Assignment 2: Unstructured Meshes

Mech 588

Due Date: March 6

1. In class, we discussed the general principles involved in setting up an unstructured mesh discretization. In this problem, you will construct a second-order accurate finite-volume discretization for the wave equation on an unstructured triangular mesh, step-by-step. You'll do this in the context of calculating the flux across a single control volume interface, because that calculation, repeated for each face, will give you the complete flux integral.

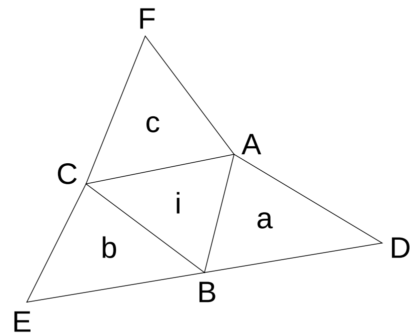
   The wave equation, in control volume form, looks like this, of course:

$$A_i \frac{d\overline{T}_i}{dt} = -\oint_i \vec{u}T \cdot \hat{n}\, ds$$

   (a) (3 marks) To achieve second-order accuracy, you need to have a second-order accurate value of the flux at the control volume boundary, so you need a second-order accurate approximation to the solution, which implies that you need to know the gradient of the solution in each CV.

   Analytically find the least-squares estimate of the gradient in control volume i in terms of the solution values in i, a, b, and c. (*Hint:* when you reach the point where you have a non-square system of equations to solve, multiply both sides of your equation by the transpose of the 2×3 matrix on the LHS to get a 2×2 system.) Find the numerical value of the gradient if $\bar{T}_i = 100$, $\bar{T}_a = 102$, $\bar{T}_b = 97$, and $\bar{T}_c = 101$.

   (b) (2 marks) Now assume that you've calculated the gradient in every cell, so that you know not only the average value of the solution $\bar{T}$, but also the gradient of the solution ∇T in each control volume. Find the values of T on both sides of interface AB at its midpoint.



Vertex	x	y
A	3.25	1.25
B	3	0.25
C	2	1
D	4.5	0.5
E	1.5	0
F	2.5	2.25

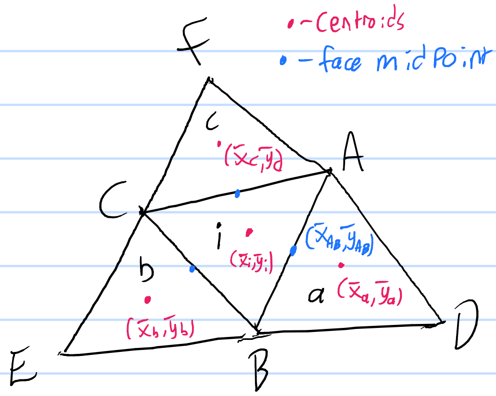
Figure 1: Unstructured mesh fragment, including vertex coordinates.

(c) (2 marks) Given $T_{AB,\text{left}}$ and $T_{AB,\text{right}}$, you need to calculate a flux at the midpoint; for conservation, that flux must be the same regardless of whether you're calculating the flux integral for CV i or a. The simplest approach, as we discussed in class, is to compute the flux once for each interface. For a given velocity field $\vec{u}(x, y)$, how would you determine the upwind flux at the midpoint of AB?

(d) (1 marks) Now that you have the flux at the midpoint of AB, how would you estimate the total flux across that interface, and how would that flux contribute to the flux integrals in control volumes i and a?

And that, as they say, is that. What you've just outlined is all the calculation needed to successfully set up a finite-volume solver for the wave equation on a unstructured mesh! Well, okay, this assignment doesn't talk about time advance, but for the wave equation you could simply choose something explicit, like Runge-Kutta. The assignment also glosses over issues of mesh connectivity information, which is "only" bookkeeping, although admittedly a fairly large amount of bookkeeping.

