1 Definitions and Notations

1.1 Quaternions

- Reference(s)
  - [1]
- Definition(s)
  - $\mathcal{A} = [\![a, \vec{A}]\!]$
  - a: a real scalar number
  - $\vec{A}$: a vector in $\mathbb{R}^3$
  - Quaternion algebra obeys to all the usual arithmetical laws except that the multiplication is non-commutative
- Notation(s)
  - $\mathcal{A}$: a quaternion
  - $[\![a, \vec{A}]\!]$: a quaternion composed of a scalar a and a vector $\vec{A}$
  - $\vec{A} = (A_x, A_y, A_z)$
  - $[\![a, (A_x, A_y, A_z)]\!]$: other representation of the scalar/vector components of the quaternion
  - $\|\mathcal{A}\|$: norm of the quaternion
  - $\mathcal{A}^*$: conjugate of the quaternion
  - $\mathcal{A}^{-1}$: inverse of the quaternion
- Advantage(s)
  - Easy to determine geometrically as $R\left(\phi\vec{n}\right)$
  - They are the only parameters for which the group multiplication rule can be given in closed form
  - They behave correctly near the identity whereas the euler angles become undetermined for $\beta = 0$ and $\beta = \pi$
  - They uniquely determine rotation poles with the convention of the positive hemi-sphere
    - $z > 0$
    - if $z = 0$, $x > 0$
    - if $z = 0$ and $x = 0$, $y > 0$
  - Keep track of 2π turns introduced on multiplying rotations

1.2 Euler Angles

- Reference(s)
  - [3]
- Definition(s)
  - From Bunge
  - Rotation performed from $\theta_1 \to \theta_2 \to \theta_3$ where:
    1. θ_1 is a counter-clockwise rotation around the $\vec{z}$ axis [001]: $R\left(\theta_1\vec{z}\right)$
    2. θ_2 is a counter-clockwise rotation around the $\vec{x}$ axis [100]: $R\left(\theta_2\vec{x}\right)$
    3. θ_3 is a counter-clockwise rotation around the $\vec{z}$ axis [001]: $R\left(\theta_3\vec{z}\right)$
  - Limits
    - $-\pi < \theta_1 \leq \pi$
    - $0 \leq \theta_2 \leq \pi$
    - $-\pi < \theta_3 \leq \pi$

2 Algebra

2.1 Product

- Reference(s)
  - [1]
- Properties
  - Non-commutative
    - $\mathcal{AB} \neq \mathcal{BA}$
  - Associative
    - $(\mathcal{AB})\,\mathcal{C} = \mathcal{A}\,(\mathcal{BC})$
  - Scalar product
    - $a[\![b, \vec{B}]\!] = [\![a, \vec{0}]\!][\![b, \vec{B}]\!] = [\![ab, a\vec{B}]\!]$
  - Quaternion product
    - $\mathcal{C} = \mathcal{AB} = [\![a, \vec{A}]\!][\![b, \vec{B}]\!] = [\![ab - \vec{A} \bullet \vec{B}, a\vec{B} + b\vec{A} + \vec{A} \times \vec{B}]\!]$
      - $c = ab - \vec{A} \bullet \vec{B}$
      - $C_x = aB_x + bA_x + \left(\vec{A} \times \vec{B}\right)_x$
      - $C_y = aB_y + bA_y + \left(\vec{A} \times \vec{B}\right)_y$
      - $C_z = aB_z + bA_z + \left(\vec{A} \times \vec{B}\right)_z$
- Derivation(s)
  - [1] or http://www.euclideanspace.com/maths/algebra/realNormedAlgebra/quaternions/arithmetic/index.htm

2.2 Division

- Reference(s)
  - [1]
- Properties
  - Scalar division
    - $\frac{[\![a, \vec{A}]\!]}{a} = \frac{1}{a}[\![a, \vec{A}]\!]$
  - Quaternion division
    - $\frac{[\![a, \vec{A}]\!]}{[\![b, \vec{B}]\!]} \equiv [\![a, \vec{A}]\!][\![b, \vec{B}]\!]^{-1}$
    - We defined this relationship, since $\frac{\mathcal{A}}{\mathcal{B}}$ could be equal to $\mathcal{AB}^{-1}$ or $\mathcal{B}^{-1}\mathcal{A}$ which doesn't respect the non-commutative rule
    - $[\![a, \vec{A}]\!][\![b, \vec{B}]\!]^{-1} = [\![ab + \vec{A} \bullet \vec{B}, b\vec{A} - a\vec{B} - \vec{A} \times \vec{B}]\!]$
- Derivation(s)
  - [1] or http://www.euclideanspace.com/maths/algebra/realNormedAlgebra/quaternions/arithmetic/index.htm

