The Finite-Volume Method on Curvilinear Meshes

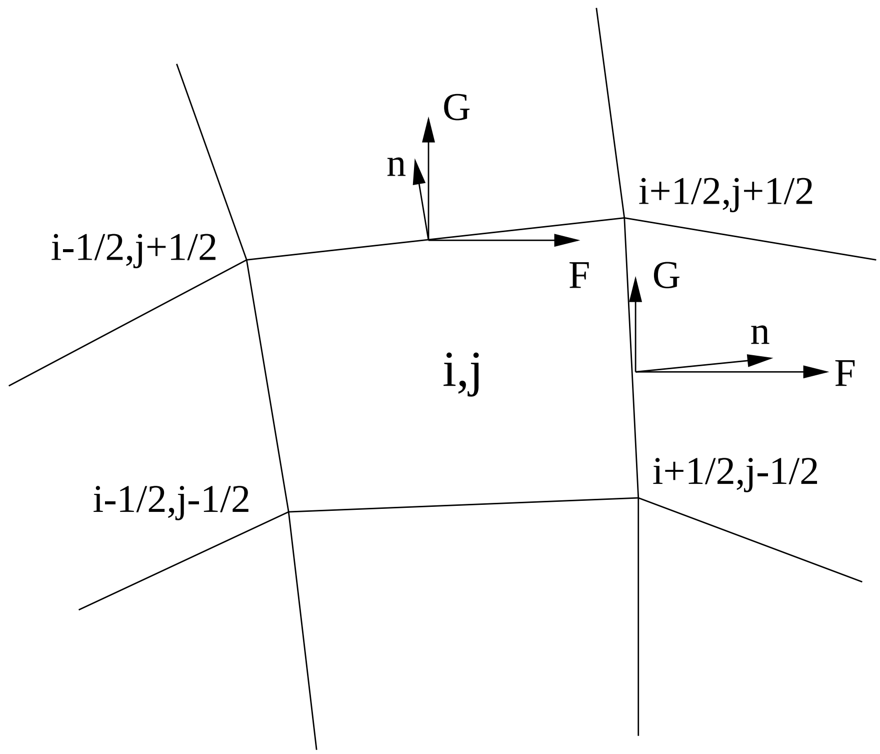

Figure 1: Non-rectangular control volume in two dimensions, with fluxes through two faces.

1 Introduction

So far, all of our calculations havebeen done on equally-spaced rectangular meshes. While this makes the design and analysis of finite volume methods much easier, such meshes are not overwhelmingly useful in practice. Fortunately, it is not too difficult to see how to apply the finite-volume method to non-regular meshes. Consider the control volume i, j shown in Figure 1. If we knew the fluxes at the interfaces, we would be in business, because we could write our usual finite-volume discretization:

$$A_{i,j}\frac{\partial \bar{U}_{i,j}}{\partial t} + \oint_{\partial CV_{i,j}} (F\,\mathbf{n_x} + G\,\mathbf{n_y})\,ds \quad = \quad 0$$

$$
\begin{aligned}
A_{i,j}\frac{\partial \bar{U}_{i,j}}{\partial t} &= -\left[(F\,\mathbf{n_x} + G\,\mathbf{n_y})\cdot l\right]_{i-\frac{1}{2},j}^{i+\frac{1}{2},j} \\
&\quad -\left[(F\,\mathbf{n_x} + G\,\mathbf{n_y})\cdot l\right]_{i,j-\frac{1}{2}}^{i,j+\frac{1}{2}}
\end{aligned}
\tag{1}
$$

where the l's are edge lengths and the $\mathbf{n}$'s are outward-pointing unit normals:

$$
\begin{aligned}
\mathbf{n}_{i\pm\frac{1}{2},j} &= \pm\frac{1}{l_{i\pm\frac{1}{2},j}}\begin{pmatrix} y_{i\pm\frac{1}{2},j+\frac{1}{2}} - y_{i\pm\frac{1}{2},j-\frac{1}{2}} \\ -x_{i\pm\frac{1}{2},j+\frac{1}{2}} + x_{i\pm\frac{1}{2},j-\frac{1}{2}} \end{pmatrix} \\
\mathbf{n}_{i,j\pm\frac{1}{2}} &= \pm\frac{1}{l_{i,j\pm\frac{1}{2}}}\begin{pmatrix} -y_{i+\frac{1}{2},j\pm\frac{1}{2}} + y_{i-\frac{1}{2},j\pm\frac{1}{2}} \\ x_{i+\frac{1}{2},j\pm\frac{1}{2}} - x_{i-\frac{1}{2},j\pm\frac{1}{2}} \end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
l_{i\pm\frac{1}{2},j} &= \sqrt{\left(x_{i\pm\frac{1}{2},j+\frac{1}{2}} - x_{i\pm\frac{1}{2},j-\frac{1}{2}}\right)^2 + \left(y_{i\pm\frac{1}{2},j+\frac{1}{2}} - y_{i\pm\frac{1}{2},j-\frac{1}{2}}\right)^2} \\
l_{i,j\pm\frac{1}{2}} &= \sqrt{\left(x_{i+\frac{1}{2},j\pm\frac{1}{2}} - x_{i-\frac{1}{2},j\pm\frac{1}{2}}\right)^2 + \left(y_{i+\frac{1}{2},j\pm\frac{1}{2}} - y_{i-\frac{1}{2},j\pm\frac{1}{2}}\right)^2}
\end{aligned}
$$

2 Calculating Fluxes on Non-rectilinear Meshes

Equation 1 is a great help, but it isn't the whole story, because we have to be able to calculate F and G at the control volume boundaries.

