The University of British Columbia
MECH 479

Module 5
Upwind Discretization & Finite Volume Method

October 31, 2022

Upwind discretization and Finite volume method

This module introduces the upwind discretization for the hyperbolic PDE and the fundamentals of the finite volume method.

1 Review: Hyperbolic PDE

In this section, we will specifically focus on the numerical discretization of the hyperbolic PDE. Recall that the hyperbolic PDE describes a class of problems where the information propagates in certain directions at finite speeds, and the solution is a superposition of multiple simple waves.

A typical first order hyperbolic PDE is the advection equation, which is given by:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

In the space-time domain $x \times t$, one find characteristic lines $x(t)$ along which the solution remains unchanged, that is, lines where $du = 0$. We can then rewrite the solution as $u(x(t), t)$, and by using the chain rule, we have

$$\frac{d}{dt}\left(u(x(t), t\right) = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} = 0.$$

The characteristics is:

$$\frac{dx}{dt} = c; \quad du = 0$$

which means that the information represented by $u(x)$ will keeps the same along the characteristic line (when $du = 0$) and propagate with velocity c, as shown if Fig. 1. Hence the characteristic speed of a scalar hyperbolic equation is its advection velocity.

A typical second order hyperbolic PDE is wave equation, which is given by:

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

It can be further written in matrix form by taking $\phi = \frac{\partial u}{\partial t}$ and $\psi = \frac{\partial u}{\partial x}$:

$$\begin{bmatrix} \frac{\partial \phi}{\partial t} \\ \frac{\partial \psi}{\partial t} \end{bmatrix} + \begin{bmatrix} 0 & -c^2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \psi}{\partial x} \end{bmatrix} = 0$$

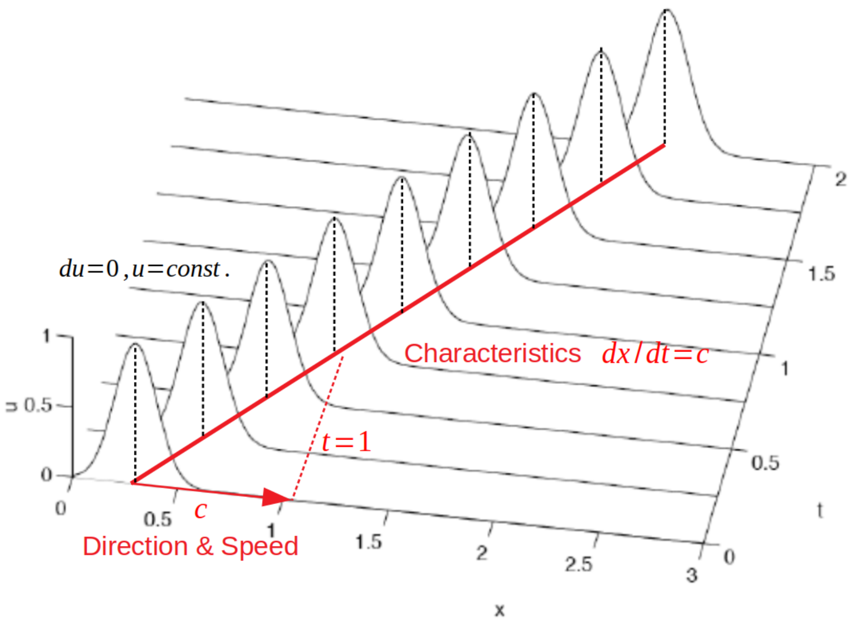

Figure 1: Typical behavior of the solution of an advection equation

Through diagonalization, we have

$$\frac{\partial \Phi_1}{\partial t} - c\frac{\partial \Phi_1}{\partial x} = 0$$
$$\frac{\partial \Phi_2}{\partial t} + c\frac{\partial \Phi_2}{\partial x} = 0$$

where $\Phi_1 = (\frac{\sqrt{1+c^2}}{2c}\phi + \frac{\sqrt{1+c^2}}{2}\psi)$, $\Phi_2 = (-\frac{\sqrt{1+c^2}}{2c}\phi + \frac{\sqrt{1+c^2}}{2}\psi)$. As a result, it reduces to two independent advection equation. The characteristics are given by:

$$\frac{dx}{dt} = -c; \quad d\Phi_1 = 0$$
$$\frac{dx}{dt} = c; \quad d\Phi_2 = 0$$

which means that the information represented by Φ_1 and Φ_2 will keep the same and propagate along the characteristics at velocity $-c$ and c respectively.

As shown above, it is essential to find a proper algorithm to solve the 1D linear advection PDE for the numerical study of hyperbolic PDEs. This will enable the numerical study of physical problems where the information propagates in certain directions with finite speeds. This physical insight into the problem is of key importance in the design of the numerical schemes, which will be shown in the following upwind schemes.

