A Practical Approach To Signals, Systems, and Controls

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Chapter 1

Introduction: Systems And The Use Of Mathematical Models

1.1 Widespread Applicability And Common Features Of Mathematical Models

With the advent of powerful computers, the use of mathematical models by mathematicians, scientists, engineers, and researchers has become commonplace. Mathematical models are used for analysis, design, and to predict the behavior of complex

- *physical systems* (e.g. airplanes, robots, submarines, economic systems, musculoskeletal systems, heat exchangers, satellites, electrical networks, weather systems, missile target intercepts, etc.),
- physical, social, economic, political, and virtual *processes* (e.g. semiconductor manufacturing processes, chemical processes, thermal diffusion, breathing, combustion, signal processing, population growth, securities trading, supply and demand, decision making, battle management, etc.), and
- physical, social, economic, and political *phenomena* (e.g. electrical, mechanical, thermal, optical, inflation, interest rates, structural vibration and deformation, ozone layer depletion, etc.).

Common Features. While the above list is intended to illustrate the widespread applicability of mathematical models, it is more important to note three common features which each element of the list possesses. These comon features include:

- *input variables* responsible for *causing* certain actions, behaviors, or phenomena to take place (e.g. fuel flow to engine, torque applied to robotic manipulator arm, etc.),
- *output variables* which are associated with some *effect*, action, or phenomena (e.g. speed of aircraft, angle of robotic manipulator arm, etc.), and
- an implicit, possibly unknown or uncertain, *cause-effect relationship* between input and output variables - each depending on some independent dynamic (changing) variable such as time.

Internal variables which are neither inputs nor outputs, or which are associated with energy (also called *state variables*), may also be involved.

1.2 A Unifying Framework: The Concept Of A Dynamical System

These common features provide a unifying framework which enables the systematic and uniform treatment of diverse subject matter which may appear unrelated (see above list). More specifically, they motivate the concept of a *dynamical system* or *system* - a concept which was developed to refer to the implicit *cause-effect relationship* between input variables internal variables, and output variables.

The concept of a system generalizes the simple notion of a *function* to *dynamical settings*. To understand this, it is useful to recall the definition of a function:

- Let U and Y denote two nonempty sets. A *function* $T(\cdot)$ from U to Y is a rule of correspondence that assigns exactly one element of Y to each element of U. The first set U is called the domain of $T(\cdot)$. The second set Y is called the co-domain of $T(\cdot)$.

For systems, the "system" represents the "function" and the natural elements of the "function domain" U and "co-domain" Y are themselves functions [50].

1.3 Linear Systems: A Good Tool For Approximation

Linear dynamical systems or *linear systems* represent a very large class of systems, or mathematical models, which are used to describe, model, simulate, emulate, or approximate systems exhibiting "linear behavior." Recall that a linear relationship $T(\cdot)$ is one which satisfies the following property

$$T(a_1u_1 + a_2u_2) = a_1T(u_1) + a_2T(u_2) \tag{1.1}$$

for all scalars a_1, a_2 and all "input elements" u_1, u_2 within the domain of $T(\cdot)$.

Linear systems are used in all fields of science and engineering (e.g. robotics, semiconductor manufacturing, missile navigation, guidance and control, economics, etc.). While the use of such models is widespread, it is important to note that they are typically used as approximations to more complex nonlinear systems or models. As such, users of system models, and especially linear models, must keep in mind the following compelling truth: "mathematical models have limitations, stupidity does not!" The point here, is that it is very important to understand how well or poorly a mathematical model approximates the behavior of a system.

Connection With Mathematical Objects. More precisely, the term linear system is generally used to refer to systems whose inputs, outputs, and internal variables (e.g. *state variables*) are related by

1. linear ordinary differential equations, linear algebraic and integral relationships, and linear partial differential equations, and/or

2. linear ordinary difference equations, linear algebraic and series relationships, and linear partial difference equations.

1.4 System Types

Continuous Time Systems. Systems described by linear ordinary differential equations, linear algebraic and integral relationships, and linear partial differential equations, are referred to as *continuous time systems* since they depend on a continuous (real valued) independent variable t - usually representing time [10], [32].

Discrete Time Systems. Systems described by linear ordinary difference equations, linear algebraic and series relationships, and linear partial difference equations, are referred to as *discrete time systems* since they depend on a discrete (integer valued) independent variable n - usually representing discrete time [33], [34], [35].

Sampled-Data Systems. Systems described by both differential and difference equations are referred to as *sampled-data systems* [9], [18].

General Outline. In what follows, the initial focus will be placed on continuous time systems described by linear ordinary differential equations with constant coefficients [32]. The development will then shift to discrete time systems described by linear ordinary difference equations with constant coefficients.

Before developing mathematical models for continuous time systems, it is essential to develop mathematical models for typically encountered continuous time test signals (functions).

Chapter 2

Mathematical Preliminaries

2.1 Complex Numbers And Complex Arithmetic

Complex numbers are particularly important in the study of linear dynamical systems. Their main utility lies in the fact that they permit engineers to *represent sinusoidal and exponential sinusoidal functions by complex exponentials*. Given this, complex exponentials greatly facilitate the study of linear systems. Moreover, their use leads naturally to new powerful system concepts (e.g. frequency response of a system, etc.). As such, complex numbers are indispensable in the study of dynamical systems.

Motivation: The Need For Complex Numbers. Because equations of the form

$$z^2 = -1 \tag{2.1}$$

were found to be present everywhere in science and engineering, it became essential to invent *complex numbers.* (Necessity is the mother of invention!)

Definition 2.1.1 Definition: Complex Numbers.

A complex number *z is an ordered pair of real numbers x and y. The complex number z is denoted $z \stackrel{\text{def}}{=} (x, y)$. The real number x is called the* real part *of the complex number $z = (x, y)$. The real number y is called the* imaginary part *of the complex number $z = (x, y)$. If $x = 0$, we say that the* *complex number* $z = (0, y)$ *is* purely imaginary. *If* $y = 0$*, we say that the complex number* $z = (x, 0)$ *is* real. *The set of all complex numbers is denoted* $\mathcal{C}$.

Equality of Complex Numbers. *Two complex numbers* $z_1 = (x_1, y_1)$ *and* $z_2 = (x_2, y_2)$ *are equal if and only if their real parts are equal and their imaginary parts are equal; i.e.*

$$z_1 = z_2 \text{ if and only if } x_1 = x_2 \text{ and } y_1 = y_2. \tag{2.2}$$

Addition of Complex Numbers. *The sum of two complex numbers* $z_1 = (x_1, y_1)$ *and* $z_2 = (x_2, y_2)$ *is denoted* $z_1 + z_2$ *and defined as follows:*

$$z_1 + z_2 \stackrel{\text{def}}{=} (\ x_1 + x_2,\ y_1 + y_2\ ). \tag{2.3}$$

That is, the sum is obtained by adding the real parts and by adding the imaginary parts to get the new real and imaginary parts, respectively. This definition implies that complex numbers add like vectors. An intuitive geometric interpretation is thus possible.

Multiplication of Complex Numbers. *The product of two complex numbers* $z_1 = (x_1, y_1)$ *and* $z_2 = (x_2, y_2)$ *is denoted* $z_1 z_2$ *and defined as follows:*

$$z_1 z_2 \stackrel{\text{def}}{=} (\ x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1\ ). \tag{2.4}$$

While this definition is easy to use, it is rather strange and arguably unmotivated. An intuitive (geometric) interpretation for complex multiplication will be given below.

∎

Commutative, Associative, and Distributive Properties. It can be shown that the addition and multiplication of complex numbers are *commutative*, *associative*, and *distributive* operations. That is, for any complex numbers z_1, z_2, and z_3, the following relationships hold:

$$\begin{array}{lcllcl} z_1 + z_2 & = & z_2 + z_1 & z_1 z_2 & = & z_2 z_1 \\ (z_1 + z_2) + z_3 & = & z_1 + (z_2 + z_3) & (z_1 z_2) z_3 & = & z_1 (z_2 z_3) \\ z_1(z_2 + z_3) & = & z_1 z_2 + z_1 z_3 & (z_1 + z_2) z_3 & = & z_1 z_3 + z_2 z_3. \end{array} \tag{2.5}$$

Most of us take these properties for granted when working with real numbers. It is very comforting that they hold for complex numbers as well.

The Symbol j. Given the above definition for complex multiplication, it follows that $(0,1)(0,1) = (-1,0) = -1$ and $(0,-1)(0,-1) = (0,1)(0,1) = (-1,0) = -1$. From this, it follows that $z = (0,1)$ and $z = (0,-1) = -(0,1)$ are each solutions to the quadratic equation $z^2 = -1$. Because of this, we assign the special symbol j to $(0,1)$; i.e.

$$j \stackrel{\text{def}}{=} (0,1). \tag{2.6}$$

Because $j^2 = -1$, it follows that $j = \sqrt{-1}$. It should be noted that while the symbol j is widely used within the engineering community, the symbol i is still the preferred symbol within the mathematics community [1], [3], [41]. The symbol i is used to represent electrical current within the engineering community [23].

Rectangular Representation. With the above "j-notation" in place, it follows that

$$\begin{aligned}
(x,y) &= (x,0) + (0,y) && (2.7)\\
&= (x,0) + (0,1)(y,0) && (2.8)\\
&= (x,0) + j(y,0) && (2.9)\\
&= x + jy && (2.10)
\end{aligned}$$

This provides motivation for the widely used notation

$$z = x + jy \tag{2.11}$$

which we adopt hereafter. This representation for z is commonly referred to as the *rectangular representation* for z. Given this, the complex number $z = x + jy$ can be visualized as indicated in Figure 2.1. The idea of representing complex numbers by points (vectors) in the plane was formulated by Gauss in his 1799 dissertation and, independently, by Argand in 1806 [3, pg. 17]. Figure 2.1 is sometimes referred to as an *Argand diagram*.

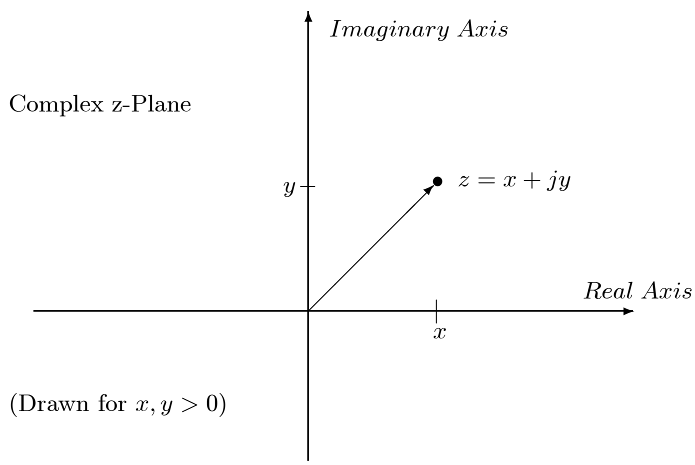

Figure 2.1: Visualizing a Complex Number z In The Complex z-Plane

With the above notation, equations (2.3) and (2.4) become

$$z_1 + z_2 \stackrel{\text{def}}{=} [x_1 + x_2] + j\,[y_1 + y_2] \tag{2.12}$$

$$z_1 z_2 \stackrel{\text{def}}{=} [x_1 x_2 - y_1 y_2] + j\,[x_1 y_1 + x_2 y_1]\,. \tag{2.13}$$

In order to properly define complex division, it is useful to introduce new terms.

Conjugate of a Complex Number. The *conjugate* of a complex number $z = x + jy$ is denoted $\overline{z}$ and defined as follows:

$$\overline{z} \stackrel{\text{def}}{=} x - jy. \tag{2.14}$$

The conjugate $\overline{z}$ of the complex number $z = x + jy$ may visualized as shown if Figure 2.2. Moreover, it can be shown that

$$\overline{\overline{z}} = z. \tag{2.15}$$

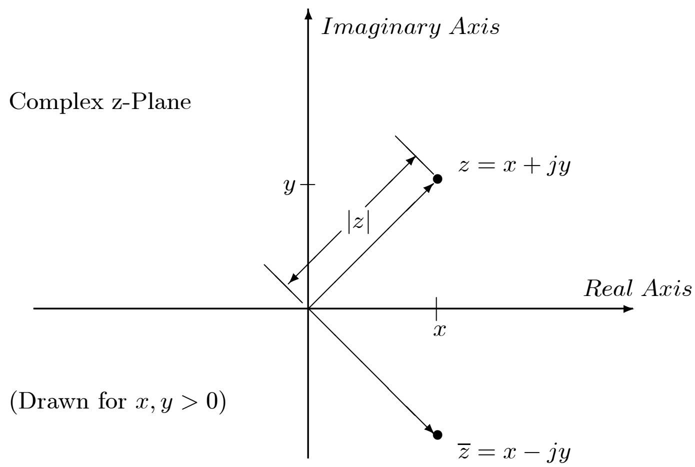

Figure 2.2: Visualizing The Conjugate $\overline{z}$ Of A Complex Number z In The Complex z-Plane

Magnitude of a Complex Number. The *magnitude* of a complex number $z = x + jy$ is denoted $|z|$ and defined as follows:

$$|z| \stackrel{\text{def}}{=} \sqrt{x^2 + y^2}. \tag{2.16}$$

The magnitude $|z|$ can be visualized as shown in Figure 2.2. From the figure, and the Pythagorean theorem, it follows that $|z|$ is precisely the length of the vector used in representing z. It is useful to note that

$$|z| = \sqrt{z\,\overline{z}} \tag{2.17}$$

Division. Given the above, the division of two complex numbers $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$ ($z_2 \neq 0$) is defined in terms of the ratio

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\,\frac{\overline{z_2}}{\overline{z_2}} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{(x_2 + jy_2)(x_2 - jy_2)} \tag{2.18}$$

as follows

$$\frac{z_1}{z_2} \stackrel{\text{def}}{=} \left[\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}\right] + j\left[\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}\right]. \tag{2.19}$$

As with complex multiplication, an intuitive (geomeric) interpretation for complex divison will be given below.

The rectangular representation for complex numbers is most useful for carrying out and visualizing the addition and subtraction of complex numbers. Rectangular representations, however, are not very intuitive when it comes to carrying out and visualizing complex multiplication and division. To obtain nicer formulae - with useful geometric interpretation, we need to motivate an alternative representation for complex numbers. Recalling that real exponentials satisfy very nice relationships for multiplication and divison, we are led naturally to *polar representations* and *complex exponentials*. To properly introduce these ideas, we require a new term - the *angle* of a complex number.

Angle Of A Complex Number. Consider the complex number $z = x + jy$ shown in Figure 2.3. The *angle* of the complex number z is defined as follows:

$$\angle z \stackrel{\text{def}}{=} \begin{cases} \tan^{-1}\frac{y}{x} & x > 0, \quad y > 0 \\ 180 - \tan^{-1}\frac{y}{|x|} & x < 0, \quad y > 0 \\ -180 + \tan^{-1}\frac{|y|}{|x|} & x < 0, \quad y < 0 \\ -\tan^{-1}\frac{|y|}{x} & x > 0, \quad y < 0 \end{cases} \tag{2.20}$$

Engineers often use the word *phase* instead of angle. From Figure 2.3, it follows that the real and imaginary parts of a complex number z may be readily computed from the magnitude and angle of z as follows:

$$Re\{z\} = |z| \cos \angle z \tag{2.21}$$

$$Im\{z\} = |z| \sin \angle z. \tag{2.22}$$

This motivates the alternative representation that we seek for complex numbers - the so-called *polar* or *exponential representation.*

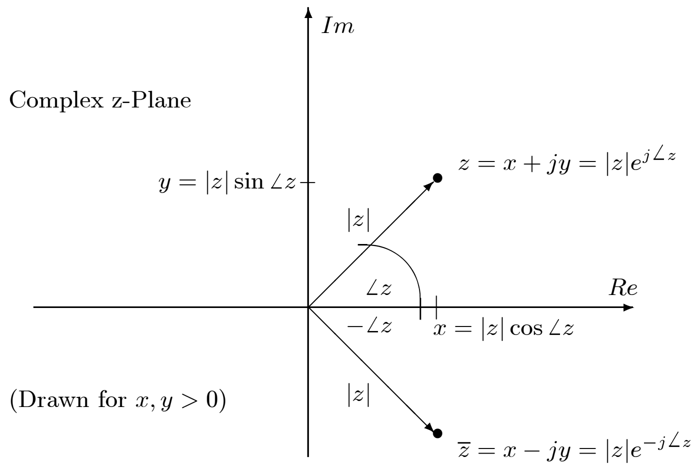

Figure 2.3: Visualizing The Polar Form Of A Complex Number

Polar (Exponential) Representation. From the above discussion, it follows that

$$z = |z|e^{j\angle z} \tag{2.23}$$

where

$$e^{j\angle z} \stackrel{\text{def}}{=} \cos \angle z + j \sin \angle z. \tag{2.24}$$

Equation (2.23) defines the the *polar* or *exponential representation* for the complex number z.

Polar Representation Properties: Geometric Interpretations For Multiplication And Division. The polar representation of a complex number offers very nice properties when it comes to complex multiplication and division. If $z_1 = |z_1|e^{\angle z_1}$ and $z_2 = |z_2|e^{\angle z_2}$, then it can be shown that

$$z_1 z_2 = |z_1||z_2|\; e^{j(\angle z_1 + \angle z_2)} \tag{2.25}$$

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}\; e^{j(\angle z_1 - \angle z_2)}. \tag{2.26}$$

Unlike their rectangular counterparts (cf. equations (2.13), (2.19)), these properties are very useful and are readily visualized geometrically. Equation (2.25) shows that the multiplication of two complex numbers involves the multiplication of their magnitudes and the addition of their angles.

Equation (2.26) shows that the division of two complex numbers involves the division of their magnitudes and the subtraction of their angles.

Euler Identities. From the above discussion and Figure 2.4, one obtains the following Euler identities:

$$z + \overline{z} = 2Re\{z\} = 2|z|\cos\angle z \tag{2.27}$$

$$z - \overline{z} = j2Im\{z\} = 2|z|\sin\angle z. \tag{2.28}$$

These identities are extensively used in science and engineering.

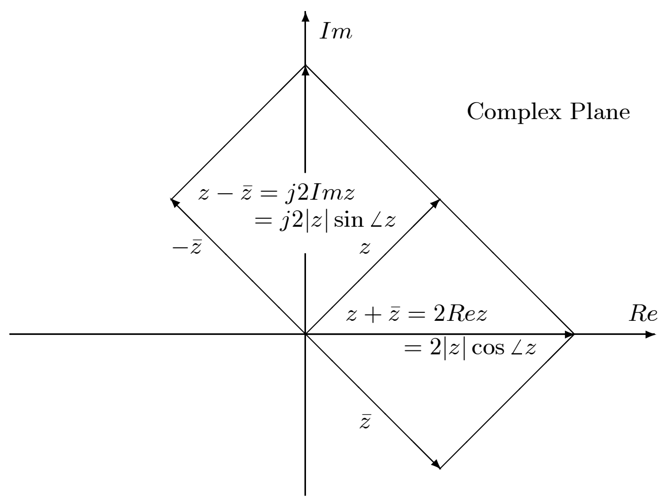

Figure 2.4: Visualizing Euler's Identities

Complex Exponential, Periodicity. Just as the polar representation of a complex number was motivated by the nice multiplication and division properties that exponentials offer, these ideas motivate the definition for the *complex exponential* e^z of a complex number $z = x + jy$:

$$e^z \stackrel{\text{def}}{=} e^x e^{jy} \tag{2.29}$$

From equation (2.24), it then follows that

$$e^z = e^x e^{jy} = e^x \cos y + je^x \sin y. \tag{2.30}$$

From this, it follows that the magnitude of e^z is given by

$$|e^z| = e^x \tag{2.31}$$

and the angle e^z is given by

$$\angle e^z = y. \tag{2.32}$$

It is very important to note that the complex exponential is periodic in the angle y with period 2π. Thus, e^0 and $e^{j2\pi}$ are different representations for the same number, namely $z = 1$.

Roots. The polar representation of a complex number z is also very useful for determining the roots of the number as follows:

$$\begin{aligned} z^{\frac{m}{n}} &= |z|^{\frac{m}{n}} e^{j\frac{m}{n}\angle z} && (2.33) \\ &= |z|^{\frac{m}{n}} e^{j\frac{m}{n}(\angle z + 2\pi k)} \quad k = 0, 1, \ldots n-1. && (2.34) \end{aligned}$$

Example 2.1.1 (Cube Root of -1.)

Since the number -1 may be expressed as $e^{-j180°}$, $e^{j180°}$, and $e^{j540°}$, it follows that the three roots which satsfy $z^3 = -1$ are given by

$$z = e^{-j60°},\ e^{j60°},\ e^{j180°}. \tag{2.35}$$

■

Example 2.1.2 (Generation of Exponential Sinusoids From Complex Exponentials.)

One very important use of complex exponentials lies in their utility to construct real exponential sinusoids. This follows from the relationship:

$$\begin{aligned} x(t) = |X|e^{\sigma_o t}\cos(\omega_o t + \angle X) &= |X|e^{\sigma_o t} Re\{e^{j(\omega_o t + \angle X)}\} && (2.36) \\ &= Re\{|X|e^{\sigma_o t} e^{j(\omega_o t + \angle X)}\} && (2.37) \\ &= Re\{|X|e^{j\angle X} e^{(\sigma_o + j\omega_o)t}\}. && (2.38) \end{aligned}$$

Similarly,

$$x(t) = |X|e^{\sigma_o t}\sin(\omega_o t + \angle X) = |X|e^{\sigma_o t} Im\{e^{j(\omega_o t + \angle X)}\} \tag{2.39}$$

$$= Im\{|X|e^{\sigma_o t}e^{j(\omega_o t + \angle X)}\} \tag{2.40}$$

$$= Im\{|X|e^{j\angle X}e^{(\sigma_o + j\omega_o)t}\}. \tag{2.41}$$

We say that the above exponential sinusoids are generated from the complex exponential

$$z = |z|e^{j\angle z} = |X|e^{\sigma_o t}e^{j(\omega_o t + \angle X)} = |X|e^{j\angle X}e^{(\sigma_o + j\omega_o)t} = Xe^{s_o t} \tag{2.42}$$

where

$$s_o = \sigma_o + j\omega_o \tag{2.43}$$

is called the complex frequency *of the exponential sinusoids and*

$$X \stackrel{\text{def}}{=} |X|e^{j\angle X} \tag{2.44}$$

is called the phasor representation *for the function $x(\cdot)$. It is often very useful to associate the above real exponential sinusoids with their phasor representations. This association is made in many areas of science and engineering. One area includes the steady state analysis of linear time invariant (LTI) continuous time dynamical systems [32] and the steady state analysis of linear shift invariant (LSI) discrete time dynamical systems [33].* ■

Example 2.1.3 (Addition Of Sinusoids Using Complex Phasors.)

Addition Of Two Sinusoids. *Consider the continuous time signal*

$$f(t) = 2\sin(\omega_o t - 30^\circ) + \cos(\omega_o t + 45^\circ). \tag{2.45}$$

This signal is the sum of two sinusoids of the same frequency ω_o. The goal in this example is to illustrate how complex phasors may be used to add the sinusoids to get a single cosine signal of frequency ω_o. We proceed algebraically using the trigonometric relationship

$$\sin(x) = \cos(x - 90^\circ) \tag{2.46}$$

and Euler's identities, as follows:

$$\begin{aligned} f(t) &= 2\cos(\omega_o t - 120°) + \cos(\omega_o t + 45°) && (2.47)\\ &= 2Re\{e^{j(\omega_o t-120°)}\} + Re\{e^{j(\omega_o t+45°)}\} && (2.48)\\ &= Re\{2e^{-j120°}e^{j\omega_o t}\} + Re\{e^{j45\deg}e^{j\omega_o t}\} && (2.49)\\ &= Re\{\left[2e^{-j120°} + e^{j45°}\right]e^{j\omega_o t}\} && (2.50)\\ &= Re\{Xe^{j\omega_o t}\} && (2.51)\\ &= Re\{|X|e^{j(\omega_o t+\angle X)}\} && (2.52)\\ &= |X|\cos(\omega_o t + \angle X) && (2.53) \end{aligned}$$

where the crtical calculation lies in the addition of complex phasors associated with each sinusoid as follows:

$$\begin{aligned} X &= 2e^{-j120°} + e^{j45°} && (2.54)\\ &= -1 - j1.7321 + 0.7071 + j0.7071 && (2.55)\\ &= -0.2929 - j1.0249 && (2.56)\\ &= 1.0660e^{-j105.9481°}. && (2.57) \end{aligned}$$

Addition Of Multiple Sinusoids. *The above calculation shows that the addition of two sinusoids of the same frequency is equivalent to the addition of complex phasors. The same is true if we are adding many sinusoids of the same frequency. This is illustrated by the following algebraic steps:*

$$\begin{aligned} f(t) &= \sum_{k=1}^{n} |X_k|\cos(\omega_o t + \angle X_k) && (2.58)\\ &= \sum_{k=1}^{n} |X_k| Re\{e^{j(\omega_o t+\angle X_k)}\} && (2.59)\\ &= \sum_{k=1}^{n} Re\{|X_k|e^{j(\omega_o t+\angle X_k)}\} && (2.60)\\ &= \sum_{k=1}^{n} Re\{|X_k|e^{j\angle X_k}e^{j\omega_o t}\} && (2.61)\\ &= Re\{\sum_{k=1}^{n} |X_k|e^{j\angle X_k}e^{j\omega_o t}\} && (2.62) \end{aligned}$$

$$= Re\{\left[\sum_{k=1}^{n}|X_k|e^{j\angle X_k}\right]e^{j\omega_o t}\} \quad (2.63)$$

$$= Re\{\left[\sum_{k=1}^{n}X_k\right]e^{j\omega_o t}\} \quad (2.64)$$

$$= Re\{Xe^{j\omega_o t}\} \quad (2.65)$$

$$= Re\{|X|e^{j\angle X}e^{j\omega_o t}\} \quad (2.66)$$

$$= Re\{|X|e^{j(\omega_o t+\angle X)}\} \quad (2.67)$$

$$= |X|\cos(\omega_o t+\angle X) \quad (2.68)$$

where

$$X = |X|e^{j\angle X} = \sum_{k=1}^{n}X_k = \sum_{k=1}^{n}|X_k|e^{j\angle X_k}. \quad (2.69)$$

■

2.2 Results From Trigonometry

The following relationships are useful in the study of dynamical systems.

$$\sin(x\pm y) = \sin x\cos y \pm \cos x \sin y \quad (2.70)$$

$$\cos(x\pm y) = \cos x\cos y \mp \sin x \sin y \quad (2.71)$$

$$\tan(x\pm y) = \frac{\tan x \pm \tan y}{1\mp \tan x\tan y} \quad (2.72)$$

$$\sin(x\pm 90^\circ) = \pm\cos(x) \quad (2.73)$$

$$\cos(x\pm 90^\circ) = \mp\sin(x) \quad (2.74)$$

$$\tan(x\pm 90^\circ) = -\tan(x) \quad (2.75)$$

$$\sin x\cos y = \frac{1}{2}[\sin(x+y)+\sin(x-y)] \quad (2.76)$$

$$\cos x\cos y = \frac{1}{2}[\cos(x+y)+\cos(x-y)] \quad (2.77)$$

$$\sin^2 x = \frac{1}{2}[1 - \cos 2x] \tag{2.78}$$
$$\cos^2 x = \frac{1}{2}[1 + \cos 2x] \tag{2.79}$$
$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x} \tag{2.80}$$

$$\sin^2 x + \cos^2 x = 1 \tag{2.81}$$

$$\tan^2 x + 1 = \sec^2 x \tag{2.82}$$

$$\cot^2 x + 1 = \csc^2 x \tag{2.83}$$

2.3 Results From Calculus

The following relationships are useful in the study of dynamical systems.

$$f'(x) \stackrel{\text{def}}{=} \lim_{\Delta \longrightarrow 0} \left[\frac{f(x + \Delta) - f(x)}{\Delta} \right] \tag{2.84}$$

$$\frac{d}{dx} \sin x = \cos x \tag{2.85}$$
$$\frac{d}{dx} \cos x = -\sin x \tag{2.86}$$
$$\frac{d}{dx} \tan x = \sec^2 x \tag{2.87}$$
$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}} \tag{2.88}$$
$$\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}} \tag{2.89}$$
$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2} \tag{2.90}$$
$$\frac{d}{dx} e^{ax} = a\, e^{x} \tag{2.91}$$
$$\frac{d}{dx} \ln x = \frac{1}{x} \tag{2.92}$$

$$\frac{d}{dx}[f(g(x))] = f'(g(x))\, g'(x) \tag{2.93}$$
$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x) \tag{2.94}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{1}{g^2(x)}\left[f'(x)g(x) - f(x)g'(x)\right] \tag{2.95}$$

$$\int u dv = uv - \int v du \tag{2.96}$$

Theorem 2.3.1 (Mean Value Theorem For Differentiation.)

