Improper Integrals

1. **Quote.** "Our knowledge can only be finite, while our ignorance must necessarily be infinite."

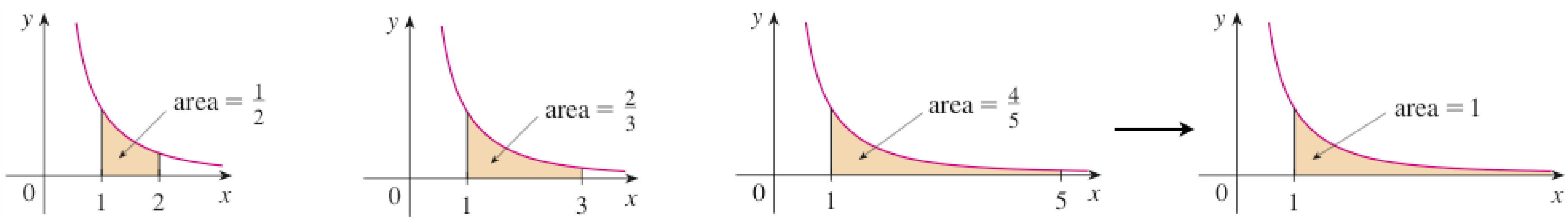
   (Karl Raimund Popper, Austrian-British philosopher, 1902-1994)

2. **Quote.** "Finite to fail, but infinite to venture."

   (Emily Elizabeth Dickinson, American poet, 1830-1886)

3. **Problem.** Evaluate the area of the region bounded by the curves

$$y = \frac{1}{x^2}, \quad y = 0, \ x = 1.$$



4. **Improper Integral of Type I.**

(a) If $\int_a^t f(x)dx$ exists for all $t \geq a$, then $\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$ provided that this limit exists (i.e. as a finite number).

(b) If $\int_t^b f(x)dx$ exists for all $t \leq b$, then $\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$ provided that this limit exists (i.e. as a finite number).

The improper integrals $\int_a^\infty f(x)dx$ and $\int_{-\infty}^b f(x)dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_{-\infty}^a f(x)dx$ and $\int_a^\infty f(x)dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx.$$

5. **Example.** Investigate the improper integrals.

(a) $\displaystyle\int_1^{\infty} \frac{dx}{x}$

(b) $\displaystyle\int_1^{\infty} \frac{dx}{x^2}$

(c) $\displaystyle\int_{-\infty}^{0} \frac{dx}{\sqrt{1-x}}$

(d) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

6. **Problem.** Evaluate the area of the region bounded by the curves

$$y = \frac{1}{\sqrt{x}}, \quad y = 0, \quad x = 0, \quad x = 1.$$

7. **Improper Integral of Type II.**

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

provided that this limit exists.

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

provided that this limit exists.

The improper integral $\int_a^b f(x)dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

8. **Example.** Investigate the improper integrals.

(a) $$\int_1^2 \frac{dx}{(x-2)^2}$$

(b) $$\int_0^2 \frac{dx}{(2x-1)^{2/3}}$$

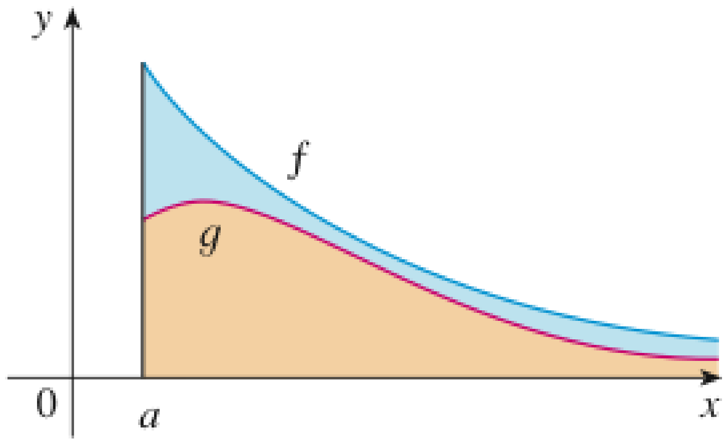
(c) $$\int_0^1 \ln x \, dx$$

9. **Comparison Theorem.**

Suppose that f and g are continuous functions with $0 \leq g(x) \leq f(x)$ for $x \geq a$.

(a) If $\displaystyle\int_a^\infty f(x)dx$ is convergent then $\displaystyle\int_a^\infty g(x)dx$ is convergent.

(b) If $\displaystyle\int_a^\infty g(x)dx$ is divergent then $\displaystyle\int_a^\infty f(x)dx$ is divergent.



10. **Example.** Use the Comparison Theorem to determine if the following integrals are convergent or divergent.

(a) $\displaystyle\int_4^\infty \frac{dx}{\ln x - 1}$

(b) $\displaystyle\int_1^\infty e^{-x^2/2}\,dx$