Problem Set 1

MECH 479/587 - Computational Fluid Dynamics
Winter Term 1

Due: September 29, 2022

1 Shallow Water Waves (15 marks)

The shallow-water equations describe a thin layer of fluid of constant density in hydrostatic balance, bounded from below by the bottom topography and from above by a free surface.

Figure 1: Shallow water at a beach with ripples by a light breeze

To explore the mathematical structure of the shallow-water equations, consider the following one-dimensional form of the time-dependent shallow water equations (Saint-Venant equations):

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0, \tag{2}$$

where h denotes the spatial distribution of height of free water surface in a stream with the velocity component u, and g represents the force acting on the fluid due to gravity.

Express the above system in a matrix form, find the eigenvalues, and show that the system is hyperbolic.

2 2D Steady, Inviscid, Incompressible flow (20 marks)

The equations governing the steady, two-dimensional motion of an inviscid, incompressible fluid ($\rho = \text{const}$) are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0, \tag{2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0. \tag{3}$$

Show that these equations always have just one real eigenvalue, and hence one characteristic equation. Find the characteristic equation.
HINT: Start by transforming the equations into the matrix form $\mathbf{u}_x + A\mathbf{u}_y = \mathbf{0}$, where $\mathbf{u} = (u, v, p)^T$.

3 Assessing Accuracy of ODE integration (15 marks)

Consider the following initial value problem of first-order ODE system:

$$\frac{du}{dt} = -2u, \tag{1}$$

$$u(0) = 1. \tag{2}$$

1. (Stability) Plot the ODE solutions until final time $t_{final} = 8$ obtained using the forward Euler, the backward Euler and the Trapezoidal time integration schemes at four representative values of time step sizes $\Delta t = \{0.1, 0.2, 0.4, 0.8\}$, and compare the solutions obtained with the exact solution on the same plot. Briefly comment on the results obtained.

2. (Order of accuracy) By varying step sizes (Δt), one can obtain different values of the absolute local error at a particular time instant. Vary the step size $\Delta t \in [0.001, 1]$, and graphically show the absolute local error at t = 4.0 for the backward Euler and Trapezoidal method. Briefly comment on the results obtained.

For the order of accuracy analysis, you need to use log-scale for both the step size (along horizontal X-axis) and the absolute error (along vertical Y-axis).

Note:

- **Please append a screenshot of your code for problem 3 in your solution.**
- **All assignments should be submitted through Canvas.**