Problem Set 2

MECH 479/587 - Computational Fluid Dynamics
Winter Term 1

Due: October 11, 2022

1 Discretizing the First Order Derivative (15 marks)

Given sufficient continuity of a function and its derivatives, Taylor's series allows for constructing derivatives of any order. Furthermore, by considering sufficient information about the function u, a derivative of k^{th} order can be constructed to an arbitrary order of accuracy. Given the stencil $\{i-2, i-1, i, i+1\}$, and the representation of the first order derivative:

$$\left.\frac{\partial u}{\partial x}\right|_i = \left(\frac{\alpha u_{i+1} + \beta u_i + \gamma u_{i-1} + \delta u_{i-2}}{\Delta x}\right) + O(\Delta x^p), \tag{1}$$

construct the family of second-order schemes and the corresponding truncation errors obtained in terms of the parameter β. Given the construction, answer the followings:

1. What will be the ordered set or parameters $(\alpha, \beta, \gamma, \delta)$ corresponding to the second-order central difference scheme?
2. What will be a second-order accurate scheme by setting $\alpha = 0$ in eq. (1)? Please provide the ordered set (β, γ, δ).
3. For the third-order accurate scheme on the given stencil, what will be the corresponding ordered set $(\alpha, \beta, \gamma, \delta)$?

2 Alternative way to Discretize the First Order Derivative (15 marks)

High gradients in momentum or density can be present in fluid flow problems. There is a need to capture these gradients with a high level of accuracy, either owing to the nature of the flow physics or due to computational resource limitations. Compact high-order finite difference schemes are generally preferred in such situations, where the information about the derivatives at two or more computational points is simultaneously unknown. A general form a compact scheme for the first derivative is given by:

$$\alpha \left.\frac{\partial u}{\partial x}\right|_{i-1} + \left.\frac{\partial u}{\partial x}\right|_i + \alpha \left.\frac{\partial u}{\partial x}\right|_{i+1} = \beta\left(\frac{u_{i+1} - u_{i-1}}{2\Delta x}\right) + \gamma\left(\frac{u_{i+2} - u_{i-2}}{4\Delta x}\right) + O(\Delta x^p). \tag{1}$$

Observe that by setting $(\alpha, \beta, \gamma) = (0, 1, 0)$, we recover the second order central difference scheme for the first order derivative. Given this construction, answer the followings:

1. Let $\alpha = 0$. Obtain (β, γ) such that a fourth-order approximation to the first-order derivative is obtained. Give the truncation error obtained.

2. Let $\gamma = 0$. Obtain (α, β) such that a fourth-order approximation to the first-order derivative is obtained. Give the truncation error obtained.

3 Assessment of Stability Limit (20 marks)

Consider the viscous Burgers' equation (a prototypical equation in 1-D for incompressible Navier-Stokes equations) in a periodic domain of length 1:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

$$u(x,0) = 1 + \sin(2\pi x), \tag{2}$$

$$u(1,t) = u(0,t), \tag{3}$$

where $\nu = 0.01$. Consider two different discrete domains with $N = 20$ and $N = 40$ points respectively.

1. Using a forward discretization (Euler explicit) for the time derivatives, and central discretization for the first and second order spatial derivatives, evaluate the solution at $t = 1.0$ using $\Delta t = 5 \times 10^{-3}$.

2. For each of the grids, conduct numerical experiments to determine the maximum value of Δt for which the solution remains stable. Present your results in Tabular form.

Note:

- **A code has been provided as an example for problem 3.**
- **All assignments should be submitted through Canvas.**

$$\left.\frac{\partial u}{\partial x}\right|_i = \left(\frac{\alpha u_{i+1} + \beta u_i + \gamma u_{i-1} + \delta u_{i-2}}{\Delta x}\right) + O(\Delta x^p)$$

$$u_{i+1} = u_i + \Delta x (u_x)_i + \frac{\Delta x^2}{2!}(u_{xx})_i + \frac{\Delta x^3}{3!}(u_{xxx})_i + \dots$$

$$u_{i-1} = u_i - \Delta x (u_x)_i + \frac{\Delta x^2}{2!}(u_{xx})_i - \frac{\Delta x^3}{3!}(u_{xxx})_i + \dots$$

1.1) To isolate $(u_x)_i$, we can perform the following calculation

$$u_{i+1} - u_{i-1} = \left(\cancel{u_i} + \Delta x (u_x)_i + \cancel{\frac{\Delta x^2}{2!}}(u_{xx})_i + \frac{\Delta x^3}{3!}(u_{xxx})_i + \dots\right)$$
$$- \left(\cancel{u_i} - \Delta x (u_x)_i + \cancel{\frac{\Delta x^2}{2!}}(u_{xx})_i - \frac{\Delta x^3}{3!}(u_{xxx})_i + \dots\right)$$

$$u_{i+1} - u_{i-1} = 2\Delta x (u_x)_i + \frac{2\Delta x^3}{3!}(u_{xxx})_i + \dots$$

$$u_{i+1} - u_{i-1} - \frac{2\Delta x^3}{3!}(u_{xxx})_i = 2\Delta x (u_x)_i$$

$$(u_x)_i = \frac{u_{i+1} - u_{i-1}}{2\Delta x} - \frac{\cancel{2}\Delta x^{\cancel{3}}}{(\cancel{2}\cancel{\Delta x})\cdot 3!}(u_{xxx})_i = \frac{u_{i+1} - u_{i-1}}{2\Delta x} - O(\Delta x^2)$$

