MECH 479/587

Computational Fluid Dynamics

Module 3 – Part B: 2D Finite Difference Approximation

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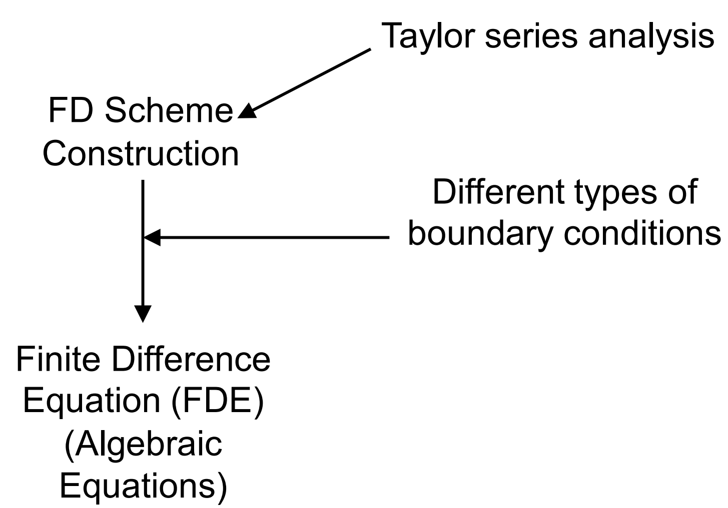
Review:

- ❑ Introduction of Taylor series
  - ▶ Point difference operators
  - ▶ Construction of finite difference equations (FDEs)
- ❑ Generalized finite difference approximation
  - ▶ Matrix difference operators
  - ▶ Boundary conditions
- ❑ Numerical Properties of FDEs
  - ▶ Order of Accuracy
  - ▶ Consistency
  - ▶ Stability
  - ▶ Lax Equivalence Theorem



Roadmap

Measurable Learning Outcomes

- Understand and apply Taylor series for derivative approximation

- Implement a finite difference discretization to solve a representative PDE (or set of PDEs) from an engineering application

- Understand basic numerical properties (accuracy, stability and convergence) of finite difference equation (FDE)

Module #3 : Part B

PDE → FDE's
(finite difference eqns)

1. 1D equations

→ Convection $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$

→ Diffusion $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$

2. Extension to 2D:

→ 2D Advection problem

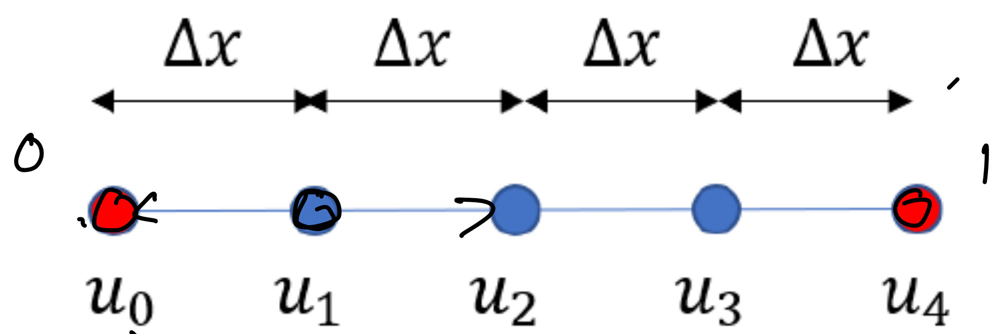
→ 2D Advection - Diffusion

3. Code demo : Stead & unsteady heat

Example: 1D Heat Conduction

For the PDE $\frac{\partial^2 u(x)}{\partial x^2} + 2u(x) = 1$,

use the Central Difference Scheme to find the values of the solution variable u at the given points.



PDE: $\boxed{\frac{\partial^2 u}{\partial x^2} + 2u(x) = 1}$

$\Downarrow$

FDE's

Given $u_0 = 0$, $u_4 = 1$ and $\Delta x = 0.5$

$$\frac{\partial^2 u}{\partial x^2} \simeq \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + O(\Delta x^2)$$

Solution:

(1) Taylor expansion to convert derivatives

(2) Express PDE into FDE counterpart

At location i

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + 2u_i = 1$$

(3) Discrete or numerical forms of FDE for finding unknowns "u" @ interior points : u_1, u_2, u_3

At $i=1$

$$\frac{u_0 - 2u_1 + u_2}{\Delta x^2} + 2u_1 = 1 \quad -①$$

At $i=2$ $$\frac{u_1 - 2u_2 + u_3}{\Delta x^2} + 2u_2 = 1 \quad -②$$

At $i=3$ $$\frac{u_2 - 2u_3 + u_4}{\Delta x^2} + 2u_3 = 1 \quad -③$$

Using $u_0 = 0$, $u_4 = 1$, $\Delta x = 0.5$

Solve for 3 eq^{n}s w/ 3 unknowns

$$\frac{-2u_1 + u_2}{0.5^2} + 2u_1 = 1$$

$$\frac{u_1 - 2u_2 + u_3}{0.5^2} + 2u_2 = 1$$

$$\frac{u_2 - 2u_3 + u_4}{0.5^2} + 2u_3 = 1$$

$$\left\{ \begin{array}{l} u_1 = \frac{1}{6} \\ u_2 = \frac{1}{2} \\ u_3 = \frac{5}{6} \end{array} \right.$$

