Inference in Bayesian networks

Chapter 14.4–5

Outline

- ♢ Exact inference by enumeration
- ♢ Approximate inference by stochastic simulation

Inference tasks

Simple queries: compute posterior marginal $\mathbf{P}(X_i|\mathbf{E}=\mathbf{e})$
e.g., $P(NoGas|Gauge=empty, Lights=on, Starts=false)$

Conjunctive queries: $\mathbf{P}(X_i, X_j|\mathbf{E}=\mathbf{e}) = \mathbf{P}(X_i|\mathbf{E}=\mathbf{e})\mathbf{P}(X_j|X_i, \mathbf{E}=\mathbf{e})$

Optimal decisions: decision networks include utility information;
probabilistic inference required for $P(outcome|action, evidence)$

Value of information: which evidence to seek next?

Sensitivity analysis: which probability values are most critical?

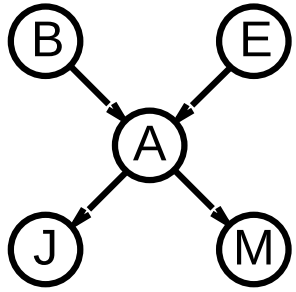
Explanation: why do I need a new starter motor?

Inference by enumeration

Slightly intelligent way to sum out variables from the joint without actually constructing its explicit representation

Simple query on the burglary network:

$$\begin{aligned}&\mathbf{P}(B|j,m)\\ &= \mathbf{P}(B,j,m)/P(j,m)\\ &= \alpha\mathbf{P}(B,j,m)\\ &= \alpha\,\textstyle\sum_e\sum_a\,\mathbf{P}(B,e,a,j,m)\end{aligned}$$



Rewrite full joint entries using product of CPT entries:

$$\begin{aligned}&\mathbf{P}(B|j,m)\\ &= \alpha\,\textstyle\sum_e\sum_a\,\mathbf{P}(B)P(e)\mathbf{P}(a|B,e)P(j|a)P(m|a)\\ &= \alpha\mathbf{P}(B)\,\textstyle\sum_e P(e)\sum_a\,\mathbf{P}(a|B,e)P(j|a)P(m|a)\end{aligned}$$

Recursive depth-first enumeration: $O(n)$ space, $O(d^n)$ time

Enumeration algorithm

function ENUMERATION-ASK($X, \mathbf{e}, bn$) **returns** a distribution over X
  inputs: X, the query variable
    $\mathbf{e}$, observed values for variables $\mathbf{E}$
    bn, a Bayesian network with variables $\{X\} \cup \mathbf{E} \cup \mathbf{Y}$

  $\mathbf{Q}(X) \leftarrow$ a distribution over X, initially empty
  for each value x_i of X **do**
    extend $\mathbf{e}$ with value x_i for X
    $\mathbf{Q}(x_i) \leftarrow$ ENUMERATE-ALL(VARS[bn], $\mathbf{e}$)
  return NORMALIZE($\mathbf{Q}(X)$)

---

function ENUMERATE-ALL($vars, \mathbf{e}$) **returns** a real number
  if EMPTY?($vars$) **then return** 1.0
  $Y \leftarrow$ FIRST($vars$)
  if Y has value y in $\mathbf{e}$
    then return $P(y \mid Pa(Y)) \times$ ENUMERATE-ALL(REST($vars$), $\mathbf{e}$)
    else return $\Sigma_y \; P(y \mid Pa(Y)) \times$ ENUMERATE-ALL(REST($vars$), $\mathbf{e}_y$)
      where $\mathbf{e}_y$ is $\mathbf{e}$ extended with $Y = y$

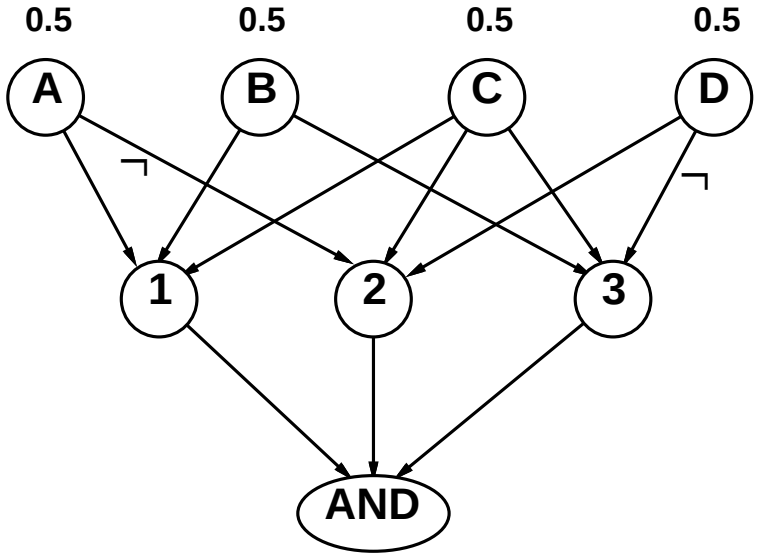
Complexity of exact inference

Multiply connected networks:

– can reduce 3SAT to exact inference $\Rightarrow$ NP-hard

– equivalent to **counting** 3SAT models $\Rightarrow$ #P-complete

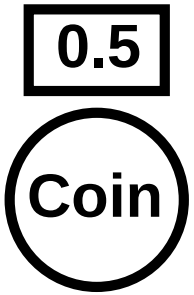
1. A v B v C
2. C v D v ¬A
3. B v C v ¬D



Inference by stochastic simulation

Basic idea:

1) Draw N samples from a sampling distribution S
2) Compute an approximate posterior probability $\hat{P}$
3) Show this converges to the true probability P



Outline:

– Sampling from an empty network
– Rejection sampling: reject samples disagreeing with evidence
– Likelihood weighting: use evidence to weight samples

Sampling from an empty network

function PRIOR-SAMPLE(bn) **returns** an event sampled from bn

inputs: bn, a belief network specifying joint distribution $\mathbf{P}(X_1, \ldots, X_n)$