If $f(\cdot)$ is differentiable on the open interval (x_1, x_2) and continuous on the closed interval $[x_1, x_2]$, then there exists $z \in (x_1, x_2)$ such that

$$f'(z) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}. \tag{2.97}$$

The number $f'(z)$ is sometimes referred to as the mean derivative *or* average rate of change *of f on the interval $[x_1, x_2]$.* ∎

Theorem 2.3.2 (Mean Value Theorem For Integration.)

If $f(\cdot)$ is continuous on a closed interval $[x_1, x_2]$, then there exists $z \in (x_1, x_2)$ such that

$$f(z) = \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} f(\tau)d\tau. \tag{2.98}$$

The number $f(z)$ is sometimes referred to as the mean value *or* average value *of f on the interval $[x_1, x_2]$.* ∎

Theorem 2.3.3 (Fundamental Theorem Of Calculus.)

Suppose f is continuous on the closed interval $[a, b]$.

(i) If

$$G(x) \stackrel{\text{def}}{=} \int_a^x f(\tau)d\tau \tag{2.99}$$

for all $x \in [a, b]$, then G is the antiderivative of f on $[a, b]$; i.e.

$$G'(x) = \frac{d}{dx}\int_a^x f(\tau)d\tau = f(x) \tag{2.100}$$

for all $x \in [a, b]$.

(ii) If F *is any antiderivative of* f, *then*

$$\int_a^b f(\tau)d\tau = F(b) - F(a). \tag{2.101}$$

∎

This theorem was discovered independently by Sir Isaac Newton (1642-1727) in England and by Gottfried Leibniz (1646-1716) in Germany. It is primarily because of this, that they are credited with the invention of calculus.

Theorem 2.3.4 (Taylor's Theorem.)

Let f *be a function and* n *a positive integer such that the derivative* $f^{(n+1)}(x)$ *exists for every* x *in an interval* I. *If* x_o *and* x *are distinct numbers in* I, *then there exists a number* $z \in (x_o, x)$ *such that*

$$f(x) = f(x_o) + \frac{f'(x_o)}{1!}(x - x_o) + \cdots + \frac{f^{(n)}(x_o)}{n!}(x - x_o)^n + \frac{f^{(n+1)}(z)}{(n+1)!}(x - x_o)^{n+1}. \tag{2.102}$$

∎

This theorem was discovered by the English mathematician Brook Taylor (1685-1731).

Algorithm 2.3.1 (Newton's Method For Solving $f(x) = 0$).

Newton's method for solving the equation

$$f(x) = 0 \tag{2.103}$$

is based on a sequence of tangent line approximations. The method is described by the following recursive algorithm:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2.104}$$

where $n = 0, 1, 2, \ldots$. *Convergence ideas are discussed in [47].* ■

Chapter 3

Mathematical Models For Continuous Time Signals

In this section, mathematical models for typically encountered continuous time signals (functions) are presented [34], [36].

Function Notation. In what follows, a function f will be denoted by the notation $f(\cdot)$ or simply f. At times the symbol $f(t)$ might be used. It must be noted, however, that strictly speaking $f(t)$ does not represent the function f but rather, the value of the f at t. Despite this, the symbol $f(t)$ is widely used to represent f.

3.1 A Model for Tall Thin Pulse-like Signals: Dirac Delta Distribution

In the study of systems, it is useful to use test signals which are pulse-like in nature. A model for tall thin pulse-like signals possessing unit area is now developed. Let $p_\epsilon(\cdot)$ denote the tall thin pulse defined as follows:

$$p_\epsilon(t) \stackrel{\text{def}}{=} \begin{cases} 0 & -\infty < t < -\epsilon/2 \\ \frac{1}{\epsilon} & -\epsilon/2 \leq t < \epsilon/2 \\ 0 & \epsilon/2 \leq t < \infty \end{cases} \tag{3.1}$$

where ϵ represents a small positive quantity. p_ϵ defines a pulse with height $\frac{1}{\epsilon}$, width ϵ, which is symmetric with respect to the origin. As ϵ approaches zero, the above pulse becomes thinner and taller, while its area remains unity (cf. Figure 3.1). In the limit p_ϵ approaches a "function" which has infinite value at $t = 0$ and is zero elsewhere. This limit certainly does not describe a traditional function. A fundamental *sifting property* of $p_\epsilon(\cdot)$ is now observed. Let $f(\cdot)$ denote a function which is continuous in a small ϵ-neighborhood of the point $t = a$. Given this, it follows that for sufficiently small ϵ

$$\int_{-\infty}^{\infty} f(\tau)\, p_\epsilon(\tau - a)\, d\tau = \int_{a-\epsilon/2}^{a+\epsilon/2} f(\tau)\, \frac{1}{\epsilon}\, d\tau \approx \frac{1}{\epsilon} \int_{a-\epsilon/2}^{a+\epsilon/2} f(a)\, d\tau = f(a) \tag{3.2}$$

Moreover,

$$\lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} f(\tau)\, p_\epsilon(\tau - a)\, d\tau = f(a) \tag{3.3}$$

This relationship is referred to as a sifting property because it states that the pulse $p_\epsilon(\cdot - a)$ with center at $t = a$, allows the value $f(a)$ to pass through the integral - in the same manner that a wiremesh sieve (pronounced "siv") allows certain material to pass through it. This sifting property motivates the introduction of a new "non-tradional function" - the unit Dirac delta distribution. Its definition is now given.

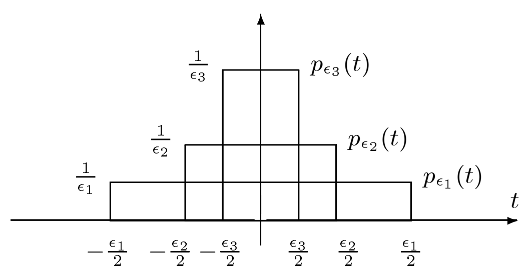

Figure 3.1: Tall Thin Pulse-Like Signals

Definition 3.1.1 (Unit Dirac Delta Distribution.)

The unit Dirac delta distribution is denoted $\delta(\cdot)$ and defined by the following three properties:

i. Integral Sifting Property:

$$\int_{-\infty}^{\infty} f(\tau)\, \delta(\tau - a)\, d\tau = \int_{a^-}^{a^+} f(\tau)\, \delta(\tau - a)\, d\tau \stackrel{\text{def}}{=} f(a) \tag{3.4}$$

ii. Even Symmetry Property:

$$\delta(t) \stackrel{\text{def}}{=} \delta(-t) \tag{3.5}$$

iii. Infinite Height Zero Width Property:

$$\delta(t) \stackrel{\text{def}}{=} \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases} \tag{3.6}$$

∎

The first property implies that δ has unit area, just like the pulse $p_\epsilon(\cdot)$. The second property implies that δ is symmetric with respect to the vertical axis at $t = 0$, just like $p_\epsilon(\cdot)$. The last property implies that δ has infinite height and zero width, just like $p_\epsilon(\cdot)$ as ϵ approaches zero. The unit Dirac delta distribution is usually depicted as shown in Figure 3.2. It is important to emphasize that the *unit Dirac delta distribution* is not a traditional function. Despite this, it is widely referred to as the unit Dirac delta function or the *unit impulse function*. Sometimes, it is referred to as a *generalized function* [49]. Finally, it should be noted that while the unit Dirac delta distribution only makes sense underneath integrals, it is often referred to without explicit reference to integrals.

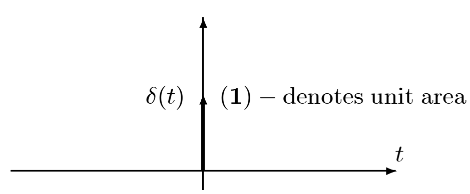

Figure 3.2: Unit Dirac Delta Distribution

3.2 A Model for Step-like Signals: Unit Step Function

Step-like test signals are also useful in the study of systems. A model for such signals is the *unit step function*. Represented by the symbol $1(\cdot)$, the unit step function is defined as follows:

$$1(t) \stackrel{\text{def}}{=} \begin{cases} 0 & -\infty < t < 0 \\ 1 & 0 \leq t < \infty. \end{cases} \tag{3.7}$$

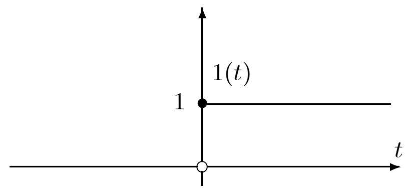

Figure 3.3: Unit Step Function

From the above discussion, it follows that the unit step function $1(\cdot)$ and the unit Dirac delta distribution $\delta(\cdot)$ are related as follows:

$$\int_{-\infty}^{t} \delta(\tau) d\tau \;=\; 1(t) \tag{3.8}$$

$$\frac{d}{dt} 1(t) \;=\; \delta(t). \tag{3.9}$$

3.3 Exponential Signals

The exponential function

$$f(t) \;=\; e^{at}\, 1(t) \tag{3.10}$$

is widely used to model exponential signals in the study of systems. The constant a is sometimes referred to as the neper frequency. For $a > 0$, the exponential grows toward ∞ as t approaches ∞. As a is made more positive, the rate at which the exponential grows increases (see Figure 3.4).

For $a = 0$, the function becomes the unit step function $1(\cdot)$ discussed earlier. Finally, for $a < 0$ the exponential decays toward zero as t approaches ∞. In such a case, we say that the exponential has a time constant $\tau \stackrel{\text{def}}{=} \frac{1}{|a|}$ and settles to zero in approximately five time constants. The latter, widely used, convention follows because $e^{a5\tau} = e^{-5} \approx 0.0067$ is “small.” Of course, how small is “small” depends on the specific application under consideration. As a is made more negative, the associated time constant decreases and the exponential decays more rapidly toward zero. It is interesting to note that the inital rate of decay (slope) is given by $-\frac{1}{\tau}$ (see Figure 3.5).

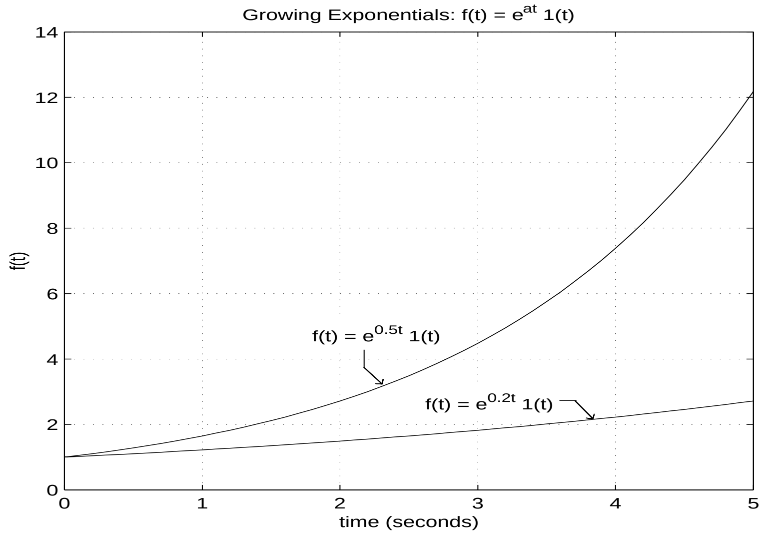

Figure 3.4: Growing Exponentials

3.4 Complex Exponential Signals: Creating Sinusoids and Exponential Sinusoids

The utility of complex exponential signals is succinctly conveyed by Euler's relationships. Letting $z = |z|e^{j\angle z}$ denote an arbitrary complex quantity, $\bar{z}$ its complex conjugate, $Re\{z\}$ its real part, and $Im\{z\}$ its imaginary part, Euler's relationships may be stated as follows [1, pg. 42], [41]:

$$|z|\cos\angle z = Re\,\{\ z\ \} = \frac{z+\bar{z}}{2} \qquad\qquad |z|\sin\angle z = Im\,\{\ z\ \} = \frac{z-\bar{z}}{j2} \tag{3.11}$$

Substituting

$$z \;=\; |X|e^{\sigma_o t}\ e^{j(\omega_o t+\angle X)}\ 1(t) \;=\; |X|e^{j\angle X}\ e^{(\sigma_o+j\omega_o)t}\ 1(t) \tag{3.12}$$

into the above formulae, yields the following useful relationships:

$$|X|e^{\sigma_o t}\cos(\omega_o t+\angle X)1(t) \;=\; Re\,\{\ |X|e^{\sigma_o t}\ e^{j(\omega_o t+\angle X)}\ \}1(t) = Re\,\{\ |X|e^{j\angle X}\ e^{(\sigma_o+j\omega_o)t}\ \}1(t) \tag{3.13}$$

$$|X|e^{\sigma_o t}\sin(\omega_o t+\angle X)1(t) \;=\; Im\,\{\ |X|e^{\sigma_o t}\ e^{j(\omega_o t+\angle X)}\ \}1(t) = Im\,\{\ |X|e^{j\angle X}\ e^{(\sigma_o+j\omega_o)t}\ \}1(t). \tag{3.14}$$

These clearly show that exponential sinusoids (and hence sinusoids) can be created from complex exponentials. We say that the above exponential sinusoids are generated by the complex exponential

$$X(t) = X\ e^{(\sigma_o+j\omega_o)t}\ 1(t) \tag{3.15}$$

where

$$X \stackrel{\rm def}{=} |X|\ e^{j\angle X} \tag{3.16}$$

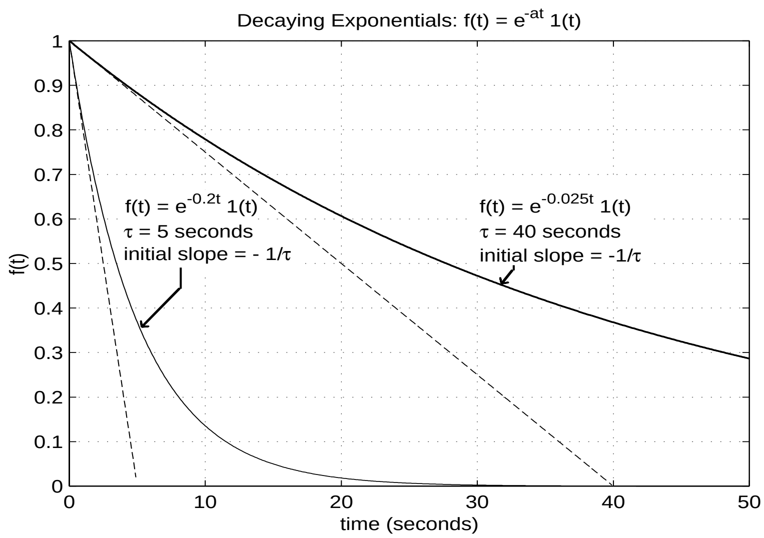

Figure 3.5: Decaying Exponentials

is called the *complex phasor representation* of the above real exponential sinusoids. The motivation behind the latter new terminology is that once the "complex frequency" $s_o = \sigma_o + j\omega_o$ is noted, the complex exponential $X(\cdot)$ and hence the original exponential sinusoids are completely determined by the complex phasor X. The above exponential sinuosids are said to have

- neper frequency $|\sigma_o|$,
- time constant $\tau \stackrel{\text{def}}{=} \frac{1}{|\sigma_o|}$ (assuming $\sigma_o < 0$ so that the exponentials decay),
- frequency of oscillation ω_o,
- cyclic frequency $f \stackrel{\text{def}}{=} \frac{\omega_o}{2\pi}$,
- period of oscillation $T \stackrel{\text{def}}{=} \frac{2\pi}{\omega_o}$,
- phase shift $-\theta$,
- time shift $-\frac{\theta}{\omega_o}$,
- complex frequency $s_o = \sigma_o + j\omega_o$,
- conjugate frequency $\bar{s}_o = \sigma_o - j\omega_o$, and a
- complex phasor representation $X \stackrel{\text{def}}{=} |X|e^{j\angle X}$.

The neper frequency σ_o defines real exponentials which bound the exponential sinusoids. The exponential sinusoids grow toward ∞ if and only if $\sigma_o = Re\{s_o\} > 0$; i.e. s_o lies in the right half complex plane (see Figure 3.6). The exponential sinusoids decay to zero if and only if $\sigma_o = Re\{s_o\} < 0$; i.e. s_o lies in the left half complex plane (see Figure 3.7).

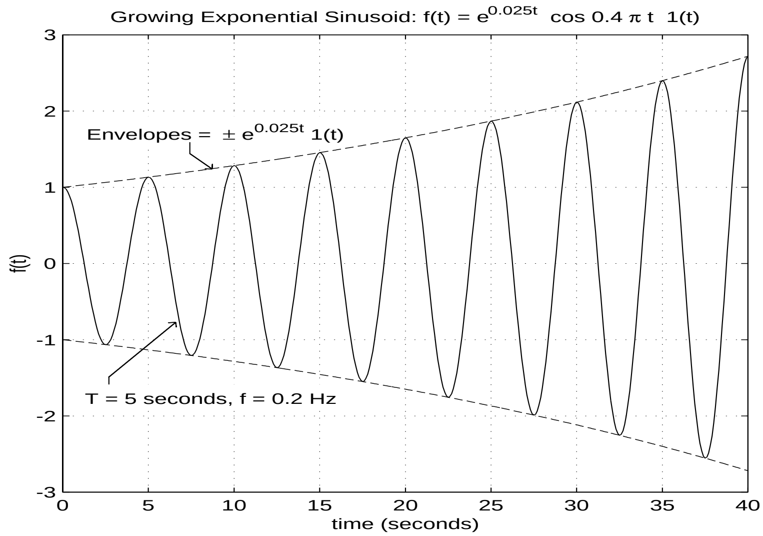

Figure 3.6: Growing Exponential Sinusoid

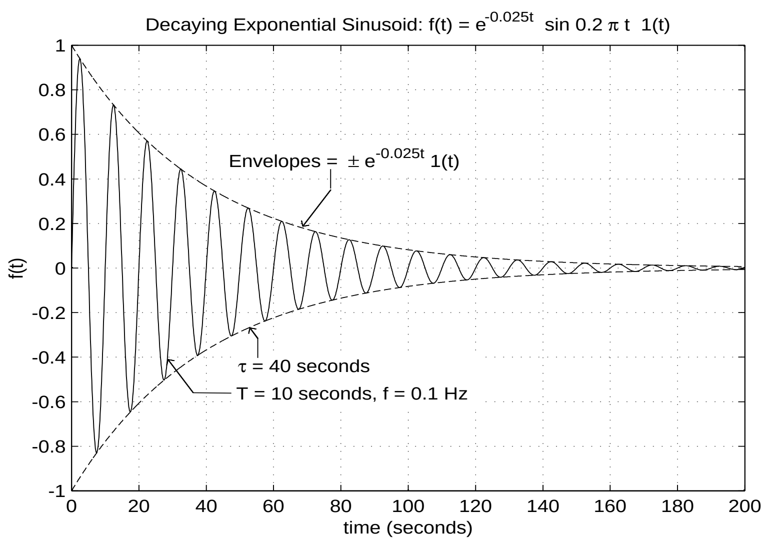

Figure 3.7: Decaying Exponential Sinusoid

In what follows, it will be shown that exponentials (real and complex) are particularly important in the study of linear systems described by linear ordinary differential equations with constant coefficients.

Chapter 4

An Introduction To Laplace Transforms

Many dynamical systems may be modeled or approximated by linear ordinary differential equations with constant coefficients. Given this, an indispensible tool for analyzing such systems is the unilateral Laplace transform [7], [12], [14], [15], [19], [28], [32], [34], [39], [43]. A definition is now provided.

4.1 The Unilateral Laplace Transform: An Introduction

Definition 4.1.1 (Unilateral Laplace Transform.) *The* unilateral Laplace transform *of a function $f(\cdot)$ is denoted $F(s) = (\mathcal{L}f)(s)$ and defined as follows:*

$$F(s) = (\mathcal{L}f)(s) \stackrel{\text{def}}{=} \int_{0^-}^{\infty} f(\tau)e^{-s\tau}d\tau. \tag{4.1}$$

The purpose of the minus sign on the lower limit is to ensure that Dirac delta distributions at the origin are captured. The set of complex numbers s for which the Laplace integral converges (i.e. makes sense, is finite) is called the region of convergence *(ROC) of $F(\cdot)$ [32].* ∎

A list of elementary Laplace transform pairs is contained in Table 4.1. A comprehensive list of Laplace transform properties is contained in Table 4.2.

	$f(t)$	$F(s)$	ROC
1.	$\delta(t)$	1	All s
2.	$\frac{t^n}{n!}e^{at}\ 1(t)$	$\frac{1}{(s-a)^{n+1}}$	$Re\ s > Re\ a$
3.	$e^{\sigma_o t}\cos(\omega_o t+\theta)\ 1(t)$	$\frac{\cos\theta(s-\sigma_o)-\omega_o\sin\theta}{s^2-2\sigma_o s+\sigma_o^2+\omega_o^2}$	$Re\ s > \sigma_o$
4.	$e^{\sigma_o t}\sin(\omega_o t+\theta)\ 1(t)$	$\frac{\sin\theta(s-\sigma_o)+\omega_o\cos\theta}{s^2-2\sigma_o s+\sigma_o^2+\omega^2}$	$Re\ s > \sigma_o$

Table 4.1: Elementary Laplace Transform Pairs

t-Description	f(t)	F(s)	s-Description
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$	Linearity
Exponentiation	$e^{at} f(t)$	$F(s-a)$	Shift
Multiplication	$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n} F(s)$	Differentiation
Scaling	$f(\frac{t}{a})$	$aF(as)$	Scaling
Shift (Delay)	$f(t-\Delta)1(t-\Delta)$	$e^{-s\Delta}F(s); \quad \Delta \geq 0$	Exponentiation
Differentiation	$\overset{(n)}{f}(t)$	$s^n F(s) - \Sigma_{k=1}^n s^{n-k} \overset{(k-1)}{f}(0^-)$	Multiplication
Integration	$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$	Division
Convolution	$(f_1 * f_2)(t) \overset{\text{def}}{=} \int_0^t f_1(t-\tau)f_2(\tau)d\tau$	$F_1(s)F_2(s)$	Function Multiplication
Function Multiplication	$f_1(t)f_2(t)$	$\frac{1}{j2\pi}\int_{c-j\infty}^{c+j\infty} F_1(s-p)F_2(p)dp$	Convolution

Table 4.2: Laplace Transform Properties

4.2 Elementary Unilateral Laplace Transforms

Example 4.2.1 (Dirac Delta Distribution.) *Show that*

$$\delta(t) \overset{\mathcal{L}}{\longleftrightarrow} 1. \tag{4.2}$$

■

Example 4.2.2 (Unit Step Function.) *Show that*

$$1(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}, \quad Re\ s > 0. \tag{4.3}$$

Solution:

Letting $f(t) = 1(t)$, *it follows that*

$$F(s) = \int_{0^-}^{\infty} e^{-s\tau} d\tau = \frac{e^{-s\tau}}{-s}|_0^{\infty} = \frac{e^{-s\tau}}{s}|_{\infty}^{0}. \tag{4.4}$$

Letting $s = \sigma + j\omega$ *in the above exponential yields*

$$F(s) = \frac{e^{-(\sigma+j\omega)\tau}}{s}|_{\infty}^{0} = \frac{e^{-(\sigma+j\omega)\tau}}{s}|_{\infty}^{0} = \frac{1}{s}\left[1 - e^{-\sigma\tau}e^{j\omega\tau}|_{\tau\to\infty}\right]. \tag{4.5}$$

From this it follows that the right hand side makes sense (i.e. is finite) if and only if $Res = \sigma > 0$. *It thus follows that*

$$F(s) = \frac{1}{s} \tag{4.6}$$

where $Re\ s > 0$ *is the region of convergence of* $F(\cdot)$; *i.e. the set of* s *values in the complex plane for which the integral*

$$F(s) = \int_{0^-}^{\infty} e^{-s\tau} d\tau \tag{4.7}$$

makes sense is given by

$$ROC_F = \{\ s \in \mathcal{C} \mid Re\ s > 0\ \}. \tag{4.8}$$

For any other values of s, *the integral makes no sense. Try, for example, the values* $s = -1, 1 + j1, j1$. *For these values, the integral makes no sense.* ∎

Example 4.2.3 (Exponential Function.) *Show that*

$$e^{at}1(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s-a}, \quad Re\ s > Re\ a \tag{4.9}$$

where $a \in \mathcal{C}$ *is an arbitrary complex number.*

Solution:

Letting $f(t) = e^{at}1(t)$, *it follows that*

$$F(s) = \int_{0^-}^{\infty} e^{at}e^{-s\tau} d\tau = \int_{0^-}^{\infty} e^{-(s-a)\tau} d\tau = \frac{e^{-(s-a)\tau}}{-(s-a)}|_0^{\infty} = \frac{e^{-(s-a)\tau}}{(s-a)}|_{\infty}^{0}. \tag{4.10}$$

Letting $s = \sigma + j\omega$ in the above exponential yields

$$F(s) = \frac{e^{-(\sigma+j\omega-a)\tau}}{s-a}|_{\infty}^{0} = \frac{e^{-(\sigma+j\omega-a)\tau}}{s-a}|_{\infty}^{0} = \frac{1}{s-a}\left[1 - e^{-(\sigma-a)\tau}e^{j\omega\tau}|_{\tau\to\infty}\right]. \tag{4.11}$$

From this it follows that the right hand side makes sense (i.e. is finite) if and only if $Re\ s = \sigma > Re\ a$. It thus follows that

$$F(s) = \frac{1}{s-a} \tag{4.12}$$

where $Res > Rea$ is the region of convergence of $F(\cdot)$; i.e. the set of s values in the complex plane for which the integral

$$F(s) = \int_{0^-}^{\infty} e^{-(s-a)\tau} d\tau \tag{4.13}$$

makes sense is given by

$$ROC_F = \{\ s \in \mathcal{C} \mid Re\ s > Re\ a\ \}. \tag{4.14}$$

For any other values of s, the integral makes no sense. ∎

In the above example, the parameter a may be real, purely imaginary, or complex.