Finite Difference for Multi-D Partial Differential Equations: 2D Advection

Advection in 2D

Linear U V

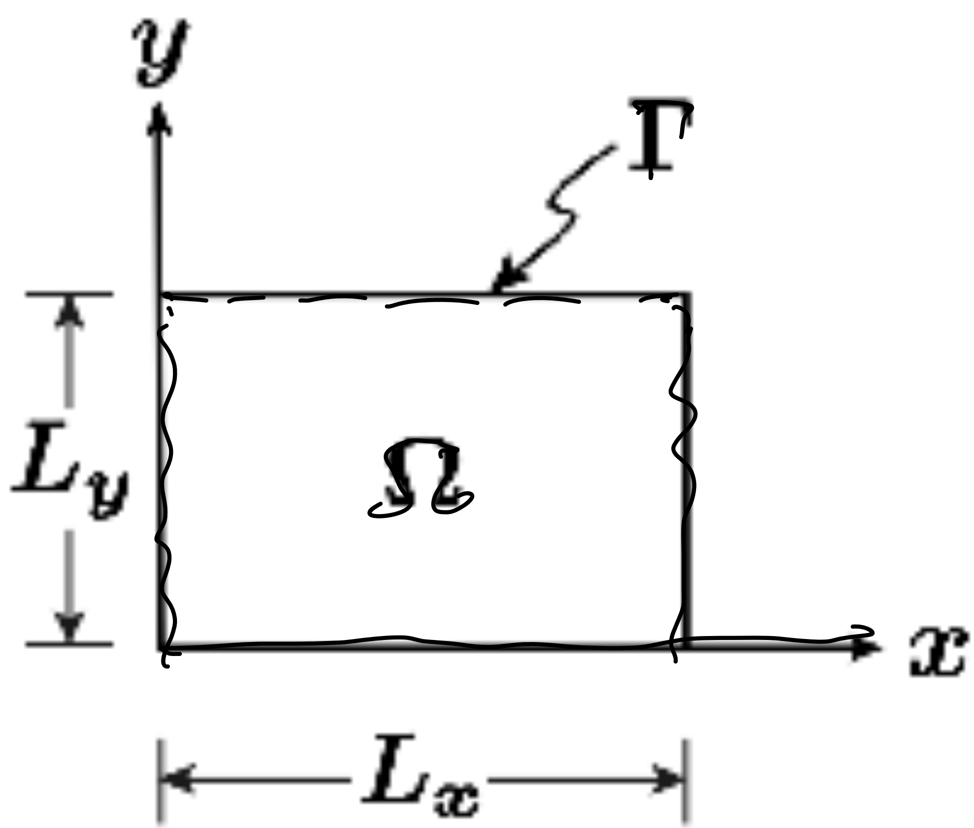
(c_x) (c_y)

