MECH 479/587

Computational Fluid Dynamics

Module 2b: PDE Classification

(x, y, z, t)

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Winter Term 1, 2022

What is PDE?

- A differential equation that contains unknown multivariable functions and their partial derivatives
  - PDEs are distinguished from ODEs by the fact that they contain derivatives more than one independent variable

Consider a scalar partial differential equation (PDE) of the general form

$\partial \Rightarrow$ partial

A general implicit form of PDE is:

$$F\left(x, y, \ldots, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \ldots, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial xy}, \frac{\partial^2 u}{\partial y^2}, \ldots\right) = 0$$

where $x, y, \ldots$ are the indepdenent variables, and $u = u(x, y, \ldots)$ is the dependent variable.

Classification of PDEs

- PDEs are classified based on the mathematical concept of characteristics that are lines (2D) or surfaces (3D) along which certain properties remain constant
  - Certain derivatives may be discontinuous
  - Characteristics are related in which "information" can be transmitted in physical system

- PDEs can be classified into hyperbolic, parabolic and elliptic ones
  - Each class of PDEs models a different kind of physical processes
  - Number of initial/boundary conditions depends on the PDE type
  - Different solution methods are required for PDEs of different type

Classification of PDEs

Information Direction } Wave

- Hyperbolic equations
  - Information propagates in certain directions at finite speeds; the solution is a superposition of multiple simple waves
    - *Wave-like* solutions
- Parabolic equations
  - Information travels downstream/forward in time; the solution can be constructed using a marching/time-stepping method
    - *Damped wave-like*

Marching (Diffusion/damping)

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

- Elliptic equations
  - Information propagates in all directions at infinite speed; describe equilibrium phenomena (unsteady problems are never elliptic)
    - *Not wave-like*

Equilibrium (Blind)

Some Examples

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0$$

$$\phi(x, t)$$

- 1D scalar equations

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0 \quad \text{Advection}$$

$$\frac{\partial \phi}{\partial t} - \nu\frac{\partial^2 \phi}{\partial x^2} = 0 \quad \text{Diffusion}$$

$$\boxed{\frac{\partial^2 \phi}{\partial t^2} - \nu\frac{\partial^2 \phi}{\partial x^2} = 0} \quad \text{Wave propagation}$$

Evolution in time

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{Laplace equation}$$

Link with the 2D Navier-Stokes Equation

- X-momentum

Diffusion

$$\underbrace{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}}_{\text{Advection part}} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

- Continuity

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}$$

$$u = \frac{\partial \phi}{\partial x}$$

$$v = \frac{\partial \phi}{\partial y}$$

Laplace-type

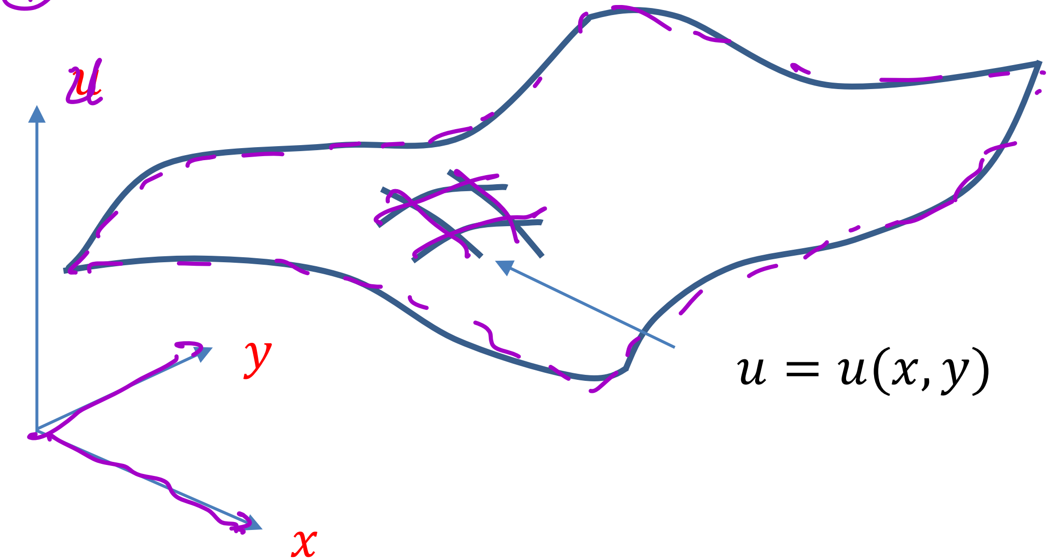
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Quasi-Linear 1st Order PDEs