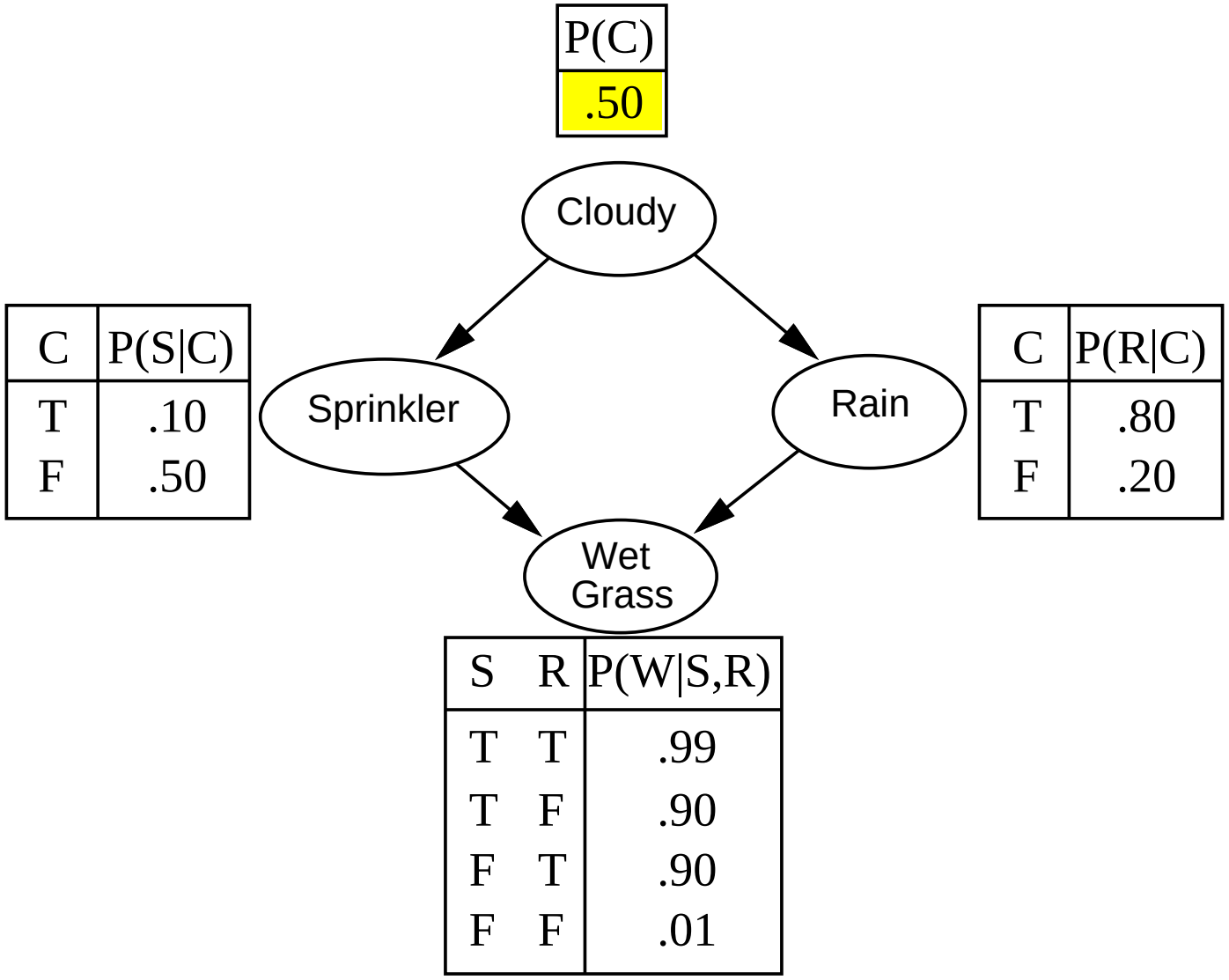
$\mathbf{x} \leftarrow$ an event with n elements

for $i = 1$ **to** n **do**

$x_i \leftarrow$ a random sample from $\mathbf{P}(X_i \mid parents(X_i))$
given the values of $Parents(X_i)$ in $\mathbf{x}$

return $\mathbf{x}$

Example


P(C)
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Cloudy
C P(S|C)
T .10
F .50
Sprinkler
Rain
C P(R|C)
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F .20
Wet Grass
S R P(W|S,R)
T T .99
T F .90
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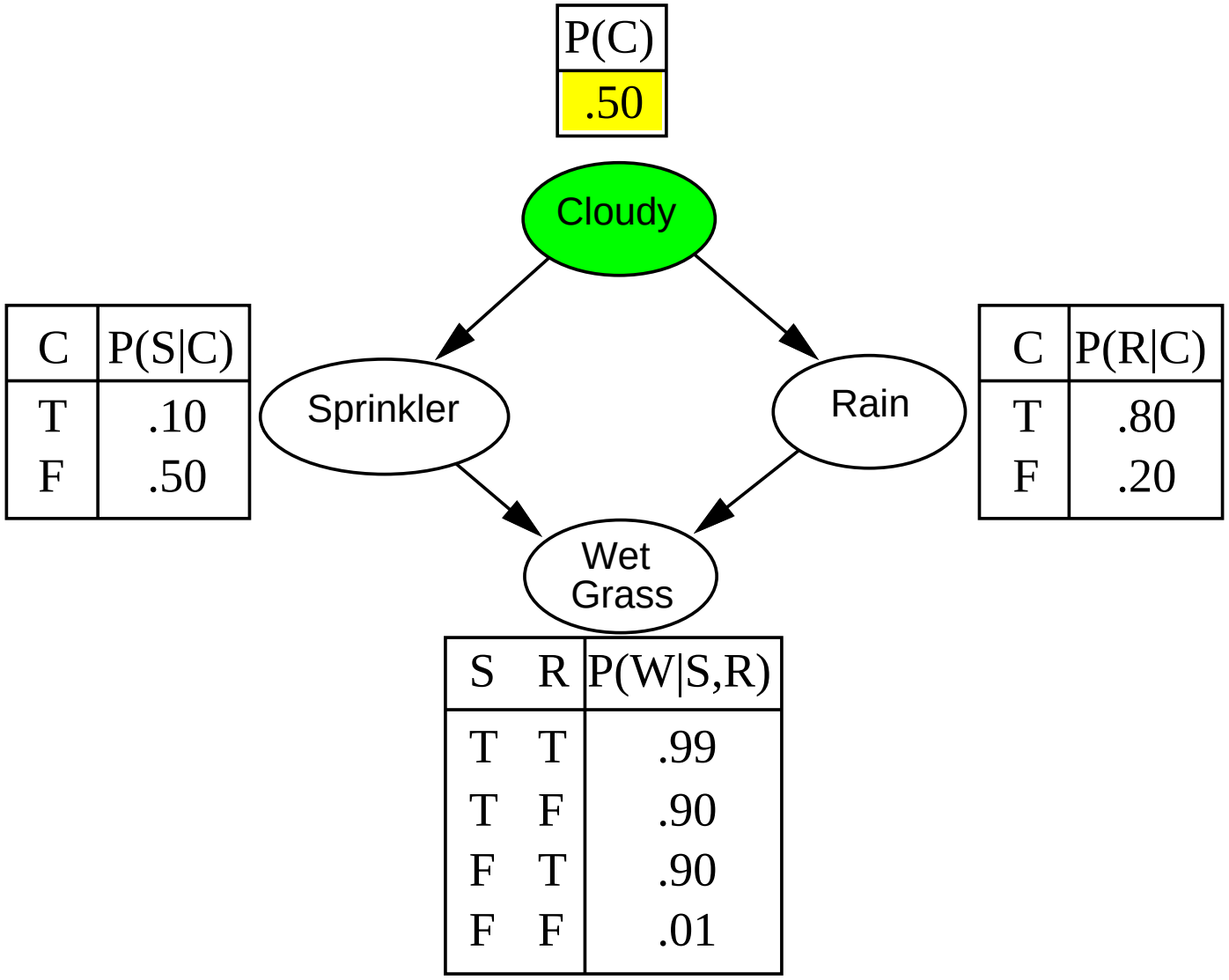
F F .01