2.1 Fluxes without derivatives

For fluxes that do not contain gradients — for example, the wave equation, the continuity equation, or the convective terms in the energy and Navier-Stokes equations — this task is straightforward. We compute $U_{i+\frac{1}{2},j}$ using a one-dimensional extrapolation along a mesh line of constant j. For example, the second-order upwind extrapolation gives as usual

$$
U_{i+\frac{1}{2},j} = \frac{3U_{i,j} - U_{i-1,j}}{2}
$$

As we already know, the solution must be smooth or this formula will produce overshoots, even on uniform meshes. On non-uniform meshes, this extrapolation can be shown to be second-order accurate unless the mesh has a discontinuity that does not go away with mesh refinement. See the mesh generation handout for an example of a mesh with such a discontinuity.

2.2 Fluxes with derivatives

Fluxes with derivatives require more effort. Let us consider the example of the heat equation, where $F = -\alpha \frac{\partial T}{\partial x}$ and $G = -\alpha \frac{\partial T}{\partial y}$. We can not calculate the derivatives in the physical (x, y) space directly. Nor can we directly use the derivatives in the computational (ξ, η) space (where the indices i and j are discrete values of ξ and η), because the dot product with the unit normal is tricky in this space. Instead, we will have to calculate the derivatives in the computational space and transform the result into the physical space.

2.2.1 Derivatives in computational space

We need to be able to calculate $\frac{\partial T}{\partial \xi}$ and $\frac{\partial T}{\partial \eta}$ at $i \pm \frac{1}{2}, j$ and $i, j \pm \frac{1}{2}$. For the $i \pm \frac{1}{2}, j$ faces, these can be written as

$$\begin{aligned}
\left.\frac{\partial T}{\partial \xi}\right|_{i+\frac{1}{2},j} &= \frac{\bar{T}_{i+1,j} - \bar{T}_{i,j}}{\Delta \xi} \\
\left.\frac{\partial T}{\partial \eta}\right|_{i+\frac{1}{2},j} &= \frac{\bar{T}_{i+1,j+1} + \bar{T}_{i,j+1} - \bar{T}_{i+1,j-1} - \bar{T}_{i,j-1}}{4\Delta \eta}
\end{aligned}$$

It is easy to show by Taylor series expansion that these are second-order accurate finite-difference approximations to the required derivatives. Since we showed last term that the difference between finite-difference and finite-volume approximations is second-order in mesh size, we're on fairly solid ground here if we simply assume that the finite-volume version is also second-order accurate. In fact, with only slightly more difficulty (caused by the integration to get CV averages), one can show that this approach is definitely second-order accurate in the finite-volume case also. The one caveat is that both the solution and the mesh itself must be smooth, just as for solution extrapolation.

The computational-space derivatives at the other interfaces are similar:

$$\begin{aligned}
\left.\frac{\partial T}{\partial \xi}\right|_{i,j+\frac{1}{2}} &= \frac{\bar{T}_{i+1,j+1} + \bar{T}_{i+1,j} - \bar{T}_{i-1,j+1} - \bar{T}_{i-1,j}}{4\Delta \xi} \\
\left.\frac{\partial T}{\partial \eta}\right|_{i,j+\frac{1}{2}} &= \frac{\bar{T}_{i,j+1} - \bar{T}_{i,j}}{\Delta \eta}
\end{aligned}$$

2.2.2 Converting derivatives from computational to physical space

We can, if we choose, write T as $T\left(\xi(x,y),\eta(x,y)\right)$ (mapping physical coordinates onto computational coordinates) and then write the gradient of T in physical space as

$$\begin{aligned}\frac{\partial T}{\partial x} &= \frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \frac{\partial T}{\partial y} &= \frac{\partial T}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial y}\end{aligned} \tag{2}$$

This gives us equations for the derivatives we want (in physical space) in terms of the derivatives we can easily compute (in computational space) and some *mesh metric* terms involving derivatives of the computational coordinates with respect to the physical coordinates. The metric terms look pretty mystical, but in fact amount more or less to dot products of the unit normals in the two coordinate systems. Despite this description, we still need a way to calculate the metrics.