Compare to $\left.\frac{\partial u}{\partial x}\right|_i = \left(\frac{\alpha u_{i+1} + \beta u_i + \gamma u_{i-1} + \delta u_{i-2}}{\Delta x}\right) + O(\Delta x^p)$

$$\alpha = \frac{1}{2},\ \beta = 0,\ \gamma = -\frac{1}{2},\ \delta = 0$$

1.2) find Taylor Series for u_{i-2}

$$u_{i-2} = u_i - 2\Delta x (u_x)_i + \frac{4\Delta x^2}{2!}(u_{xx})_i - \frac{8\Delta x^3}{3!}(u_{xxx})_i + \dots$$

$$\left.\frac{\partial u}{\partial x}\right|_i = \left(\frac{\alpha u_{i+1} + \beta u_i + \gamma u_{i-1} + \delta u_{i-2}}{\Delta x}\right) + O(\Delta x^{p})$$

$$\Delta x \left.\frac{\partial u}{\partial x}\right|_i = \alpha u_{i+1} + \beta u_i + \gamma u_{i-1} + \delta u_{i-2} + O(\Delta x^{p+1})$$

For 1.2, $\alpha = 0$

$$\Delta x \left.\frac{\partial u}{\partial x}\right|_i = \beta u_i + \gamma u_{i-1} + \delta u_{i-2} + O(\Delta x^{p+1})$$

$$\Delta x \left.\frac{\partial u}{\partial x}\right|_i = \beta u_i + \gamma\left(u_i - \Delta x (u_x)_i + \frac{\Delta x^2}{2!}(u_{xx})_i - \frac{\Delta x^3}{3!}(u_{xxx})_i + \dots\right) + \delta\left(u_i - 2\Delta x (u_x)_i + \frac{4\Delta x^2}{2!}(u_{xx})_i - \frac{8\Delta x^3}{3!}(u_{xxx})_i\right)$$

$$\Delta x (u_x)_i = \beta u_i + \gamma u_i - \gamma \Delta x (u_x)_i + \gamma\frac{\Delta x^2}{2!}(u_{xx})_i - \gamma\frac{\Delta x^3}{3!}(u_{xxx})_i + \dots + \delta u_i - 2\delta\Delta x(u_x)_i + + \frac{4}{2!}\delta\Delta x^2(u_{xx})_i - \frac{8}{3!}\delta \Delta x^3(u_{xxx})_i + \dots$$

$$\beta u_i + \gamma u_i + \delta u_i = 0 \quad [1]$$

$$-\gamma\Delta x(u_x)_i - 2\delta\Delta x(u_x)_i = \Delta x (u_x)_i \rightarrow$$

$$-\gamma\Delta x(u_x)_i - 2\delta\Delta x(u_{x i}) - \Delta x(u_x)_i = 0 \quad [2]$$

$$\gamma\frac{\Delta x^2}{2!}(u_{xx})_i + \frac{4}{2!}\delta\Delta x^2(u_{xx})_i = 0 \quad [3]$$

$$\beta u_i + \gamma u_i + \delta u_i = 0 \quad [1]$$

$$-\gamma \Delta x (u_x)_i - 2\delta \Delta x (u_x)_i = \Delta x (u_x)_i \rightarrow$$

$$-\gamma \Delta x (u_x)_i - 2\delta \Delta x (u_x)_i - \Delta x (u_x)_i = 0 \quad [2]$$

$$\gamma \frac{\Delta x^2}{2!} (u_{xx})_i + \frac{4}{2!}\delta \Delta x^2 (u_{xx})_i = 0 \quad [3]$$

$$\begin{array}{c} \begin{array}{ccc} \beta & \gamma & \delta \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -\Delta x (u_x)_i & -2\Delta x (u_x)_i & \Delta x (u_x)_i \\ 0 & \frac{\Delta x^2}{2!}(u_{xx})_i & \frac{4}{2!}\Delta x^2 (u_{xx})_i & 0 \end{array}\right] \end{array} \xrightarrow{-\frac{1}{\Delta x (u_x)_i} R_2 \to R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & \frac{\Delta x^2}{2!}(u_{xx})_i & \frac{4}{2!}\Delta x^2 (u_{xx})_i & 0 \end{array}\right] \xrightarrow{\frac{2!}{\Delta x^2 (u_{xx})_i} R_3 \to R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 4 & 0 \end{array}\right] \xrightarrow{-R_2 + R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 4 & 0 \end{array}\right] \xrightarrow{-R_2 + R_3 \to R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & -1 \end{array}\right] \xrightarrow{\frac{1}{2} R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 1/2 \end{array}\right] \xrightarrow{R_3 + R_1 \to R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 1/2 \end{array}\right] \xrightarrow{-2R_3 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1/2 \end{array}\right]$$

$$\beta = 3/2, \quad \gamma = -2, \quad \delta = 1/2$$

1.3) Same as 1.2) but $\alpha \neq 0$

$$U_{i+1} = U_i + \Delta x (U_x)_i + \frac{\Delta x^2}{2!}(U_{xx})_i + \frac{\Delta x^3}{3!}(U_{xxx})_i + \ldots$$

$$U_{i-1} = U_i - \Delta x (U_x)_i + \frac{\Delta x^2}{2!}(U_{xx})_i - \frac{\Delta x^3}{3!}(U_{xxx})_i + \ldots$$

$$U_{i-2} = U_i - 2\Delta x (U_x)_i + \frac{4\Delta x^2}{2!}(U_{xx})_i - \frac{8\Delta x^3}{3!}(U_{xxx})_i + \ldots$$

$$\Delta x (U_x)_i = \alpha U_{i+1} + \beta U_i + \gamma U_{i-1} + \delta U_{i-2} + O(\Delta x^{p+1})$$

$$\Delta x (U_x)_i = \alpha\left(U_i + \Delta x (U_x)_i + \frac{\Delta x^2}{2!}(U_{xx})_i + \frac{\Delta x^3}{3!}(U_{xxx})_i + \ldots\right)$$
$$+ \beta U_i + \gamma\left(U_i - \Delta x (U_x)_i + \frac{\Delta x^2}{2!}(U_{xx})_i - \frac{\Delta x}{3!}(U_{xxx})_i + \ldots\right)$$
$$+ \delta\left(U_i - 2\Delta x (U_x)_i + \frac{4\Delta x^2}{2!}(U_{xx})_i - \frac{8\Delta x^3}{3!}(U_{xxx})_i + \ldots\right)$$

$$\Delta x (U_x)_i = \alpha U_i + \alpha \Delta x (U_x)_i + \frac{\alpha \Delta x^2}{2!}(U_{xx})_i + \frac{\alpha \Delta x^3}{3!}(U_{xxx})_i + \ldots$$
$$+ \beta U_i + \gamma U_i - \gamma \Delta x (U_x)_i + \frac{\gamma \Delta x^2}{2!}(U_{xx})_i - \frac{\gamma \Delta x^3}{3!}(U_{xxx})_i + \ldots$$
$$+ \delta U_i - 2\delta \Delta x (U_x)_i + \frac{4\delta \Delta x^2}{2!}(U_{xx})_i - \frac{8\delta \Delta x^3}{3!}(U_{xxx})_i + \ldots$$

$$\alpha \cancel{U_i} + \beta \cancel{U_i} + \gamma \cancel{U_i} + \delta \cancel{U_i} = 0 \quad [1]$$

$$\alpha \cancel{\Delta} x (\cancel{U}_x)_i - \gamma \cancel{\Delta} x (\cancel{U}_x)_i - 2\delta \cancel{\Delta} x (\cancel{U}_x)_i = \cancel{\Delta} x (\cancel{U}_x)_i \quad [2]$$

$$\alpha \frac{\cancel{\Delta x^2}}{2!}(U_{\cancel{xx}})_i + \frac{\gamma \cancel{\Delta x^2}}{2!}(U_{\cancel{xx}})_i + \frac{4\delta \cancel{\Delta x^2}}{\cancel{2!}}(U_{\cancel{xx}})_i = 0 \quad [3]$$

$$\alpha \frac{\cancel{\Delta x^3}}{\cancel{3!}}(U_{\cancel{xxx}})_i - \gamma \frac{\cancel{\Delta x^3}}{\cancel{3!}}(U_{\cancel{xxx}})_i - \delta \frac{8\cancel{\Delta x^3}}{\cancel{3!}}(U_{\cancel{xxx}})_i = 0 \quad [4]$$

$$\alpha + \beta + \gamma + \delta = 0 \quad [1]$$

$$\alpha - \gamma - 2\delta = 1 \quad [2]$$

$$\alpha + \gamma + 4\delta = 0 \quad [3]$$

$$\alpha - \gamma - 8\delta = 0 \quad [4]$$

$$\begin{array}{cccc} \alpha & \beta & \gamma & \delta \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & -2 & 1 \\ 1 & 0 & 1 & 4 & 0 \\ 1 & 0 & -1 & -8 & 0 \end{array}\right] \quad \begin{array}{l} -R_1 + R_2 \to R_2 \\ -R_1 + R_3 \to R_3 \\ -R_1 + R_4 \to R_4 \end{array} \quad \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & -3 & 1 \\ 0 & -1 & 0 & 3 & 0 \\ 0 & -1 & -2 & -9 & 0 \end{array}\right] \quad \begin{array}{l} R_2 + R_1 \to R_1 \\ -R_2 + R_3 \to R_3 \\ -R_2 + R_4 \to R_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 1 \\ 0 & -1 & -2 & -3 & 1 \\ 0 & 0 & 2 & 6 & -1 \\ 0 & 0 & 0 & -6 & -1 \end{array}\right] \quad \begin{array}{l} R_3 + R_2 \to R_2 \\ \frac{1}{2} R_3 + R_1 \to R_1 \end{array} \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1/2 \\ 0 & -1 & 0 & 3 & 0 \\ 0 & 0 & 2 & 6 & -1 \\ 0 & 0 & 0 & -6 & -1 \end{array}\right] \quad \begin{array}{l} 1/6\, R_4 + R_1 \to R_1 \\ R_4 + R_3 \to R_3 \\ \frac{1}{2} R_4 + R_2 \to R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1/3 \\ 0 & -1 & 0 & 0 & -1/2 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & -6 & -1 \end{array}\right] \quad \begin{array}{l} -R_2 \to R_2 \\ 1/2\, R_3 \to R_3 \\ -1/6\, R_4 \to R_4 \end{array} \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1/6 \end{array}\right]$$