2 Upwind scheme

Let us consider a 1D advection PDE with positive velocity ($c > 0$):

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

Recall the FTCS discretization:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{c}{2\Delta x}(u_{i+1}^n - u_{i-1}^n)$$

The stencil of the scheme is shown in Fig. 2: The amplification factor is

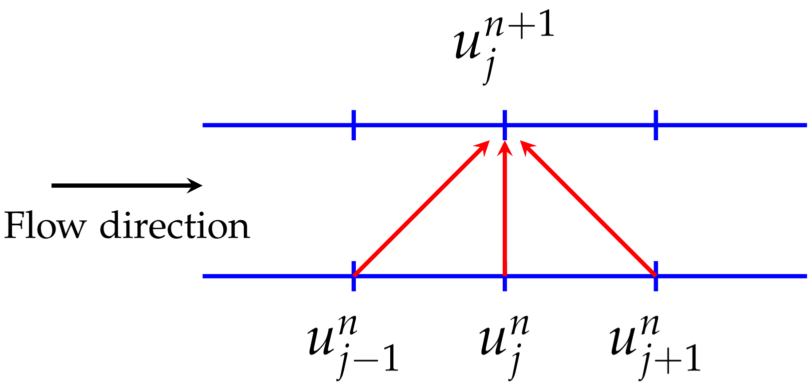

Figure 2: Stencil for the FTCS scheme

$$G_m = 1 - \frac{c}{\Delta x} i \sin(k_m \Delta x)$$

$$|G_m| = \sqrt{1 + (\frac{c}{\Delta x}\sin(k_m\Delta x))^2} > 1$$

The accuracy of the scheme is $O(\Delta t, \Delta x^2)$. The scheme is unconditionally unstable. This means that FTCS is not a proper discretization for the advection equation.

Instead, let's consider forward in time and backward in space scheme (FTBS):

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{c}{\Delta x}\underbrace{(u_i^n - u_{i-1}^n)}_{\text{Backward}} = 0$$

which takes the information from the upwind direction only, as shown in Fig. 2: To investigate its stability, we use von Neumann analysis:

$$\frac{v_m^{n+1} - v_m^n}{\Delta t} + \frac{c}{\Delta x}(v_m^n - v_m^n e^{-ik_m\Delta x}) = 0$$

$$G_m = \frac{v_m^{n+1}}{v_m^n} = 1 - \frac{c\Delta t}{\Delta x}(1 - \cos(k_m\Delta x) + i\sin(k_m\Delta x))$$

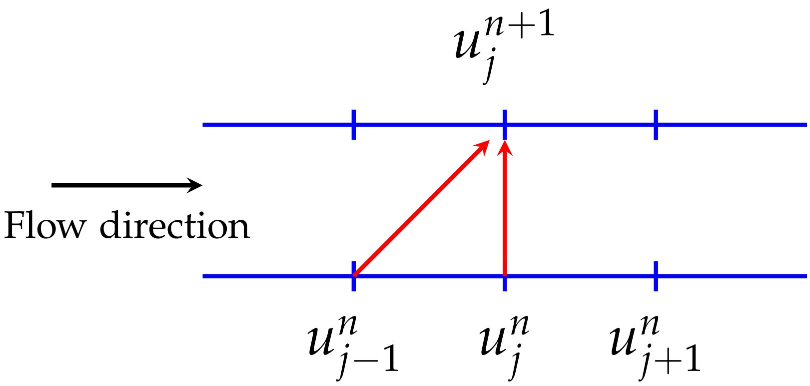

Figure 3: Stencil for upwind scheme

Denoting $\lambda = \frac{c\Delta t}{\Delta x}$, $\theta = k_m \Delta x$, we have

$$\begin{aligned}|G_m| &= \sqrt{(1-\lambda+\lambda\cos\theta)^2 + (\lambda\sin\theta)^2} \\ &= \sqrt{(1-\lambda)^2 + 2(1-\lambda)\lambda\cos\theta + \lambda^2}\end{aligned}$$

Consider extreme cases $\cos\theta = 1$ and $\cos\theta = -1$. For $\cos\theta = 1$, we have

$$\begin{aligned}|G_m| &= \sqrt{(1-\lambda)^2 + 2(1-\lambda)\lambda + \lambda^2} \\ &= 1 \leq 1\end{aligned}$$

which is always stable. For $\cos\theta = -1$, we have

$$\begin{aligned}|G_m| &= \sqrt{(1-\lambda)^2 - 2(1-\lambda)\lambda + \lambda^2} \\ &= |1-2\lambda|\end{aligned}$$

According to the stabilization condition, we have

$$\begin{aligned}|1-2\lambda| &\leq 1 \\ -1 \leq 1-2\lambda &\leq 1 \\ 0 \leq \lambda &\leq 1 \\ c\Delta t &\leq \Delta x\end{aligned}$$

Which means that the signal has to travel less than one grid space in one time step. Because only for these signals, at their originates of the characteristics, the spatial gradient approximation $\frac{\partial u^n}{\partial x} = \frac{u_i^n - u_{i-1}^n}{\Delta x} + O(h)$ is allowed, as shown in Fig. 2

