MECH 479/587

Computational Fluid Dynamics

Module 1: Fluids Review

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Review of Basic Fluid Flow Principles

- ❑ Overview
  - ► Notation and fluid flow properties
  - ► Hydrostatics and fluid dynamics
  - ► Control volumes
  - ► Fluid flow classification

- ❑ Conservation laws
  - ► Mass, momentum and energy
  - ► The Navier-Stokes equations

Recommended Readings

- ❑ Handout
- ❑ F.M. White, Chapters 1-3
- ❑ Çengel and Cimbala, Chapters 1-4

Basic Principles and Definitions

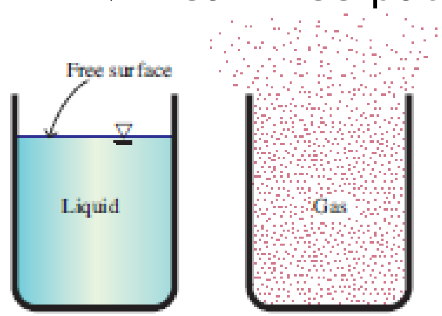
- A fluid is a body of matter that can flow; i.e. continues to deform under a shearing force.
  - Fluids may be liquids (definite volume; free surface) or gases (expand to fill any container).

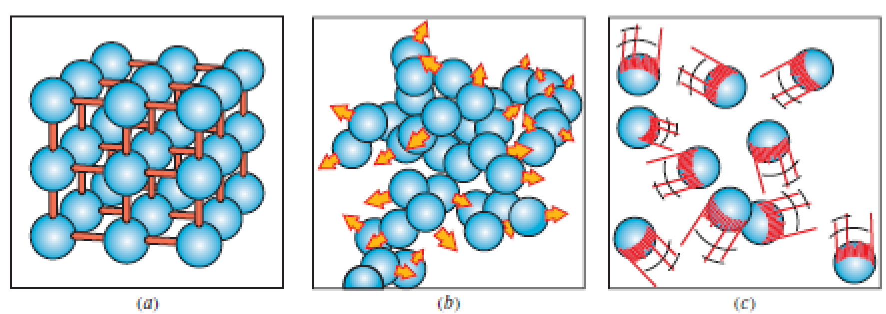
- Fluids obey the usual laws of Newtonian mechanics, but as a continuum. Unlike rigid bodies, fluid particles may move relative to each other, interacting via internal forces.
  - A small volume element in the fluid contains a very large number of molecules

- Fluid dynamics concerns fluids in motion
  - Continuity (mass conservation)
  - Momentum
  - Energy

Properties of Fluids

- Liquids and gases are made up of a large number of molecules undergoing constant motion and collision
  - Is this discrete nature of the fluid important for us? In a liquid, the answer is clearly NO. We are generally interested in the *gross behavior* of the fluid, i.e. the *average manifestation* of molecular motion.
  - The molecules are in contact as they slide past each other, and overall act like a uniform fluid material at macroscopic scales.
  - In a gas, the molecules are not in immediate contact. Thus we must look at the mean free path, which is the distance the average molecule travels before colliding with another. Some known data for air:
    - Mean free path at 0 km (sea level) : 0.0001 mm
    - Mean free path at 20 km (U2 flight) : 0.001 mm





The arrangement of atoms in different phases

Fluid Quantities and Notation

Key flow variables are:

$\mathbf{u}$ or $\vec{\mathrm{V}}$: flow velocity

p : pressure (force/area)

ρ : density (mass/volume)

The flow field variables are functions of position and time t; e.g. $p(\mathbf{x},t)$, $\mathbf{u}=\mathbf{u}(\mathbf{x},t)$

Vector quantities like position and velocity are often written in components:

$$\mathbf{x}=(x,y,z)$$

$$\mathbf{u}=(u,v,w)$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x}+j\frac{\partial\phi}{\partial y}+k\frac{\partial\phi}{\partial z}=\left[\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial z}\right]^T$$

Nabla operator: $\nabla = i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}$

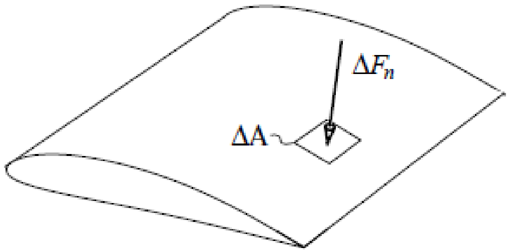
Divergence: $\nabla\cdot\mathbf{u}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$ Velocity gradient: $\nabla\mathbf{u}=[\nabla u,\nabla v,\nabla w]^T=\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y}\\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z}\end{bmatrix}^T$

Multivariable calculus and gradient (Khan Academy)