$$\alpha = 1/3, \quad \beta = 1/2, \quad \gamma = -1, \quad \delta = 1/6$$

2)

$$\alpha (u_x)_{i-1} + (u_x)_i + \alpha (u_x)_{i+1} = \beta \left(\frac{u_{i+1} - u_{i-1}}{2\Delta x} \right) + \gamma \left(\frac{u_{i+2} - u_{i-2}}{4\Delta x} \right) + O(\Delta x^p)$$

2.1) $\alpha = 0$

$$(u_x)_i = \beta \left(\frac{u_{i+1} - u_{i-1}}{2\Delta x} \right) + \gamma \left(\frac{u_{i+2} - u_{i-2}}{4\Delta x} \right) + O(\Delta x^p)$$

$$u_{i\pm 1} = u_i \pm \Delta x (u_x)_i + \frac{\Delta x^2}{2!}(u_{xx})_i \pm \frac{\Delta x^3}{3!}(u_{xxx})_i + \dots$$

$$u_{i\pm 2} = u_i \pm 2\Delta x (u_x)_i + \frac{4\Delta x^2}{2!}(u_{xx})_i \pm \frac{8\Delta x^3}{3!}(u_{xxx})_i + \dots$$

$$\Delta x (u_x)_i = \frac{\beta}{2}(u_{i+1} - u_{i-1}) + \frac{\gamma}{4}(u_{i+2} - u_{i-2}) + O(\Delta x^{p+1})$$

$$\Delta x (u_x)_i - \frac{\beta}{2}(u_{i+1} - u_{i-1}) - \frac{\gamma}{4}(u_{i+2} - u_{i-2}) = O(\Delta x^{p+1})$$

$$\Delta x (u_x)_i - \frac{\beta}{2} u_{i+1} + \frac{\beta}{2} u_{i-1} - \frac{\gamma}{4} u_{i+2} + \frac{\gamma}{4} u_{i-2} = O(\Delta x^{p+1})$$

	u_i	$(u_x)_i$	$(u_{xx})_i$	$(u_{xxx})_i$	$(u_4)_i$	$(u_5)_i$
$\Delta x (u_x)_i$	Δx	0	0	0	0	0
$-\frac{\beta}{2} u_{i+1}$	$-\frac{\beta}{2}$	$-\frac{\beta}{2}\Delta x$	$-\frac{\beta}{2} \cdot \frac{\Delta x^2}{2!}$	$-\frac{\beta}{2} \cdot \frac{\Delta x^3}{3!}$	$-\frac{\beta}{2} \cdot \frac{\Delta x^4}{4!}$	$-\frac{\beta}{2} \cdot \frac{\Delta x^5}{5}$
$\frac{\beta}{2} u_{i-1}$	$\frac{\beta}{2}$	$\frac{\beta}{2} \cdot -\Delta x$	$\frac{\beta}{2} \cdot \frac{\Delta x^2}{2!}$	$\frac{\beta}{2} \cdot \frac{-\Delta x^3}{3!}$	$\frac{\beta}{2} \cdot \frac{\Delta x^4}{4!}$	$\frac{\beta}{2} \cdot \frac{-\Delta x^5}{5!}$
$-\frac{\gamma}{4} u_{i+2}$	$-\frac{\gamma}{4}$	$-\frac{\gamma}{4} \cdot 2\Delta x$	$-\frac{\gamma}{4} \cdot \frac{4\Delta x^2}{2!}$	$-\frac{\gamma}{4} \cdot \frac{8\Delta x^3}{3!}$	$-\frac{\gamma}{4} \cdot \frac{16\Delta x^4}{4!}$	$-\frac{\gamma}{4} \cdot \frac{32\Delta x^5}{5!}$
$\frac{\gamma}{4} u_{i-2}$	$\frac{\gamma}{4}$	$\frac{\gamma}{4} \cdot -2\Delta x$	$\frac{\gamma}{4} \cdot \frac{4\Delta x^2}{2!}$	$\frac{\gamma}{4} \cdot \frac{-8\Delta x^3}{3!}$	$\frac{\gamma}{4} \cdot \frac{16\Delta x^4}{4!}$	$\frac{\gamma}{4} \cdot \frac{32\Delta x^5}{5!}$
LHS	RHS1	RHS2	RHS3	RHS4	RHS5	RHS6