To get this, we write T as $T(x(\xi,\eta),y(\xi,\eta))$ (mapping computational coordinates onto physical coordinates). Then

$$\begin{aligned}\frac{\partial T}{\partial \xi} &= \frac{\partial T}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial \xi} \\ \frac{\partial T}{\partial \eta} &= \frac{\partial T}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial \eta}\end{aligned} \tag{3}$$

Solving Equations 3 for $\frac{\partial T}{\partial x}$ and $\frac{\partial T}{\partial y}$, and combining with Equations 2, we get

$$\begin{pmatrix}\frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y}\end{pmatrix} = \begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}\end{bmatrix}^{-1} \begin{pmatrix}\frac{\partial T}{\partial \xi} \\ \frac{\partial T}{\partial \eta}\end{pmatrix} \tag{4}$$

$$= \begin{bmatrix}\frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y}\end{bmatrix} \begin{pmatrix}\frac{\partial T}{\partial \xi} \\ \frac{\partial T}{\partial \eta}\end{pmatrix} \tag{5}$$

The form in Equation 4 is easier to calculate than that in Equation 5 in that the metric terms can be calculated in computational space (the same space where $\frac{\partial T}{\partial \xi}$ and $\frac{\partial T}{\partial \eta}$ are calculated, although the formulae are different). In principle, care must be taken to use the centroidal values of x and y to compute these derivatives.

The transformations of Equations 4 and 5 also allow us to determine the other set of metrics, which are used much more often in finite-difference formulations than in finite-volume formulations, but are included here for completeness:

$$\begin{aligned} \xi_x &= y_\eta J \\ \xi_y &= -x_\eta J \\ \eta_x &= -y_\xi J \\ \eta_y &= x_\xi J \end{aligned} \tag{6}$$

$$\begin{aligned} \frac{1}{J} &= \begin{vmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{vmatrix} \\ &= x_\xi y_\eta - x_\eta y_\xi \end{aligned} \tag{7}$$

The Jacobian term $1/J$ is the ratio of the size of an infinitesimal area in the computational space and the area of the mapping of that infinitesimal area into the physical space.

2.2.3 The case of three dimensions

The same approach to derivative calculation can be used in three dimensions. In this case, we would have, for example,

$$\begin{aligned} \frac{\partial T}{\partial \xi}_{i+\frac{1}{2},j,k} &= \frac{\bar{T}_{i+1,j,k} - \bar{T}_{i,j,k}}{\Delta \xi} \\ \frac{\partial T}{\partial \eta}_{i+\frac{1}{2},j,k} &= \frac{\bar{T}_{i+1,j+1,k} + \bar{T}_{i,j+1,k} - \bar{T}_{i+1,j-1,k} - \bar{T}_{i,j-1,k}}{4\Delta \eta} \\ \frac{\partial T}{\partial \zeta}_{i+\frac{1}{2},j,k} &= \frac{\bar{T}_{i+1,j,k+1} + \bar{T}_{i,j,k+1} - \bar{T}_{i+1,j,k-1} - \bar{T}_{i,j,k-1}}{4\Delta \zeta} \end{aligned}$$

$$\begin{pmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \\ \frac{\partial T}{\partial z} \end{pmatrix} = \begin{bmatrix} x_\xi & y_\xi & z_\xi \\ x_\eta & y_\eta & z_\eta \\ x_\zeta & y_\zeta & z_\zeta \end{bmatrix}^{-1} \begin{pmatrix} \frac{\partial T}{\partial \xi} \\ \frac{\partial T}{\partial \eta} \\ \frac{\partial T}{\partial \zeta} \end{pmatrix} \tag{8}$$

$$= \begin{bmatrix} \xi_x & \eta_x & \zeta_x \\ \xi_y & \eta_y & \zeta_y \\ \xi_z & \eta_z & \zeta_z \end{bmatrix} \begin{pmatrix} \frac{\partial T}{\partial \xi} \\ \frac{\partial T}{\partial \eta} \\ \frac{\partial T}{\partial \zeta} \end{pmatrix} \tag{9}$$

Once again, we can find the second set of metrics analytically, but once again they are useful primarily in finite-difference settings.

$$\begin{aligned}
\xi_x &= J\left(y_\eta z_\zeta - z_\eta y_\zeta\right)\\
\xi_y &= J\left(z_\eta x_\zeta - x_\eta z_\zeta\right)\\
\xi_z &= J\left(x_\eta y_\zeta - y_\eta x_\zeta\right)\\
\eta_x &= J\left(y_\zeta z_\xi - z_\zeta y_\xi\right)\\
\eta_y &= J\left(z_\zeta x_\xi - x_\zeta z_\xi\right)\\
\eta_z &= J\left(x_\zeta y_\xi - y_\zeta x_\xi\right)\\
\zeta_x &= J\left(y_\xi z_\eta - z_\xi y_\eta\right)\\
\zeta_y &= J\left(z_\xi x_\eta - x_\xi z_\eta\right)\\
\zeta_z &= J\left(x_\xi y_\eta - y_\xi x_\eta\right)
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
\frac{1}{J} &= \begin{vmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{vmatrix} \qquad (11)\\
&= x_\xi\left(y_\eta z_\zeta - z_\eta y_\zeta\right) + x_\eta\left(y_\zeta z_\xi - z_\zeta y_\xi\right) + x_\zeta\left(y_\xi z_\eta - z_\xi y_\eta\right)
\end{aligned}$$

In 3D, the Jacobian term $1/J$ is the ratio of the size of an infinitesimal volume in the computational space and the volume of the mapping of that infinitesimal volume into the physical space.