To summarize, the upwind scheme is valid for solving linear advection equation when: (1) only the information from the upwind direction is used (2) the location of the origin of the characteristics falls into the range where the spatial derivative can be approximated by the spatial discretization properly. When these conditions are satisfied, the numerical domain of dependence is

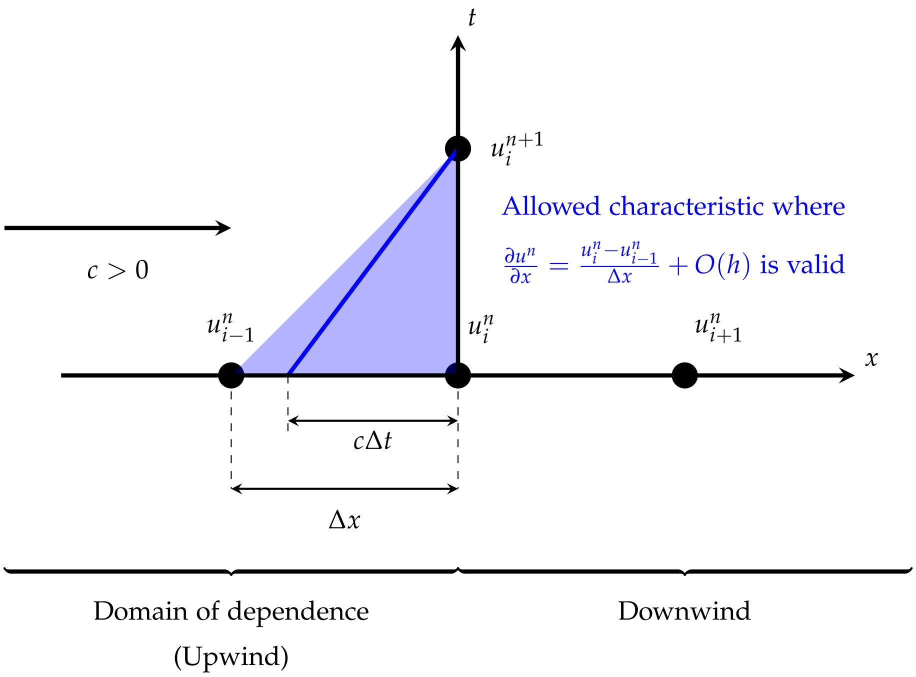

Figure 4: Allowed characteristics for upwind scheme

able to model the real physical domain of dependence. These conditions were derived by Courant-Friedrichs-Levy. It is famously called CFL or Courant condition.

To further validate the above statement, let's consider the Forward time forward space scheme, which takes the information from the downwind direction:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{c}{\Delta x} \underbrace{(u_{i+1}^n - u_i^n)}_{\text{Forward}} = 0$$

The stencil of the scheme is shown in Fig. 2: Though von Neumann analysis:

$$\frac{v_m^{n+1} - v_m^n}{\Delta t} + \frac{c}{\Delta x}(v_m^n e^{ik_m \Delta x} - v_m^n) = 0$$

$$G_m = \frac{v_m^{n+1}}{v_m^n} = 1 - \frac{c\Delta t}{\Delta x}(\cos(k_m \Delta x) + i \sin(k_m \Delta x) - 1)$$

Denoting $\lambda = \frac{c\Delta t}{\Delta x}$, $\theta = k_m \Delta x$, we have

$$\begin{aligned} |G_m| &= \sqrt{(1 + \lambda - \lambda \cos\theta)^2 + (\lambda \sin\theta)^2} \\ &= \sqrt{(1+\lambda)^2 - 2(1-\lambda)\lambda\cos\theta + \lambda^2} \end{aligned}$$

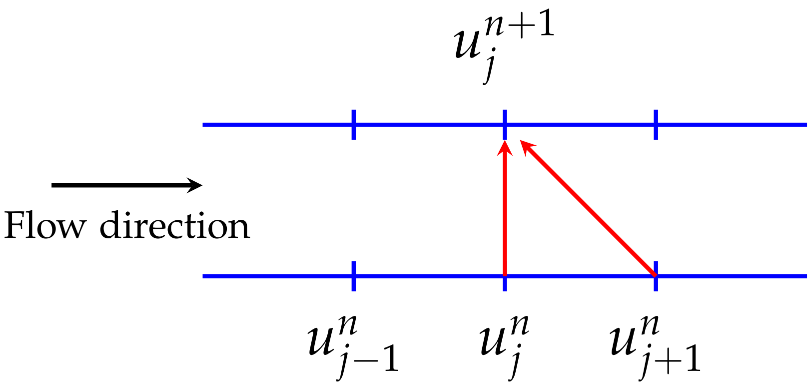

Figure 5: Stencil for downwind scheme

Consider extreme cases $\cos\theta = 1$ and $\cos\theta = -1$. For $\cos\theta = 1$, we have

$$\begin{aligned} |G_m| &= \sqrt{(1+\lambda)^2 - 2(1+\lambda)\lambda + \lambda^2} \\ &= 1 \leq 1 \end{aligned}$$

which is always stable. For $\cos\theta = -1$, we have

$$\begin{aligned} |G_m| &= \sqrt{(1-\lambda)^2 + 2(1-\lambda)\lambda + \lambda^2} \\ &= |1 + 2\lambda| \end{aligned}$$

Note that $c > 0$, $|1 + 2\lambda| > 1$. The scheme is unconditionally unstable because we take the information from the downwind direction. However, for $c < 0$, which means that the information propagates from the right to the left, FTFS becomes the upwind scheme which is stable.