Example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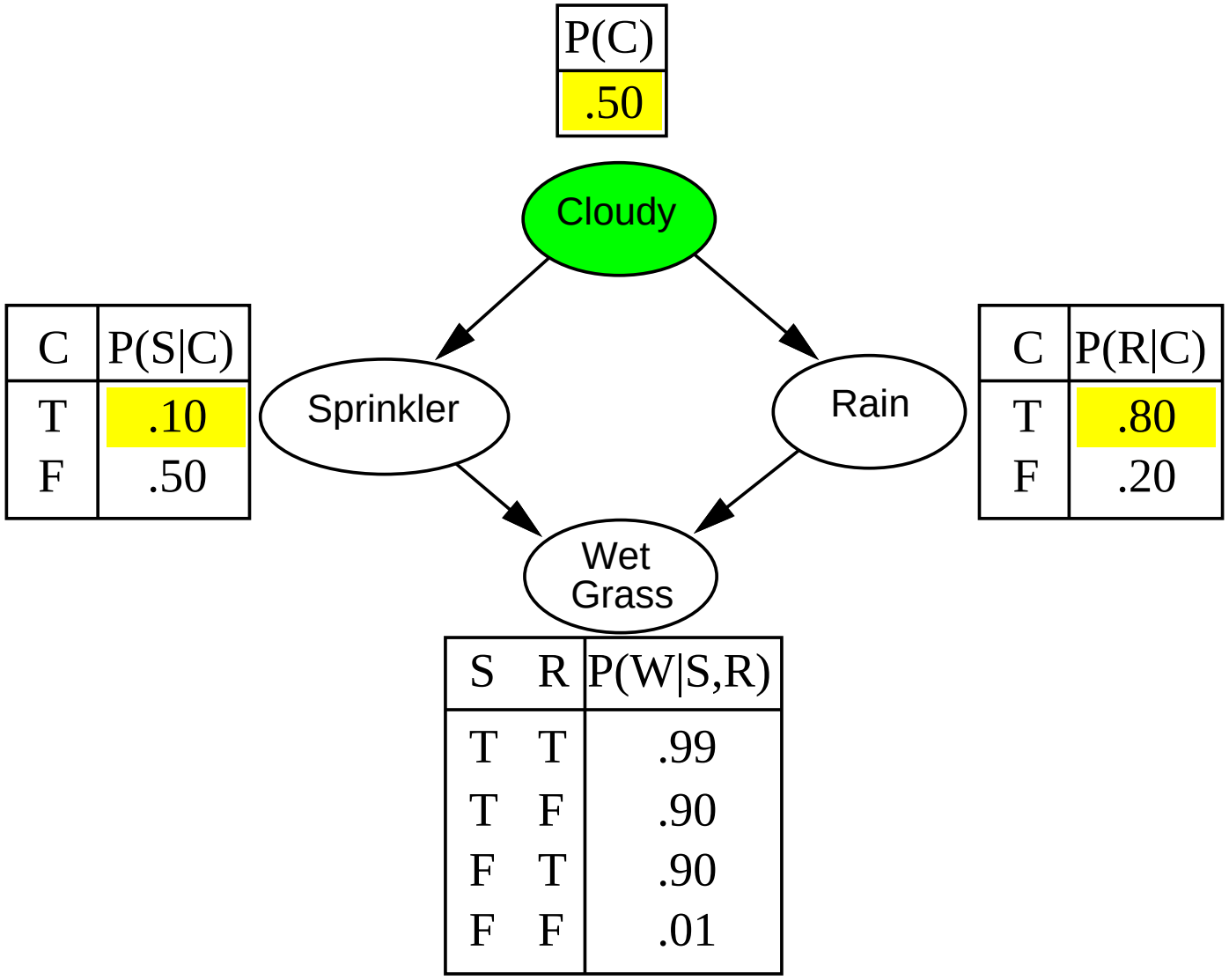
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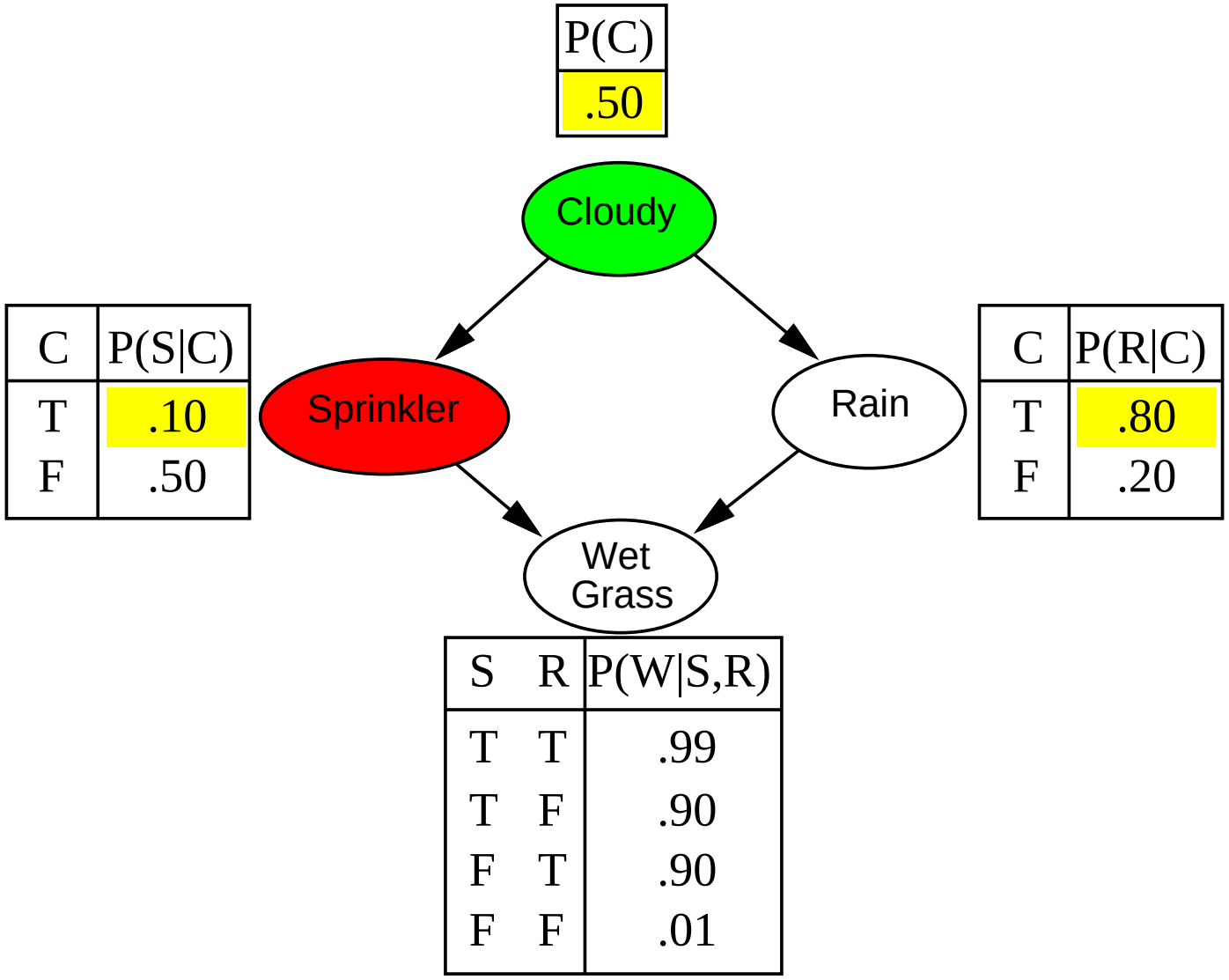
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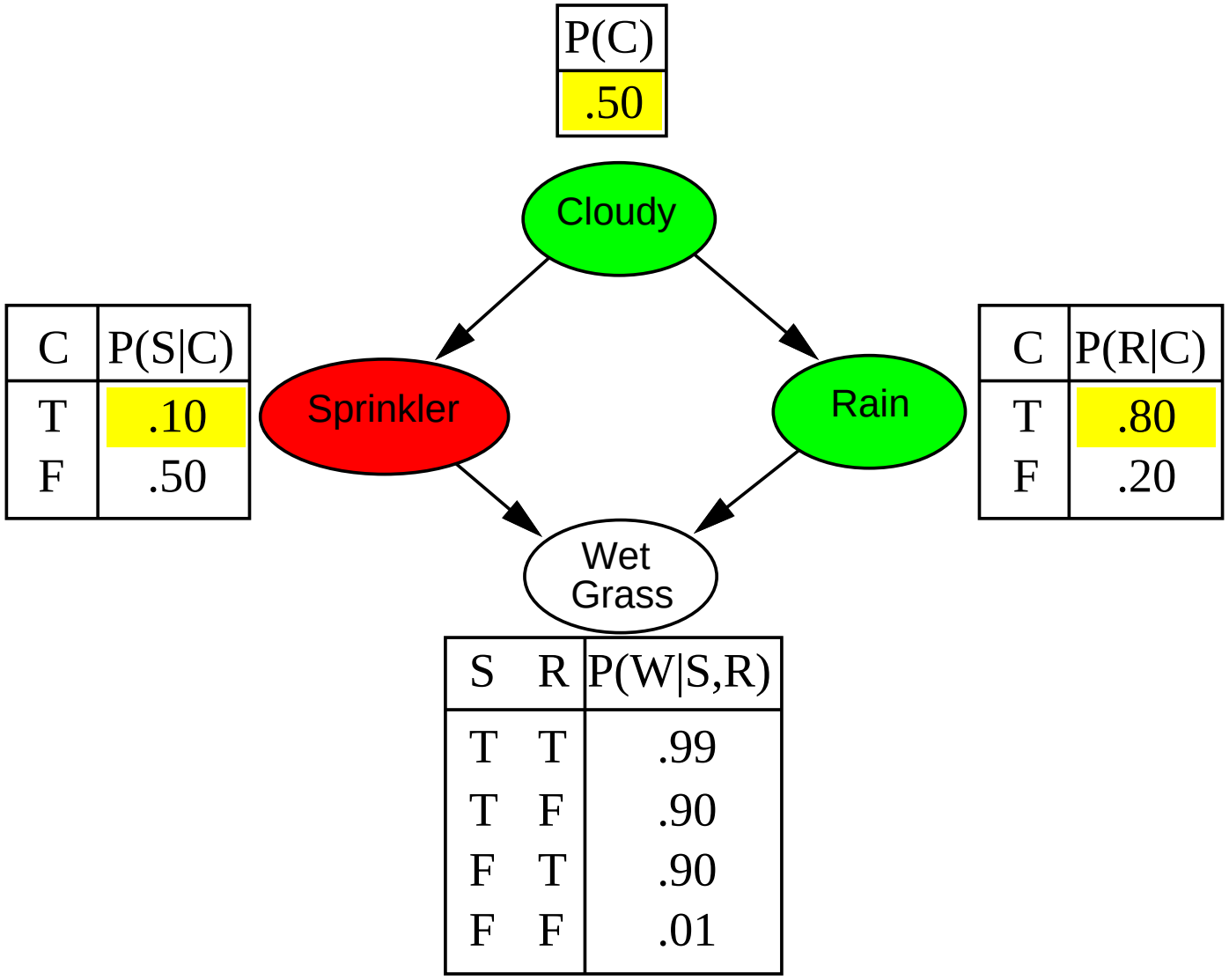
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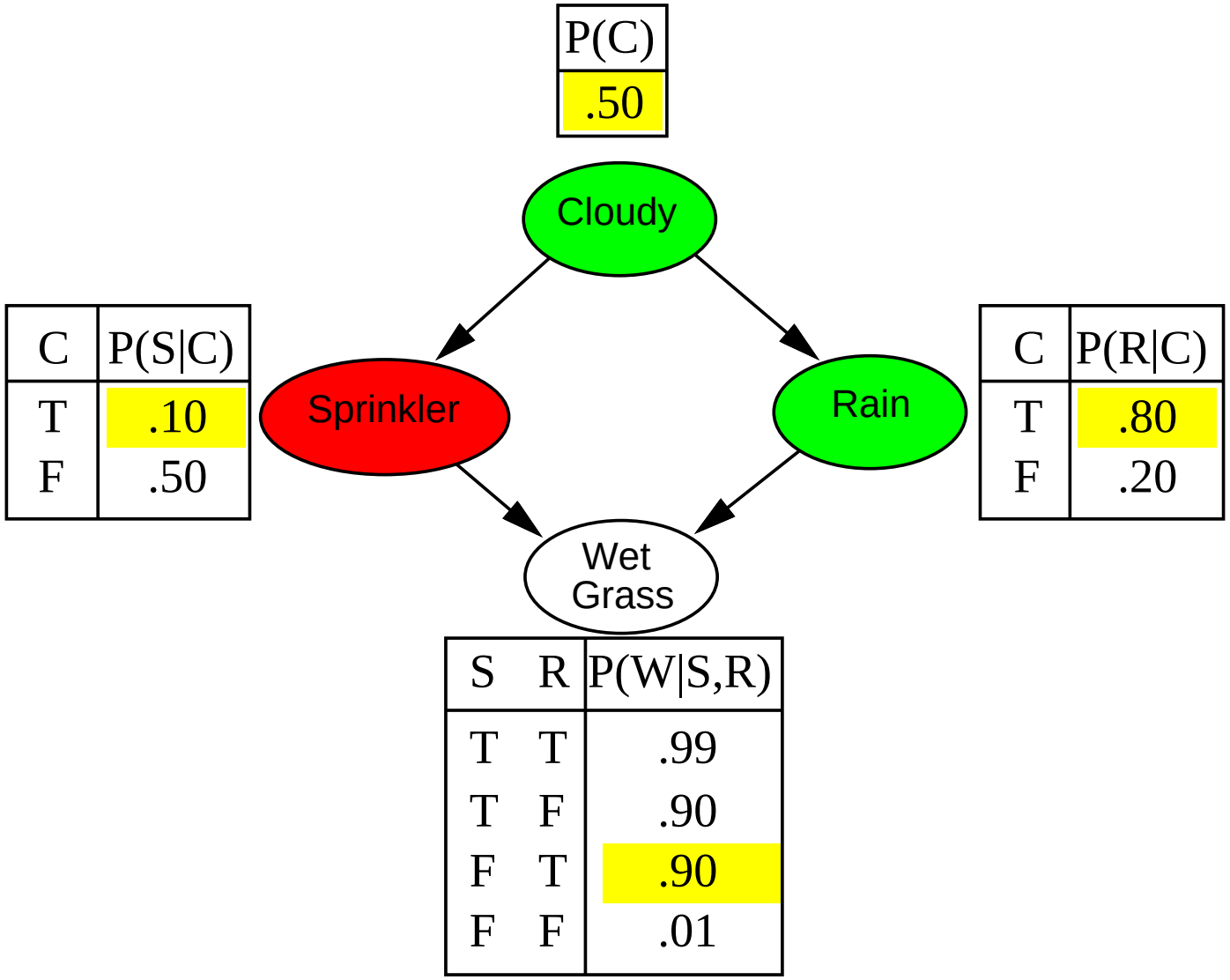
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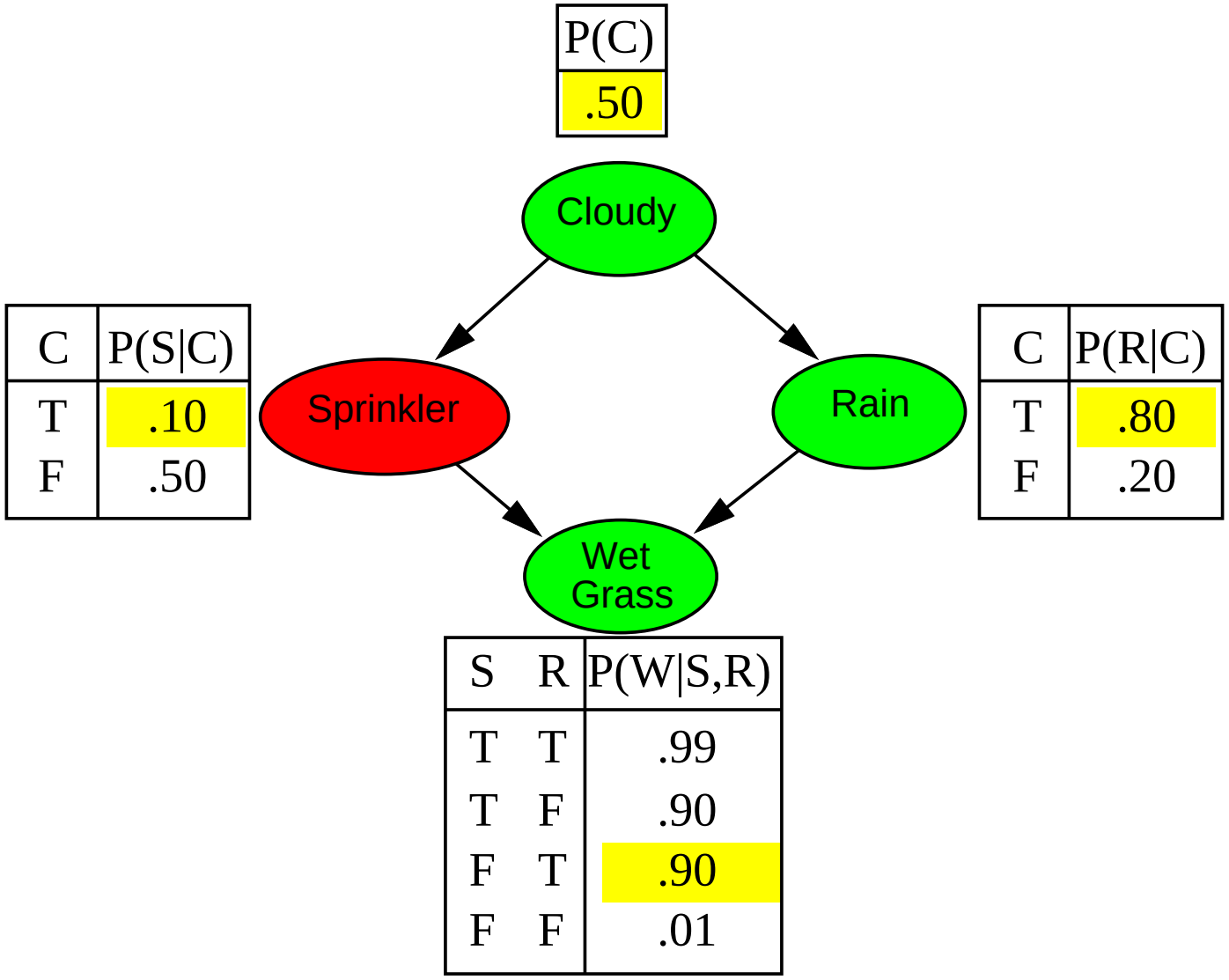
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Sampling from an empty network contd.

