Assignment 1: Dynamical Systems

Due Feb 1st at 11:59pm

This assignment is to be done individually.

Important Note: The university policy on academic dishonesty (cheating) will be taken very seriously in this course. You may not provide or use any solution, in whole or in part, to or by another student.

You are encouraged to discuss the concepts involved in the questions with other students. If you are in doubt as to what constitutes acceptable discussion, please ask! Further, please take advantage of office hours offered by the instructor and the TA if you are having difficulties with this assignment.

DO NOT:

- Give/receive code or proofs to/from other students
- Use Google to find solutions for assignment

DO:

- Meet with other students to discuss assignment (it is best not to take any notes during such meetings, and to re-work assignment on your own)
- Use online resources (e.g. Wikipedia) to understand the concepts needed to solve the assignment.

Submitting Your Assignment

The assignment must be submitted online at `https://canvas.sfu.ca/`. You must submit one zip file (student_number_hs1.zip) containing:

1. An assignment report in **PDF format**, named `student_number_hw1.pdf`. This report should contain your solutions to questions 1-4.
2. Your code for question 1, named `rk4.py`.
3. Your code for question 4, named `ode.py`.

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1 Satellite Orbiting Earth (13 marks)

a) Let $x_1 = r, x_2 = \dot{r}, x_3 = \theta, x_4 = \dot{\theta}$. Then we have

$$\dot{x}_1 = x_2 \tag{1}$$
$$\dot{x}_2 = x_1 x_4^2 - \frac{k}{x_1^2} + u_1 \tag{2}$$
$$\dot{x}_3 = x_4 \tag{3}$$
$$\dot{x}_4 = -\frac{2x_2x_4}{x_1} + \frac{u_2}{x_1} \tag{4}$$

Alternatively, let $x_1 = r, x_2 = \dot{r}, x_3 = \theta$. Then we have

$$\dot{x}_1 = x_2 \tag{5}$$
$$\dot{x}_2 = x_1 x_3^2 - \frac{k}{x_1^2} + u_1 \tag{6}$$
$$\dot{x}_3 = -\frac{2x_2x_3}{x_1} + \frac{u_2}{x_1} \tag{7}$$

b) Setting $u_1 = u_2 = 0$, we have

$$\dot{x}_1 = x_2 \tag{8}$$
$$\dot{x}_2 = x_1 x_4^2 - \frac{k}{x_1^2} \tag{9}$$
$$\dot{x}_3 = x_4 \tag{10}$$
$$\dot{x}_4 = -\frac{2x_2x_4}{x_1} \tag{11}$$

Equilibrium points are given by $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = \dot{x}_4 = 0$. From $\dot{x}_1 = \dot{x}_3 = 0$, we get $x_2 = x_4 = 0$. From $\dot{x}_2 = 0$ and $x_4 = 0$, the second component of the dynamics becomes $\dot{x}_2 = \frac{k}{x_1^2}$, which is never zero for any positive k.

Therefore, under zero control input, the system does not have any equilibrium points.

This is because the $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = \dot{x}_4 = 0$ means that the satellite is staying a fixed distance from earth and is not rotating. However, to keep such a configuration, some thrust input is needed to balance out gravity.

Alternatively, with a three-dimensional state space, after setting $u_1 = u_2 = 0$, we have

$$\dot{x}_1 = x_2 \tag{12}$$
$$\dot{x}_2 = x_1 x_3^2 - \frac{k}{x_1^2} \tag{13}$$
$$\dot{x}_3 = -\frac{2x_2x_3}{x_1} \tag{14}$$

From $\dot{x}_1 = 0$, we get $x_2 = 0$. From $\dot{x}_2 = 0$, we have, $x_1^3 x_3^2 = k$, which is the third law of Kepler: `https://en.wikipedia.org/wiki/Kepler's_laws_of_planetary_motion#Third_law_of_Kepler`.

c) What are the equilibrium points of the state space model, under $u_1 = k/x_1^2, u_2 = 0$? Give a physical interpretation of this control set point and of the equilibrium points.