Consider the following scheme:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{c}{2\Delta x}(u_i^n - u_{i-2}^n) = 0$$

Intuitively, the current spatial discretization provides a correct approximation of the spatial derivative at time n for the characteristics originating from u_{i-2}^n to u_i^n, thus the stability condition should relax to $c\Delta t \leq 2\Delta x$. You can prove it easily via von Neumann's analysis.

Besides the von Neumann analysis, we can analyze the upwind scheme in the modified equation approach. Let us consider the upwind scheme for $c > 0$:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{c}{\Delta x}(u_i^n - u_{i-1}^n) = 0$$

The modified equation for the upwind scheme is given by:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = -\frac{\Delta t}{2}u_{tt} + \frac{c\Delta x}{2}u_{xx} - \frac{\Delta t^2}{6}u_{ttt} - \frac{c\Delta x^2}{6}u_{xxx} + O(\Delta t^3, \Delta x^3)$$

We refer to this as an unresolved modified equation. It is clearly seen from the above equation that the right-hand side vanishes when Δt and Δx tend to zero, and therefore the scheme is consistent. As expected, the accuracy of the scheme is first order in time and first order in space. The derivation of the modified equation is closely related to the calculation of the local truncation error for a given PDE.

In order to understand the behavior of the solution, we need to change the high-order time derivatives to spatial derivatives, thus the concepts of dissipation and dispersion are applicable. Let's say the leading dissipation term (u_{xx}) and dispersion term (u_{xxx}) are desired in the analysis. To achieve this, we need to eliminate u_{tt} and u_{ttt}. This can be accomplished by differentiating the unresolved equation with respect to x and t:

$$u_{tt} + cu_{xt} = -\frac{\Delta t}{2}u_{ttt} + \frac{c\Delta x}{2}u_{xxt} - \frac{\Delta t^2}{6}u_{tttt} - \frac{c\Delta x^2}{6}u_{xxxt} + O(\Delta t^3, \Delta x^3)$$
$$u_{tx} + cu_{xx} = -\frac{\Delta t}{2}u_{ttx} + \frac{c\Delta x}{2}u_{xxx} - \frac{\Delta t^2}{6}u_{tttx} - \frac{c\Delta x^2}{6}u_{xxxx} + O(\Delta t^3, \Delta x^3)$$

Eliminating the mixed derivative, we have:

$$u_{tt} = c^2u_{xx} - \frac{\Delta t}{2}(u_{ttt} - cu_{ttx}) + \frac{c\Delta x}{2}(u_{xxt} - cu_{xxx}) + O(\Delta t^2, \Delta x^2)$$

Similarly, by taking higher order derivative of the unresolved modified equation, we can eliminate higher order temporal derivatives and mixed derivatives:

$$u_{ttt} + cu_{xtt} = -\frac{\Delta t}{2}u_{tttt} + \frac{c\Delta x}{2}u_{xxtt} - \frac{\Delta t^2}{6}u_{ttttt} - \frac{c\Delta x^2}{6}u_{xxxtt} + O(\Delta t^3, \Delta x^3) \tag{1}$$

$$u_{ttx} + cu_{xtx} = -\frac{\Delta t}{2}u_{tttx} + \frac{c\Delta x}{2}u_{xxtx} - \frac{\Delta t^2}{6}u_{ttttx} - \frac{c\Delta x^2}{6}u_{xxxtx} + O(\Delta t^3, \Delta x^3) \tag{2}$$

$$u_{txt} + cu_{xxt} = -\frac{\Delta t}{2}u_{ttxt} + \frac{c\Delta x}{2}u_{xxxt} - \frac{\Delta t^2}{6}u_{tttxt} - \frac{c\Delta x^2}{6}u_{xxxxt} + O(\Delta t^3, \Delta x^3) \tag{3}$$

$$u_{txx} + cu_{xxx} = -\frac{\Delta t}{2}u_{ttxx} + \frac{c\Delta x}{2}u_{xxxx} - \frac{\Delta t^2}{6}u_{tttxx} - \frac{c\Delta x^2}{6}u_{xxxxx} + O(\Delta t^3, \Delta x^3) \tag{4}$$

Since we can freely arrange the sequence of the derivative, eliminating the mixed derivative, we have:

$$\begin{aligned} u_{ttt} &= -c^3u_{xxx} + O(\Delta t, \Delta x) && (1) - c(3) + c^2(4)) \\ u_{ttx} &= c^2u_{xxx} + O(\Delta t, \Delta x) && (3) - c(4) \\ u_{xxt} &= -cu_{xxx} + O(\Delta t, \Delta x) && (4) \end{aligned}$$

