Lecture 7:
Other undecidable languages, Rice's theorem, and Reductions

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1 Semi-decidable vs. decidable

We know that a language may be semi-decidable but not decidable. However, if both L and its complement $\bar{L}$ are semi-decidable then L must in fact be decidable.

Claim 1. *If a language L and its complement $\bar{L}$ are both semi-decidable, then L is decidable.*

Proof. Let M_L be a TM accepting L, and let $M_{\bar{L}}$ be a TM accepting $\bar{L}$. On input x, run both TMs "in parallel", until one of them accepts. (At some finite point in time, one of the machines must accept as every input x is either in L or in $\bar{L}$.) If M_L accepted, then halt and accept. If $M_{\bar{L}}$ accepted, then halt and reject. □

As a corollary, we get that the *complement* of A_{TM} is *not* semi-decidable! Do you see why?

We also have the following.

Theorem 1. *The class of decidable languages is closed under complementation. On the other hand, the class of semi-decidable languages is not closed under complementation.*

Proof. Given a DTM M deciding a language $L = L(M)$, construct a new DTM M' by taking M and swapping q_{accept} and q_{reject} states. It's easy to see that the new DTM M' accepts exactly those strings that are rejected by M, abd rejects exactly those strings that are accepted by M. So, we have $L(M') = \bar{L}$, as required.

On the other hand, A_{TM} is semi-decidable, but, as observed above, its complement is not semi-decidable. □

2 Examples of undecidable languages

Theorem 2. *The language*

$$E_{TM} = \{\langle M\rangle \mid L(M) \text{ is empty}\}$$

is undecidable.

Proof. Proof by reduction from A_{TM}. Given input $\langle M, w\rangle$, design a TM M' as follows:

M': "On input x, simulate M on input w. If M accepts, then Accept."

Observe that

1. if M accepts w, then $L(M') = \Sigma^*$ (i.e., M' accepts every input x),
2. if M does not accept w, then $L(M') = \emptyset$.

Now, if we have a decider TM R for the language E_{TM}, we can decide A_{TM} as follows:

"On input $\langle M, w \rangle$,

1. Construct the TM M' for this pair $\langle M, w \rangle$, as explained above.
2. Run R on input $\langle M' \rangle$.
3. If R accepts $\langle M' \rangle$, then Reject. If R rejects $\langle M' \rangle$, then Accept."

□

Theorem 3. *The language*

$$ALL_{TM} = \{\langle M \rangle \mid L(M) = \Sigma^*\}$$

is undecidable.

Proof. Suppose that ALL_{TM} is decidable by R. Show how to decide A_{TM}.

On input $\langle M, w \rangle$, construct TM M' as follows:

M': "On input x, simulate M on w, accepting if M accepts w".

Now, if M accepts w, then $L(M') = \Sigma^*$; and if M does not accept w, then $L(M') = \emptyset$.

So to decide A_{TM}, do the following:

"On input $\langle M, w \rangle$, construct TM M' defined above. Run R on input $\langle M' \rangle$. If R accepts $\langle M' \rangle$, then Accept; otherwise, Reject."

Since A_{TM} is undecidable, we conclude that R cannot exist. □

3 Another example of undecidability

Theorem 4. *The language*

$$EQ_{TM} = \{\langle M_1, M_2 \rangle \mid L(M_1) = L(M_2)\}$$

is undecidable.

Proof. Suppose it is decidable by some decider R. We reduce E_{TM} to EQ_{TM}.

Given $\langle M \rangle$, construct $M_1 = M$ and $M_2 = M_\emptyset$, where $M_\emptyset$ is some fixed TM such that $L(M_\emptyset) = \emptyset$. Clearly, we have $L(M) = \emptyset$ iff $L(M_1) = L(M_2)$.

So, to decide E_{TM}, we do the following:

"On input $\langle M \rangle$, construct M_1 and M_2, as described above. Run R on $\langle M_1, M_2 \rangle$. If R accepts, then Accept; otherwise, Reject."