Probability that PriorSample generates a particular event

$$S_{PS}(x_1 \ldots x_n) = \Pi_{i=1}^{n} P(x_i | parents(X_i)) = P(x_1 \ldots x_n)$$

i.e., the true prior probability

E.g., $S_{PS}(t, f, t, t) = 0.5 \times 0.9 \times 0.8 \times 0.9 = 0.324 = P(t, f, t, t)$

Let $N_{PS}(x_1 \ldots x_n)$ be the number of samples generated for event $x_1, \ldots, x_n$

Then we have

$$\begin{aligned} \lim_{N \to \infty} \hat{P}(x_1, \ldots, x_n) &= \lim_{N \to \infty} N_{PS}(x_1, \ldots, x_n)/N \\ &= S_{PS}(x_1, \ldots, x_n) \\ &= P(x_1 \ldots x_n) \end{aligned}$$

That is, estimates derived from PriorSample are consistent

Shorthand: $\hat{P}(x_1, \ldots, x_n) \approx P(x_1 \ldots x_n)$

Rejection sampling

$\hat{\mathbf{P}}(X|\mathbf{e})$ estimated from samples agreeing with $\mathbf{e}$

function REJECTION-SAMPLING($X, \mathbf{e}, bn, N$) **returns** an estimate of $P(X|\mathbf{e})$
  local variables: $\mathbf{N}$, a vector of counts over X, initially zero

  for $j = 1$ to N **do**
    $\mathbf{x} \leftarrow$ PRIOR-SAMPLE(bn)
    if $\mathbf{x}$ is consistent with $\mathbf{e}$ **then**
      $\mathbf{N}[x] \leftarrow \mathbf{N}[x]+1$ where x is the value of X in $\mathbf{x}$
  return NORMALIZE($\mathbf{N}[X]$)

E.g., estimate $\mathbf{P}(Rain|Sprinkler = true)$ using 100 samples

27 samples have $Sprinkler = true$

Of these, 8 have $Rain = true$ and 19 have $Rain = false$.

$\hat{\mathbf{P}}(Rain|Sprinkler = true) = \text{NORMALIZE}(\langle 8, 19\rangle) = \langle 0.296, 0.704\rangle$

Similar to a basic real-world empirical estimation procedure

Analysis of rejection sampling

$\hat{\mathbf{P}}(X|\mathbf{e}) = \alpha \mathbf{N}_{PS}(X,\mathbf{e})$ (algorithm defn.)
$= \mathbf{N}_{PS}(X,\mathbf{e})/N_{PS}(\mathbf{e})$ (normalized by $N_{PS}(\mathbf{e})$)
$\approx \mathbf{P}(X,\mathbf{e})/P(\mathbf{e})$ (property of PriorSample)
$= \mathbf{P}(X|\mathbf{e})$ (defn. of conditional probability)

Hence rejection sampling returns consistent posterior estimates

Problem: hopelessly expensive if $P(\mathbf{e})$ is small

$P(\mathbf{e})$ drops off exponentially with number of evidence variables!

Likelihood weighting

Idea: fix evidence variables, sample only nonevidence variables, and weight each sample by the likelihood it accords the evidence

function LIKELIHOOD-WEIGHTING($X, \mathbf{e}, bn, N$) **returns** an estimate of $P(X|\mathbf{e})$

  local variables: $\mathbf{W}$, a vector of weighted counts over X, initially zero

  for $j = 1$ to N **do**

    $\mathbf{x}, w \leftarrow$ WEIGHTED-SAMPLE(bn)

    $\mathbf{W}[x] \leftarrow \mathbf{W}[x] + w$ where x is the value of X in $\mathbf{x}$

  return NORMALIZE($\mathbf{W}[X]$)

---

function WEIGHTED-SAMPLE($bn, \mathbf{e}$) **returns** an event and a weight

  $\mathbf{x} \leftarrow$ an event with n elements; $w \leftarrow 1$

  for $i = 1$ **to** n **do**

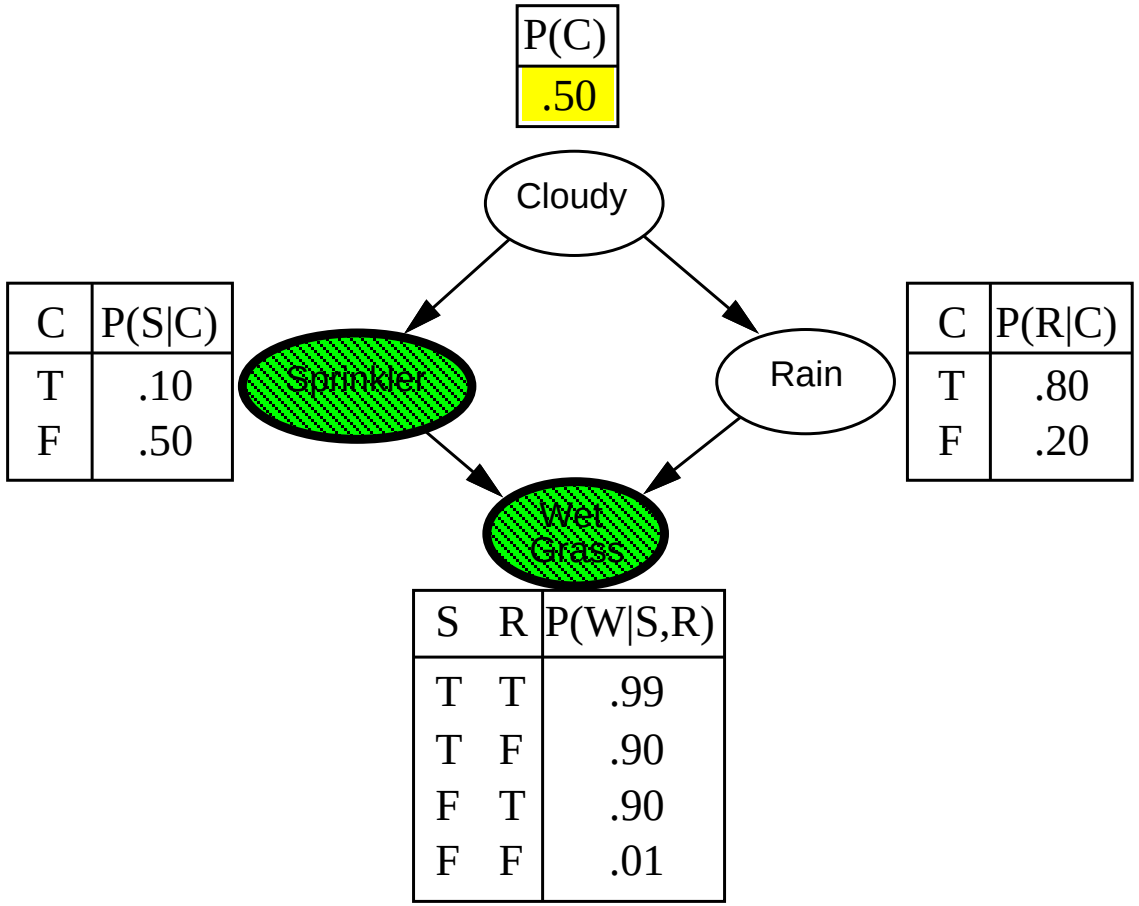
    if X_i has a value x_i in $\mathbf{e}$

