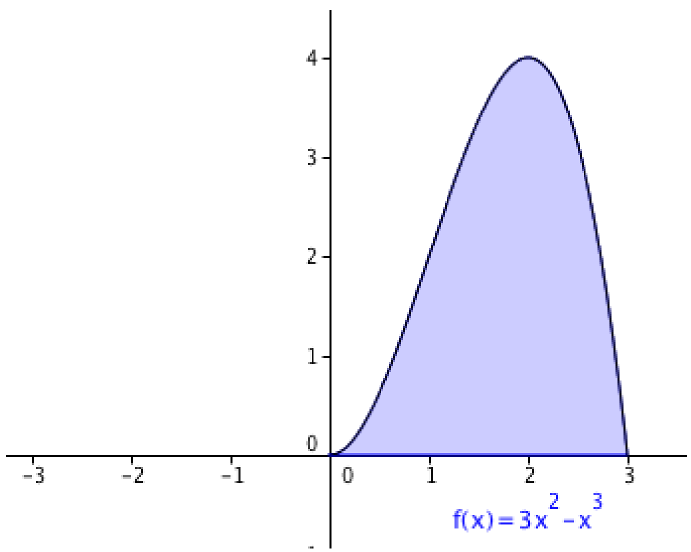
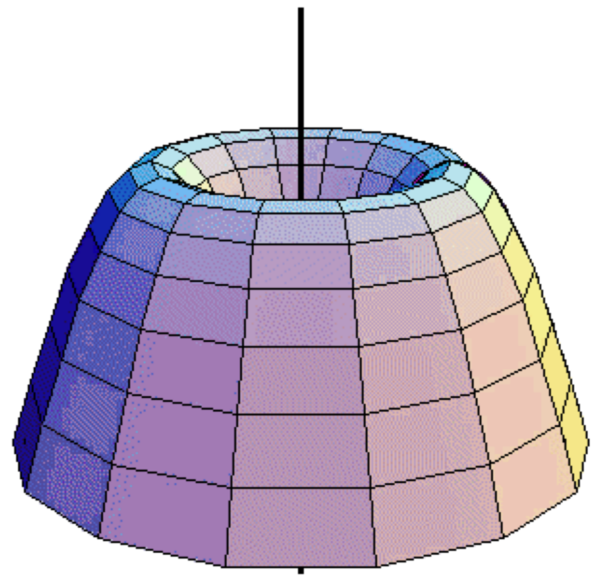
Volumes by Cylindrical Shells

1. **Problem.** Consider the region in the xy-plane bounded by the curves

$$y = 3x^2 - x^3 \text{ and } y = 0.$$

Imagine this region rotated about the y-axis. How do we find the volume of the resulting solid?