Setting $u_1 = k/x_1^2, u_2 = 0$. we have

$$\dot{x}_1 = x_2 \tag{15}$$
$$\dot{x}_2 = x_1 x_4^2 \tag{16}$$
$$\dot{x}_3 = x_4 \tag{17}$$
$$\dot{x}_4 = -\frac{2x_2 x_4}{x_1} \tag{18}$$

As before, Equilibrium points are given by $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = \dot{x}_4 = 0$. From $\dot{x}_1 = \dot{x}_3 = 0$, we get $x_2 = x_4 = 0$. Next, observe that $x_2 = x_4 = 0$ implies $\dot{x}_2 = \dot{x}_4 = 0$ for any x_1 and x_3 .

The control set point represents thrust input that balances out gravity. As long as gravity can be counteracted, the system would be able to remain at any distance x_1 from the earth, at any phase angle x_3.

Alternatively, with a three-dimensional state space, after setting $u_1 = k/x_1^2, u_2 = 0$. we have

$$\dot{x}_1 = x_2 \tag{19}$$
$$\dot{x}_2 = x_1 x_3^2 \tag{20}$$
$$\dot{x}_3 = -\frac{2x_2 x_3}{x_1} \tag{21}$$

From $\dot{x}_1 = 0$, we require $x_2 = 0$, which leads to $\dot{x}_3 = 0$ for any x_1 and x_3. From $\dot{x}_2 = 0$, we get $x_3 = 0$ since $x_1 \neq 0$, with the same physical interpretation as for the four-dimensional model.

d) Linearize the model with respect to a reference orbit given by $r(t) \equiv \rho, \theta(t) = \omega t, u_1 = u_2 = 0$.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ x_4^2 + \frac{2k}{x_1^3} & 0 & 0 & 2x_1 x_4 \\ 0 & 0 & 0 & 1 \\ \frac{2x_2 x_4}{x_1^2} - \frac{u_2}{x_1^2} & -\frac{2x_4}{x_1} & 0 & -\frac{2x_2}{x_1} \end{bmatrix}$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} \\ \frac{\partial f_4}{\partial u_1} & \frac{\partial f_4}{\partial u_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{x_1} \end{bmatrix}$$

The set point implies $x_1 = \rho, x_2 = 0, x_3 = \omega t, x_4 = \omega, u_1 = u_2 = 0$. Evaluating at the set point, we have

$$\frac{\partial f}{\partial x}_{x_1=\rho,x_2=0,x_3=\omega t,x_4=\omega,u_1=u_2=0} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \omega^2 + \frac{2k}{\rho^3} & 0 & 0 & 2\rho\omega \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2\omega}{\rho} & 0 & 0 \end{bmatrix}$$

We can make a further simplification by observing that since $x_2 = 0$, we also have $\dot{x}_2 = 0$, so

$$x_1 x_4^2 - \frac{k}{x_1^2} + u_1 = 0 \tag{22}$$

$$x_1^3 x_4^2 = k \tag{23}$$

$$\omega^2 = \frac{k}{\rho^3} \tag{24}$$

Now, substituting $2\omega^2$ for $\frac{2k}{\rho^3}$ in $\frac{\partial f_2}{\partial x_1}$, we get

$$\frac{\partial f}{\partial x}_{x_1=\rho,x_2=0,x_3=\omega t,x_4=\omega,u_1=u_2=0} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\rho\omega \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2\omega}{\rho} & 0 & 0 \end{bmatrix} \tag{25}$$

Lastly,

$$\frac{\partial f}{\partial u}_{x_1=\rho,x_2=0,x_3=\omega t,x_4=\omega,u_1=u_2=0} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{\rho} \end{bmatrix} \tag{26}$$

So the linearized system is

$$\dot{\tilde{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\rho\omega \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2\omega}{\rho} & 0 & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{\rho} \end{bmatrix} \tilde{u} \tag{27}$$

where $\tilde{x} = \begin{bmatrix} x_1 - \rho \\ x_2 \\ x_3 - \omega t \\ x_4 - \omega \end{bmatrix}$, and $\tilde{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.