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial y} = 0$$

Time derivative

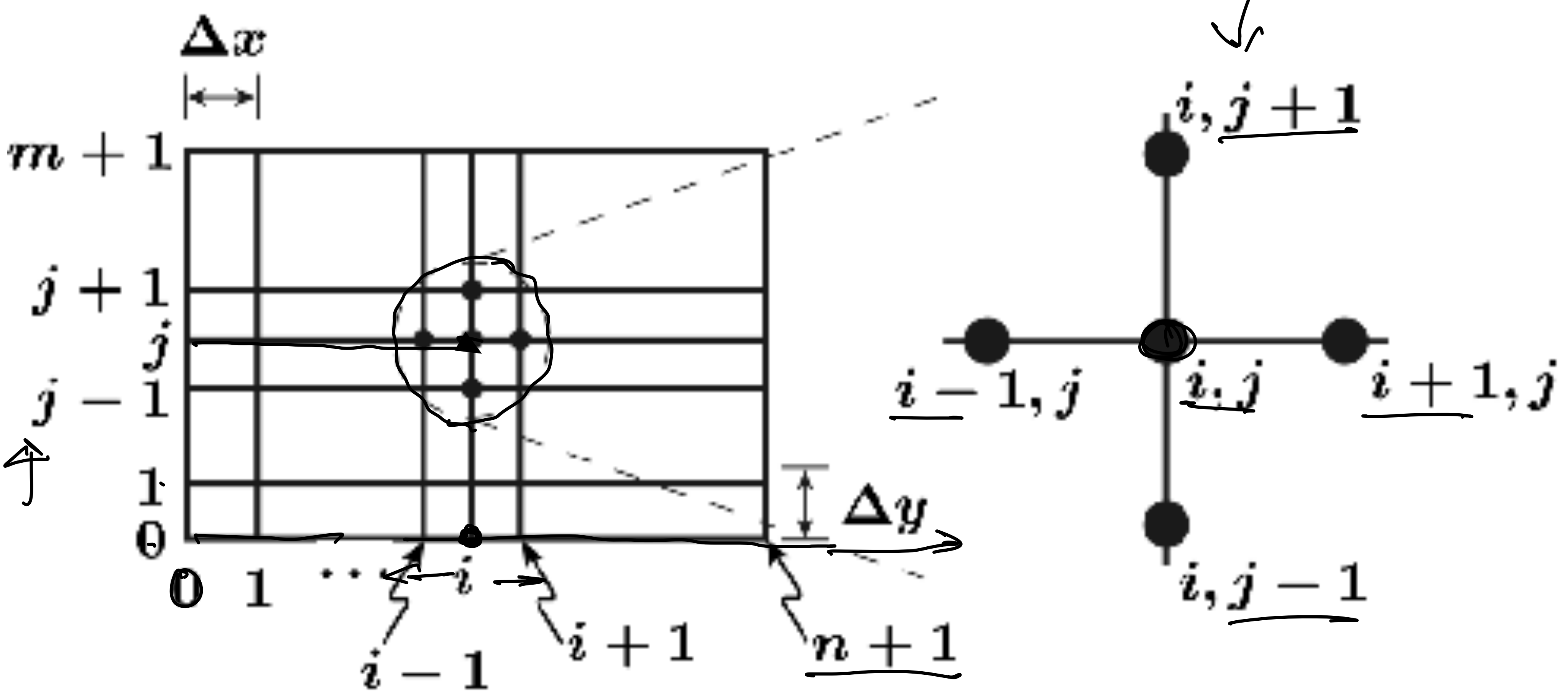
X-dir

Y-dir



FD Discretization in 2D

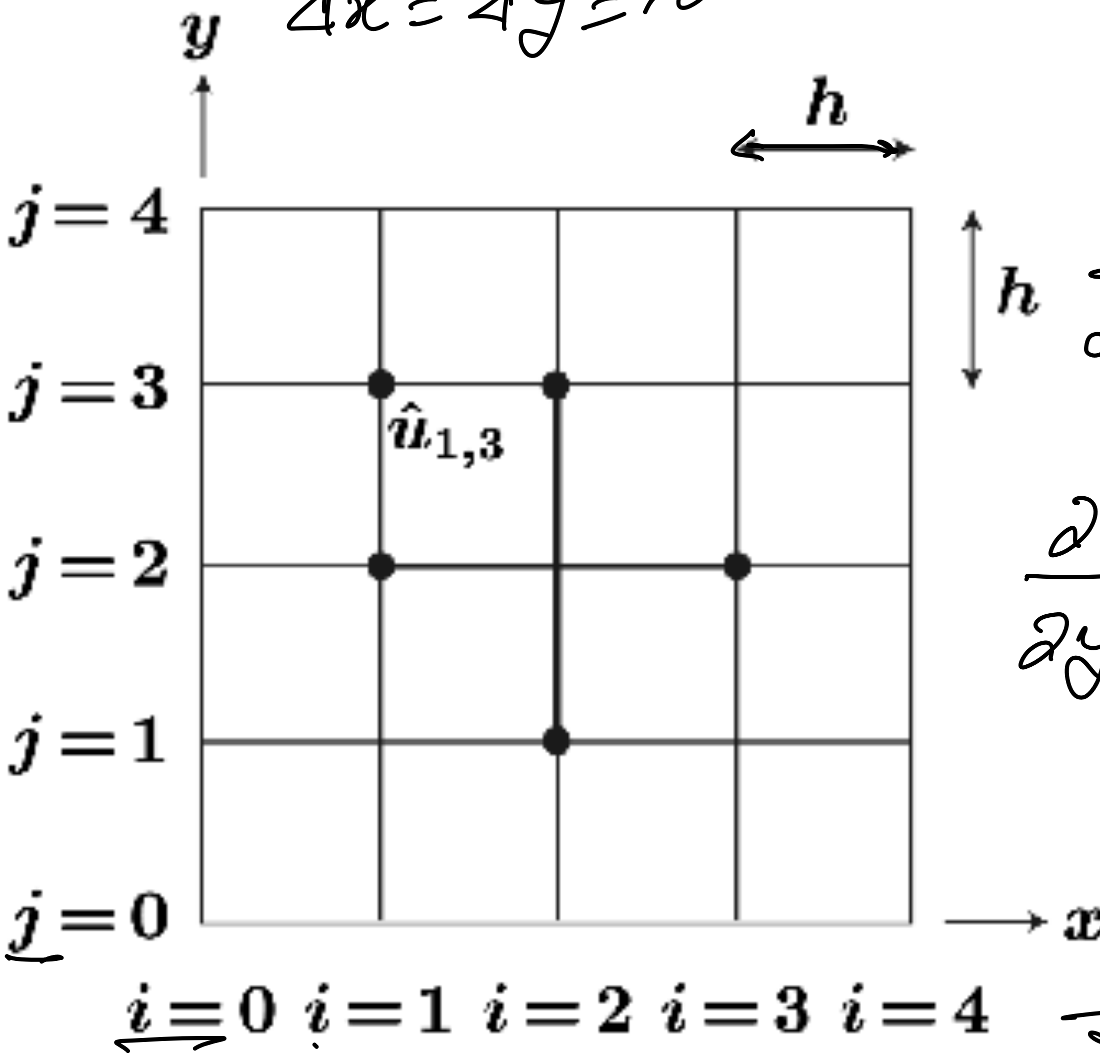
Stencil



$$u_{ij}^{n} \approx u(x_i, y_j, t_n)$$

FD for 2D Advection Equation

$\Delta x = \Delta y = h$



Backward difference

$$\left.\frac{\partial u}{\partial x}\right|_{i,j} \approx \frac{u_{i,j} - u_{i-1,j}}{\Delta x}$$

$$\left.\frac{\partial u}{\partial y}\right|_{i,j} \approx \frac{u_{i,j} - u_{i,j-1}}{\Delta y}$$