Example 4.2.4 (Exponentiation and Frequency Shifting Property.) *Show that*

$$e^{at} f(t) \ \stackrel{\mathcal{L}}{\longleftrightarrow}\ F(s-a) \tag{4.15}$$

where $a \in \mathcal{C}$ is an arbitrary complex number. ∎

Example 4.2.5 (Exponentiation and Frequency Shifting Property.) *Show that*

$$e^{\sigma_o t}\sin(\omega_o t + \theta)\ 1(t) \ \stackrel{\mathcal{L}}{\longleftrightarrow}\ \frac{\sin\theta(s-\sigma_o) + \omega_o\cos\theta}{s^2 - 2\sigma_o s + \sigma_o^2 + \omega^2} \qquad Re\ s > \sigma_o \tag{4.16}$$

∎

Example 4.2.6 (Scaling Property.) *Show that*

$$f(\frac{t}{a}) \stackrel{\mathcal{L}}{\longleftrightarrow} aF(as) \tag{4.17}$$

for any real constant a. ∎

Example 4.2.7 (Time Multiplication and Frequency Differentiation Property.) *Show that*

$$tf(t) \stackrel{\mathcal{L}}{\longleftrightarrow} -\frac{d}{ds}F(s). \tag{4.18}$$

∎

Example 4.2.8 (Powers of t.) *Show that*

$$\frac{t^n}{n!}1(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s^{n+1}} \tag{4.19}$$

for $n = 0, 1, 2, \ldots$. ∎

Example 4.2.9 (Time Differentiation Property.) *Show that*

$$\frac{d}{dt}f(t) \stackrel{\mathcal{L}}{\longleftrightarrow} sF(s) - f(0^-) \tag{4.20}$$

$$\frac{d^2}{dt^2}f(t) \stackrel{\mathcal{L}}{\longleftrightarrow} s^2F(s) - sf(0^-) - \dot{f}(0^-) \tag{4.21}$$

$$\frac{d^3}{dt^3}f(t) \stackrel{\mathcal{L}}{\longleftrightarrow} s^3F(s) - s^2f(0^-) - s\dot{f}(0^-) - \ddot{f}(0^-). \tag{4.22}$$

∎

Example 4.2.10 (Higher Order Deltas.) *Show that*

$$\overset{(n)}{\delta}(t) \stackrel{\mathcal{L}}{\longleftrightarrow} s^n \tag{4.23}$$

■

Example 4.2.11 (Time Delay Property.) *Show that*

$$f(t-\Delta)1(t-\Delta) \stackrel{\mathcal{L}}{\longleftrightarrow} e^{-s\Delta}F(s) \tag{4.24}$$

for any $\Delta \geq 0$. ■

Example 4.2.12 (Time Convolution-Frequency Multiplication Property.) *Show that*

$$\int_0^t f(t-\tau)g(\tau)d\tau \stackrel{\mathcal{L}}{\longleftrightarrow} F(s)G(s). \tag{4.25}$$

■

Example 4.2.13 (Integration Property.) *Show that*

$$\int_0^t f(\tau)d\tau \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{F(s)}{s}. \tag{4.26}$$

■

Example 4.2.14 (Initial Value Theorem.) *Suppose that* $F(\cdot)$ *is a real-rational function. Show that*

$$f(0^+) = \lim_{s\longrightarrow\infty} sF(s). \tag{4.27}$$

■

Example 4.2.15 (Final Value Theorem.) *Suppose that* $F(\cdot)$ *is a real-rational function. Show that if the limit*

$$f(\infty) \stackrel{\text{def}}{=} \lim_{t\longrightarrow\infty} f(t) \tag{4.28}$$

exists, then

$$f(\infty) = \lim_{s \longrightarrow 0} sF(s). \tag{4.29}$$

■

4.3 Applications

The following example is used to illustrate how Laplace transform techniques are applied to analyze systems described by linear ordinary differential equations with constant coefficients. It is also used to initiate the introduction of many fundamental system concepts.

Example 4.3.1 (Analysis Of A First Order System.)

Consider a single-input single-output (SISO) system with input $u(\cdot)$ and output $y(\cdot)$, related by the following first order linear ordinary differential equation with constant coefficients:

$$\dot{y}(t) = -ay(t) + bu(t) \qquad y(0^-) = y_o \qquad t > 0 \tag{4.30}$$

where $a, b > 0$. The purpose of this example is to illustrate elementary Laplace transform techniques and to initiate the introduction of fundamental system concepts.

I. Input-Output Modeling. *To provide motivation, it is useful to note that the above input-output model may be used to approximate the input-output behavior of many physical systems. These might include*

- *an RC series circuit with y and u representing voltages,*
- *an autonomous vehicle with y representing vehicle speed and u representing an applied force,*
- *a mechanical shaft with y representing the angular speed of the shaft and u representing an applied torque,*

- *a semiconductor diffusion process with y representing substrate temperature and u representing an applied power.*

II. System Order. *Since our system is modeled by a first order differential equation, we say that we have a* first order system.

III. Analysis Via Laplace Transforms. *Let $Y(\cdot) = (\mathcal{L}y)(\cdot)$ and $U(\cdot) = (\mathcal{L}u)(\cdot)$. Taking the Laplace transform of each side of equation (4.30) then yields*

$$sY(s) - y_o = -aY(s) + bU(s) \tag{4.31}$$

or

$$Y(s) = \frac{y_o}{s+a} + \left[\frac{b}{s+a}\right] U(s). \tag{4.32}$$

This shows that $Y(\cdot)$ consists of two terms. The first term is due to initial conditions. The second term is due to the system input (or forcing function) $u(\cdot)$.

IV. Characteristic Equation, System Poles, And Natural Modes. *Both terms in equation (4.32) have the common denominator*

$$\Phi(s) = s + a. \tag{4.33}$$

This polynomial is caled the characteristic equation *or* characterisitc polynomial *of the system. The roots of this equation are called the* system poles. *Given this, it follows that our system has a single pole at*

$$s = -a \tag{4.34}$$

in the left half complex s-plane (since $a > 0$). From Laplace transforms, we know that this pole corresponds to a decaying exponential

$$e^{-at}\, 1(t) \tag{4.35}$$

in the time domain. We thus say that our system possesses a natural exponential decaying mode.

V. Linear System. *Equation (4.32) shows that* $Y(\cdot)$ *is linear in* $[\, y_o \quad U(\cdot)\,]$. *That is, if* $[\, y_{o_i} \quad U_i(\cdot)\,]^T$ *produces* $Y_i(\cdot)$ $(i = 1, 2)$, *then*

$$\begin{bmatrix} y_o \\ U(\cdot) \end{bmatrix} = a_1 \begin{bmatrix} y_{o_1} \\ U_1(\cdot) \end{bmatrix} + a_2 \begin{bmatrix} y_{o_2} \\ U_2(\cdot) \end{bmatrix} \overset{produces}{\Rightarrow} \quad Y(\cdot) = a_1 Y_1(\cdot) + a_2 Y_2(\cdot). \tag{4.36}$$

Since the inverse Laplace operator $\mathcal{L}^{-1}$ *is also a linear operator, it follows that* $y(\cdot)$ *is linear in* $[\, y_o \quad u(\cdot)\,]$; *i.e. if* $[\, y_{o_i} \quad u_i(\cdot)\,]^T$ *produces* $y_i(\cdot)$ $(i = 1, 2)$, *then*

$$\begin{bmatrix} y_o \\ u(\cdot) \end{bmatrix} = a_1 \begin{bmatrix} y_{o_1} \\ u_1(\cdot) \end{bmatrix} + a_2 \begin{bmatrix} y_{o_2} \\ u_2(\cdot) \end{bmatrix} \overset{produces}{\Rightarrow} \quad y(\cdot) = a_1 y_1(\cdot) + a_2 y_2(\cdot). \tag{4.37}$$

Because of this, we say the the system under consideration is a linear system.

VI. Zero Input Response. *The first term in equation (4.32) is identified as the Laplace transform of the* zero input response

$$y_{zir}(t) = y_o e^{-at}\, 1(t); \tag{4.38}$$

i.e. $Y_{zir}(s) \overset{\text{def}}{=} (\mathcal{L} y_{zir})(s) = \frac{y_o}{s+a}$. *The function* $y_{zir}(\cdot)$ *represents the* natural response *of the system to an initial condition* y_o *with no input* $(u = 0)$; *i.e. it is a solution to the* homogeneous equation

$$\dot{y}(t) = -a y(t) \qquad y(0^-) = y_o \qquad t > 0. \tag{4.39}$$

The term "homogeneous equation" comes from differential equations. The natural response exhibits the natural modes of the system. Our system, as mentioned above, has a natural exponential decaying mode.

VII. Equilibrium And Exponential Stability. *From the original system differential equation (equation (4.30)), one notes that when* $u = 0$, *and hence* $\dot{y} = -ay$, *the system has a natural* equilibrium *of* $y = 0$. *That is, when left alone with* $y_o = u = 0$, *the system remains at* $y = 0$. *From equation (4.38), we see that if the system is given an offset* y_o *then it naturally returns back (exponentially) to its* *equilibrium* $y = 0$. *Because of this, we say that the system, or more precisely the equilibrium* $y = 0$ *is* exponentially stable.

VIII. System Transfer Function. *From equation (4.32), it follows that*

$$H(s) \stackrel{\text{def}}{=} \frac{(\mathcal{L}y)(s)}{(\mathcal{L}u)(s)}\Big|_{\substack{\text{zero initial}\\ \text{conditions}}} \tag{4.40}$$

$$= \frac{b}{s+a}. \tag{4.41}$$

The region of convergence of $H(\cdot)$ *is given by*

$$ROC_H = \{s \in \mathcal{C} \mid Re\ s > -a\}. \tag{4.42}$$

This region defines a half plane which lies to the left of the system pole at $s = -a$. *This zero initial condition ratio is called the* transfer function of the system. *It is important to note that the system transfer function completely determines the system input-output linear differential equation model and vice versa. That is, knowledge of one implies knowledge of the other. This follows from the very important zero inital condition transform pair*

$$\dot{f}(t) \stackrel{\mathcal{L}}{\longleftrightarrow} sF(s). \tag{4.43}$$

IX. System Impulse Response. *Now suppose that* $y_o = 0$ *and that the input is a unit Dirac delta distribution; i.e.* $u(t) = \delta(t)$. *Under these conditions,* $Y(s) = H(s)$ *and hence* $y(t) = h(t) \stackrel{\text{def}}{=} (\mathcal{L}^{-1}H)(t)$. *Because of this, we call*

$$h(t) \stackrel{\text{def}}{=} (\mathcal{L}^{-1}H)(t) = be^{-at}1(t) \tag{4.44}$$

the impulse response of the system. *As with the system transfer function* $H(\cdot)$, *the system impulse response* $h(\cdot)$ *completely specifies the system differential equation. Moreover, knowledge of any one of the following:*

- *system ordinary differential equation,*
- *system transfer function* $H(\cdot)$,

- *system impulse response* $h(\cdot)$,

implies knowledge of the other two.

X. General Solution And Convolution. *From equation (4.32), it follows that if* y_o *and* $U(\cdot)$ *are known, then partial fraction expansion inversion methods can be used to determine* $y(\cdot)$*. While this powerful technique is available, it is also very useful to note that*

$$y(t) = y_{zir}(t) + (h * u)(t) \tag{4.45}$$

$$= y_{zir}(t) + \int_0^t h(t-\tau)u(\tau)d\tau \qquad t > 0 \tag{4.46}$$

where $(h * u)(\cdot)$ *denotes the* convolution *of the system impulse response* $h(\cdot)$ *with the system input* $u(\cdot)$*. This shows that convolution occurs naturally for systems described by linear ordinary differential equations with constant coefficients. Equation (4.46) gives us another method for determining* $y(\cdot)$ *given* y_o *and* $u(\cdot)$ *(and knowledge of* $h(\cdot)$*). While the first term is called the zero input response, the second term is called the* forced response. *The second term is also* a particular solution *to the system differential equation (assuming zero initial conditions). The term "particular solution" comes from differential equations.*

XI. Casual System. *From equation (4.46), it follows that the current output* $y(t)$ *depends on current and past values of the input* $u(\cdot)$*. This implies that the system cannot produce an output* $y(\cdot)$ *until an input* $u(\cdot)$ *is applied. Because of this, we say that the system under consideration is a* causal system.

XII. Bounded-Input Bounded-Output (BIBO) Stability. *Suppose that the system is initially inert (i.e.* $y_o = 0$*). This implies that*

$$y(t) = \int_0^t h(t-\tau)u(\tau)d\tau = \int_0^t u(t-\tau)h(\tau)d\tau. \tag{4.47}$$

Suppose that a bounded input $u(\cdot)$ *is applied to the system. Specifically, suppose that*

$$|u(t)| \le M \tag{4.48}$$

for all t. Given this, it follows that

$$\begin{aligned} |y(t)| &\leq \int_0^t |u(t-\tau)h(\tau)|d\tau && (4.49)\\ &\leq \int_0^t M|h(\tau)|d\tau && (4.50)\\ &\leq M\int_0^t |h(\tau)|d\tau && (4.51)\\ &\leq M\int_0^\infty |h(\tau)|d\tau && (4.52)\\ &\leq M\int_0^\infty be^{-a\tau}d\tau && (4.53)\\ &\leq M\frac{b}{a}. && (4.54)\end{aligned}$$

This implies that a bounded-input $u(\cdot)$ results in a bounded-output $y(\cdot)$. We thus say that the system under consideration is bounded-input bounded-output (BIBO) stable.

XIII. Time Invariance. *Suppose that $y_o = 0$. It then follows that $Y(s) = H(s)U(s)$. Suppose $u_1(\cdot)$ produces $y_1(\cdot)$. What is the output $y(\cdot)$ when a delayed input $u(t) = u_1(t-\Delta)$ ($\Delta \geq 0$) is applied to the system? This input results in*

$$Y(s) = H(s)U_1(s) = H(s)U(s)e^{-s\Delta} = Y_1(s)e^{-s\Delta}. \qquad (4.55)$$

and hence $y(t) = y_1(t-\Delta)$; i.e. shifted inputs result in shifted outputs. Because of this, we say that our system is a time invariant system. *Fundamentally, our system is time invariant becuase the coefficients in the differential equation are constant. Because they are constant, the "shape" of the output doesn't depend on when the input is applied.*

XIV. Analysis At $t = t_o$. *Suppose that the input $u(\cdot)$ is applied at some time $t = t_o \in \mathcal{R}$. That is, suppose that we are interested in*

$$\dot{y}(t) = -ay(t) + bu(t) \qquad y(t_o^-) = y_o \qquad t > t_o. \qquad (4.56)$$

In such a case, we have

$$y(t) = e^{-a(t-t_o)}y_o + \int_{t_o}^t h(t-\tau)u(\tau)d\tau \qquad t > t_o. \qquad (4.57)$$

XV. Concept Of State, State Variables, State Equation. *The* state of the system at time t_o *is defined to be the minimum amount of information required to compute the future evolution of the system output* $y(\cdot)$*, given knowledge of the input* $u(\cdot)$*. With this definition, it follows from equation (4.57) that the state of the system at* $t = t_o$ *is precisely the inital condition* y_o*. (One cannot, for example, compute the speed of a vehicle given knowledge of the applied force if the inital speed of the vehicle is not known!) Since the state of the system at time* t_o *is* $y(t_o^-) = y_o$ *for any* t_o*, we call* $y(\cdot)$ *a* state variable. *If we let* $x \overset{\text{def}}{=} y$ *denote the state of the system, then equation (4.57) may be rewritten as follows:*

$$\dot{x}(t) = -ax(t) + bu(t) \tag{4.58}$$

$$y(t) = x(t). \tag{4.59}$$

Since the first equation involves the state x*, the equation is called a* state equation. *Since the second equation relates the output* y *to the system state* x *(and the system input* u*), the equation is called an* output equation. *The two equations constitute a* state space representation for the system.

XVI. Zero State Response. *Just as* $y_{zir}(\cdot)$ *denotes the zero input response, the above terminology leadsone to call the second term (forced response,* $(h * u)(\cdot)$*) in equation (4.57), the* zero state response (ZSR)*; i.e. it is the response of the system to the input* $u(\cdot)$ *assuming zero initial conditions (*$y_o = 0$*, zero initial state).*

XVII. Step Response. *The response to a step input of size* k *is found by setting* $U(s) = \frac{k}{s}$ *and using partial fraction expansion methods as follows:*

$$Y(s) = H(s)U(s) = \left[\frac{b}{s+a}\right]\frac{k}{s} = H(0)k\left[\frac{1}{s} - \frac{1}{s+a}\right] = \frac{b}{a}k\left[\frac{1}{s} - \frac{1}{s+a}\right]. \tag{4.60}$$

From this it follows that the step response is given by

$$y(t) = H(0)k\left[1 - e^{-at}\right]1(t) = \frac{b}{a}k\left[1 - e^{-at}\right]1(t). \tag{4.61}$$

The exponential portion of equation (4.61) is due to the system's natural exponential mode. This portion is called the transient *portion of the response or the* transient response. *The constant portion of equation (4.61) is due to the system input. This portion is called the* steady state *portion of the response or the* steady state response.

The step response may also be computed using

$$y(t) = \int_0^t u(t-\tau)h(\tau)d\tau = k\int_0^t h(\tau)d\tau. \tag{4.62}$$

Given this, we note further that the unit impulse response $h(\cdot)$ and the unit step response $s(\cdot)$ $(k = 1)$ are related by the following formulae:

$$s(t) = \int_0^t h(\tau)d\tau \tag{4.63}$$

$$h(t) = \frac{d}{dt}s(t). \tag{4.64}$$

Figure 4.1 illustrates how the step response of the system depends on the parameter a with $b = a$. As a increases, the response time decreases.

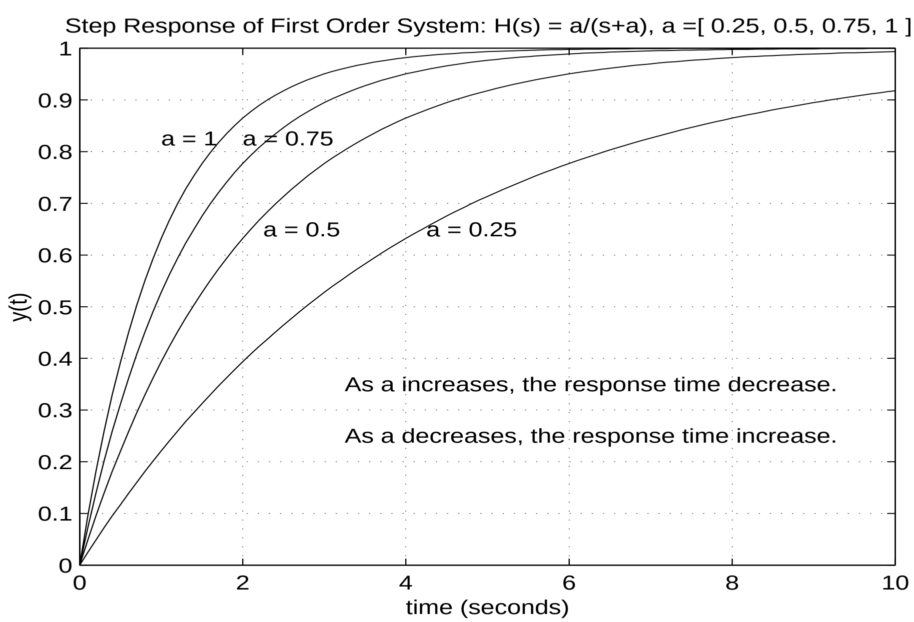

Figure 4.1: Step Response Of A First Order System

XVIII. Steady State Response To Step Input. *From equation (4.61), it follows that as* $t \to \infty$*, the output* $y(\cdot)$ *approaches the* steady state *value:*

$$y_{ss} \stackrel{\text{def}}{=} \lim_{t \longrightarrow \infty} y(t) H(0) k = \frac{b}{a} k \tag{4.65}$$

where

$$H(s)|_{s=j0} = \frac{b}{a} \tag{4.66}$$

is called the dc gain of the system.

XIX. Response To A Sinusoid. *Now suppose that an input* $u(t) = |X| \cos(\omega_o t + \angle X)$ *is applied to the system. Setting*

$$U(s) = |X| \left[\frac{\cos \angle X s - \omega_o \sin \angle X}{s^2 + \omega_o^2} \right] \tag{4.67}$$

yields

$$Y(s) = \left[\frac{b}{s+a} \right] |X| \left[\frac{\cos \angle X s - \omega_o \sin \angle X}{s^2 + \omega_o^2} \right] = \frac{A}{s+a} + \frac{B}{s - j\omega_o} + * \tag{4.68}$$

where $*$ *denotes the conjugate of the preceeding term. From this it follows that*

$$y(t) = \left[A e^{-at} + 2|B| \cos(\omega_o t + \angle B) \right] 1(t) \tag{4.69}$$

where

$$A = \lim_{s \longrightarrow -a} (s+a) Y(s) = -|X| b \, \frac{(a \cos \angle X + \omega_o \sin \angle X)}{a^2 + \omega_o^2} \tag{4.70}$$

$$B = \lim_{s \longrightarrow j\omega_o} (s - j\omega_o) Y(s) = \frac{H(j\omega_o)}{2} |X| e^{j\angle X}. \tag{4.71}$$

The first term in equation (4.69) is due to the system's natural decaying exponential mode. This term is called the transient response since it goes away as $t \longrightarrow \infty$*. The second term in equation (4.69) is due to the sinusoidal forcing function* $u(\cdot)$ *and is called the* steady state response.

XX. Steady State Response To A Sinusoid. *From equation (4.69), one obtains the steady state response y_{ss} to a sinusoid $u(\cdot)$ as follows:*

$$\begin{array}{rcl} u(t) & = & |X|\cos(\omega_o t + \angle X) \\ & = & Re\ \{\ |X|e^{j\angle X}e^{j\omega_o t}\ \} \\ & = & Re\ \{\ Xe^{j\omega_o t}\ \} \end{array} \quad \overset{produces}{\Longrightarrow} \quad \begin{array}{rcl} y_{ss} & = & |H(j\omega_o)||X|\cos(\omega_o t + \angle X + \angle H(j\omega_o)) \\ & = & Re\ \{\ H(j\omega_o)|X|e^{j\angle X}e^{j\omega_o t}\ \} \\ & = & Re\ \{\ H(j\omega_o)Xe^{j\omega_o t}\ \} \end{array} \tag{4.72}$$

where

$$X \stackrel{\text{def}}{=} |X|e^{j\angle X} \tag{4.73}$$

is called the phasor representation *of the input signal $u(\cdot)$. Equation (4.72) implies that in the steady state, the input cosine with frequency ω_o, amplitude $|X|$, and angle $\angle X$, comes out of the system as a cosine signal with the same frequency ω_o, with new amplitude $|H(j\omega_o)||X|$, and new angle $\angle X + \angle H(j\omega_o)$. That is, the amplitude of the input is modified by a multiplicative factor $|H(j\omega_o)|$ and the angle is modified by an additve factor $\angle H(j\omega_o)$.*

XXI. Continuous Time Fourier Transform. *The unilateral continuous time* Fourier transform *of a function $f(\cdot)$ is denoted $(\mathcal{F}f)(j\cdot)$ and defined as follows:*

$$(\mathcal{F}f)(j\omega) \stackrel{\text{def}}{=} \int_0^\infty f(\tau)e^{-j\omega\tau}d\tau. \tag{4.74}$$

In our example, the region of convergence for $H(\cdot) \stackrel{\text{def}}{=} (\mathcal{L}h)(\cdot)$ includes the imaginary axis. Given this, it follows that for the system under consideration $H(j\cdot)$ is the Fourier transform of $h(\cdot)$. In general, however,

$$H(j\omega) \stackrel{\text{def}}{=} (\mathcal{L}h)(s)|_{s=j\omega} \neq (\mathcal{F}h)(j\omega) \tag{4.75}$$

because $s = j\omega$ may not lie within the region of convergence of $H(\cdot) \stackrel{\text{def}}{=} (\mathcal{L}h)(\cdot)$. An example illustrating this point is easily obtained [34, pg. 307], [36]:

$$h(t) = u(t) \qquad H(s) \stackrel{\text{def}}{=} (\mathcal{L}h)(s) = \frac{1}{s} \qquad H(j\omega) = \frac{1}{j\omega} \neq (\mathcal{F}h)(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega). \tag{4.76}$$

In general,

$$(\mathcal{L}h)(s)|_{s=j\omega} = (\mathcal{F}h)(j\omega) \tag{4.77}$$

if and only if the $s = j\omega$ lies within the region of convergence of $(\mathcal{L}h)(s)$. The above statements also apply when working with bilateral Laplace and Fourier transforms:

$$(\mathcal{L}h)(s) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \tag{4.78}$$

$$(\mathcal{F}h)(j\omega) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau. \tag{4.79}$$

XXII. Frequency Response. *The function $H(j\omega) \stackrel{\text{def}}{=} (\mathcal{L}h)(s)|_{s=j\omega}$ is called the* frequency response *of the system. The function $|H(j\omega)|$ is called the* magnitude response *of the system. Its relevance is succinctly conveyed by equation (4.72). For our system, the magnitude response function is given as follows:*

$$|H(j\omega)| = \frac{b}{\sqrt{\omega^2 + a^2}}. \tag{4.80}$$

The function $\angle H(j\omega)$ is called the phase response of the system. *Its relevance is also succinctly conveyed by equation (4.72). For our system, the magnitude response function is given as follows:*

$$\angle H(j\omega) = -\tan^{-1}\frac{\omega}{a}. \tag{4.81}$$

Figure 4.2 illustrates how the frequency response of the system depends on the parameter a with $b = a$. In this figure, $20log_{10}|H(j\omega)|$ is plotted on a linear vertical scale versus ω in radians/second on a horizontal logarithmic (base 10) scale. We say that $|H(j\omega)|$ is plotted in unit of decibels (dB) versus ω on a semilog grid. As a is increased, the system magnitude response is seen to shift towards the right. In the same figure, $\angle H(j\omega)$ is plotted in degrees on a linear vertical scale versus ω in radians/second on a horizontal logarithmic (base 10) scale. As a is increased, the angle response is seen to shift towards the right.