$u(x,y)$

- ❑ Quasi-linear first order equation

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c \quad \text{—} (1)$$



$$a = a(x,y,u)$$

$$b = b(x,y,u)$$

$$c = c(x,y,u)$$

Quasi-Linear PDE

- Arbitrary change in u is given by

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \quad \text{---} (2)$$

- The original equation and the condition for an infinitesimal change

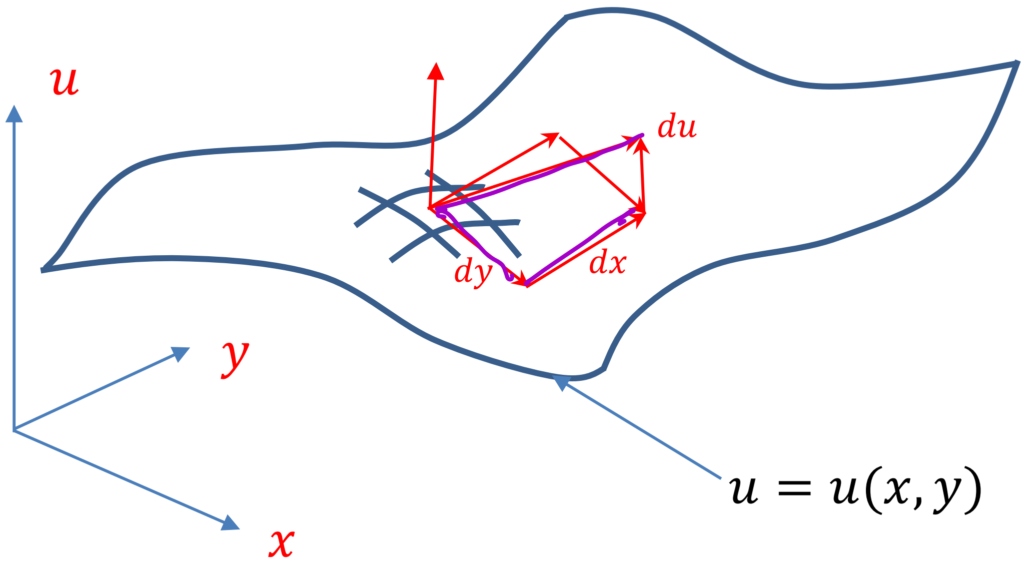
$$a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} = c \Rightarrow \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, -1\right) \cdot \underbrace{(a, b, c)}_{} = 0$$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \Rightarrow \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, -1\right) \cdot \underbrace{(dx, dy, du)}_{} = 0$$

- Both (a, b, c) and (dx, dy, du) lie on the surface
  - ▶ The vector field (a, b, c) is tangent to the surface u

Visual Inspection

- The solution of this equation defines a single valued surface *u(x,y)* in three-dimensional space:



- Total derivative

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

Characteristics

→ Comparing (a, b, c)

$$(dx, dy, du) = \underline{ds}\,(a, b, c)$$

$$\Rightarrow \quad \underbrace{\frac{dx}{ds} = a}, \quad \underbrace{\frac{dy}{ds} = b}, \quad \frac{du}{ds} = c$$