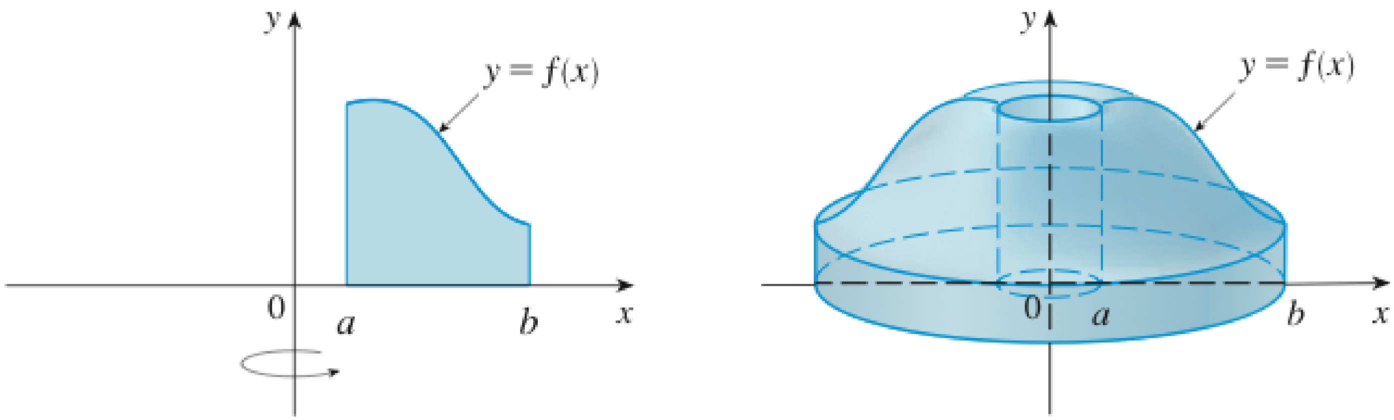


2. Exercise your imagination!

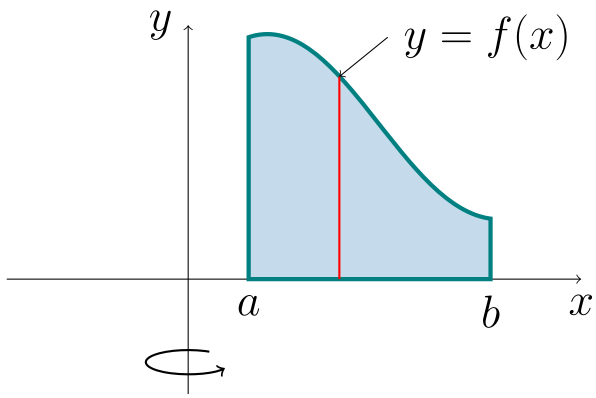
Let $0 \leq a < b$ and let a function f be continuous on $[a, b]$ with $f(x) \geq 0$. Let R be the region bounded by

$$y = f(x),\ y = 0,\ x = a,\ \text{and } x = b.$$

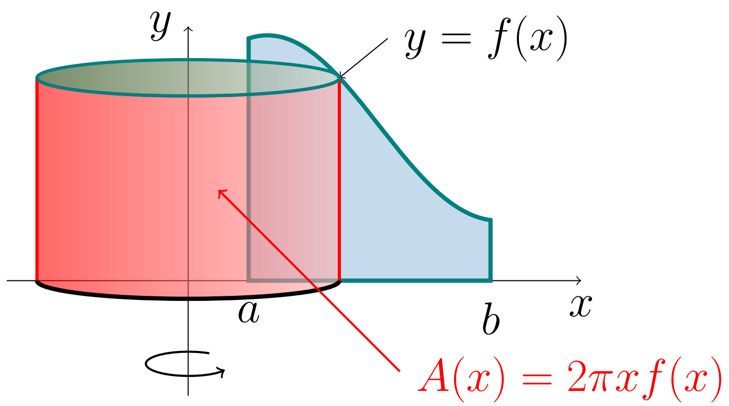
If we rotate R about the y-axis, we get a solid volume S.



Next, take an $x \in [a, b]$. Let L_x be the line segment inside the region R, between the points $(x, 0)$ and $(x, f(x))$. Imagine that L_x is colored red. Now rotate L_x about the y-axis. Do this slowly so that you can see how a red cylinder with the radius x and the height $f(x)$ emerges. This is your cylindrical shell, called C_x. The shell C_x is made of ”skin” only. To calculate its surface we cut it along the line segment L_x and then flatten it to obtain a rectangle with the width $2\pi x$ and the height $f(x)$. Thus the surface of C_x equals $A_x = 2\pi x f(x)$.



y
$y = f(x)$
a
b
x



y
$y = f(x)$
a
b
x
$A(x) = 2\pi x f(x)$

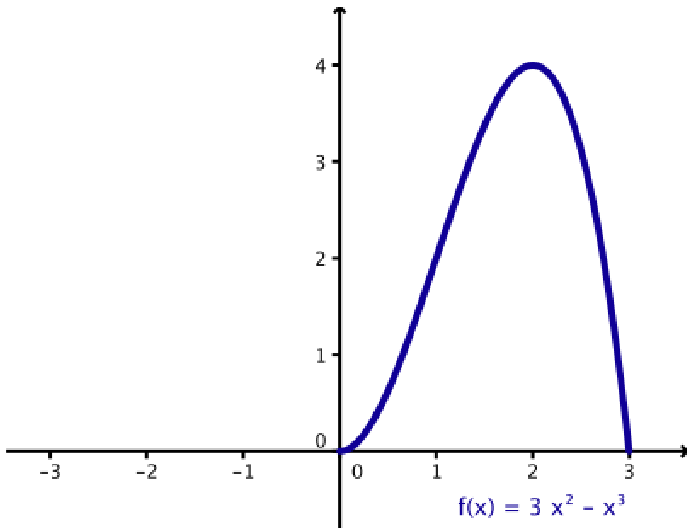
Almost there...