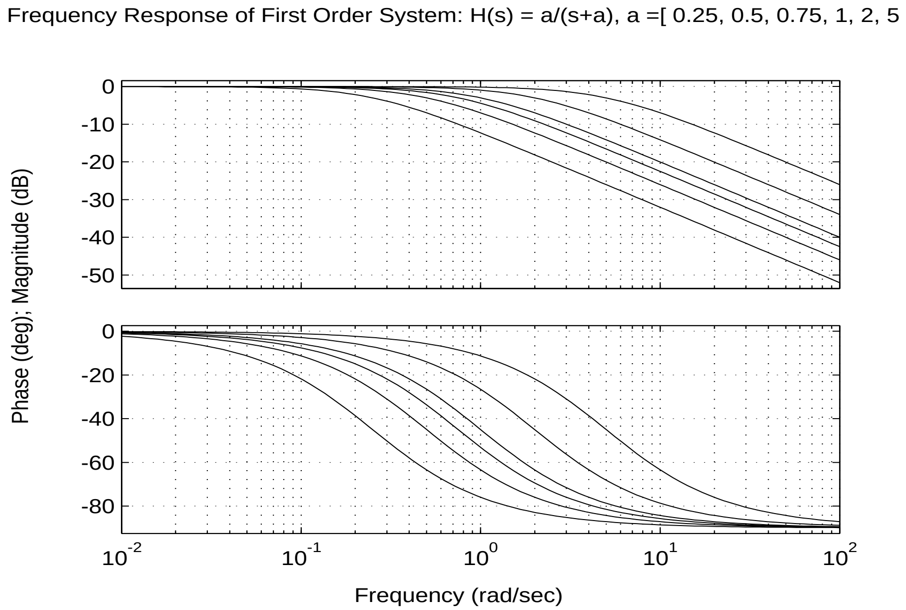

Figure 4.2: Frequency Response Of A First Order System

XXIII. Bandwidth. *The bandwidth of a system is a term which is aimed at quantifying the ability or inability of a system to amplify rapidly varying signals. The most widely used definition for bandwidth is as follows. The bandwidth of a system is defined to be that frequency BW at which*

$$|H(j\omega)| = H(0)\frac{1}{\sqrt{2}}. \tag{4.82}$$

For our system, we have

$$BW = a. \tag{4.83}$$

Given this, Figure 4.2 shows that as a is increased the system bandwidth increases. This explains why the step response looks more and more like a step as a is increased. ∎

The following example illustrates how elementary transform techniques may be used to address linear systems described by linear ordinary differential equations with constant coefficients.

Example 4.3.2 (5^{th} Order Linear Ordinary Differential Equation.)

Suppose that a system with input $u(\cdot)$ and output $y(\cdot)$ is described by the following 5^{th} order linear ordinary differential equation with constant coefficients

$$\overset{(5)}{y}(t) + a_4 \overset{(4)}{y}(t) + a_3 \overset{(3)}{y}(t) + a_2 \overset{(2)}{y}(t) + a_1 \dot{y}(t) + a_o y(t) = u(t) \tag{4.84}$$

where

$$s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_o = s(s+1)^2(s^2+4s+13). \tag{4.85}$$

In this example, we illustrate elementary transform techniques and we introduce fundamental system concepts. We proceed in several steps.

I. System Transfer Function. *Assuming zero initial conditions, we begin by determining the ratio*

$$H(s) \stackrel{\text{def}}{=} \frac{(\mathcal{L}y)(s)}{(\mathcal{L}u)(s)}\Big|_{\substack{\text{zero initial}\\ \text{conditions}}}. \tag{4.86}$$

This ratio is called the transfer function of the system. *Taking Laplace transforms of both sides of equation (4.84) yields the real-rational s-domain function*

$$H(s) = \frac{1}{s(s+1)^2(s^2+4s+13)}. \tag{4.87}$$

II. Sinusoidal Response. *The goal now is to determine $y(\cdot)$ when $u(t) = |X|\sin(\omega_o t + \angle X)\ 1(t)$. This requires that we take the Laplace transform of equation (4.84) and use the fact that*

$$U(s) = \frac{-\sin\angle X s + \omega_o \cos\angle X}{s^2+\omega_o^2} \tag{4.88}$$

to obtain

$$Y(s) = H(s)U(s) = \frac{-\sin\angle X s + \omega_o \cos\angle X}{s(s+1)^2(s^2+4s+13)(s^2+\omega_o^2)}. \tag{4.89}$$

To determine the inverse transform $y(\cdot)$ it is essential to note that this $Y(\cdot)$ has poles *at $s = 0, -1, -1, -2 \pm j3, \pm j\omega_o$. These poles are critical for determining y. Next, expand Y in a partial fraction expansion [32] as follows:*

$$Y(s) = \frac{A}{s} + \frac{B}{(s+1)^2} + \frac{C}{s+1} + \frac{D}{s+2-j3} + * + \frac{E}{s-j\omega_o} + * \tag{4.90}$$

where A, B, C, D, E *are constants to be determined below and* $*$ *denotes the complex conjugate of the previous term in the expansion. In performing the above partial fraction expansion, we have used the fact that the sum is a rational function with real coefficients and hence the complex quantities must occur in complex conjugate pairs. Taking the inverse Laplace transform of each term yields:*

$$y(t) = [\ A + Bte^{-t} + Ce^{-t} + De^{(-2+j3)t} + \ * \ + Ee^{j\omega_o t} + \ * \ ]\ 1(t) \tag{4.91}$$

Substituting $D = |D|e^{j\angle D}$ *and* $E = |E|e^{j\angle E}$ *into the above expression yields:*

$$y(t) = [\ A + Bte^{-t} + Ce^{-t} + |D|e^{-2t}e^{j(3t+\angle D)} + \ * \ + |E|e^{j(\omega_o t+\angle E)} + \ * \ ]\ 1(t) \tag{4.92}$$

Simple application of Euler's formula then yields:

$$y(t) = [\ A + Bte^{-t} + Ce^{-t} + 2Re\ \{|D|e^{-2t}e^{j(3t+\angle D)}\} + 2Re\ \{|E|e^{j(\omega_o t+\angle E)}\}\ ]\ 1(t) \tag{4.93}$$

or

$$y(t) = [\ A + Bte^{-t} + Ce^{-t} + 2|D|e^{-2t}\cos(3t+\angle D) + 2|E|\cos(\omega_o t+\angle E)\ ]\ 1(t) \tag{4.94}$$

It now remains to determine the coefficients A, B, C, D, E*. These can be determined using the following standard partial fraction expansion formulae [32, pp. 35-44]:*

$$\begin{aligned}
A &= \lim_{s\to 0} sY(s) = \frac{-\sin\angle X s + \omega_o\cos\angle X}{(s+1)^2(s^2+4s+13)(s^2+\omega_o^2)}\ |_{s=0} && (4.95)\\
B &= \lim_{s\to -1}(s+1)^2Y(s) = \frac{-\sin\angle X s + \omega_o\cos\angle X}{s(s^2+4s+13)(s^2+\omega_o^2)}\ |_{s=-1} && (4.96)\\
C &= \lim_{s\to -1}\frac{d}{ds}[(s+1)^2Y(s)] = \frac{d}{ds}[\ \frac{-\sin\angle X s + \omega_o\cos\angle X}{s(s^2+4s+13)(s^2+\omega_o^2)}\ ]\ |_{s=-1} && (4.97)\\
D &= \lim_{s\to -2+j3}(s+2-j3)Y(s) = \frac{-\sin\angle X s + \omega_o\cos\angle X}{s(s+1)^2(s+2+j3)(s^2+\omega_o^2)}\ |_{s=-2+j3} && (4.98)\\
E &= \lim_{s\to j\omega_o}(s-j4)Y(s) = \frac{-\sin\angle X s + \omega_o\cos\angle X}{s(s+1)^2(s^2+4s+13)(s+j\omega_o)}\ |_{s=j\omega_o}. && (4.99)
\end{aligned}$$

III. Sinusoidal Steady State Analysis. *It is useful to note that for large* t*,* y *may be approximated as follows:*

$$y(t) \approx [\ A + 2|E|\cos(\omega_o t+\angle E)\ ]\ 1(t) \tag{4.100}$$

The righthand side is called the steady state *component of y and is denoted*

$$y_{ss} = [\ A + 2|E|\cos(\omega_o t + \angle E)\ ]\ 1(t) \tag{4.101}$$

Specifically, it is useful to note that the steady state component *due to the input sinusoid $u(\cdot)$ is also a sinusoid. Specifically, it can be shown that*

$$E \;=\; \lim_{s \to j\omega_o} (s - j\omega_o) Y(s) = \frac{-\sin\angle X s + \omega_o \cos\angle X}{s(s+1)^2(s^2+4s+13)(s+j\omega_o)} \Big|_{s=j\omega_o} = \frac{H(j\omega_o)}{2} e^{j90\,\mathrm{deg}} \tag{4.102}$$

This, however implies that

$$\begin{aligned} u(t) &= |X|\sin(\omega_o t + \angle X) \\ &= Im\ \{\ |X| e^{j\angle X} e^{j\omega_o t}\ \} \\ &= Im\ \{\ X e^{j\omega_o t}\ \} \end{aligned} \quad \overset{produces}{\Longrightarrow} \quad \begin{aligned} y_{ss_{sinusoid}} &= |H(j\omega_o)||X|\cos(\omega_o t + \angle X + \angle H(j\omega_o) - 90\,\mathrm{deg}) \\ &= |H(j\omega_o)||X|\sin(\omega_o t + \angle X + \angle H(j\omega_o)) \\ &= Im\ \{\ H(j\omega_o)|X| e^{j\angle X} e^{j\omega_o t}\ \} \\ &= Im\ \{\ H(j\omega_o) X e^{j\omega_o t}\ \} \end{aligned} \tag{4.103}$$

where

$$X \overset{\mathrm{def}}{=} |X| e^{j\angle X} \tag{4.104}$$

is called the phasor representation *of the input signal $u(\cdot)$. Equation (4.103) implies that in the steady state, the input sine wave with frequency ω_o, amplitude $|X|$, and angle $\angle X$, results in a steady state sinusoidal component with the same frequency ω_o, with new amplitude $|H(j\omega_o)||X|$, and new angle $\angle X + \angle H(j\omega_o)$. That is, the amplitude of the input is modified by a multiplicative factor $|H(j\omega_o)|$ and the angle is modified by an additive factor $\angle H(j\omega_o)$.*

V. Frequency Response. *The function $H(j\omega)$ is called the* frequency response *of the system. The function $|H(j\omega)|$ is called the* magnitude response *of the system. Its relevance is succinctly conveyed by equation (4.103). The function $\angle H(j\omega)$ is called the* phase or angle response of the system. *Its relevance is also succinctly conveyed by equation (4.103).*

■

Chapter 5

SISO Input-Output Models: LTI Systems

5.1 An Introduction To Linear Systems: An Illustrative Example

The following simple RLC series circuit example will be used to motivate, introduce, and illustrate many of the linear system concepts to follow.

Example 5.1.1 (Series RLC Circuit - A Single-Input Single-Output (SISO) System.)

Consider the series RLC circuit shown in Figure 5.1. The circuit, or system, input (forcing function) is the voltage source u. The circuit, or system, output is the voltage measurement y. In this circuit, the resistor, inductor, and capacitor values are denoted R, L, and C respectively. It is assumed that they are nonnegative and constant. The symbol i_L is used to represent the current flowing through the inductor and y represents the voltage across the capacitor.

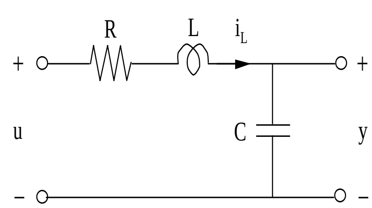

Figure 5.1: Single-Input Single-Output (SISO) Series RLC Circuit

This simple single-input single-output (SISO) example will be used to motivate/illustrate the following concepts/techniques: input-output model, system order, analysis via Laplace transforms, *natural response, stability, characteristic equation, characteristic roots, poles, damping factor, system transfer function, general RLC circuit impedance methods, impulse response, general solution, convolution, linearity, causality, step response, overshoot, undershoot, time invariance, linear time invariance (LTI), bounded-input bouded-output stability, sinusoidal steady state analysis, zeros, frequency response, magnitude response, phase response, Fourier transform, descriptor representations, state, state variables, and state space representations. As such, the example provides an excellent "fuel-rich" vehicle for embarking on our study of linear systems. Moreover, it will be shown that* ALL of the principles and concepts to be illustrated using this very simple example extend effortlessly to arbitrarily complex systems described by linear ordinary differential equations with constant coefficients. ∎

5.2 LTI System Concepts

The following example is used to motivate and introduce various important system concepts.

Example 5.2.1 (Series RLC Circuit Input-Output Model Analysis.)

Consider the series RLC circuit in Figure 5.1.

Input-Output Linear Differential Equation Model. *One can use Kirchhoff's voltage and current laws [23] to relate the output $y(\cdot)$ to the input $u(\cdot)$. Doing so, yields the following second order linear ordinary differential equation with constant coefficients:*

$$\ddot{y}(t) + \frac{R}{L}\dot{y}(t) + \frac{1}{LC}y(t) = \frac{1}{LC}u(t). \tag{5.1}$$

This equation is subject to the two initial conditions: $y(0^-) = y_o$, $\dot{y}(0^-) = \dot{y}_o$. Equation (5.1) relates the output $y(\cdot)$ to the input $u(\cdot)$ for $t > 0$. As such, we say that it defines a linear differential equation input-output model *for the system. Because the system is described by a second order differential equation, we say that the system is a* second order system*. It should be noted* *that we have obtained a second order model because we have assumed the presence of two energy storage elements, namely L and C. It should also be noted that modeling other (perhaps difficult to measure) parasitic inductances and capacitances would increase the system order.*

Analysis Via Laplace Transforms. *It is now shown how Laplace transform techniques may be used to analyze systems described by linear ordinary differential equations with constant coefficients such as equation (5.1). The opportunity is taken to introduce fundamental terminology as well.*

Let $Y(\cdot)$ and $U(\cdot)$ denote the Laplace transforms of the output $y(\cdot)$ and the input $u(\cdot)$, respectively. Now apply the transform properties in Table (4.2) (time differentiation property) to the differential equation (5.1). Doing so yields:

$$[\, s^2Y(s) - sy_o - \dot{y}_o \,] \; + \; \frac{R}{L}\,[\, sY(s) - y_o \,] \; + \; \frac{1}{LC}\,[\, Y(s) \,] \; = \; \frac{1}{LC}\,[\, U(s) \,]. \tag{5.2}$$

Solving for $Y(s)$ yields:

$$Y(s) \quad = \quad \left[\frac{sy_o + \dot{y}_o + \frac{R}{L}y_o}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \right] \; + \; \left[\frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \right] U(s). \tag{5.3}$$

Natural (Zero Input) Response. *At this point, it is useful to note that the first term is due to initial conditions. We therefore identify*

$$Y_{\text{zir}}(s) = \frac{sy_o + \dot{y}_o + \frac{R}{L}y_o}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \;\; = \;\; \frac{(s + \frac{R}{L})y_o + \dot{y}_o}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \tag{5.4}$$

as the Laplace transform of the zero input response $y_{\text{zir}}(\cdot)$*; i.e. the response to initial conditions with no input ($u = 0$). The zero input response captures the* natural tendencies *or* natural modes *of the system. A detailed zero input response partial fraction analysis for our RLC circuit will be given subsequently.*

System Transfer Function. *It is useful to write equation (5.3) as*

$$Y(s) \quad = \quad Y_{\text{zir}}(s) + H(s)U(s) \tag{5.5}$$

where

$$H(s) \stackrel{\text{def}}{=} \frac{(\mathcal{L}y)(s)}{(\mathcal{L}u)(s)}\Big|_{\substack{\text{zero initial}\\ \text{conditions}}} \tag{5.6}$$

$$= \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \tag{5.7}$$

is called the system transfer function.

System Output and Convolution. *Given values for the initial conditions* $(y_o, \dot{y}_o)$ *and the input* $u(\cdot)$, *one can (in principle) always use partial fraction techniques to determine the system output* $y(\cdot)$ *- assuming, of course, that all relevant Laplace transforms and inverse transforms are computable.*

Despite the availability of powerful partial fraction techniques, it is instructive to note that the system output $y(\cdot)$ *is related to system initial conditions* $(y_o, \dot{y}_o)$ *and the system input* $u(\cdot)$ *as follows:*

$$y(t) = y_{\text{zir}}(t) + (Tu)(t) \tag{5.8}$$

$$= y_{\text{zir}}(t) + (h * u)(t) \tag{5.9}$$

$$= y_{\text{zir}}(t) + \int_0^t h(t-\tau)u(\tau)d\tau \tag{5.10}$$

where

$$(Tu)(t) \stackrel{\text{def}}{=} (h * u)(t) \tag{5.11}$$

$$= \int_0^t h(t-\tau)u(\tau)d\tau. \tag{5.12}$$

denotes the system's convolution operator, $(h * u)(\cdot)$ *represents the* convolution *of* $h(\cdot)$ *with* $u(\cdot)$, *and*

$$h(t) \stackrel{\text{def}}{=} (\mathcal{L}^{-1}H)(t). \tag{5.13}$$

System Impulse Response. *Suppose that all initial conditions are set to zero* $(y_o = \dot{y}_o = 0)$ *and that a unit Dirac delta distribution is applied to the system; i.e.* $u(t) = \delta(t)$. *In such a case, it follows that* $Y(s) = H(s)$ *and hence* $y(t) = h(t)$. *Because of this, we call*

$$h(t) \stackrel{\text{def}}{=} (\mathcal{L}^{-1}H)(t) \tag{5.14}$$

the impulse response *of the system. It should be noted that* $h(t) = 0$ *for all* $t < 0$*. The impulse response for our RLC circuit may be computed using partial fraction techniques.*

Causal System. *Many systems do not exhibit an output until some input is applied to the system. Such systems are said to be* causal. *More precisely, our RLC circuit is a* causal system *because the current output* $y(t)$ *depends on the present and/or previous inputs* $u(\tau)$, $\tau \in [0, t]$*. This follows from the integral limits in equation (5.10). The lower integral limit follows from the assumption that the input is applied at* $t = 0$*. The upper limit of* t *follows from the fact that the impulse response* $h(\cdot)$ *is zero for all* $t < 0$*. This fact implies that* $h(t-\tau) = 0$ *for* $t < \tau$*. This, in turn, forces the upper limit in equation (5.10) to be* t *rather than* ∞*.*

Linear System. *From equation (5.3), it follows that* $Y(\cdot)$ *is linear in* $(y_o, \dot{y}_o, U(\cdot))$*. Since the inverse Laplace operator* $\mathcal{L}^{-1}$ *is also a linear operator, it follows that the output* $y(\cdot)$ *is linear in* $(y_o, \dot{y}_o, u(\cdot))$*; i.e.*

$$\textit{The output } y(\cdot) \textit{ is linear in the initial conditions and the input } u(\cdot).$$

That is, if $[y_{o_1}\ \dot{y}_{o_1}\ u_1(\cdot)]^T$ *produces an output* $y_1(\cdot)$ *and* $[y_{o_2}\ \dot{y}_{o_1}\ u_2(\cdot)]^T$ *produces an output* $y_2(\cdot)$*, then the linear combination* $[y_o\ \dot{y}_o\ u(\cdot)]^T = a_1[y_{o_1}\ \dot{y}_{o_1}\ u_1(\cdot)]^T + a_2[y_{o_2}\ \dot{y}_{o_2}\ u_2(\cdot)]^T$ *produces the output* $y(\cdot) = a_1 y_1(\cdot) + a_2 y_2(\cdot)$*. For this reason, we say that our RLC circuit is a* linear system.

Time Invariant System. *For many systems, the response to a given input* u *is "time invariant," i.e. shifting the input results in a shifted output. To show that this is true for our RLC circuit, define the the* delay operator

$$(S_\Delta w)(t) \stackrel{\text{def}}{=} w(t-\Delta). \tag{5.15}$$

Now consider the following (zero initial condition) calculation:

$$(TS_\Delta u)(t) \quad = \quad \int_0^t h(t-\tau)u(\tau-\Delta)d\tau \tag{5.16}$$

$$
\begin{aligned}
&= \int_{-\Delta}^{t-\Delta} h(t-\Delta-\alpha)u(\alpha)d\alpha \quad \alpha \stackrel{\text{def}}{=} \tau - \Delta \quad d\alpha = d\tau && (5.17)\\
&= \int_{0}^{t-\Delta} h(t-\Delta-\alpha)u(\alpha)d\alpha && (5.18)\\
&= \int_{0}^{t-\Delta} h(t-\Delta-\alpha)u(\alpha)d\alpha && (5.19)\\
&= y(t-\Delta) && (5.20)\\
&= (S_\Delta T u)(t) && (5.21)
\end{aligned}
$$

In this calculation, we have assumed that the input has been applied at $t = 0$*. It shows that our RLC circuit is a* time invariant system.

Linear Time Invariant (LTI) System. *Because our RLC circuit is linear and time invariant, we say that it is a* linear time invariant (LTI) system*. We will study LTI systems in great detail.*

Observations: Structure of Output. *At this point, some general observations are in order.* $Y(\cdot)$ *and its inverse* $y(\cdot)$ *consists of two components (cf. equation (5.10)):*

1. Natural Response. *The first term (*$Y_{zir}(\cdot)$ *or* $y_{zir}(\cdot)$*) is due to initial conditions and hence captures the* natural response *of the system; i.e. the response of the system to initial conditions with no input applied (*$u = 0$*). The denominator of* $Y_{\text{zir}}(\cdot)$ *(or that of* $H(\cdot)$*) is called the* characteristic equation *of the system. Its roots are called the* system poles*. These poles correspond to time domain signals which represent the* natural tendencies *or* natural modes *of the system.*

2. Forced Response. *The second term is due to the system input (forcing function)* $u(\cdot)$*. More precisely, it is due to the system input "exciting" the system transfer function* $H(\cdot)$ *and hence the system's natural modes. The idea being conveyed here is best understood in the s-domain. Both* $U(\cdot)$ *and* $H(\cdot)$ *have poles. These poles correspond to time domain functions which help make up the second term (forced response). The second term thus has a contribution due to the system transfer function* $H(\cdot)$ *and a contribution due to the input* $U(\cdot)$*.*

$\blacksquare$

SISO LTI Systems Described By Linear Differential Equation Models. The significance of the above observations lies in the fact that they apply to arbitrarily complex LTI systems. In what follows, we will examine LTI systems described by linear ordinary differential equations with constant coefficients as follows:

$$\sum_{k=0}^{n_1} a_k \overset{(k)}{y}(t) = \sum_{k=0}^{n_2} b_k \overset{(k)}{u}(t) \tag{5.22}$$

$$Y(s) = Y_{\text{zir}}(s) + H(s)U(s) \tag{5.23}$$

$$Y_{\text{zir}}(s) = \frac{\sum_{k=0}^{n_1} a_k \left[s^{k-1} y_o + s^{k-2} \dot{y}_o + \cdots + s \overset{(k-2)}{y_o} + \overset{(k-1)}{y_o} \right]}{\sum_{k=0}^{n_1} a_k s_k} \tag{5.24}$$

$$H(s) \overset{\text{def}}{=} \frac{(\mathcal{L}y)(s)}{(\mathcal{L}u)(s)} \Big|_{\substack{\text{zero initial} \\ \text{conditions}}} = \frac{\sum_{k=0}^{n_2} b_k s^k}{\sum_{k=0}^{n_1} a_k s^k} \tag{5.25}$$

$$h(t) \overset{\text{def}}{=} (\mathcal{L}^{-1} H)(t) \tag{5.26}$$

$$y(t) = y_{\text{zir}}(t) + (h * u)(t) \tag{5.27}$$

$$= y_{\text{zir}}(t) + \int_0^t h(t-\tau) u(\tau) d\tau \tag{5.28}$$

where $\overset{(k)}{y}(0^-) = \overset{(k)}{y_o}$ denote initial time derivatives. In equation (5.24), it has been assumed that all initial conditions associated with the input u are zero; i.e. $\{\overset{(k)}{u_o} = 0\}_k$. If the latter initial input derivatives are not zero, then another (similar) term must be added in equation (5.24).

System Order. For systems described by linear differential equation models (cf. equation (5.22)), the order of the system is defined to be the order of the highest derivative appearing in the differential equation model; i.e. $n \overset{\text{def}}{=} \max(n_1, n_2)$.

SISO LTI Systems Described By Linear Integral Models. In what follows, we will also consider LTI (possibly noncausal) systems described by *linear integral models* as follows:

$$y(t) = (Tu)(t) \overset{\text{def}}{=} \int_{-\infty}^{\infty} h(t-\tau) u(\tau) d\tau. \tag{5.29}$$

For such (possibly noncausal) systems, the transfer function is defined by a bilateral Laplace transform as follows:

$$H(s) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} u(\tau) d\tau. \tag{5.30}$$

5.3 Natural Response, Stability, Characteristic Equation, And Poles

To deeply understand any system, it is important to understand the *natural tendencies* or *natural modes* of the system. Two physical examples are given to illustrate these and related ideas (e.g. equilibrium, stability, etc.).

Example 5.3.1 (An Upright Broomstick.)

Consider an upright broomstick. When left completely alone, it remains upright - assuming a perfectly flat,albeit infinitesimally thin, bottom. We say that the upright position is an equilibrium *for the broomstick. When the broomstick is given a small initial offset (initial condition), its* natural tendency *is to fall (assuming no human input is applied). One can use Newton's laws to show that its angle with respect to the vertical grows exponentially as*

$$e^{\sqrt{\frac{g}{l}}t} \tag{5.31}$$

where g is the acceleration due to gravity and l is the length of the broomstick. This shows that the broomstick possesses a natural growing exponential mode. *Moreover, since the broomstick does not return to its vertical equilibrium and actually moves further away from it, we say that the vertical equilibrium is an* unstable equilibrium. *We also say that the broomstick possesses a* natural mode which is unstable. *From Laplace transforms we know that this* instability *has associated with it a* right half plane (unstable) pole *at*

$$s = \sqrt{\frac{g}{l}} \tag{5.32}$$

since

$$\mathcal{L}\{ e^{\sqrt{\frac{g}{l}}t} \} = \frac{1}{s - \sqrt{\frac{g}{l}}}. \tag{5.33}$$

∎

Example 5.3.2 (Swinging Pendulum Within Grandfather Clock.)

Now consider the pendulum which swings within a grandfather clock. When left alone, the pendulum naturally remains downward. We say that the downward position is an equilibrium *for the pendulum. When given a small inital offset (initial condition), the pendulum's* natural tendency *is to return to its downward equilibrium position. Newton's laws can be used to show that the angle of the pendulum with respect to the vertical decays like an exponential sinusoid*

$$e^{-\frac{k}{2}t}\cos\left(\sqrt{\frac{g}{l} - \left(\frac{k}{2}\right)^2}\; t + \theta\right) \tag{5.34}$$

where k represents a friction coefficient, g the acceleration due to gravity, and l the length of the pendulum. This shows that the pendulum possesses a natural decaying exponential sinusoidal mode. *Moreover, since the pendulum returns to its downward equilibrium exponentially, we say that the downward equilibrium is an* exponentially stable equilibrium. *We also say that the pendulum possesses a* natural mode which is exponentially stable. *From Laplace transforms we know that this mode has associated with it a* left half plane (stable) pole *at*

$$s = -\frac{k}{2} \pm j\sqrt{\frac{g}{l} - \left(\frac{k}{2}\right)^2}. \tag{5.35}$$

∎

The above examples suggest that in order to examine the natural tendencies or natural modes of a system, it is essential to study the natural (unforced) response (zero input response) of the system; i.e. the response due to inital conditions with no input ($u = 0$). Doing so also tells us about the stability of the system.

Stability. For systems described by linear differential equations (with coefficients that are possibly time varying), we say that the system is

- *unstable* if initial conditions produce time responses which do not remain bounded,
- *stable* or *marginally stable* if initial conditions produce bounded time responses,
- *asymptotically stable* if initial conditions decay to zero,
- *exponentially stable* if initial conditions decay to zero exponentially.

Stability For Finite-Dimensional LTI Systems. For systems described by linear ordinary differential equations with constant coefficients (and hence real-rational transfer functions), asymptotic and exponential stability are equivalent. Such systems are

exponentially stable if and only if all of the system poles

lie in the right half complex s-plane; i.e. have negative real parts.

Other notions of stability will be presented subsequently.

