Differential Equations and the Exponential function. Models for Population Growth

1. **Quote.** A finite world can support only a finite population; therefore, population growth must eventually equal zero. (Garrett James Hardin, 1915 - 2003. American ecologist)

2. We are looking at (closely related) differential equations whose solutions involve the exponential function $y(t) = e^{\alpha t}$.

   (a) Constant growth rate - Exponential Growth or Decay (radio activity, bank account, population)
   $y'(t) = \alpha y(t) \qquad \Rightarrow \qquad y(t) = Ce^{\alpha t}$

   (b) Constant source term (mixing problem, mortgage payment)
   $y'(t) = \alpha y(t) - \beta \qquad \Rightarrow \qquad y(t) = \frac{\beta}{\alpha}(1 - e^{\alpha t}) + Ce^{\alpha t}$

   (c) Logistic growth model - limits exponential growth.
   $y'(t) = \alpha y(t) - \gamma y^2(t) = \alpha y(t)(1 - \frac{\gamma}{\alpha} y(t)) \qquad \Rightarrow \quad y(t) = \frac{Ce^{\alpha t}}{1 + C\frac{\gamma}{\alpha}e^{\alpha t}}$ **(or** $\frac{Ce^{\alpha t}}{-1 + C\frac{\gamma}{\alpha}e^{\alpha t}}$**)**

   (d) Exponential source term $y'(t) = \alpha y(t) + e^{\beta t} \quad \rightarrow$ Homework 10, A1.
   $y(t) = Ce^{\alpha t} + \frac{e^{\beta t} - e^{\alpha t}}{\beta - \alpha}$

3. **Example: A Mixing Problem**

A tank contains 1000L of pure water. Brine that contains 0.05 kg of salt per litre of water enters the tank at a rate of 5 L/min. Brine that contains 0.04 kg of salt per litre of water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at rate of 15 L/min. How much salt is in the tank (a) after t minutes and (b) after one hour?

Identify variables, parameters, and use correct units

Let $y(t)$ be the amount of salt (in kg) in the tank.

$\dfrac{dy}{dt}$ = rate of inflow minus rate of outflow

Inflow: 0.05kg/L $\times$ 5L/min

total $\beta = 0.65$ kg/min.

Outflow: ...., $\alpha =$ ________

Equation: $y'(t) = -\alpha y(t) + \beta, \quad y(0) = 0.$

Mixing Problem continued

4. **Example.** Mortgage payment. Let's use differential equations to estimate a mortgage or loan payment.

Amount outstanding at time t: $A(t)$
Amount borrowed: A_0 ($A_0 = \$100,000$)
Interest rate: α ($\alpha = 4\%$, i.e., $\alpha = .04$)
Amortization period: N ($N = 25$)
Yearly payment: yp (monthly payment $= yp/12$); set $\beta = -yp$

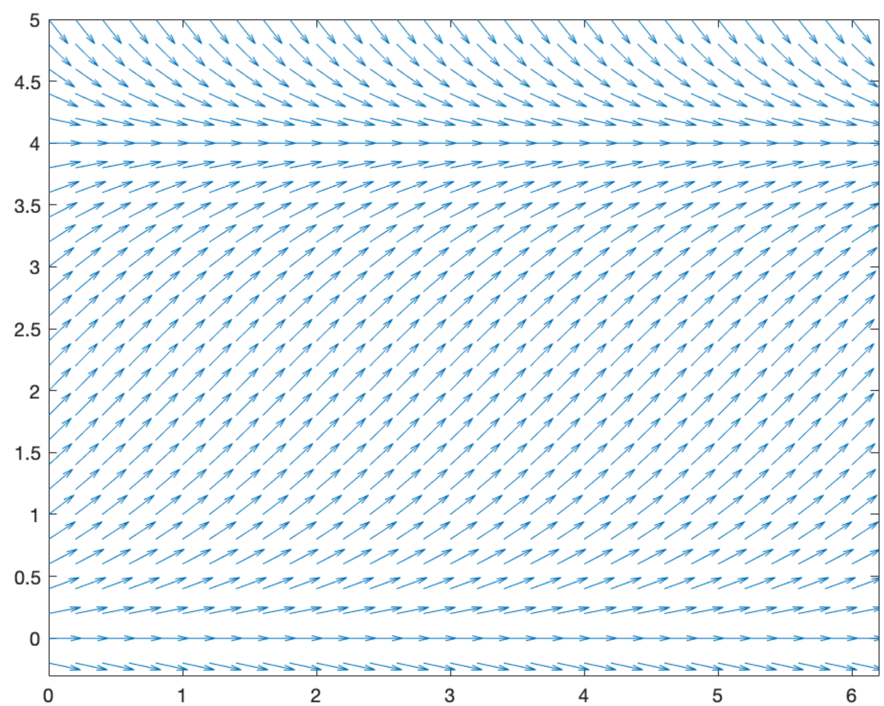
Note: actual mortgage payment calculation uses semi-annual compounding (or monthly compounding on a HELOC - home equity line of credit). This results in (slightly) different effective interest rates. This computation assumes continuous compounding.

Mortgage Problem continued

5. Add quadratic term to $y' = \alpha y$: A "simple" nonlinear first-order equation

$$\frac{dy}{dt} = \alpha y - \gamma y^2 = \alpha y(1 - \frac{\gamma}{\alpha} y)$$

(a) What are the equilibria, i.e., when do we have solutions $y(t) = \text{constant}$?



(b) Direction field ($\frac{\gamma}{\alpha} = \frac{1}{4}$)

(c) Separable equation: Solve $\dfrac{dy}{dt} = \alpha y(1 - \dfrac{\gamma}{\alpha} y)$ for $y(t)$.

6. **Natural Growth Model** - revisited

The **Natural Growth Model** for population growth assumes that the population P at time t changes at a rate proportional to its size at any given time t. This can be written as

$$\frac{dP}{dt} = kP$$

where k is a constant.

7. **Logistic Growth Model**

The Natural Growth Model implies that the population would grow exponentially indefinitely. However the model must break down at some point since the population would eventually outstrip the food supply. In searching for an improvement we should look for a model whose solution is approximately an exponential function for small values of the population, but which levels off later.

The **Logistic Growth Model** for a population $P(t)$ at time t is based on the following assumptions:

- The growth rate is initially close to being proportional to size:

$$\frac{dP}{dt} \approx kP \qquad \text{if } P \text{ is small}$$

- The environment is only capable of a maximum population in the long run, this is called the **carrying capacity**, usually denoted by M:

$$\lim_{t\to\infty} P(t) = M.$$

The simplest expression for the growth rate that incorporates these assumptions is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) \qquad \text{Logistic Growth Model}$$

where k is a constant.

We can also write this equation as

$$\frac{dP}{dt} = kP - \frac{k}{M}P^2,$$

the quadratic term can be interpreted as a competition term.

Notes.