Since E_{TM} is undecidable (as shown above), we conclude that R cannot exist. □

4 Rice's Theorem

Generalizing the arguments above, we will prove that essentially every nontrivial property of TM languages is undecidable. More precisely,

Theorem 5 (Rice's theorem)**.** *Any nontrivial property P of TMs is undecidable.*

Here a *property* is a collection of TM descriptions $\langle M \rangle$ such that, for any two M_1 and M_2, if $L(M_1) = L(M_2)$ then either $\langle M_1 \rangle, \langle M_2 \rangle \in P$, or $\langle M_1 \rangle, \langle M_2 \rangle \notin P$.[1]

Nontrivial means that it is neither empty nor everything: some TM M_1 exists such that $\langle M_1 \rangle \in P$, and some TM M_2 exists such that $\langle M_2 \rangle \notin P$.

Before we do the proof, consider the language $E_{TM} = \{\langle M \rangle \mid L(M) = \emptyset\}$. Verify that E_{TM} satisfies the definition of a nontrivial property. Thus, Rice's theorem implies that E_{TM} is undecidable!

For an example of a non-property, consider the set of TM descriptions $\langle M \rangle$ such that the length of the description $|\langle M \rangle| > 100$. This set is not a property in the above sense because we can have two TMs M_1 and M_2 with $L(M_1) = L(M_2) = \emptyset$, but $|\langle M_1 \rangle| < 100$ while $|\langle M_2 \rangle| > 100$. Can you see how to design such M_1 and M_2?

Proof of Rice's Theorem. Towards a contradiction, suppose some nontrivial property P is decidable by R. We'll show how to decide A_{TM}.

Assume, without loss of generality, that $\langle M_1 \rangle \in P$ for a TM M_1 such that $L(M_1) = \emptyset$. Let M_2 be any TM such that $\langle M_2 \rangle \notin P$. (Such M_2 exists since P is nontrivial.)

Given an instance $\langle M, w \rangle$ of A_{TM}, do the following:

1. Construct TM A:

   A: "On input x, run M on w. If M accepts w, then simulate M_2 on x, accepting if M_2 accepts."

2. Run R on input $\langle A \rangle$.

3. If R accepts $\langle A \rangle$, then Reject; If R rejects $\langle A \rangle$, then Accept.

Note on the construction of TM A:

- If M accepts w, then $L(A) = L(M_2)$, and so $\langle A \rangle \notin P$ (since $\langle M_2 \rangle \notin P$).

- If M does not accept w, then $L(A) = L(M_1) = \emptyset$, and so $\langle A \rangle \in P$ (since $\langle M_1 \rangle \in P$).

Thus, by being able to decide whether $\langle A \rangle$ is in P or is not in P, we can decide whether M accepts w, or not. In other words, we can decide A_{TM}. A contradiction. □

Justification of "without loss of generality, can assume $\langle M_1 \rangle \in P$, with $L(M_1) = \emptyset$":

Take some fixed M_1 such that $L(M_1) = \emptyset$. Either $\langle M_1 \rangle \in P$, or $\langle M_1 \rangle \notin P$. If it is in P, then we're done. If $\langle M_1 \rangle \notin P$, then $\langle M_1 \rangle \in \bar{P}$, where $\bar{P}$ is the complement of P. It's easy to see that $\bar{P}$ is also a property if P is a property, and that $\bar{P}$ is nontrivial if P is nontrivial. Then we argue about the property $\bar{P}$ as before, reaching the conclusion that $\bar{P}$ is undecidable. Since P is decidable iff its complement is decidable, we get that P is undecidable as well.

[1] One can define an *equivalence relation* $\equiv$ on TM descriptions: $\langle M_1 \rangle \equiv \langle M_2 \rangle$ iff $L(M_1) = L(M_2)$. Then a property P can be thought of as a collection of *equivalence classes* under the equivalence relation $\equiv$.