Example 5.3.3 (Natural Modes Of Standard Second Order System.)

Consider the RLC series circuit in Figure 5.1. An input-output model for this circuit is given by equation (5.1).

Characteristic Equation, Poles. *To understand the natural modes of this system, and its stability, one begins with an examination of its* characteristic equation*:*

$$\Phi(s) \stackrel{\text{def}}{=} s^2 + \frac{R}{L}s + \frac{1}{LC}. \tag{5.36}$$

It's roots are called the characteristic roots *of the system, the* poles *of the system, or simply the system poles. These poles completely determine the natural tendencies or modes of the system.*

Standard Second Order Form. *At this point, it is convenient to write the characteristic equation* $\Phi(\cdot)$ *as follows:*

$$\Phi(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 \tag{5.37}$$

where

$$\omega_n \stackrel{\text{def}}{=} \frac{1}{\sqrt{LC}} \tag{5.38}$$

is called the undamped natural frequency *and*

$$\zeta \stackrel{\text{def}}{=} \frac{R}{2}\sqrt{\frac{C}{L}} \tag{5.39}$$

is called the damping factor. *When written in this manner, we say that the characterisitc equation is in* standard second order form.

System Poles, Natural Response Dependence On Damping Factor ζ. *With this notation, the poles of the system are given by*

$$s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \qquad s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \tag{5.40}$$

for $\zeta \geq 1$ *and*

$$s_1 = -\zeta\omega_n + j\omega_n\sqrt{1 - \zeta^2} \qquad s_2 = -\zeta\omega_n - j\omega_n\sqrt{1 - \zeta^2} \tag{5.41}$$

for $\zeta \in [0, 1)$. *The natural (unforced) response of this system to initial conditions depends greatly on the nature of these poles. Given this, it is useful to examine how the system poles depend on the damping factor* ζ *(pronounced "zeta"). Four cases arise.*

Case 1: Undamped ($\zeta = 0$). *When* $\zeta = 0$, *the system is said to be* undamped. *In such a case the system poles are distinct, purely imaginary, complex conjugates of one another, and correspond to* *complex exponentials* $e^{\pm j\omega_n t}$ *or a sinusoid. In such a case, the natural response of the system is an undamped oscillation. More precisely,*

$$y_{\text{zir}}(t) = [\ 2|A|\cos(\omega_n t + \angle A)\ ]\ 1(t) \tag{5.42}$$

where

$$A = \frac{sy_o + \dot{y}_o + \frac{R}{L}y_o}{s + j\omega_n}\ |_{s=j\omega_n}. \tag{5.43}$$

In such a case, the system is said to be marginally stable.

Case 2: Underdamped ($0 < \zeta < 1$). *When* $0 < \zeta < 1$, *the system is said to be* underdamped. *In such a case the system poles are distinct, complex, complex conjugates of one another, and correspond to decaying (exponentially stable) complex exponentials* $e^{(-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2})t}$ *or an exponential sinusoid. More precisely,*

$$y_{\text{zir}}(t) = [\ 2|A|e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2}t + \angle A)\ ]\ 1(t) \tag{5.44}$$

where

$$A = \frac{sy_o + \dot{y}_o + \frac{R}{L}y_o}{s + \zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}}\ |_{s=-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}}. \tag{5.45}$$

For this case

- $\sigma = -\zeta\omega_n$ *is the neper frequency,*
- $\tau = \frac{1}{\zeta\omega_n}$ *is the time constant,*
- $\omega_d \stackrel{\text{def}}{=} \omega_n\sqrt{1-\zeta^2}$ *is the* damped natural frequency,
- $T = \frac{2\pi}{\omega_d}$ *is the period of oscillation,*
- $s = \sigma + j\omega_d$ *is the complex frequency, and*
- $s = \sigma - j\omega_d$ *is the conjugate complex frequency.*

As ζ *is made slightly positive, the system poles move into the left half s-plane, the system becomes underdamped, and the system's natural response becomes less oscillatory than for the* $\zeta = 0$ *(undamped) case. For small* ζ, *the natural response of the system is still very oscillatory. As* ζ *is* *increased, the response becomes less oscillatory.*

Case 3: Critically Damped ($\zeta = 1$). *When* $\zeta = 1$, *the system is said to be* critically damped. *In such a case the system poles are repeated (both at* $s = -\omega_n$*), real, and correspond to decaying (exponentially stable) exponentials* $e^{-\omega_n t}$, $te^{-\omega_n t}$. *More precisely,*

$$y_{\rm zir}(t) = [\, tAe^{-\omega_n t} + Be^{-\omega_n t} \,]\, 1(t) \tag{5.46}$$

where

$$A = -\omega_n y_o + \dot{y}_o + \frac{R}{L} y_o \qquad\qquad B = y_o. \tag{5.47}$$

For this ζ, *oscillations are no longer present in the systems's natural response and the system time constant is* $\tau = \frac{1}{\omega_n}$.

Case 4: Overdamped ($\zeta > 1$). *When* $\zeta > 1$, *the system is said to be* overdamped. *In such a case, the system poles are distinct, real, and correspond to decaying (exponentially stable) exponentials* $e^{(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1})t}$, $e^{(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})t}$. *More precisely,*

$$y_{\rm zir}(t) = [\, Ae^{s_1 t} + Be^{s_2 t} \,]\, 1(t) \tag{5.48}$$

where

$$A = \frac{s_1 y_o + \dot{y}_o + \frac{R}{L} y_o}{s_1 - s_2} \qquad\qquad B = \frac{s_2 y_o + \dot{y}_o + \frac{R}{L} y_o}{s_2 - s_1}. \tag{5.49}$$

As ζ *is increased above unity, the system poles split - one moving to the right, the other to the left. Because* $\tau_1 = \frac{1}{|s_1|} > \tau_2 = \frac{1}{|s_2|}$, *it follows that* s_1 *is the* slow pole *and* s_2 *is the* fast pole. *Generally, left half plane poles near the orign are slower than poles which are further to the left. As* ζ *is increased significantly, the slow pole* s_1 *dominates the natural response. For* $\zeta > 0$, *the system is exponentially stable.*

If one defines the critically damped resisitance *as follows:*

$$R_c \stackrel{\text{def}}{=} 2\sqrt{\frac{L}{C}} \tag{5.50}$$

then our RLC circuit is undamped/underdamped for $R \in [0, R_c)$, *critically damped for* $R = R_c$, *and overdamped for* $R \in (R_c, \infty)$.

The above discussion shows how the system poles and natural response depend on the damping factor ζ. *The dependence of the poles on* ζ *may be visualized as indicated in the root locus plot shown in Figure 5.2. It is interesting to note that for* $0 \leq \zeta \leq 1$, *the system poles lie on a circle of radius* ω_n *with center at* $s = 0$.

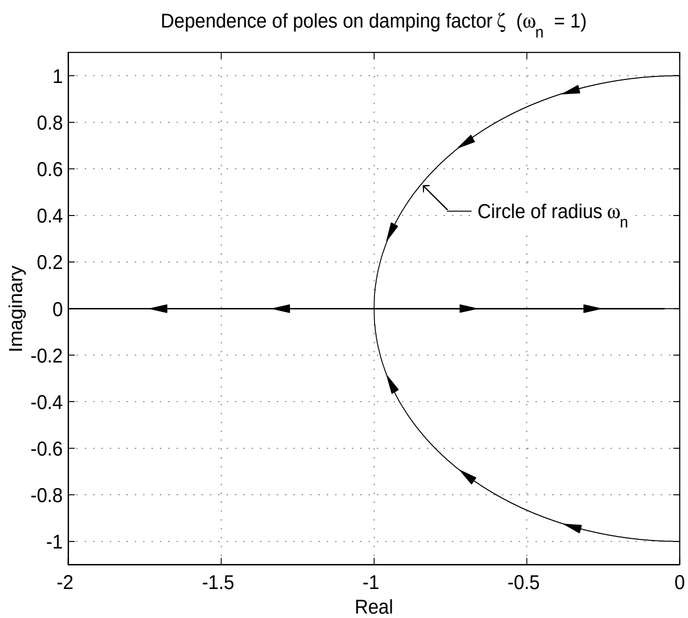

Figure 5.2: Pole Dependence on Damping Factor: Standard Second Order Polynomial

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Natural Response: General Finite-Dimensional System Modal and Stability Analysis. Consider a system whose input-output model yields the following general n^{th} order characteristic equation:

$$\Phi(s) \stackrel{\text{def}}{=} \sum_{k=0}^{n} a_k s^k \qquad (a_n \stackrel{\text{def}}{=} 1) \tag{5.51}$$

$$= \prod_{k=1}^{K_1}(s - p_k) \prod_{k=1}^{K_2}(s^2 + 2\zeta_k \omega_{n_k} s + \omega_{n_k}^2) \tag{5.52}$$

where $n = K_1 + 2K_2$, $a_k, p_k \in \mathcal{R}$, $\zeta_k \in (-1, 1)$, and $\omega_{n_k} \in (0, \infty)$. The roots of $\Phi(\cdot)$ are precisely the poles of the system.

Associated with the first product are K_1 *real poles* $\{p_k\}_{k=1}^{K_1}$ and hence K_1 *real exponential modes*

$$\{\ A_k e^{p_k t}\ \}_{k=1}^{K_1}. \tag{5.53}$$

Associated with the second product are $2K_2$ *complex poles* $\{-\zeta_k\omega_{n_k} \pm j\omega_{n_k}\sqrt{1-\zeta_k^2}\}_{k=1}^{K_2}$. These are associated with $2K_2$ *complex exponential modes* which occur in complex conjugate pairs or K_2 *real exponential sinusoidal modes*

$$\{\ |B_k|e^{-\zeta_k\omega_{n_k}t}\cos(\omega_{n_k}\sqrt{1-\zeta_k^2}\ t + \angle B_k)\ \}_{k=1}^{K_2}. \tag{5.54}$$

Given the above, it follows that the above general system is

- *exponentially stable* if and only if all of the poles lie in the left half s-plane;
- *marginally stable* if and only if all of the poles lie in the left half s-plane with the possible exception of unrepeated poles on the imaginary axis.
- *unstable* if and only if any one of its poles lies in the right half plane and/or if there is a repeated pole on the imaginary axis.

5.4 System Transfer Function

In this section, we consider transfer functions for SISO LTI systems.

Standard Second Order Transfer Function. The transfer function

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5.55}$$

is said to be in *standard second order form.* This standard form appears everywhere in the study of more complex SISO LTI systems described by linear ordinary differential equations with constant coefficients.

Example: RLC Series Circuit, Standard Second Order Transfer Function.

Consider the RLC series circuit in Figure 5.1. Assuming zero initial conditions, the ratio $Y(s)/U(s)$ is given by

$$H(s) \stackrel{\text{def}}{=} \frac{(\mathcal{L}y)(s)}{(\mathcal{L}u)(s)}\Big|_{\substack{\text{zero initial}\\ \text{conditions}}} \tag{5.56}$$

$$= \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}. \tag{5.57}$$

This ratio is called the *system transfer function*. This transfer function may be written in *standard second order form* as follows:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5.58}$$

where $\omega_n \stackrel{\text{def}}{=} \frac{1}{\sqrt{LC}} \in (0, \infty)$ and $\zeta \stackrel{\text{def}}{=} \frac{R}{2}\sqrt{\frac{C}{L}} \in [0, \infty)$. ∎

General Real-Rational Transfer Function. Consider a SISO LTI system with real-rational transfer function

$$H(s) \stackrel{\text{def}}{=} \frac{\mathcal{L}(y(t))}{\mathcal{L}(u(t))}\Big|_{\substack{\text{zero initial}\\ \text{conditions}}} = \frac{\sum_{k=0}^{n_2} b_k s^k}{\sum_{k=0}^{n_1} a_k s^k} \qquad (a_{n_1} \stackrel{\text{def}}{=} 1) \tag{5.59}$$

When written in this form, we will say that it is written in *standard polynomial form*. It is often useful to write this transfer function in the following *standard pole-zero-gain form*:

$$H(s) = g \; \frac{\prod_{k=1}^{NRZ}(s - z_k)}{\prod_{k=1}^{NRP}(s - p_k)} \; \frac{\prod_{k=1}^{NCZP}(s^2 + 2\zeta_{z_k}\omega_{n_{z_k}} s + \omega_{n_{z_k}}^2)}{\prod_{k=1}^{NCPP}(s^2 + 2\zeta_{p_k}\omega_{n_{p_k}} s + \omega_{n_{p_k}}^2)} \tag{5.60}$$

where $a_k, b_k \in \Re$,

- NRP is the number of finite real poles and the $\{p_k\}$ are real poles ($p_k \in \Re$),
- $NCPP$ is the number of finite complex pole pairs and the $\{-\zeta_{p_k}\omega_{n_{p_k}} \pm j\omega_{n_{p_k}}\sqrt{1 - \zeta_{p_k}^2}\}$ are complex poles ($\zeta_{p_k} \in (-1, 1)$, $\omega_{n_{p_k}} \in (0, \infty)$),

- NRZ is the number of finite real zeros and the z_k are real zeros ($z_k \in \Re$),

- $NCZP$ is the number of finite complex zero pairs and the $\{-\zeta_{z_k}\omega_{n_{z_k}} \pm j\omega_{n_{z_k}}\sqrt{1-\zeta_{z_k}^2}\}$ are complex zeros ($\zeta_{z_k} \in (-1,1)$, $\omega_{n_{z_k}} \in (0,\infty)$), and

- the system's *leading gain* $g \in \Re$ is given by

$$g \stackrel{\text{def}}{=} \lim_{s \longrightarrow \infty} s^{n_1-n_2} H(s) = b_{n_2}. \tag{5.61}$$

If $n_2 \leq n_1$, then the above system is said to have $n_1 - n_2$ zeros at ∞. If $n_2 > n_1$, then the above system is said to have $n_2 - n_1$ poles at ∞. With this method of "accounting" (counting poles and zeros at ∞) the number of poles is always equal to the number of system zeros.

Because the above SISO LTI systems possess a finite number of poles, they are referred to as *finite-dimensional systems.* They are sometimes referred to as *lumped parameter systems.*

It should be noted that for SISO LTI systems with a real-rational transfer function, the system transfer function is readily obtainable from the system input-output differential equation and vice versa. This is always the case - for any SISO LTI system described by linear ordinary differential equations with constant coefficients. Specifically, this "equivalence" between time domain differential equations with constant coefficients and s-domain transfer functions follows from the following (zero initial condition) transform pair:

$$\dot{y}(t) \longleftrightarrow sY(s). \tag{5.62}$$

This pair suggests that the Laplace transform variable s is fundamentally a *differential operator.*

Example: Analysis Of General RLC Circuits Via Impedance Methods.

For general RLC circuits, transfer functions and hence differential equations, are readily obtainable if one exploits the circuit concept of *impedance.* This concept generalizes the concept of resistance. The impedance of a resistor R is

$$Z_R \stackrel{\text{def}}{=} \frac{V_R(s)}{I_R(s)}\Big|_{\substack{\text{zero initial}\\ \text{conditions}}} = R. \tag{5.63}$$

The impedance of an inductor L is

$$Z_L \stackrel{\text{def}}{=} \frac{V_L(s)}{I_L(s)}\Big|_{\substack{\text{zero initial}\\ \text{conditions}}} = sL. \tag{5.64}$$

The impedance of a capacitor C is

$$Z_C \stackrel{\text{def}}{=} \frac{V_C(s)}{I_C(s)}\Big|_{\substack{\text{zero initial}\\ \text{conditions}}} = \frac{1}{sC}. \tag{5.65}$$

With this impedance concept, arbitrary RLC circuits may be analyzed in the s-domain by merely treating the impedances as if they were resistances.

Consider the RLC series circuit in Figure 5.1. Given the above impedance ideas, one can use voltage division (in the s-domain) to readily obtain the system transfer function from the input $u(\cdot)$ to the output voltage $y(\cdot)$ across the capacitor. Doing so yields the following:

$$\frac{Y(s)}{U(s)}\Big|_{\substack{\text{zero initial}\\ \text{conditions}}} = \frac{\frac{1}{sC}}{R + sL + \frac{1}{sC}} \tag{5.66}$$

$$= \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}. \tag{5.67}$$

Analogous ideas exist for other dynamical systems. ∎

Irrational Transfer Functions. Not all LTI systems are associated with real-rational transfer functions. Transfer functions which are not real-rational are said to be *irrational*. Such systems are often referred to as *infinite-dimensional systems* or *distributed parameter systems* [10].

Example 5.4.1 (A Delay System.)

Consider the SISO LTI system described by the linear ordinary (delay) differential equation with constant coefficients:

$$\dot{y}(t) = -y(t) + u(t-\Delta) \tag{5.68}$$

where $\Delta > 0$ *represents a time delay. The system transfer function is given by*

$$H(s) \quad \stackrel{\text{def}}{=} \quad \frac{\mathcal{L}(y(t))}{\mathcal{L}(u(t))}|_{\substack{\text{zero initial}\\ \text{conditions}}} = \frac{e^{-s\Delta}}{s+1}. \tag{5.69}$$

This is an irrational transfer function. ∎

Example 5.4.2 (An Irrational Transfer Function.)

Consider the LTI system described by the following linear integral convolution equation:

$$y(t) = (Tu)(t) \stackrel{\text{def}}{=} \int_{-\infty}^{t} h(t-\tau)u(\tau) \tag{5.70}$$

with impulse response $h(\cdot)$ *given by the* zero$^{\text{th}}$ *order Bessel function*

$$h(t) = J_o(t) \stackrel{\text{def}}{=} \left[1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 4^2} - \frac{t^6}{2^2 4^2 6^2} + \cdots\right] 1(t). \tag{5.71}$$

It can be shown that the system transfer function is given by [43, pg. 23]

$$H(s) \quad \stackrel{\text{def}}{=} \quad \frac{\mathcal{L}(y(t))}{\mathcal{L}(u(t))}|_{\substack{\text{zero initial}\\ \text{conditions}}} = \frac{1}{\sqrt{s^2+1}}. \tag{5.72}$$

This is an irrational transfer function. ∎

5.5 Impulse Response

It has been shown that for an LTI system with transfer function $H(\cdot)$, the impulse response $h(\cdot)$ is given by

$$h(t) = (\mathcal{L}^{-1}H)(t). \tag{5.73}$$

The impulse response of an LTI system is important for various reasons.

- The impulse response $h(\cdot)$ determines the transfer function $H(\cdot)$;
- The impulse response $h(\cdot)$ determines the systems' input-output model (e.g. linear ordinary differential equation with constant coefficients, convolution integral, etc.).

The impulse response $h(\cdot)$ of a system corresponding to a real-rational transfer function $H(\cdot)$ can be readily found using standard partial fraction expansion techniques [32].

Example 5.5.1 (RLC Series Circuit Impulse Response.)

Consider the RLC circuit in Figure 5.1. It should be noted that the qualitative statements which were made for the circuit's zero input response in an earlier example, also apply to the circuit's impulse response. This is because the roots of the characteristic polynomial are also poles of the system transfer function. ∎

5.6 General Solution And Convolution

It has been shown that the general solution for an LTI system described by a linear ordinary differential equation with constant coefficients, takes the form:

$$\begin{aligned} y(t) &= y_{\text{zir}}(t) + (h * u)(t) && (5.74) \\ &= y_{\text{zir}}(t) + \int_0^t h(t-\tau)u(\tau)d\tau && (5.75) \end{aligned}$$

where $y_{\text{zir}}(\cdot)$ denotes the system's zero input (or natural) response and $(h * u)(\cdot)$ represents the *convolution* of $h(\cdot)$ with $u(\cdot)$.

A substitution of variables shows that convolution is *commutative*; i.e.

$$(h * u)(t) = \int_0^t h(t-\tau)u(\tau)d\tau = \int_0^t u(t-\tau)h(\tau)d\tau = (u * h)(t). \tag{5.76}$$

From this, it follows that

$$y(t) = y_{\text{zir}}(t) + (u * h)(t) \tag{5.77}$$

$$= y_{\rm zir}(t) + \int_0^t u(t-\tau)h(\tau)d\tau. \tag{5.78}$$

In principle, once $h(\cdot)$ is known, equation (5.75) or (5.78) may be used to determine the output $y(\cdot)$ to any input $u(\cdot)$ - given the inital conditions. $y_{\rm zir}(\cdot)$ may be found using partial fraction Laplace inversion techniques.

Convolution Integral Evaluation Methods. It has been shown how convolution integrals may be evaluated using Laplace transform techniques. It must be noted that *graphical convolution methods* [34] may be used to evaluate convolution integrals. If the integrands in equations (5.75) and (5.78) are too complex, then the convolution integrals may have to be evaluated using numerical methods [20], [22].

Equations (5.75) and (5.78) show that convolution occurs naturally for LTI systems described by linear ordinary differential equations with constant coefficients.

5.7 Linearity

From equations (5.24) and (5.28) one sees why SISO LTI systems described by a linear ordinary differential equation with constant coefficients are called *linear systems.* Specifically, these equations show that the system output $y(\cdot)$ is linear in the (joint initial condition - input) vector

$$\left[\begin{array}{c} y_o \\ \vdots \\ \overset{(n_1-1)}{y_o} \\ \hline u(\cdot) \end{array}\right]. \tag{5.79}$$

That is, if

$$\left[\begin{array}{c} y_{o_i} \\ \vdots \\ \overset{(n_1-1)}{y_{o_i}} \\ \hline u_i(\cdot) \end{array}\right] \overset{produces}{\Rightarrow} \quad y_i(\cdot) \qquad\qquad i = 1, 2 \tag{5.80}$$

then the linear combination

$$\left[\begin{array}{c} y_o \\ \vdots \\ \overset{(n_1-1)}{y_o} \\ \hline u(\cdot) \end{array}\right] = a_1 \left[\begin{array}{c} y_{o1} \\ \vdots \\ \overset{(n_1-1)}{y_{o1}} \\ \hline u_1(\cdot) \end{array}\right] + a_2 \left[\begin{array}{c} y_{o2} \\ \vdots \\ \overset{(n_1-1)}{y_{o2}} \\ \hline u_2(\cdot) \end{array}\right] \overset{produces}{\Rightarrow} \quad y(\cdot) = a_1 y_1(\cdot) + a_2 y_2(\cdot). \tag{5.81}$$

Given this, it follows that such systems are linear in the input assuming zero initial conditions and linear in the initial conditions assuming zero input.

It should be noted that sometimes the above linearity principle is referred to as the *principle of superposition* [23].

Nonlinear Systems. Systems that are not linear are said to be *nonlinear systems* [27], [47].

Example 5.7.1 (Saturation Nonlinearity.)

Consider the system defined by the static input-output model

$$y = \begin{cases} -1 & \phantom{-1 \leq {}} u < -1 \\ u & -1 \leq u < 1 \\ 1 & 1 \leq u \end{cases}. \tag{5.82}$$

This system defines a saturation nonlinearity . *This type of nonlinearity is used, for example, to model the fact that the elevator on an aircraft can not rotate an arbitrary amount.* ∎

5.8 Time Invariance

For many systems the response to a given input is "time invariant," i.e. it does not depend on when the input is applied. This notion motivates the following definition.

Time Invariance. Suppose that all initial conditions are set to zero and that an input $u(\cdot)$ results in an output $y(\cdot)$. A system is said to be *time invariant* if the delayed input $u(\cdot - \Delta)$ results in the corresponding delayed output $y(\cdot - \Delta)$ for any input $u(\cdot)$ and any delay $\Delta \geq 0$. This definition may be visualized via system block diagram as indicated in Figure 5.3 where T represents the system in question and S_Δ is a delay operator defined by the relationship

$$(S_\Delta w)(t) \stackrel{\text{def}}{=} w(t-\Delta). \tag{5.83}$$

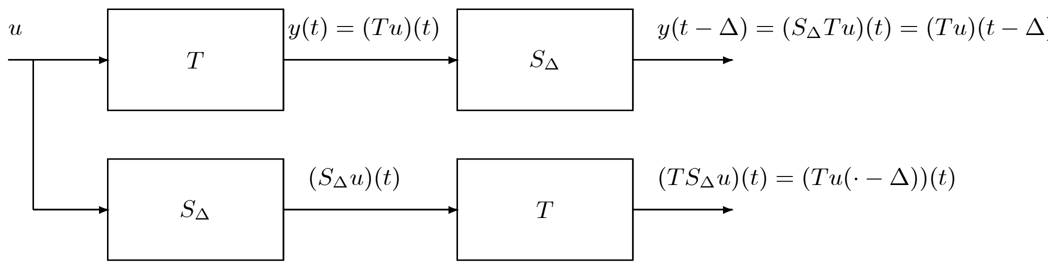

Figure 5.3: Block Diagram Visualization of Time Invariance Test: $(S_\Delta Tu)(t) = (TS_\Delta u)(t)$

From Figure 5.3, it follows that a system T is *time invariant* if and only if the operator defined by T commutes with the delay operator S_Δ. That is, the system T is time invariant if and only if

$$y(t-\Delta) = (Tu)(t-\Delta) = (S_\Delta Tu)(t) = (TS_\Delta u)(t) = (Tu(\cdot - \Delta))(t) \tag{5.84}$$

for any input u and any delay $\Delta \geq 0$.

Convolution Operators. If T defines a convolution operator, then it can be shown that T defines a time invariant system (cf. equations (5.17)-(5.21)). All systems described by linear ordinary differential equations with constant coefficients define time invariant systems.

Linear Time Invariant (LTI) Systems. Systems which are linear and time invariant are called *linear time invariant (LTI)* systems. Practically speaking, most LTI systems define a convolution operation, an impulse response, and a transfer function. Only "pathological" (i.e. weird) LTI systems [26, pg. 3] do not define a convolution operation. All systems described by linear ordinary differential equations with constant coefficients define an LTI system.

Time Varying Systems. Systems which are not time invariant are said to be time varying systems. Linear differential equations with coefficents that depend on the independent variable t are an example of linear time varyting systems.

5.9 Causality

A *causal system* is one whose current output depends on the present and/or previous inputs. Based on this definition, it follows that the RLC circuit in Figure 5.1 is a causal system.

More general (not necesarily causal) LTI systems may be described by the convolution operation:

$$y(t) = (h * u)(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau. \tag{5.85}$$

For more general systems, the current output may depend on future inputs as well as past and present inputs. Such descriptions might occur in applications where t does not represent real time. For example, $h(\cdot)$ might represent the impulse response of a linear system whose purpose is to process a spatially distributed image $u(\cdot)$ [17].

Example 5.9.1 (A Noncausal Linear Space Invariant System.)

Consider the system described by the following second order linear ordinary differential equation with constant coefficients

$$\ddot{y}(x) + y(x) = -2u(x) \qquad x \in (-\infty, \infty) \tag{5.86}$$

where x represents a spatial independent variable - *not time. We seek a solution $y(\cdot)$ which depends on $u(\cdot)$ and holds for all $x \in (-\infty, \infty)$. This goal is very different from that which we have pursued in the initial value problems which we have focussed on thus far.*

Suppose one defines a bilateral Laplace transform

$$F(s) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(x)e^{-sx}dx. \tag{5.87}$$

With this definition, it can be shown that

$$e^{-ax}1(x) \quad \stackrel{\mathcal{L}}{\longleftrightarrow} \quad \frac{1}{s+a} \qquad Re\ s > -Re\ a \tag{5.88}$$

$$e^{ax}1(-x) \quad \stackrel{\mathcal{L}}{\longleftrightarrow} \quad \frac{-1}{s-a} \qquad Re\ s < Re\ a \tag{5.89}$$

$$\overset{(n)}{f}(x) \quad \stackrel{\mathcal{L}}{\longleftrightarrow} \quad s^n F(s) \qquad n = 0, 1, 2, \ldots . \tag{5.90}$$

While the first transform pair is not new, the latter transform pairs do differ from the unilateral transform pairs we have seen earlier (cf. Tables 4.1, 4.2).

