Phil 320

Chapter 10: Semantics

An *interpretation* $\mathcal{M}$ of $\mathcal{L}$ is:

(1) A non-empty *domain* $|\mathcal{M}|$ of objects.

(2) Denotations.

(a) *Constants*. An assignment of an object $c^{\mathcal{M}}$ in $|\mathcal{M}|$ to each constant c.

(b) *Predicates*. An assignment of an n-place relation $R^{\mathcal{M}}$ on $|\mathcal{M}|$ to each n-place predicate R.

(c) *Function symbols*. An assignment of a n-place *function* $f^{\mathcal{M}}$ on $|\mathcal{M}|$ to each n-place function symbol f. The function must be total.

1. Definition of truth on an interpretation

We want to define, for any sentence F of $\mathcal{L}$,

$\mathcal{M} \models F$ [$\mathcal{M}$ *satisfies* F; $\mathcal{M}$ makes F true; F is true on the interpretation $\mathcal{M}$.]

This symbol, $\models$, is NOT part of the language $\mathcal{L}$. It's really just an abbreviation for 'makes true'. We define this *inductively* on the sentences (closed formulas) of $\mathcal{L}$..

1. *Atomic sentences*.

(1a) $\mathcal{M} \models R(t_1,\ldots t_n)$ iff $R^{\mathcal{M}}(t_1^{\mathcal{M}},\ldots, t_n^{\mathcal{M}})$

(1b) $\mathcal{M} \models t_1 = t_2$ iff $t_1^{\mathcal{M}} = t_2^{\mathcal{M}}$

If function symbols are absent, the closed terms t_i are just constants and the denotations are well-defined as part of the interpretation.

If function symbols are present, both (1a) and (1b) are still fine, but we need to define the denotation $t^{\mathcal{M}}$ of a closed term t as follows (the denotation will always be an object in $|\mathcal{M}|$):

i) If t is a constant c, then $t^{\mathcal{M}} = c^{\mathcal{M}}$ is already defined.

ii) Suppose $t_1^{\mathcal{M}}, t_2^{\mathcal{M}}, .. , t_n^{\mathcal{M}}$ are defined and f is an n-place function symbol. Then

$(f(t_1,\ldots t_n))^{\mathcal{M}} = f^{\mathcal{M}}(t_1^{\mathcal{M}},\ldots,t_n^{\mathcal{M}})$

2. *Nonatomic sentences*, part 1 (connectives.)

(2a) $\mathcal{M} \models \sim F$ if and only if not $\mathcal{M} \models F$

(2b) $\mathcal{M} \models (F \,\&\, G)$ if and only if $\mathcal{M} \models F$ and $\mathcal{M} \models G$

(2c) $\mathcal{M} \models (F \vee G)$ if and only if $\mathcal{M} \models F$ or $\mathcal{M} \models G$

[If we want clauses for the other two connectives, they would be:

$\mathcal{M} \models F \rightarrow G$ if and only if not $\mathcal{M} \models F$ or $\mathcal{M} \models G$ or both.

$\mathcal{M} \models F \leftrightarrow G$ if and only if both $\mathcal{M} \models F$ and $\mathcal{M} \models G$, or neither.]

3. *Nonatomic sentences*, part 2 (quantifiers)

Rough idea: $\forall xF$ means 'F holds of every object in the domain'. How can we say this?

First attempt: the *substitutional approach* [fails].

Recall: $F(x)$ means that x is free; $F(t)$ is the result of putting t for x.

$\mathcal{M} \models \forall xF(x)$ if and only if for every closed term t, $\mathcal{M} \models F(t)$.

$\mathcal{M} \models \exists xF(x)$ if and only if for some closed term t, $\mathcal{M} \models F(t)$.

Problem: We want to guarantee that F holds of every object in the domain; and there may be objects in the domain that are not named by *any* closed term.

Second attempt: the objectual approach.

For every object m in the domain, if we *extend the language* by adding new constant c and *extend the interpretation* to make c denote m, the extended interpretation makes $F(c)$ true.

Def (Shift Interpretation): If c is a constant and m is in $\mathcal{M}$, then $\mathcal{M}^* = \mathcal{M}^c_m$ is the interpretation which is just like $\mathcal{M}$ except that $c^{\mathcal{M}^*} = m$. [The constant c could be new or already part of $\mathcal{L}$.]

Def. $\mathcal{M} \models F[m]$ (m satisfies $F(x)$) if and only if $\mathcal{M}^c_m \models F(c)$.