5 Reductions

We will consider a special kind of reductions: *mapping reductions.*

Definition 1. *Language A is m-reducible to B (denoted $A < B$) if there is a computable function $f : \Sigma^* \to \Sigma^*$ such that, for every $x \in \Sigma^*$,*

$$x \in A \Leftrightarrow f(x) \in B.$$

Theorem 6. *If $A < B$, and B is decidable, then so is A. If $A < B$, and B is semi-decidable, then so is A.*

The contrapositive: If $A < B$, and A is not (semi-) decidable, then neither is B.

Remark 1. *You can interpret "$<$" as saying "less hard than".*

5.1 Examples

$E_{TM} < EQ_{TM}$ via the reduction f such that $f(\langle M \rangle) = \langle M, M_\emptyset \rangle$, where $M_\emptyset$ is some fixed TM such that $L(M_\emptyset) = \emptyset$. (Check that this is indeed a reduction!)

Theorem 7. *EQ_{TM} is not semi-decidable.*

Proof. We reduce the complement of A_{TM} to EQ_{TM}. Given $\langle M, w \rangle$, define $f(\langle M, w \rangle) = \langle M_1, M_2 \rangle$ where

- M_1: "On input x, simulate TM M on input w, accepting if M accepts".
- $M_2 = M_\emptyset$ (where $M_\emptyset$ accepts the empty language).

Note that $L(M_1) = \Sigma^*$, if M accepts w; and $L(M_1) = \emptyset$, if M does not accept w. So this is indeed a reduction. Since we know that the complement of A_{TM} is not semi-decidable, we conclude that EQ_{TM} is not semi-decidable as well. $\square$

6 Hardness of INF

Consider the language $INF = \{\langle M \rangle \mid L(M) \text{ is infinite}\}$.

We will prove the following:

1. INF is undecidable.
2. INF is not semi-decidable.
3. The complement of INF is not semi-decidable.

To prove INF is undecidable we can either refer to Rice's theorem (arguing that INF is a non-trivial property), or give a direct reduction, e.g., $A_{TM} < INF$ as follows.

Theorem 8. *$A_{TM} < INF$.*

Proof. Given $\langle M, w \rangle$, construct M': "On input x, simulate M on w. If M accepts w, then Accept."

Clearly, M accepts w iff $L(M') = \Sigma^*$ is infinite. (If M does not accept w, then $L(M') = \emptyset$.) $\square$

To prove INF is not semi-decidable, we reduce from $\bar{A}_{TM}$ which is known to be non-semi-decidable.

Theorem 9. $\bar{A}_{TM} < INF$.

Proof. Given $\langle M, w \rangle$, construct

> M': "On input x, simulate M on w for $|x|$ steps. If M accepts w within $|x|$ steps, then Reject x. If M does not accept w within $|x|$ steps, then Accept x."

If M does not accept w, then M' will accept every x, and so $L(M') = \Sigma^*$ is infinite.

Suppose M accept w. Then M accepts w within some t number of steps, where t is a constant dependent on M and w. We get that for every input x of length $|x| < t$, our simulation of M on w for $|x|$ steps will not accept, and so M' accepts x. On the other hand, for every x of length $|x| \geq t$, our simulation of M on w for $|x|$ steps will complete with success, and so M' will reject x. Note that the number of x's that M' accepts is finite (all strings of length less than t, which is $2^t - 1$, a constant dependent on M and w). So, $L(M')$ is finite in this case.

Thus we get that M accepts w iff $L(M')$ is finite. $\square$

Finally, to prove that the complement of INF is not semi-decidable, we need to give a reduction from $\bar{A}_{TM}$ to $\overline{INF}$. By the following easy result (Theorem 10 below), this is equivalent to giving a reduction from A_{TM} to INF. We have given such a reduction earlier (see Theorem 8)! So we're done.

Theorem 10. *Let A and B be any two languages. We have $A < B$ iff $\bar{A} < \bar{B}$.*

The proof is a simple exercise!