With these equations, we can eliminate the higher order time derivative to form the modified equation:

$$\begin{aligned}
\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} &= -\frac{\Delta t}{2}u_{tt} + \frac{c\Delta x}{2}u_{xx} - \frac{\Delta t^2}{6}u_{ttt} - \frac{c\Delta x^2}{6}u_{xxx} + O(\Delta t^3, \Delta x^3) \\
&= -\frac{\Delta t}{2}\left(c^2 u_{xx} - \frac{\Delta t}{2}(u_{ttt} - cu_{ttx}) + \frac{c\Delta x}{2}(u_{xxt} - cu_{xxx})\right) + \frac{c\Delta x}{2}u_{xx} \\
&\quad - \frac{\Delta t^2}{6}\left(-c^3 u_{xxx} + O(\Delta t, \Delta x)\right) - \frac{c\Delta x^2}{6}u_{xxx} + O(\Delta t^3, \Delta x^3) \\
&= -\frac{\Delta t}{2}\left(c^2 u_{xx} - \frac{\Delta t}{2}(-c^3 u_{xxx} - c^3 u_{xxx} + O(\Delta x, \Delta t)) + \frac{c\Delta x}{2}(-cu_{xxx} - cu_{xxx} + O(\Delta t, \Delta x))\right) \\
&\quad + \frac{c\Delta x}{2}u_{xx} - \frac{\Delta t^2}{6}\left(-c^3 u_{xxx} + O(\Delta t, \Delta x)\right) - \frac{c\Delta x^2}{6}u_{xxx} + O(\Delta t^3, \Delta x^3)
\end{aligned}$$

which can be further simplified as:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \frac{c\Delta x}{2}(1-\lambda)u_{xx} - \frac{c\Delta x^2}{6}(2\lambda^2 - 3\lambda + 1)u_{xxx} + O(\Delta t^3, \Delta t^2\Delta x, \Delta t\Delta x^2, \Delta x^3)$$

and can be written in the following form

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = Du_{xx} + O\left(\Delta t^2, \Delta x^2\right)$$

with a diffusion constant given by $D = \frac{c\Delta x}{2}(1-\lambda)$. As we can see, the upwind scheme is essentially solving a convection-diffusion equation. We expect the solution of these equations to become smeared out as time evolves. The stability condition is that there is no negative diffusion, which is given by $\lambda < 1$.

3 Generalization of the upwind scheme

3.1 Comparison between the central difference scheme and upwind scheme

Consider the upwind scheme:

$$\begin{aligned}
u_i^{n+1} &= u_i^n - \frac{c\Delta t}{\Delta x}\left(u_i^n - u_{i-1}^n\right),\ c > 0 \\
u_i^{n+1} &= u_i^n - \frac{c\Delta t}{\Delta x}\left(u_{i+1}^n - u_i^n\right),\ c < 0
\end{aligned}$$

Define

$$\begin{aligned}
c^+ &= \frac{1}{2}\left(c + |c|\right); c^+ = c, c \geq 0; c^+ = 0, c < 0; \\
c^- &= \frac{1}{2}\left(c - |c|\right); c^- = 0, c \geq 0; c^- = c, c < 0;
\end{aligned}$$

The two cases of the upwind scheme can be combined into:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(c^+\left(u_i^n - u_{i-1}^n\right) + c^-\left(u_{i+1}^n - u_i^n\right)\right)$$
$$u_i^{n+1} = u_i^n - \underbrace{\frac{c\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right)}_{\text{Central difference}} + \underbrace{\frac{|c|\Delta t}{h}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)}_{\text{Numerical viscosity}}$$

As we can see, compared to the central difference method, the upwind scheme is stable because the added numerical viscosity. The price is that the order of accuracy decrease from $O(\Delta t, \Delta x^2)$ to $O(\Delta t, \Delta x)$.

3.2 Second order schemes

Leap-Frog method

For the linear convection equation, the Leap-Frog method is given by:

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} + \frac{c}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) = 0 \tag{5}$$

The modified equation is given by:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \frac{c\Delta x^2}{6}(\lambda^2 - 1)u_{xxx} + O(\Delta x^4, \Delta t^4).$$

It can be observed that the behavior of the leading order error is dispersion. As a result, we will obtain a strongly oscillatory solution in time. The stability condition through von Neumann analysis is:

$$\frac{c\Delta t}{h} \leq 1 \tag{6}$$

Lax-Wendroff's method (LW-I)

Knowing that the leap-frog method is strongly oscillatory, we want to improve the scheme by adding a diffusion term. In Lax-Wendroff's method, diffusion is added by expanding the time derivative to the second term and transferring the temporal derivative to the spatial derivative utilizing the original equation (similar thinking process with the modified equation). We can start with the Taylor expansion:

$$u(t + \Delta t) = u(t) + \Delta t\frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + O(\Delta t^3)$$