Centered in Space

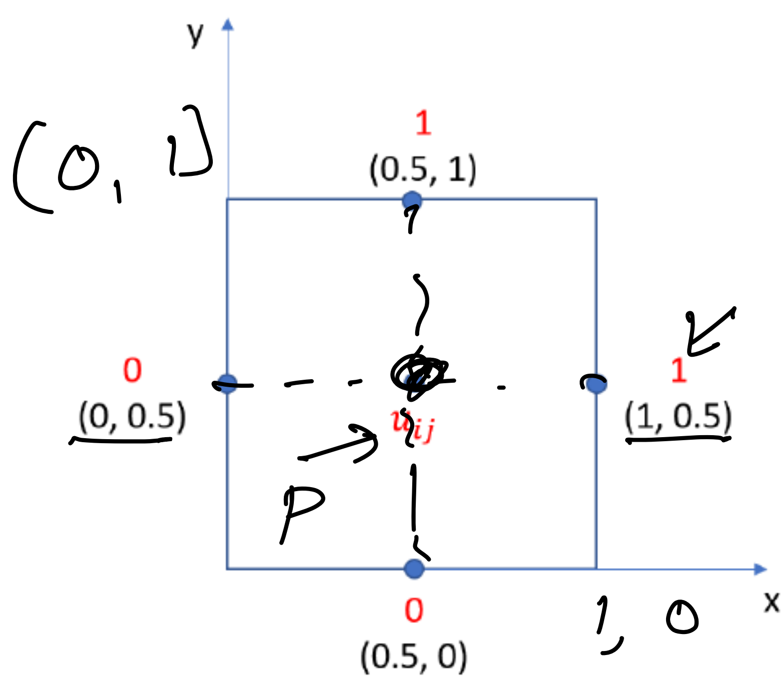
$$\left.\frac{\partial u}{\partial x}\right|_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}$$

$$\left.\frac{\partial u}{\partial y}\right|_{i,j} = \ldots$$

Checkpoint 1

$$\frac{\partial^2 u}{\partial x^2} \qquad \frac{\partial^2 u}{\partial y^2}$$

Consider a finite difference solution of the Poisson equation: $\boxed{u_{xx} + u_{yy} = x + y}$ on the unit square using the boundary conditions and the mesh shown in the drawing. Use a second-order accurate, centered finite difference scheme to compute the approximate value of the solution at the center of the square.



Find the solution value at the center.

u_{xx}

$\downarrow$

FDE:

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2}$$

$$= x_{i,j} + y_{i,j}$$

$$\frac{0 - 2u_p + 1}{0.5^2} + \frac{0 - 2u_p + 1}{0.5^2} = 0.5 + 0.5$$

$$\Rightarrow \boxed{u_p = 0.4375}$$

Multi-D FD Discretization: 2D Advection-Diffusion

Consider 2D model equation for unsteady heat or mass transfer

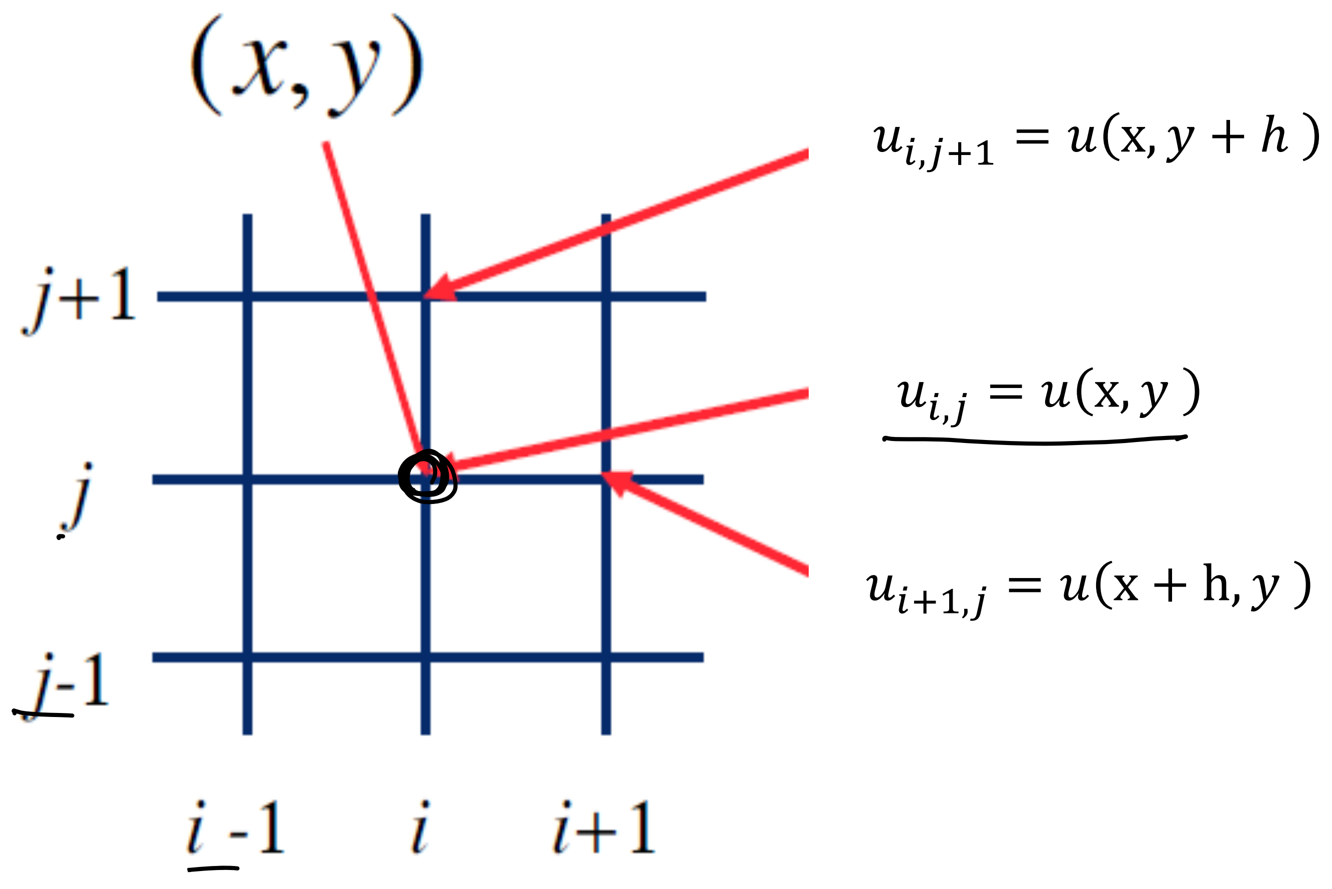
- ▶ Include diffusion term into the advection form
  - ◇ This can be also considered as a model problem for the Navier-Stokes equations

$$\underbrace{\frac{\partial u}{\partial t}}_{\text{Time derivative}} + \underbrace{U\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial y}}_{\text{Convection effect}} = \underbrace{\nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}_{\text{Diffusion}}$$

