Binary Search Trees

Data Structures and Algorithms
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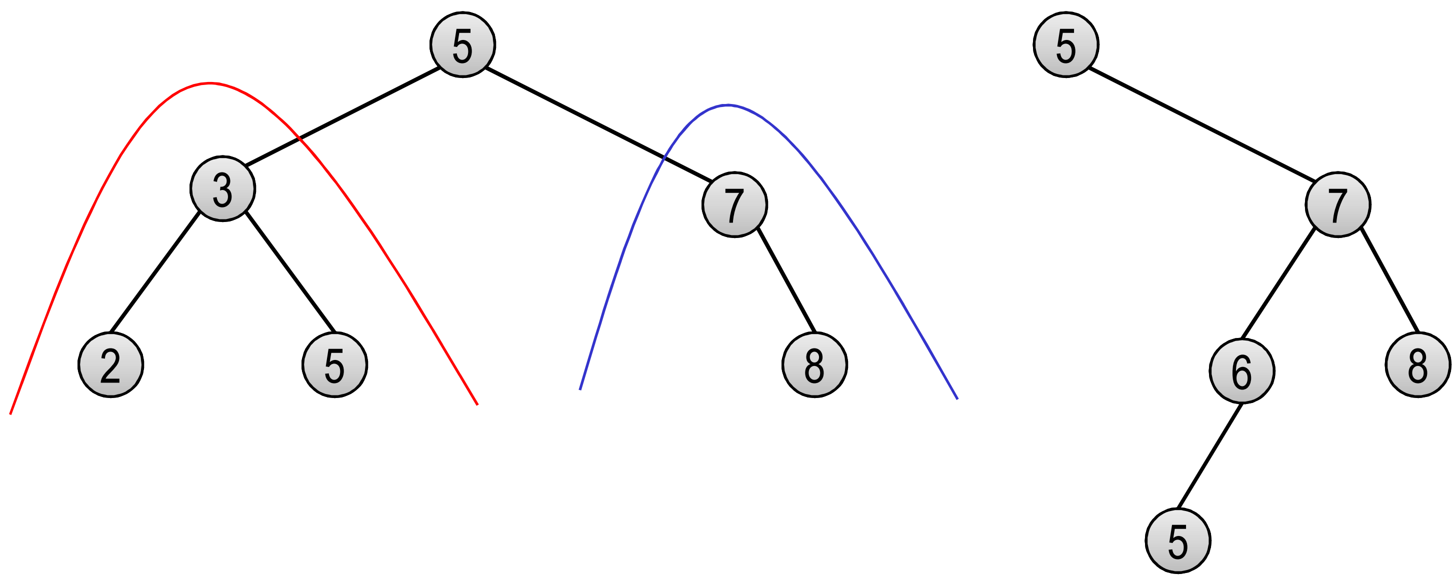
Binary Search Tree

A binary search tree is a binary rooted tree, in which keys satisfy the Binary Search Tree property:

Let x be a node in a binary search tree. If y is a node in the left subtree of x, then key[y] ≤ key[x].

If y is a node in the right subtree of x, then key[x] ≤ key[y]



Inorder Tree Walk

Having a binary search tree one can print its content in sorted order

Inorder-Tree-Walk(x)

```
if x≠Nil then do
     Inorder-Tree-Walk(left[x])
     print key[x]
     Inorder-Tree-Walk(right[x])
```

To print the entire tree just call Inorder-Tree-Walk(root[T])

It is called inorder because the root is printed between the subtrees

A tree walk can also be preorder and postorder

Inorder Tree Walk (cntd)

Lemma

If x is the root of an n-node subtree, then the call Inorder-Tree-Walk(x) takes $\Theta(n)$ time

Proof

Let $T(n)$ denote the running time

If $x = Nil$ then $T(0) = c$, a constant

If $n > 0$ then $T(n) = T(k) + T(n - k - 1) + d$ where k is the number of nodes in the left subtree

We prove that $T(n) = (c+d)\, n + c$

For $n = 0$ we have $(c + d) \cdot 0 + c = c = T(0)$

Inorder Tree Walk (cntd)

Proof

For $n > 0$ we have

$$
\begin{aligned}
T(n) &= T(k) + T(n - k - 1) + d \\
&= ((c + d)\,k + c) + ((c + d)\,(n - k - 1) + c) + d \\
&= (c + d)\,n + c - (c + d) + c + d \\
&= (c + d)\,n + c
\end{aligned}
$$