3 Discretization of the Heat Equation on a Curvilinear Mesh

Now it's time to put together all of the solution derivative and mesh metric terms to discretize the heat equation. We're going to do this for an interior control volume, but with appropriate use of ghost cells we can of course apply this even to control volumes adjacent to the boundary.

3.1 Spatial Discretization

This is a long process, conceptually straightforward but algebraically complicated. The easiest way to approach it is face by face, using the general principle that the flux integral I

across a given face is

$$I = \nabla T \cdot \vec{n} \tag{12}$$

where the normal vector $\vec{n} \equiv \hat{n} \cdot l$. Also, we can assume without loss of generality that the mesh is defined so that $\Delta\xi = \Delta\eta = 1$.

3.1.1 Derivative definitions

Derivatives at faces are defined as follows. For the $i + \frac{1}{2}, j$ faces

$$\begin{aligned} T_\xi &= T_{i+1,j} - T_{i,j} \\ T_\eta &= \frac{T_{i+1,j+1} + T_{i,j+1} - T_{i+1,j-1} - T_{i,j-1}}{4} \end{aligned} \tag{13}$$

and

$$\begin{aligned} \vec{x}_\xi &= \frac{\vec{x}_{i+\frac{3}{2},j+\frac{1}{2}} + \vec{x}_{i+\frac{3}{2},j-\frac{1}{2}} - \vec{x}_{i-\frac{1}{2},j+\frac{1}{2}} - \vec{x}_{i-\frac{1}{2},j-\frac{1}{2}}}{4} \\ \vec{x}_\eta &= \vec{x}_{i+\frac{1}{2},j+\frac{1}{2}} - \vec{x}_{i+\frac{1}{2},j-\frac{1}{2}} \end{aligned} \tag{14}$$

The derivatives at $i - \frac{1}{2}, j$ differ only by decreasing the i index by one in each case.

For the $i, j + \frac{1}{2}$ faces, the derivatives are given by

$$\begin{aligned} T_\xi &= \frac{T_{i+1,j+1} + T_{i+1,j} - T_{i-1,j+1} - T_{i-1,j}}{4} \\ T_\eta &= T_{i,j+1} - T_{i,j} \end{aligned} \tag{15}$$

and

$$\begin{aligned} \vec{x}_\xi &= \vec{x}_{i+\frac{1}{2},j+\frac{1}{2}} - \vec{x}_{i-\frac{1}{2},j+\frac{1}{2}} \\ \vec{x}_\eta &= \frac{\vec{x}_{i+\frac{1}{2},j+\frac{3}{2}} + \vec{x}_{i-\frac{1}{2},j+\frac{3}{2}} - \vec{x}_{i+\frac{1}{2},j-\frac{1}{2}} - \vec{x}_{i-\frac{1}{2},j-\frac{1}{2}}}{4} \end{aligned} \tag{16}$$

Again, the derivatives at $i, j - \frac{1}{2}$ follow directly from these.

Finally, at each face, the derivatives in physical space and computational space are related by

$$\begin{pmatrix} T_x \\ T_y \end{pmatrix} = \frac{1}{x_\xi y_\eta - x_\eta y_\xi} \begin{bmatrix} y_\eta & -y_\xi \\ -x_\eta & x_\xi \end{bmatrix} \begin{pmatrix} T_\xi \\ T_\eta \end{pmatrix} \tag{17}$$

3.1.2 Right-hand face of the control volume

In terms of metrics at the control volume interface,

$$\vec{n} = \begin{pmatrix} y_\eta \, \Delta\eta \\ -x_\eta \, \Delta\eta \end{pmatrix} = \begin{pmatrix} y_\eta \\ -x_\eta \end{pmatrix}$$

Combining this with Equations 12, 13, 14, and 17, we get

$$I_{i+\frac{1}{2},j} = \frac{1}{x_\xi y_\eta - x_\eta y_\xi} \left[T_\xi \left(x_\eta^2 + y_\eta^2 \right) - T_\eta \left(x_\eta x_\xi + y_\eta y_\xi \right) \right]$$

where all derivatives are evaluated at $i + \frac{1}{2}, j$.

3.1.3 Left-hand face of the control volume

In terms of metrics at the control volume interface,

$$\vec{n} = \begin{pmatrix} -y_\eta \\ x_\eta \end{pmatrix}$$

Combining as before, we get

$$I_{i-\frac{1}{2},j} = -\frac{1}{x_\xi y_\eta - x_\eta y_\xi} \left[T_\xi \left(x_\eta^2 + y_\eta^2 \right) - T_\eta \left(x_\eta x_\xi + y_\eta y_\xi \right) \right]$$

where all derivatives are evaluated at $i - \frac{1}{2}, j$.