	u_i	$(u_x)_i$	$(u_{xx})_i$	$(u_{xxx})_i$	$(u_4)_i$	$(u_5)_i$
$\Delta x (u_x)_i$	0	Δx	0	0	0	0
$-\frac{\beta}{2} u_{i+1}$	$-\frac{\beta}{2}$	$-\frac{\beta}{2} \Delta x$	$-\frac{\beta}{2} \cdot \frac{\Delta x^2}{2!}$	$-\frac{\beta}{2} \cdot \frac{\Delta x^3}{3!}$	$-\frac{\beta}{2} \cdot \frac{\Delta x^4}{4!}$	$-\frac{\beta}{2} \cdot \frac{\Delta x^5}{5}$
$\frac{\beta}{2} u_{i-1}$	$\frac{\beta}{2}$	$\frac{\beta}{2} \cdot -\Delta x$	$\frac{\beta}{2} \cdot \frac{\Delta x^2}{2!}$	$\frac{\beta}{2} \cdot \frac{-\Delta x^3}{3!}$	$\frac{\beta}{2} \cdot \frac{\Delta x^4}{4!}$	$\frac{\beta}{2} \cdot \frac{-\Delta x^5}{5!}$
$-\frac{\gamma}{4} u_{i+2}$	$-\frac{\gamma}{4}$	$-\frac{\gamma}{4} \cdot 2\Delta x$	$-\frac{\gamma}{4} \cdot \frac{4\Delta x^2}{2!}$	$-\frac{\gamma}{4} \cdot \frac{8\Delta x^3}{3!}$	$-\frac{\gamma}{4} \cdot \frac{16\Delta x^4}{4!}$	$-\frac{\gamma}{4} \cdot \frac{32\Delta x^5}{5!}$
$\frac{\gamma}{4} u_{i-2}$	$\frac{\gamma}{4}$	$\frac{\gamma}{4} \cdot -2\Delta x$	$\frac{\gamma}{4} \cdot \frac{4\Delta x^2}{2!}$	$\frac{\gamma}{4} \cdot \frac{-8\Delta x^3}{3!}$	$\frac{\gamma}{4} \cdot \frac{16\Delta x^4}{4!}$	$\frac{\gamma}{4} \cdot \frac{32\Delta x^5}{5!}$
LHS	RHS1	RHS2	RHS3	RHS4	RHS5	RHS6

RHS1: $-\frac{\beta}{2} + \frac{\beta}{2} - \frac{\gamma}{4} + \frac{\gamma}{4} = 0$ ✓

RHS2: $\cancel{\Delta x} - \frac{\beta}{2}\cancel{\Delta x} - \frac{\beta}{2}\cancel{\Delta x} - \frac{\gamma}{\cancel{4}} \cdot \cancel{2}\cancel{\Delta x} - \frac{\gamma}{\cancel{4}} \cdot \cancel{2}\cancel{\Delta x} = 0$

$1 - \frac{\beta}{2} - \frac{\beta}{2} - \frac{\gamma}{2} - \frac{\gamma}{2} = 0$

$1 - \beta - \gamma = 0$ [1]

RHS3: $-\frac{\beta}{2} \cdot \frac{\Delta x^2}{2!} + \frac{\beta}{2} \frac{\Delta x^2}{2!} - \frac{\gamma}{4} \cdot \frac{4\Delta x^2}{2!} + \frac{\gamma}{4} \cdot \frac{4\Delta x^2}{2!} = 0$ ✓

RHS4: $-\frac{\beta}{2} \cdot \frac{\cancel{\Delta x^3}}{\cancel{3!}} - \frac{\beta}{2} \cdot \frac{\cancel{\Delta x^3}}{\cancel{3!}} - \frac{\gamma}{\cancel{4}} \cdot \frac{\cancel{8}\cancel{\Delta x^3}}{\cancel{3!}} - \frac{\gamma}{\cancel{4}} \cdot \frac{8\cancel{\Delta x^3}}{\cancel{3!}} = 0$

$-\frac{\beta}{2} - \frac{\beta}{2} - 2\gamma - 2\gamma = 0 \rightarrow -\beta - 4\gamma = 0$ [2]

RHS5: $-\frac{\beta}{2} \frac{\Delta x^4}{4!} + \frac{\beta}{2} \frac{\Delta x^4}{4!} - \frac{\gamma}{4} \cdot \frac{16\Delta x^4}{4!} + \frac{\gamma}{4} \cdot \frac{16\Delta x^4}{4!} = 0$ ✓

RHS6: $-\frac{\beta}{2} \cdot \frac{\Delta x^5}{5!} - \frac{\beta}{2} \cdot \frac{\Delta x^5}{5!} - \frac{\gamma}{4} \cdot \frac{32\Delta x^5}{5!} - \frac{\gamma}{4} \cdot \frac{32\Delta x^5}{5!} =$

