MECH 479/587

Computational Fluid Dynamics

Module 4 (a): Stability Analysis

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Module #4: Stability Analysis

Today's Plan

0. Recap

1. Modified FDE
   (Equivalent Eq^n)

2. 2D Von Neumann Analysis
   → Basic Steps
   → Example

3. Physical intepretation

2D incompressible NS Eqns:

Continuity: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

X-Mom $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Y-Mom $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Independent variables : (x, y, t)

Dependent variables : (u, v, p)

1D Convection: $$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

1D Diffusion: $$\boxed{\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial x^2}}$$

1D Convection-Diffusion: $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$
(Burger Eqn)

Energy / Heat Eq^n:

$$\rho C_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = K \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

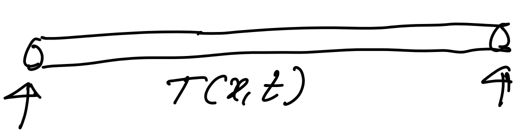
$$\Rightarrow \underbrace{\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}}_{\text{2D Convection}} = \alpha \underbrace{\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)}_{\text{2D Diffusion}}$$

1D Diffusion: $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$

2D Convection: $\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = 0$

1D Convection - Diffusion:

$$\underbrace{\frac{\partial T}{\partial t}}_{\text{Unsteady}} + \underbrace{u \frac{\partial T}{\partial x}}_{\text{Convection}} = \underbrace{K \frac{\partial^2 T}{\partial x^2}}_{\text{Diffusion}}$$



Review: Analysis of Numerical Scheme

- General steps

  (a) Write down the finite difference equation

  (b) Write down the truncation (modified) equation

  (c) Find the accuracy of the scheme

  (d) Use the von Neumann's method to derive an equation for the amplification factor

Consider 1D diffusion equation

$$\boxed{\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}}$$

(a) Approximate both terms by centered differences

$\boxed{\text{Modified Eq}^n}$ (Equivalent Eq^n)

$$PDE \neq FDE$$

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} \cong \left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right)_j^n$$

$$\Delta t \to 0 , \quad \Delta x \to 0$$

$$\underbrace{FDE}_{\Delta t \to 0,\ \Delta x \to 0} \longrightarrow PDE$$

(Consistency)

Using the FDE, we are effectively solving an equation that is somewhat different than the original PDE!

Neumann Stability Condition :

Solution $t^n \xrightarrow{u^n}$ Fourier Analyzer $\xrightarrow{u^{n+1}} t^{n+1}$

$$|G| = \left|\frac{u^{n+1}}{u^n}\right| \leq 1$$

Example : 1D Diffusion :

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

Three choices :

Forward Euler : Conditional Stability

$$\Delta t \leq \frac{1}{2} \frac{\Delta x^2}{\alpha}$$

Backward Euler : Unconditional Stable

Central Difference : $\frac{\partial u}{\partial t} \cong \frac{u_j^{n+1} - u_j^{n-1}}{\Delta t}$

Analysis of Numerical Scheme (1)

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

(b) Write down modified equation

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = \frac{\alpha}{\Delta x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

$$(1)\ u_j^{n+1} = u_j^n + \frac{\partial u_j^n}{\partial t}\Delta t + \frac{\partial^2 u_j^n}{\partial t^2}\frac{\Delta t^2}{2} + \frac{\partial^3 u_j^n}{\partial t^3}\frac{\Delta t^3}{6} + \cdots$$

$$(2)\ u_j^{n-1} = u_j^n - \frac{\partial u_j^n}{\partial t}\Delta t + \frac{\partial^2 u_j^n}{\partial t^2}\frac{\Delta t^2}{2} - \frac{\partial^3 u_j^n}{\partial t^3}\frac{\Delta t^3}{6} + \cdots$$

$$(3)\ u_{j+1}^n = u_j^n + \frac{\partial u_j^n}{\partial x}h + \frac{\partial^2 u_j^n}{\partial x^2}\frac{h^2}{2} + \frac{\partial^3 u_j^n}{\partial x^3}\frac{h^3}{6} + \cdots$$