$u \to T$

Two-Dimensional Grid

- For 2D flow physics, discretize the variables on a two-dimensional grid



2D Discretization

$$\left(\frac{\partial u}{\partial t}\right)^n_{i,j} + U\left(\frac{\partial u}{\partial x}\right)^n_{i,j} + V\left(\frac{\partial u}{\partial y}\right)^n_{i,j} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)^n_{i,j}$$

$\downarrow$

FE

$$\frac{u^{n+1}_{i,j} - u^n_{i,j}}{\Delta t} = -U\left[\frac{u^n_{i+1,j} - u^n_{i-1,j}}{2h}\right] - V\left[\frac{u^n_{i,j+1} - u^n_{i,j-1}}{2h}\right] + \nu\left[\frac{u^n_{i+1,j} - 2u^n_{i,j} + u^n_{i-1,j}}{h^2} + \cdots\right]$$

$\nearrow$

$O(\Delta t)$

Formula for Coding:

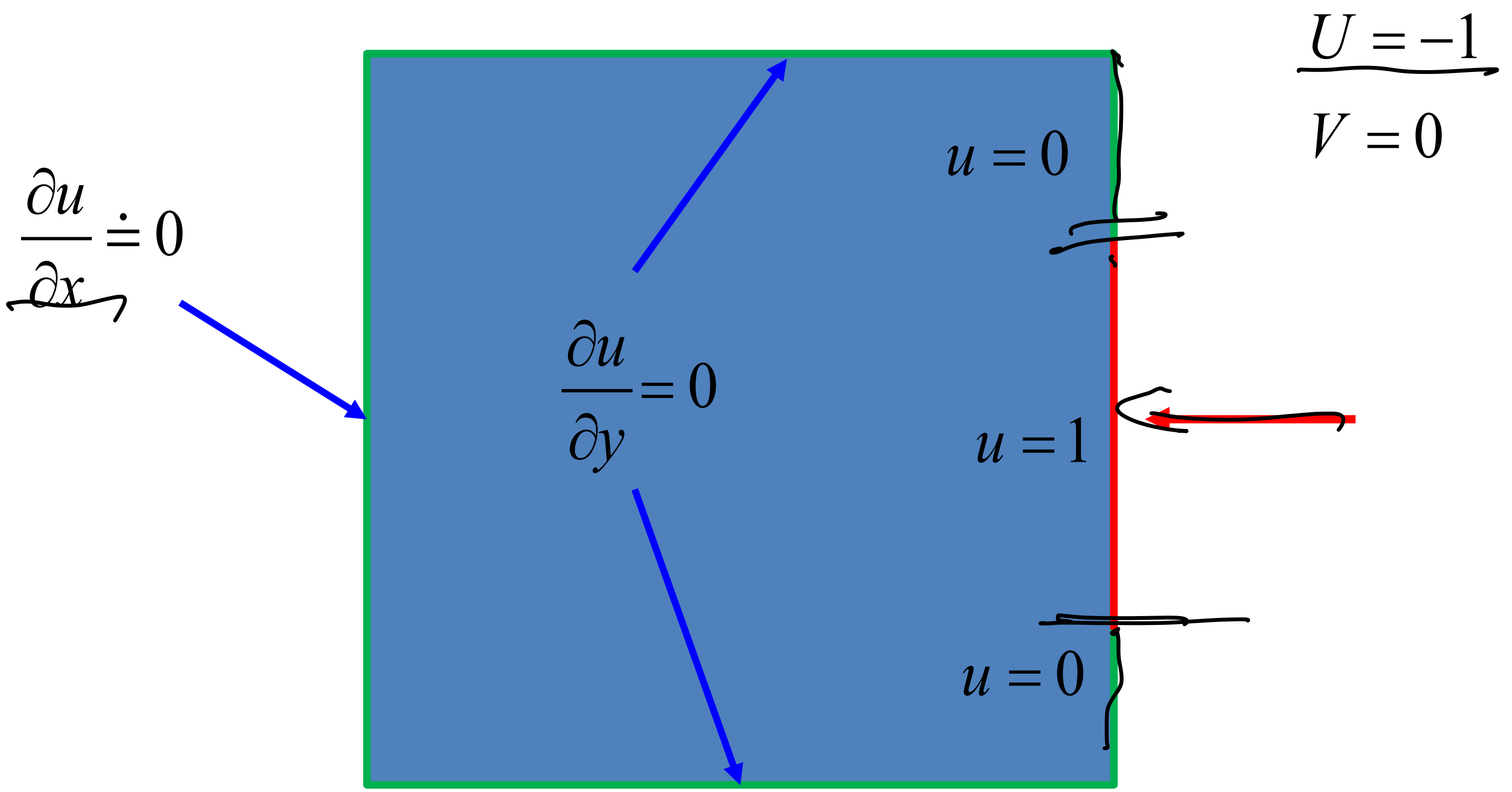
$$u_{i,j}^{n+1} = u_{i,j}^{n} - \frac{U\Delta t}{2h}\left[u_{i+1,j}^{n} - u_{i-1,j}^{n}\right]$$

$$- \frac{V\Delta t}{2h}\left[u_{i,j+1}^{n} - u_{i,j-1}^{n}\right]$$

$$+ \frac{\nu\Delta t}{h^2}\left[\cdots\cdots\right]$$

Example

$$\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}+V\frac{\partial u}{\partial y}=\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$