Now we transfer the temporal derivative to spatial derivative through:

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}$$
$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

which leads to:

$$u(t+\Delta t) = u(t) - c\Delta t\frac{\partial u}{\partial x} + c^2\frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial x^2} + O(\Delta t^3)$$

Using the central difference for spatial derivative, we get the Lax-Wendroff's method:

$$u_i^{n+1} = u_i^n - \frac{c\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) + \frac{c^2\Delta t^2}{2\Delta x^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

The scheme is second order accurate in space and time. The stability condition is given as $\frac{c\Delta t}{\Delta x} < 1$

Two-step Lax-Wendroff's method (LW-II)

The two-step lax-Wendroff's method requires the evaluation at half time step $n+1/2$ and half grids $i+1/2$:

$$\frac{u_{i+1/2}^{n+1/2} - (u_{i+1}^n + u_i^n)}{\Delta t/2} + c\frac{u_{i+1}^n - u_i^n}{h} = 0 \text{ (Lax method)}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\frac{u_{i+1/2}^{n+1/2} - u_{i-1/2}^{n+1/2}}{h} = 0 \text{ (Leap} - \text{Frog method)}$$

This scheme is second order in space and time. It is stable when $\frac{c\Delta t}{\Delta x} < 1$. For linear equations, LW-II is identical to LW-I

MacCormack method

MacCormack method is similar to LW-II method without $i+1/2$, $i-1/2$:

$$u_i^* = u_i^n - \frac{c\Delta t}{\Delta x}(u_{i+1}^n - u_i^n) \text{ (Predictor : Forward differencing)}$$

$$u_j^{n+1} = \frac{1}{2}\left(u_i^n + u_i^* - \frac{c\Delta t}{\Delta x}(u_i^* - u_{i-1}^*)\right) \text{ (Corrector : Backward differencing)}$$

For linear equations, accuracy and stability are identical to lax-Wendroff's method.

Second order upwind scheme

Beam-warming implicit scheme is given by:

$$u_i^* = u_i^n - \frac{c\Delta t}{\Delta x}(u_i^n - u_{i-1}^n) \text{ (Predictor : Upwinding)}$$

$$u_i^{n+1} = \frac{1}{2}\left(u_i^n + u_i^* - \frac{c\Delta t}{\Delta x}(u_i^* - u_{i-1}^*) - \frac{c\Delta t}{\Delta x}(u_i^n - 2u_{i-1}^n + u_{i-2}^n)\right) \text{ (Corrector : Upwinding)}$$

Combining the two:

$$u_i^{n+1} = u_i^n - \lambda(u_i^n - u_{i-1}^n) + \frac{1}{2}\lambda(\lambda-1)(u_i^n - 2u_{i-1}^n + u_{i-2}^n)$$

The scheme is stable for $0 \leq \lambda \leq 2$ and second order accurate in space and time.

4 Finite Volume method

In fluid mechanics, the first principles are the conservation laws, including mass, momentum and energy conservation. When continuum and isotherm assumptions apply, the conservation laws in a control volume become the integral form of the Navier-Stokes equations. When we proceed from the continuum level to the discrete level, it is a natural desire to take a control volume as the basic element for numerical analysis so that we can utilize the well-established knowledge about the control volume. This directly leads to the finite volume method.

In the finite volume method, we discretize the computational domain as a series of control volumes. In each control volume, we apply the conservation law, which means that the rate of change of the variable equals the difference between the flux that comes in and goes out of the control volume plus the rate of generation/elimination of the variable. We can write down a general form of conservation law: suppose we have a continuous function$u(x,t)$ with flux vector $\overrightarrow{f}(x,t)$ and volumetric source $q(x,y)$. The differential form of the conservation of u is given by:

$$\frac{\partial u}{\partial t} + \nabla \cdot \overrightarrow{f} = q \tag{7}$$

In one space dimension, a finite volume method is based on subdividing the spatial domain into intervals (i.e. grid cells) and keeping track of an approximation to the integral of over each of these cells. In each time step we update these values using approximations to the flux through the endpoints of the intervals. Considering the cell or control volume between $x_{i-1/2}$ and $x_{i+1/2}$, we have:

$$\frac{\partial}{\partial t}\int_{x_{i-1/2}}^{x_{i+1/2}} u(x,t)dx + \int_{x_{i-1/2}}^{x_{i+1/2}} \nabla \cdot \overrightarrow{f}(x,t)dx = \int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)dx \tag{8}$$

In 1D, the flux vector reduces to a scalar:

$$\frac{\partial}{\partial t}\int_{x_{i-1/2}}^{x_{i+1/2}} u(x,t)dx + \int_{x_{i-1/2}}^{x_{i+1/2}} \frac{\partial f(x,t)}{\partial x}dx = \int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)dx \tag{9}$$

Applying the divergence theorem:

$$\frac{\partial}{\partial t}\int_{x_{i-1/2}}^{x_{i+1/2}} u(x,t)dx + f(x_{i+1/2},t) - f(x_{i-1/2},t) = \int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)dx$$