Note that each point of the solid S belongs to only one cylindrical shell C_x, for some $x \in [a, b]$. So we can imagine that S is obtained by gluing all cylindrical shells together. Each cylindrical shell contributes its surface (or ”skin”!) to the volume of S, or, in other words, the volume is the ”sum” of all surfaces. Each $x \in [a, b]$ gives one shell C_x with a surface area A_x, and so the ”sum” of all of them is given by

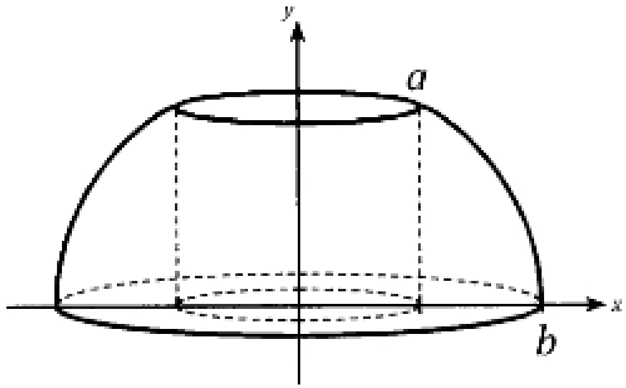
$$V = \int_a^b A_x dx = 2\pi \int_a^b x f(x) dx.$$

3. **Example.** Find the volume of the solid obtained by rotating about the y-axis the region bounded by curves

$$y = 3x^2 - x^3 \text{ and } y = 0.$$

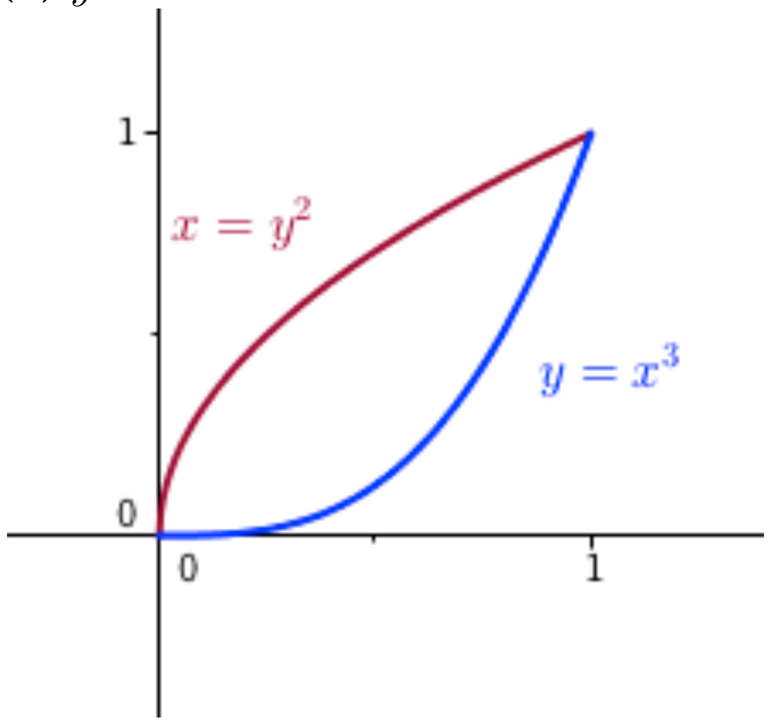


4. **Example.** Find the volume of the solid that remains after you bore a circular hole of radius a through the center of a solid sphere of radius $b > a$.