2.3 Addition / Substraction

- Reference(s)
  - ◇ [1]
- Properties
  - ◇ Addition
    - $[\![a, \vec{A}]\!] + [\![b, \vec{B}]\!] = [\![a + b, \vec{A} + \vec{B}]\!]$
  - ◇ Subtraction
    -
    - $[\![a, \vec{A}]\!] - [\![b, \vec{B}]\!] = [\![a - b, \vec{A} - \vec{B}]\!]$
- Derivation(s)
  - ◇ [1] or http://www.euclideanspace.com/maths/algebra/realNormedAlgebra/quaternions/arithmetic/index.htm

2.4 Conjugate

- Reference(s)
  - ◇ [1]
- Definition(s)
  - ◇ $\mathcal{A}^* = [\![a, \vec{A}]\!]^* = [\![a, -\vec{A}]\!]$
- Derivation(s)
  - ◇ [1] or http://www.euclideanspace.com/maths/algebra/realNormedAlgebra/quaternions/functions/index.htm

2.5 Norm

- Reference(s)
  - ◇ [1] and [2]
- Definition(s)
  - ◇ $\|\mathcal{A}\| = \mathcal{A}\mathcal{A}^*$
  - ◇ $\left\|[\![a, \vec{A}]\!]\right\| = [\![a, \vec{A}]\!][\![a, \vec{A}]\!]^*$
  - ◇ $\left\|[\![a, \vec{A}]\!]\right\| = \sqrt{a^2 + A_x^2 + A_y^2 + A_z^2}$
- Properties
  - ◇ $\mathcal{A} = [\![\cos\alpha, \sin\alpha\vec{n}]\!] = \cos\alpha + \sin\alpha\mathcal{N},\ \|\vec{n}\| = 1$
  - ◇ or $\mathcal{A} = [\![\cos\frac{\phi}{2}, \sin\frac{\phi}{2}\vec{n}]\!]$ (as in Euler-Rodrigues)
  - ◇ The product of 2 normalized quaternions is itself a normalized quaternion
- Derivation(s)
  - ◇ [1] or http://www.euclideanspace.com/maths/algebra/realNormedAlgebra/quaternions/functions/index.htm

2.6 Inverse

- Reference(s)
  - ◇ [1]
- Definition(s)
  - ◇ $\mathcal{A}^{-1} = \mathcal{A}^* \left\|\mathcal{A}\right\|^{-2}$
  - ◇ $\mathcal{C} = \mathcal{A}\mathcal{B}^{-1} = \mathcal{A}\mathcal{B}^* \left\|\mathcal{B}\right\|^{-2}$ if $\mathcal{B} \neq [\![0, \vec{0}]\!]$
  - ◇ For normalized quaternion, $\mathcal{A}^{-1} = \mathcal{A}^*$
- Derivation(s)
  - ◇ [1] or http://www.euclideanspace.com/maths/algebra/realNormedAlgebra/quaternions/functions/index.htm

2.7 Equality

- Definition(s)
  - ◇ Two quaternions ($\mathcal{A}$ and $\mathcal{B}$) are equal if and only if
    - ∗ $a = b$
    - ∗ $A_x = B_x$
    - ∗ $A_y = B_y$
    - ∗ $A_z = B_z$

3 Conversion

3.1 Axis Angle to Quaternion

- Reference(s)
  - [2]
- Definition(s)
  - ϕ: Rotation angle
  - $\vec{n}$: Rotation axis
- Equation(s)
  - $(\phi, \vec{n}) \rightarrow [\![\cos\frac{1}{2}\phi, \frac{\sin\frac{1}{2}\phi}{\|\vec{n}\|}\vec{n}]\!]$
- Derivation(s)
  - http://www.euclideanspace.com/maths/geometry/rotations/conversions/angleToQuaternion/index.htm

3.2 Matrix to Quaternion

- Reference(s)
  - [2]
- Definition(s)
  - m: a 3×3 orthogonal matrix (i.e. SO3)
    - $\mathrm{Det}(m) = 1$
    - $\mathrm{Tr}(m) > 0$
  - m_{ij}: element of matrix m in row i and column j
- Equation(s)
  - $a = \frac{1}{2}\sqrt{1 + m_{00} + m_{11} + m_{22}} = \frac{1}{2}\sqrt{1 + \mathrm{Tr}(m)}$
  - $A_x = \frac{m_{21} - m_{12}}{4a}$
  - $A_y = \frac{m_{02} - m_{20}}{4a}$
  - $A_z = \frac{m_{10} - m_{01}}{4a}$
- Derivation(s)
  - http://www.euclideanspace.com/maths/geometry/rotations/conversions/matrixToQuaternion/index.htm