We want to solve the wave equation on an unstructured mesh, with 2^{nd} order accuracy. We will look at the following control volume i, with neighbouring cells $a, b,$ & c



We will be using a cell centered control volume. We also are aiming for 2^{nd} order accuracy, so we need 1 point for our polynomial.

We wish to compute a linear approximation of T_i, which we can write as follows.

$$T_i^R(x,y) = T_i + \frac{\partial T}{\partial x}\Big|_i (x - x_i) + \frac{\partial T}{\partial y}\Big|_i (y - y_i) + O(h^2)$$

Where $T_i^R(x,y)$ represents our linear approximation of the solution, & the R.H.S. represents a Taylor series expansion about our control volume reference location (x_i, y_i), truncated to second order accuracy. As we are using a cell centered volume, we will select our reference location to be the cell centroid.

We must constrain our reconstruction such that the average of our linear reconstruction matches the control volume average

$$\bar{T}_i = \frac{1}{A_i}\int_i T_i^R(x,y)\,dx\,dy$$

$$\bar{T}_i = \frac{1}{A_i}\int T_i + \frac{\partial T}{\partial x}\Big|_i (x-x_i) + \frac{\partial T}{\partial y}\Big|_i (y-y_i)\,dx\,dy$$

$$\bar{T}_i = T_i + \frac{1}{A_i}\frac{\partial T}{\partial x}\Big|_i \iint_i x - x_i\,dx\,dy + \frac{1}{A_i}\frac{\partial T}{\partial y}\Big|_i \int_i y - y_i\,dx\,dy$$

$$\bar{T}_i = T_i + \frac{\partial T}{\partial x}\Big|_i \bar{X}_i + \frac{\partial T}{\partial y}\Big|_i \bar{y}_i \quad \text{where } \bar{X}_i = \frac{1}{A_i}\int_i x - x_i\,dx\,dy \ \& \ \bar{y}_i = \frac{1}{A_i}\int_i y - y_i\,dx\,dy$$

But, remember, we have selected the centroid of the control volume as our reference location! Therefore $x = x_i$ & $y = y_i$. As such $\bar{X}_i = \bar{y}_i = 0$. Finally we get

$$\bar{T}_i = T_i$$

We also need the control volume averages for our neighbouring cells a, b, & c. If our reconstructed function is accurate, it will predict the control volume averages in all cells in our stencil for i.

$$\frac{1}{A_a}\iint_{CV_a} T_i^R(x,y)\,dA \approx \bar{T}_a = T_i + \left.\frac{\partial T}{\partial x}\right|_i \underbrace{(\bar{x}_a - \bar{x}_i)}_{\Delta x_a} + \left.\frac{\partial T}{\partial y}\right|_i \underbrace{(\bar{y}_a - \bar{y}_i)}_{\Delta y_a}$$

$$\frac{1}{A_b}\iint_{CV_b} T_i^R(x,y)\,dA \approx \bar{T}_b = T_i + \left.\frac{\partial T}{\partial x}\right|_i \Delta x_b + \left.\frac{\partial T}{\partial y}\right|_i \Delta y_b$$

$$\frac{1}{A_c}\iint_{CV_c} T_i^R(x,y)\,dA \approx \bar{T}_c = T_i + \left.\frac{\partial T}{\partial x}\right|_i \Delta x_c + \left.\frac{\partial T}{\partial y}\right|_i \Delta y_c$$

We can write this in matrix form

$$\begin{pmatrix} 1 & \Delta x_a & \Delta y_a \\ 1 & \Delta x_b & \Delta y_b \\ 1 & \Delta x_c & \Delta y_c \end{pmatrix} \begin{pmatrix} T_i \\ \left.\frac{\partial T}{\partial x}\right|_i \\ \left.\frac{\partial T}{\partial y}\right|_i \end{pmatrix} = \begin{pmatrix} \bar{T}_a \\ \bar{T}_b \\ \bar{T}_c \end{pmatrix}$$