Applying these to our differential equations yields

$$Y(s) = \frac{-2}{s^2-1}U(s). \tag{5.91}$$

From this, it follows that the system transfer function

$$H(s) = \frac{-2}{s^2-1} = \frac{1}{s+1} + \frac{-1}{s-1} \qquad -1 < Re\ s < 1. \tag{5.92}$$

The corresponding impulse response is given by the two-sided function

$$h(x) = e^{-x}1(x) + e^{x}1(-x) = e^{-|x|}. \tag{5.93}$$

The output is now given by the following convolution operation

$$y(x) = \int_{-\infty}^{\infty} h(x-z)u(z)dz \qquad x \in (-\infty, \infty). \tag{5.94}$$

Here, $H(\cdot)$ can be thought of as an linear space invariant *system (filter or image processor) which processes the two-sided image intensity specified by the input $u(\cdot)$. Because the system impulse response decays to zero exponentially as the spatial variable $|x| \longrightarrow \infty$, it is meaningful to say that the system is* exponentially stable. ■

5.10 Bounded-Input Bounded-Output (BIBO) Stability

In an earlier section, we considered one notion of stability. Here we consider another notion.

Bounded-Input Bounded-Output (BIBO) Stability. A system is said to be *bounded-input bounded-output (BIBO) stable* if bounded inputs result in bounded outputs.

Suppose that $h(\cdot)$ is the impulse response of a SISO LTI system. Given this, the LTI system defined by $h(\cdot)$ defines a BIBO system [13, pg. 23] if and only if

$$\|h\|_{\mathcal{L}^1} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty. \tag{5.95}$$

For such a system the following relationship holds:

$$\max_t |y(t)| \le \|h\|_{\mathcal{L}^1} \max_t |u(t)|. \tag{5.96}$$

Similar results hold for general (not necessarily causal) multiple-input multiple-output (MIMO) LTI systems with well defined impulse response matrices and convolution representations [13, pg. 26].

5.11 Step Response

Once the impulse response $h(\cdot)$ of an LTI system is known, equation (5.28) may be used to determine the *step response* of the system. Denoted by the symbol $s(\cdot)$, the step response is given by

$$s(t) = \int_0^t h(t-\tau) d\tau = \int_0^t h(\tau) d\tau \tag{5.97}$$

where the latter expression follows from a substitution of variables. For very complex $h(\cdot)$ - for which closed form integral expressions are unknown - this relationship may be used to determine $s(\cdot)$ using some numerical integration routine [20], [22]. From equation (5.97), it follows that

$$h(t) = \frac{d}{dt} s(t). \tag{5.98}$$

These relationships between $h(\cdot)$ and $s(\cdot)$ follow from equations (3.8)-(3.9) and the fact that the systems under consideration are linear. One can also use standard partial fraction expansion techniques to determine the step response. This is illustrated in the following example.

Example 5.11.1 (Step Response For Standard Second Order System.)

Consider a SISO system whose transfer function is in standard second order form as follows:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}. \tag{5.99}$$

The step response of this system can be found by finding the inverse Laplace transform of

$$Y(s) = H(s)\frac{1}{s}. \tag{5.100}$$

Using partial fraction expansion techniques gives rise to several cases which depend on the damping factor ζ *as follows.*

Undamped/Underdamped ($0 \leq \zeta < 1$):

$$s(t) = [\ A + 2|B|e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2}t + \angle B)\ ]\ 1(t) \tag{5.101}$$

where $A = H(0) = 1$ *and* $B = \frac{\omega_n^2}{s+\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}}|_{s=-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}}$*;*

Critically Damped ($\zeta = 1$):

$$s(t) = [\ A + Bte^{-\omega_n t} + Ce^{-\omega_n t}\ ]\ 1(t) \tag{5.102}$$

where $A = H(0) = 1,\ B = \omega_n^2,$ *and* $C = 0$*;*

Overdamped ($\zeta > 1$):

$$s(t) = [\ A + Be^{s_1 t} + Ce^{s_2 t}\ ]\ 1(t) \tag{5.103}$$

where $A = H(0) = 1,\ B = \frac{\omega_n^2}{s(s_1 - s_2)},$ *and* $C = \frac{\omega_n^2}{s(s_2 - s_1)}$.

Each of these may be visualized as shown in Figure (5.4). It should be noted that in each case, the steady state output is given by $s(\infty) = H(0) = 1$.

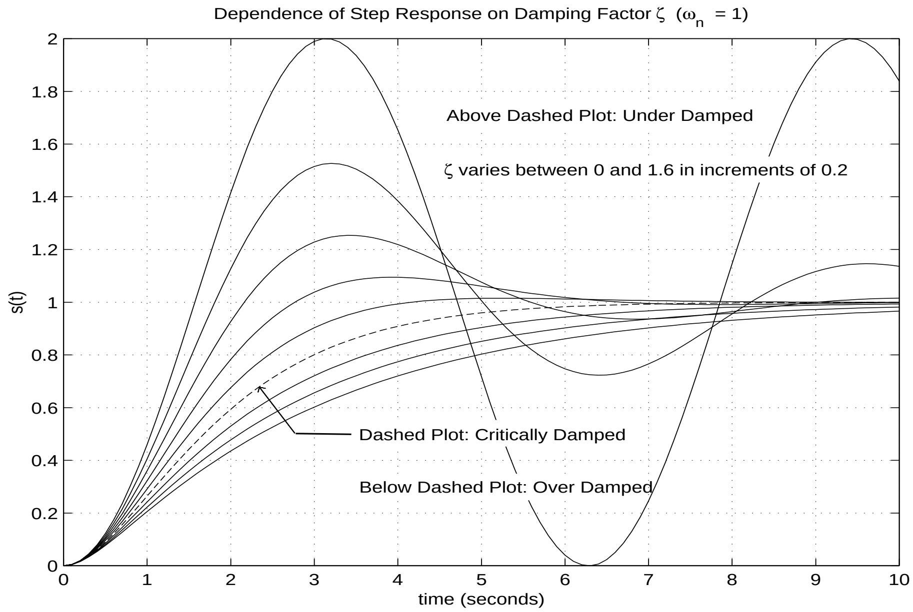

Figure 5.4: Step Response For Standard Second Order System: Dependence on Damping Factor

Overhoot, Undershoot, Time-To-Peak. *From the above discussion, one sees that for small* ζ*, the step response of our standard second order system (5.55) exhibits oscillatory behavior. When the response goes above unity - the steady state value - we say that the response exhibits an overshoot. When it dips below unity, we say that we have an* undershoot. *More specifically, one observes that for small enough* ζ*, the step response exhibits a rapid rise to a well defined maximum. In such a case, we say that the response exhibits* overshoot. *See Figure 5.5.*

The maximum percent overshoot M_p *is defined as follows [32, pp. 150-156]:*

$$M_p \stackrel{\text{def}}{=} \frac{\max_{t\geq 0} s(t) - s(\infty)}{s(\infty)}. \tag{5.104}$$

It can be shown via elementary calculus that overshoot occurs for $\zeta \in [0,1)$*. Moreover,*

$$M_p = e^{-\zeta\omega_n t_p} = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \tag{5.105}$$

where

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d} \tag{5.106}$$

is the time at which y *achieves its maximum overshoot (peak); i.e.* $\max_{t\geq 0} y(t) = y(t_p)$.

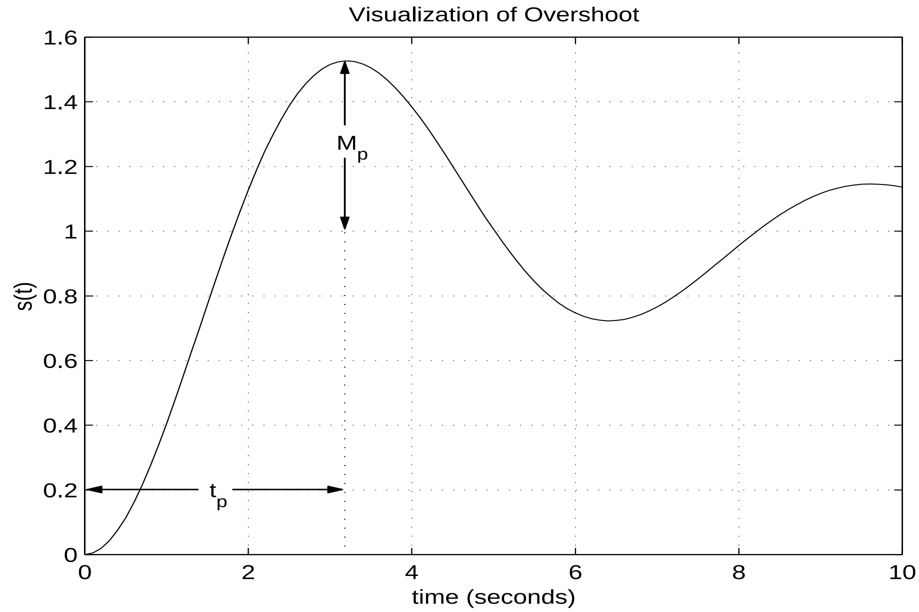

Figure 5.5: Step Response For Standard Second Order System: Visualization of Overshoot

Equation (5.105) demonstrates that M_p *depends on* ζ*, is independent of* ω_n*, and lies in the interval* $M_p \in (0,1]$ *for* $\zeta \in [0,1)$*. Specifically,* M_p *approaches unity as* ζ *approaches zero (undamped system).* M_p *approaches zero as* ζ *approaches unity (crtically damped system). Equation (5.105) for* M_p *may be visualized as shown in Figure 5.5. Equation (5.106) demonstates that the peak time* t_p *varies inversely with the damped natural frequency* $\omega_d = \omega_n\sqrt{1-\zeta^2}$*. As* ω_d *is increased, the time to reach the peak* t_p *decreases.*

∎

Example 5.11.2 (Effect Of Numerator Zero On Step Response Of Standard Second Order System.)

Consider the underdamped SISO LTI system

$$H(s) = \frac{2}{a}\left[\frac{s+a}{s^2+2s+2}\right]. \tag{5.107}$$

with poles at $s = -1 \pm j1$ *and a zero at* $s = -a$*. We would like to understand how the numerator zero affects the step response of the system. The sytem's step response is plotted in Figure 5.6* *for different values of the parameter a. For large a, the system responds like the underdamped ($\zeta = 0.7071$) second order system*

$$H(s) = \frac{2}{s^2 + 2s + 2}. \tag{5.108}$$

As a is decreased, the derivative action of the zero becomes more pronounced and the system's step response exhibits more overshoot. The derivative action becomes most pronounced when the numerator zero lies to the right of the real part of the system pole; i.e. $a < 1$. ■

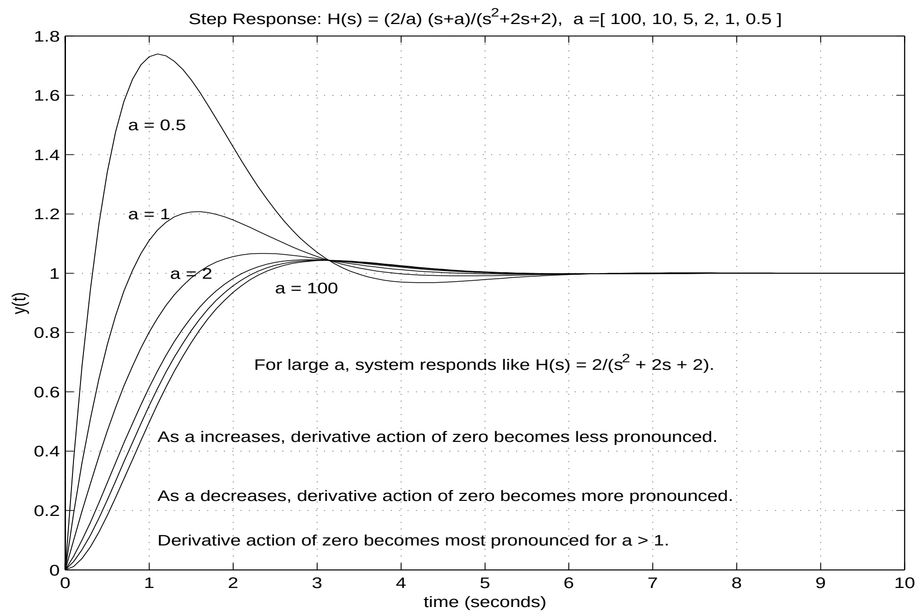

Figure 5.6: Step Response For Standard Second Order System: Effect Of Numerator Zero

5.12 Sinusoidal Steady State Analysis And System Zeros

Sinusoidal signals are arguably amongst the most widely used test signals in the analysis of dynamical systems. Understanding their steady state impact on LTI system is of fundamental importance.

Steady State Response To An Exponential. Let H denote the transfer function of an asymptotically stable system (e.g. RLC series circuit in Figure 5.1). Suppose that s_o denotes an arbitrary complex number in the region of convergence of $H(\cdot)$ and that the input

$$u(t) = e^{s_o t} \tag{5.109}$$

is applied to H. The output $y(\cdot)$ is given by

$$y(t) = y_{\text{zir}}(t) + (u * h)(t) = y_{\text{zir}}(t) \ + \int_0^t u(t-\tau)h(\tau)d\tau = y_{\text{zir}}(t) \ + \int_0^t e^{s_o(t-\tau)}h(\tau)d\tau \tag{5.110}$$

or

$$y(t) = y_{\text{zir}}(t) \ + \ [\int_0^t h(\tau)e^{-s_o\tau}d\tau \ ] \ e^{s_o t} \tag{5.111}$$

For t large, y approaches the *steady state* function

$$y_{ss} = \ H(s_o) \ e^{s_o t}. \tag{5.112}$$

Steady State Response To An Exponential Sinusoid. In what follows, it will be seen how this fundamental relationship establishes a foundation for the study of LTI systems. Toward this end, suppose that

$$u(t) = |X|e^{\sigma_o t}\cos(\omega_o t + \angle X) = \frac{|X|}{2}[\ e^{(\sigma_o + j\omega_o)t}e^{j\angle X} \ + e^{(\sigma_o - j\omega_o)t}e^{-j\angle X}]. \tag{5.113}$$

This produces

$$y_{ss} = \frac{|X|}{2}H(\sigma_o + j\omega_o)e^{(\sigma_o + j\omega_o)t}e^{j\angle X} \ + \frac{|X|}{2}H(\sigma_o - j\omega_o)e^{(\sigma_o - j\omega_o)t}e^{-j\angle X} \tag{5.114}$$

Using the fact that $H(\overline{s}_o) = \overline{H(s_o)}$ when $h(\cdot)$ is real, yields the following:

$$y_{ss} = Re \ [\ H(\sigma_o + j\omega_o)|X|e^{j\angle X}e^{(\sigma_o + j\omega_o)t} \ ] \tag{5.115}$$

or

$$y_{ss} = |H(\sigma_o + j\omega_o)||X| \ e^{\sigma_o t} \ \cos(\omega_o t + \angle X + \angle H(j\omega_o)). \tag{5.116}$$

This result implies that exponential sinusoidal inputs result in a steady state exponential sinusoid.

Steady State Response To A Sinusoid. From the above relationship, one obtains the main result for sinusoidal steady state analysis of LTI systems:

$$u(t) = |X|\cos(\omega_o t + \angle X) \overset{produces}{\longrightarrow} y_{ss} = |H(j\omega_o)||X|\ \cos(\omega_o t + \angle X + \angle H(j\omega_o)). \qquad (5.117)$$

From this result, it follows that the steady state output of an LTI system to a sinusoidal input is a sinusoid. The input sinusoid is modified in amplitude by the multiplicative factor $|H(j\omega_o)|$ and in phase by the additive factor $\angle H(j\omega_o)$. Equation (5.117) arguably establishes the main result for the study of LTI systems.

Interpretation of Zeros. The zeros of a transfer function are those values of s such that $H(s) = 0$. From this, it follows that if $H(\cdot)$ represents an asymptotically stable LTI system with zeros at $s = \pm j\omega_o$, then the application of an input $u(t) = |X|\cos(\omega_o + \angle X)$ will result in a steady state output $y_{ss} = 0$ since $H(j\omega_o) = 0$.

Steady State DC Analysis and DC Gain. A special case of equation (5.117) arises when the input u is a constant. Such an input corresponds to an input frequency $\omega_o = 0$. In such a case, we say that the input u is a *dc signal*. The term "dc" is widely used and has its roots in circuit analysis (i.e. dc for direct current [23]). For such a signal, equation (5.117) becomes

$$u(t) = k \overset{produces}{\longrightarrow} y_{ss} = H(0)\ k \qquad (5.118)$$

for any real constant k. The quantity $H(0)$ is called the *dc gain of the system*. Since many systems are designed to operate at low frequencies, it follows that the above *dc analysis* is of great importance in the analysis and design of systems.

5.13 Frequency Response, Magnitude Response, Phase Response, And Fourier Transform

In studying the steady state response of LTI systems to sinusoidal signals, it becomes immediately evident that the system transfer function H, and specifically, the complex function of ω

$$H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)} \tag{5.119}$$

is of fundamental importance.

Frequency Response, Magnitude Response, Phase Response. The two functions $|H(j\omega)|$ and $\angle H(j\omega)$ are referred to as the *frequency response* of the system. Sometimes $H(j\omega)$ is called the *system frequency response* or *frequency response*. Sometimes $H(j\omega)$ is called the *spectrum* of the signal $h(\cdot)$. The function $|H(j\omega)|$ is called the *magnitude response* of the system. The function $\angle H(j\omega)$ is called the *phase response* of the system.

Magnitude and Phase Response Properties. It can be shown that if the impulse response $h(\cdot)$ is real (i.e. $\overline{h}(\cdot) = h(\cdot)$) - which is typically the case in most applications, then

$$|H(j\omega)| = |H(-j\omega)| \tag{5.120}$$

$$\angle H(j\omega) = -\angle H(-j\omega). \tag{5.121}$$

That is, $|H(j\omega)|$ is an even function of ω and $\angle H(j\omega)$ is an odd function of ω.

Continuous Time Fourier Transform. The unilateral continuous time *Fourier transform* of $h(\cdot)$ [36] is defined as follows:

$$(\mathcal{F}h)(j\omega) \stackrel{\text{def}}{=} \int_{0^-}^{\infty} h(\tau)e^{-j\omega\tau}d\tau. \tag{5.122}$$

It is important to note that this function is not always equal to the function obtained by substituting $s = j\omega$ into the Laplace transform of $h(\cdot)$ [36]. This is because $s = j\omega$ need not lie within the region of convergenece of $H(\cdot)$. In general,

$$(\mathcal{L}h)(s)|_{s=j\omega} = (\mathcal{F}h)(j\omega) \tag{5.123}$$

if and only if $s = j\omega$ lies within the region of convergence of $(\mathcal{L}h)(\cdot)$. The above statements also apply when working with bilateral Laplace and Fourier transforms:

$$(\mathcal{L}h)(s) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \tag{5.124}$$

$$(\mathcal{F}h)(j\omega) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau. \tag{5.125}$$

System Bandwidth. It is often useful to discuss a system's ability or inability to amplify or attenuate rapidly changing input signals. The concept of *bandwidth* attempt to quantify this notion. The most widely accepted definition for bandwidth is one which applies to LTI systems with a low pass magnitude response; i.e. one which is approximately zero above some frequency. The bandwidth, denoted BW, of an LTI system with transfer function $H(\cdot)$ is defined to be that frequency at which the magnitude response is down 3 db (a factor of $1/\sqrt{2}$) from the dc gain; i.e.

$$|H(jBW)| = \frac{1}{\sqrt{2}}\,|H(j0)|. \tag{5.126}$$

If a system is to amplify rapidly varying signals, then it should have a high bandwidth. Scaling a system's input or output by a gain greater than unity increases the bandwidth of the system. Scaling a system's input or output by a gain less than unity decreases the bandwidth of the system.

Example 5.13.1 (Magnitude And Phase For Standard Second Order System.)

The dependence of the magnitude and phase responses for the standard second order system

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5.127}$$

on the damping factor ζ are shown in Figure 5.7.

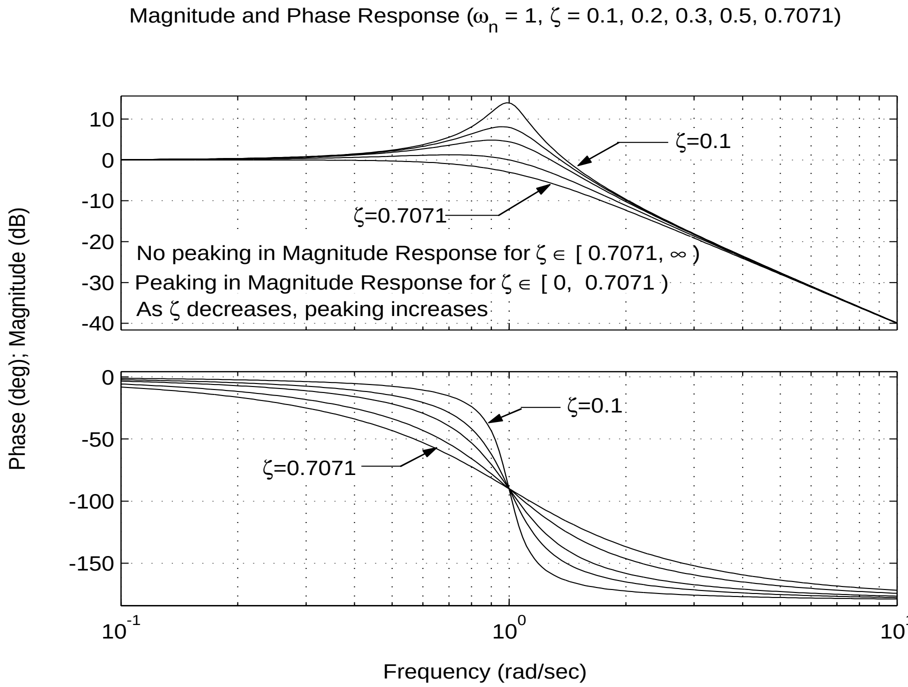

Figure 5.7: Frequency Response For Standard Second Order System: Dependence on Damping Factor

In this figure, the independent variable ω *is plotted on a logarithmic (base 10) horizontal scale. The magnitude response* $|H(j\omega)|$ *is plotted on a linear vertical scale in* decibels (db)*; i.e.* $20\log_{10}|H(j\omega)|$ *is plotted on a linear vertical scale [32]. The angle response* $\angle H(j\omega)$ *is plotted in degrees on a linear vertical scale. Methods for systematically sketching* asymptotic approximations - *called* Bode plots - *to these were developed by H.W. Bode [5]. A discussion of Figure 5.7 now follows.*

We know that for small ζ*, our standard second order system has poles near the imaginary axis. For* $\zeta = 0$ *both poles are on the imaginary axis at* $s = \pm j\omega$ *and* $H(\pm j\omega) = \infty$*. Based on this, one expects by continuity that the magnitude response will exhibit a large peak for small* ζ*. If one lets* ω_r *denote the frequency at which the maginitude response peaks and* M_r *denote the peak magnitude; i.e.*

$$M_r \stackrel{\text{def}}{=} \max_{\omega} |H(j\omega)| = |H(j\omega_r)|, \tag{5.128}$$

then one obtains the following relationships for ω_r *and* M_r *[32, pp. 482-484]:*

$$\omega_r = \begin{cases} w_n\sqrt{1-2\zeta^2} & \zeta \in [0, \frac{1}{2}] \\ 0 & \zeta > \frac{1}{2} \end{cases} \tag{5.129}$$

$$M_r = \begin{cases} \frac{1}{2\zeta\sqrt{1-\zeta^2}} & \zeta \in [0, \frac{1}{2}] \\ 1 & \zeta > \frac{1}{2}. \end{cases} \tag{5.130}$$

■

Chapter 6

Introduction To Bode Plots

(1) Consider the SISO LTI system

$$L = \frac{1}{s}. \tag{6.1}$$

This system represents an integrator. The magnitude and phase response for L is plotted in Figure 6.1. The slope of the magnitude plot is -20 *db/dec* or -6 *db/oct* (db - decibels; dec - decade; oct - octave). The phase contributes -90 *degrees* at all frequencies.

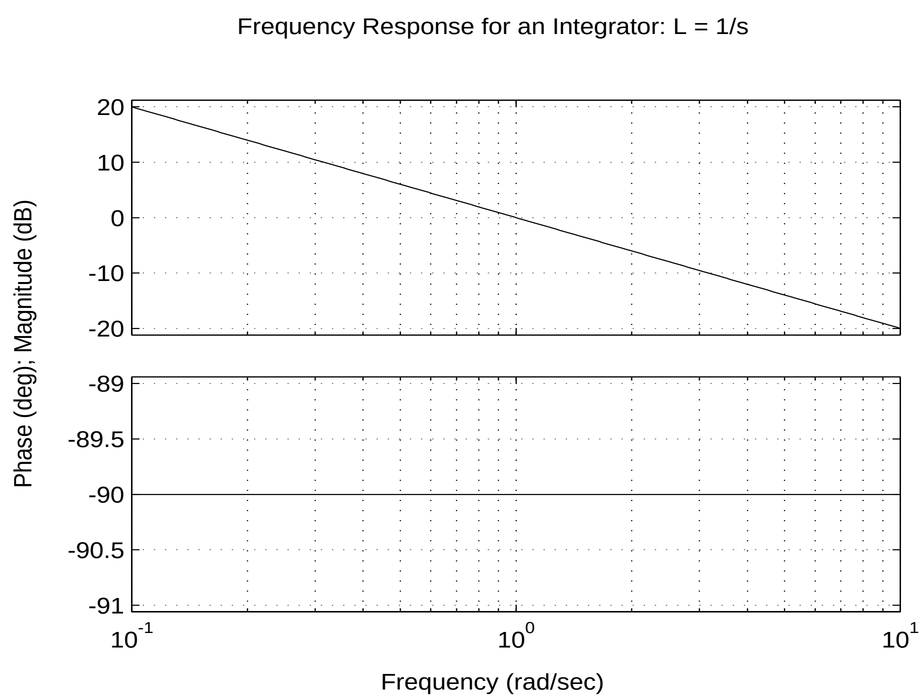

Figure 6.1: Frequency Response For Integrator: $L = \frac{1}{s}$.

(2) Consider the SISO LTI system

$$L = \frac{1}{s + 0.4}. \tag{6.2}$$

This system represents an approximate integrator. The magnitude and phase response for L is plotted in Figure 6.2. It approximates an integrator well at high frequencies. It approximates the integrator well above $\omega = 4\ rad/sec$. Above $\omega = 4\ rad/sec$, the magnitude response rolls-off at a slope of $-20\ db/dec$.