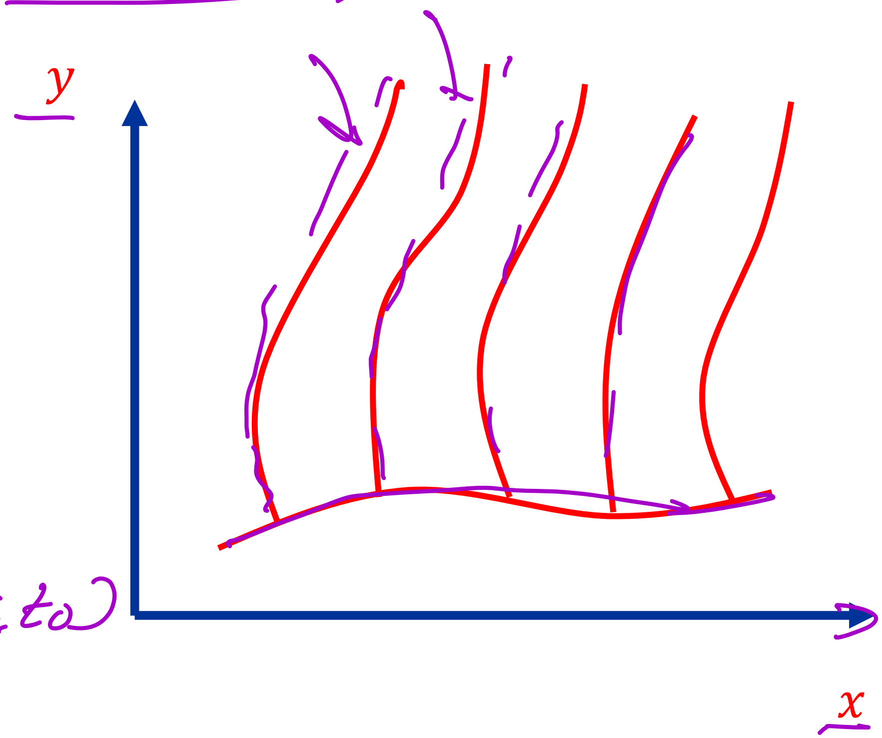
$$\longrightarrow \boxed{\frac{dx}{dy} = \frac{a}{b}} \quad \longrightarrow \quad \boxed{\frac{du}{ds} = c}$$

$\partial \rightarrow d$

Characteristics Cont'd

- ❑ Three equations specify in the *x*-*y* plane

Characteristics



$$\frac{dx}{ds} = a; \quad \frac{dy}{ds} = b; \quad \frac{du}{ds} = c$$

Given initial conditions:

$$x = x(s, t_0), \quad y = y(s, t_0), \quad u = u(s, t_0)$$

$$\boxed{\frac{dx}{dy} = \frac{a}{b}}$$

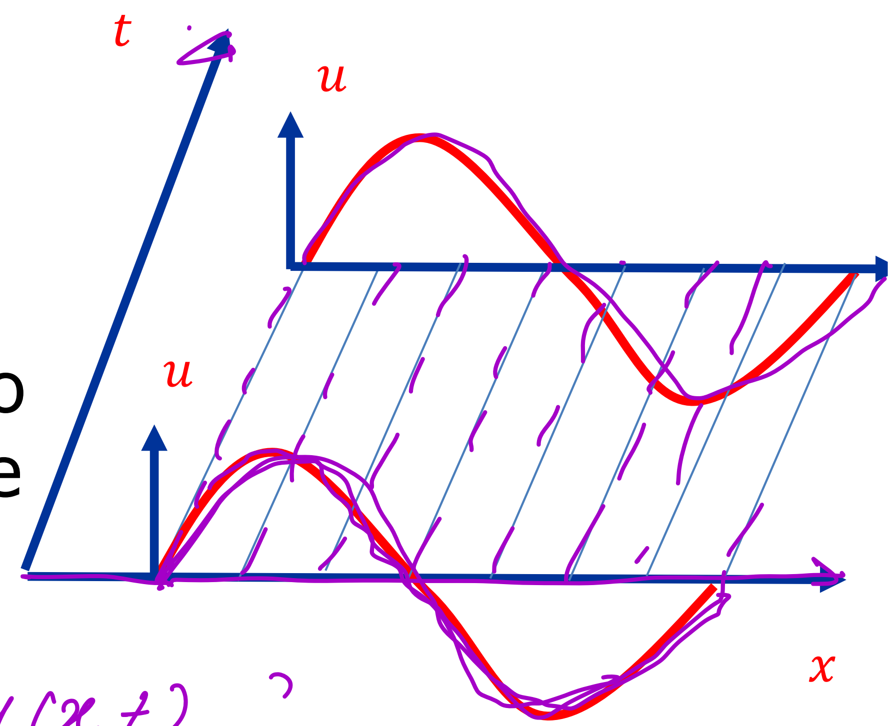
- ❑ Along the characteristics, the original PDE becomes the ODE

Example: Linear Advection Equation

- Goal: We want to transform this linear first-order PDE into an ODE along the appropriate curve

speed

$$\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = 0$$



$u(x,t)$?