Alternatively, with the three-dimensional model, simply ignore the third row and column in the analysis above, and we obtain

$$\dot{\tilde{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 3\omega^2 & 0 & 2\rho\omega \\ 0 & -\frac{2\omega}{\rho} & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{\rho} \end{bmatrix} \tilde{u} \tag{28}$$

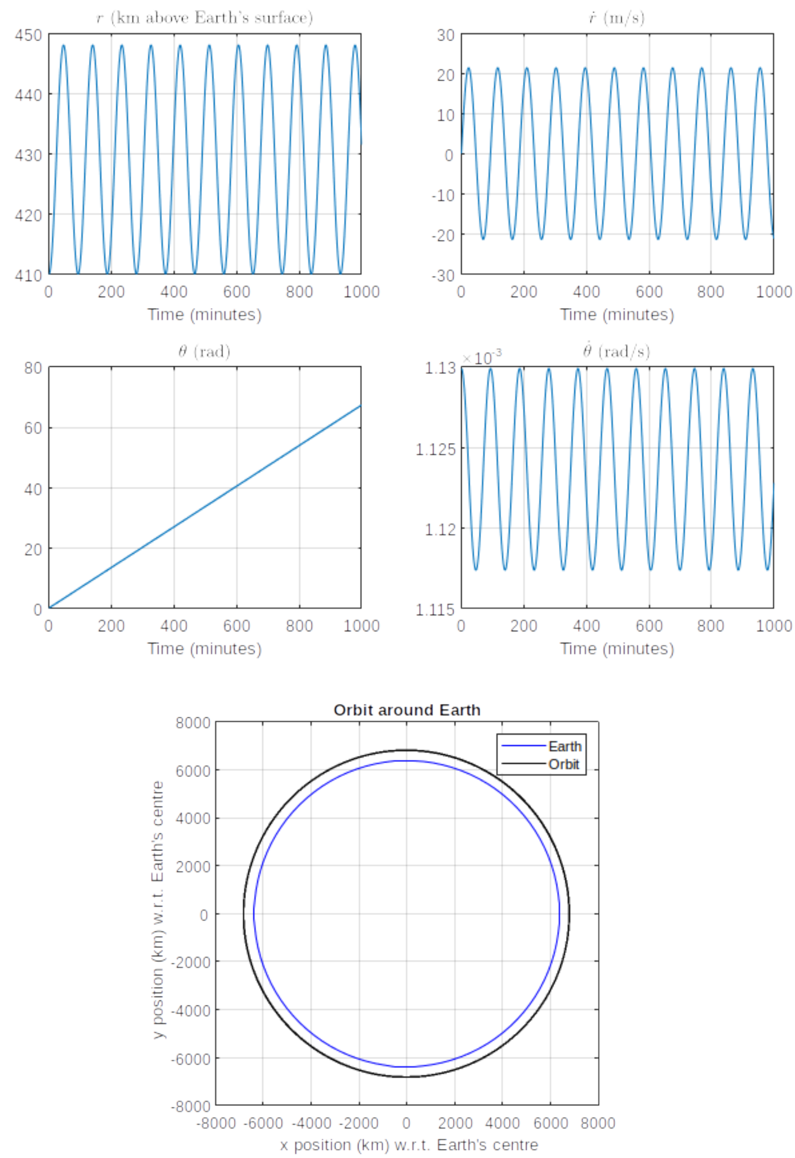
r (km above Earth's surface)
Time (minutes)
$\dot{r}$ (m/s)
Time (minutes)
θ (rad)
Time (minutes)
$\dot{\theta}$ (rad/s)
$\times 10^{-3}$
Time (minutes)
Orbit around Earth
Earth
Orbit
y position (km) w.r.t. Earth's centre
x position (km) w.r.t. Earth's centre

Figure 1:

where $\tilde{x} = \begin{bmatrix} x_1 - \rho \\ x_2 \\ x_3 - \omega \end{bmatrix}$, and $\tilde{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.

e) Numerically integrate the ODE, with $u_1(t), u_2(t) \equiv 0$, using your own implementation of RK4. Plot the state trajectory and intuitively explain the behaviour. Please attach your code.

