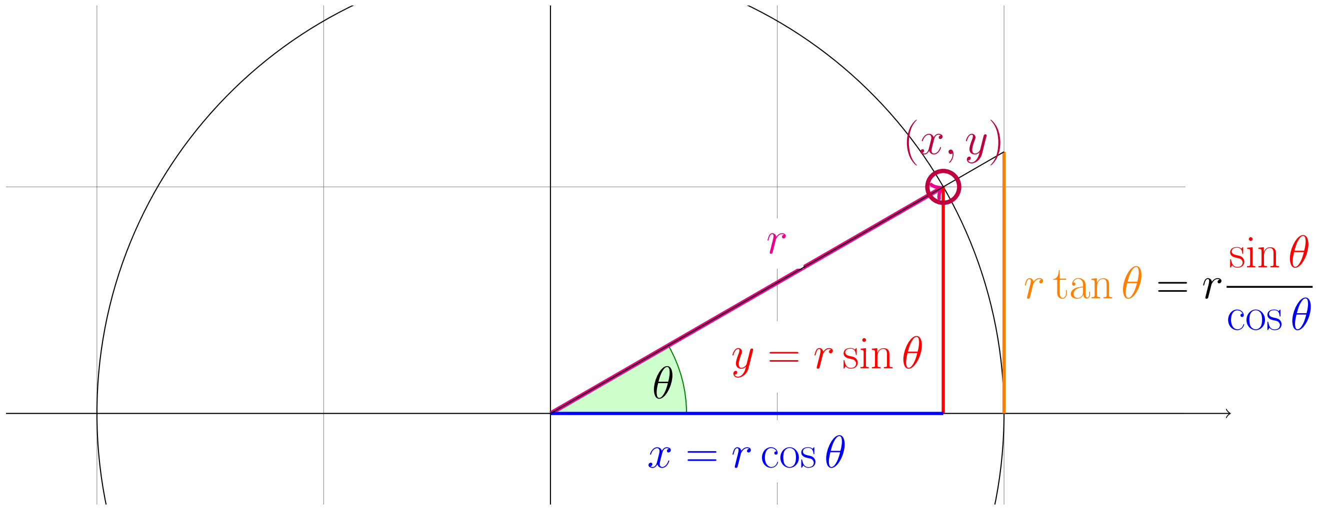
Areas in Polar Coordinates

1. **Reminder.** Polar coordinates.



Cartesian coordinates x, y (go x to the right, and y up) versus
polar coordinates r, θ (go distance r into the direction given by angle θ).

$$\begin{aligned} x &= r\cos\theta & \qquad r &= \sqrt{x^2+y^2} \geq 0 \\ y &= r\sin\theta & \qquad \theta &= \tan^{-1}\left(\tfrac{y}{x}\right) \end{aligned}$$

Note (COMPLEX NUMBERS): $z = x + iy$.
i is the imaginary unit, $i^2 = -1$; it corresponds to $(x, y) = (0, 1)$.

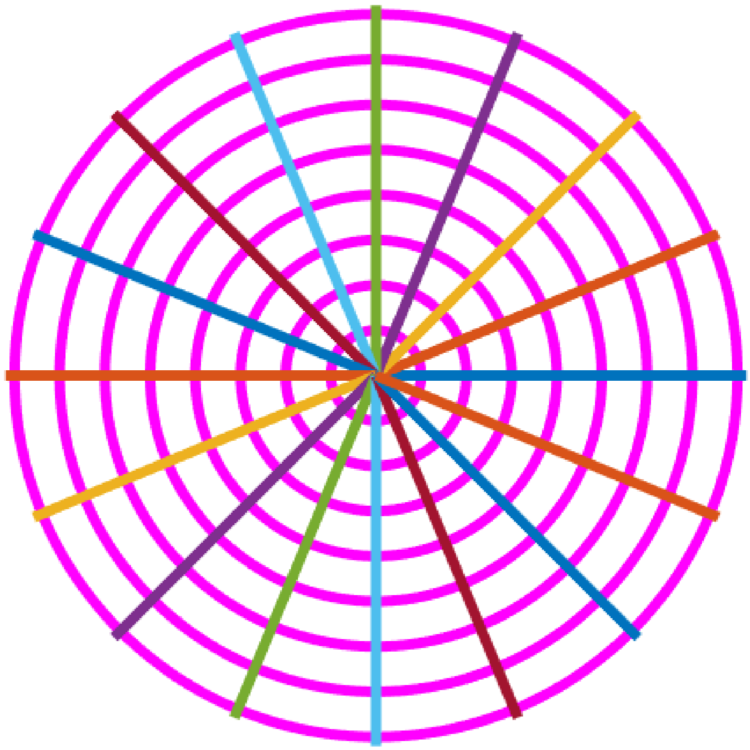
Exponential function for complex arguments:
$e^{i\theta} = \cos\theta + i\sin\theta$; $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$
A complex number $z = x + iy$ can be written in terms of polar coordinates as $z = re^{i\theta}$.
Note that $z = e^{i\theta}$ has r=1, so $|z| = 1$.