Denoting the control volume average of $u(x,t)$ as $\overline{U}_i$:

$$\overline{U}_i = \left(\int_{x_{i-1/2}}^{x_{i+1/2}} u(x,t)dx\right) / \Delta x \tag{10}$$

where $\Delta x = x_{i+1/2} - x_{i-1/2}$. Similarly,

$$\overline{Q}_i = \left(\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t)dx\right) / \Delta x \tag{11}$$

Besides, we approximate the flux as

$$f(x_{i+1/2}, t) \approx F_{i+1/2}(\cdots, \overline{U}_{i-1}, \overline{U}_i, \overline{U}_{i+1}, \cdots)$$
$$f(x_{i-1/2}, t) \approx F_{i-1/2}(\cdots, \overline{U}_{i-1}, \overline{U}_i, \overline{U}_{i+1}, \cdots)$$

As a result, the equation can be further simplified as:

$$\frac{\partial}{\partial t}\overline{U}_i \Delta x + F_{i+1/2} - F_{i-1/2} = \overline{Q}_i \Delta x$$
$$\frac{\partial}{\partial t}\overline{U}_i + \frac{F_{i+1/2} - F_{i-1/2}}{\Delta x} = \overline{Q}_i$$

which is the semi-discrete form of the FVM, where the spatial discretization has been completed yet temporal discretization to be done. The control volume average of the variable and fluxs are illustrated in Fig. 4

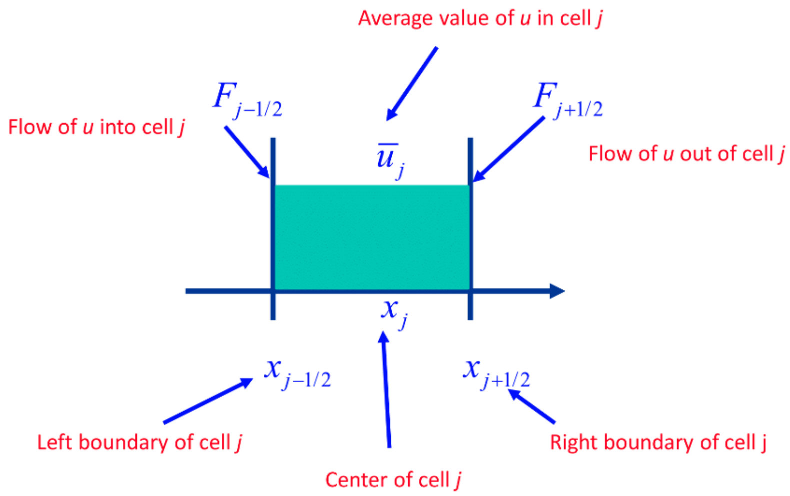

Figure 6: Illustration of a control volume in FVM

Note that once the conservation is satisfied at each control volume, it will be satisfied as well at any larger control volume composed of these small control volumes. Consider the control volume composed of the control volumes i and $i+1$:

$$\frac{\partial}{\partial t}\overline{U}_i + \frac{F_{i+1/2} - F_{i-1/2}}{\Delta x} + \frac{\partial}{\partial t}\overline{U}_{i+1}\frac{F_{i+3/2} - F_{i+1/2}}{\Delta x} = \overline{Q}_i + \overline{Q}_{i+1}$$
$$\frac{\partial}{\partial t}\left(\overline{U}_i + \overline{U}_{i+1}\right) + \frac{F_{i+3/2} - F_{i-1/2}}{\Delta x} = \left(\overline{Q}_i + \overline{Q}_{i+1}\right)$$

At the surface $x_{i+1/2}$, what flows out of the control volume i goes into control volume $i+1$. There is no generation/elimination of u due to the discretization. Thus the conservation is kept.

To further clarify the conservation of the FVM at the discrete level, we consider the 1D inviscid Burger's equation, which is essentially the momuntum conservation equation of 1D inviscid Navier-Stokes equations. We write it in a form which we can discretize strictly following the steps of FVM method. To achieve this, the equation should only contain the time derivative term and the divergence of a flux term. In 1D, the divergence term reduces to the spatial derivative. As a result, we have:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0.$$

We refer to this form as the *conservative form*. When the upwind scheme is employed to approximate the flux term for the control volume i, we have:

$$\begin{aligned} F_{i+1/2} &= \left(\frac{1}{2}\overline{U}_i^2\right) \\ F_{i-1/2} &= \left(\frac{1}{2}\overline{U}_{i-1}^2\right) \end{aligned}$$

which leads to the FVM discretization:

$$\frac{\partial}{\partial t}\overline{U}_i + \frac{1}{2}\left(\overline{U}_i^2 - \overline{U}_{i-1}^2\right) = 0$$

Now let's consider a larger control volume composed of control volume i and $i+1$:

$$\begin{aligned} \frac{\partial}{\partial t}\overline{U}_i + \frac{1}{2}\left(\overline{U}_i^2 - \overline{U}_{i-1}^2\right) + \frac{\partial}{\partial t}\overline{U}_{i+1} + \frac{1}{2}\left(\overline{U}_{i+1}^2 - \overline{U}_i^2\right) &= 0 \\ \frac{\partial}{\partial t}\left(\overline{U}_i + \overline{U}_{i+1}\right) + \frac{1}{2}\left(\overline{U}_{i+1}^2 - \overline{U}_{i-1}^2\right) &= 0 \end{aligned}$$

where the fluxes at $x_{i+1/2}$ cancels each other exactly and the conservation is ensured.