$g(x_0) = \sin(x_0)$

$$du = \frac{\partial u}{\partial t}\,dt + \frac{\partial u}{\partial x}\,dx$$

- Characteristics are given by

$$* \quad \frac{dt}{ds} = 1 \; ; \quad \frac{dx}{ds} = c \; ; \quad \frac{du}{ds} = 0$$

$$\boxed{\frac{dx}{dt} = c} \; , \quad du = 0$$

$$(dt, dx, du) = ds\,(1, c, 0)$$

Example: Linear Advection Equation

- Along the characteristics, the solution remains constant
  - Along the the characteristics $(x(s), t(s))$, the original PDE becomes the ODE
- To determine the general solution, it is enough to find the characteristics by solving the characteristic system of ODEs

$\frac{dt}{ds} = 1$ let $t(0) = 0 \Rightarrow t = s$

$\frac{dx}{ds} = c$ let $x(0) = x_0$ we know: $x = cs + x_0$

$\frac{du}{ds} = 0$ let $u(0) = g(x_0)$, we know

$u(x(t), t) = g(x_0)$
$= g(x - ct)$

Linear Advection Equation

- Graphically

$$u(x,t) = \underline{g(x-ct)} \quad \text{where } g(x) = u(x, t=0)$$

Verify:

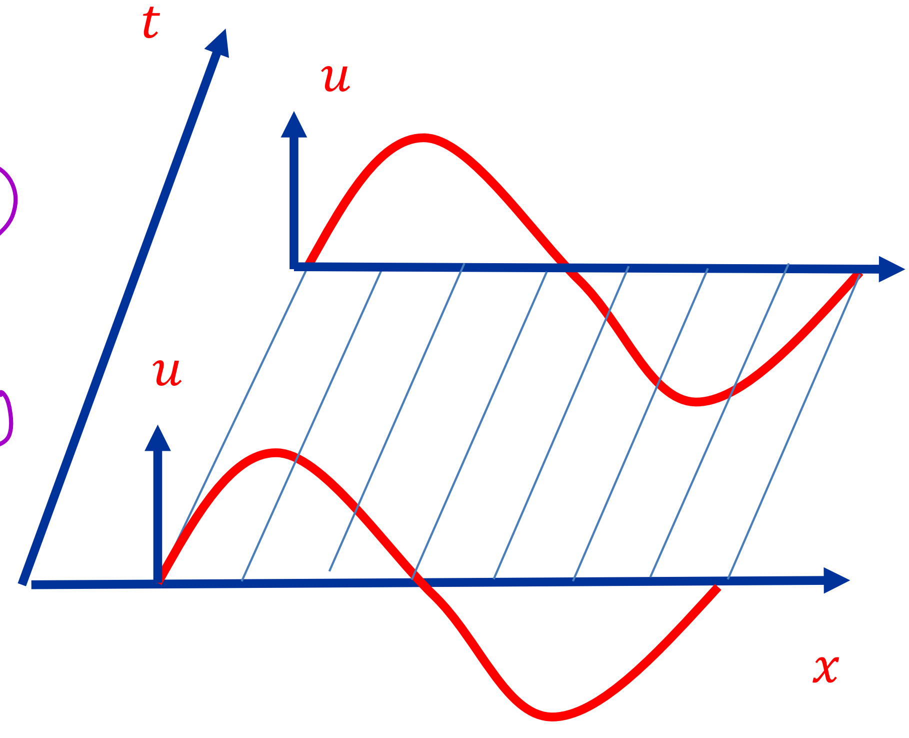
$$\alpha(x,t) = x - ct$$

Then

$$\frac{\partial u}{\partial t} = \frac{\partial g}{\partial \alpha}\frac{\partial \alpha}{\partial t} = \frac{\partial g}{\partial \alpha}(-c)$$

$$\frac{\partial u}{\partial x} = \frac{\partial g}{\partial \alpha}\frac{\partial \alpha}{\partial x} = \frac{\partial g}{\partial x}\frac{\partial}{\partial x}(x-ct)$$

$$= \frac{\partial g}{\partial \alpha}$$



- Solution:
  - ► Waves travel with constant speed
  - ► Preserve the initial waveform

Some Examples

- Add a source term

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = u$$

- Nonlinear (quasi-linear) advection equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

- Advection-diffusion (Burger) equation

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \epsilon\frac{\partial^2 u}{\partial x^2} = 0}$$

Partial Differential Eq^{n}s (PDE's)

→ Why we care for CFD ?

