Assignment 1: Dynamical Systems

Due Feb 1st at 11:59pm

This assignment is to be done individually.

Important Note: The university policy on academic dishonesty (cheating) will be taken very seriously in this course. You may not provide or use any solution, in whole or in part, to or by another student.

You are encouraged to discuss the concepts involved in the questions with other students. If you are in doubt as to what constitutes acceptable discussion, please ask! Further, please take advantage of office hours offered by the instructor and the TA if you are having difficulties with this assignment.

DO NOT:

- Give/receive code or proofs to/from other students
- Use Google to find solutions for assignment

DO:

- Meet with other students to discuss assignment (it is best not to take any notes during such meetings, and to re-work assignment on your own)
- Use online resources (e.g. Wikipedia) to understand the concepts needed to solve the assignment.

Submitting Your Assignment

The assignment must be submitted online at `https://coursys.cs.sfu.ca`. You must submit one zip file (student_number_hs1.zip) containing:

1. An assignment report in **PDF format**, named `student_number_hw1.pdf`. This report should contain your solutions to questions 1-4.
2. Your code for question 1, named `rk4.py`.
3. Your code for question 4, named `ode.py`.

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1 Satellite Orbiting Earth (13 marks)

A simplified set of equations of motion for the Earth and an orbiting satellite is given by

$$\ddot{r} = \theta^2 - \frac{k}{r^2} + u_1, \tag{1}$$

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} + \frac{1}{r}u_2, \tag{2}$$

where r represents the Earth-satellite distance measured from their centres, and θ represents the phase of the orbit. k is a positive constant.

a) Derive a state space model of this system in the form of a first-order ordinary differential equation.

Slides #2
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b) What are the equilibrium points of the state space model, under zero control input, $u_1 = u_2 = 0$? Give a physical interpretation of the result.

c) What are the equilibrium points of the state space model, under $u_1 = k/x_1^2, u_2 = 0$? Give a physical interpretation of this control set point and of the equilibrium points.

d) Linearize the model with respect to a reference orbit given by $r(t) \equiv \rho, \theta(t) = \omega t, u_1 = u_2 = 0$.

e) Numerically integrate the ODE, with $u_1(t), u_2(t) \equiv 0$, using your own implementation of RK4. Plot the state trajectory and intuitively explain the behaviour. Please attach your code.

   Use parameters for the international space station orbiting the Earth:

   - $r(0) = 410$ km $+ 6378$ km, which represents an orbit 410 km above the earth's surface,
   - $\dot{r}(0) = 0$ m/s,
   - $\theta(0) = 0$,
   - $\dot{\theta}(0) = 2\pi/T$, where $T = 92.68$ minutes, the orbital period
   - $k = GM$, where $G = 6.67 \times 10^{-11}$ m^{3}kg^{-1}s^{-2}, $M = 5.97 \times 10^{24}$ kg.

2 The Lotka-Volterra Predator-Prey Model (7 marks)

The Lotka-Volterra Predator-Prey Model describes the interaction between a population of predators and a population of prey. Let x be the number of prey, and y be the number of predators. The model describes the evolution of x and y as follows:

$$\dot{x} = ax - bxy \tag{3}$$

$$\dot{y} = -dy + cxy \tag{4}$$

The interpretation of the parameters a, b, c, d are as follows:

- a represents the birth rate of prey,
- d represents the death rate of predators,
- b is the prey's susceptibility to predators, and
- c is the ability of predators to hunt prey.

Note that all parameters and states are strictly positive.

a) Find the (non-trivial) equilibrium of the system.

b) Let $V(x,y) = y^a e^{-by} x^d e^{-cx}$. Show that $\dot{V}(x,y) = 0$.

c) Prove that the system is stable around the equilibrium point. linearization

Hint: Find the maximum of $V(x,y)$, by using the fact that $\log V(x,y)$ has the same maximum. In addition, consider the convexity of $\log V(x,y)$.

3 Stabilization via Linear Feedback (4 marks)

Given the system $\dot{x} = Ax + Bu$, with $A = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, construct a linear state feedback controller so that the closed loop system is stable.

4 Numerical Solutions to ODEs (8 marks)

Consider the linear system $\dot{x} = Ax$, with $A = \begin{bmatrix} 0 & 1 \\ -500 & -501 \end{bmatrix}$.

a) Determine conditions on the time step size that must be satisfied for the forward Euler method to be stable.

b) Repeat the above two steps for the backward Euler method.

c) Write a python script to plot the linear system around the stable Δt value using both forward and backward Euler methods. Try this for 4 different Δt values around the stable Δt you found on section a.

- Running "`python3 ode.py` Δt_1" should plot both forward and backward Euler with the following parameters: $x_0 = (1,1)$, total time duration = 2 seconds, $\Delta t = \Delta t_1$.
- Print the output vector from both backward and forward Euler methods (similar to the provided script).