Pressure and Density

- Pressure p is defined as the force/area acting normal to a surface

$$p = \lim_{\Delta A \to 0} \frac{\Delta F_n}{\Delta A}$$



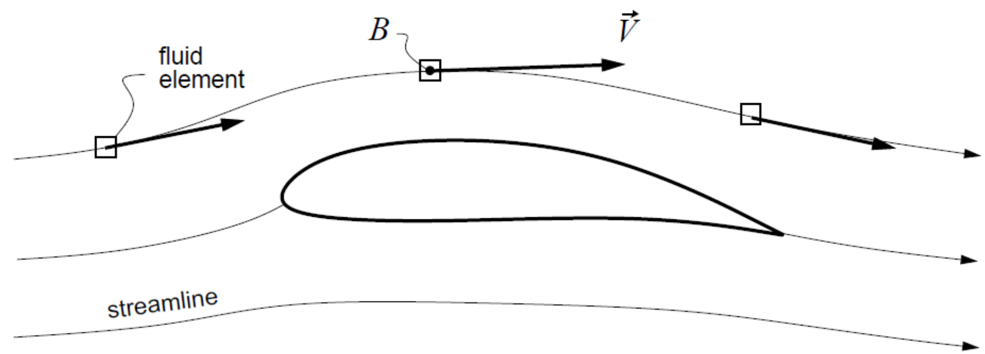
Normal force on area element due to pressure

- Density ρ is defined as the mass/volume, for an infinitesimally small volume.

$$\rho = \lim_{\Delta \mathcal{V} \to 0} \frac{\Delta m}{\Delta \mathcal{V}}$$

Velocity

- We are interested in motion of fluids, hence velocity is important. Two ways to look at this:
  - Body is moving in stationary fluid – e.g. airplane in flight
  - Fluid is moving past a stationary body – e.g. airplane in wind tunnel



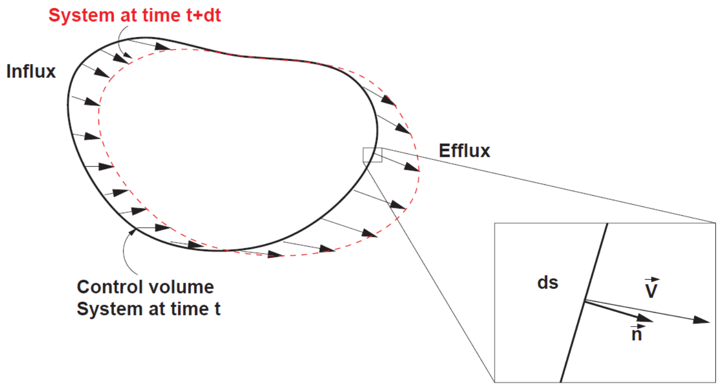
Velocity at point B

$\vec{V}$ at a point = velocity of fluid element as it passes that point

- The pressure fields and forces in these two cases will be the same if all else is equal.
- The governing equations we will develop are unchanged by a *Galilean Transformation*, such as the switch from a fixed to a moving frame of reference.

Control Volumes

- Reynolds' Transport Theorem
  - General way of formulating conservation laws for system to use control volumes
  - For fixed control volume:



$$\frac{d}{dt}\left(B_{\text{syst}}\right) = \frac{d}{dt}\left(\int_{\text{CV}} \beta\rho\, d\mathcal{V}\right) + \oint_{\text{CS}} \beta\rho(\vec{V}\cdot\vec{n})\,dA$$

where B is some conserved quantity for the system and β is the amount of B per unit mass

Change in *B* for the system = change in *B* inside the control volume + net amount of *B* carried out of the CV

Conservation of Mass (1)

- Conservation law for a system

$$\frac{dm}{dt} = 0$$

Here $B = m; \ \beta = 1$. Applying the thorem:

$$\frac{dm}{dt} = 0$$

$$\frac{d}{dt}\left(\int_{\mathrm{CV}} \rho d\mathcal{V}\right) + \oint_{\mathrm{CS}} \rho(\vec{V} \cdot \vec{n}) dA = 0$$

- In MECH222/280, you commonly employed velocity perpendicular to inflow and outflow. For this case, we get the familiar form for incompressible flow

$$\sum_{\mathrm{out}} V_{\mathrm{out}} A_{\mathrm{out}} - \sum_{\mathrm{in}} V_{\mathrm{in}} A_{\mathrm{in}} = 0$$

Conservation of Mass (2)