$$(4)\ u_{j-1}^n = u_j^n - \frac{\partial u_j^n}{\partial x}h + \frac{\partial^2 u_j^n}{\partial x^2}\frac{h^2}{2} - \frac{\partial^3 u_j^n}{\partial x^3}\frac{h^3}{6} + \cdots$$

$$\frac{(1)-(2)}{2\Delta t} = \alpha\frac{(3) - 2u_j^n + (4)}{h^2}$$

Yielding

$$\frac{\partial u}{\partial t} + \underbrace{\frac{\partial^3 u}{\partial t^3}\frac{\Delta t^2}{6}} = \alpha\frac{\partial^2 u}{\partial x^2} + \underbrace{\alpha\frac{\partial^4 u}{\partial x^4}\frac{h^2}{12}} + \cdots$$

Rearrange

Modified equation:

$$\boxed{\frac{\partial u}{\partial t} - \alpha\frac{\partial^2 u}{\partial x^2} = \alpha\frac{\partial^4 u}{\partial x^4}\frac{h^{2}}{12} - \frac{\partial^3 u}{\partial t^3}\frac{\Delta t^{2}}{6}}$$

Analysis of Numerical Scheme (2)

(c) Find the accuracy of the scheme

$$O(\Delta t^2, h^2)$$

Second order in space & time!

(d) Use von Neumann (Fourier series) for the stability analysis

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = \frac{\alpha}{\Delta x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n \right)$$

$$u_j^{n+1} - u_j^{n-1} = s\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n \right) \qquad s = \alpha \frac{2\Delta t}{\Delta x^2}$$

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = \frac{\alpha}{\Delta x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

$$s = \alpha \frac{2\Delta t}{\Delta x^2}$$

(*)

$$\Rightarrow u_j^{n+1} - u_j^{n-1} = s\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

Substituting: $$u_j^n = \underbrace{v^n}_{\text{amplitude}} \underbrace{e^{ikx_j}}_{\text{spatial error}}$$

$$u_{j\pm 1}^n = v^n e^{ik(x_j \pm h)}$$

$$u_j^{n\pm 1} = v^{n\pm 1} e^{ikx_j}$$

$\Rightarrow$ From (*)

$$v^{n+1} e^{ikx_j} - v^{n-1} e^{ikx_j} = s\left(v^n e^{ik(x_j+h)} - v^n e^{ikx_j} + v^n e^{ik(x_j-h)}\right)$$

Goal: Amplification factor G

$$G = \frac{v^{n+1}}{v^n}, \quad G = \frac{v^n}{v^{n-1}}$$

Some algebraic arrangement

$$v^{n+1} - v^{n-1} = s v^n \left(e^{ikh} - 2 + e^{-ikh} \right)$$

$$\Rightarrow \frac{v^{n+1}}{v^n} - \frac{v^{n-1}}{v^n} = s \left(e^{i\theta} - 2 + e^{-i\theta} \right)$$

$$\Rightarrow G - \frac{1}{G} = s(2\cos\theta - 2)$$

$$= -2s \times 2\sin^2\frac{\theta}{2}$$

$kh = \theta$

$$\boxed{\begin{array}{l} e^{i\theta} + e^{-i\theta} = 2\cos\theta \\ 2\sin^2\theta = 1 - \cos 2\theta \end{array}}$$

$$\Rightarrow G - \frac{1}{G} = \underbrace{-4s\sin^2\frac{\theta}{2}}_{B}$$

$$G - \frac{1}{G} = -B$$

$$\Rightarrow G^2 + BG - 1 = 0$$

$$G = -\frac{B}{2} \pm \sqrt{\left(\frac{B}{2}\right)^2 + 1}$$

Consider worst case scenario

$$|G| = \left| \frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + 1} \right|$$

$$> 1$$

Hence this discretization or numerical scheme (Central in time & Central in space) is unconditionally unstable!

Von Neumann Stability Analysis

Substituting $u_j^n = v^n\, e^{ik\cdot x_j};\qquad u_{j\pm 1}^n = v^n\, e^{ik(x_j \pm \Delta x)};\qquad u_j^{n\pm 1} = v^{n\pm 1}\, e^{ik\cdot x_j}$

$$v^{n+1}\, e^{ik\, x_j} - v^{n-1}\, e^{ik\, x_j} = s\left(v^n\, e^{ik(x_j+\Delta x)} - 2v^n\, e^{ik\, x_j} + v^n\, e^{ik(x_j-\Delta x)} \right)$$

Algebraic manipulation

$$G - \frac{1}{G} = -s\ \left(2\cos\theta - 2\right) = -2s * 2\sin^2\frac{\theta}{2}$$

$$\boxed{\begin{aligned} &e^{i\theta} + e^{-i\theta} = 2\cos\theta \\ &2\sin^2\theta = 1 - \cos 2\theta \end{aligned}}$$

$$\Rightarrow G^2 - 1 = -BG \qquad \text{where } B = 4s\sin^2\frac{\theta}{2}$$

$$\Rightarrow G^2 + BG - 1 =$$

$$G = -\frac{B}{2} \pm \sqrt{\left(\frac{B}{2}\right)^2 + 1} \qquad \Rightarrow G > 1$$

Hence this scheme is **unconditionally** unstable.

Summary on von Neumann Method…

- Application to the combination of time and spatial discretization for linear equations.
  - Periodic boundary condition (other Boundary conditions cannot be included)
- Considers a component of the Fourier series expansion for error.
  - For stability make sure the amplitude does not amplify.
- For explicit scheme – conditionally stable;

  Courant number: $\boxed{c\dfrac{\Delta t}{\Delta x} \le 1}$

  $$\underbrace{c\frac{\Delta t}{\Delta x}}_{\text{Courant number}} \le 1$$

  Diffusion number: $\boxed{\nu\dfrac{\Delta t}{\Delta x^2} \le 1/2}$ $\rightarrow$ $\nu\dfrac{\Delta t}{\Delta x^2} \le \dfrac{1}{2}$