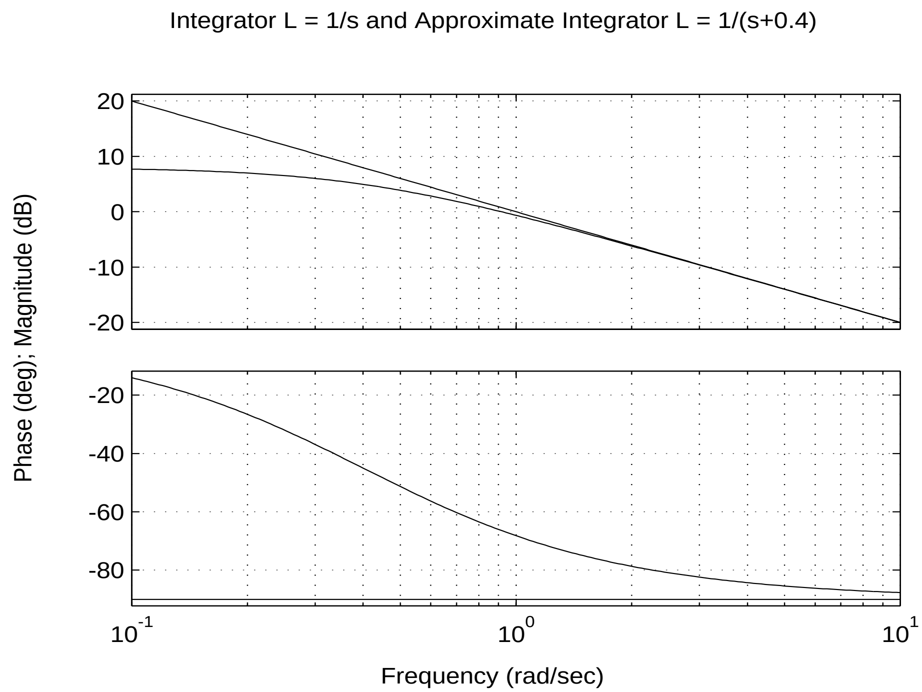

Figure 6.2: Frequency Response For Integrator $L = \frac{1}{s}$ and Approximate Integrator $L = \frac{1}{s+0.4}$.

(3) Consider the SISO LTI system

$$L_3 = s. \tag{6.3}$$

This system represents a differentiator. The magnitude and phase response for L is plotted in Figure 6.3. The magnitude response has a slope of 20 db/dec. The phase response contributes 90 $degrees$ at all frequencies.

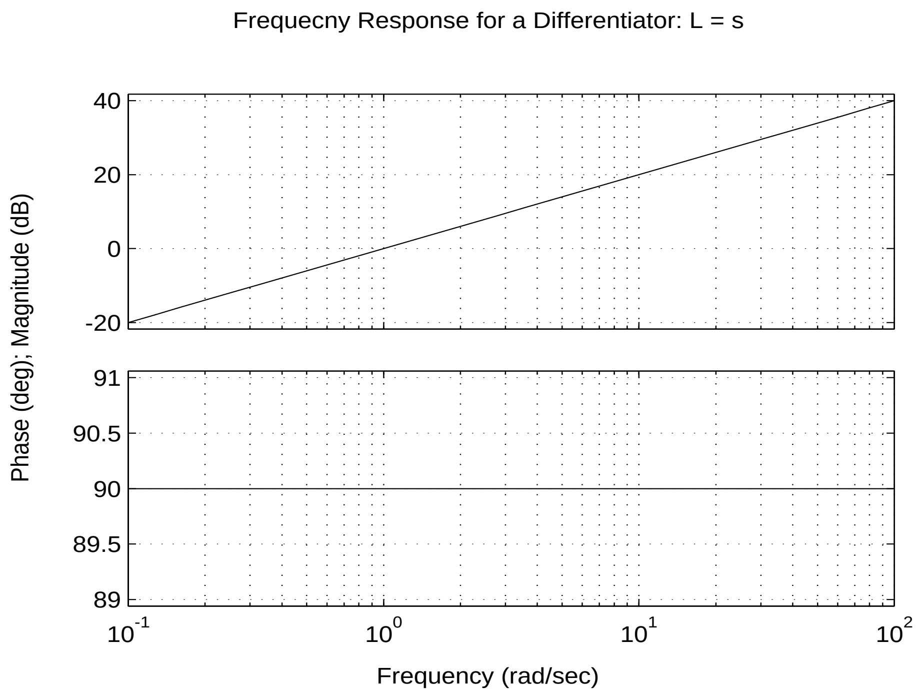

Figure 6.3: Differentiator: $L = s$.

(4) Consider the SISO LTI system

$$L_4 = \frac{10s}{s+10}. \tag{6.4}$$

This system represents an approximate differentiator. The magnitude and phase response for L is plotted in Figure 6.4. It approximates the differentiator well at low frequencies. Below $\omega = 1\ rad/sec$, it approximates the differentiator well.

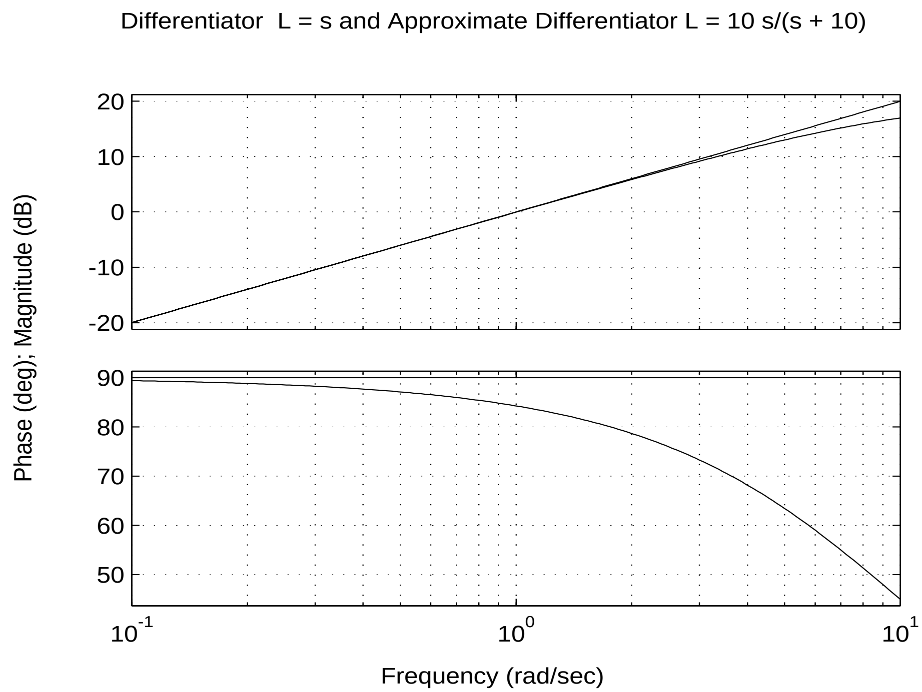

Figure 6.4: Differentiator $L = s$ and Approximate Differentiator $L = \frac{10s}{s+10}$

(5) Consider the SISO LTI system

$$L = \left[\frac{1}{s(s+1)} \right] \left[\frac{10^3}{s+10^3} \right]. \tag{6.5}$$

This system represents a second order lag-integral system with a high frequency pole. The magnitude and phase response for L are plotted in Figure 6.5. The high frequency pole contributes an additional 90 *degrees* of phase lag by $\omega \approx 10^4$ *rad/sec*.

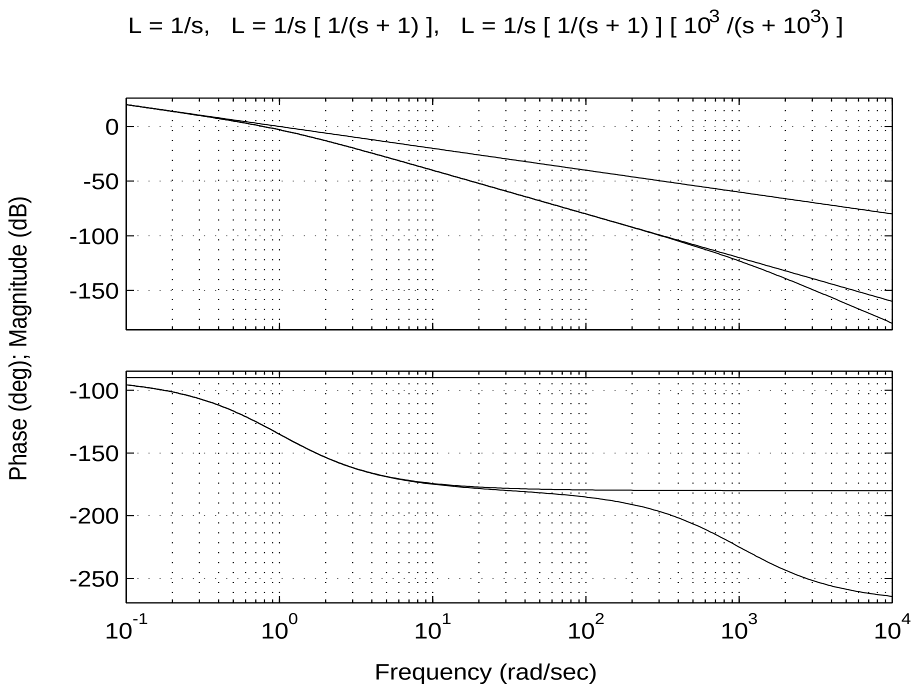

Figure 6.5: 3rd Order System $L = \left[\frac{1}{s(s+1)}\right]\left[\frac{10^3}{s+10^3}\right]$.

(6) Consider the SISO LTI system

$$L = \left[\frac{1}{s(s+1)}\right] \left[\frac{10^3}{s+10^3}\right] \left[\frac{s+10^6}{10^6}\right]. \tag{6.6}$$

This system represents a second order lag-integral system with a high frequency pole and zero. The magnitude and phase response for L is plotted in Figure 6.6. The high frequency zero contributes an additional 90 *degrees* of phase lead by $\omega \approx 10^7$ *rad/sec*.

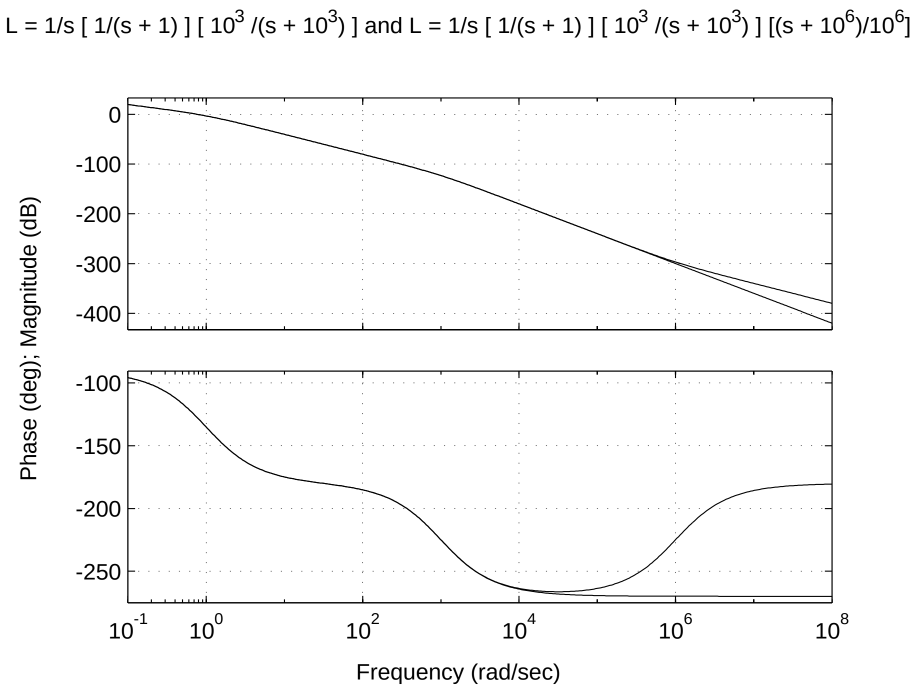

Figure 6.6: 3rd Order System With Numerator Zero $L = \left[\frac{1}{s(s+1)}\right] \left[\frac{10^3}{s+10^3}\right] \left[\frac{s+10^6}{10^6}\right]$.

(7) Consider the SISO LTI system

$$L = \left[\frac{1}{s(s+1)}\right] \left[\frac{10^3}{s+10^3}\right] \left[\frac{s+10^6}{10^6}\right]^2. \tag{6.7}$$

This system represents a second order lag-integral system with a high frequency pole and two zeros. The magnitude and phase response for L is plotted in Figure 6.7. The two high frequency zeros contribute an additional 180 *degrees* of phase lead by $\omega \approx 10^7$ *rad/sec*.

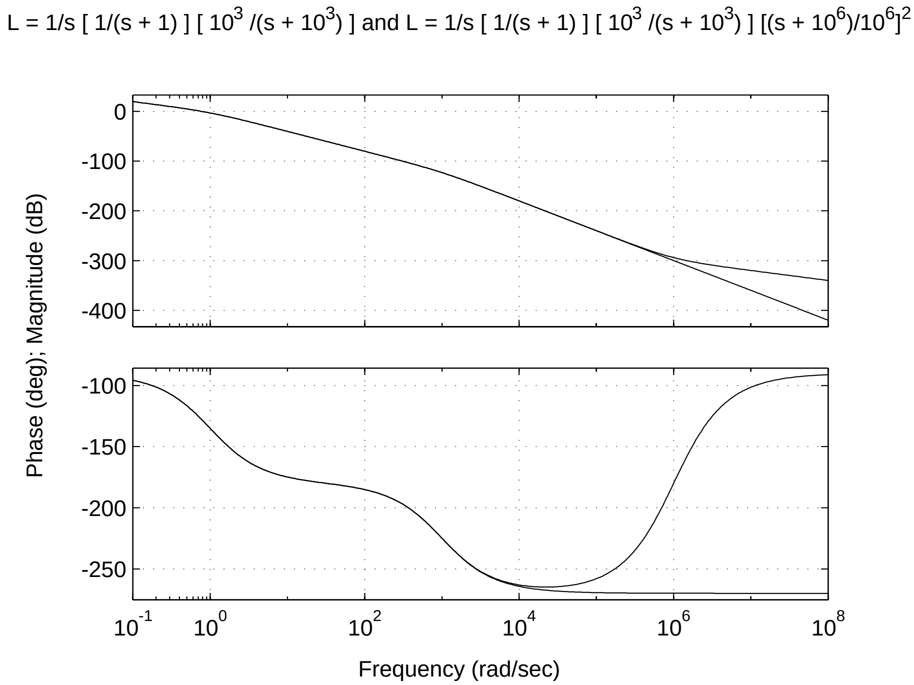

Figure 6.7: 3rd Order System With Two Numerator Zeros: $L = \left[\frac{1}{s(s+1)}\right] \left[\frac{10^3}{s+10^3}\right] \left[\frac{s+10^6}{10^6}\right]^2$.

(8) Consider the SISO LTI system

$$L = \frac{5}{s-1} \tag{6.8}$$

This system represents a first order unstable system. The magnitude and phase response for L is plotted in Figure 6.8. While the magnitude response is identical to that of the stable system $\frac{5}{s+1}$, the phase response looks very different. The phase contributes 180 *degrees* of phase lag at dc. As we go up in frequency, L contributes phase lead.

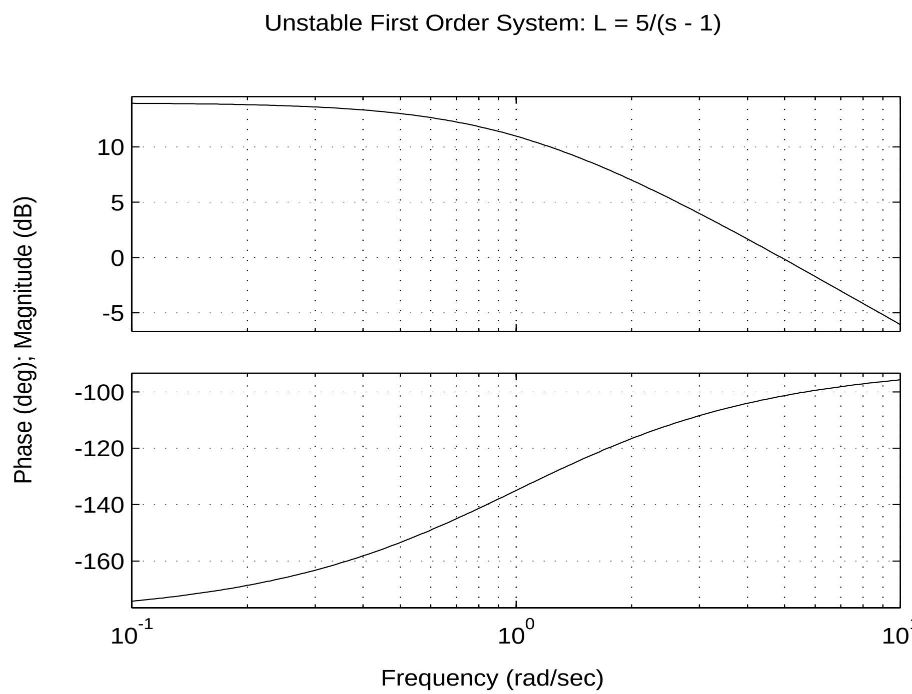

Figure 6.8: Frequency Response For Unstable First Order System: $L = \frac{5}{s-1}$.

(9) Consider the SISO LTI system

$$L = \left[\frac{10}{s-1}\right] \left[\frac{10^3}{s+10^3}\right]. \tag{6.9}$$

This system represents a first order unstable system with a high frequency pole. The magnitude and phase response for L is plotted in Figure 6.9. The high frequency pole contributes an additional 90 degrees of lag by $\omega \approx 10^4$ rad/sec.

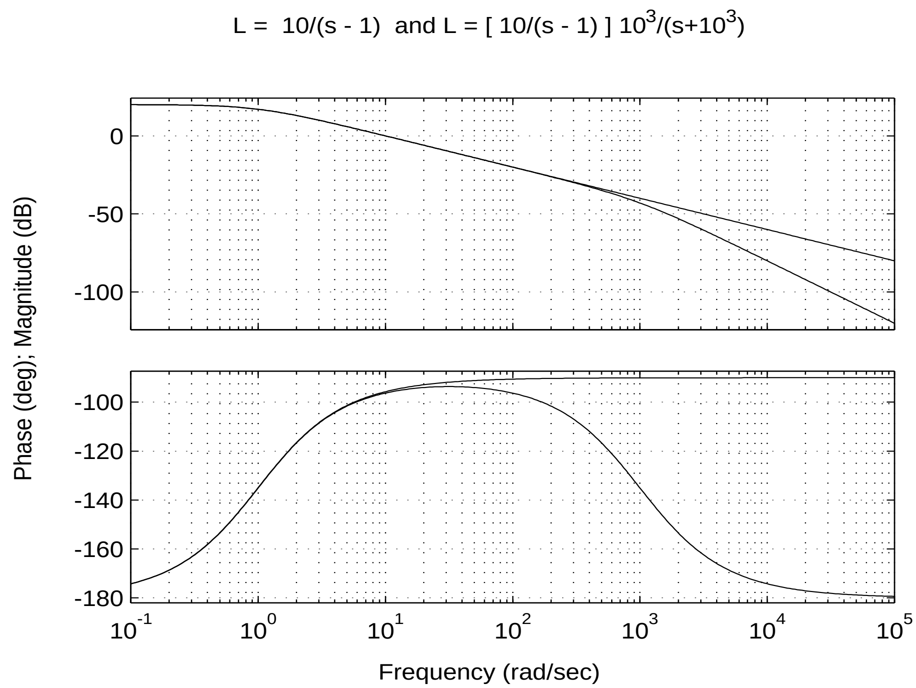

Figure 6.9: $L = \left[\frac{10}{s-1}\right] \left[\frac{10^3}{s+10^3}\right]$.

(10) Consider the SISO LTI system

$$L = \left[\frac{10}{s-1}\right]\left[\frac{10^3}{s+10^3}\right]\left[\frac{10^6}{s+10^6}\right]. \tag{6.10}$$

The magnitude and phase response for L is plotted in Figure 6.10. The high frequency pole contributes an additional 90 *degrees* of lag by $\omega \approx 10^7$ *rad/sec*.

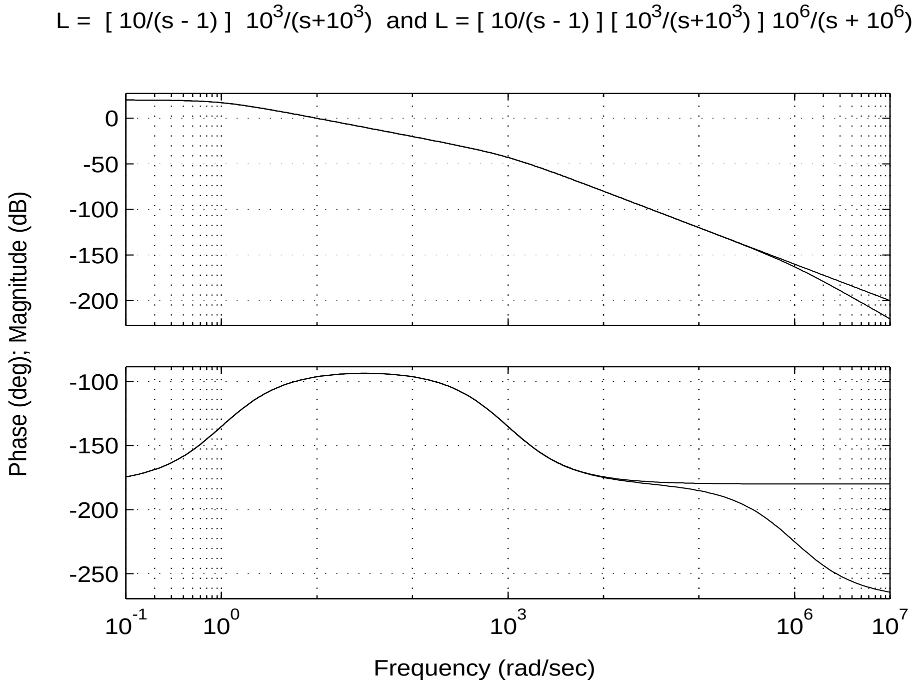

Figure 6.10: $L = \left[\frac{10}{s-1}\right]\left[\frac{10^3}{s+10^3}\right]\left[\frac{10^6}{s+10^6}\right]$.

Chapter 7

An Introduction To State Space Concepts

7.1 State Of A System And State Variables

State Of A System. The *state of a system* is the minimum information required to predict the system's future output $y(\cdot)$ given knowledge of current and future inputs $u(cdot)$.

Example 7.1.1 (State and State Variables For Series RLC Circuit.)

Consider the series RLC circuit in Figure 5.1. From equation (5.3), it follows that the state of the circuit at $t = 0$ *(or at any initial time* $t = t_o$*) is the initial condition vector* $[\ y_o \ \ \dot{y}_o\ ]$*. Because of this, we say that* y *and* $\dot{y}$ *are* state variables*. It must be noted that the The state of a system is not unique. For the RLC circuit it suffices, for example, to know any two independent pieces of information (e.g. initial capacitor voltage and inductor voltage) in order to determine* $[\ y_o \ \ \dot{y}_o\ ]$ *and predict the future output with knowledge of future inputs. Given this, it follows that state variables are also not unique. For the RLC circuit, another set of state variables consists of the capacitor voltage* y *and the inductor current* i_L*.* ■

Selection Of State Variables. In general, state variables are typically selected to be variables associated with *energy storage* (e.g. velocities, spring displacements, etc.).

For RLC circuits, the capacitor voltages and inductor currents

are typically chosen as state variables [23].

Any n^{th} order system has n (not unique) state variables associated with it. Similarly, any system which is described by n state variables may be represented by an n^{th} order (vector) differential equation. This will be further explored below.

7.2 Introduction to Descriptor Models, State Space Models, and MIMO LTI Systems

Many dynamical systems are described by very high order differential equations (e.g. a flexible space structure [25]). Rather than writing high order differential equations, it is often much easier to write many simple equations. This motivates the new system descriptions presented in this section.

Descriptor Representation. Consider systems which are desribed by linear algebraic and ordinary differential equations with constant coefficients. Such systems may be written in *descriptor form* as follows:

$$M\dot{x} = Nx + Pu \qquad x(0^-) = x_o \qquad t > 0 \tag{7.1}$$

$$y = Cx + Du \tag{7.2}$$

where $M \in \mathcal{R}^{n\times n}$, $N \in \mathcal{R}^{n\times n}$, $P \in \mathcal{R}^{n\times m}$, $u = [u_1, \ldots, u_m]^T \in \mathcal{R}^m$ denotes a vector of m input variables, $x = [x_1, \ldots, x_n]^T \in \mathcal{R}^n$ denotes a vector of n state variables, $y = [y_1, \ldots, y_p]^T \in \mathcal{R}^p$ denotes a vector of p output variables, and $x_o = [x_{1_o}, \ldots, x_{n_o}]^T \in \mathcal{R}^n$ denotes a vector of n initial conditions for the state.

For systems described by linear differential and algebraic equations, it is always easy to write system equations in descriptor form.

State Space Representation: Multiple-Input Multiple-Output (MIMO) LTI Systems. When the matrix M is invertible, equation (7.1) may be rewritten in so-called *state space form*. Doing so yields:

$$\text{State Equation :} \quad \dot{x} \quad = \quad Ax + Bu; \quad x(0^-) = x_o \quad t > 0 \tag{7.3}$$

$$\text{Output Equation :} \quad y \quad = \quad Cx + Du. \tag{7.4}$$

where

$$A = M^{-1}N \tag{7.5}$$

$$B = M^{-1}P. \tag{7.6}$$

For general MIMO LTI systems, $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, $C \in \mathcal{R}^{p\times n}$, $D \in \mathcal{R}^{p\times m}$, $u = [u_1, \ldots, u_m]^T \in \mathcal{R}^m$ denotes a vector of m input variables, $x = [x_1, \ldots, x_n]^T \in \mathcal{R}^n$ denotes a vector of n state variables, $y = [y_1, \ldots, y_p]^T \in \mathcal{R}^p$ denotes a vector of p output variables, and $x_o = [x_{1_o}, \ldots, x_{n_o}]^T \in \mathcal{R}^n$ denotes a vector of n initial conditions for the state. The quadruple (A, B, C, D) is said to define a *state space representation* for the system. The pair (A, B) define the *state equation* (7.3) while the pair (C, D) define the *output equation* (7.4). Because the state equation contains n differential equations, we say that the system is an n^{th} *order system*. A general MIMO LTI system with state space representation (A, B, C, D) may be visualized as shown in Figure 7.1.

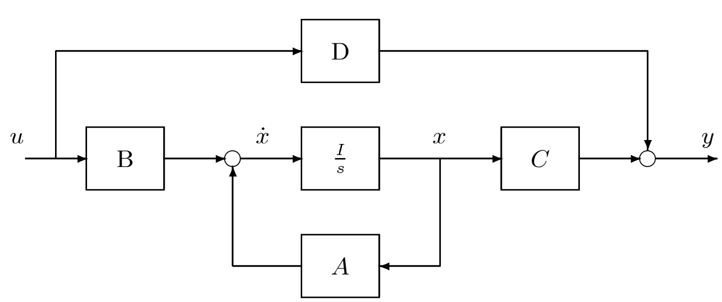

Figure 7.1: Block Diagram Visualization of General LTI System State Space Representation

Utility of State Space Representations. State space representations (A, B, C, D) are very useful for a variety of reasons:

- Many systems may be approximated well by a state space representation (A, B, C, D);
- A rich set of analysis and design techniques exist;
- State-of-the-art support software is readily available [11], [21], [30].

