Problem Set 2

MECH 479/587 - Computational Fluid Dynamics
Winter Term 1

Due: October 11, 2022

1 Discretizing the First Order Derivative (15 marks)

Given sufficient continuity of a function and its derivatives, Taylor's series allows for constructing derivatives of any order. Furthermore, by considering sufficient information about the function u, a derivative of k^{th} order can be constructed to an arbitrary order of accuracy. Given the stencil $\{i-2, i-1, i, i+1\}$, and the representation of the first order derivative:

$$\left.\frac{\partial u}{\partial x}\right|_i = \left(\frac{\alpha u_{i+1} + \beta u_i + \gamma u_{i-1} + \delta u_{i-2}}{\Delta x}\right) + O(\Delta x^p), \tag{1}$$

construct the family of second-order schemes and the corresponding truncation errors obtained in terms of the parameter β. Given the construction, answer the followings:

1. What will be the ordered set or parameters $(\alpha, \beta, \gamma, \delta)$ corresponding to the second-order central difference scheme?
2. What will be a second-order accurate scheme by setting $\alpha = 0$ in eq. (1)? Please provide the ordered set (β, γ, δ).
3. For the third-order accurate scheme on the given stencil, what will be the corresponding ordered set $(\alpha, \beta, \gamma, \delta)$?

2 Alternative way to Discretize the First Order Derivative (15 marks)

High gradients in momentum or density can be present in fluid flow problems. There is a need to capture these gradients with a high level of accuracy, either owing to the nature of the flow physics or due to computational resource limitations. Compact high-order finite difference schemes are generally preferred in such situations, where the information about the derivatives at two or more computational points is simultaneously unknown. A general form a compact scheme for the first derivative is given by:

$$\alpha \left.\frac{\partial u}{\partial x}\right|_{i-1} + \left.\frac{\partial u}{\partial x}\right|_i + \alpha \left.\frac{\partial u}{\partial x}\right|_{i+1} = \beta\left(\frac{u_{i+1} - u_{i-1}}{2\Delta x}\right) + \gamma\left(\frac{u_{i+2} - u_{i-2}}{4\Delta x}\right) + O(\Delta x^p). \tag{1}$$

Observe that by setting $(\alpha, \beta, \gamma) = (0, 1, 0)$, we recover the second order central difference scheme for the first order derivative. Given this construction, answer the followings:

1. Let $\alpha = 0$. Obtain (β, γ) such that a fourth-order approximation to the first-order derivative is obtained. Give the truncation error obtained.

2. Let $\gamma = 0$. Obtain (α, β) such that a fourth-order approximation to the first-order derivative is obtained. Give the truncation error obtained.

3 Assessment of Stability Limit (20 marks)

Consider the viscous Burgers' equation (a prototypical equation in 1-D for incompressible Navier-Stokes equations) in a periodic domain of length 1:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

$$u(x,0) = 1 + \sin(2\pi x), \tag{2}$$

$$u(1,t) = u(0,t), \tag{3}$$

where $\nu = 0.01$. Consider two different discrete domains with $N = 20$ and $N = 40$ points respectively.

1. Using a forward discretization (Euler explicit) for the time derivatives, and central discretization for the first and second order spatial derivatives, evaluate the solution at $t = 1.0$ using $\Delta t = 5 \times 10^{-3}$.

2. For each of the grids, conduct numerical experiments to determine the maximum value of Δt for which the solution remains stable. Present your results in Tabular form.

Note:

- **A code has been provided as an example for problem 3.**
- **All assignments should be submitted through Canvas.**