→ Approximation / discretization based on underlying character or behavior !

→ Boundary conditions

Characteristics :

Can we come up with some curves or surfaces (pathways) where PDE's tend to behave like ODE's

→ Lines / curves represent connection or relationship among independent variables !

$$\underbrace{f(x, y, z, t)} = 0$$

1D wave/ advection Eq^n:

① → $$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \quad u = u_0(x) \;@\; t=0$$

② → $$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt}$$

Compare ① & ②:

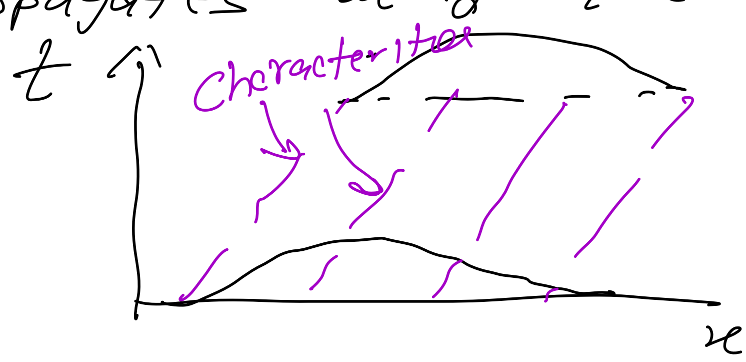
$$\boxed{\frac{dx}{dt} = c}$$

$\Rightarrow \quad x = ct + \text{const.}$ (Characteristic Eq^n)

Along these lines:

$$\frac{du}{dt} = 0 \quad (\text{ODE})$$

$\underset{\sim}{u}$ solution propagates along the lines



Classification of PDE using eigenvalue method

(u, v)

(x, y)

Set of First-order PDE's :

$$\rightarrow a_1 \frac{\partial u}{\partial x} + b_1 \frac{\partial u}{\partial y} + c_1 \frac{\partial v}{\partial x} + d_1 \frac{\partial v}{\partial y} = 0 \quad — (1)$$

$$a_2 \frac{\partial u}{\partial x} + b_2 \frac{\partial u}{\partial y} + c_2 \frac{\partial v}{\partial x} + d_2 \frac{\partial v}{\partial y} = 0 \quad — (2)$$

We can write in matrix form

$$\underbrace{\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}}\ \underbrace{\frac{\partial}{\partial x}\begin{Bmatrix} u \\ v \end{Bmatrix}} + \underbrace{\begin{bmatrix} b_1 & d_1 \\ b_2 & d_2 \end{bmatrix}}\ \frac{\partial}{\partial y}\underbrace{\begin{Bmatrix} u \\ v \end{Bmatrix}} = 0$$

<u>Denote</u>:

$$\underset{\sim}{W} = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\Rightarrow \quad \frac{\partial \underset{\sim}{W}}{\partial x} + \underbrace{\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}^{-1} \begin{bmatrix} b_1 & d_1 \\ b_2 & d_2 \end{bmatrix}}_{A} \frac{\partial \underset{\sim}{W}}{\partial y} = 0$$

$$\Rightarrow \quad \frac{\partial \underset{\sim}{W}}{\partial x} + [A] \frac{\partial \underset{\sim}{W}}{\partial y} = 0$$

Determinant Condition to find eigenvalues:

$$\det(A - \lambda I) = 0$$

$$A \underset{\sim}{X} = \lambda \underset{\sim}{X}$$

eigenvectors (characteristics)