- The net inflow through the boundary of the control volume is

Inflow to the CV

$$\oint_{CS} \rho \mathbf{V} \cdot \mathbf{n} \, dA$$

The integral over the boundary

Mass flow normal to boundary

- For incompressible flow (constant density):

$$\sum_{\text{out}} V_{\text{out}} A_{\text{out}} - \sum_{\text{in}} V_{\text{in}} A_{\text{in}} = 0$$

$$\textit{mass flow rate out} = \textit{mass flow rate in}$$

Conservation of Momentum

For a system

$$\frac{d}{dt}(m\vec{V}) = \sum F$$

where the forces involved include pressure and viscous forces at the CV boundary and body forces in the interior

So now B is momentum and β is velocity:

$$\frac{d}{dt}\left(\int_{\mathrm{CV}} \rho \vec{V} d\mathcal{V}\right) + \oint_{\mathrm{CS}} \rho \vec{V}(\vec{V} \cdot \vec{n}) dA = \sum F$$

Simplifying like mass balance and assuming steady flow

$$\sum_{\mathrm{out}} \dot{m}_{\mathrm{out}} \vec{V}_{\mathrm{out}} - \sum_{\mathrm{in}} \dot{m}_{\mathrm{in}} \vec{V}_{\mathrm{in}} = \sum F$$

Conservation of Energy

- General form:

$$\dot{Q} - \dot{W}_s - \dot{W}_v = \frac{\partial}{\partial t}\left[\int_{\mathrm{CV}}\left(\hat{u} + \frac{1}{2}V^2 + gz\right)\rho d\mathcal{V}\right]$$

$$+ \oint_{\mathrm{CS}}\left(\hat{h} + \frac{1}{2}V^2 + gz\right)\rho(\vec{V}\cdot\vec{n})dA$$

enthalpy $\hat{h} = \hat{u} + p/\rho$

$\hat{u} =$ internal energy

where $\dot{Q}$ heat addition, $\dot{W}_s$ and $\dot{W}_v$ are shaft and viscous work done.

- For steady flow, with uniform flow at each inflow and outflow

$$q - w_s - w_v = \sum_{\mathrm{out}}\left(\hat{h} + \frac{1}{2}V^2 + gz\right)\dot{m}_{\mathrm{out}}$$

$$-\sum_{\mathrm{in}}\left(\hat{h} + \frac{1}{2}V^2 + gz\right)\dot{m}_{\mathrm{in}}$$

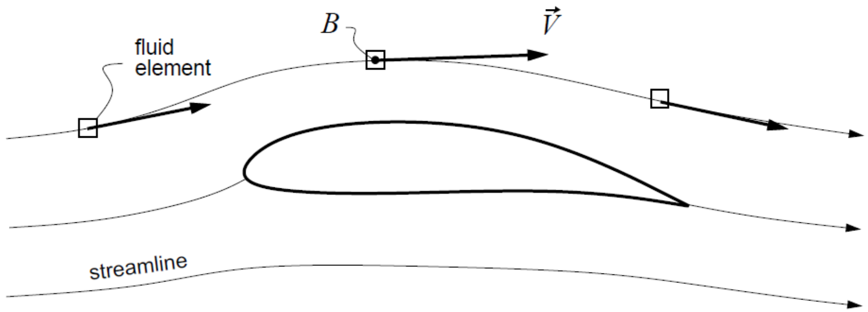
$$\boxed{\textit{change of energy} = \textit{work done} + \textit{heat input}}$$

Some Nondimensional Parameters

Name	Definition	Physics	Applicability
Reynolds	$\mathrm{Re} = \frac{\rho U L}{\mu} = \frac{\rho U^2}{\mu U/L}$	$\frac{\text{Inertia}}{\text{Viscosity}}$	Always
Mach	$\mathrm{Ma} = \frac{U}{c}$	$\frac{\text{Flow speed}}{\text{Sound speed}}$	Compressible flow
Froude	$\mathrm{Fr} = \frac{U^2}{gL} = \frac{\rho U^2}{\rho g L}$	$\frac{\text{Inertia}}{\text{Gravity}}$	Free surface flows
Weber	$\mathrm{We} = \frac{\rho U^2}{\Upsilon/L}$	$\frac{\text{Inertia}}{\text{Surface tension}}$	Free surface flows
Roughness	$\frac{\varepsilon}{L}$	$\frac{\text{Roughness height}}{\text{Body size}}$	Turbulent flow
Pressure coeff	$C_P = \frac{p - p_\infty}{\frac{1}{2}\rho U^2}$	$\frac{\text{Static pressure}}{\text{Dynamic pressure}}$	Aero/hydrodynamics
Force coeff	$C_F = \frac{F}{\frac{1}{2}\rho U^2 \cdot A}$	$\frac{\text{Applied force}}{\text{Dyn pressure force}}$	Aero/hydrodynamics

Fluid Flow Classification: Steady vs. Unsteady Flows

- If the flow is steady, then $p, \rho, \vec{V}$ don't change in time for any point, and hence can be given as $p(x, y, z), \rho(x, y, z), V(x, y, z)$. If the flow is unsteady, then these quantities do change in time at some or all points.