      then $w \leftarrow w \times P(X_i = x_i \mid parents(X_i))$

      else $x_i \leftarrow$ a random sample from $\mathbf{P}(X_i \mid parents(X_i))$

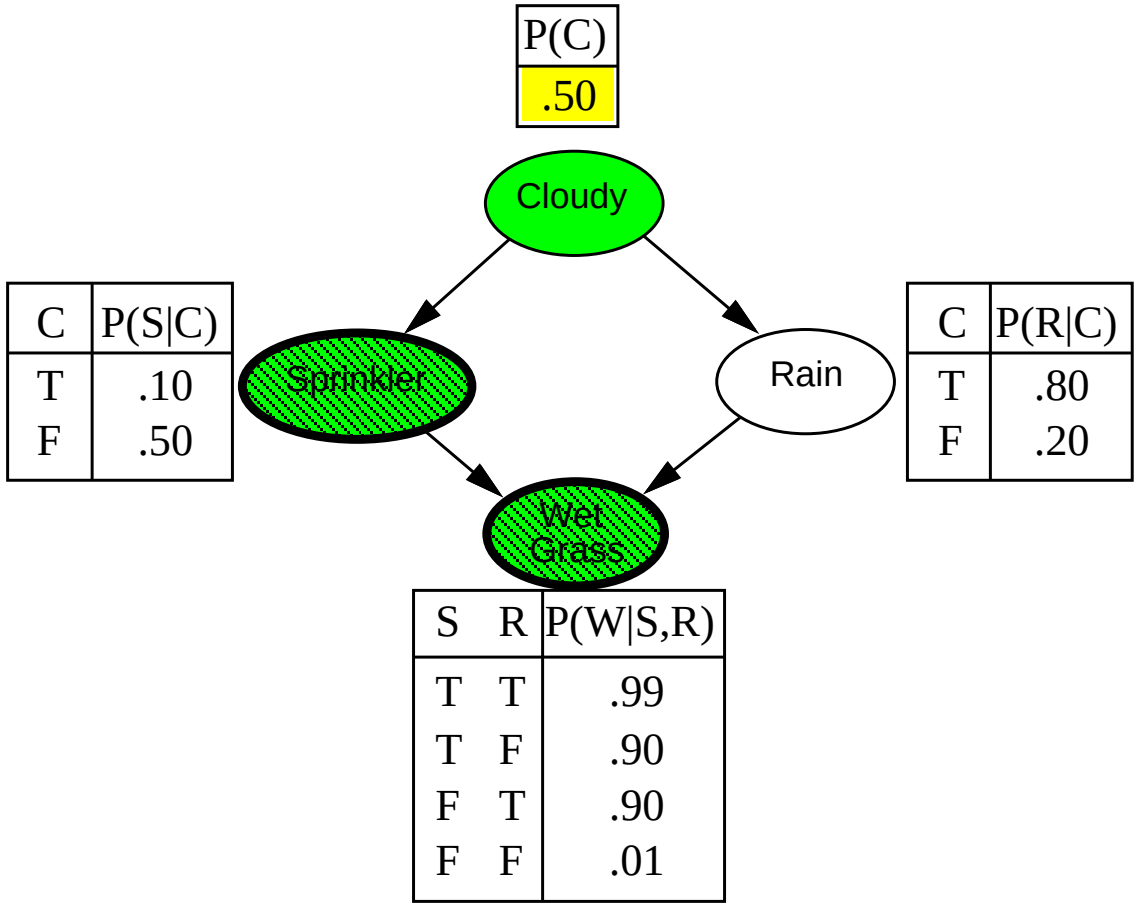
  return $\mathbf{x}$, w

Likelihood weighting example



$w = 1.0$

Likelihood weighting example



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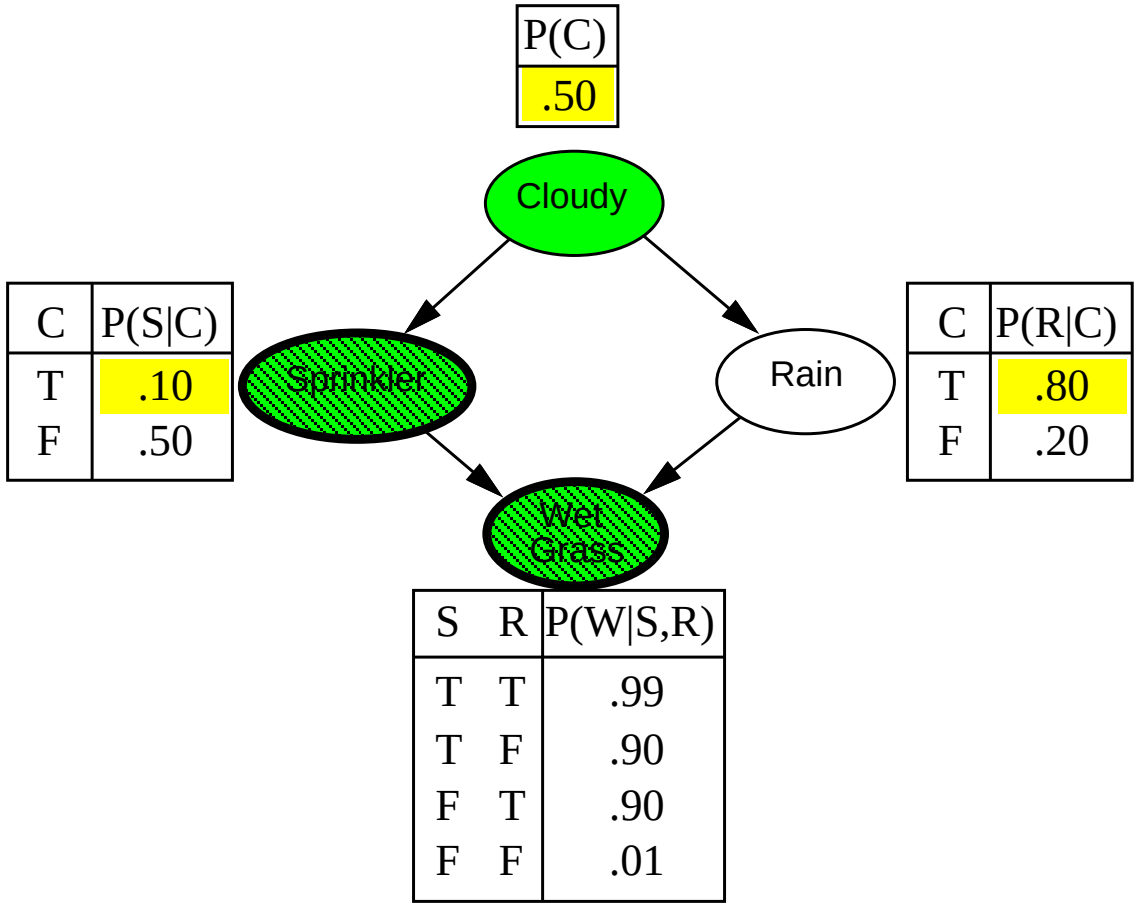
Likelihood weighting example