3.1.4 Top face of the control volume

In terms of metrics at the control volume interface,

$$\vec{n} = \begin{pmatrix} -y_\xi \\ x_\xi \end{pmatrix}$$

Using Equations 12, 15, 16, and 17 we get

$$I_{i,j+\frac{1}{2}} = \frac{1}{x_\xi y_\eta - x_\eta y_\xi} \left[-T_\xi \left(x_\eta x_\xi + y_\eta y_\xi \right) + T_\eta \left(x_\xi^2 + y_\xi^2 \right) \right]$$

where all derivatives are evaluated at $i, j + \frac{1}{2}$.

3.1.5 Bottom face of the control volume

In terms of metrics at the control volume interface,

$$\vec{n} = \begin{pmatrix} y_\xi \\ -x_\xi \end{pmatrix}$$

Proceeding as before, we get

$$I_{i,j-\frac{1}{2}} = -\frac{1}{x_\xi y_\eta - x_\eta y_\xi} \left[-T_\xi \left(x_\eta x_\xi + y_\eta y_\xi \right) + T_\eta \left(x_\xi^2 + y_\xi^2 \right) \right]$$

where all derivatives are evaluated at $i, j - \frac{1}{2}$.

3.1.6 A Few More Definitions

Before combining these into one big expression, let's define

$$\begin{aligned} J &\equiv x_\xi y_\eta - x_\eta y_\xi \\ S &\equiv \frac{x_\eta x_\xi + y_\eta y_\xi}{J} \\ D_{i+\frac{1}{2},j} &= \frac{\left(x_\eta^2 + y_\eta^2 \right)_{i+\frac{1}{2},j}}{J_{i+\frac{1}{2},j}} \\ D_{i,j+\frac{1}{2}} &= \frac{\left(x_\xi^2 + y_\xi^2 \right)_{i,j+\frac{1}{2}}}{J_{i,j+\frac{1}{2}}} \end{aligned}$$

3.1.7 Putting It All Together

Then the flux integral for control volume i, j is

$$\begin{aligned} \oint \left(F\hat{\imath} + G\hat{\jmath} \right) \cdot \hat{n}\, ds &= I_{i+\frac{1}{2},j} + I_{i-\frac{1}{2},j} + I_{i,j+\frac{1}{2}} + I_{i,j+\frac{1}{2}} \\ &= + \left[T_{\xi;i+\frac{1}{2},j} D_{i+\frac{1}{2},j} - T_{\eta;i+\frac{1}{2},j} S_{i+\frac{1}{2},j} \right] \\ &\quad - \left[T_{\xi;i-\frac{1}{2},j} D_{i-\frac{1}{2},j} - T_{\eta;i-\frac{1}{2},j} S_{i-\frac{1}{2},j} \right] \\ &\quad + \left[-T_{\xi;i,j+\frac{1}{2}} S_{i,j+\frac{1}{2}} + T_{\eta;i,j+\frac{1}{2}} D_{i,j+\frac{1}{2}} \right] \\ &\quad - \left[-T_{\xi;i,j-\frac{1}{2}} S_{i,j-\frac{1}{2}} + T_{\eta;i,j-\frac{1}{2}} D_{i,j-\frac{1}{2}} \right] \end{aligned}$$

This can, of course, be written as a linear combination of temperatures in nearby control volumes. For example, $T_{i+1,j+1}$ appears in $T_{\eta;i+\frac{1}{2},j}$ and in $T_{\xi;i,j+\frac{1}{2}}$. The coefficient for this term is

$$-\frac{1}{4}S_{i+\frac{1}{2},j} - \frac{1}{4}S_{i,j+\frac{1}{2}}$$

Writing out all of these terms in full is tedious, but not difficult. The result is

$$\begin{aligned}
\oint (F\hat{\imath} + G\hat{\jmath}) \cdot \hat{n}\, ds \;=\; & \bar{T}_{i+1,j+1}\left(\frac{-S_{i+\frac{1}{2},j} - S_{i,j+\frac{1}{2}}}{4}\right) \\
& + \bar{T}_{i+1,j}\left(D_{i+\frac{1}{2},j} + \frac{-S_{i,j+\frac{1}{2}} + S_{i,j-\frac{1}{2}}}{4}\right) \\
& + \bar{T}_{i+1,j-1}\left(\frac{S_{i+\frac{1}{2},j} + S_{i,j-\frac{1}{2}}}{4}\right) \\
& + \bar{T}_{i,j+1}\left(D_{i,j+\frac{1}{2}} + \frac{-S_{i+\frac{1}{2},j} + S_{i-\frac{1}{2},j}}{4}\right) \\
& + \bar{T}_{i,j}\left(-D_{i+\frac{1}{2},j} - D_{i,j+\frac{1}{2}} - D_{i-\frac{1}{2},j} - D_{i,j-\frac{1}{2}}\right) \qquad (18) \\
& + \bar{T}_{i,j-1}\left(D_{i,j-\frac{1}{2}} + \frac{S_{i+\frac{1}{2},j} - S_{i-\frac{1}{2},j}}{4}\right) \\
& + \bar{T}_{i-1,j+1}\left(\frac{S_{i-\frac{1}{2},j} + S_{i,j+\frac{1}{2}}}{4}\right) \\
& + \bar{T}_{i-1,j}\left(D_{i-\frac{1}{2},j} + \frac{S_{i,j+\frac{1}{2}} - S_{i,j-\frac{1}{2}}}{4}\right) \\
& + \bar{T}_{i-1,j-1}\left(\frac{-S_{i-\frac{1}{2},j} - S_{i,j-\frac{1}{2}}}{4}\right)
\end{aligned}$$