Use parameters for the international space station orbiting the Earth:

- $r(0) = 410$ km $+ 6378$ km, which represents an orbit 410 km above the earth's surface,
- $\dot{r}(0) = 0$ m/s,
- $\theta(0) = 0$,
- $\dot{\theta}(0) = 2\pi/T$, where $T = 92.68$ minutes, the orbital period
- $k = GM$, where $G = 6.67 \times 10^{-11}$ m^{3}kg^{-1}s^{-2}, $M = 5.97 \times 10^{24}$ kg.

Plots of the evolution of all four state variables are shown in Figure 1. In addition, the actual orbit in coordinates a round the Earth is shown as well. Even though the state trajectories show noticeable oscillations, the overall orbit is actually very circular, since the Earth and the orbit are so large.

2 The Lotka-Volterra Predator-Prey Model (7 marks)

a) Equilibrium points satisfy $\dot{x} = \dot{y} = 0$.

From $\dot{x} = 0$, we have $ax - bxy = 0$. Since $x > 0$, we have $y = \frac{a}{b}$.

From $\dot{y} = 0$, we have $-dy + cxy = 0$. Since $y > 0$, we have $x = \frac{d}{c}$.

Therefore the equilibrium is at $(\frac{d}{c}, \frac{a}{b})$.

b) Let $V(x, y) = y^a e^{-by} x^d e^{-cx}$, then

$$\dot{V}(x,y) = x^d e^{-cx}(ay^{a-1}e^{-by} - by^a e^{-by})\dot{y} + y^a e^{-by}(dx^{d-1}e^{-cx} - cx^d e^{-cx})\dot{x} \tag{29}$$
$$= x^d e^{-cx} y^{a-1} e^{-by}(a - by)(cxy - dy) + y^a e^{-by} x^{d-1} e^{-cx}(d - cx)(ax - bxy) \tag{30}$$
$$= x^d e^{-cx} y^a e^{-by}(a - by)(cx - d) + y^a e^{-by} x^d e^{-cx}(d - cx)(a - by) \tag{31}$$

Plugging in $x = \frac{d}{c}, y = \frac{a}{b}$, we get $\dot{V}(\frac{d}{c}, \frac{a}{b}) = 0$.

c) First, we write put the exponentials into the same base for V:

$$V(x,y) = y^a e^{-by} x^d e^{-cx} = e^{-by + a\log y} e^{-cx + d\log x} = e^{-by + a\log y - cx + d\log x} \tag{32}$$

Note that $\log V(x, y) = -by + a\log y - cx + d\log x$ is a concave function, and has the same maximum as $V(x, y)$. Since $\log V(x, y)$ is concave, its maximum, and the maximum of $V(x, y)$, occur at where $\nabla \log V(x, y) = 0$.

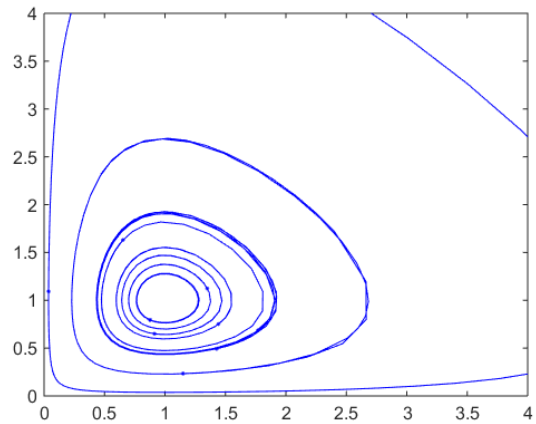

Figure 2:

$$\frac{\partial \log V(x,y)}{\partial x} = -c + \frac{d}{x} \tag{33}$$

$$\frac{\partial \log V(x,y)}{\partial x} = 0 \Rightarrow x = \frac{d}{c} \tag{34}$$

$$\frac{\partial \log V(x,y)}{\partial y} = -b + \frac{a}{y} \tag{35}$$

$$\frac{\partial \log V(x,y)}{\partial y} = 0 \Rightarrow y = \frac{a}{b} \tag{36}$$

Since $V(x,y)$ attains its maximum at $(\frac{d}{c}, \frac{a}{b})$, the equilibrium point, and $V(x,y) = 0$ everywhere, $(\frac{d}{c}, \frac{a}{b})$ must be a nonlinear center for all a, b, c, d. This means all trajectories around the equilibrium point are bounded, and therefore stable.