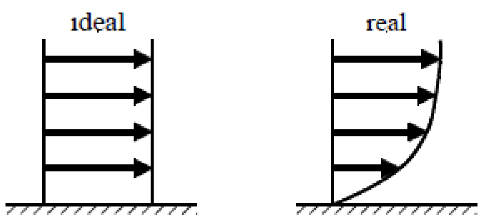


Velocity at point B

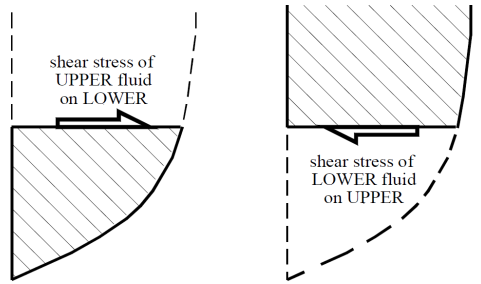
- For a steady flow, we can define a streamline, which is the path followed by some chosen fluid element.

Fluid Flow Classification: Real and Ideal Fluids

- Ideal fluids have no viscosity – there is no internal friction or loss of mechanical energy.
  - No such fluid exists, but many flows can be approximated as ideal if viscous forces are small
- Real fluids have non-zero viscosity. This has two important consequences:
  - They satisfy the no-slip condition at solid boundaries



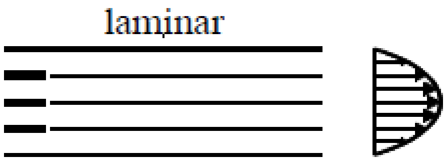
  - Frictional forces between adjacent layers of fluid moving at different speeds and between the fluid and a boundary.

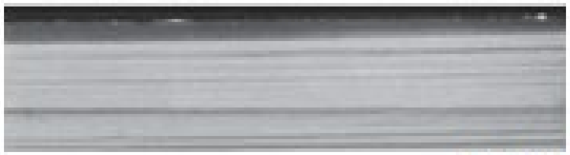


Fluid Flow Classification: Laminar vs. Turbulent

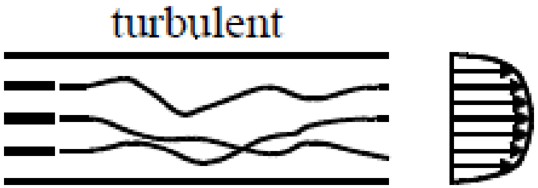
Osborne Reynolds

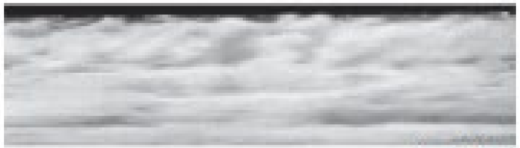
- At low flow speeds viscosity ensures that adjacent layers of fluid slide smoothly over one another in a steady fashion without intermingling. This flow regime is called laminar.

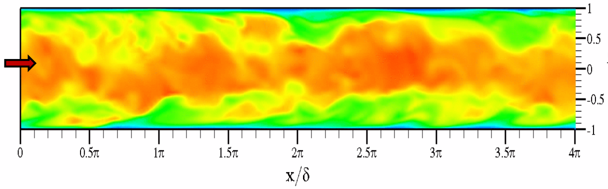




- At higher speeds viscosity is insufficient to smooth out minor perturbations to the flow, which grow rapidly to create unsteadiness and eddying motions. This flow regime is called turbulent.