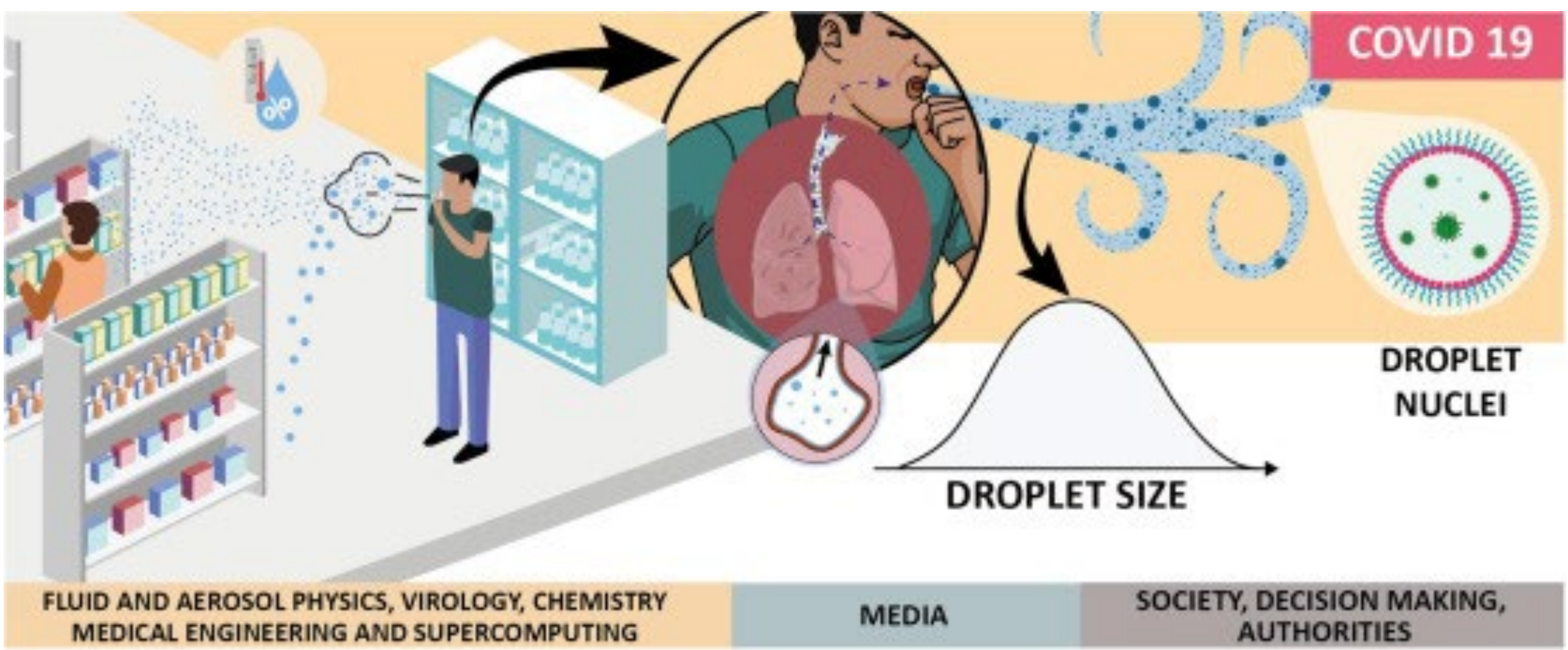
- Can be extended to <u>multidimensional problems</u>
  - Easy application to practical use

Multi-dimensional System

- Aerosol-based transmission of SARS-CoV-2 virus in public place

$$\text{2D Convection: } \frac{\partial u}{\partial t} + V_x \frac{\partial u}{\partial x} + V_y \frac{\partial u}{\partial y} = 0$$

$$\text{2D Diffusion: } \frac{\partial u}{\partial t} = D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$



Multi-dimensional Case (2D and 3D)

$e^{x+y} = e^x e^y$

For the multi-dimensional case, the error can also satisfy the homogenous equation. The error is approximated by the higher dimension Fourier series expansion in 2-D as below:

$$\epsilon^n_{l,m} = \sum_{k_x} \sum_{k_y} v^n \, e^{i k_x x_l} \, e^{i k_y y_m}$$

$l \to x$, $m \to y$

temporal amplitude

spatial error

and in 3-D;

$$\epsilon^n_{l,m,p} = \sum_{k_x} \sum_{k_y} \sum_{k_z} v^n \, e^{i k_x x_l} \, e^{i k_y y_m} \, e^{i k_z z_p}$$

$l \to x$, $m \to y$, $p \to z$

Multi-dimensional Case

The application of Fourier stability analysis in the multi-dimensional case is the same as in the 1-D case. That is, we only need to consider an arbitrary component of the series expansion.

$$v^n e^{ik_x x_l} \cdot \boxed{e^{ik_y y_m}}$$ in 2-D

$$v^n e^{ik_x x_l} \cdot e^{ik_y y_m} \cdot e^{ik_z z_p}$$ in 3-D

$$u^n \longrightarrow \boxed{\text{Fourier Analyzer}} \longrightarrow u^{n+1}$$

$$G = \left| \frac{v^{n+1}}{v^n} \right|$$

The stability condition is still the same $\boxed{|G| \le 1}$

2-D Example…

$u(x, y, t)$

Our model equation is the 2-D convection equation

$v = (v_x, v_y)$

$$\frac{\partial u}{\partial t} + \upsilon_x \frac{\partial u}{\partial x} + \upsilon_y \frac{\partial u}{\partial y} = 0$$