QED

Searching

Elements of a binary search tree can be found efficiently

Tree-Search(x,k)

```
if x=Nil or k=key[x] then
   return x
if  k<key[x] then
   return Tree-Search(left[x],k)
else
   return Tree-Search(right[x],k)
```

Minimum and Maximum

We can find minimum and maximum keys in the tree

Tree-Minimum(x)

```
while left[x]≠Nil do
   set x:=left[x]
endwhile
return x
```

Tree-Maximum(x)

```
while right[x]≠Nil do
   set x:=right[x]
endwhile
return x
```

Successor

We can find the successor of a key in the tree

Tree-Successor(x)

```
if right[x]≠Nil then
   return Tree-Minimum(right[x])
endif
set y:=parent[x]
while y≠Nil and x=right[y] do
   set x:=y
   set y:=parent[y]
endwhile
return y
```

Running Time

Theorem

The operations Search, Minimum, Maximum, Successor, and Predecessor can be made to run in O(h) time on a binary search tree of height h.

Insertion

We insert a new value v into a binary search tree T.

The procedure is passed a node z for which key[z] = v, left[z] = Nil, and right[z] = Nil.

Tree-Insert(T,z)

```
set y:=Nil,   x:=root[T]
while x≠Nil do
   set y:=x
   if key[z]<key[x] then set x:=left[x]
                    else set x:=right[x]
endwhile
set parent[z]:=y
if y=Nil then set root[T]:=z
         else if key[z]<key[y] then set left[y]:=z
                               else set right[y]:=z
```

Deletion

We delete a given node z
Consider 3 cases:

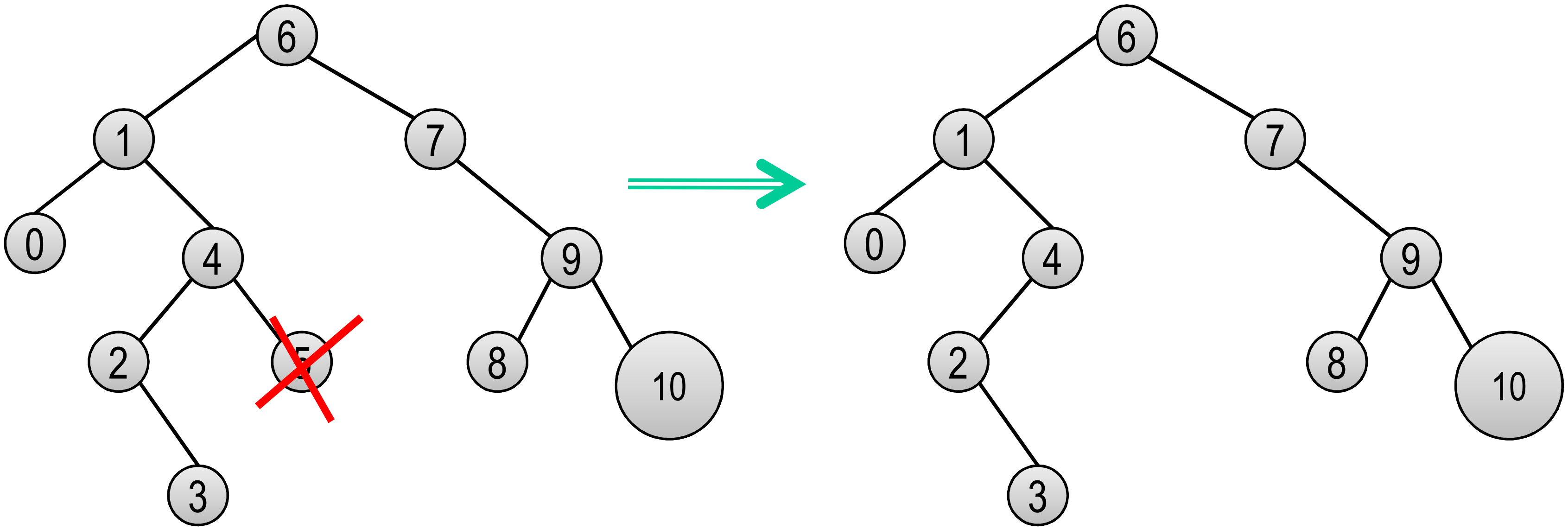
(i) z has no children

(ii) z has one child

(iii) z has 2 children

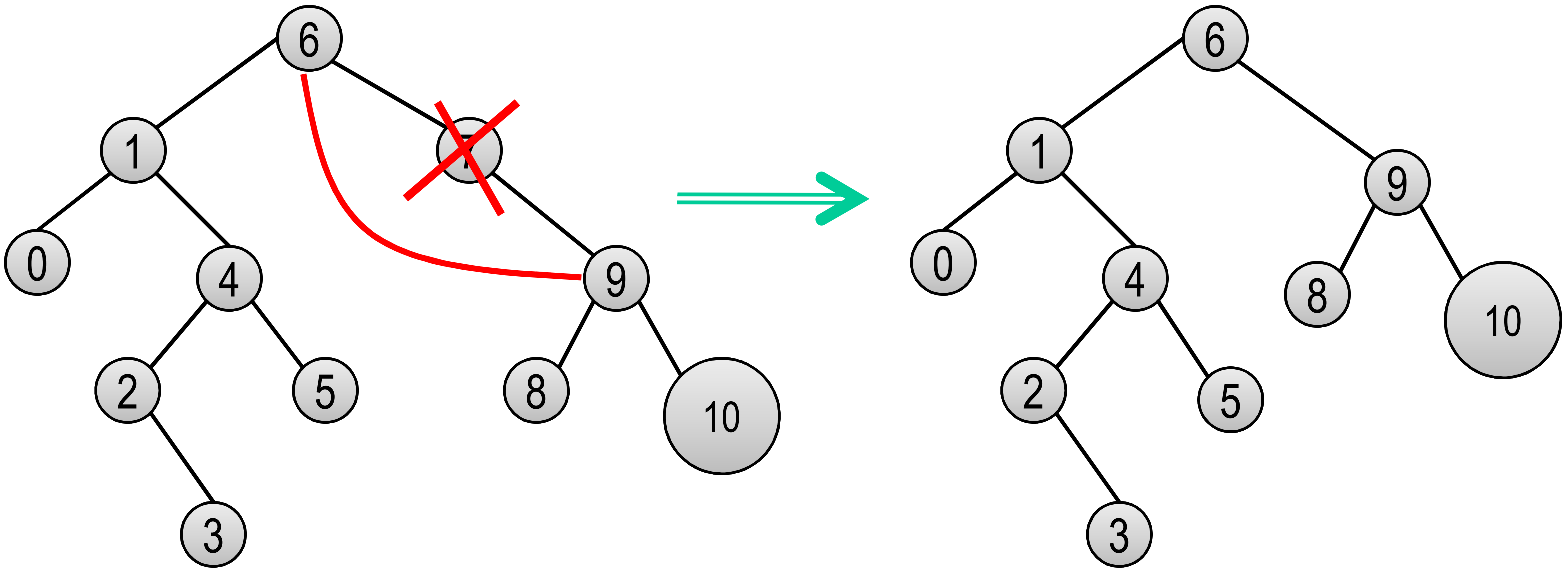
Deletion (cntd)

