

1 Definitions and Notations

1.1 Quaternions

- Reference(s)
 - ◊ [1]
- Definition(s)
 - ◊ $\mathcal{A} = \llbracket a, \vec{A} \rrbracket$
 - ◊ a : a real scalar number
 - ◊ \vec{A} : a vector in \mathbb{R}^3
 - ◊ Quaternion algebra obeys to all the usual arithmetical laws except that the multiplication is non-commutative
- Notation(s)
 - ◊ \mathcal{A} : a quaternion
 - ◊ $\llbracket a, \vec{A} \rrbracket$: a quaternion composed of a scalar a and a vector \vec{A}
 - ◊ $\vec{A} = (A_x, A_y, A_z)$
 - ◊ $\llbracket a, (A_x, A_y, A_z) \rrbracket$: other representation of the scalar/vector components of the quaternion
 - ◊ $\|\mathcal{A}\|$: norm of the quaternion
 - ◊ \mathcal{A}^* : conjugate of the quaternion
 - ◊ \mathcal{A}^{-1} : inverse of the quaternion
- Advantage(s)
 - ◊ Easy to determine geometrically as $R(\phi\vec{n})$
 - ◊ They are the only parameters for which the group multiplication rule can be given in closed form
 - ◊ They behave correctly near the identity whereas the euler angles become undetermined for $\beta = 0$ and $\beta = \pi$
 - ◊ They uniquely determine rotation poles with the convention of the positive hemi-sphere
 - * $z > 0$
 - * if $z = 0$, $x > 0$
 - * if $z = 0$ and $x = 0$, $y > 0$
 - ◊ Keep track of 2π turns introduced on multiplying rotations

1.2 Euler Angles

- Reference(s)
 - ◊ [3]
- Definition(s)
 - ◊ From Bunge
 - ◊ Rotation performed from $\theta_1 \rightarrow \theta_2 \rightarrow \theta_3$ where:
 1. θ_1 is a counter-clockwise rotation around the \vec{z} axis [001]: $R(\theta_1\vec{z})$
 2. θ_2 is a counter-clockwise rotation around the \vec{x} axis [100]: $R(\theta_2\vec{x})$
 3. θ_3 is a counter-clockwise rotation around the \vec{z} axis [001]: $R(\theta_3\vec{z})$
 - ◊ Limits
 - * $-\pi < \theta_1 \leq \pi$
 - * $0 \leq \theta_2 \leq \pi$
 - * $-\pi < \theta_3 \leq \pi$

2 Algebra

2.1 Product

- Reference(s)
 - ◊ [\[1\]](#)
- Properties
 - ◊ Non-commutative
 - * $\mathcal{AB} \neq \mathcal{BA}$
 - ◊ Associative
 - * $(\mathcal{AB})\mathcal{C} = \mathcal{A}(\mathcal{BC})$
 - ◊ Scalar product
 - * $a[[b, \vec{B}]] = [[a, \vec{0}]][[b, \vec{B}]] = [[ab, a\vec{B}]]$
 - ◊ Quaternion product
 - * $\mathcal{C} = \mathcal{AB} = [[a, \vec{A}]][[b, \vec{B}]] = [[ab - \vec{A} \bullet \vec{B}, a\vec{B} + b\vec{A} + \vec{A} \times \vec{B}]]$
 - $c = ab - \vec{A} \bullet \vec{B}$
 - $C_x = aB_x + bA_x + (\vec{A} \times \vec{B})_x$
 - $C_y = aB_y + bA_y + (\vec{A} \times \vec{B})_y$
 - $C_z = aB_z + bA_z + (\vec{A} \times \vec{B})_z$
- Derivation(s)
 - ◊ [\[1\]](#) or <http://www.euclideanspace.com/maths/algebra/realNormedAlgebra/quaternions/arithmetic/index.htm>

2.2 Division

- Reference(s)
 - ◊ [\[1\]](#)
- Properties
 - ◊ Scalar division
 - * $\frac{[[a, \vec{A}]]}{a} = \frac{1}{a}[[a, \vec{A}]]$
 - ◊ Quaternion division
 - * $\frac{[[a, \vec{A}]]}{[[b, \vec{B}]]} \equiv [[a, \vec{A}]][[b, \vec{B}]]^{-1}$
 - * We defined this relationship, since $\frac{\mathcal{A}}{\mathcal{B}}$ could be equal to \mathcal{AB}^{-1} or $\mathcal{B}^{-1}\mathcal{A}$ which doesn't respect the non-commutative rule
 - * $[[a, \vec{A}]][[b, \vec{B}]]^{-1} = [[ab + \vec{A} \bullet \vec{B}, b\vec{A} - a\vec{B} - \vec{A} \times \vec{B}]]$
- Derivation(s)
 - ◊ [\[1\]](#) or <http://www.euclideanspace.com/maths/algebra/realNormedAlgebra/quaternions/arithmetic/index.htm>

2.3 Addition / Substraction

- Reference(s)
 - ◊ [1]
- Properties
 - ◊ Addition
 - * $\llbracket a, \vec{A} \rrbracket + \llbracket b, \vec{B} \rrbracket = \llbracket a + b, \vec{A} + \vec{B} \rrbracket$
 - ◊ Subtraction
 - * $\llbracket a, \vec{A} \rrbracket - \llbracket b, \vec{B} \rrbracket = \llbracket a - b, \vec{A} - \vec{B} \rrbracket$
- Derivation(s)
 - ◊ [1] or <http://www.euclideanspace.com/maths/algebra/realNormedAlgebra/quaternions/arithmetic/index.htm>

2.4 Conjugate

- Reference(s)
 - ◊ [1]
- Definition(s)
 - ◊ $\mathcal{A}^* = \llbracket a, \vec{A} \rrbracket^* = \llbracket a, -\vec{A} \rrbracket$
- Derivation(s)
 - ◊ [1] or <http://www.euclideanspace.com/maths/algebra/realNormedAlgebra/quaternions/functions/index.htm>

2.5 Norm

- Reference(s)
 - ◊ [1] and [2]
- Definition(s)
 - ◊ $\|\mathcal{A}\| = \mathcal{A}\mathcal{A}^*$
 - ◊ $\|\llbracket a, \vec{A} \rrbracket\| = \llbracket a, \vec{A} \rrbracket \llbracket a, \vec{A} \rrbracket^*$
 - ◊ $\|\llbracket a, \vec{A} \rrbracket\| = \sqrt{a^2 + A_x^2 + A_y^2 + A_z^2}$
- Properties
 - ◊ $\mathcal{A} = \llbracket \cos \alpha, \sin \alpha \vec{n} \rrbracket = \cos \alpha + \sin \alpha \mathcal{N}$, $\|\vec{n}\| = 1$
 - ◊ or $\mathcal{A} = \llbracket \cos \frac{\phi}{2}, \sin \frac{\phi}{2} \vec{n} \rrbracket$ (as in Euler-Rodrigues)
 - ◊ The product of 2 normalized quaternions is itself a normalized quaternion
- Derivation(s)
 - ◊ [1] or <http://www.euclideanspace.com/maths/algebra/realNormedAlgebra/quaternions/functions/index.htm>

2.6 Inverse

- Reference(s)

◊ [1]

- Definition(s)

◊ $\mathcal{A}^{-1} = \mathcal{A}^* \|\mathcal{A}\|^{-2}$

◊ $\mathcal{C} = \mathcal{A}\mathcal{B}^{-1} = \mathcal{A}\mathcal{B}^* \|\mathcal{B}\|^{-2}$ if $\mathcal{B} \neq \llbracket 0, \vec{0} \rrbracket$

◊ For normalized quaternion, $\mathcal{A}^{-1} = \mathcal{A}^*$

- Derivation(s)