$u \rightarrow T$

Using the Lax scheme,

$$u_{j,k}^{n+1} = \frac{1}{4}\left(u_{j+1,k}^{n} + u_{j-1,k}^{n} + u_{j,k+1}^{n} + u_{j,k-1}^{n} \right)$$

Substituting $u_{j,k}^{n} = v^{n} e^{ik_x x_j} \cdot e^{ik_y y_k}$ into the above equation gives

$$u_{j,k}^{n+1} = \frac{1}{4}\left(u_{j+1,k}^{n} + u_{j-1,k}^{n} + u_{j,k+1}^{n} + u_{j,k-1}^{n}\right)$$

$$- \frac{\Delta t}{2h} v_x \underbrace{\left(u_{j+1,k}^{n} - u_{j-1,k}^{n}\right)}_{\text{central for } x}$$

$$- \frac{\Delta t}{2h} v_y \underbrace{\left(u_{j,k+1}^{n} - u_{j,k-1}^{n}\right)}_{\text{central for } y}$$

$$v^{n+1} e^{i k_x x_j} \cdot e^{i k_y y_k}$$

$$= \frac{1}{4} v^n e^{i k_x x_j} e^{i k_y y_k} \left(e^{i k_x h} + e^{-i k_x h} + e^{i k_y h} + e^{-i k_y h} \right)$$

$$- \frac{\Delta t}{2h} \cdots \underbrace{\left(e^{i k_x h} - e^{-i k_x h} \right)}$$

$$- \frac{\Delta t}{2h} \left(e^{i k_y h} - e^{-i k_y h} \right)$$

$$\Rightarrow \quad G = \frac{v^{n+1}}{v^n}, \qquad \theta_x = v_x \frac{\Delta t}{h} \quad — (1)$$

$$\theta_y = v_y \frac{\Delta t}{h} \quad — (2)$$

$$\rightarrow \alpha = k_x h$$

$$\beta = k_y h$$

$$\boxed{\frac{\partial u}{\partial t} + v_x \frac{\partial u}{\partial x} = 0}$$

2-D Example…

$$\boxed{G=\frac{1}{4}\left(e^{i\alpha}+e^{-i\alpha}+\left(e^{i\beta}+e^{-i\beta}\right)\right)-\frac{1}{2}\theta_x\left(e^{i\alpha}-e^{-i\alpha}\right)-\frac{1}{2}\theta_y\left(\left(e^{i\beta}-e^{-i\beta}\right)\right)}$$

$$=\frac{1}{2}\left(\cos\alpha+\cos\beta\right)-i\left(\theta_x\sin\alpha+\theta_y\sin\beta\right)$$

$$\frac{e^{i\alpha}-e^{-i\alpha}}{2}=\sin\alpha$$

$$(a+ib)^2=a^2+b^2$$

$$|G|^2=\frac{1}{4}\left(\cos\alpha+\cos\beta\right)^2+\left(\theta_x\sin\alpha+\theta_y\sin\beta\right)^2=\frac{1}{4}\left(\cos^2\alpha+\cos\alpha\cdot\cos\beta+\cos^2\beta\right)$$

$$+\theta_x^2\sin^2\alpha+2\theta_x\theta_y\sin\alpha\cdot\sin\beta+\theta_y^2\sin^2\beta$$

$$=1-\underbrace{\left(\sin^2\alpha+\sin^2\beta\right)}_{+\text{ve}}\boxed{\left[\frac{1}{2}-\left(\theta_x^2+\theta_y^2\right)\right]}-\underbrace{\frac{1}{4}\left(\cos\alpha-\cos\beta\right)^2}_{+\text{ve}}-\underbrace{\left(\theta_y\sin\alpha-\theta_x\sin\beta\right)^2}_{+\text{ve}}$$