3.3 Euler Angles to Quaternion

- Reference(s)
  - Adaptation from [2], [3] and [4]
- Definition(s)
  - $\theta_1, \theta_2, \theta_3$: see 1.2
  - $c_i \equiv \cos\left(\frac{1}{2}\theta_i\right)$ and $s_i \equiv \sin\left(\frac{1}{2}\theta_i\right)$
- Equation(s)

  - $a = c_1c_2c_3 - s_1c_2s_3$
  - $A_x = c_1s_2c_3 + s_1s_2s_3$
  - $A_y = c_1s_2s_3 - s_1s_2c_3$
  - $A_z = c_1c_2s_3 + s_1c_2c_3$

- Derivation(s)
  - Three rotations
    1. $Q_1 = [\![c_1, (0, 0, s_1)]\!]$
    2. $Q_2 = [\![c_2, (s_2, 0, 0)]\!]$
    3. $Q_3 = [\![c_3, (0, 0, s_3)]\!]$
  - Overall rotation
    * $Q_T = Q_3Q_2Q_1 = (Q_3Q_2)\,Q_1 = Q_3\,(Q_2Q_1)$

3.4 Quaternion to Axis Angle

- Reference(s)
  - [2]
- Equation(s)
  - $\|[\![a, A]\!]\| = 1$ (the quaternion has to be normalized)
  - $d = \sqrt{1 - a^2}$
  - $\phi = 2 \arccos a$
  - If $d \neq 0$: $\vec{n} = \frac{1}{d}\vec{A}$
  - If $d \neq 0$: $\vec{n} = \vec{A}$
- Derivation(s)
  - http://www.euclideanspace.com/maths/geometry/rotations/conversions/quaternionToAngle/index.htm

3.5 Quaternion to Matrix

- Reference(s)
  - [2]
- Equation(s)
  - $\|[\![a, A]\!]\| = 1$ (the quaternion has to be normalized)
  - $m = \begin{pmatrix} 1 - 2A_y^2 - 2A_z^2 & 2A_xA_y - 2A_za & 2A_xA_z + 2A_ya \\ 2A_xA_y + 2A_za & 1 - 2A_x^2 - 2A_z^2 & 2A_yA_z - 2A_xa \\ 2A_xA_z - 2A_ya & 2A_yA_z + 2A_xa & 1 - 2A_x^2 - 2A_y^2 \end{pmatrix}$
- Derivation(s)
  - http://www.euclideanspace.com/maths/geometry/rotations/conversions/quaternionToMatrix/index.htm

3.6 Quaternion to Euler Angles

- Reference(s)
  - ◇ Adaptation from [2], [3] and [4]
- Definition(s)
  - ◇ $\theta_1, \theta_2, \theta_3$: see 1.2
  - ◇ $c_i \equiv \cos\left(\frac{1}{2}\theta_i\right)$ and $s_i \equiv \sin\left(\frac{1}{2}\theta_i\right)$
- Equation(s)
  - ◇ $\|[\![a, A]\!]\| = 1$ (the quaternion has to be normalized)
  - ◇ Three cases
    1. $\theta_2 = 0$ $(A_x = A_y = 0)$
       - ∗ $\theta_1 = 2\arctan\left(\frac{A_z}{a}\right)$
       - ∗ $\theta_2 = 0$
       - ∗ $\theta_3 = 0$
    2. $\theta_2 = \pi$ $(A_x^2 + A_y^2 = 1)$
       - ∗ $\theta_1 = 2\arctan\left(\frac{A_y}{A_x}\right)$
       - ∗ $\theta_2 = \pi$
       - ∗ $\theta_3 = 0$
    3. $0 < \theta_2 < \pi$
       - ∗ $\theta_1 = \arctan\left(\frac{A_xA_z - A_ya}{A_yA_z + A_xa}\right)$
       - ∗ $\theta_2 = \arccos\left(1 - 2A_x^2 - A_y^2\right)$
       - ∗ $\theta_3 = \arctan\left(\frac{A_xA_z + A_ya}{A_xa - A_yA_z}\right)$
- Derivation(s)
  - ◇ From the Euler to Matrix we get
    - ∗ $m_{\text{euler}} = \begin{pmatrix} c_1c_3 - s_1c_2s_3 & -s_1c_3 - c_1c_2s_3 & s_2s_3 \\ s_1c_2c_3 + c_1s_3 & c_1c_2c_3 - s_1s_3 & -s_2c_3 \\ s_1s_2 & c_1s_2 & c_2 \end{pmatrix}$
    - ∗ We can get the euler angles from these relationships
      - · $\tan\theta_1 = \frac{m_{20}}{m_{21}} = \frac{s_1}{c_1} = \frac{s_1s_2}{c_1s_2}$
      - · $\cos\theta_2 = m_{22}$
      - · $\tan\theta_3 = -\frac{m_{02}}{m_{12}} = -\frac{s_3}{c_3} = -\frac{s_2s_3}{s_2c_3}$
  - ◇ By comparing with the quaternion matrix (see 3.5), we get replace the m_{ij} by the quaternion coefficients
  - ◇ For the 2 special cases, where the c_2 is not defined
    1. $\theta_2 = 0$
       - ∗ This condition happens when
         - · $\arccos 0 = 1 = 1 - 2A_x^2 - 2A_y^2$
         - · $A_x^2 = -A_y^2$
         - · $A_x = A_y = 0$
       - ∗ $\theta_2 = 0$ implies that $\cos\left(\frac{1}{2}0\right) = c_2 = 1$ and $\sin\left(\frac{1}{2}0\right) = s_2 = 0$
       - ∗ From the Euler to quaternion conversion (see 3.3)
         - · $a = c_1c_2c_3 - s_1c_2s_3 = c_1c_3 - s_1s_3$
         - · $A_x = 0$