This can be extended to formulas with more than one free variable. With this idea, we can define the clauses for $\forall x$ and $\exists x$:

(3a) $\mathcal{M} \models \forall xFx$ if and only if for every m in the domain $|\mathcal{M}|$, $\mathcal{M} \models F[m]$

(3b) $\mathcal{M} \models \exists xFx$ if and only if for some m in the domain $|\mathcal{M}|$, $\mathcal{M} \models F[m]$.

2. Examples

Ex. 1: $a^{\mathcal{M}}$= Romeo, $b^{\mathcal{M}}$ = Juliet, $L^{\mathcal{M}}$ is ‘likes’, $f^{\mathcal{M}}$ is ‘father of’

$\mathcal{M} \models f(a)Lb$ iff $L^{\mathcal{M}}(f^{\mathcal{M}}(a^{\mathcal{M}}), b^{\mathcal{M}})$
iff $L^{\mathcal{M}}$(Romeo’s father, Juliet)
iff Romeo’s father likes Juliet.

So not $\mathcal{M} \models f(a)Lb$

$\mathcal{M} \models \forall x\, xLa$ iff $\mathcal{M}^c_m \models cLa$, all $m \in |\mathcal{M}|$.
iff $L^{\mathcal{M}}(m, a^{\mathcal{M}})$, all $m \in |\mathcal{M}|$ *Note: $L^{\mathcal{M}}$ and $a^{\mathcal{M}}$ unchanged.
iff m likes Romeo, all $m \in |\mathcal{M}|$.

But this is not true (e.g., if m = Juliet’s father), so not $\mathcal{M} \models \forall x\, xLa$.

Ex. 2: $\mathcal{M}$ = N*, the standard interpretation of arithmetic.

$\mathcal{M} \models \forall x \exists y\, y = x'$ iff for all $m \in$ N, $\mathcal{M}^a_m \models \exists y\, y = a'$ is true.
iff for all $m \in$ N, for some $n \in$ N, $\mathcal{M}^{a\,b}_{m\,n} \models b = a'$
iff for all $m \in$ N, there is some $n \in$ N, $n = m'$.

True: let $n = m+1$.

10.2 Extensionality lemma

(a) Whether $\mathcal{M} \models A$ depends only upon the domain $|\mathcal{M}|$ and the denotation of the nonlogical symbols in A.
- not dependent on the particular choice of nonlogical symbols in A, but only upon their denotations
- not dependent on denotations of nonlogical symbols not in A: if $\mathcal{M}$ and $\mathcal{M}'$ agree on all nonlogical symbols in A, then $\mathcal{M} \models A$ iff $\mathcal{M}' \models A$.

(b) Whether a formula $F(x)$ is satisfied by an element m of the domain depends only upon the domain, the denotations of nonlogical symbols in F, and the element m.

(c) Whether $\mathcal{M} \models F(t)$ depends only upon the domain $|\mathcal{M}|$, the denotations of the nonlogical symbols in $F(x)$, and the denotation of the closed term t.

Philosophical remark: The extensionality lemma says that truth or falsity of a sentence depends entirely on the domain and what the nonlogical symbols in the sentence denote – not at all upon the particular choice or form of those symbols.

Extensionality fails in ordinary language. For example:

Lois Lane believes that Superman will save her

may be true, while

Lois Lane believes that Clark Kent will save her

is false. Yet Superman and Clark Kent denote the same individual.

10.2. Metalogical notions

"Metalogical": in contrast to logical notions like negation and conjunction, which are represented *within* our formal language, these notions are part of the higher-level language (English) used to talk *about* the formal language.

The text contains examples here (10.3-10.5) that are not of much intrinsic interest, but are needed for proofs in a later chapter. I'll just prove some of these, but you should look all of them over.

1. Implication.

Def. A set Γ of sentences *implies* the sentence D (Γ implies D) if there is no interpretation $\mathcal{M}$ that makes every sentence in Γ true, but makes D false. [Here, we assume that the interpretations provide denotation for all nonlogical symbols in Γ and in D.]

Examples:

(i) $\sim\sim A$ implies A.
Proof: If $\mathcal{M} \models \sim\sim A$, then $\sim A$ must be false by (2a) and hence A is true by (2a).

Similarly, A implies $\sim\sim A$: if $\mathcal{M} \models A$, then not $\mathcal{M} \models \sim A$, so $\mathcal{M} \models \sim\sim A$.