So to guarantee $|G|^2\le 1;\quad \frac{1}{2}-\left(\theta_x^2+\theta_y^2\right)\ge 0 \;\Rightarrow\; \frac{1}{2}-\upsilon_x^2\frac{\Delta t^2}{\Delta^2}-\upsilon_y^2\frac{\Delta t^2}{\Delta^2}\ge 0$

$$\frac{1}{2}-\left(\theta_x^2+\theta_y^2\right)\ge 0$$

$$\boxed{\Delta t\le\frac{\Delta}{\sqrt{2}\cdot\sqrt{\upsilon_x^2+\upsilon_y^2}}}$$

Goal : $|G| \leq 1$

For given speed (v_x, v_y)

& discretization $(\Delta t, h)$

$$(\theta_x, \theta_y) \rightarrow \underline{|G| \leq 1}$$

$$(\theta_x^2 + \theta_y^2) \leq \frac{1}{2}$$

$$\theta_x = \frac{v_x \Delta t}{h}$$

$$\Rightarrow \boxed{\Delta t \leq \frac{h}{\sqrt{2}\sqrt{v_x^2 + v_y^2}}}$$

$$\theta_y = \frac{v_y \Delta t}{h}$$

$\boxed{\text{Problem set \#3}}$

$$\frac{\partial \phi}{\partial t} = \nu \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right)$$

Continous / Physical Form

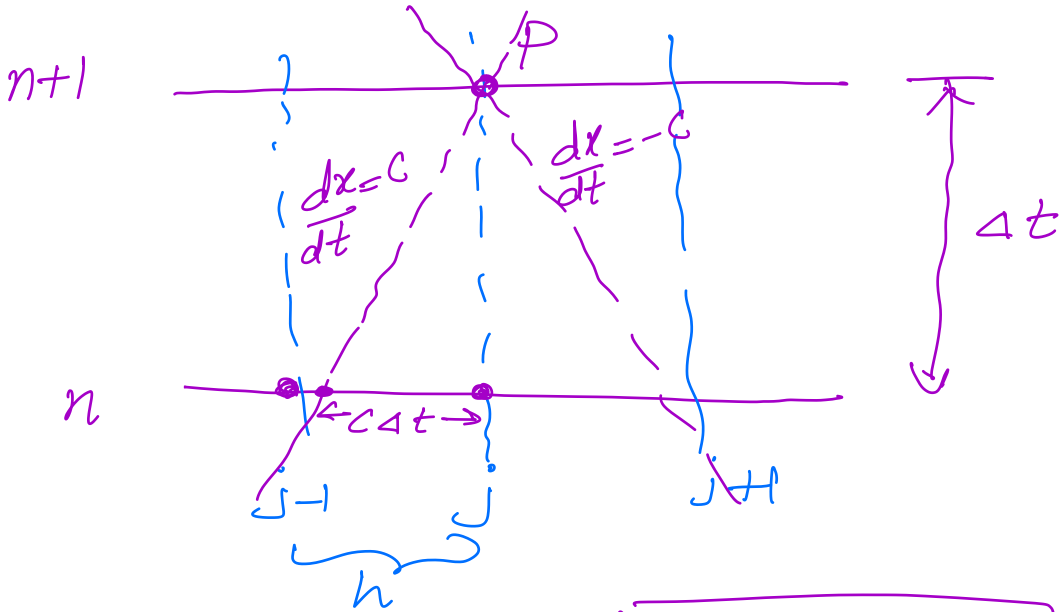
Consider 1D Advection (wave) Eqn:

$$\frac{\partial u}{\partial t} + \underline{c}\,\frac{\partial u}{\partial x} = 0\,;$$