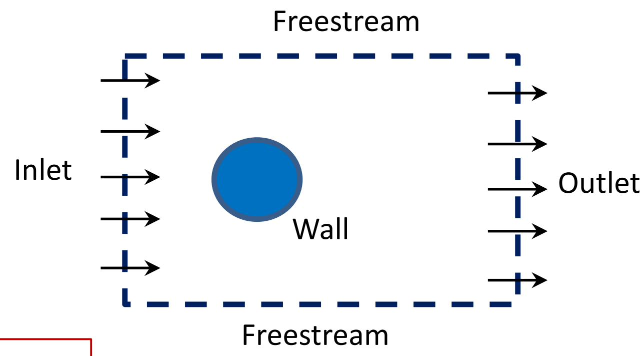




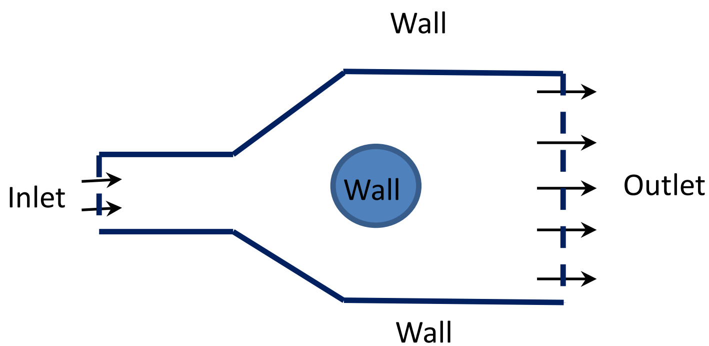


Fluid Flow Classification: Internal vs External Flow

- Fluid flow is classified as being internal or external, depending on whether the fluid flows in a confined space or over a surface
  - The flow in a pipe or duct is internal flow if the fluid is completely bounded by solid surfaces
    - Internal flows are dominated by the influence of viscosity throughout
    - Water flow in a pipe, for example, is internal flow
  - The flow of an unbounded fluid over a surface such as a plate, a wing or a car or ship is external flow
    - Viscous effects are limited to boundary layers near solid surfaces and to wake regions downstream of bodies



External flow



Internal flow

Fluid Flow Classification: Compressible vs Incompressible Flow

- Fluid flow is categorized as being as being compressible or incompressible, depending on the level of variation of density during flow
  - <u>Incompressibility</u> is an approximation, in which the flow is said to be incompressible if the density remains nearly constant throughout
    - Densities of liquids are essentially constant, and thus the flow of liquids is typically incompressible.
  - Compressible flow defined as variable density flow with significant <u>compressibility</u> effects

Compressible effects are determined by the magnitude of the Mach number

$M \equiv \frac{u}{c}$ where u is the speed of flow and c is the speed of sound

$c^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_s$, where s signifies partial derivative is taken at constant entropy

$$(i)\ \textit{Incompressible}: M < 0.3$$
$$(ii)\ \textit{Subsonic flow}: 0.3 < M < 1$$
$$(iii)\ \textit{Transonic flow}: 0.8 < M < 1.2$$
$$(iv)\ \textit{Supersonic flow}: 1 < M < 3$$
$$(v)\ \textit{Hypersonic flow}: M > 3$$

Ernest Mach

The Navier-Stokes Equations

CONSERVATION LAWS

Integral vs. Differential Form (Mass/Continuity)

$$\frac{\partial}{\partial t}(\text{mass}) = (\text{mass } \mathit{flux})_{in} - (\text{mass } \mathit{flux})_{\text{out}}$$

$$\underbrace{\frac{\partial}{\partial t}\int_V \rho dV}_{\text{Rate of change of mass}} = \underbrace{-\oint_S \rho \mathbf{u}\cdot\mathbf{n}d\mathbf{S}}_{\text{Net inflow of mass}}$$

- Using Gauss's Theorem:

$$\oint_S \rho\mathbf{u}\cdot\mathbf{n}dS = \int_V \nabla\cdot(\rho\mathbf{u})\,dV$$

- The mass conservation:

$$\frac{\partial}{\partial t}\int_V \rho dV = -\int_V \nabla\cdot(\rho\mathbf{u})\,dV$$

- Rearranging:

$$\int_V \left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\right)dV = 0$$

$$\Rightarrow \boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0}$$

Differential Form (Mass/Continuity Equation)

- Expanding the divergence

$$\nabla \cdot (\rho \mathbf{u}) = \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u}$$

- Mass conservation (convective form)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = 0$$

$$\Rightarrow \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0$$

- Incompressible fluid by taking density constant:

$$\boxed{\nabla \cdot \mathbf{u} = 0}$$

Momentum Principle

- Consideration of momentum is necessary whenever one is considering forces. Two mechanical principles are involved.
  - Force = Rate of change of momentum (Newton's 2^{nd} Law)
  - The force exerted by a fluid on its containment is equal and opposite to that exerted by its containment on the fluid

- Force is the sum of all external forces on the control volume, including
  - Reaction from solid boundaries
  - Fluid forces from adjacent fluid (pressure, viscous forces etc.)

$$\text{force} = (\text{ momentum } flux)_{\text{out}} - (\text{ momentum } flux)_{\text{in}}$$

$$\mathbf{F} = (\dot{m}\mathbf{u})_2 - (\dot{m}\mathbf{u})_1$$

  - Force, velocity and momentum are all vector quantities and each direction must be considered.

Conservation of Momentum

- Momentum per volume: $\rho\mathbf{u}$

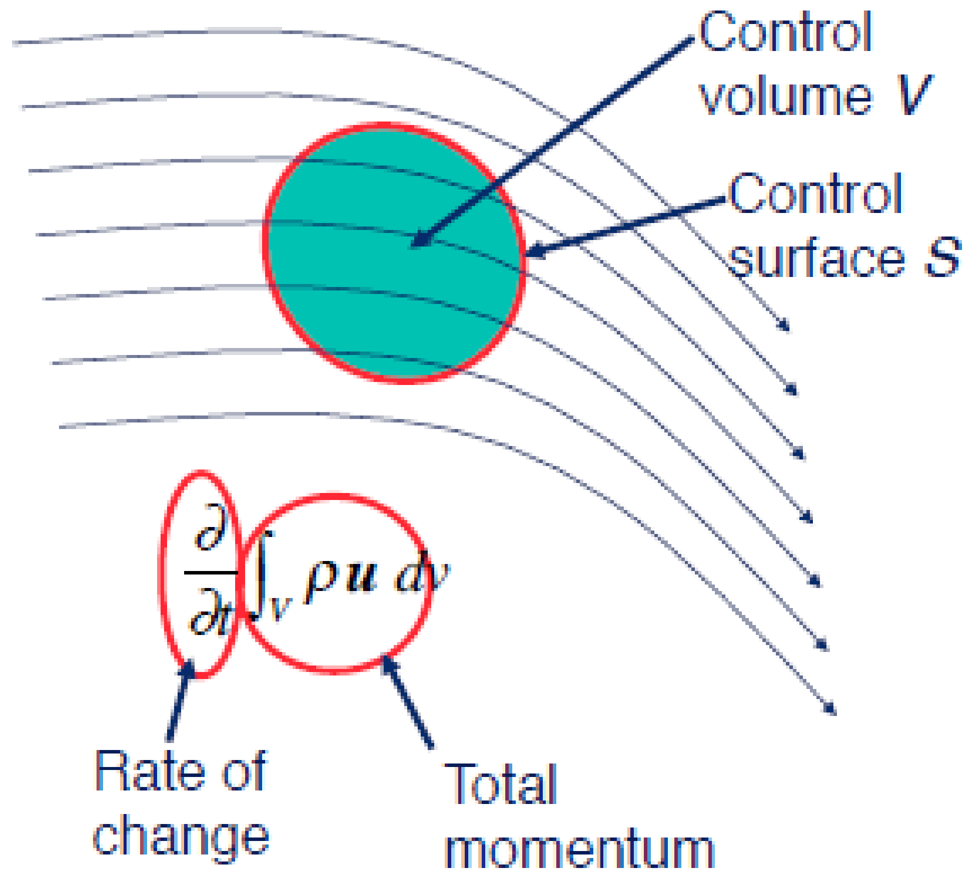
- Momentum in control volume:

$$\int_V \rho\mathbf{u}\, dV$$

Rate of change of momentum = (Net influx of momentum) + (Surface forces) + (Body forces)

Conservation of Momentum (1)

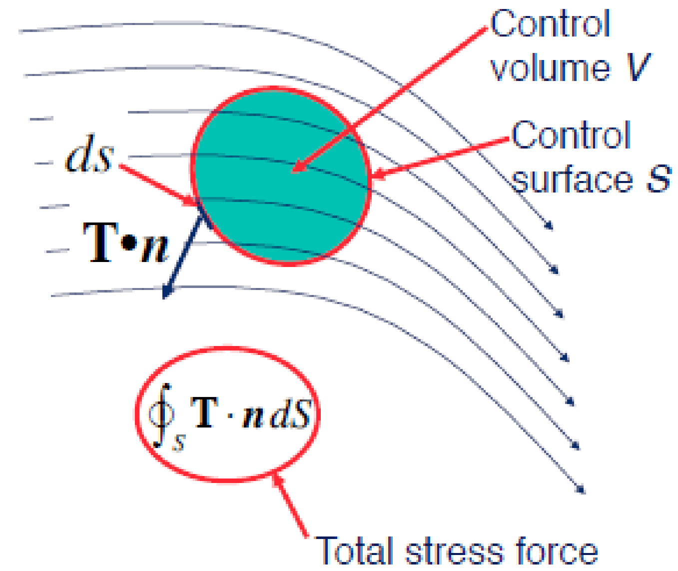
The rate of change of momentum in the control volume can be given as:



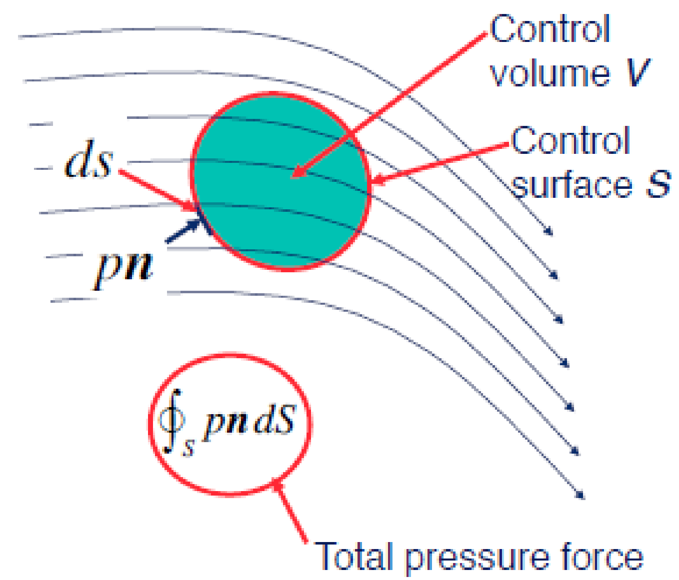
Conservation of Momentum (2)