The function $e^{i\theta}$: This slide has some useful information, in particular, if you are taking Math 232; it is not required for this course!

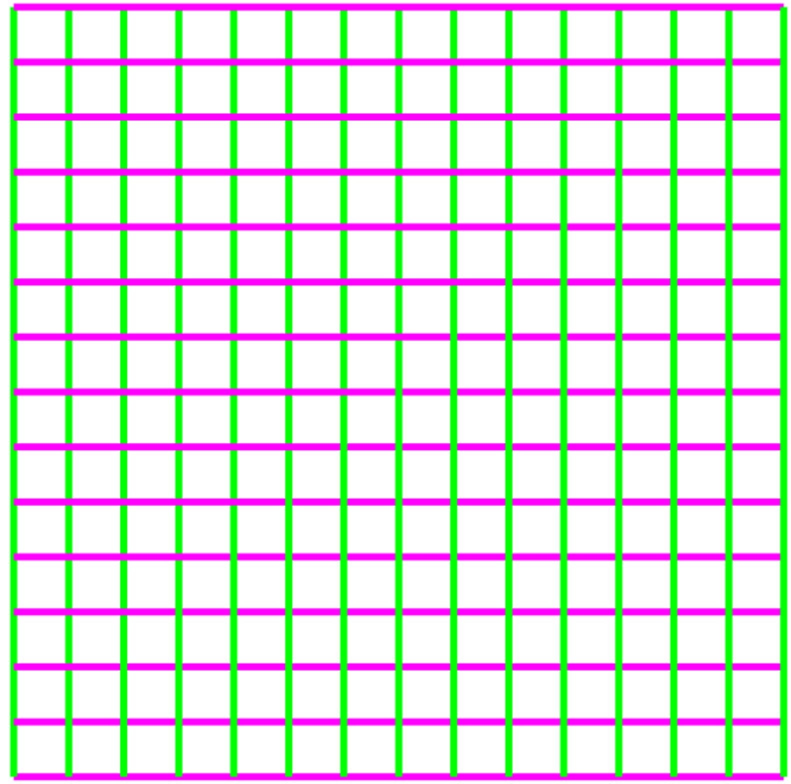
An essential property of the exponential function is $e^{a+b} = e^a e^b$.
If we are comfortable computing with exponential functions, it is easy to find a number of trigonometric identities. Just two examples:

- $\cos^2\theta + \sin^2\theta = 1$ follows from $e^{i\theta} = 1$.
- $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$
  From
  $$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}$$
  we can conclude
  $$\begin{aligned} \cos\theta &= \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \\ \sin\theta &= \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \end{aligned}$$

$$\begin{aligned} \cos^2\theta &= \left(\frac{1}{2}(e^{i\theta} + e^{-i\theta})\right)^2 = \frac{1}{4}(e^{2i\theta} + 2 + e^{-2i\theta}) \\ &= \frac{1}{4}(e^{2i\theta} + e^{-2i\theta}) + \frac{1}{2} = \frac{1}{2}\cos(2\theta) + \frac{1}{2} = \frac{1}{2}\left(\cos(2\theta) + 1\right). \end{aligned}$$



Equispaced Polar grid lines
$r = \text{const}, \theta = \text{const}$



Equispaced Cartesian grid lines
$x = \text{const}, y = \text{const}$

*2

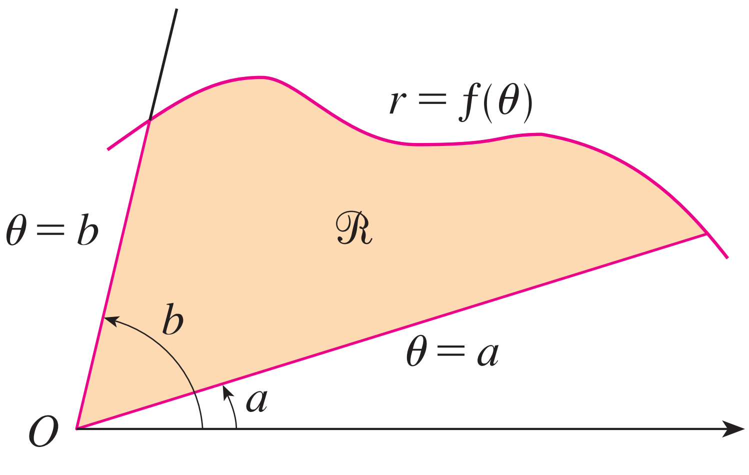
2. **Problem.** Sketch the curve and find the area that it encloses:

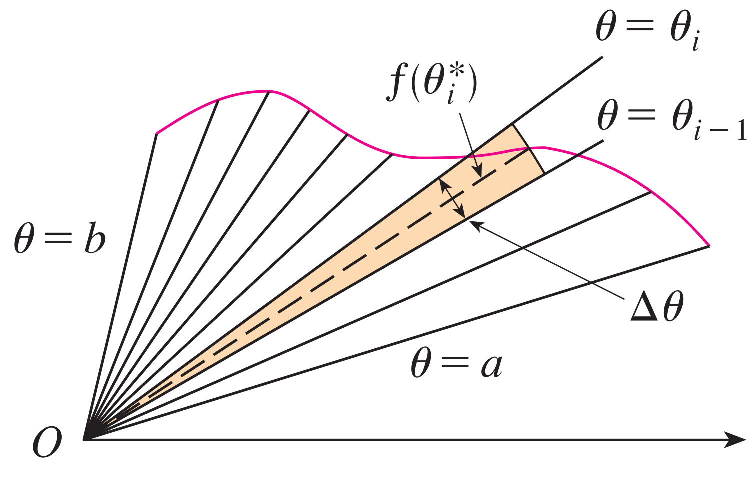
$$r = 1 + \cos\theta.$$

3. **Area bounded by polar curves.**

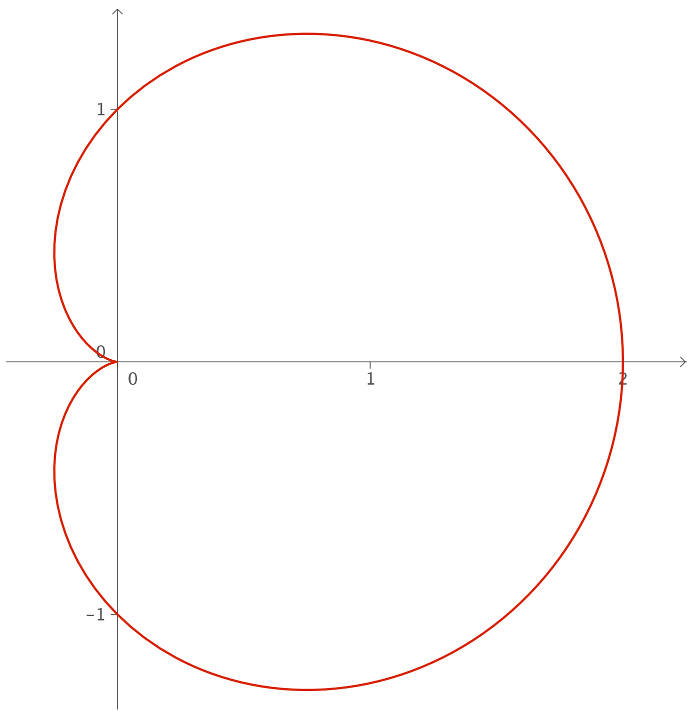
The area of a polar region $\mathcal{R}$ bounded by the curve $r = f(\theta)$, for $\theta \in [a, b]$, is given by

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta = \int_a^b \frac{1}{2} r^2 d\theta.$$

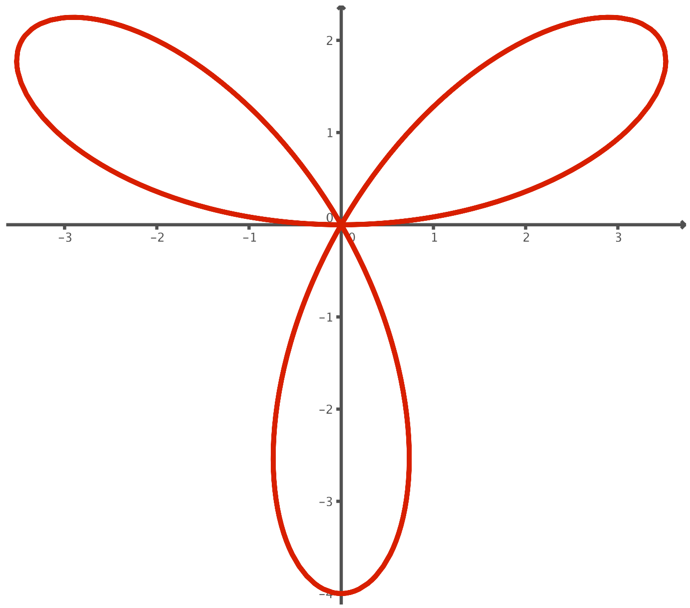




4. **Example.** Find the area enclosed by $r = 1 + \cos\theta$.



5. **Example.** Find the area of the region enclosed by the 3-leaved rose $r = 4\sin(3\theta)$.



*2?

Notes