The University of British Columbia

MECH 479
Computational Fluid Dynamics
(Review of Second-Year Fluids)

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What is a Fluid?

A fluid is a state of matter that can't resist shear stress. That is, for a given fixed amount of shear stress, no matter how small, a fluid will continue to deform, unlike a solid, which for small amounts of shear deform a small amount and reach equilibrium.

At the microscopic level, solids can resist shear stress while fluids can't because the interatomic / intermolecular forces are stronger in a solid than in a fluid. Liquids have larger intermolecular forces than gases, which is why liquids stick together well enough to fill the bottom of a container instead of expanding to fill the whole container.

Examples of fluids: water, air, gasoline, oil, paint, etc.

1 Fluid Properties

1.1 Density

Mass per unit volume ($1/V$, where V is specific volume from thermo).

- Typical density of water: 998 kg/m^3
- Typical density of air: 1.2 kg/m^3

Example My mass is about 94 kg. About what mass of air do I displace?
I float (more or less), so my volume is $94 \text{ kg}/998 \text{ kg/m}^3 = 0.094 \text{ m}^3$. Mass of air displaced is $0.094 \text{ m}^3 \times 1.2 \text{ kg/ m}^3 = 0.113 \text{ kg}$.

1.2 Viscosity

Quantifies rate of transfer of momentum between slower and faster moving fluid.

Dynamic (μ) Units of $\frac{\text{kg}}{\text{m}\cdot\text{sec}}$. Relates shear stress to velocity gradient for a Newtonian fluid: $\tau = \mu \frac{\partial u}{\partial y}$ in the simplest case.

Kinematic (ν) Units of $\frac{\text{m}^2}{\text{sec}}$ [only length and time, like in particle kinematics (Mech 221)]. Related to μ by $\nu = \frac{\mu}{\rho}$

Values at 20 C, atmospheric pressure:

	$\mu \frac{\text{kg}}{\text{m}\cdot\text{sec}}$	$\nu \frac{\text{m}^2}{\text{sec}}$
Water	$1.003 \cdot 10^{-3}$	$1.005 \cdot 10^{-6}$
Air	$1.8 \cdot 10^{-5}$	$1.5 \cdot 10^{-5}$

Water and air are Newtonian fluids: viscous shear stress is proportional to the rate of strain. Things like toothpaste are called viscoplastic fluids: they have a yield stress, like solids, so they can hold their shape up to a point, but beyond that point they flow like liquids. Paint is a shear-thinning fluid: its viscosity decreases with increasing rate of strain. Other fluids, including concentrated suspensions of cornstarch in water, are shear-thickening: their viscosity increases with increasing rate of strain.

2 Hydrostatics

Pressure in a stationary fluid varies with depth, because of the weight of the fluid

$$\Delta P = -\rho g \Delta z$$

Pressure always acts perpendicular to a surface.

Units: Pascals (Newtons/square meter). Other units used: pounds (force) per square inch or square foot; or equivalent height of a column of some fluid. Atmospheric pressure is still commonly given in inches (or millimeters) of mercury, for instance.

Several useful definitions to remember:

Gauge pressure Pressure relative to atmospheric. This is the pressure that an order pressure gauge or manometer will measure.

Vacuum pressure Also relative to atmospheric, but measured downwards with atmospheric pressure being 0 vacuum pressure.

Specific weight: weight of a fluid per volume (ρg)

Specific gravity: density relative to a reference density.

- For liquids: specific gravity is $\rho / \rho_{\text{H}_2\text{O}}$
- For gases: specific gravity is ρ / ρ_{Air}

Example Find pressure at the bottom of the Mariana Trench (depth 11,034 m). The specific weight of sea water is $10,050 \frac{\text{N}}{\text{m}^3}$ at the surface and $10,520 \frac{\text{N}}{\text{m}^3}$ at the bottom.

We'll use the average of the specific weights at the surface and the bottom. (In principle we should integrate for a variable density, but here the variation is quite small, so the error introduce by using the average is small.)

$$\Delta P = 11,034 \times (10,050 + 10,520)/2 = 113.5\text{MPa absolute}$$

2.1 Manometry Example

Gas in new plastic manometer tubing can be 25% more dense than air. Suppose the gas in the tubing has a density of $\rho_t = 1.5\frac{\text{kg}}{\text{m}^3}$, the liquid in the U-tube has a density of $\rho_m = 860\frac{\text{kg}}{\text{m}^3}$, $H = 1.32\text{m}$, and $h = 0.58\text{cm}$.