$$\frac{dx}{dt} = c$$

c = speed of information

Discrete Form



$$\underline{c\,\Delta t} < h$$

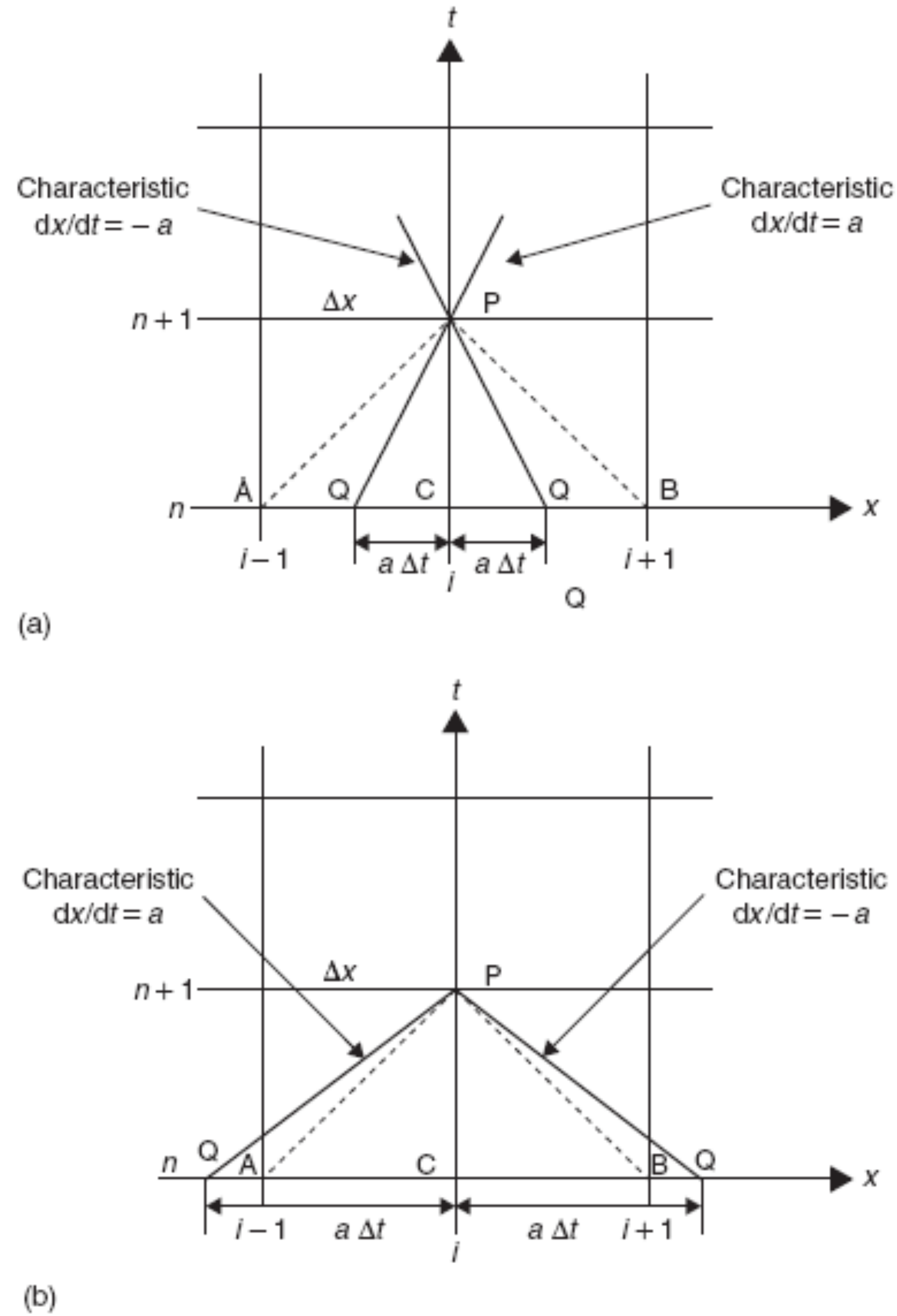
$$\boxed{\frac{c\,\Delta t}{h} < 1}$$

The distance covered (physically) during Δt, the physical information should be lower than the minimum distance between the two mesh points!

Characteristic Interpretation of the CFL Condition

- The domain of dependence of the differential equation should be entirely contained in the numerical domain of dependence of the discretized equations.

- This interpretation is generally applied for two- and three-dimensional problems when it appears difficult to express analytically



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