Checkpoint 2

(Top)

$$\frac{\partial T}{\partial y} = 10$$

→ Neumann (Top)

→ Dirichlets Left, Right, Bottom

$T = 50$ (Left)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$T = 0$ (Right)

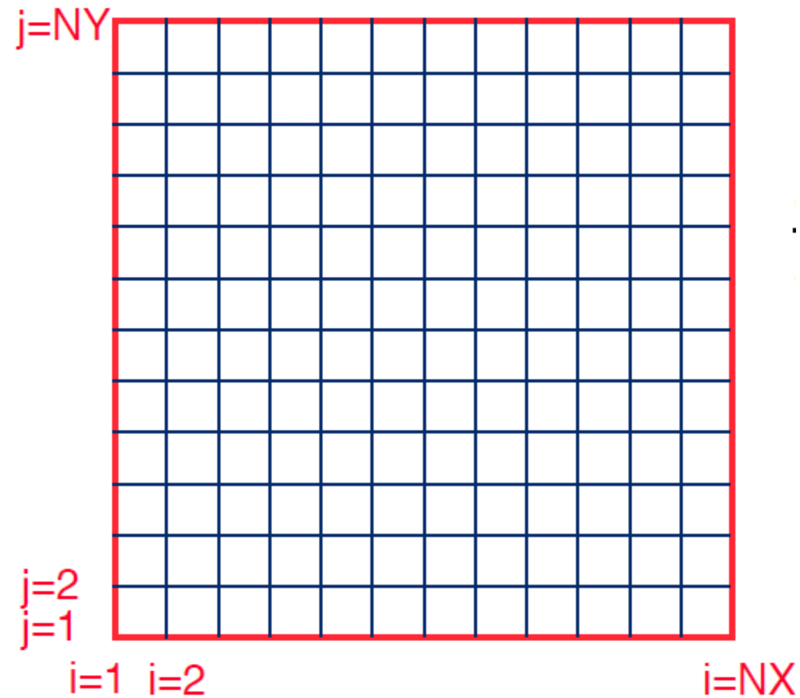
$T = 50$ (Bottom)

Boundary Conditions

$u_{i,j} = u(\mathrm{x}, y)$
stored at each grid point

- When the solution u is given, we simply specify (*Dirichlet condition*)

- Where the normal derivative (*Neumann condition*) is specified, we approximate the value at the boundary by one-sided differences

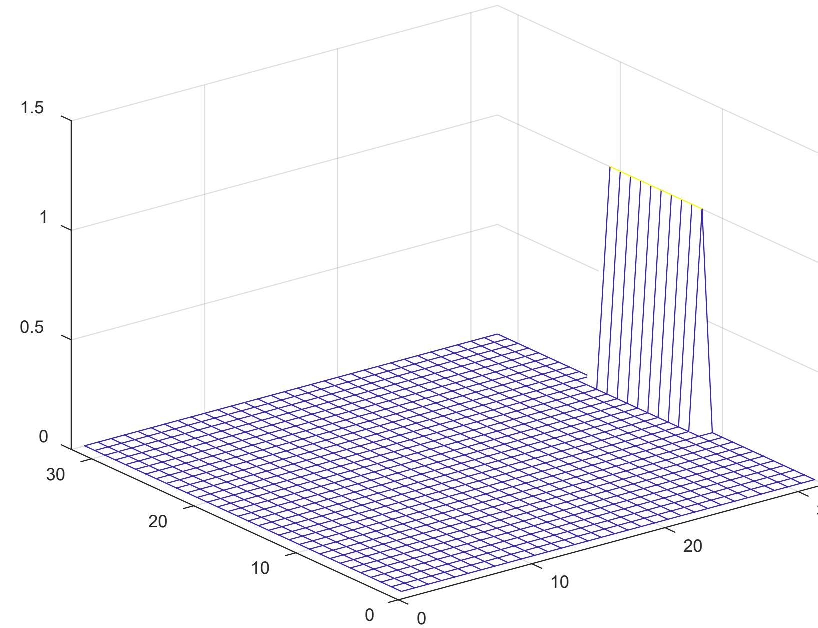


At $i=1$ boundary, $\frac{\partial u}{\partial y} = 0$

By using $\frac{\partial u}{\partial y} = \frac{u^n_{i,2} - u^n_{i,1}}{\Delta y} = 0$

$\Rightarrow \boxed{u^n_{i,2} = u_{i,1}}$

```
% MECH 479 - CFD
% EX3: Two-dimensional unsteady diffusion by the FTCS scheme
%-------------------------------------------------------------
n=32;
m=32;
nstep=120;
D=0.025;
length=2.0;
h=length/(n-1);
dt=1.0*0.125*h*h/D;
u=zeros(n,m);
uo=zeros(n,m);
time=0.0;
U=-0.0; V=-1.0; u(12:21,n)=1.0;
for l=1:nstep,l,time
hold off;mesh(u); axis([0 n 0 m 0 1.5]);pause;
uo=u;
for i=2:n-1, for j=2:m-1
u(i,j)=uo(i,j)-(0.5*dt*U/h)*(uo(i+1,j)-uo(i-1,j))-...
(0.5*dt*V/h)*(uo(i,j+1)-uo(i,j-1))+...
(D*dt/h^2)*(uo(i+1,j)+ uo(i,j+1)+uo(i-1,j)+uo(i,j -1)-4*uo(i,j));
end,end
for i=1:n,
    u(i,1)=u(i,2);
end
for j=1:m,
    u(1,j)=u(2,j);
    u(m,j)=u(m-1,j);
end
time=time+dt;
end
```



