Programming Assignment - 1

In this assignment, your final objective is to numerically solve the unsteady 2-D heat equation in a unit domain. The governing PDE in a domain Ω is given by,

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \text{in} \quad \Omega, \tag{1}$$

$$u = u_0 \quad \text{on} \quad \partial\Omega_D, \tag{2}$$

$$\frac{\partial u}{\partial n} = g \quad \text{on} \quad \partial\Omega_N, \tag{3}$$

where $\partial\Omega_D$ and $\partial\Omega_N$ denote the regions of the boundary over which Dirichlet and Neumann boundary conditions are applied respectively. The final objective can be achieved by solving the following sub-problems.

1 Laplace's equation

In this section, you will solve the Laplace's equation over a unit domain given by

$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0 \quad \text{in} \quad (0,1) \times (0,1),$$

$$u = \begin{cases} y^4, & x = 0, \quad 0 \le y \le 1 \\ x^4, & 0 \le x \le 1, \quad y = 0 \\ 1 - 6y^2 + y^4, & x = 1, \quad 0 \le y \le 1 \\ 1 - 6x^2 + x^4, & 0 \le x \le 1, \quad y = 1. \end{cases}$$

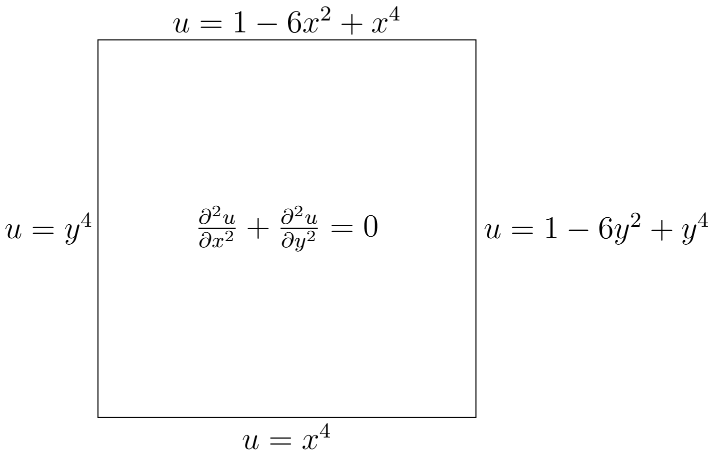

Figure 1: Computational domain and boundary conditions for Laplace's equation

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For the problem, your parameters are:

- Discretize the spatial derivatives using a 2^{nd} order central differencing scheme.
- Use a uniform and isotropic Cartesian grid, i.e. $\Delta x = constant$ and $\Delta x = \Delta y$.
- Solution on 3 different grids with 17, 33 and 65 grid points along each direction.

For the solution, your tasks are:

1. Write two functions/subroutines, which separately construct A and b corresponding to the Finite Difference Equations arising from the modified form of the PDE.
2. Compare the solution obtained with the exact solution given by $u_e(x, y) = x^4 + y^4 - 6x^2y^2$, by plotting the log-log plot of L_2 Norm of the error against grid-size, and compute the slope of the line obtained. Determine the order of accuracy of your solution.
3. Tabulate the time taken to obtain a solution u against grid-size.

2 Unsteady Heat Equation

From section 1, you have constructed a code for applying the discrete Laplacian operator on a variable defined on Ω. In this section, you will use the ideas developed earlier to solve the Unsteady Heat Equation in an explicit and implicit manner upto a time level T. The PDE to solve in $(0, T) \times \Omega$, now becomes

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$
$$u(x, y, 0) = \exp(-100(x^2 + y^2)).$$

Using parameters and boundary conditions identical to those of the Laplace's equation, your tasks for the current problem are:

1. Derive the error-residual form ($\boldsymbol{A\delta u} = \boldsymbol{b}, \boldsymbol{u}^{n+1} = \boldsymbol{u}^n + \boldsymbol{\delta u}$) for the Euler explicit method.
2. The steady state occurs at time step n if $||u^{n+1} - u^n||_{L2} < 1e - 8$. Integrate the equation to steady state using (a) Euler explicit method and (b) Trapezoidal method. Write functions/subroutines which execute the time-integration process.
3. For the Euler explicit method, obtain an upper bound on the time-step for which your solution does not diverge. Express your answer as $\frac{\Delta t}{\Delta x^2}$. Do the same for the trapezoidal scheme.
4. Does the steady state solution match that obtained in section 1? Tabulate the values of $||u_{Laplace} - u_{Unsteady}||_{L2}$. Do you observe the initial condition influencing the steady state solution? In this regard, comment on the behaviour of the PDE?
5. Tabulate the time taken to obtain a solution u against grid-size. Between solving the steady state equation, and time-integrating to steady state, which method was faster?

Some points to consider:

- The function/subroutine which constructs b, can compute the diffusion field for any variable defined over the domain Ω.
- Memory allocation for the construction of A and b has been implemented in the Matrix and Vector classes. This aspect has been addressed in the handout and the tutorial.