But wait! Remember, we had 1 other constraint. The average of our reconstruction must match our control volume average so $\bar{T}_i = T_i$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & \Delta x_a & \Delta y_a \\ 1 & \Delta x_b & \Delta y_b \\ 1 & \Delta x_c & \Delta y_c \end{pmatrix} \begin{pmatrix} T_i \\ \left.\frac{\partial T}{\partial x}\right|_i \\ \left.\frac{\partial T}{\partial y}\right|_i \end{pmatrix} \approx \begin{pmatrix} \bar{T}_i \\ \bar{T}_a \\ \bar{T}_b \\ \bar{T}_c \end{pmatrix}$$

We can't solve this system of equations exactly because we have more equations than unknowns. We can solve the mean constraint ($\bar{T}_i = T_i$) exactly, & approximate the others. Using Gauss elimination, we get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \Delta x_a & \Delta y_a \\ 0 & \Delta x_b & \Delta y_b \\ 0 & \Delta x_c & \Delta y_c \end{pmatrix} \begin{pmatrix} T_i \\ \left.\frac{\partial T}{\partial x}\right|_i \\ \left.\frac{\partial T}{\partial y}\right|_i \end{pmatrix} = \begin{pmatrix} \bar{T}_i \\ \bar{T}_a \\ \bar{T}_b \\ \bar{T}_c \end{pmatrix}$$

Seperating out the equations wrapped in red

$$\underbrace{\begin{pmatrix} \Delta x_a & \Delta y_a \\ \Delta x_b & \Delta y_b \\ \Delta x_c & \Delta y_c \end{pmatrix}}_{A} \begin{pmatrix} \frac{\partial T}{\partial x}\Big|_i \\ \frac{\partial T}{\partial y}\Big|_i \end{pmatrix} = \begin{pmatrix} \bar{T}_a - \bar{T}_i \\ \bar{T}_b - \bar{T}_i \\ \bar{T}_c - \bar{T}_i \end{pmatrix}$$

Multiply both sides by $A^T = \begin{pmatrix} \Delta x_a & \Delta x_b & \Delta x_c \\ \Delta y_a & \Delta y_b & \Delta y_c \end{pmatrix}$

$$\begin{pmatrix} \Delta x_a & \Delta x_b & \Delta x_c \\ \Delta y_a & \Delta y_b & \Delta y_c \end{pmatrix} \begin{pmatrix} \Delta x_a & \Delta y_a \\ \Delta x_b & \Delta y_b \\ \Delta x_c & \Delta y_c \end{pmatrix} \begin{pmatrix} \frac{\partial T}{\partial x}\Big|_i \\ \frac{\partial T}{\partial y}\Big|_i \end{pmatrix} = \begin{pmatrix} \Delta x_a & \Delta x_b & \Delta x_c \\ \Delta y_a & \Delta y_b & \Delta y_c \end{pmatrix} \begin{pmatrix} \bar{T}_a - \bar{T}_i \\ \bar{T}_b \quad \bar{T}_i \\ \bar{T}_c \quad \bar{T}_i \end{pmatrix}$$

$$\begin{pmatrix} \Delta x_a^2 + \Delta x_b^2 + \Delta x_c^2 & \Delta x_a \Delta y_a + \Delta x_b \Delta y_b + \Delta x_c \Delta y_c \\ \Delta y_a \Delta x_a + \Delta y_b \Delta x_b + \Delta y_c \Delta x_c & \Delta y_a^2 + \Delta y_b^2 + \Delta y_c^2 \end{pmatrix} \begin{pmatrix} \frac{\partial T}{\partial x}\Big|_i \\ \frac{\partial T}{\partial y}\Big|_i \end{pmatrix} = \begin{pmatrix} \Delta x_a(\bar{T}_a - \bar{T}_i) + \Delta x_b(\bar{T}_b - \bar{T}_i) + \Delta x_c(\bar{T}_c - \bar{T}_i) \\ \Delta y_a(\bar{T}_a - \bar{T}_i) + \Delta y_b(\bar{T}_b - \bar{T}_i) + \Delta y_c(\bar{T}_c - \bar{T}_i) \end{pmatrix}$$

Let $\Delta x_a^2 + \Delta x_b^2 + \Delta x_c^2 = \Delta X^2$, & $\Delta y_a^2 + \Delta y_b^2 + \Delta y_c^2 = \Delta Y^2$ Let $\Delta x_a(\bar{T}_a - \bar{T}_i) + \Delta x_b(\bar{T}_b - \bar{T}_i) + \Delta x_c(\bar{T}_c - \bar{T}_i) = \Delta X \bar{T}$

& $\Delta x_a \Delta y_a + \Delta x_b \Delta y_b + \Delta x_c \Delta y_c = \Delta X Y$ Let $\Delta y_a(\bar{T}_a - \bar{T}_i) + \Delta y_b(\bar{T}_b - \bar{T}_i) + \Delta y_c(\bar{T}_c - \bar{T}_i) = \Delta Y \bar{T}$

$$\underbrace{\begin{pmatrix} \Delta X^2 & \Delta X Y \\ \Delta X Y & \Delta Y^2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \frac{\partial T}{\partial x}\Big|_i \\ \frac{\partial T}{\partial y}\Big|_i \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} \Delta X \bar{T} \\ \Delta Y \bar{T} \end{pmatrix}}_{b}$$

We now have a simple 2×2 system of equations $Ax = b$

We can rewrite this as $x = A^{-1} b$ where

$$A^{-1} = \frac{1}{\Delta X^2 \Delta Y^2 - (\Delta X Y)^2} \begin{pmatrix} \Delta Y^2 & -\Delta X Y \\ \Delta X Y & \Delta X^2 \end{pmatrix}$$

$$\begin{pmatrix} \left.\frac{\partial T}{\partial x}\right|_i \\ \left.\frac{\partial T}{\partial y}\right|_i \end{pmatrix} = \frac{1}{\Delta X^2 \Delta Y^2 - (\Delta XY)^2} \begin{pmatrix} \Delta Y^2 & -\Delta XY \\ \Delta XY & \Delta X^2 \end{pmatrix} \begin{pmatrix} \Delta XT \\ \Delta YT \end{pmatrix}$$

$$\begin{pmatrix} \left.\frac{\partial T}{\partial x}\right|_i \\ \left.\frac{\partial T}{\partial y}\right|_i \end{pmatrix} = \frac{1}{\Delta X^2 \Delta Y^2 - (\Delta XY)^2} \begin{pmatrix} \Delta Y^2(\Delta XT) - \Delta XY(\Delta YT) \\ \Delta XY(\Delta XT) + \Delta X^2(\Delta YT) \end{pmatrix}$$

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a) Knowing $\bar{T}_i = 100$, $\bar{T}_a = 102$, $\bar{T}_b = 97$ & $\bar{T}_c = 101$

calculate the gradient

Remember, the following coordinates

Table 1

Vertex	x	y
A	3.25	1.25
B	3	0.25
C	2	1
D	4.5	0.5
E	1.5	0
F	2.5	2.25
$\bar{i}$	2.75	0.8333
$\bar{a}$	3.5833	0.667
$\bar{b}$	2.167	0.4167
$\bar{c}$	2.583	1.5
$\overline{AB}$	3.125	0.75

Using the information found on the previous page, the gradient can be calculated to be

$$\begin{pmatrix} \left.\frac{\partial T}{\partial x}\right|_i \\ \left.\frac{\partial T}{\partial y}\right|_i \end{pmatrix} \simeq \begin{pmatrix} 3.0751 \\ 2.4847 \end{pmatrix}$$

b) In part a), our reference location $\bar{X}_i$ was at the centroid of i. Now, we can calculate our gradient based on the reference location being at cell a. The only difference will be the reference locations.

$$\frac{1}{A_a}\iint_{CV_i} T_a^R(x,y)\,dA \approx \bar{T}_i = T_a + \left.\frac{\partial T}{\partial x}\right|_i \underbrace{(\bar{x}_i - \bar{x}_a)}_{\Delta x_i} + \left.\frac{\partial T}{\partial y}\right|_i \underbrace{(\bar{y}_i - \bar{y}_a)}_{\Delta y_i}$$