Unsteady evolution of the solution

2D Steady Boundary Value Problem

- Consider Steady State Poisson Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = S(x,y)$$

$$* \quad \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} = S_{i,j}$$

Solve $u_{i,j}$:

$$\rightarrow u_{i,j} = \frac{1}{4}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - h^2 S_{i,j}\right]$$

Iterative Solution Procedure

Solve $u_{i,j}$ & use right hand side

All previous values α, new values will be represented @ $\alpha+1$

$$u_{i,j}^{\alpha+1} = \frac{1}{4}\left[u_{i+1,j}^{\alpha} + u_{i-1,j}^{\alpha} + u_{i,j+1}^{\alpha} + \dots \right]$$

$$R_{i,j} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}}{h^2} - S_{i,j}$$

Jacobi Iteration

- The iteration must be carried out until the solution is sufficiently accurate. To measure the error, define the residual:

$$R_{i,j} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}}{h^2} - S_{i,j}$$

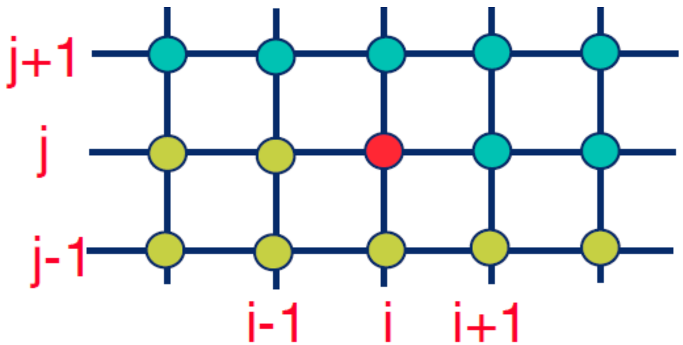
- At steady-state the residual should be zero.
  - The pointwise residual or average absolute residual can be used, depending on the problem.

Gauss Seidel Iteration

- Jacobi iteration is generally robust but many iterations are required to reach an accurate solution
  - ► Need a way to accelerate the convergence

- Using Gauss-Seidel, the Jacobi iteration can be improved somewhat by using new values as soon as they become available.

```
for j=1:m
for i=1:n
    iterate
end
end
```



$$u_{i,j}^{\alpha+1} = \frac{1}{4}\left(u_{i+1,j}^{\alpha} + u_{i-1,j}^{\alpha+1} + u_{i,j+1}^{\alpha} + u_{i,j-1}^{\alpha+1} - h^2 S_{i,j}\right)$$

- Gauss-Seidel iteration can be further improved by SOR treatment

Successive Over Relaxation

- Gauss-Seidel iteration can be further improved by SOR treatment

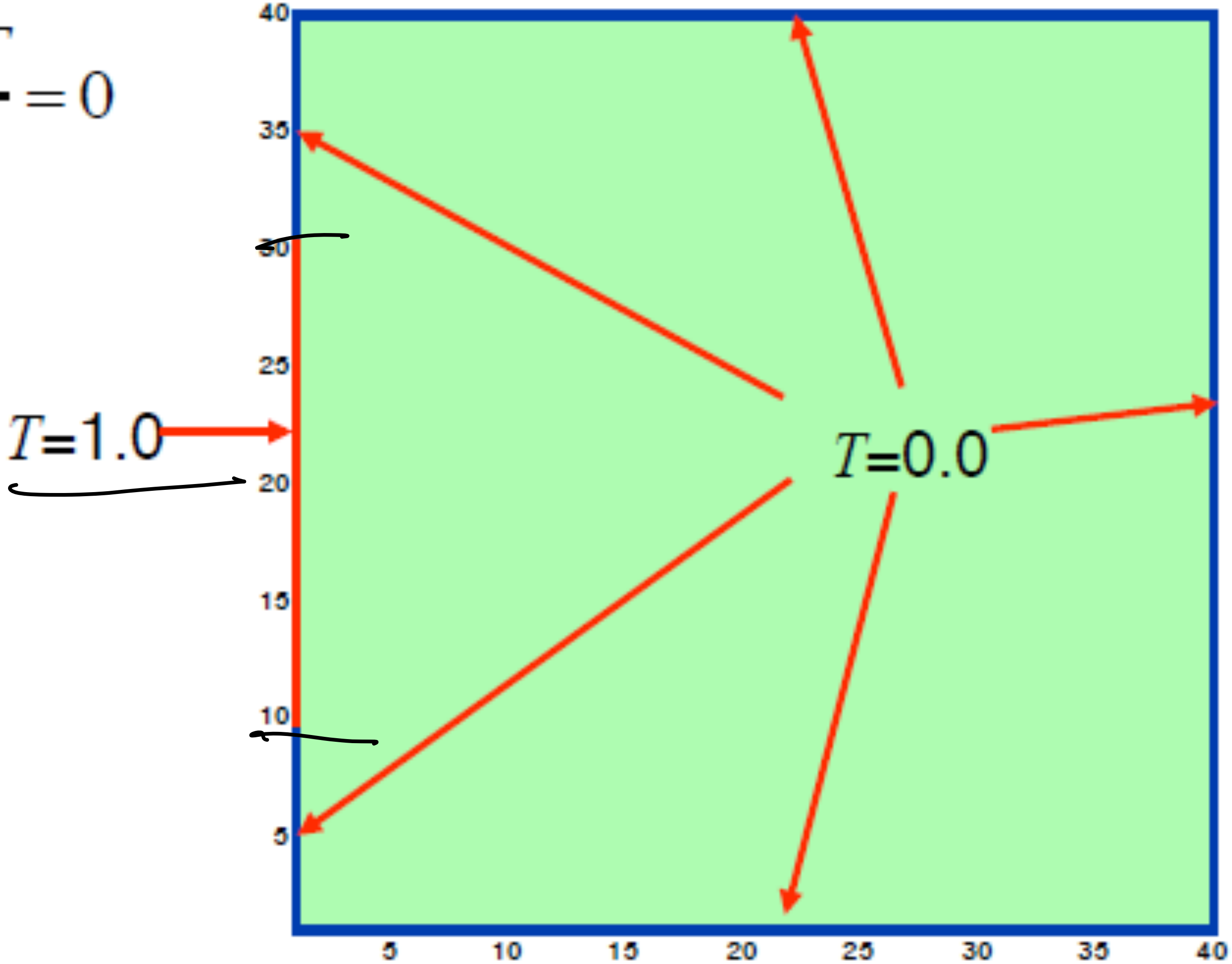
$$u_{i,j}^{\alpha+1} = \frac{\beta}{4}\left(\underline{u_{i+1,j}^{\alpha} + u_{i-1,j}^{\alpha+1} + u_{i,j+1}^{\alpha} + u_{i,j-1}^{\alpha+1} - h^2 S_{i,j}}\right)$$
$$+\left(1-\beta\right)u_{i,j}^{\alpha}$$