3.1.8 Checking the Result

The result in Equation 18 was derived for a general mesh, so it should certainly apply to the simple case of a uniform, coordinate-aligned mesh. For this special case, we have

$$\begin{aligned}
x_\xi &= \Delta x \\
x_\eta &= 0 \\
y_\xi &= 0 \\
y_\eta &= \Delta y
\end{aligned}$$

for all faces. This makes all of the S terms zero. The D terms reduce to

$$\begin{aligned} D_{i\pm\frac{1}{2},j} &\equiv \frac{x_\eta^2+y_\eta^2}{x_\xi y_\eta - x_\eta y_\xi} = \frac{\Delta y^2}{\Delta x \Delta y} = \frac{\Delta y}{\Delta x} \\ D_{i,j\pm\frac{1}{2}} &\equiv \frac{x_\xi^2+y_\xi^2}{x_\xi y_\eta - x_\eta y_\xi} = \frac{\Delta x^2}{\Delta x \Delta y} = \frac{\Delta x}{\Delta y} \end{aligned}$$

So the flux integral reduces to

$$\begin{aligned} \oint (F\hat{\imath} + G\hat{\jmath}) \cdot \hat{n}\, ds &= \frac{\Delta y}{\Delta x}\left(\bar{T}_{i+1,j} - 2\bar{T}_{i,j} + \bar{T}_{i-1,j}\right) \\ &\quad + \frac{\Delta x}{\Delta y}\left(\bar{T}_{i,j+1} - 2\bar{T}_{i,j} + \bar{T}_{i,j-1}\right) \end{aligned}$$

This is correct (recall that we have not divided the flux integral by area yet).

3.2 Implicit Time Discretization

Now we have a spatial discretization of the heat equation for 2D curvilinear meshes, but that's only half the battle: we also need to discretize in time. Typically for the heat equation we would use an implicit time discretization, because time step restrictions would make an explicit scheme impractical. After applying trapezoidal integration in time and defining some coefficients (R's below), we have:

For illustration here, we'll use the trapezoidal time advance scheme

$$\begin{aligned} A_{i,j}\frac{\bar{T}^{n+1} - \bar{T}^{n}}{\Delta t} &= \frac{\bar{T}^{n+1}_{i+1,j+1} + \bar{T}^{n}_{i+1,j+1}}{2}\left(\frac{-S_{i+\frac{1}{2},j} - S_{i,j+\frac{1}{2}}}{4}\right) \\ &\quad + \frac{\bar{T}^{n+1}_{i+1,j} + \bar{T}^{n}_{i+1,j}}{2}\left(D_{i+\frac{1}{2},j} + \frac{-S_{i,j+\frac{1}{2}} + S_{i,j-\frac{1}{2}}}{4}\right) \\ &\quad + \frac{\bar{T}^{n+1}_{i+1,j-1} + \bar{T}^{n}_{i+1,j-1}}{2}\left(\frac{S_{i+\frac{1}{2},j} + S_{i,j-\frac{1}{2}}}{4}\right) \\ &\quad + \frac{\bar{T}^{n+1}_{i,j+1} + \bar{T}^{n}_{i,j+1}}{2}\left(D_{i,j+\frac{1}{2}} + \frac{-S_{i+\frac{1}{2},j} + S_{i-\frac{1}{2},j}}{4}\right) \\ &\quad + \frac{\bar{T}^{n+1}_{i,j} + \bar{T}^{n}_{i,j}}{2}\left(-D_{i+\frac{1}{2},j} - D_{i,j+\frac{1}{2}} - D_{i-\frac{1}{2},j} - D_{i,j-\frac{1}{2}}\right) \qquad (19) \\ &\quad + \frac{\bar{T}^{n+1}_{i,j-1} + \bar{T}^{n}_{i,j-1}}{2}\left(D_{i,j-\frac{1}{2}} + \frac{S_{i+\frac{1}{2},j} - S_{i-\frac{1}{2},j}}{4}\right) \end{aligned}$$

$$
\begin{aligned}
&+\frac{\bar{T}^{n+1}_{i-1,j+1}+\bar{T}^{n}_{i-1,j+1}}{2}\left(\frac{S_{i-\frac{1}{2},j}+S_{i,j+\frac{1}{2}}}{4}\right)\\
&+\frac{\bar{T}^{n+1}_{i-1,j}+\bar{T}^{n}_{i-1,j}}{2}\left(D_{i-\frac{1}{2},j}+\frac{S_{i,j+\frac{1}{2}}-S_{i,j-\frac{1}{2}}}{4}\right)\\
&+\frac{\bar{T}^{n+1}_{i-1,j-1}+\bar{T}^{n}_{i-1,j-1}}{2}\left(\frac{-S_{i-\frac{1}{2},j}-S_{i,j-\frac{1}{2}}}{4}\right)
\end{aligned}
$$