On the contrary, let's consider the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$

which is equal to the conservative form of 1D inviscid Burger's equation at the continuum level. Similarly, we approximate the spatial derivative with the upwind scheme:

$$\frac{\partial}{\partial t}\overline{U}_i + \overline{U}_i\frac{\overline{U}_i - \overline{U}_{i-1}}{\Delta x} = 0$$

In the current discretization, the concepts of the flux doesn't exist. As a result, we lose the conservation. Furthermore, when a larger control volume composed of control volume i and $i+1$ is considered:

$$\begin{aligned} \frac{\partial}{\partial t}\overline{U}_i + \overline{U}_i\frac{\overline{U}_i - \overline{U}_{i-1}}{\Delta x} + \frac{\partial}{\partial t}\overline{U}_{i+1} + \overline{U}_{i+1}\frac{\overline{U}_{i+1} - \overline{U}_i}{\Delta x} &= 0 \\ \frac{\partial}{\partial t}\left(\overline{U}_i + \overline{U}_{i+1}\right) + \frac{\overline{U}_{i+1}^2 - \overline{U}_{i+1}\overline{U}_i + \overline{U}_i^2 - \overline{U}_i\overline{U}_{i-1}}{\Delta x} &= 0 \end{aligned}$$

There is no clear relation between the variation of $\overline{U}_i$ and $\overline{U}_{i+1}$ either.

5 FVM approximation for advection-diffusion equation

In this section, we apply the FVM approximation for the advection-diffusion equation. The differential form of the advection-diffusion equation is given by:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2}$$

which can be written as:

$$\frac{\partial u}{\partial t} + \frac{\partial}{x}\left(cu - \alpha \frac{\partial u}{\partial x}\right) = 0 \tag{12}$$

in 1D case, the conservation law in FVM can be written as:

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = q \tag{13}$$

Through comparison, we find that for the advection-diffusion equation,

$$F = cu - \alpha \frac{\partial u}{\partial x}$$

As a result,

$$F_{i+1/2} = cu_{i+1/2} - \alpha \left.\frac{\partial u}{\partial x}\right|_{i+1/2}$$

Through approximation:

$$\begin{aligned} u_{i+1/2} &\approx \frac{1}{2}(\overline{U}_{i+1} + \overline{U}_i) \\ \left.\frac{\partial u}{\partial x}\right|_{i+1/2} &\approx \frac{1}{\Delta x}(\overline{U}_{i+1} - \overline{U}_i) \end{aligned}$$

we have the flux function

$$\begin{aligned} F_{i+1/2} &= \frac{c}{2}\left(\overline{U}_{i+1} + \overline{U}_i\right) - \frac{\alpha}{\Delta x}\left(\overline{U}_{i+1} - \overline{U}_i\right) \\ F_{i-1/2} &= \frac{c}{2}\left(\overline{U}_i + \overline{U}_{i-1}\right) - \frac{\alpha}{\Delta x}\left(\overline{U}_i - \overline{U}_{i-1}\right) \end{aligned}$$

Approximating the temporal discretization with forward Euler method, we have:

$$\frac{\partial u_i}{\partial t} = \frac{1}{\Delta t}\left(\overline{U}_i^{n+1} - \overline{U}_i^n\right)$$

Note that other schemes can be employed to approximate the flux and the temporal derivative.

Substituting all the above equations, we have the fully discretized advection-diffusion equation:

$$\frac{\overline{U}_i^{n+1} - \overline{U}_i^n}{\Delta t} + \frac{c}{2\Delta x}(\overline{U}_{i+1} - \overline{U}_{i-1}) = \frac{\alpha}{\Delta x^2}\left(\overline{U}_{i+1} - 2\overline{U}_i + \overline{U}_{i-1}\right)$$

which is exactly the same as the finite difference equation (FTCS) if we take the average value to be the same as the value in the center of the cell.

Compared with the finite difference method, the FVM method has three main advantages:

1. Working directly with the conservation principles (no implicit assumption of smoothness and differentiability)
2. Easier visualization of how the solution is updated (the fluxes have a physical meaning)
3. Easier to incorporate unstructured grids

However, the analysis of accuracy and stability is usually easier using the finite difference method.

Materials from:

Hirsch C. 1988-1990 Numerical Computation of Internal and External Flows, Volumes 1 and 2, Wiley.
LeVeque R. J. 2002. Finite Volume Methods for Hyperbolic Problems, Cambridge University Press.