where $1 < \beta < 2$. In general, $\underline{\beta = 1.5}$ is a good starting value.

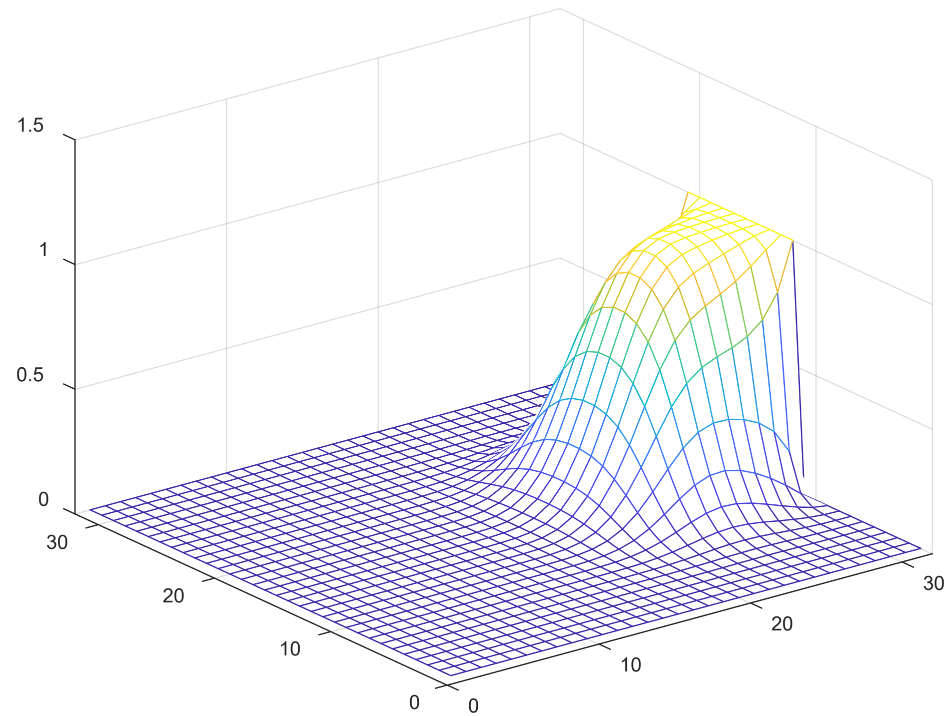
- The SOR iteration is very simple to program, just as the Gauss-Seidler iteration.

Example

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$



```
% MECH 479 - CFD
% EX4: Two-dimensional steady heat problem by SOR
%-------------------------------------------------------------
-
n=40;
m=40;
iterations=5000;
length=2.0;
h=length/(n-1);
T=zeros(n,m);
bb=1.7;
T(10:n-10,1)=1.0;
for l=1:iterations,
for i=2:n-1, for j=2:m-1
T(i,j)=bb*0.25*(T(i+1,j)+...
T(i,j+1)+T(i-1,j)+T(i,j-1))+(1.0-bb)*T(i,j);
end,end
% find residual
res=0;
for i=2:n-1,
    for j=2:m-1
        res=res+abs(T(i+1,j)+...
        T(i,j+1)+T(i-1,j)+T(i,j-1)-4*T(i,j))/h^2;
    end
end
l,res/((m-2)*(n-2)) % Print iteration and residual
if (res/((m-2)*(n-2)) < 0.001)
    break
end
end;
contour(T);
```



Summary

- FDEs for multi-dimensional advection-diffusion are similar to 1D problem

- Iterative methods for boundary value problems. Elementary approaches to steady state problems
  - Jacobi iteration
  - Gauss-Seidel iteration
  - Successive Over-Relaxation

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