- What is P_1?
- What would h be if the gas in the tubing were the same density as air?

Find pressure at 1 by working your way through the manometer.

$$\begin{aligned} P_1 - P_{\text{atm;1}} &= \rho_{\text{air}} g\,(H-h) + \rho_m g h - \rho_t g H \\ P_{1;\text{guage}} &= 1.2 \cdot 9.81 \cdot 1.3142 + 860 \cdot 9.81 \cdot 0.0058 - 1.5 \cdot 9.81 \cdot 1.32 \\ &= 45.0 \text{ Pa} \end{aligned}$$

Find the value of h with air in tube:

$$\begin{aligned} P_{1;\text{guage}} &= 45.0 \text{ Pa} \\ &= \rho_{\text{air}} g\,(H-h) + \rho_m g h - \rho_{\text{air}} g H \\ &= g h \left(\rho_m - \rho_{\text{air}}\right) \qquad (1) \\ h &= 0.534 \text{ cm} \end{aligned}$$

An error of 8%! This error is much smaller than the error introduced by neglecting variation in air pressure outside the tubing (equivalent to ignoring ρ_{air} in Equation 1). Why?

2.2 Forces on Submerged Surfaces

- For a planar surface,
$$F = P_{\text{CG}} A$$
where CG is the center of gravity / centroid of the surface, and the force is directed perpendicular to the surface.
- Alternately, horizontal force is $P_{\text{CG}} A_{\text{proj}}$. The vertical force is the weight of fluid above the surface, plus the surface pressure times planform area.
- Line of action of force passes through the *center of pressure*, which is offset from the centroid by:

$$\begin{aligned} x_{\text{CP}} &= -\frac{I_{xy} \sin\theta}{h_{\text{CG}} A} \\ y_{\text{CP}} &= -\frac{I_{xx} \sin\theta}{h_{\text{CG}} A} \end{aligned}$$

x lies along surface parallel to free surface; y lies in surface, $\perp$ to x, positive toward free surface.

2.3 Buoyancy

Regardless of body shape:

Floating object displaces its own *weight* in fluid.

Submerged object displaces its own *volume* in fluid.

2.4 Reynolds' Transport Theorem

General way of converting conservation laws for systems to use with control volumes. We'll stick with the fixed control volume form:

$$\frac{d}{dt}\left(B_{\text{syst}}\right) = \frac{d}{dt}\left(\int_{\text{CV}} \beta\rho d\mathcal{V}\right) + \oint_{\text{CV}} \beta\rho\left(\vec{V}\cdot\vec{n}\right) dA$$

where B is some conserved quantity for the system and β is the amount of B per unit mass. In words:

change in B for the system	=	change in B inside the control volume
		+ net amount of B carried out of the CV

Let's look at particular examples.

2.5 Conservation of Mass (Continuity)

Conservation law for a system:

$$\frac{dm}{dt} = 0$$

Here, $B = m$; $\beta = 1$. Applying the theorem,

$$\begin{aligned} \frac{dm}{dt} &= 0 \\ \frac{d}{dt}\left(\int_{\mathrm{CV}} \rho d\mathcal{V}\right) + \oint_{\mathrm{CV}} \rho \left(\vec{V} \cdot \vec{n}\right) dA &= 0 \end{aligned}$$

Still looks pretty imposing. In Mech 222/280, you commonly assumed constant velocity perpendicular to inflow and outflow (and got away with it by picking the right CV). For this case, with incompressible flow, we get the familiar

$$\sum_{\mathrm{out}} V_{\mathrm{out}} A_{\mathrm{out}} - \sum_{\mathrm{in}} V_{\mathrm{in}} A_{\mathrm{in}} = 0$$

By the way, this case illustrates one of the reasons why you pick a particular location for a control volume boundary: because you happen to know something about the flow there (in this case, that the velocity is uniform there). In general, you should never put a control volume boundary anywhere where you don't either (a) know something about the velocity, pressure, etc or (b) *want* to know something about the flow or about forces induced by the flow. Otherwise, you'll waste effort working out stuff that you don't care about.