Solve for β & γ

$1-\beta-\gamma=0$ [1], $-\beta-4\gamma=0$ [2], $-\beta-16\gamma=0$ [3]

$$\begin{array}{cc} \beta & \gamma \\ \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & -4 & 0 \end{array}\right] \xrightarrow{R_1+R_2\to R_2} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -3 & 1 \end{array}\right] \xrightarrow{-\frac{1}{3}R_2\to R_2} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{3} \end{array}\right] \xrightarrow{-R_2+R_1\to R_1}$$

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{4}{3} \\ 0 & 1 & -\frac{1}{3} \end{array}\right] \longrightarrow \beta = 4/3,\ \gamma = -1/3$$

find truncation error

$$RHS\ 6:\ -\frac{\beta}{2}\cdot\frac{\Delta x^5}{5!}-\frac{\beta}{2}\cdot\frac{\Delta x^5}{5!}-\frac{\gamma}{4}\cdot\frac{32\Delta x^5}{5!}-\frac{\gamma}{4}\cdot\frac{32\Delta x^5}{5!}$$

Plus β & γ found below in

$$-\frac{4/3}{2}\cdot\frac{\Delta x^5}{5!}-\frac{4/3}{2}\cdot\frac{\Delta x^5}{5!}-\frac{(-1/3)}{4}\cdot\frac{32\Delta x^5}{5!}-\frac{(-1/3)}{4}\cdot\frac{32\Delta x^5}{5!}$$

$$-\frac{4}{6}\frac{\Delta x^5}{5!}-\frac{4}{6}\frac{\Delta x^2}{5!}+\frac{1}{12}\cdot\frac{32\Delta x^5}{5!}+\frac{1}{12}\cdot\frac{32\Delta x^5}{5!}$$

$$-\frac{4}{3}\frac{\Delta x^5}{5!}+\frac{8}{3}\frac{\Delta x^5}{5!}+\frac{8}{3}\frac{\Delta x^5}{5!}=-\frac{4}{3}\frac{\Delta x^5}{5!}+\frac{16}{3}\frac{\Delta x^5}{5!}$$

$$=\frac{12}{3}\frac{\Delta x^5}{5!}=4\frac{\Delta x^5}{5!}=\frac{\cancel{4}}{1\cdot2\cdot3\cdot\cancel{4}\cdot5}\Delta x^5=\frac{\Delta x^5}{30}(u_5)$$

2.2) $\gamma = 0 \rightarrow \alpha \frac{\partial u}{\partial x}\Big|_{i-1} + \frac{\partial u}{\partial x}\Big|_{i} + \alpha \frac{\partial u}{\partial x}\Big|_{i+1} = \beta\left(\frac{u_{i+1} - u_{i-1}}{2\Delta x}\right) + O(\Delta x^p)$

$$\alpha (u_x)_{i-1} + (u_x)_i + \alpha (u_x)_{i+1} = \beta\left(\frac{u_{i+1} - u_{i-1}}{2\Delta x}\right) + O(\Delta x^p)$$

$$\alpha (u_x)_{i-1} + (u_x)_i + \alpha (u_x)_{i+1} - \frac{\beta}{2\Delta x}(u_{i+1} - u_{i-1}) = O(\Delta x^p)$$

$$\alpha (u_x)_{i-1} + (u_x)_i + \alpha (u_x)_{i+1} - \frac{\beta}{2\Delta x} u_{i+1} + \frac{\beta}{2\Delta x} u_{i-1} = O(\Delta x^p)$$

Write Taylor Expansions

$$u_{i\pm1} = u_i \pm \Delta x (u_x)_i + \frac{\Delta x^2}{2!}(u_{xx})_i \pm \frac{\Delta x}{3!}(u_{xxx})_i + \ldots$$

$$(u_x)_{i\pm1} = (u_x)_i \pm \Delta x (u_{xx})_i + \frac{\Delta x^2}{2!}(u_{xxx})_i \pm \frac{\Delta x^3}{3!}(u_4)_i + \ldots$$

	u_i	$(u_x)_i$	$(u_{xx})_i$	$(u_{xxx})_i$	$(u_4)_i$	$(u_5)_i$
$(u_x)_i$	0	1	0	0	0	0
$\alpha(u_x)_{i-1}$	0	α	$-\alpha\Delta x$	$\alpha\frac{\Delta x^2}{2!}$	$-\frac{\alpha\Delta x^3}{3!}$	$\frac{\alpha\Delta x^4}{4!}$
$\alpha(u_x)_{i+1}$	0	α	$\alpha\Delta x$	$\alpha\frac{\Delta x^2}{2!}$	$\frac{\alpha\Delta x^3}{3!}$	$\frac{\alpha\Delta x^4}{4!}$
$\frac{-B}{2\Delta x}u_{i+1}$	$\frac{-B}{2\Delta x}$	$\frac{-B}{2}$	$\frac{-B}{2}\cdot\frac{\Delta x}{2!}$	$\frac{-B}{2}\cdot\frac{\Delta x^2}{3!}$	$\frac{-B}{2}\cdot\frac{\Delta x^3}{4!}$	$\frac{-B}{2}\cdot\frac{\Delta x^4}{5!}$
$\frac{B}{2\Delta x}u_{i-1}$	$\frac{B}{2\Delta x}$	$\frac{-B}{2}$	$\frac{B}{2}\frac{\Delta x}{2!}$	$\frac{-B}{2}\cdot\frac{\Delta x^2}{3!}$	$\frac{B}{2}\cdot\frac{\Delta x^3}{4!}$	$\frac{-B}{2}\cdot\frac{\Delta x^5}{5!}$
LHS	RHS1	RHS2	RHS3	RHS4	RHS5	RHS6