- Surface Forces

The viscous force is given by the dot product of the normal with the stress tensor **T**



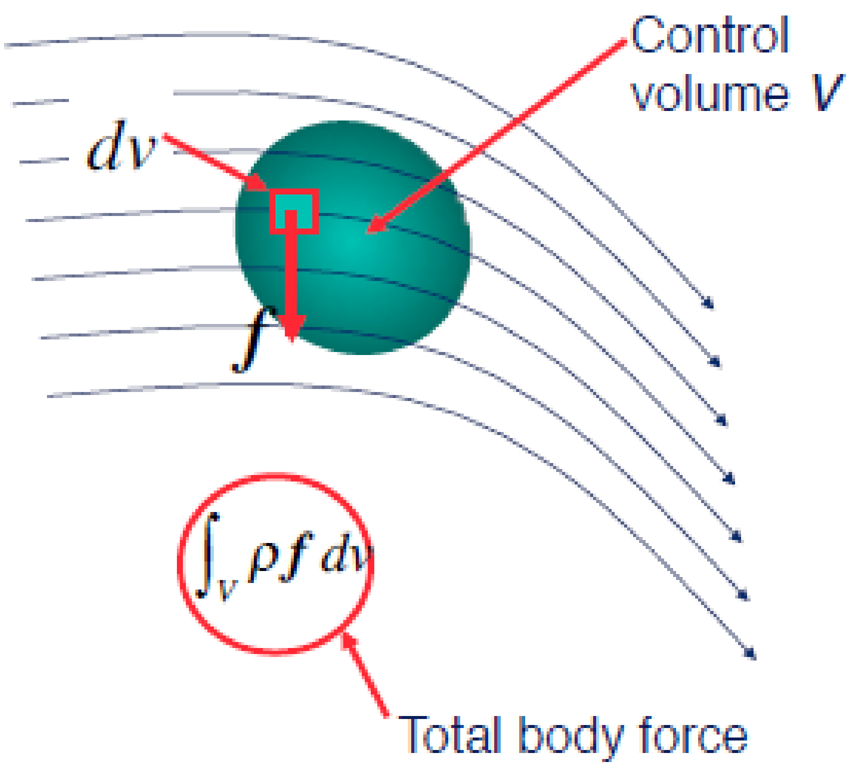
The pressure produces a normal force on the control surface.



Conservation of Momentum (3)

- Body forces

Body forces (e.g., gravity) Acting on the fluid control volume



Viscous Flux of Momentum (4)

- Stress Tensor

$$\mathbf{T} = \left(-p + \lambda \nabla \cdot \mathbf{u}\right)\mathbf{I} + \mathbf{2}\mu\mathbf{D}$$

where $\mathbf{I}$ identity tensor,

μ denotes dynamic viscosity and λ is the second viscosity.

- Deformation Tensor
  - ▶ For a Newtonian fluid, viscous stress is proportional to velocity gradients

$$\mathbf{D} = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}}\right) \quad \text{and } \lambda \approx -\frac{2}{3}\mu \quad \text{(Stokes' hypothesis)}$$

- For incompressible flow:

$$\nabla \cdot \mathbf{u} = 0 \qquad \mathbf{T} = -p\mathbf{I} + \mathbf{2}\mu\mathbf{D}$$

Conservation of Momentum (5)

$$\frac{\partial}{\partial t}\int_V \rho \mathbf{u}\, dV = -\oint_S \rho \mathbf{u}\mathbf{u}\cdot \mathbf{n}\, dS + \oint \mathbf{T}\mathbf{n}\, dS + \int \rho \mathbf{f}\, dV$$

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}$$

- Combining:

$$\underbrace{\frac{\partial}{\partial t}\int \rho \mathbf{u}\, dV}_{\text{Rate of change of momentum}} = \underbrace{-\oint \rho \mathbf{u}\mathbf{u}\cdot \mathbf{n}\, dS}_{\text{Net inflow of momentum}} - \underbrace{\oint p\mathbf{n}\, dS}_{\text{Total pressure}} + \underbrace{\oint 2\mu \mathbf{D}\cdot \mathbf{n}\, dS}_{\text{Total viscous force}} + \underbrace{\int \rho \mathbf{f}\, dV}_{\text{Total body force}}$$