2.6 Conservation of Momentum (Newton's Second Law)

For a system:

$$\frac{d}{dt}\left(m\vec{V}\right) = \sum F$$

where the forces involved include pressure and viscous forces at the CV boundary and body forces in the interior

So now B is momentum and β is velocity:

$$\frac{d}{dt}\left(\int_{\mathrm{CV}} \rho \vec{V} d\mathcal{V}\right) + \oint_{\mathrm{CV}} \rho \vec{V} \left(\vec{V} \cdot \vec{n}\right) dA = \sum F$$

In the second term, the $\rho\vec{V}$ is momentum being transported and the $\vec{V} \cdot \vec{n}$ is the signed flow velocity across the boundary (+ for out, - for in).

Simplifying as with conservation of mass, and also assuming steady flow:

$$\sum_{\mathrm{out}} \dot{m}_{\mathrm{out}} \vec{V}_{\mathrm{out}} - \sum_{\mathrm{in}} \dot{m}_{\mathrm{in}} \vec{V}_{\mathrm{in}} = \sum F$$

Here's a useful way to think about this, in terms of getting signs right on momentum fluxes when you assemble a momentum balance. Think about all the flow coming in to the control volume getting stopped; this obviously requires a force in the direction opposite to the motion of the incoming fluid. Then all the outgoing fluid has to be accelerated from rest, which requires a force in the direction of motion of the outgoing fluid. You may (or may not) find that this physical approach is easier to work with than having a vector momentum that you multiply by the result of a dot product, and then add or subtract depending on whether it's in or out (you can end up with three minus signs for a single term).

2.7 Mass/Momentum Example

Given:
- Dry weight of tank: 500 N
- Water in tank: 600 liters, 20C
- Pipe diameters: 6 cm
- Flow rates: 300 $\frac{\text{m}^3}{\text{hr}}$ for each pipe, steady

Find:
- Scale reading in Newtons.
- Minimum coefficient of friction between tank and scale to prevent slipping.

Solution:

- Flow velocity:

$$\begin{aligned} VA &= 300\frac{\text{m}^3}{\text{hr}} \cdot \frac{\text{hr}}{3600\text{ sec}} \\ V &= 29.5\frac{\text{m}}{\text{sec}} \end{aligned}$$

- Scale reading (from vertical momentum balance). Add up the weight of the tank and the water in the tank, and then add the momentum flux from the incoming flow. (The outgoing flow has no vertical momentum.)

$$\begin{aligned} W &= 500 + \frac{600}{1000} \cdot 998 \cdot 9.81 - \left(998\frac{\text{kg}}{\text{m}^3} \cdot 300\frac{\text{m}^3}{\text{hr}} \cdot \frac{\text{hr}}{3600\text{sec}}\right)\left(-29.5\frac{\text{m}}{\text{sec}}\right) \\ &= 8830\text{ N} \end{aligned}$$

- Coefficient of friction. First, find horizontal momentum flux to get horizontal reaction:

$$\begin{aligned} F_y &= 998 \cdot \frac{300}{3600} \cdot 29.5 \\ &= 2450\text{ N} \end{aligned}$$

  To prevent slipping, the coefficient of friction must be large enough to produce at least this large a frictional force:

$$\mu \geq \frac{2450}{8830} = 0.278$$

2.8 Conservation of Energy

General form:

$$
\begin{aligned}
\dot{Q} - \dot{W}_s - \dot{W}_v &= \frac{\partial}{\partial t}\left[\int_{\mathrm{CV}} \left(\hat{u} + \frac{1}{2}V^2 + gz\right)\rho d\mathcal{V}\right] \\
&\quad + \oint_{\mathrm{CV}} \left(\hat{h} + \frac{1}{2}V^2 + gz\right)\rho\left(\vec{V}\cdot\vec{n}\right) dA
\end{aligned}
$$

For steady flow, with uniform flow at each inflow and outflow,

$$
\begin{aligned}
q - w_s - w_v &= \sum_{\mathrm{out}} \left(\hat{h} + \frac{1}{2}V^2 + gz\right)\dot{m}_{\mathrm{out}} \\
&\quad - \sum_{\mathrm{in}} \left(\hat{h} + \frac{1}{2}V^2 + gz\right)\dot{m}_{\mathrm{in}}
\end{aligned}
$$

2.9 Bernoulli's Equation

$$P + \frac{1}{2}\rho V^2 + \rho g z = \text{constant}$$

Requirements to apply Bernoulli's Equation in this form:

1. Along a streamline
2. Steady flow
3. Incompressible flow
4. Inviscid flow (no viscous losses)
5. No work done on/by the fluid
6. No heat transfer to/from the fluid

Ignore these restrictions at your peril!!