◊ [1] or <http://www.euclideanspace.com/maths/algebra/realNormedAlgebra/quaternions/functions/index.htm>

2.7 Equality

- Definition(s)

◊ Two quaternions (\mathcal{A} and \mathcal{B}) are equal if and only if

* $a = b$

* $A_x = B_x$

* $A_y = B_y$

* $A_z = B_z$

3 Conversion

3.1 Axis Angle to Quaternion

- Reference(s)

◇ [2]

- Definition(s)

◇ ϕ : Rotation angle

◇ \vec{n} : Rotation axis

- Equation(s)

◇ $(\phi, \vec{n}) \rightarrow \llbracket \cos \frac{1}{2} \phi, \frac{\sin \frac{1}{2} \phi}{\|\vec{n}\|} \vec{n} \rrbracket$

- Derivation(s)

◇ <http://www.euclideanspace.com/maths/geometry/rotations/conversions/angleToQuaternion/index.htm>

3.2 Matrix to Quaternion

- Reference(s)

◇ [2]

- Definition(s)

◇ m : a 3×3 orthogonal matrix (i.e. SO3)

* $\text{Det}(m) = 1$

* $\text{Tr}(m) > 0$

◇ m_{ij} : element of matrix m in row i and column j

- Equation(s)

◇ $a = \frac{1}{2} \sqrt{1 + m_{00} + m_{11} + m_{22}} = \frac{1}{2} \sqrt{1 + \text{Tr}(m)}$

◇ $A_x = \frac{m_{21} - m_{12}}{4a}$

◇ $A_y = \frac{m_{02} - m_{20}}{4a}$

◇ $A_z = \frac{m_{10} - m_{01}}{4a}$

- Derivation(s)

◇ <http://www.euclideanspace.com/maths/geometry/rotations/conversions/matrixToQuaternion/index.htm>

3.3 Euler Angles to Quaternion

- Reference(s)

◇ Adaptation from [2], [3] and [4]

- Definition(s)

◇ $\theta_1, \theta_2, \theta_3$: see 1.2

◇ $c_i \equiv \cos(\frac{1}{2}\theta_i)$ and $s_i \equiv \sin(\frac{1}{2}\theta_i)$

- Equation(s)

- ◇ $a = c_1 c_2 c_3 - s_1 c_2 s_3$
- ◇ $A_x = c_1 s_2 c_3 + s_1 s_2 s_3$
- ◇ $A_y = c_1 s_2 s_3 - s_1 s_2 c_3$
- ◇ $A_z = c_1 c_2 s_3 + s_1 c_2 c_3$

- Derivation(s)

- ◇ Three rotations
 1. $Q_1 = \llbracket c_1, (0, 0, s_1) \rrbracket$
 2. $Q_2 = \llbracket c_2, (s_2, 0, 0) \rrbracket$
 3. $Q_3 = \llbracket c_3, (0, 0, s_3) \rrbracket$
- ◇ Overall rotation
 - * $Q_T = Q_3 Q_2 Q_1 = (Q_3 Q_2) Q_1 = Q_3 (Q_2 Q_1)$

3.4 Quaternion to Axis Angle

- Reference(s)

- ◇ [\[2\]](#)

- Equation(s)

- ◇ $\|\llbracket a, A \rrbracket\| = 1$ (the quaternion has to be normalized)
- ◇ $d = \sqrt{1 - a^2}$
- ◇ $\phi = 2 \arccos a$
- ◇ If $d \neq 0$: $\vec{n} = \frac{1}{d} \vec{A}$
- ◇ If $d = 0$: $\vec{n} = \vec{A}$

- Derivation(s)

- ◇ <http://www.euclideanspace.com/maths/geometry/rotations/conversions/quaternionToAngle/index.htm>

3.5 Quaternion to Matrix

- Reference(s)

- ◇ [\[2\]](#)

- Equation(s)

- ◇ $\|\llbracket a, A \rrbracket\| = 1$ (the quaternion has to be normalized)
- ◇ $m = \begin{pmatrix} 1 - 2A_y^2 - 2A_z^2 & 2A_x A_y - 2A_z a & 2A_x A_z + 2A_y a \\ 2A_x A_y + 2A_z a & 1 - 2A_x^2 - 2A_z^2 & 2A_y A_z - 2A_x a \\ 2A_x A_z - 2A_y a & 2A_y A_z + 2A_x a & 1 - 2A_x^2 - 2A_y^2 \end{pmatrix}$

- Derivation(s)

- ◇ <http://www.euclideanspace.com/maths/geometry/rotations/conversions/quaternionToMatrix/index.htm>

3.6 Quaternion to Euler Angles

- Reference(s)
 - ◊ Adaptation from [2], [3] and [4]
- Definition(s)
 - ◊ $\theta_1, \theta_2, \theta_3$: see 1.2
 - ◊ $c_i \equiv \cos\left(\frac{1}{2}\theta_i\right)$ and $s_i \equiv \sin\left(\frac{1}{2}\theta_i\right)$
- Equation(s)
 - ◊ $\|a, A\| = 1$ (the quaternion has to be normalized)
 - ◊ Three cases
 1. $\theta_2 = 0$ ($A_x = A_y = 0$)
 - * $\theta_1 = 2 \arctan\left(\frac{A_z}{a}\right)$
 - * $\theta_2 = 0$
 - * $\theta_3 = 0$
 2. $\theta_2 = \pi$ ($A_x^2 + A_y^2 = 1$)
 - * $\theta_1 = 2 \arctan\left(\frac{A_y}{A_x}\right)$
 - * $\theta_2 = \pi$
 - * $\theta_3 = 0$
 3. $0 < \theta_2 < \pi$
 - * $\theta_1 = \arctan\left(\frac{A_x A_z - A_y a}{A_y A_z + A_x a}\right)$
 - * $\theta_2 = \arccos(1 - 2A_x^2 - A_y^2)$
 - * $\theta_3 = \arctan\left(\frac{A_x A_z + A_y a}{A_x a - A_y A_z}\right)$
- Derivation(s)
 - ◊ From the Euler to Matrix we get

$$* m_{\text{euler}} = \begin{pmatrix} c_1 c_3 - s_1 c_2 s_3 & -s_1 c_3 - c_1 c_2 s_3 & s_2 s_3 \\ s_1 c_2 c_3 + c_1 s_3 & c_1 c_2 c_3 - s_1 s_3 & -s_2 c_3 \\ s_1 s_2 & c_1 s_2 & c_2 \end{pmatrix}$$
 - * We can get the euler angles from these relationships
 - $\tan \theta_1 = \frac{m_{20}}{m_{21}} = \frac{s_1}{c_1} = \frac{s_1 s_2}{c_1 s_2}$
 - $\cos \theta_2 = m_{22}$
 - $\tan \theta_3 = -\frac{m_{02}}{m_{12}} = -\frac{s_3}{c_3} = -\frac{s_2 s_3}{s_2 c_3}$
 - ◊ By comparing with the quaternion matrix (see 3.5), we get replace the m_{ij} by the quaternion coefficients
 - ◊ For the 2 special cases, where the c_2 is not defined
 1. $\theta_2 = 0$
 - * This condition happens when
 - $\arccos 0 = 1 = 1 - 2A_x^2 - 2A_y^2$
 - $A_x^2 = -A_y^2$
 - $A_x = A_y = 0$
 - * $\theta_2 = 0$ implies that $\cos\left(\frac{1}{2}0\right) = c_2 = 1$ and $\sin\left(\frac{1}{2}0\right) = s_2 = 0$
 - * From the Euler to quaternion conversion (see 3.3)
 - $a = c_1 c_2 c_3 - s_1 c_2 s_3 = c_1 c_3 - s_1 s_3$
 - $A_x = 0$