(i) z has no children



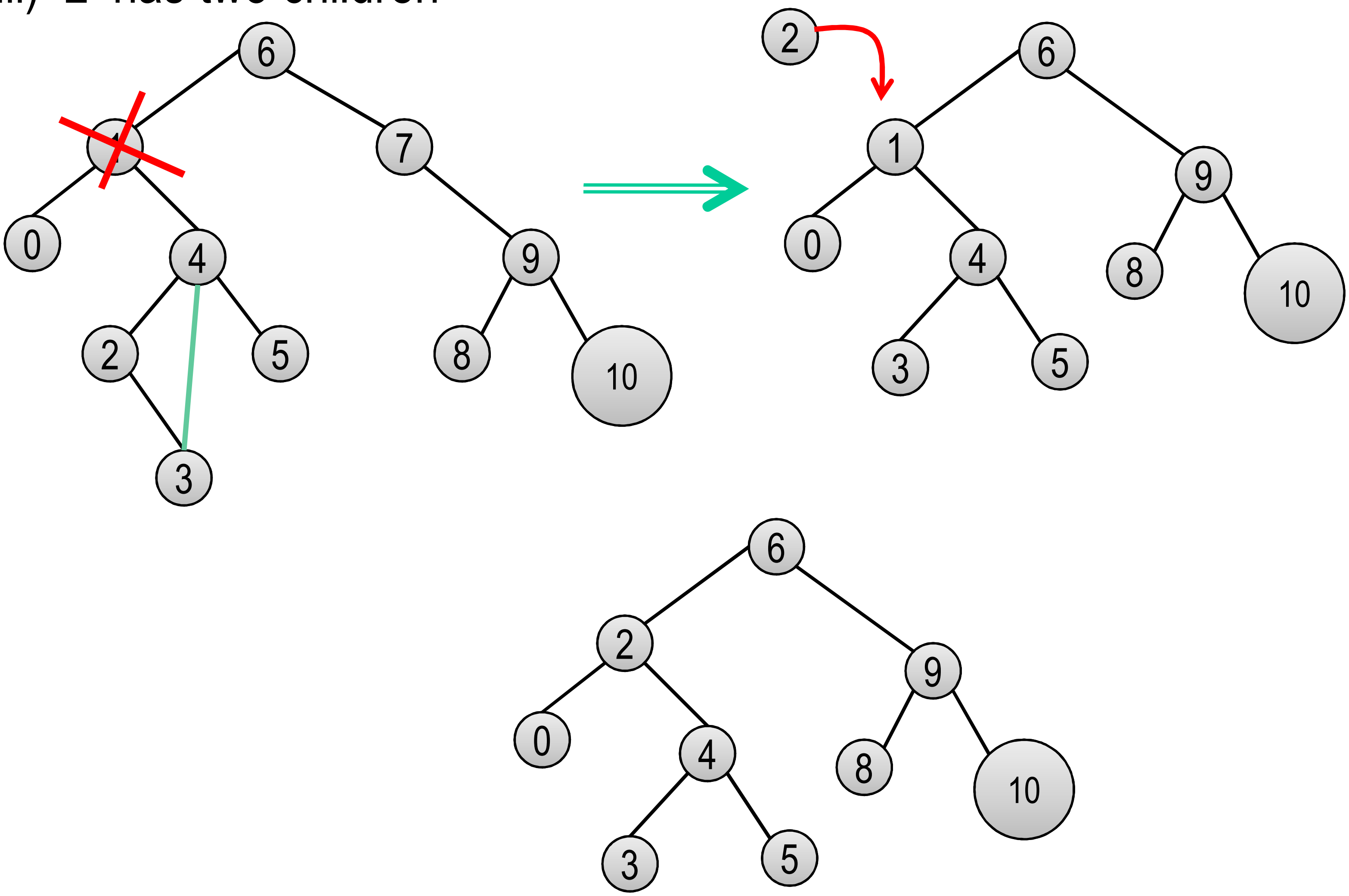
Deletion (cntd)

(ii) z has one child



Deletion (cntd)

(iii) z has two children



Deletion

Tree-Delete(T,z)

```
if left[z]=Nil or right[z]=Nil then set y:=z
                              else set y:=Tree-Successor(z)
if left[y]≠Nil then set x:=left[y]
               else set x:=right[y]
if x≠Nil then set parent[x]:=parent[y]
                    else set x:=right[x]
if parent[y]=Nil then set root[T]:=x
 else if y=left[parent[y]]:=x then left[parent[y]:=x
                            else right[parent[y]]:=x
if y≠z then do
  set key[z]:=key[y]
  copy y's data into z
return y
```

Red-Black Trees

All binary search tree operations take O(h) time, where h is the height of the tree

Therefore, it is important to `balance' the tree so that its height is as small as possible