If all eigenvalues are real $\Rightarrow$ System is hyperbolic

If all eigenvalues are complex $\Rightarrow$ Elliptic

If eigenvalues are mixed (real & complex):

mixed PDE's

Quasi-linear 2nd Order PDEs

variable : u

(x, y) independent

$$\boxed{a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial x \partial y} + c \frac{\partial^2 u}{\partial y^2} = d}$$

$$d = d\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \ldots\right)$$

$$a = a(x, y, u, u_x, u_y)$$

$$b = b(\quad)$$

$$c = c(\quad)$$

$$u_x = \frac{\partial u}{\partial x}$$

$$u_y = \frac{\partial u}{\partial y}$$

Second-Order PDE

Define : $\phi = \dfrac{\partial u}{\partial x}$, $\psi = \dfrac{\partial u}{\partial y}$

$$a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial x \partial y} + c \frac{\partial^2 u}{\partial y^2} = d$$

$$\Rightarrow a \frac{\partial \phi}{\partial x} + b \frac{\partial \phi}{\partial y} + c \frac{\partial \psi}{\partial y} = d \quad - (1)$$

Other Eqn: $\dfrac{\partial \psi}{\partial x} = \dfrac{\partial \phi}{\partial y} \Rightarrow \dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial^2 u}{\partial x \partial y}$

(2)

Transformed Equations (optional)

[See handout for derivation]

- We can transform 2nd order PDE

into first order PDE

$$\left.\begin{aligned} a\frac{\partial \phi}{\partial x} + b\frac{\partial \phi}{\partial y} + c\frac{\partial \psi}{\partial y} &= d \\ \frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} &= 0 \end{aligned}\right\}$$

In the matrix form

$$\begin{pmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \psi}{\partial x} \end{pmatrix} + \underbrace{\begin{pmatrix} \frac{b}{a} & \frac{c}{a} \\ -1 & 0 \end{pmatrix}}_{A} \begin{Bmatrix} \frac{\partial \phi}{\partial y} \\ \frac{\partial \psi}{\partial y} \end{Bmatrix} = \begin{pmatrix} \frac{d}{a} \\ 0 \end{pmatrix}$$

$\Rightarrow$ Find the existence of characteristics!

Determinant Condition (2)

- Determinant

$$\det(A - \lambda I) = 0$$

$$-\lambda\left(\frac{b}{a} - \alpha\right) + \frac{c}{a} = 0$$

- Solving:

$$\lambda = \frac{1}{2a}\left(b \pm \underbrace{\sqrt{b^2 - 4ac}}\right)$$

- PDE types:

Hyperbolic $\leftarrow$ $\sqrt{b^2 - 4ac} > 0$ Two characteristics

Parabolic $\leftarrow$ $\sqrt{b^2 - 4ac} = 0$ one characteristic

Elliptic $\leftarrow$ $\sqrt{b^2 - 4ac} < 0$ No real characteristics

Example 1:

$$\frac{\partial^2 u}{\partial x^2} - c^2 \frac{\partial^2 u}{\partial y^2} = 0$$

- Comparing $a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = d$

- Gives: $a = 1$; $b = 0$; $c = -c^2$; $d = 0$;

$$b^2 - 4ac = 0^2 + 4 \cdot 1 \cdot c^2 = 4c^2 > 0$$

Hyperbolic !

Example 2:

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

- Comparing

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = d$$

- Gives: $a = 1;\quad b = 0;\quad c = 1;\quad d = 0;$

$$\underline{b^2 - 4ac} = 0^2 - 4.1.1 = \underline{-4 < 0}$$

Elliptic

Example 3:

First order derivatives play no role !

$$\underbrace{\frac{\partial u}{\partial x}} - k\frac{\partial^2 u}{\partial y^2} = 0$$

$-K\dfrac{\partial^2 u}{\partial y^2} = \cancel{-\dfrac{\partial y}{\partial x}}$

- Comparing

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = d$$

$a=0$ $b=0$

- Gives: $\underline{a = 0}$; $\underline{b = 0}$; $\underline{c = -k}$; $d = \cancel{0}$;