Example 7.2.1 (Series RLC Circuit Descriptor and State Space Representations.)

Consider the series RLC circuit in Figure 5.1. It may be represented by two first order linear differential equations as follows:

$$u = Ri_L + L\frac{di_L}{dt} + y \tag{7.7}$$

$$i_L = C\frac{dy}{dt}. \tag{7.8}$$

These may be written in so called descriptor form *as follows:*

$$L\frac{di_L}{dt} = -y - Ri_L + u \tag{7.9}$$

$$C\frac{dy}{dt} = i_L \tag{7.10}$$

or in matrix-vector descriptor form as follows:

$$M\dot{x} = Nx + Pu \tag{7.11}$$

where

$$M = \begin{bmatrix} 0 & L \\ C & 0 \end{bmatrix} \qquad N = \begin{bmatrix} -1 & -R \\ 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad x = \begin{bmatrix} y \\ i_L \end{bmatrix} \tag{7.12}$$

Since the matrix M is invertible, the above equation may be written in state space form *as follows:*

$$\dot{x} = Ax + Bu \tag{7.13}$$

where

$$A = M^{-1}N = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \qquad B = M^{-1}P = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}. \tag{7.14}$$

An output equation *may be written as follows:*

$$y = Cx + Du \tag{7.15}$$

where

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \end{bmatrix}. \tag{7.16}$$

■

7.3 Linear Transformation of System Varaibles

In many applications, it is desirable to

- change the units of the variables being used or
- change the coordinate system being used.

In either case, a *transformation of variables* is required. Given this, suppose that we are given an LTI system with state space representation (A_1, B_1, C_1, D_1):

$$\dot{x}_1 = A_1 x_1 + B_1 u_1 \tag{7.17}$$

$$y_1 = C_1 x_1 + D_1 u_1. \tag{7.18}$$

Suppose that we desire to transform the variables u_1, x_1, y_1. This can be accomplished via the following linear transformations:

$$u_2 = S_u u_1 \tag{7.19}$$

$$x_2 = S_x x_1 \tag{7.20}$$

$$y_2 = S_y y_1 \tag{7.21}$$

where $S_u \in \mathcal{R}^{m\times m}$, $S_x \in \mathcal{R}^{n\times n}$, $S_y \in \mathcal{R}^{p\times p}$ are invertible (nonsingular) matrices. Given this, the new state space representation (A_2, B_2, C_2, D_2) is given by:

$$\dot{x}_2 = A_2 x_2 + B_2 u_2 \tag{7.22}$$

$$y_2 = C_2 x_2 + D_2 u_2 \tag{7.23}$$

where

$$A_2 = S_x A_1 S_x^{-1} \quad B_2 = S_x B_1 S_u^{-1} \tag{7.24}$$

$$C_2 = S_y C_1 S_x^{-1} \quad D_2 = S_y D_1 S_u^{-1}. \tag{7.25}$$

The transformation matrix S_x is often called a *similarity transformation* [44] and the matrices A_1 and A_2 are said to be *similar* to one another. As might be expected, it can be shown that changing units does not change the fundamental properties of a system.

7.3.1 State Space Representation Analysis Via Laplace: System Resolvent

The matrix-vector equations (7.3) and (7.4) may be analyzed using Laplace Transforms. Taking the transform of each equation yields:

$$X(s) = (sI - A)^{-1} x_o + (sI - A)^{-1} B U(s) \tag{7.26}$$

$$Y(s) = C(sI - A)^{-1} x_o + \left[\, C(sI - A)^{-1} B + D \,\right] U(s). \tag{7.27}$$

The matrix $(sI - A)^{-1}$ is called the *resolvent of the system.*

7.4 Transfer Function Matrix, Matrix Exponential, And Impulse Response Matrix

For SISO systems, the concept of a transfer function is important. For MIMO systems, the analogous concept is the *transfer function matrix.* From equation (7.27), one sees that the transfer function matrix $H(\cdot)$ from the input vector $u = [u_1, \ldots, u_m]^T$ to the output vector $y = [y_1, \ldots, y_p]^T$ is given by

$$H(s) = C(sI - A)^{-1}B + D. \tag{7.28}$$

$H(\cdot)$ is a $p \times m$ matrix of scalar transfer functions. Its ij$^{\text{th}}$ entry represents the transfer function from the j$^{\text{th}}$ input $u_j(\cdot)$ to the i$^{\text{th}}$ output $y_i(\cdot)$. At this point, it is useful to identify the *matrix exponential* e^{At} as the inverse Laplace transform of $(sI - A)^{-1}$; i.e.

$$e^{At} \stackrel{\text{def}}{=} \sum_{i=0}^{\infty} \frac{(At)^n}{n!} = \mathcal{L}^{-1}\{\ (sI - A)^{-1}\ \}. \tag{7.29}$$

This implies that the matrix exponential e^{At} may be computed as the term-by-term inverse Laplace transform of $(sI - A)^{-1}$. Given the above, the *impulse response matrix* h is given by

$$h(t) = Ce^{At}B\ 1(t) + D\ \delta(t) \tag{7.30}$$

and the general solutions for $x(\cdot)$ and $y(\cdot)$ are given by

$$\begin{aligned} x(t) &= e^{At}x_o + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau && (7.31)\\ y(t) &= Ce^{At}x_o + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t). && (7.32) \end{aligned}$$

Linearity. Equations (7.31)-(7.32) show that both the state $x(\cdot)$ and the output $y(\cdot)$ are linear in the vector $[x_o^T \quad u^T(\cdot)]^T$ - a confirmation that systems described by a state space representation (A, B, C, D) do indeed define a *linear system*.

Time Invariance. Because the integral in equation (7.32) defines a *convolution operator*, it follows that systems described by a state space representation (A, B, C, D) do indeed define a *time invariant system*. See equations (5.17)-(5.21).

Computational Approaches. Several approaches exist for evaluating $x(\cdot)$ and $y(\cdot)$. Two commonly used approaches are (i) s-domain matrix algebra [44] with term-by-term Laplace transform inversion and (ii) eigenvalue-eigenvector techniques - to be discussed subsequently. While the first approach is a straight forward extension of classical SISO techniques, the second method offers new useful insight regarding the natural modes of a system. Numerical integration [22], of course, represents one more approach.

Structure And Terminology: State Transition Matrix. If the input $u(\cdot)$ is applied at $t = t_o$, then equation (7.31) may be written as

$$x(t) = \Phi(t, t_o)x_o + \int_{t_o}^{t} \Phi(t, \tau)Bu(\tau)d\tau \tag{7.33}$$

where $x(t_0^-) = x_o$ and

$$\Phi(t, \tau) \stackrel{\text{def}}{=} e^{A(t-\tau)} \tag{7.34}$$

is called the *state transition matrix* for the system.

Properties of State Transition Matrix. It can be shown that the state transition matrix satisfies the following properies:

$$\Phi(t, t) = I \qquad (Identity\ Property) \tag{7.35}$$

$$\Phi(t_1, t_3) = \Phi(t_1, t_2)\Phi(t_2, t_3) \qquad (Semi-Group\ Property) \tag{7.36}$$

$$\frac{d}{dt}\Phi(t, \tau) = A\Phi(t, \tau) \qquad (Derivative\ Property) \tag{7.37}$$

for all $t, t_1, t_2, t_3 \in \mathcal{R}$.

Looking Ahead To Time Varying Systems. In subsequent sections, time varying systems will be studied in greater detail. Given this, it is useful to note that

- The structure exhibited in equation (7.33) will be seen to hold when the matrices (A, B) are functions of the independent variable t;

- Properties (7.35)-(7.37) will hold when the matrix A is a function of the independent variable t. In fact, it can be shown [6, pg. 33] that if

$$A(t)\left[\int_{t_o}^{t} A(\sigma)d\sigma\right] = \left[\int_{t_o}^{t} A(\sigma)d\sigma\right] A(t) \tag{7.38}$$

for all t, then

$$\Phi(t,t_o) = e^{\left[\int_{t_o}^{t} A(\sigma)d\sigma\right]}. \tag{7.39}$$

7.5 Obtaining A State Space Representation For A SISO Input-Output (Transfer Function) Description

It is often useful to describe a system with a given transfer function (input-output model), by a state space model. To demonstrate how this can be done, consider the following arbitrary 3$^{\text{rd}}$ order transfer function:

$$H(s) = \frac{b_2 s^2 + b_1 s + b_o}{s^3 + a_2 s^2 + a_1 s + a_o} + d \tag{7.40}$$

where the a_i, b_i, and d are arbitrary real (scalar) constants. The goal is to obtain a 3$^{\text{rd}}$ order state space representation (A, B, C, D) whose transfer function is $H(\cdot)$. To do this, first note that H may be rewritten as follows:

$$H(s) = \frac{\frac{b_2}{s} + \frac{b_1}{s^2} + \frac{b_o}{s^3}}{1 + \frac{a_2}{s} + \frac{a_1}{s^2} + \frac{a_o}{s^3}} + d \tag{7.41}$$

Now consider the block diagram shown in Figure 7.2. In this figure, the integrator states (their outputs) have been labeled x_1, x_2, and x_3, respectively. From this figure, one can "walk around" the diagram to obtain the following s-domain (zero initial condition) relationships:

$$\begin{aligned}
\dot{x}_3 &= u - [\, a_2 x_3 + a_1 x_2 + a_o x_1 \,]; \quad x_1 = \frac{\dot{x}_3}{s^3}, \;\; x_2 = \frac{\dot{x}_3}{s^2}, \;\; x_3 = \frac{\dot{x}_3}{s}, && (7.42)\\
&= u - [\, \frac{a_2}{s} + \frac{a_1}{s^2} + \frac{a_o}{s^3} \,]\, \dot{x}_3 && (7.43)\\
&= \frac{1}{1 + \frac{a_2}{s} + \frac{a_1}{s^2} + \frac{a_o}{s^3}}\, u && (7.44)\\
y &= [\, \frac{b_2}{s} + \frac{b_1}{s^2} + \frac{b_o}{s^3} \,]\, \dot{x}_3 + d\, u && (7.45)
\end{aligned}$$

$$= \left[\frac{b_2}{s} + \frac{b_1}{s^2} + \frac{b_o}{s^3} \right] \frac{1}{1 + \frac{a_2}{s} + \frac{a_1}{s^2} + \frac{a_o}{s^3}} u + d\, u \quad (7.46)$$

$$= H(s)\, u. \quad (7.47)$$

This shows that the 3rd order system shown in Figure 7.2 has the desired transfer function.

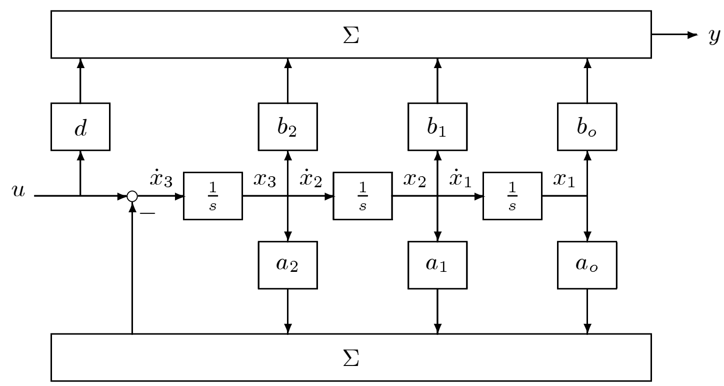

Figure 7.2: General 3rd Order Transfer Function: Obtaining a State Space Representation

From Figure 7.2, one immediately obtains the following state space representation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_o & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad (7.48)$$

$$y = \begin{bmatrix} b_o & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} d \end{bmatrix} u \quad (7.49)$$

for the general 3rd order SISO transfer function

$$H(s) = \frac{b_2 s^2 + b_1 s + b_o}{s^3 + a_2 s^2 + a_1 s + a_o} + d. \quad (7.50)$$

The above state space repressentation is said to be in *controller canonical form* [26]. Its 'A' matrix, specifically, is said to be in *companion form* [26]. Other canonical forms (i.e. standard block diagram structures) are discussed in [26]. The ideas presented in this example may be extended to higher order systems and to MIMO LTI systems [26] as well.

7.6 Obtaining An Input-Output (Transfer Function Matrix) Description From An LTI State Space Representation

Given a MIMO LTI system with state space representation

$$\dot{x} = Ax + Bu \tag{7.51}$$

$$y = Cx + Du, \tag{7.52}$$

we would like to find an input-output description.

Markov Parameters. To do this, it is helpful to define the so called *Markov parameters* as follows:

$$M_k \stackrel{\text{def}}{=} \begin{cases} 0 & k = -1, -2, \ldots \\ D & k = 0 \\ CA^k B & k = 1, 2, \ldots \end{cases} \tag{7.53}$$

Without loss of generality, suppose that A has an n^{th} order characteristic polynomial given by

$$\det(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_o \tag{7.54}$$

where the a_i are real (scalar) constants.

Cayley-Hamilton Theorem. Next, it is important to recall the *Cayley-Hamilton Theorem* [44]. This fundamental result from linear algebra states that every matrix satisfies its characteristic polynomial; i.e.

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_o I. \tag{7.55}$$

Using the Cayley-Hamilton Theorem, one can show that the output vector $y \in \mathcal{R}^p$ is related to the input vector $u \in \mathcal{R}^m$ as follows:

$$\overset{(n)}{y(t)} + a_{n-1} \overset{(n-1)}{y(t)} + \cdots + a_1 \dot{y}(t) + a_o y(t) = B_n \overset{(n)}{u(t)} + B_{n-1} \overset{(n-1)}{u(t)} + \cdots + B_1 \dot{u}(t) + B_o u(t) \tag{7.56}$$

where

$$B_k = M_{n-k} + \sum_{i=k}^{n-1} a_i M_{i-k}. \tag{7.57}$$

Example 7.6.1 (A Two-Input Two-Output (TITO) RLC Circuit.)

Consider the RLC circuit in Figure 7.3. The circuit, or system, inputs (forcing functions) are the voltage sources u_1 *and* u_2*. The circuit, or system, outputs are the voltage measurements* y_1 *and* y_2*. The circuit, or system, state variables are the capacitor voltage* x_1 *and the inductor current* x_2*.*

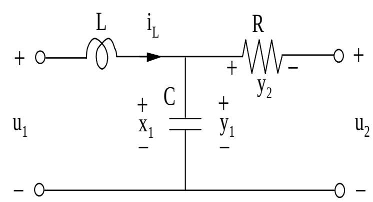

Figure 7.3: Two-Input Two-Output (TITO) RLC Circuit

In this two-input two-output (TITO) circuit, the resistor, inductor, and capacitor values are denoted R, L, *and* C *respectively. It is assumed that they are nonnegative and constant.*

Using impedance methods, one obtains the following s-domain zero inital condition relationships between the s-domain functions Y_1, $,Y_2$, U_1, *and* U_2*:*

$$Y_1(s) \quad = \quad [\,\frac{\frac{1}{LC}}{s^2+\frac{1}{RC}s+\frac{1}{LC}}\,]\,U_1(s) \;+\; [\,\frac{\frac{1}{RC}s}{s^2+\frac{1}{RC}s+\frac{1}{LC}}\,]\,U_2(s). \tag{7.58}$$

$$Y_2(s) \quad = \quad Y_1(s)-U_2(s) = [\,\frac{\frac{1}{LC}}{s^2+\frac{1}{RC}s+\frac{1}{LC}}\,]\,U_1(s) \;+\; [\,\frac{-(s^2+\frac{1}{LC})}{s^2+\frac{1}{RC}s+\frac{1}{LC}}\,]\,U_2(s). \tag{7.59}$$

From these, one can obtain an input-output model relating y_1, y_2, u_1, *and* u_2*. More specifically, one obtains the following second order ordinary differential equations with constant coefficients:*

$$\ddot{y}_1+\frac{1}{RC}\dot{y}_1+\frac{1}{LC}y_1 \quad = \quad \frac{1}{LC}u_1+\frac{1}{RC}\dot{u}_2. \tag{7.60}$$

$$\ddot{y}_2+\frac{1}{RC}\dot{y}_2+\frac{1}{LC}y_2 \quad = \quad \frac{1}{LC}u_1-\ddot{u}_2-\frac{1}{LC}u_2. \tag{7.61}$$

The state space representation which relates u_1, u_2, x_1, x_2, y_1, *and* y_2 *is as follows:*

$$\dot{x} \quad = \quad Ax+Bu \tag{7.62}$$

$$y \quad = \quad Cx+Du \tag{7.63}$$

where

$$A = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & \frac{1}{RC} \\ \frac{1}{L} & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \tag{7.64}$$

The system transfer function matrix is given by:

$$H(s) = C(sI - A)^{-1}B + D = \begin{bmatrix} \frac{\frac{1}{LC}}{s^2+\frac{1}{RC}s+\frac{1}{LC}} & \frac{\frac{1}{RC}s}{s^2+\frac{1}{RC}s+\frac{1}{LC}} \\ \frac{\frac{1}{LC}}{s^2+\frac{1}{RC}s+\frac{1}{LC}} & \frac{-(s^2+\frac{1}{LC})}{s^2+\frac{1}{RC}s+\frac{1}{LC}} \end{bmatrix} \tag{7.65}$$

This is expected from equations (7.58) and (7.59).

From the above transfer function, one obtains (by inspection) the following vector differential equation input-output model:

$$\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \frac{1}{RC}\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} + \frac{1}{LC}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{RC} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{LC} & 0 \\ \frac{1}{LC} & -\frac{1}{LC} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{7.66}$$

which relates the outputs y_1 and y_2 to the inputs u_1 and u_2. ■

7.7 Natural Modes, Exponential Stability, Eigenvalues, Eigenvectors, Matrix Exponential, And Modal Equations

As discussed earlier, every system has *natural tendencies* or *natural modes.* Each mode corresponds to a system pole (e.g. real exponential mode) or complex poles (e.g. exponential sinusoidal mode). As for SISO systems, a MIMO LTI system with state space representation (A, B, C, D) is said to be *exponentially stable* if all of the eigenvalues of A (i.e. system poles) lie in the left half plane (i.e. have negative real parts).

The modes of an LTI system are best examined by applying standard tools from linear algebra, namely *eigenvalues* and *eigenvectors*, to the system matrix A.

Eigenvalues and Eigenvectors. The *eigenvalues* $\{s_i\}_{i=1}^n$ of a square matrix $A \in \mathcal{R}^{n\times n}$ are the n roots of the characteristic equation

$$\phi(s) \stackrel{\text{def}}{=} \det(sI - A) = 0. \tag{7.67}$$

Since [44]

$$(sI - A)^{-1} = \frac{[\, cof(sI - A) \,]^T}{\det(sI - A)}, \tag{7.68}$$

it follows that the $\{s_i\}_{i=1}^n$ are precisely the n system poles. The *right eigenvectors* v_i corresponding to the eigenvalues s_i of A are found by solving

$$(s_i I - A)v_i = 0 \tag{7.69}$$

for a nonzero v_i for each $i = 1, 2, \ldots, n$. The s_i and v_i satisfy the matrix equation

$$AV = V\Lambda \tag{7.70}$$

where

$$V \stackrel{\text{def}}{=} \begin{bmatrix} | & \cdots & | \\ v_1 & \cdots & v_n \\ | & \cdots & | \end{bmatrix} \qquad \Lambda \stackrel{\text{def}}{=} \begin{bmatrix} s_1 & & \\ & \ddots & \\ & & s_n \end{bmatrix}. \tag{7.71}$$

In what follows, it will be assumed that the matrix A has n *linearly independent* right eigenvectors v_i. In such a case, the $n \times n$ matrix V is invertible. It is convenient to define its inverse W as follows:

$$W \stackrel{\text{def}}{=} \begin{bmatrix} - & w_1^H & - \\ & \vdots & \\ - & w_n^H & - \end{bmatrix} = V^{-1} \tag{7.72}$$

where w_i^H denotes the i^{th} row of W and $(\,\cdot\,)^H \stackrel{\text{def}}{=} \overline{(\,\cdot\,)}^T$ denotes a conjugate-transpose operation [44]. Since V is invertible, it follows that $WA = \Lambda W$ and that the w_i satisfy

$$w_i^H(s_i I - A) = 0 \tag{7.73}$$

for each $i = 1, 2, \ldots, n$. Because of this, the w_i are called *left eigenvectors* of A corresponding to the eigenvalues s_i.

Dyadic Decomposition, Matrix Exponential. Given the above, it follows that [44]

$$A = V\Lambda W = \sum_{i=1}^{n} s_i \; v_i w_i^H. \tag{7.74}$$

This shows that A may be written as a linear combination of n rank one matrices. The sum in equation (7.74) is referred to as a *dyadic decomposition* for A. From this, it follows that [44]

$$e^{At} = Ve^{\Lambda t}W = \sum_{i=1}^{n} e^{s_i t}\, v_i w_i^H \tag{7.75}$$

and

$$(sI - A)^{-1} = V(sI - \Lambda)^{-1}W = \sum_{i=1}^{n} \frac{1}{s - s_i}\, v_i w_i^H \tag{7.76}$$

$$\tag{7.77}$$

Modal Relationships. Substituting these relationships into equations (7.26), (7.27), (7.28), (7.30), (7.31), (7.32) yields the following *modal relationships*:

$$\begin{aligned}
X(s) &= \sum_{i=1}^{n} [\; w_i^H x_o + w_i^H BU(s) \;] \frac{1}{s - s_i}\, v_i && (7.78)\\
Y(s) &= \sum_{i=1}^{n} [\; w_i^H x_o + w_i^H BU(s) \;] \frac{1}{s - s_i}\, Cv_i + DU(s) && (7.79)\\
H(s) &= \sum_{i=1}^{n} \frac{Cv_i w_i^H B}{s - s_i} + D && (7.80)\\
h(t) &= \sum_{i=1}^{n} Cv_i w_i^H B\, e^{s_i t}\, 1(t) + D\,\delta(t) && (7.81)\\
x(t) &= \sum_{i=1}^{n} [\; e^{s_i t}\, w_i^H x_o + \int_0^t e^{s_i(t-\tau)} w_i^H Bu(\tau)d\tau \;]\, v_i && (7.82)\\
y(t) &= \sum_{i=1}^{n} [\; e^{s_i t}\, w_i^H x_o + \int_0^t e^{s_i(t-\tau)} w_i^H Bu(\tau)d\tau \;]\, Cv_i + Du(t) && (7.83)
\end{aligned}$$

These relationships are very useful for understanding the manner in which the natural modes of an LTI system impact the input-output and internal behavior of the system in the presence of initial conditions x_o and inputs $u(\cdot)$.

7.8 Modal Analysis

The above modal relationships are very useful for understanding the natural tendencies of an LTI system. To examine the natural modes of the system, one sets the input u to zero and examine the state $x(\cdot)$. Doing so yields,

$$x(t) \quad = \quad \sum_{i=1}^{n} [\; w_i^H x_o \;] \; e^{s_i t} \; v_i \tag{7.84}$$

Next, it is essential to realize that $WV = I$ implies that the v_j and w_i are *mutually orthonormal*; i.e.

$$w_i^H v_j \stackrel{\text{def}}{=} \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases} \tag{7.85}$$

With this, it follows that

$$x_o = v_i \quad \stackrel{\text{produces}}{\Longrightarrow} \quad x(t) = e^{s_i t} \; v_i. \tag{7.86}$$

This is the fundamental relationship for examining and understanding the natural modes of an LTI system. For a complex mode $s_i = \sigma_i + j\omega_i$, it follows from equation (7.86) that

$$x_o = a \; Re\{v_i\} + b \; Im\{v_i\} \quad \stackrel{\text{produces}}{\Longrightarrow} \quad x(t) = a \; Re\{e^{s_i t} \; v_i\} + b \; Im\{e^{s_i t} \; v_i\}. \tag{7.87}$$

7.9 Transmission Zeros And Energy Absorption

Just as some systems "give off" energy at certain frequencies - in the form of natural modes, some systems "absorb" certain frequencies. This motivates the concept of a *zero*. A SISO transfer function has a zero at $s = s_o$ if $H(s_o) = 0$. Analogously, a MIMO transfer function matrix H has a zero at $s = s_o$ if $H(s_o)$ loses rank. While this definition represents a direct extension of the classical SISO definition, we now give a definition for MIMO systems - a definition which captures our energy absorption notion of zeros.

Transmission Zeros. A dynamical system with input u, state x, and output y, is said to have a *transmission zero* at $s = s_o$ if there exists a pair of vectors (u_o, x_o), such that $[x_o^T \quad u_o^T]^T \neq 0$ and such that

$$u(t) = u_o e^{s_o t} \qquad x(0) = x_o \quad \stackrel{\text{produces}}{\Longrightarrow} \quad x(t) = e^{s_o t} \; x_o \qquad y(t) = 0 \tag{7.88}$$

for all $t \geq 0$. This definition shows that the concept of a transmissoin zero is an *energy absorption* concept. The vectors u_o and x_o are said to define the *direction of the transmisson zero*. The vector u_o defines the *input direction*. The vector x_o defines the *state direction*.

It can be shown that a MIMO LTI system with state space representation (A, B, C, D) has a *transmission zero* at $s = s_o$ if and only if there exists $[\ x_o^T \ \ u_o^T\ ]^T \neq 0$ such that

$$\begin{bmatrix} s_o I - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x_o \\ u_o \end{bmatrix}. \tag{7.89}$$

It is important to note that for SISO LTI systems, the definition given above for transmission zeros corresponds precisely the classical definition for zeros. For MIMO LTI systems, however, the transmission zeros of the system need not be zeros of any of the individual transfer functions within the system transfer function matrix! Examples which illustrate this point are readily constructed.

Example 7.9.1 (Transmission Zeros And Transfer Function Matrix Entries.)

Consider the LTI system

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{7.90}$$

$$C = \begin{bmatrix} 1 & -3 & 1 \\ 5 & 0 & k \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{7.91}$$

This system has a transfer function matrix

$$H(s) = \begin{bmatrix} \frac{s}{(s+1)(s+3)} & \frac{1}{s+2} \\ \frac{5}{s+1} & \frac{k}{s+2} \end{bmatrix}. \tag{7.92}$$

It can be shown that for $k = 10$, the system has a transmission zero at $s_o = 3$ with input direction $u_o = [\ 24 \quad -15\ ]^T$ and state direction $x_o = [\ 6 \quad 1 \quad -3\ ]^T$. We note that for $k = 10$, the transmission zero at $s_o = 3$ is NOT a zero of any of the individual transfer function entries within the transfer function matrix $H(\cdot)$. ∎

If $x_o = 0$*, then we say that the system has a transmission zero at every* s_o *in the s-plane. If* $u_o = 0$*, then* s_o *is a system pole. If* s_o *is not a system pole, then* $s_oI - A$ *is invertible and it follows that* $x_o = (s_oI - A)^{-1}Bu_o$ *and*

$$H(s_o)\, u_o = 0 \tag{7.93}$$

where $H(s) \stackrel{\text{def}}{=} C(sI - A)^{-1}B + D$ *is the system transfer function matrix. If the number of oututs* p *is equal to the number of inputs* m*, then we say that the system is a* square system*. In such a case, the matrix in is square and the transmission zeros may be found by solving for the roots of the following polynomial:*

$$\det \begin{bmatrix} s_oI - A & -B \\ C & D \end{bmatrix} = det(s_oI - A) \det H(s_o) = 0. \tag{7.94}$$

$$\tag{7.95}$$

■