Figure c show the state trajectories, which also correspond to the level sets of $V(x,y)$.

3 Stabilization via Linear Feedback (4 marks)

First, we compute the eigenvalues of A so that we can obtain the characteristic equation.

$$\det(sI - A) = \det(\begin{bmatrix} s-1 & 0 \\ -1 & s+2 \end{bmatrix}) \tag{37}$$

$$(s-1)(s+2) = 0 \tag{38}$$

Characteristic polynomial: $s^2 + s - 2$

Current eigenvalues assuming zero input ($u(t) = 0$): $s = 1, -2$

This means that the zero-input system is not stable since one of the eigenvalues is positive.

With the controller $u = -Kx$, the closed-loop system is $\dot{x} = (A - BK)x$. To stabilize the system, we need to choose K so that $A - BK$ has eigenvalues in the open left half plane. We will pick $-1, -2$ for the eigenvalues, but any two eigenvalues with negative real part would work.

$$\begin{aligned}
A - BK &= \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} k_0 & k_1 \end{bmatrix} && (39)\\
&= \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} k_0 & k_1 \\ k_0 & k_1 \end{bmatrix} && (40)\\
&= \begin{bmatrix} 1 - k_0 & -k_1 \\ 1 - k_0 & -2 - k_1 \end{bmatrix} && (41)
\end{aligned}$$

The eigenvalues are given by

$$\begin{aligned}
\det\left(\begin{bmatrix} 1 - k_0 & -k_1 \\ 1 - k_0 & -2 - k_1 \end{bmatrix} - sI\right) &= 0 && (42)\\
\det\left(\begin{bmatrix} 1 - k_0 - s & -k_1 \\ 1 - k_0 & -2 - k_1 - s \end{bmatrix}\right) &= 0 && (43)\\
(1 - k_0 - s)(-2 - k_1 - s) + k_1(1 - k_0) &= 0 && (44)\\
s^2 + (k_0 + k_1 + 1)s + k_0k_1 + 2k_0 - k_1 - 2 + k_1 - k_0k_1 &= 0 && (45)\\
s^2 + (k_0 + k_1 + 1)s + 2k_0 - 2 &= 0 && (46)
\end{aligned}$$

The desired characteristic equation is $s^2 + 3s + 2 = 0$. Matching coefficients, we have

$$\begin{aligned}
2k_0 - 2 = 2 &\Rightarrow k_0 = 2 && (47)\\
k_0 + k_1 + 1 = 3 &\Rightarrow k_1 = 0 && (48)
\end{aligned}$$

4 Numerical Solutions to ODEs (8 marks)

a) First, note that the eigenvalues of A are $\lambda = -1, -500$.

Discretizing with the forward Euler method, we have

$$\begin{aligned}
x^{k+1} &= x^k + \Delta t A x^k && (49)\\
&= (I + \Delta t A)x^k && (50)
\end{aligned}$$

Let λ represent an eigenvalue of A, then the eigenvalue of $I + \Delta t A$ would be represented by $I + \lambda\Delta t$. For stability, we need $|1 + \lambda\Delta t| < 1$ for all eigenvalues λ.

For $\lambda = -1$, we have $|1 - \Delta t| < 1$, so $\Delta t < 2$.