- $A_y = 0$
- $A_z = c_1c_2s_3 + s_1c_2c_3 = c_1s_3 + s_1c_3$

* Using the trigonometry identities:
  - $\cos(A+B) = \cos A \cos B - \sin A \sin B$
  - $\sin(A+B) = \sin A \cos B + \cos A \sin B$
* We obtained
  - $a = \cos(1+3)$
  - $A_z = \sin(1+3)$
* By dividing A_z by a:
  - $\frac{A_z}{a} = \frac{\sin(1+3)}{\cos(1+3)} = \tan(1+3)$
  - $\frac{\theta_1}{2} + \frac{\theta_3}{2} = \arctan\left(\frac{A_z}{a}\right)$
  - $\theta_1 + \theta_3 = 2\arctan\left(\frac{A_z}{a}\right)$
* Since "only $(\theta_1 + \theta_3)$ is uniquely defined (not the individual values)"[4] when $\theta_2 = 0$
  - $\theta_1 = 2\arctan\left(\frac{A_z}{a}\right)$
  - $\theta_2 = 0$ (from the special case)
  - $\theta_3 = 0$

2. $\theta_2 = \pi$
   * This condition happens when
     - $\arccos\pi = -1 = 1 - 2A_x^2 - 2A_y^2$
     - $A_x^2 + A_y^2 = 1$
     - Since the quaternion is normalized $(a^2 + A_x^2 + A_y^2 + A_z^2 = 1)$, $a = A_z = 0$
   * $\theta_2 = \pi$ implies that $\cos\left(\frac{1}{2}\pi\right) = c_2 = 0$ and $\sin\left(\frac{1}{2}\pi\right) = s_2 = 1$
   * From the Euler to quaternion conversion (see 3.3)
     - $a = 0$
     - $A_x = c_1s_2c_3 + s_1s_2s_3 = c_1c_3 + s_1s_3$
     - $A_y = c_1s_2s_3 - s_1s_2c_3 = c_1s_3 - s_1c_3$
     - $A_z = 0$
   * Using the trigonometry identities:
     - $\cos(A-B) = \cos A \cos B + \sin A \sin B$
     - $\sin(A-B) = \sin A \cos B - \cos A \sin B$
   * We obtained
     - $A_x = \cos(1-3)$
     - $A_y = -\sin(1-3)$
   * By dividing A_y by A_x:
     - $\frac{A_y}{A_x} = \frac{-\sin(1-3)}{\cos(1-3)} = -\tan(1-3)$
     - $\frac{\theta_1}{2} - \frac{\theta_3}{2} = \arctan\left(\frac{-A_y}{A_x}\right)$
     - $\theta_1 - \theta_3 = 2\arctan\left(\frac{-A_y}{A_x}\right)$
   * Since "only $(\theta_1 - \theta_3)$ is uniquely defined (not the individual values)"[4] when $\theta_2 = \pi$
     - $\theta_1 = 2\arctan\left(\frac{-A_y}{A_x}\right)$
     - $\theta_2 = \pi$ (from the special case)
     - $\theta_3 = 0$

References

[1] Simon L. Altmann. *Rotations, Quaternions, and Double Groups*. Oxford University Press, 1986.

[2] Martin Baker. Euclidean space. http://www.euclideansplace.com, 2008.

[3] Tony A.D. Rollett and Peter Kalu. Advanced characterization and microstructural analysis: Texture and its effect on anisotropic properties, 2008. Course notes.

[4] Wikipedia. Euler angles — wikipedia, the free encyclopedia. http://en.wikipedia.org/w/index.php?title=Euler_angles&oldid=238123918, 2008. [Online; accessed 13-September-2008].