The Fundamental Theorem of Calculus

1. **Problem.** Does every continuous function f have an antiderivative? That is, does there exist a function F such that

$$F'(x) = f(x)?$$

2. **Problem.** What is the antiderivative of $f(x) = \dfrac{\sin x}{x}$? Or, $f(x) = e^{-x^2}$?

3. **The Fundamental Theorem of Calculus, Part 1.**
   If f is a continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)dt, \;\; a \leq x \leq b$$

   is continuous on $[a, b]$ and differentiable on (a, b), and

$$g'(x) = f(x).$$

4. **Example.** Apply the Fundamental Theorem of Calculus, Part 1, to find the derivative of the following functions (don't forget about chain rule!):

(a) $g_1(x) = \displaystyle\int_1^x \frac{\sin t}{t}\, dt$

(b) $g_2(x) = \displaystyle\int_0^{x^2} \sin t\; dt$

(c) $g_3(x) = \displaystyle\int_0^{h(x)} f(t)\; dt$

(d) $g_4(x) = \displaystyle\int_{-3x}^{e^x} \ln(1+t^2)dt$

5. **The Fundamental Theorem of Calculus, Part 2.**

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f. That is, a function such that $F' = f$.

6. **Example.** Evaluate the following integrals:

(a) $\displaystyle\int_0^1 x dx$

(b) $\displaystyle\int_2^3 e^x dx$

(c) $\displaystyle\int_0^\pi \sin x \, dx$

(d) $\displaystyle\int_0^1 \frac{dx}{1+x^2}$

7. **A Piecewise Example.** Let

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x \le 1 \\ 2 - x & \text{if } 1 < x \le 2 \\ 0 & \text{if } x > 2 \end{cases}$$

and let $g(x) = \int_0^x f(t)dt$.

(a) Find an expression for $g(x)$ similar to the one for $f(x)$.

(b) Sketch the graphs of f and g.

(c) Where is f differentiable? Where is g differentiable?