2.10 Example

Given:
- Specific gravity of alcohol is 0.79
- Force $F = 425\text{N}$
- Neglect losses

Find:
- Mass flow rate of alcohol
- Pressure in the alcohol at the gauge

Solution: • Find the jet velocity from control volume momentum at plate (control volume include the entire plate and cuts the jet just downstream of the nozzle outlet):

$$\begin{aligned} F &= \rho V^2 A \\ 425\text{N} &= \left(0.79 \cdot 998 \frac{\text{kg}}{\text{m}^3}\right) V^2 \left(\pi \cdot 0.01^2 \text{m}^2\right) \\ V &= 41.4 \frac{\text{m}}{\text{sec}} \end{aligned}$$

- Mass flow rate $\dot{m} = \rho V A = 10.25 \frac{\text{kg}}{\text{sec}}$
- V_1 from continuity: $V_1 A_1 = V_2 A_2$, so $V_1 = 6.62 \frac{\text{m}}{\text{sec}}$
- Pressure from Bernoulli:

$$\begin{aligned} P_1 + \frac{1}{2}\rho V_1^2 &= P_2 + \frac{1}{2}\rho V_2^2 \\ P_1 - P_2 &= \frac{1}{2}\rho\left(V_2^2 - V_1^2\right) \\ &= 658 \text{ kPa} \end{aligned}$$

3 Dimensional Analysis

Dimensional analysis can help Make sense of experimental data

Reduce the amount of experimental work needed

Relate experimental *model* data to real-world *prototype* data

Apply data to similar problems

Buckingham Π-Theorem. If a physical process involves n dimensional variables, it can be reduced to a relationship between k dimensionless variables, or Π's. The maximum number of dimensional variables which do not form a Π among them is $j \equiv n - k$, and is always less than or equal to the number of dimensions describing the variables.

This theorem tells us that we can *reduce* the number of independent variables in a problem by non-dimensionalizing. This makes our experiment easier and cheaper (fewer things to

3.1 How Does Dimensional Analysis Work?

The Buckingham Π-Theorem tells us what to expect, but not how to do it. How do we do this systematically?

1. Write down all dimensional parameters relevant to the problem (n of these)
2. Write down the dimensions (mLtT) of those parameters
3. Pick one dimensional variable (density, for instance)
4. Use it to eliminate one dimension from the problem (mass, for instance)
5. Repeat until all remaining terms are dimensionless.
6. Write the final dimensionless function

Example: Pipe Flow

Δp	ρ	μ	L	D	V
$\frac{m}{Lt^2}$	$\frac{m}{L^3}$	$\frac{m}{Lt}$	L	L	$\frac{L}{t}$

Can pick any variable and still get a valid non-dimensionalization. It's generally easiest to pick a variable that has only one dimension, raised to the first power.

I'll pick D first, and I'm going to eliminate length from every variable:

$\Delta p \cdot D$	$\rho \cdot D^3$	$\mu \cdot D$	L/D	V/D
$\frac{m}{t^2}$	m	$\frac{m}{t}$	1	$\frac{1}{t}$

Note that now I can't pick L, in the form L/D, because it's dimensionless! I can't use it to eliminate a dimension from the problem.

Next, I'll pick ρD^3 and eliminate mass:

$\frac{\Delta p}{\rho \cdot D^2}$	$\frac{\mu}{\rho \cdot D^2}$	L/D	V/D
$\frac{1}{t^2}$	$\frac{1}{t}$	1	$\frac{1}{t}$

So now only one more dimension to go. I'll choose to repeat velocity (V/D) instead of viscosity $\left(\frac{\mu}{\rho D^2}\right)$ to keep viscosity out of my non-dimensional pressure. Again, the other way isn't wrong, but it's certainly unconventional, and it makes your dimensionless groups unfamiliar to others.