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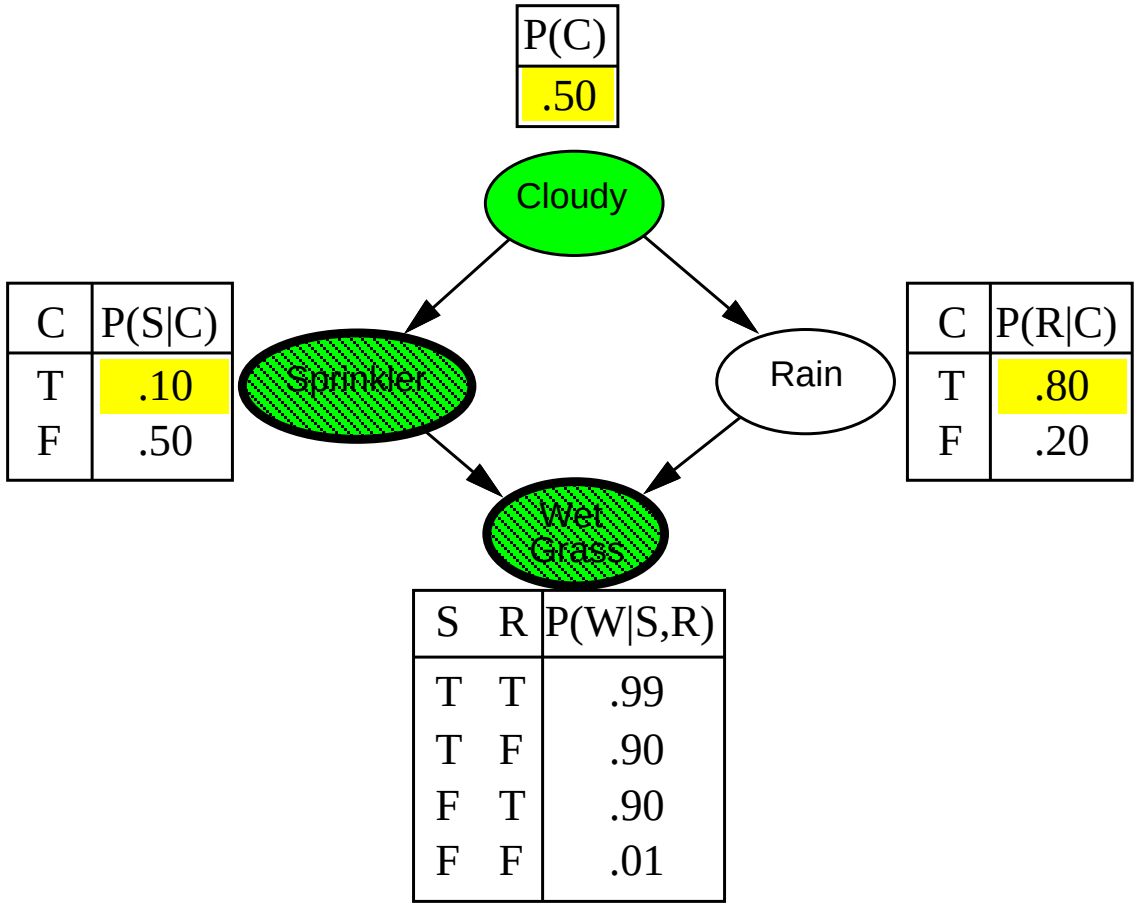
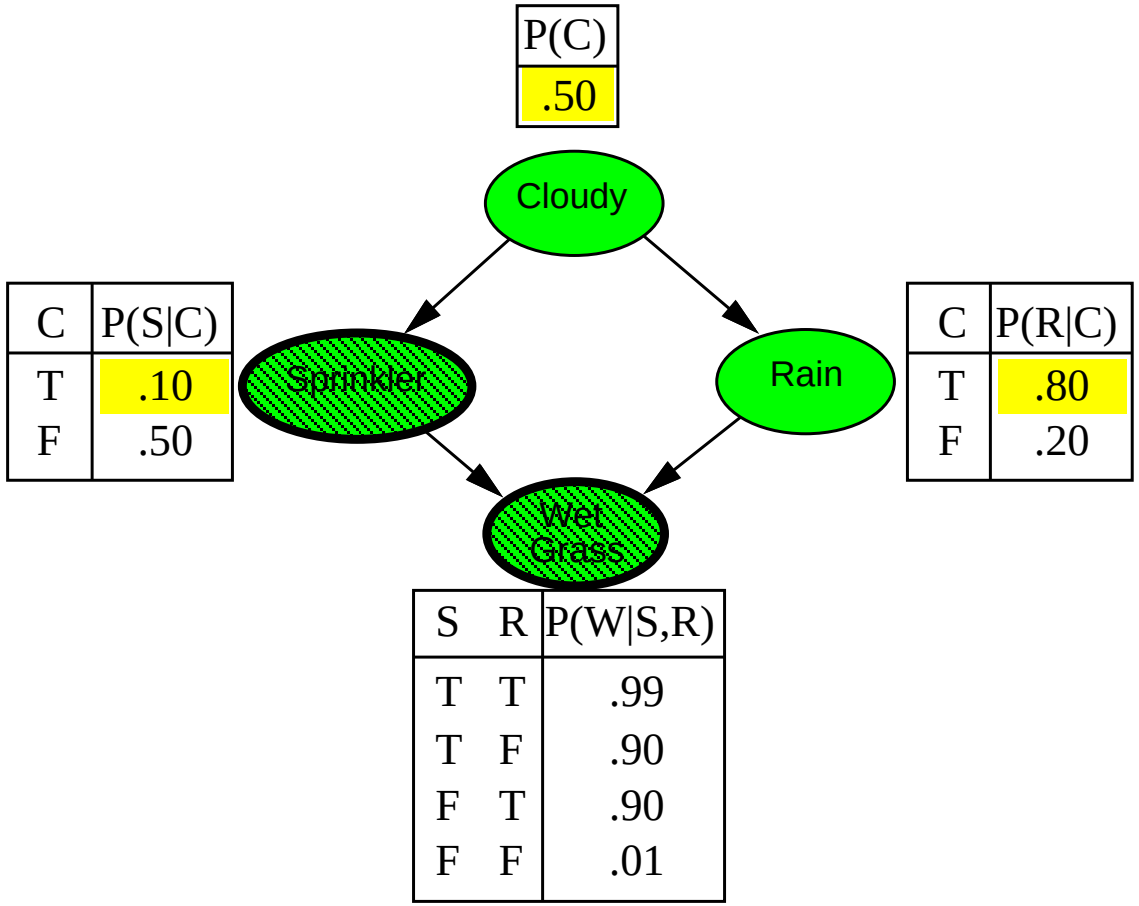
Likelihood weighting example