Writing this in δ-form and moving all the δT terms to the left-hand side, we get:

$$
\begin{aligned}
-\frac{\Delta t}{2}\delta\bar{T}^{n}_{i+1,j+1}\left(\frac{-S_{i+\frac{1}{2},j}-S_{i,j+\frac{1}{2}}}{4A_{i,j}}\right) & & \Delta t\,\bar{T}^{n}_{i+1,j+1}\left(\frac{-S_{i+\frac{1}{2},j}-S_{i,j+\frac{1}{2}}}{4A_{i,j}}\right)\\
-\frac{\Delta t}{2}\delta\bar{T}^{n}_{i+1,j}\left(\frac{D_{i+\frac{1}{2},j}}{A_{i,j}}+\frac{-S_{i,j+\frac{1}{2}}+S_{i,j-\frac{1}{2}}}{4A_{i,j}}\right) & & +\Delta t\,\bar{T}^{n}_{i+1,j}\left(\frac{D_{i+\frac{1}{2},j}}{A_{i,j}}+\frac{-S_{i,j+\frac{1}{2}}+S_{i,j-\frac{1}{2}}}{4A_{i,j}}\right)\\
-\frac{\Delta t}{2}\delta\bar{T}^{n}_{i+1,j-1}\left(\frac{S_{i+\frac{1}{2},j}+S_{i,j-\frac{1}{2}}}{4A_{i,j}}\right) & & +\Delta t\,\bar{T}^{n}_{i+1,j-1}\left(\frac{S_{i+\frac{1}{2},j}+S_{i,j-\frac{1}{2}}}{4A_{i,j}}\right)\\
-\frac{\Delta t}{2}\delta\bar{T}^{n}_{i,j+1}\left(\frac{D_{i,j+\frac{1}{2}}}{A_{i,j}}+\frac{-S_{i+\frac{1}{2},j}+S_{i-\frac{1}{2},j}}{4A_{i,j}}\right) & & +\Delta t\,\bar{T}^{n}_{i,j+1}\left(\frac{D_{i,j+\frac{1}{2}}}{A_{i,j}}+\frac{-S_{i+\frac{1}{2},j}+S_{i-\frac{1}{2},j}}{4A_{i,j}}\right)\\
+\delta\bar{T}^{n}_{i,j}-\frac{\Delta t}{2}\delta\bar{T}^{n}_{i,j}\left(\frac{-D_{i+\frac{1}{2},j}-D_{i,j+\frac{1}{2}}-D_{i-\frac{1}{2},j}-D_{i,j-\frac{1}{2}}}{A_{i,j}}\right) & = & +\Delta t\,\bar{T}^{n}_{i,j}\left(\frac{-D_{i+\frac{1}{2},j}-D_{i,j+\frac{1}{2}}-D_{i-\frac{1}{2},j}-D_{i,j-\frac{1}{2}}}{A_{i,j}}\right)\\
-\frac{\Delta t}{2}\delta\bar{T}^{n}_{i,j-1}\left(\frac{D_{i,j-\frac{1}{2}}}{A_{i,j}}+\frac{S_{i+\frac{1}{2},j}-S_{i-\frac{1}{2},j}}{4A_{i,j}}\right) & & +\Delta t\,\bar{T}^{n}_{i,j-1}\left(\frac{D_{i,j-\frac{1}{2}}}{A_{i,j}}+\frac{S_{i+\frac{1}{2},j}-S_{i-\frac{1}{2},j}}{4A_{i,j}}\right)\\
-\frac{\Delta t}{2}\delta\bar{T}^{n}_{i-1,j+1}\left(\frac{S_{i-\frac{1}{2},j}+S_{i,j+\frac{1}{2}}}{4A_{i,j}}\right) & & +\Delta t\,\bar{T}^{n}_{i-1,j+1}\left(\frac{S_{i-\frac{1}{2},j}+S_{i,j+\frac{1}{2}}}{4A_{i,j}}\right)\\
-\frac{\Delta t}{2}\delta\bar{T}^{n}_{i-1,j}\left(\frac{D_{i-\frac{1}{2},j}}{A_{i,j}}+\frac{S_{i,j+\frac{1}{2}}-S_{i,j-\frac{1}{2}}}{4A_{i,j}}\right) & & +\Delta t\,\bar{T}^{n}_{i-1,j}\left(\frac{D_{i-\frac{1}{2},j}}{A_{i,j}}+\frac{S_{i,j+\frac{1}{2}}-S_{i,j-\frac{1}{2}}}{4A_{i,j}}\right)\\
-\frac{\Delta t}{2}\delta\bar{T}^{n}_{i-1,j-1}\left(\frac{-S_{i-\frac{1}{2},j}-S_{i,j-\frac{1}{2}}}{4A_{i,j}}\right) & & +\Delta t\,\bar{T}^{n}_{i-1,j-1}\left(\frac{-S_{i-\frac{1}{2},j}-S_{i,j-\frac{1}{2}}}{4A_{i,j}}\right)
\end{aligned}
\tag{20}
$$

