MECH479 - QUIZ 2 SOLUTIONS

1.) We have

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

Using a second-order central difference scheme for the spatial derivative, our semi-discrete equation is of the form

$$\frac{\partial u_i}{\partial t} = \frac{\alpha}{\Delta x^2}\left(u_{i-1} - 2u_i + u_{i+1}\right) \qquad \text{(at } i^{th} \text{ point)}$$

Writing this in matrix form for the 10 interior points,

$$\underbrace{\frac{\partial}{\partial t}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_9 \\ u_{10} \end{bmatrix}_{(10\times1)}}_{\vec{u}} = \underbrace{\frac{\alpha}{\Delta x^2}\begin{bmatrix} -2 & 1 & 0 & & 0 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ 0 & & 1 & -2 & 1 \\ & & 0 & 1 & -2 \end{bmatrix}_{(10\times10)}}_{A} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_9 \\ u_{10} \end{bmatrix}_{(10\times1)}}_{\vec{u}} + \frac{\alpha}{\Delta x^2}\underbrace{\begin{bmatrix} u_{BC\ left} \\ 0 \\ \vdots \\ 0 \\ u_{BC\ right} \end{bmatrix}_{(10\times1)}}_{\vec{BC}}$$

thus we see that A is a band tridiagonal matrix of size (10×10). with 72 zero-valued entries and 28 non zero-valued entries.

Using the definition of sparsity given in the question, we calculate

$$\text{sparsity} = \frac{72}{100}$$

2.) We have the truncation error terms for the given discretization (hypothetical) of $\frac{\partial u}{\partial x}$ as

$$-\left(\frac{\partial^2 u}{\partial x^2}\right)\frac{\Delta x}{2} - \frac{\partial^3 u}{\partial x^3}\frac{\Delta x^3}{6}.$$

the leading term is $-\left(\frac{\partial^2 u}{\partial x^2}\right)\frac{\Delta x}{2}$ $\Rightarrow$ 1^{st} order accurate

3.) Given PDE —

$$\frac{\partial^2 u}{\partial x^2} + 2u = 1$$

Using the central-difference scheme for the derivative, we get the discretized form of the equation at the i^{th} point as —

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{\Delta x^2} + 2u_i = 1$$

Thus we can form the following 3 equations at the 3 interior points —

1) $i = 1$

$$\frac{u_0 - 2u_1 + u_2}{\Delta x^2} + 2u_1 = 1$$

2) $i = 2$

$$\frac{u_1 - 2u_2 + u_3}{\Delta x^2} + 2u_2 = 1$$

3) $i = 3$

$$\frac{u_2 - 2u_3 + u_4}{\Delta x^2} + 2u_3 = 1$$

Using values of u_0, u_4 and Δx given, we simplify as

$$\frac{-2u_1 + u_2}{(0.5)^2} + 2u_1 = 1 \qquad \text{—— (1)}$$

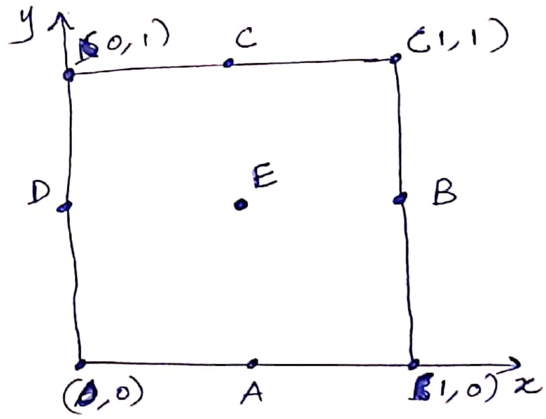
$$\frac{u_1 - 2u_2 + u_3}{(0.5)^2} + 2u_2 = 1 \qquad \text{—— (2)}$$

$$\frac{u_2 - 2u_3 + 1}{(0.5)^2} + 2u_3 = 1 \qquad \text{—— (3)}$$

3 equations with 3 unknowns. Solving, we obtain

$u_1 = 1/6$, $u_2 = 1/2$, $u_3 = 5/6$

4) We have the Poisson equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x + y$ on the given unit square



Using the central difference scheme, we write the discretized form of the equation at point (i,j) as

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\Delta x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\Delta y^2} = x_{i,j} + y_{i,j}$$

marking the points as A, B, C, D, E for convenience (as in diagram)

we can write

$$\frac{u_D - 2u_E + u_B}{\Delta x^2} + \frac{u_C - 2u_E + 2u_A}{\Delta y^2} = x_E + y_E$$

Using boundary values given in question we simplify as

$$\frac{0 - 2u_E + 1}{(0.5)^2} + \frac{0 - 2u_E + 1}{(0.5)^2} = x_E + y_E \qquad \text{—— (1)}$$

(x_E, y_E) are the cartesian coordinates of the point E.

Using $(0.5, 0.5)$ as the values of (x_E, y_E) we get the value of u_E to be 0.4375.

5)
$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} \alpha_{-2} \\ \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{-6}{\Delta x^3} \end{bmatrix}$$

(A)

These can be written as the following four equations —

$A_{11}\alpha_{-2} + A_{12}\alpha_{-1} + A_{13}\alpha_0 + A_{14}\alpha_1 = 0$ —— (1) (RHS 1)

$A_{21}\alpha_{-2} + A_{22}\alpha_{-1} + A_{23}\alpha_0 + A_{24}\alpha_1 = 0$ —— (2) (RHS 2)

$A_{31}\alpha_{-2} + A_{32}\alpha_{-1} + A_{33}\alpha_0 + A_{34}\alpha_1 = 0$ —— (3) (RHS 3)

$A_{41}\alpha_{-2} + A_{42}\alpha_{-1} + A_{43}\alpha_0 + A_{44}\alpha_1 = \cancel{-}\frac{6}{\Delta x^3}$ —— (4) (RHS 4)

from the Taylor table we have the (column) summations as

$$\alpha_{-2} + \alpha_{-1} + \alpha_0 + \alpha_1 = 0 \quad \text{— (5)} \quad (RHS\ 1)$$

$$-2\Delta x\, \alpha_{-2} - \Delta x\, \alpha_{-1} + 0 + \Delta x\, \alpha_1 = 0 \quad \text{— (6)} \quad (RHS\ 2)$$

$$\frac{4\Delta x^2}{2}\alpha_{-2} + \frac{\Delta x^2}{2}\alpha_{-1} + 0 + \frac{\Delta x^2}{2}\alpha_1 = 0 \quad \text{— (7)} \quad (RHS\ 3)$$

$$\frac{-8\Delta x^3}{6}\alpha_{-2} - \frac{\Delta x^3}{6}\alpha_{-1} + 0 + \frac{\Delta x^3}{6}\alpha_1 = 1 \quad \text{— (8)} \quad (RHS\ 4)$$

Comparing the coefficients in equations (1)-(5), (2)-(6), (3)-(7) and (4)-(8)

we obtain the values of the unknowns as —

$$A = \begin{bmatrix} 1_{A_{11}} & 1_{A_{12}} & 1_{A_{13}} & 1_{A_{14}} \\ -2_{A_{21}} & -1_{A_{22}} & 0_{A_{23}} & 1_{A_{24}} \\ 2_{A_{31}} & 1/2_{A_{32}} & 0_{A_{33}} & 1/2_{A_{34}} \\ -8_{A_{41}} & -1_{A_{42}} & 0_{A_{43}} & 1_{A_{44}} \end{bmatrix}$$

<u>Note</u>: the value of one entry in each row was given to avoid any confusions regarding signs (+, -) and fractional values