$\frac{\Delta p}{\rho V^2}$	$\frac{\mu}{\rho V D}$	L/D
1	1	1

In the end, we get:

$$\frac{\Delta p}{\frac{1}{2}\rho V^2} = g_1\left(\frac{L}{D}, \frac{1}{\mathrm{Re}}\right) = g_2\left(\frac{L}{D}, \mathrm{Re}\right)$$

Well, actually, we got $1/\mathrm{Re}$, but if we can write something as a function of x, we can just as easily write it as a function of $1/x$, so the two versions are equivalent. That extra factor of $\frac{1}{2}$ is traditionally used (it's the dynamic pressure in the flow); since the constant is non-dimensional, all it does is change what the function g_1 (or g_2) looks like.

3.2 Do These Dimensionless Parameters *Mean* Anything?

Name	Definition	Physics	Applicability
Reynolds	$\mathrm{Re} = \frac{\rho U L}{\mu} = \frac{\rho U^2}{\mu U/L}$	$\frac{\text{Inertia}}{\text{Viscosity}}$	Always
Mach	$\mathrm{Ma} = \frac{U}{c}$	$\frac{\text{Flow speed}}{\text{Sound speed}}$	Compressible flow
Froude	$\mathrm{Fr} = \frac{U^2}{gL} = \frac{\rho U^2}{\rho g L}$	$\frac{\text{Inertia}}{\text{Gravity}}$	Free surface flows
Weber	$\mathrm{We} = \frac{\rho U^2}{Y/L}$	$\frac{\text{Inertia}}{\text{Surface tension}}$	Free surface flows
Roughness	$\frac{\epsilon}{L}$	$\frac{\text{Roughness height}}{\text{Body size}}$	Turbulent flow
Pressure coeff	$C_P = \frac{p-p_\infty}{\frac{1}{2}\rho U^2}$	$\frac{\text{Static pressure}}{\text{Dynamic pressure}}$	Aero/hydrodynamics
Force coeff	$C_F = \frac{F}{\frac{1}{2}\rho U^2 \cdot A}$	$\frac{\text{Applied force}}{\text{Dyn pressure force}}$	Aero/hydrodynamics

3.3 Notes on Physical Implications of Dimensionless Parameters

Reynolds $Re \equiv \frac{\rho V L}{\mu}$, $\frac{\textbf{inertia}}{\textbf{viscosity}}$ **ALWAYS important.** High Re for external flows implies inertial effects are dominant except very near boundaries which in turn implies the presence of a "boundary layer"; see Mech 380. For internal flows, viscous effects matter everywhere even for high Re, except possibly for a small entrance length.

Mach $M \equiv \frac{V}{c}$, $\frac{\text{flow speed}}{\text{sound speed}}$ Compressibility parameter. For compressible flow, ρ and T vary with changes in pressure; the size of these variations is linked to Ma^2. Implies Ma¡0.3 is (effectively) incompressible. Ma=1 is flow at speed of sound. Commercial jets cruise at Ma=0.85–0.88. Concorde cruised around Ma=2, the SR-71 at about Ma=3, Space Shuttle re-entry is at about Ma=25.

Froude $Fr \equiv \frac{V^2}{gL} = \frac{\rho V^2}{\rho g L}$, $\frac{\text{inertia}}{\text{gravity}}$ Free-surface parameter that correlates well with wave drag for ships (for instance). This is the dominant parameter for large-scale free surface flows (ships, open channels, etc). Unimportant otherwise (subs, planes, cars, pipes, etc).

Weber $We \equiv \frac{\rho V^2}{Y/L}$, $\frac{\text{inertia}}{\text{surface tension}}$ Surface tension parameter. Important for highly-curved free surfaces (drops, narrow jets (like a sink at low flow), small surface waves), where We $\lesssim$ 1. Surface tension force scales with $1/R$, so for free surfaces without large curvature, surface tension is basically irrelevant.

Roughness ratio $\frac{\epsilon}{L}$, $\frac{\text{roughness height}}{\text{body length}}$ Important for estimating drag in turbulent flow. Hard to scale experimentally.

Pressure coeff $C_p \equiv \frac{P-P_\infty}{\frac{1}{2}\rho V_\infty^2}$, $\frac{\text{static pressure}}{\text{dynamic pressure}}$ $C_p = 1$ is non-dimensional stagnation pressure (incompressible; the number varies with free stream Mach number for compressible flow), $C_p = 0$ is non-dimensional far-field pressure. $C_p < 0$ always indicates flow that's faster than far field flow.