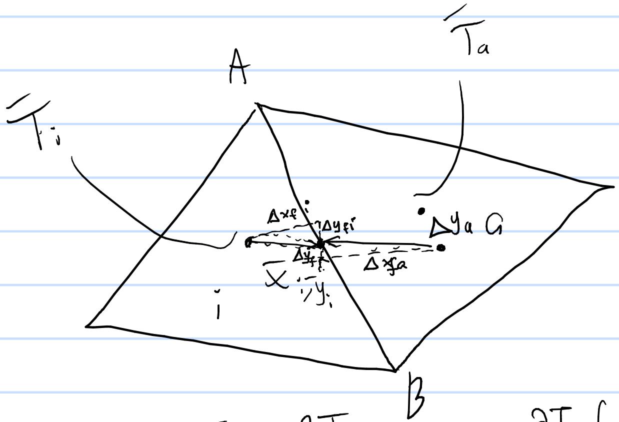
$$\frac{1}{A_b}\iint_{CV_b} T_a^R(x,y)\,dA \approx \bar{T}_b = T_a + \left.\frac{\partial T}{\partial x}\right|_i \Delta x_b + \left.\frac{\partial T}{\partial y}\right|_i \Delta y_b$$

$$\frac{1}{A_c}\iint_{CV_c} T_a^R(x,y)\,dA \approx \bar{T}_c = T_a + \left.\frac{\partial T}{\partial x}\right|_i \Delta x_c + \left.\frac{\partial T}{\partial y}\right|_i \Delta y_c$$

The above equations can be transformed into matrix form & solved in a similar manner to what was just done. With this, the following gradient can be determined

$$\begin{pmatrix} \left.\frac{\partial T}{\partial x}\right|_a \\ \left.\frac{\partial T}{\partial y}\right|_a \end{pmatrix} = \begin{pmatrix} 3.0535 \\ 2.5111 \end{pmatrix}$$

Now that we have the gradients on both sides, we can calculate the temperature at the midpoint of AB from both sides.



Using the values in table 1, & the gradients we have calculated.

$$T_i = \bar{T}_i + \frac{\partial T}{\partial x}\Big|_i \Delta x_{fi} + \frac{\partial T}{\partial y}\Big|_i \Delta y_{fi} = 100.95$$

$$T_a = \bar{T}_a + \frac{\partial T}{\partial x}\Big|_a \Delta x_{fa} + \frac{\partial T}{\partial y}\Big|_a \Delta y_{fa} = 100.85$$

C) In order to determine the upwind flux at the mid point of AB, we can compute the dot product of the velocity & the normal pointing from cell i into cell a.

$$\hat{n}_{ia} = \frac{1}{l_{AB}}\begin{pmatrix} y_A - y_B \\ -x_A + x_B \end{pmatrix}, \quad \vec{u}_{AB} = \vec{u}\left(\frac{x_A + x_B}{2}, \frac{y_A + y_B}{2}\right)$$

If this dot product is positive, we know that the flow goes from cell i into cell a. We can then compute the flux using the normal velocity multiplied by the solution in the upwind cell.

$$F_{AB} = \begin{cases} (\hat{n}_{ia} \cdot \vec{u}_{AB})\, T_i & \text{if } \hat{n}_{ia} \cdot \vec{u}_{AB} > 0 \\ (\hat{n}_{ia} \cdot \vec{u}_{AB})\, T_a & \text{if } \hat{n}_{ia} \cdot \vec{u}_{AB} < 0 \end{cases}$$

d) In order to estimate the total flux across the interface, we simply multiply our calculated flux by the length of the cell interface. To determine the contributions of t e flux to the total flux integral for each cell, we will add it to the flux integral of the cell that the flow enters, & subtract it from the flux integral of the cell that flow exits.