Differential Form of Momentum Equation (6)

- Using momentum equation

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = \mathbf{u}\frac{\partial \rho}{\partial t} + \mathbf{u}\nabla \cdot \mathbf{u}\rho + \rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} =$$

$$\mathbf{u}\underbrace{\left(\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{u}\rho\right)}_{0\text{ (mass balance)}} + \rho \frac{D\mathbf{u}}{Dt} \qquad \left(\text{where } \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right)$$

- Differential form (Convective form)

$$\Rightarrow \boxed{\rho \frac{D\mathbf{u}}{Dt} = \nabla \cdot \mathbf{T} + \rho \mathbf{f}} \qquad \mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}$$

Conservation of Energy

- Integral form:

$$\overbrace{\frac{\partial}{\partial t}\int \rho\left(e+\frac{1}{2}u^{2}\right)dV}^{\text{Rate of change of internal + kinetic energy}} = \overbrace{-\oint \rho\left(e+\frac{1}{2}u^{2}\right)\mathbf{u}\cdot\mathbf{n}\,dS}^{\text{Net inflow of internal + kinetic energy}}$$

$$-\underbrace{\oint_{V} \mathbf{n}\cdot\mathbf{q}\,dS}_{\text{Net heat flow}} + \underbrace{\oint \mathbf{n}\cdot(\mathbf{u}\mathbf{T})\,dS}_{\text{Net work done by the stress tensor}} + \underbrace{\int \mathbf{u}\cdot\mathbf{f}\,dV}_{\text{Work done by body forces}}$$

$$\boxed{\begin{array}{l} \mathbf{T} = \left(-p+\lambda\nabla\cdot\mathbf{u}\right)\mathbf{I} + 2\mu\mathbf{D} \\ \mathbf{D} = \dfrac{1}{2}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^{\mathrm{T}}\right) \\ \mathbf{I} \quad \textit{is identity tensor} \end{array}}$$

Energy principle: change of energy = work done + heat input

Differential Form of Energy Equation

- Mechanical Energy

$$\mathbf{u}\cdot\left(\rho\frac{\partial\mathbf{u}}{\partial t}+\rho\mathbf{u}\cdot\nabla\mathbf{u}=\rho\mathbf{f}+\nabla\cdot\mathbf{T}\right)$$

$$\Rightarrow \rho\frac{\partial}{\partial t}\left(\frac{u^2}{2}\right)=-\rho\mathbf{u}\cdot\nabla\left(\frac{u^2}{2}\right)+\rho\mathbf{u}\cdot\mathbf{f}+\mathbf{u}\cdot\left(\nabla\cdot\mathbf{T}\right)$$

- Subtract the mechanical energy from the energy equation:

$$\boxed{\rho\frac{De}{Dt}-\mathbf{T}\cdot\nabla\mathbf{u}+\nabla\cdot\mathbf{q}=0}$$

$$\mathbf{T}=\left(-p+\lambda\nabla\cdot\mathbf{u}\right)\mathbf{I}+\mathbf{2\mu D}$$

$$\mathbf{q}=-\mathrm{k}\nabla T$$

For incompressible fluids we need only consider the *mechanical energy principle*:

change of kinetic energy = work done

Checkpoint: Fluid Flow Equations

Derivation of the equations governing fluid flow in integral form

- ► Conservation of Mass
- ► Conservation of Momentum
- ► Conservation of Energy

Differential form: Conservative vs. convective form

Special cases:

Incompressible flow

Inviscid compressible flow

Inviscid Compressible (Euler) Flow

- Inviscid and compressible flow

$$\mu = 0 \text{ and } \lambda = 0$$

$$\mathbf{T} = \left(-p + \lambda \nabla \cdot \mathbf{u}\right)\mathbf{I} + \mathbf{2}\mu\mathbf{D} = -p\mathbf{I}$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0$$

$$\frac{D\mathbf{u}}{Dt} = \rho \mathbf{f} - \nabla p$$

$$\rho \frac{De}{Dt} = -p \nabla \cdot \mathbf{u}$$

Incompressible Flow

- Incompressible flow

$$\frac{D\rho}{Dt} \equiv 0$$

$$\Rightarrow \frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{u} = 0 \qquad \rightarrow \nabla\cdot\mathbf{u} = 0$$

- The Navier-Stokes (NS) equations:

$$\rho\frac{\partial\mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \mu\nabla^2\mathbf{u} + \rho\mathbf{f_b}$$

- 2D incompressible NS:

$$\boxed{\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\\
&\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\end{aligned}}$$

Module 1

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