$w = 1.0 \times 0.1$

Likelihood weighting example



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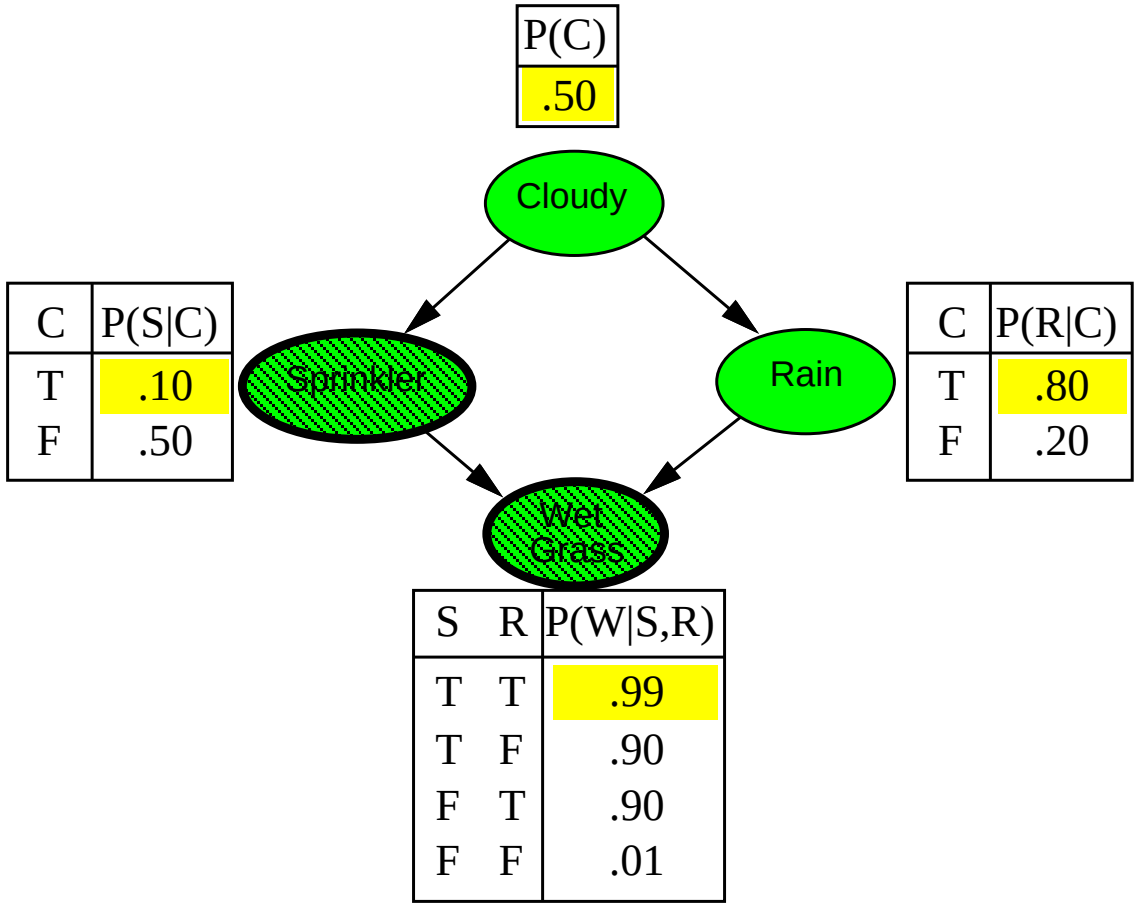
Likelihood weighting example



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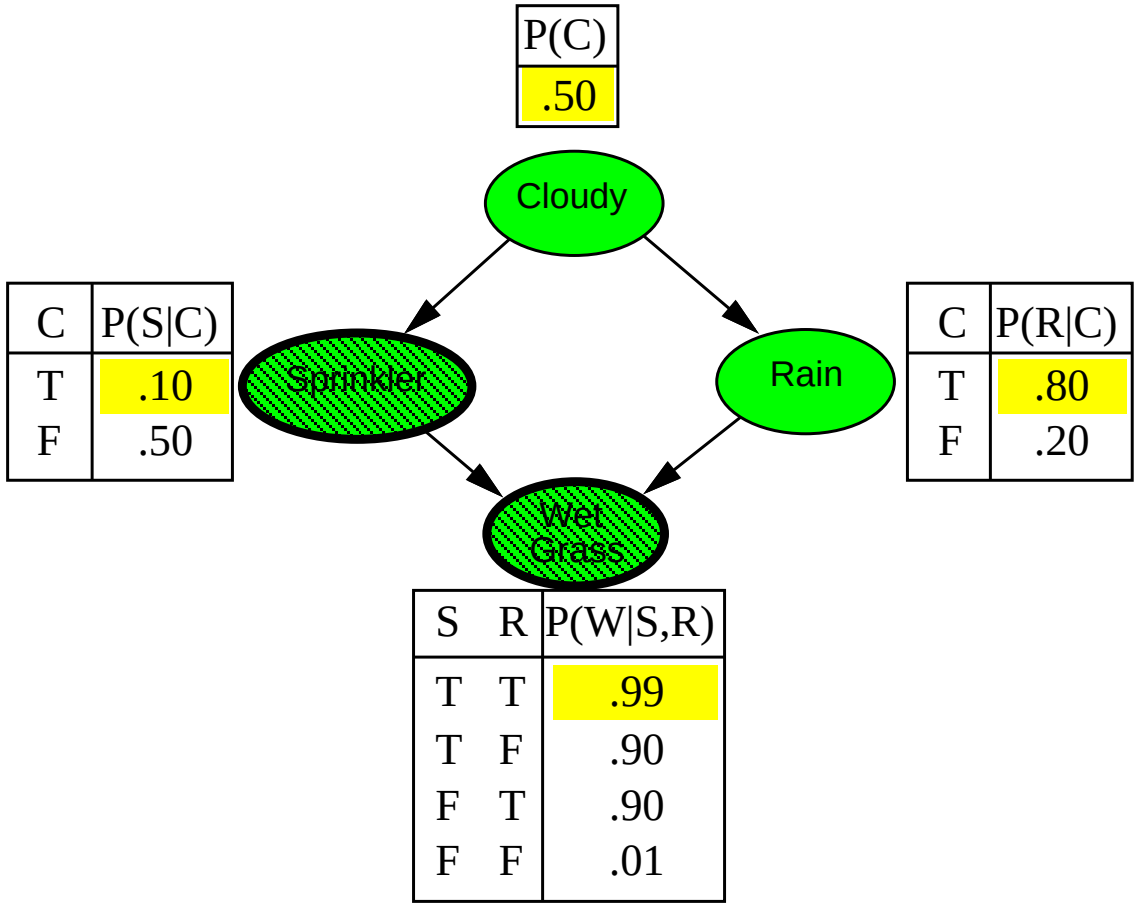
Likelihood weighting example



$w = 1.0 \times 0.1 \times 0.99 = 0.099$

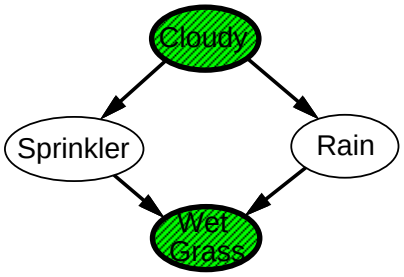
Likelihood weighting analysis

Sampling probability for WeightedSample is

$$S_{WS}(\mathbf{z}, \mathbf{e}) = \Pi_{i=1}^{l} P(z_i | parents(Z_i))$$

Note: pays attention to evidence in **ancestors** only

$\Rightarrow$ somewhere "in between" prior and posterior distribution



Weight for a given sample $\mathbf{z}, \mathbf{e}$ is

$$w(\mathbf{z}, \mathbf{e}) = \Pi_{i=1}^{m} P(e_i | parents(E_i))$$

Weighted sampling probability is

$$\begin{aligned} & S_{WS}(\mathbf{z}, \mathbf{e}) w(\mathbf{z}, \mathbf{e}) \\ & = \Pi_{i=1}^{l} P(z_i | parents(Z_i)) \ \ \Pi_{i=1}^{m} P(e_i | parents(E_i)) \\ & = P(\mathbf{z}, \mathbf{e}) \text{ (by standard global semantics of network)} \end{aligned}$$

Hence likelihood weighting returns consistent estimates
but performance still degrades with many evidence variables
because a few samples have nearly all the total weight

Summary

Exact inference by enumeration:

- NP-hard on general graphs

Approximate inference by LW:

- LW does poorly when there is lots of (downstream) evidence
- LW, generally insensitive to topology
- Convergence can be very slow with probabilities close to 1 or 0
- Can handle arbitrary combinations of discrete and continuous variables