For $\lambda = -500$, we have $|1 - 500\Delta t| < 1$, so $\Delta t < \frac{1}{250}$. Since this is a stricter requirement than $\Delta t < 2$, it takes precedence, and therefore for stability we need $\Delta t < \frac{1}{250}$.

b) The backward Euler method is given by

$$x^{k+1} = x^{k+1} + \Delta t A x^{k+1} \tag{51}$$
$$x^{k+1} - \Delta t A x^{k+1} = x^k \tag{52}$$
$$(I - \Delta t A)x^{k+1} = x^k \tag{53}$$
$$x^{k+1} = (I - \Delta t A)^{-1} x^k \tag{54}$$

Now, the eigenvalues of $(I - \Delta t A)^{-1}$ are $\{(1 - \Delta t\lambda)^{-1}\}$, where $\lambda = -1, -500$. Again, for stability we need $|(1 - \Delta t\lambda)^{-1}| < 1$, or $|1 - \Delta t\lambda| > 1$. This is satisfied by any $\Delta t > 0$.

Therefore, our choice of Δt only needs to account for integration error, and does not need to be as small as with forward Euler.

c) We plot the solutions approximated by forward and backward Euler with time steps around $\Delta t = \frac{1}{250}$, $x_0 = (1, 1)$, and a total time duration of 2 seconds on x_1 vs. x_0 plots:

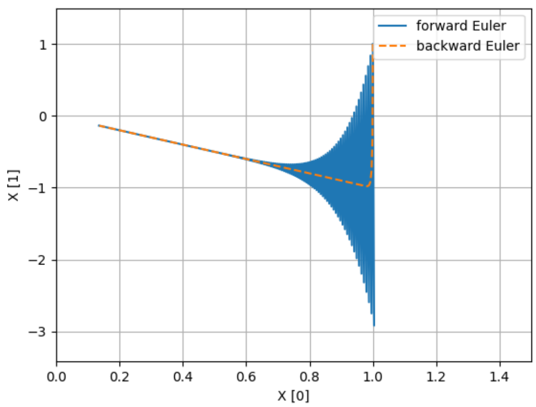

Figure 3: $\Delta t = 0.0039$

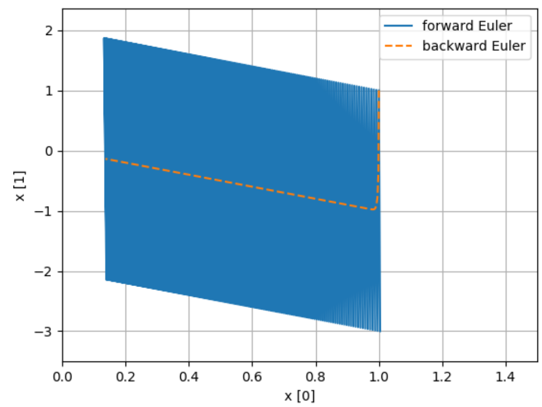

Figure 4: $\Delta t = 0.004$

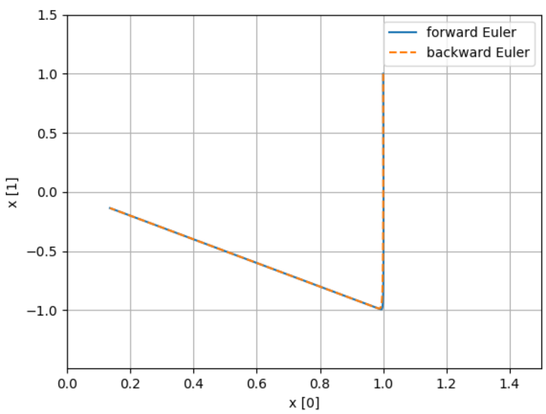

Figure 5: $\Delta t = 0.001$

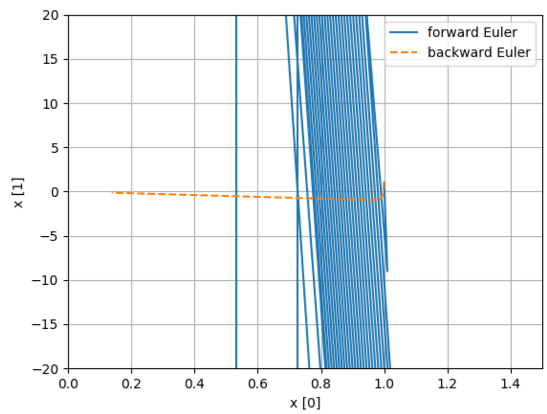

Figure 6: $\Delta t = 0.01$