There are many ways to achieve this

One of them: Red-Black trees

Every node of such a tree contains one extra bit, its color

Another agreement: the Nil pointer is treated as a leaf, an extra node

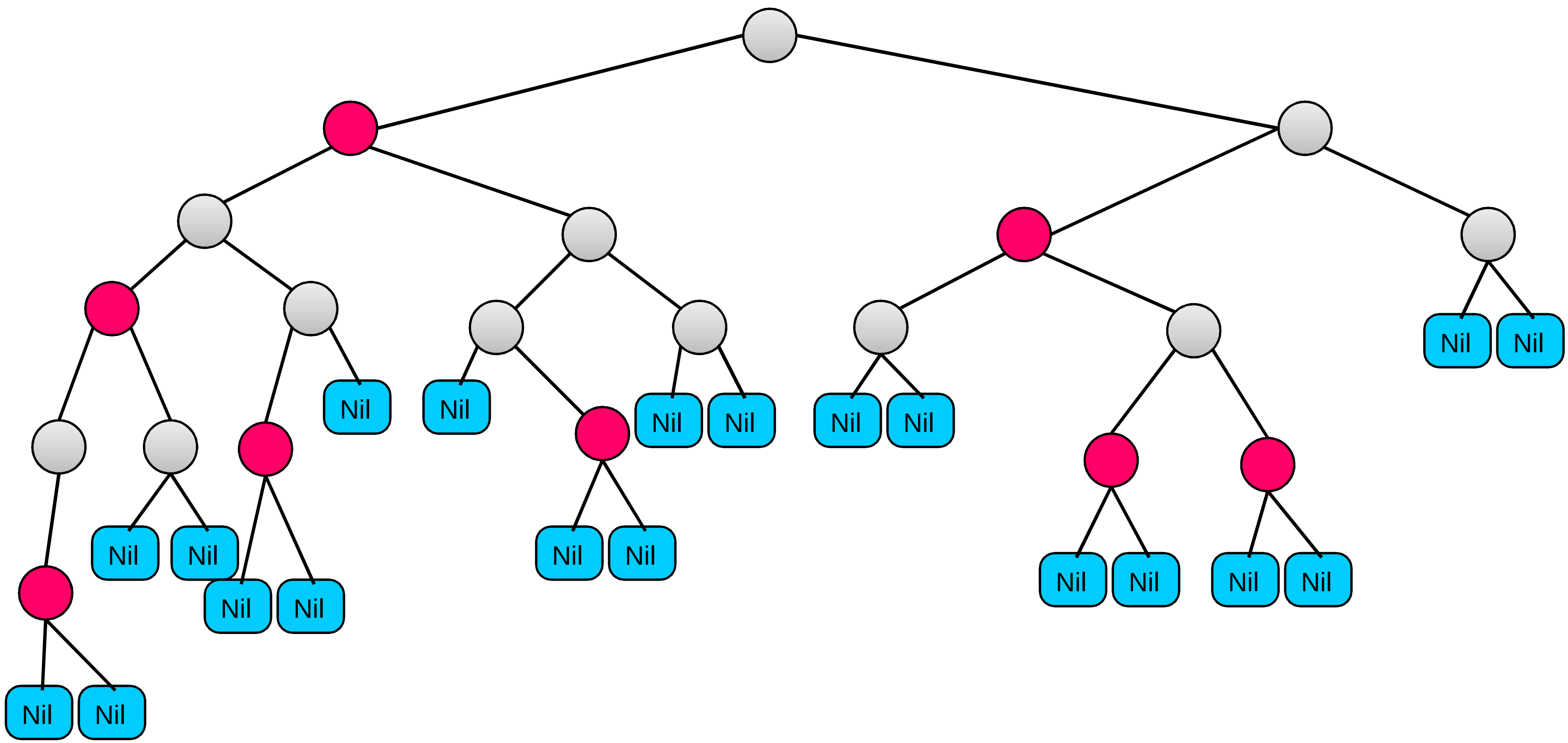
The rest of the nodes are called internal

Red-Black Properties

A binary search tree is a red-black tree if it satisfies the following red-black properties:

- Every node is either red or black
- The root is black
- Every leaf (Nil) is black
- If a node is red, then both its children are black
- For each node, all paths from the node to descendant leaves contain the same number of black nodes

Example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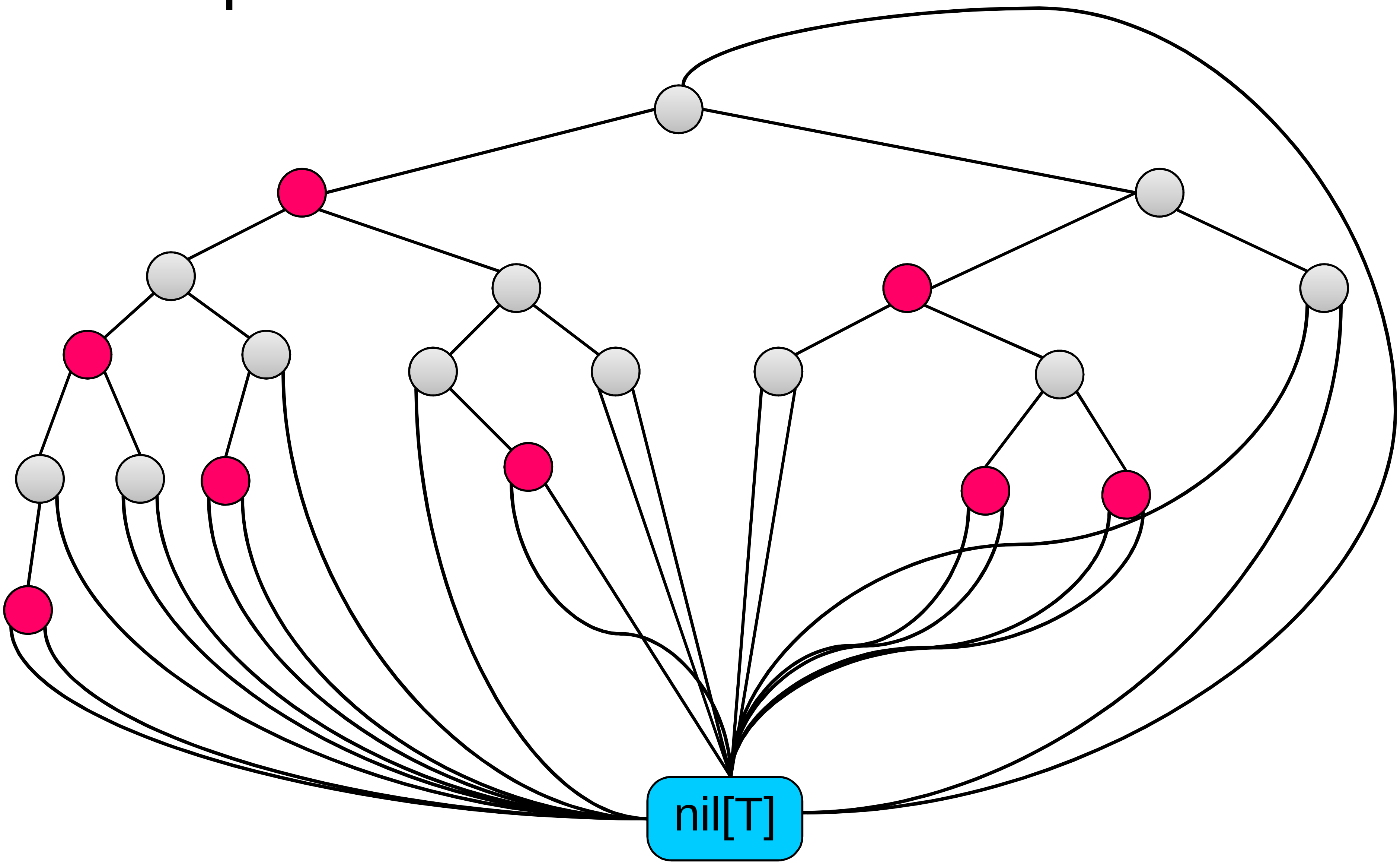
Example



Black Height

The number of black nodes on paths from node x to its descendant leaves in a red-black tree is called its black height, denoted bh(T)

Lemma

A red-black tree with n internal nodes has height at most 2·log(n + 1)

Proof

We show first that the subtree rooted at x contains at least $2^{bh(x)} - 1$ nodes

Induction on bh(x)

Base Case: If bh(x) = 0, then x is a leaf, nil[T]

In this case, $2^{bh(x)} - 1 = 2^0 - 1 = 0$ internal nodes

Black Height (cntd)

Inductive hypothesis: the claim is true for any y with height less than that of x

Inductive Case: Let bh(x) > 0 and has two children

The black height of the children is either bh(x) or bh(x) – 1, depending on its color

Since the children have smaller height, we can apply the induction hypothesis

Thus the subtree rooted at x contains at least

$$(2^{bh(x)-1}-1)+(2^{bh(x)-1}-1)+1=2^{bh(x)}-1$$

nodes

Black Height (cntd)

Suppose h is the height of the tree

By the red-black property at least half of nodes on every root-to-leaf path are black (not including the root)

Therefore the black height of the root is at least h/2

Thus

$$n \geq 2^{h/2} - 1$$
$$\log(n+1) \geq h/2$$

QED