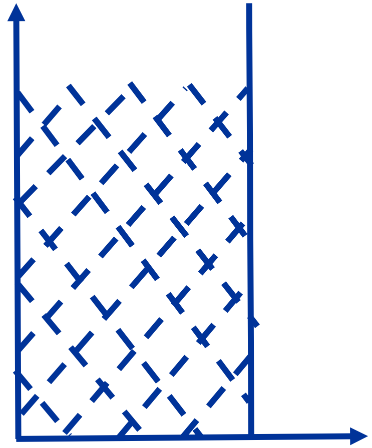
$-\dfrac{\partial y}{\partial x} \Rightarrow$ (doesn't matter) both sign or value!

$$b^2 - 4ac = 0^2 \underline{+ 4.0.1} = 0$$

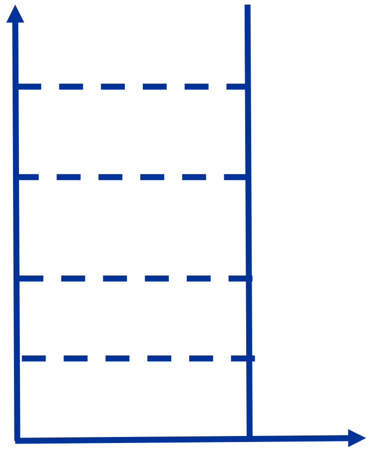
Parabolic

Relevance of PDE Classification

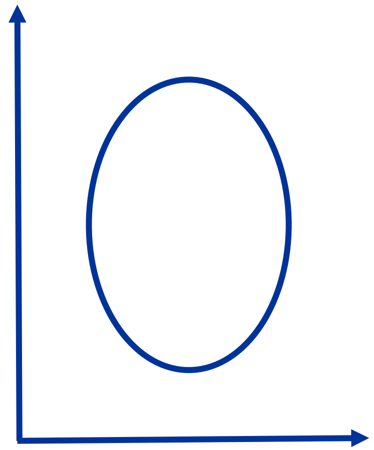
- Proper specifications of initial and boundary conditions
- Exploring and understanding different physical behaviors
- Development of appropriate numerical techniques



Hyperbolic



Parabolic



Elliptic

Link with the 2D Navier-Stokes Equation

- X-momentum

Parabolic

$$\underbrace{\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}}_{\text{Hyperbolic part}}+\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

- Continuity

$$\boxed{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0}$$

Elliptic

Note on the Navier-Stokes Equations

- The Navier-Stokes equations are <u>mixed</u> and they contain the parabolic, hyperbolic and elliptic characteristics behavior

- The different types of the equation require specialized techniques.

- Based on the underlying physical parameters, one behavior can dominant
  - ►For example, for inviscid compressible flow, only the hyperbolic part sustains

Example: The Navier-Stokes Equations

- Model problem for incompressible flow equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \bullet \nabla \mathbf{u} + \frac{\nabla p}{\rho} = \nu \Delta \mathbf{u}$$

$$\nabla \bullet \mathbf{u} = \mathbf{0}$$

These equations can be split into three pieces

$$Hyperbolic: \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \bullet \nabla \mathbf{u} = \mathbf{0}$$

$$Parabolic: \frac{\partial \mathbf{u}}{\partial t} = \nu \Delta \mathbf{u}$$

$$Elliptic: \ \Delta p = \nabla \bullet \left(-\mathbf{u} \bullet \nabla \mathbf{u} + \nu \Delta \mathbf{u} \right)$$

Boundary Conditions (B.C.):

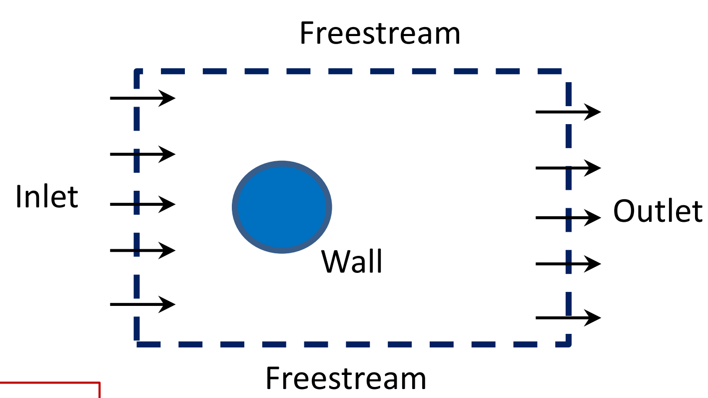
For **viscous** flow, we have $\mathbf{u}_{\text{fluid}} = \mathbf{u}_{\text{wall}}$ for the fluid on the solid wall.

This condition is known as no-slip condition.

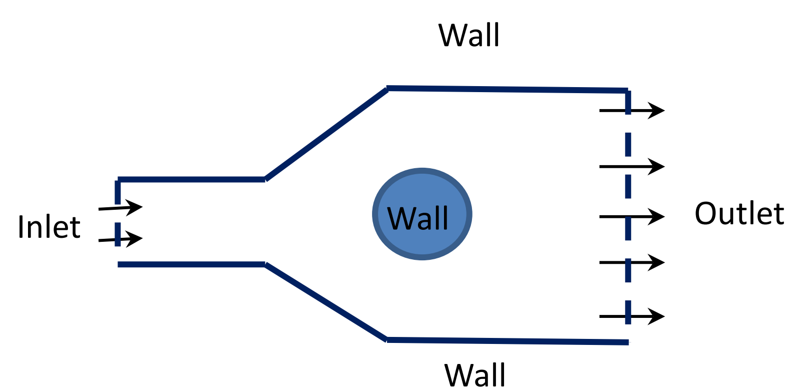
For **inviscid** flow, $(\mathbf{u}_{\text{fluid}}) \cdot (\mathbf{n}_{\text{wall}}) = 0$. This is called the no-penetration condition, or slip condition.

Initial Conditions (I.C.):

We need to provide $\mathbf{u}$ at time $t = 0$.



External flow



Internal flow

Well-posed Problem

Definition: A mathematical problem is well-posed when there exists one solution to the problem (Existence), which must also be the only solution (Uniqueness) and depends continuously on all the given data (Continuity).

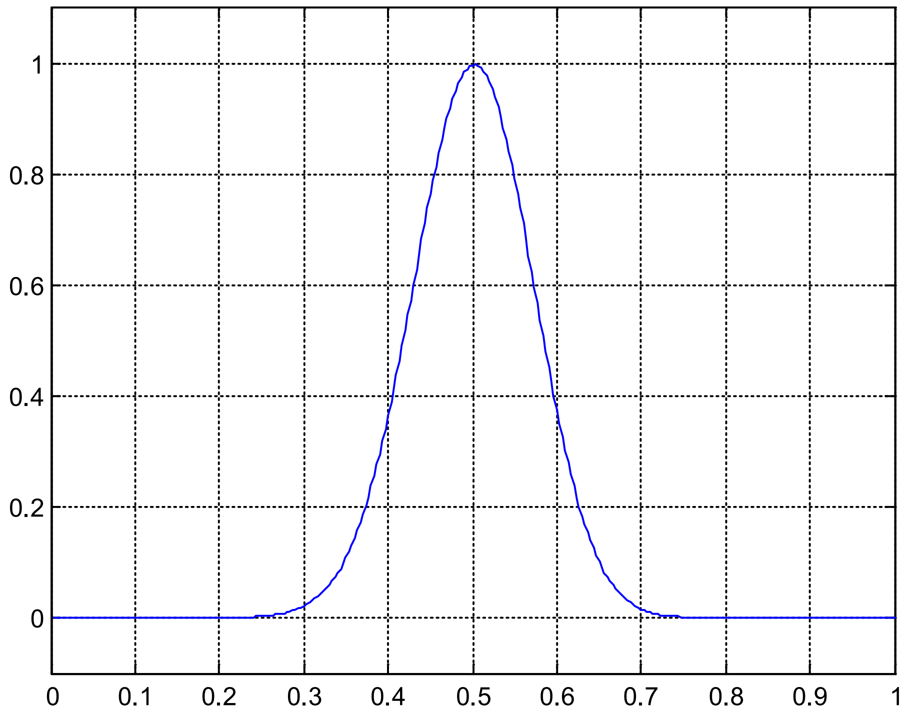
To determine whether or not a problem is well-posed, we must consider the governing equations, boundary conditions and initial conditions.

Demo: PDE Classification

- Matlab code

```
% PDE demo - advection, diffusion, dissipation

% Discretization
m = 300;
h = 1 / m;
x = h * (1:m)';
…
…
```



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