RHS 1: $\frac{-B}{2\Delta x} + \frac{B}{2\Delta x} = 0$ ✓

RHS 2: $1 + \alpha + \alpha - \frac{B}{2} - \frac{B}{2} = 0$ [1]

RHS 3: $-\alpha\Delta x + \alpha\Delta x - \frac{B}{2}\cdot\frac{\Delta x}{2!} + \frac{B}{2}\cdot\frac{\Delta x}{2!} = 0$ ✓

	u_i	$(u_x)_i$	$(u_{xx})_i$	$(u_{xxx})_i$	$(u_4)_i$	$(u_5)_i$
$(u_x)_i$	0	1	0	0	0	0
$\alpha(u_x)_{i-1}$	0	α	$-\alpha\Delta x$	$\alpha\frac{\Delta x^2}{2!}$	$-\frac{\alpha\Delta x^3}{3!}$	$\frac{\alpha\Delta x^4}{4!}$
$\alpha(u_x)_{i+1}$	0	α	$\alpha\Delta x$	$\alpha\frac{\Delta x^2}{2!}$	$\frac{\alpha\Delta x^3}{3!}$	$\frac{\alpha\Delta x^4}{4!}$
$-\frac{\beta}{2\Delta x}u_{i+1}$	$-\frac{\beta}{2\Delta x}$	$-\frac{\beta}{2}$	$-\frac{\beta}{2}\cdot\frac{\Delta x}{2!}$	$-\frac{\beta}{2}\cdot\frac{\Delta x^2}{3!}$	$-\frac{\beta}{2}\cdot\frac{\Delta x^3}{4!}$	$-\frac{\beta}{2}\cdot\frac{\Delta x^4}{5!}$
$\frac{\beta}{2\Delta x}u_{i-1}$	$\frac{\beta}{2\Delta x}$	$-\frac{\beta}{2}$	$\frac{\beta}{2}\frac{\Delta x}{2!}$	$-\frac{\beta}{2}\cdot\frac{\Delta x^2}{3!}$	$\frac{\beta}{2}\cdot\frac{\Delta x^3}{4!}$	$-\frac{\beta}{2}\cdot\frac{\Delta x^5}{5!}$
LHS	RHS1	RHS2	RHS3	RHS4	RHS5	RHS6

RHS 4: $\alpha\frac{\Delta x^2}{2!}+\alpha\frac{\Delta x^2}{2!}-\frac{\beta}{2}\frac{\Delta x^2}{3!}-\frac{\beta}{2}\frac{\Delta x^2}{3!}=0$ [2]

RHS 5: $-\alpha\frac{\Delta x^3}{3!}+\alpha\frac{\Delta x^3}{3!}-\frac{\beta}{2}\cdot\frac{\Delta x^3}{4!}+\frac{\beta}{2}\cdot\frac{\Delta x^3}{4!}=0$ ✓

RHS 6: $\alpha\frac{\Delta x^4}{4!}+\alpha\frac{\Delta x^4}{4!}-\frac{\beta}{2}\frac{\Delta x^4}{5!}-\frac{\beta}{2}\cdot\frac{\Delta x^4}{5!}$ [3]

$1+\alpha+\alpha-\frac{\beta}{2}-\frac{\beta}{2}=0$ [1]

$\alpha\frac{\Delta x^2}{2!}+\alpha\frac{\Delta x^2}{2!}-\frac{\beta}{2}\frac{\Delta x^2}{3!}-\frac{\beta}{2}\frac{\Delta x^2}{3!}=0$ [2]

$\alpha\frac{\Delta x^4}{4!}+\alpha\frac{\Delta x^4}{4!}-\frac{\beta}{2}\frac{\Delta x^4}{5!}-\frac{\beta}{2}\cdot\frac{\Delta x^4}{5!}$ [3]

Use [1] & [2] to solve for α & β

[1] $1+\alpha+\alpha-\frac{\beta}{2}-\frac{\beta}{2}=0\rightarrow 2\alpha-\beta=-1$