Force coeff. $C_F \equiv \frac{F}{\frac{1}{2}\rho V_\infty^2 A}$, $\frac{\text{applied force}}{\text{dynamic pressure force}}$ Lift and drag coefficients are the most common flavors. Good way to compare drag of different objects (perhaps even of different sizes).

3.4 Scaling Experimental Results to the Real World

Suppose we know that the non-dimensional drag on a ship can be written as:

$$C_D = \mathcal{G}\left(\text{Re}, \text{Fr}, \frac{\epsilon}{L}\right)$$

We want to build an experimental model with the same Re, Fr, and $\frac{\epsilon}{L}$ as the full-scale ship. Then C_D will be the same for both.

$$D \equiv C_D \cdot \frac{1}{2}\rho V^2 \cdot A$$

This is the essence of scaling: match all the dimensionless parameters between model and prototype. **NOTE: Model and prototype must be perfect scale models, including surface roughness height, radii of curvature, etc.**

3.5 Pitfalls of Scaling

- Neglected effects
  - Faster speed in wind tunnel tests $\longrightarrow$ compressibility effects, which won't be present for larger, lower-speed real-world object
- Can't match all dimensionless numbers; fairly common
  - For free-surface flows, choose to match Fr rather than Re
  - For compressible flows, choose to match Ma rather than Re
- Can't match *any* dimensionless numbers
  - Reynolds number unattainable for small scale models; hope to show that (for example) drag depends only slightly on Re in the range of interest

3.6 Example

Forced convection heat transfer coefficient, h $\left(\text{in } \frac{\text{N}}{\text{sec}\cdot\text{m}\cdot\text{K}}\right)$, depends on flow velocity U, body size L, and fluid properties ρ, μ, c_p, and k. Write as a non-dimensional function.

h	U	L	ρ	μ	c_p	k
$\frac{\mathrm{M}}{\mathrm{T}^3\Theta}$	$\frac{\mathrm{L}}{\mathrm{T}}$	L	$\frac{\mathrm{M}}{\mathrm{L}^3}$	$\frac{\mathrm{M}}{\mathrm{LT}}$	$\frac{\mathrm{L}^2}{\mathrm{T}^2\Theta}$	$\frac{\mathrm{ML}}{\mathrm{T}^3\Theta}$

- What variables to repeat, and why?ρ, U, L, k. The stuff we choose is going to end up being in lots of different dimensionless groups, so picking things like μ to repeat, while permissible, gives weird-looking dimensionless groups. Generally speaking, pick density, velocity and length if you have them readily available. In this case, we also needed something with temperature. I chose k because k happens to appear in the definition of the non-dimensional heat transfer coefficient (Stanton number). Choosing c_p instead would have gotten us something like $\mathrm{St}/(\mathrm{Pr}\cdot\mathrm{Re})$ in the end.

- Group from h:

$$[h][U]^a[L]^b[\rho]^c[k]^d = 0$$

$$\begin{aligned} 1 + c + d &= 0 \\ a + b - 3c + d &= 0 \\ -3 - a - 3d &= 0 \\ -1 - d &= 0 \end{aligned}$$

  Leads to Stanton number:

$$\Pi_1 = \frac{h}{Lk}$$

- Group from μ leads easily to the Reynolds number: $\Pi_2 = \frac{\rho UL}{\mu}$

- Group from c_p:

$$[c_p][U]^a[L]^b[\rho]^c[k]^d = 0$$

$$\begin{aligned} c + d &= 0 \\ 2 + a + b - 3c + d &= 0 \\ -2 - a - 3d &= 0 \\ -1 - d &= 0 \end{aligned}$$

  leads to:

$$\Pi_3 = \frac{\rho U L c_p}{k}$$

  More commonly, we would replace Π_3 by $\frac{\Pi_3}{\Pi_2} = \frac{\mu c_p}{k} \equiv \mathrm{Pr}$. Could have gotten this directly by choosing μ to repeat instead of U or ρ.

- Final result:

$$\mathrm{St} = f(\mathrm{Re}, \mathrm{Pr})$$