- $A_y = 0$
 - $A_z = c_1 c_2 s_3 + s_1 c_2 c_3 = c_1 s_3 + s_1 c_3$
 - * Using the trigonometry identities:
 - $\cos(A + B) = \cos A \cos B - \sin A \sin B$
 - $\sin(A + B) = \sin A \cos B + \cos A \sin B$
 - * We obtained
 - $a = \cos(1 + 3)$
 - $A_z = \sin(1 + 3)$
 - * By dividing A_z by a :
 - $\frac{A_z}{a} = \frac{\sin(1+3)}{\cos(1+3)} = \tan(1 + 3)$
 - $\frac{\theta_1}{2} + \frac{\theta_3}{2} = \arctan\left(\frac{A_z}{a}\right)$
 - $\theta_1 + \theta_3 = 2 \arctan\left(\frac{A_z}{a}\right)$
 - * Since “only $(\theta_1 + \theta_3)$ is uniquely defined (not the individual values)” [4] when $\theta_2 = 0$
 - $\theta_1 = 2 \arctan\left(\frac{A_z}{a}\right)$
 - $\theta_2 = 0$ (from the special case)
 - $\theta_3 = 0$
2. $\theta_2 = \pi$
- * This condition happens when
 - $\arccos \pi = -1 = 1 - 2A_x^2 - 2A_y^2$
 - $A_x^2 + A_y^2 = 1$
 - Since the quaternion is normalized ($a^2 + A_x^2 + A_y^2 + A_z^2 = 1$), $a = A_z = 0$
 - * $\theta_2 = \pi$ implies that $\cos(\frac{1}{2}\pi) = c_2 = 0$ and $\sin(\frac{1}{2}\pi) = s_2 = 1$
 - * From the Euler to quaternion conversion (see 3.3)
 - $a = 0$
 - $A_x = c_1 s_2 c_3 + s_1 s_2 s_3 = c_1 c_3 + s_1 s_3$
 - $A_y = c_1 s_2 s_3 - s_1 s_2 c_3 = c_1 s_3 - s_1 c_3$
 - $A_z = 0$
 - * Using the trigonometry identities:
 - $\cos(A - B) = \cos A \cos B + \sin A \sin B$
 - $\sin(A - B) = \sin A \cos B - \cos A \sin B$
 - * We obtained
 - $A_x = \cos(1 - 3)$
 - $A_y = -\sin(1 - 3)$
 - * By dividing A_y by A_x :
 - $\frac{A_y}{A_x} = \frac{-\sin(1-3)}{\cos(1-3)} = -\tan(1 - 3)$
 - $\frac{\theta_1}{2} - \frac{\theta_3}{2} = \arctan\left(\frac{-A_y}{A_x}\right)$
 - $\theta_1 - \theta_3 = 2 \arctan\left(\frac{-A_y}{A_x}\right)$
 - * Since “only $(\theta_1 - \theta_3)$ is uniquely defined (not the individual values)” [4] when $\theta_2 = \pi$
 - $\theta_1 = 2 \arctan\left(\frac{-A_y}{A_x}\right)$
 - $\theta_2 = \pi$ (from the special case)
 - $\theta_3 = 0$

References

- [1] Simon L. Altmann. *Rotations, Quaternions, and Double Groups*. Oxford University Press, 1986.
- [2] Martin Baker. Euclidean space. <http://www.euclideanspace.com>, 2008.
- [3] Tony A.D. Rollett and Peter Kalu. Advanced characterization and microstructural analysis: Texture and its effect on anisotropic properties, 2008. Course notes.
- [4] Wikipedia. Euler angles — wikipedia, the free encyclopedia. http://en.wikipedia.org/w/index.php?title=Euler_angles&oldid=238123918, 2008. [Online; accessed 13-September-2008].