[2] $\alpha\frac{\cancel{\Delta x^2}}{2!}+\alpha\frac{\cancel{\Delta x^2}}{2!}-\frac{\beta}{2}\frac{\cancel{\Delta x^2}}{3!}-\frac{\beta}{2}\frac{\cancel{\Delta x^2}}{3!}=0\rightarrow\frac{\alpha}{2}+\frac{\alpha}{2}-\frac{\beta}{12}-\frac{\beta}{12}=0$

$$\frac{\alpha}{2} + \frac{\alpha}{2} - \frac{\beta}{12} - \frac{\beta}{12} = \alpha - \frac{\beta}{6} = 0$$

$$\begin{array}{cc} \alpha & \beta \end{array}$$

$$\left[\begin{array}{cc|c} 2 & -1 & -1 \\ 1 & -\frac{1}{6} & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -\frac{1}{6} & 0 \end{array}\right] \xrightarrow{-R_1 + R_2 \to R_2}$$

$$\left[\begin{array}{cc|c} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1/3 & 1/2 \end{array}\right] \xrightarrow{3R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -1/2 & -1/2 \\ 0 & 1 & 3/2 \end{array}\right] \xrightarrow{\frac{1}{2}R_2 + R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & 1/4 \\ 0 & 1 & 3/2 \end{array}\right]$$

$$\alpha = 1/4, \ \beta = 3/2$$

Use 3 to find truncation error

$$\alpha \frac{\Delta x^4}{4!} + \alpha \frac{\Delta x^4}{4!} - \frac{\beta}{2} \frac{\Delta x^4}{5!} - \frac{\beta}{2} \cdot \frac{\Delta x^4}{5!}$$

$$= \frac{2\alpha}{4!} \Delta x^4 - \frac{\beta}{5!} \Delta x^4 = \Delta x^4 \left(\frac{2\alpha}{4!} - \frac{\beta}{5!} \right)$$

$$= \Delta x^4 \left(\frac{2}{4!} \cdot \frac{1}{4} - \frac{1}{5!} \cdot \frac{3}{2} \right) = \Delta x^4 \left(\frac{2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 4} - \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{3}{2} \right)$$

$$= \Delta x^4 \left(\frac{1}{48} - \frac{1}{80} \right) = \frac{\Delta x^4}{120} (u_5);$$

$$\frac{1}{48} - \frac{1}{80} = \frac{80}{3840} - \frac{48}{3840} = \frac{32}{3840} = \frac{1}{120}$$

3) $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$

$u(x,0) = 1 + \sin(2\pi x)$

$u(1,t) = u(0,t)$

$\nu = 0.01$, consider $N=20$ & $N=40$

For time use forward Euler explicit & central differencing for 1st & 2nd order derivatives

Take Taylor Series using 3 point stencil

$u_j = u(x_j)$

$$u_{j+1} = u(x_j + \Delta x) = u_j + \Delta x\, u_j' + \frac{\Delta x^2}{2} u_j'' + \frac{\Delta x^3}{3!} u_j''' + \dots$$

$$u_{j-1} = u(x_j - \Delta x) = u_j - \Delta x\, u_j' + \frac{\Delta x^2}{2} u_j'' - \frac{\Delta x^3}{3!} u_j''' + \dots$$

find u_j'

$$u(x_j + \Delta x) - u(x_j - \Delta x) = \left(\cancel{u_j} + \Delta x\, u_j' + \cancel{\frac{\Delta x^2}{2}} u_j'' + \frac{\Delta x^3}{3!} u_j''' + \dots\right) - \left(\cancel{u_j} - \Delta x\, u_j' + \cancel{\frac{\Delta x^2}{2}} u_j'' - \frac{\Delta x^3}{3!} u_j''' + \dots\right)$$

$$u(x_j + \Delta x) - u(x_j - \Delta x) = 2\Delta x\, u_j' + \frac{\Delta x^3}{3} u_j'''$$

$$u_j' = \frac{u(x_j + \Delta x) - u(x_j - \Delta x)}{2\Delta x} + O(\Delta x^2)$$

find u_j''

$$u(x_j + \Delta x) + u(x_j - \Delta x) = \left(u_j + \Delta x\, u_j' + \frac{\Delta x^2}{2} u_j'' + \frac{\Delta x^3}{3!} u_j''' + \dots\right) + \left(u_j - \Delta x\, u_j' + \frac{\Delta x^2}{2} u_j'' - \frac{\Delta x^3}{3!} u_j''' + \dots\right)$$

$$= 2u_j + \Delta x^2 u_j''$$

$$u'' = \frac{u(x_j + \Delta x) - 2u_j + u(x_j - \Delta x)}{\Delta x^2} + O(\Delta x^2)$$

find time derivative

forward Euler $\rightarrow \dfrac{u_j^{n+1} - u_j^n}{\Delta t}$

Time derivative

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + O(\Delta t)$$

Spatial Derivatives

$$u_j^{n'} = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + O(\Delta x^2)$$

$$u_j^{n''} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + O(\Delta x^2$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + u_j^n\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x)}\right) = \nu\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}\right)$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \nu\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}\right) - u_j^n\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}\right)$$

$$u_j^{n+1} = \Delta t\left[\nu\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}\right) - u_j^n\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}\right)\right] + u_j^n$$

$$u_j^{n+1} = \Delta t\nu\left(\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}\right) - \Delta t u_j^n\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}\right) + u_j^n$$

$$u_j^{n+1} = \frac{\Delta t \nu}{\Delta x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) - \frac{\Delta t}{2\Delta x}u_j^n\left(u_{j+1}^n - u_{j-1}^n\right) + u_j^n$$