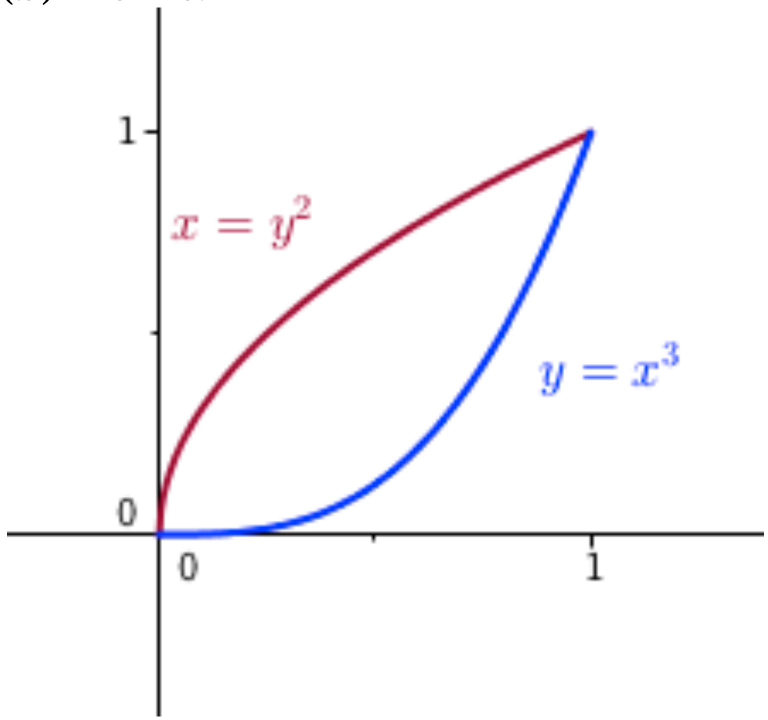


5. **Example.** Consider the region in the first quadrant bounded by the curves $y^2 = x$ and $y = x^3$. Use the method of cylindrical shells to compute the volume of the solid obtained by revolving this region around

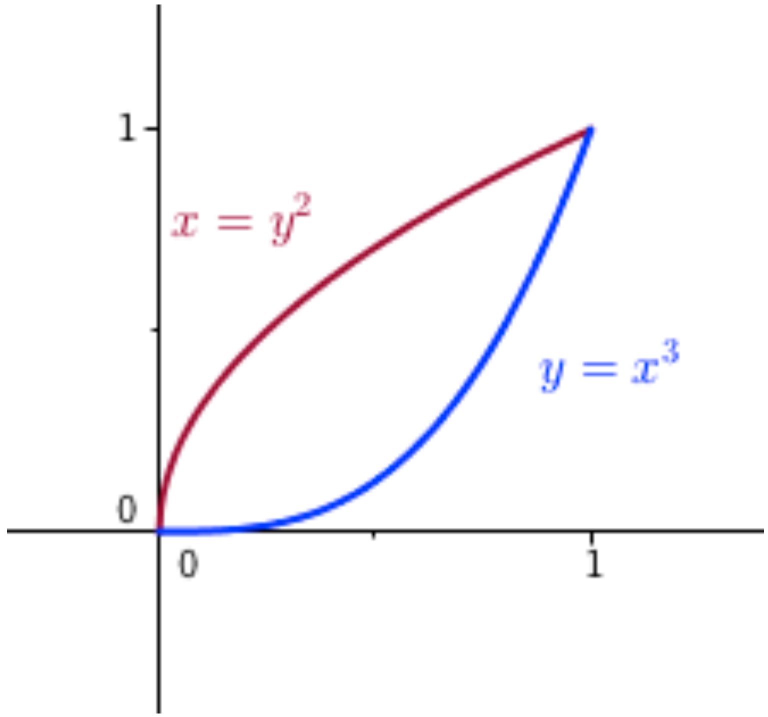
(a) y-axis.



(b) x-axis.



(c) line $x = 1$.



6. **Summary:** A good rule of thumb for which method to use is the following.

- If the area section (strip) is **parallel** to the axis of rotation, use the **shell method**.

- If the area section (strip) is **perpendicular** to the axis of rotation, use the **washer method**.

Notes: