$G = \frac{1 + S(\cos\theta - 1)}{1 - S(\cos\theta - 1)}, \quad 2\sin^2\theta = 1 - \cos(2\theta)$$

$$G = \frac{1 + 2S(-2\sin^2\theta + 1 - 1)}{1 - 2S(-2\sin^2\theta + 1 - 1)} = \frac{1 - 4S(\sin^2\theta)}{1 + 4S(\sin^2\theta)} \leq 1$$

$\sin^2\theta \in [0, 1]$, & S is positive, then this is always true, thus the top is always smaller than the bottom so $|G| \leq 1$

Unconditionally Stable.

C) Because the discretization is unconditionally stable, & consistent, then it is convergent.

3) $\frac{\partial u}{\partial t} = \frac{u_{i+1} - u_i}{\Delta t}$ - forward Euler

$$\frac{\partial^2 u}{\partial x} = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}$$ - central difference

$$\frac{\partial \phi}{\partial t} = \alpha\left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right]$$

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \alpha\left[\frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{\Delta x^2} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{\Delta y^2}\right]$$

$\Delta x = \Delta y = h$

$$u_{i,j}^{n+1} - u_{i,j}^{n} = \frac{\alpha \Delta t}{h^2}\left[u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n} + u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}\right]$$

$$u_{i,j}^{n+1} = \frac{\alpha \Delta t}{h^2}\left[u_{i+1,j}^{n} + u_{i-1,j}^{n} + u_{i,j+1}^{n} + u_{i,j-1}^{n} - 4u_{i,j}^{n}\right] + u_{i,j}^{n}$$

$$u_{i,j}^{n+1} = \frac{\alpha \Delta t}{h^2}\left[u_{i+1,j}^{n} + u_{i-1,j}^{n} + u_{i,j+1}^{n} + u_{i,j-1}^{n} - u_{i,j}^{n}\left(4 + \frac{h^2}{\alpha \Delta t}\right)\right]$$

Let $\frac{\alpha \Delta t}{h^2} = S$

$$u_{i,j}^{n+1} = S\left[u_{i+1,j}^{n} + u_{i-1,j}^{n} + u_{i,j+1}^{n} + u_{i,j-1}^{n} - u_{i,j}^{n}\left(4 + \frac{1}{S}\right)\right]$$

Change i to l for simplicity

$$u_{l,j}^{n+1} = S\left[u_{l+1,j}^{n} + u_{l-1,j}^{n} + u_{l,j+1}^{n} + u_{l,j-1}^{n} - u_{l,j}^{n}\left(4 + \frac{1}{S}\right)\right]$$

$$u_{l,j}^{n+1} = S\left[u_{l+1,j}^{n} + u_{l-1,j}^{n} + u_{l,j+1}^{n} + u_{l,j-1}^{n} - u_{l,j}^{n}\left(4+\frac{1}{S}\right)\right]$$

Substitute $u_{l,j}^{n} = \upsilon^n e^{ik_x x_l} e^{ik_y y_j}$

$$\upsilon^{n+1} e^{ik_x x_l} e^{ik_y y_j} = S\left[\upsilon^n e^{ik_x(x_l+h)} e^{ik_y y_j} + \upsilon^n e^{ik_x(x_l-h)} e^{ik_y j} + \upsilon^n e^{ik_x x_l} e^{ik_y(y_j+h)} + \upsilon^n e^{ik_x x_l} e^{ik(y_j-h)} - \upsilon^n e^{ik_x x_l} e^{ik_y y_j}\left(4+\frac{1}{S}\right)\right]$$

$$\upsilon^{n+1} e^{ik_x x_l} e^{ik_y y_j} = S\left[\upsilon^n e^{ik_x x_l} e^{ik_y y_j} e^{ik_x h} + \upsilon^n e^{ik_x x_l} e^{ik_x y_j} e^{-ik_x h} + \upsilon^n e^{ik_x x_l} e^{ik_y y_j} e^{ik_y h} + \upsilon^n e^{ik_x x_j} e^{ik_y y_j} e^{-ik_y h} - \upsilon^n e^{ik_x x_j} e^{ik_y y_j}\left(4+\frac{1}{S}\right)\right]$$

$$\upsilon^{n+1} \cancel{e^{ik_x x_l}} \cancel{e^{ik_y y_j}} = S\upsilon^n \cancel{e^{ik_x x_l}} \cancel{e^{ik_y y_j}}\left(e^{ik_x h} + e^{-ik_x h} + e^{ik_y h} + e^{-ik_y h} - 4 - \frac{1}{S}\right)$$

$$\frac{\upsilon^{n+1}}{\upsilon^n} = S\left[e^{ik_x h} + e^{-ik_x h} + e^{ik_y h} + e^{-ik_y h} - 4 - \frac{1}{S}\right]$$

Let $\theta_x = k_x h$, $\theta_y = k_y h$

$$\frac{\upsilon^{n+1}}{\upsilon^n} = S\left[e^{i\theta_x} + e^{-i\theta_x} + e^{i\theta_y} + e^{-i\theta_y} - 4 - \frac{1}{S}\right]$$

$$\frac{\upsilon^{n+1}}{\upsilon^n} = S\left[e^{i\theta_x} - 2 + e^{-i\theta_x} + e^{i\theta_y} - 2 + e^{-i\theta_y}\right] + 1$$

$$\frac{\upsilon^{n+1}}{\upsilon^n} = 1 + S\left[2\cos(\theta_x) + 2\cos(\theta_y) - 4\right]$$

Let $G = \dfrac{\upsilon^{n+1}}{\upsilon^n}$

$$G = 1 + S\left[2\cos(\theta_x) + 2\cos(\theta_y) - 4\right]$$

$$G = 1 + S\left[-4\left(\sin^2(\theta_x) + \sin^2(\theta_y)\right)\right]$$

$$\sin^2(\theta) \in [0, 1]$$

The worst case is $\sin^2(\theta_x) + \sin^2(\theta_y) = 2$

$$G = 1 + \frac{\alpha \Delta t}{h^2}\left[-4(2)\right] = 1 - 8\frac{\alpha \Delta t}{h^2} \leq 1$$

$$8\frac{\alpha \Delta t}{h^2} \leq 2$$

$$\frac{\alpha \Delta t}{h^2} \leq \frac{1}{4}$$

for the equation to be stable, $\frac{\alpha \Delta t}{h^2} \leq \frac{1}{4}$

2 Might need

$$= \Bigg[\Big(2\Big[U_i^n + \Delta t (U_t)_i^n + \frac{\Delta t^2}{2!}(U_{tt})_i^n + \frac{\Delta t^3}{3!}(U_{ttt})_i^n + \frac{\Delta t^4}{4!}(U_{tttt})_i^n\Big]$$

$$+ \Delta x^2 \Big[(U_{xx})_i^n + \Delta t (U_{xxt})_i^n + \frac{\Delta t^2}{2!}(U_{xxtt})_i^n + \frac{\Delta t^3}{3!}(U_{xxttt})_i^n$$

$$+ \frac{\Delta t^4}{4!}(U_{xxtttt})_i^n\Big]$$

$$+ \frac{2\Delta x^4}{4!}\Big[U_{xxxx}^n + \Delta t (U_{xxxxt})_i^n + \frac{\Delta t^2}{2!}(U_{xxxxtt})_i^n + \frac{\Delta t^3}{3!}(U_{xxxxttt})_i^n$$

$$+ \frac{\Delta t^4}{4!}(U_{xxxxtttt})_i^n\Big] -$$

$$2\Big[U_i^n + \Delta t (U_t)_i^n + \frac{\Delta t^2}{2!}(U_{tt})_i^n + \frac{\Delta t^3}{3!}(U_{ttt})_i^n + \frac{\Delta t^4}{4!}(U_{tttt})_i^n\Big]$$

$$+ \Big(2U_i^n + \Delta x^2 (U_{xx})_i^{n+1} + \frac{2\Delta x^4}{4!}(U_{xxxx})_i^n\Big)\Bigg]$$