(ii) A & B implies B.
Proof: If $\mathcal{M} \models A \& B$, then by (2b), $\mathcal{M} \models B$.

(iii) $\forall x A(x)$ implies $A(t)$ for any closed term t.
Proof: If $\mathcal{M} \models \forall x A(x)$, then $\mathcal{M} \models A[m]$ for any m in the domain, and in particular this is true for the m denoted by t. But then by the Extensionality Lemma, $\mathcal{M} \models A(t)$.

(iv) s=t and $A(s)$ imply $A(t)$.
Proof: If $\mathcal{M} \models A(s)$ and $\mathcal{M} \models s=t$, then $s^{\mathcal{M}} = t^{\mathcal{M}}$ and we can let m be this object. Since $\mathcal{M} \models A(s)$, we know that $\mathcal{M} \models A[m]$ and hence $\mathcal{M} \models A(t)$.

2. Validity.

Def. D is *valid* if no interpretation makes D false.

Note: if D is *valid*, then Γ implies D for any Γ.

The best way to show that a sentence is valid is to suppose it is false on some interpretation $\mathcal{M}$, and derive a contradiction.

Examples:

(i) $A \vee {\sim}A$.

Proof: Suppose this is false in $\mathcal{M}$. Then not $\mathcal{M} \models A$ and not $\mathcal{M} \models {\sim}A$; but the second implies $\mathcal{M} \models A$, a contradiction.

(ii) $\forall x F(x) \rightarrow F(t)$.

Proof: Suppose false in $\mathcal{M}$. Then $\mathcal{M} \models \forall x F(x)$ and not $\mathcal{M} \models F(t)$. Proceeding as in Example (iii) for Implication, we get a contradiction.

3. Satisfiability and unsatisfiability

Def. A set of sentences Γ is *satisfiable* if some intepretation makes (every sentence in) Γ true. Γ is *unsatisfiable* if no interpretation makes (every sentence in) Γ true.

Note: if Γ is unsatisfiable, then Γ implies D for any D. And if Γ implies every D, then Γ is unsatisfiable.

The best way to show that a set Γ is satisfiable is to produce an interpretation that makes (every sentence in) Γ true. The best way to show that a set Γ is unsatisfiable is to suppose that some interpretation makes every sentence in Γ true, and derive a contradiction.

Examples:

(i) $\{A, {\sim}A\}$ is unsatisfiable. For if $\mathcal{M} \models A$ and $\mathcal{M} \models {\sim}A$, by (2a) we have not-$\mathcal{M} \models A$, a contradiction.

(ii) $\{\forall \mathrm{x}(A(x) \,\&\, B(x)), \exists x{\sim}A(x)\}$ is unsatisfiable. For if $\mathcal{M} \models \forall x(A(x) \,\&\, B(x))$ and $\mathcal{M} \models \exists x{\sim}A(x)$, then by (3b) we have $\mathcal{M} \models {\sim}A[m]$ for some m in the domain, but also $\mathcal{M} \models A[m] \,\&\, B[m]$ and hence $\mathcal{M} \models A[m]$, a contradiction.

Some useful results about satisfiability.

(a) If $\Gamma \cup \{(A \vee B)\}$ is satisfiable, then either $\Gamma \cup \{A\}$ or $\Gamma \cup \{B\}$ is satisfiable.

(b) Γ implies A if and only if $\Gamma \cup \{{\sim}A\}$ is unsatisfiable.

(c) If $\Gamma \cup \{\exists x A(x)\}$ is satisfiable, then for any constant c not occurring in Γ or $\exists x A(x)$, $\Gamma \cup \{A(c)\}$ is satisfiable.

4. Equivalence

Def. Two sentences A and B are *equivalent over an interpretation* $\mathcal{M}$ if they have the same truth value on $\mathcal{M}$.
Two formulas $F(x)$ and $G(x)$ are *equivalent over* $\mathcal{M}$ if, for any constant c not occurring in either formula, $F(c)$ and $G(c)$ are equivalent over $\mathcal{M}^c_m$ for each object m in $|\mathcal{M}|$ (i.e., each possible denotation of c).

Def. Two sentences A and B are *logically equivalent* if they are equivalent over all interpretations.
Two formulas $F(x)$ and $G(x)$ are *logically equivalent* if, for any constant c occurring in neither formula, $F(c)$ and $G(c)$ are logically equivalent.