That's quite a mouthful! To make it a little more manageable, let's replace each of the coefficients with a single symbol and replace the entire right-hand side with notation for the flux integral:

$$
\begin{aligned}
&-\frac{\Delta t}{2}R_{1;i,j}\delta T^{n}_{i-1,j+1}-\frac{\Delta t}{2}R_{2;i,j}\delta T^{n}_{i,j+1}-\frac{\Delta t}{2}R_{3;i,j}\delta T^{n}_{i+1,j+1}\\
&-\frac{\Delta t}{2}R_{4;i,j}\delta T^{n}_{i-1,j}+\left(1-\frac{\Delta t}{2}R_{5;i,j}\right)\delta T^{n}_{i,j}-\frac{\Delta t}{2}R_{6;i,j}\delta T^{n}_{i+1,j} \quad = \quad \Delta t\,(FI)^{n}_{i,j}\\
&-\frac{\Delta t}{2}R_{7;i,j}\delta T^{n}_{i-1,j-1}-\frac{\Delta t}{2}R_{8;i,j}\delta T^{n}_{i,j-1}-\frac{\Delta t}{2}R_{9;i,j}\delta T^{n}_{i+1,j-1}
\end{aligned}
\tag{21}
$$

where for the heat equation each of the R's is a function only of geometric quantities (mesh metrics and control volume sizes). In general, the R's might also contain solution data, but not for this linear problem.

3.3 Sub-iteration to Solve the Linear System

Because of the extra terms involving $i \pm 1, j \pm 1$, we can not apply approximate factorization directly to Equation 21. Instead, we resort to a frequently-used numerical technique called *sub-iteration.* First, we move the extra terms to the RHS of the equation.

$$\begin{aligned}
& & -\frac{\Delta t}{2} R_{2;i,j} \delta T^n_{i,j+1} & & & +\frac{\Delta t}{2} R_{1;i,j} \delta T^n_{i-1,j+1} + \frac{\Delta t}{2} R_{3;i,j} \delta T^n_{i+1,j+1} \\
-\frac{\Delta t}{2} R_{4;i,j} \delta T^n_{i-1,j} + \left(1 - \frac{\Delta t}{2} R_{5;i,j}\right) \delta T^n_{i,j} & & -\frac{\Delta t}{2} R_{6;i,j} \delta T^n_{i+1,j} & & = & +\Delta t \, (FI)^n_{i,j} \\
& & -\frac{\Delta t}{2} R_{8;i,j} \delta T^n_{i,j-1} & & & +\frac{\Delta t}{2} R_{7;i,j} \delta T^n_{i-1,j-1} + \frac{\Delta t}{2} R_{9;i,j} \delta T^n_{i+1,j-1}
\end{aligned} \tag{22}$$

Now we can factor the LHS using our usual approximate factorization techniques. For the RHS, we begin by guessing that $\delta T^{(0)}_{i,j} = 0$ for all i, j; this is as good a guess as any, and better than most because we know that this guess is exactly right as we approach steady state. Then we solve Equation 22 to find $\delta T^{(1)}_{i,j}$.

$$\begin{aligned}
& & -\frac{\Delta t}{2} R_{2;i,j} \delta T^{(k)}_{i,j+1} & & & +\frac{\Delta t}{2} R_{1;i,j} \delta T^{(k-1)}_{i-1,j+1} + \frac{\Delta t}{2} R_{3;i,j} \delta T^{(k-1)}_{i+1,j+1} \\
-\frac{\Delta t}{2} R_{4;i,j} \delta T^{(k)}_{i-1,j} + \left(1 - \frac{\Delta t}{2} R_{5;i,j}\right) \delta T^{(k)}_{i,j} & & -\frac{\Delta t}{2} R_{6;i,j} \delta T^{(k)}_{i+1,j} & & = & +\Delta t \, (FI)_{i,j} \\
& & -\frac{\Delta t}{2} R_{8;i,j} \delta T^{(k)}_{i,j-1} & & & +\frac{\Delta t}{2} R_{7;i,j} \delta T^{(k-1)}_{i-1,j-1} + \frac{\Delta t}{2} R_{9;i,j} \delta T^{(k-1)}_{i+1,j-1}
\end{aligned} \tag{23}$$

We can repeat this process, using the most recent approximation to $\delta T_{i,j}$ on the RHS at each sub-iteration, until we converge to a final value of $\delta T_{i,j}$ (that is, until $\delta T^{(k)}$ and $\delta T^{(k-1)}$ are the same to within some small tolerance). Under most circumstances, the effects of the "corner" terms is small (the mesh is fairly close to being orthogonal), and the subiteration process converges within only a few inner iterations. Subiteration gives us the solution update needed to advance for t to $t + \Delta t$; we have to repeat the entire process for each time step as we continue to march in time.