Phil 320

Chapter 13: The Existence of Models

[Note: Omit 13.5]

The objective of the chapter is to prove the **Compactness Theorem** [and along the way the **Lowenheim-Skolem Theorem**]. Throughout, Γ is a set of sentences.

> **Compactness:** Suppose every finite subset of Γ is satisfiable (has a model). Then Γ is satisfiable (has a model).
>
> **Lowenheim-Skolem:** If Γ has a model, then Γ has an enumerable model.

For convenience, we work with a stripped-down language: connectives ~, ∨ and ∃ (others are treated as abbreviations).

The proof is very long. Best approach: (a) get a sense for the overall structure of the argument, by breaking the proof into parts (section 13.1), and (b) work through each part (13.2 - 13.4).

13.1 Outline of the proof

This picture illustrates the logical structure of the results of chapter 13.

13.5 (Term models lemma) → **13.3 (Model existence lemma)**

13.6 (Closure Lemma) → **13.3 (Model existence lemma)**

13.3 (Model existence lemma) → **Compactness + Lowenheim-Skolem**

13.2 (Finite character lemma) → **Compactness + Lowenheim-Skolem**

However, it's easier to use the following sequence of steps. Note: this changes the order of presentation from that of the textbook.

Step 1) Start with Γ and $\mathscr{L}$. The sentences of Γ belong to $\mathscr{L}$. Our only assumption is that every finite subset of Γ is satisfiable.

Step 2) Γ is *extendable*. That is, Γ belongs to a family S* that has the *satisfaction properties*. This is the **Finite Character lemma**. (**Section 13.1**)

Step 3) Extend Γ to Γ*. Extend $\mathscr{L}$ to $\mathscr{L}$+ by adding denumerably many extra constants. Add sentences to Γ until you get a set Γ* of sentences in $\mathscr{L}$+ that has the *closure properties* (contains 'enough of' its consequences). This is the **Closure Lemma**. (**Section 13.4**)

Step 4) Construct a model for Γ*. Given any Γ* with the closure properties, we can construct a model $\mathscr{M}$ for Γ* — this is the **Term Models Lemma**. This model has a special feature: every element of the domain is denoted by some closed term. In particular, this implies the domain is enumerable. (**Sections 13.2 and 13.3**)

Step 5) $\mathscr{M}$ is also a model for Γ. This is the **Model Existence Lemma**, which just summarizes steps 3 and 4.

Conclusion: Γ has a model with enumerable domain. This gives **Compactness** and **Lowenheim-Skolem** as well. If Γ has a model, we can produce an enumerable model.

<u>Three important concepts</u>

i) $\mathscr{L}$ and $\mathscr{L}$+. If $\mathscr{L}$ is a language, $\mathscr{L}$+ is obtained by adding denumerably many *new* constants c_1, c_2, … to $\mathscr{L}$. We will eventually construct a model for Γ each element of whose domain is denoted by some term in $\mathscr{L}$+.

ii) The satisfaction properties. Let S be a family of sets of sentences. [Each member of S is a set of sentences.] Then S has the *satisfaction properties* if the following hold (roughly, they say that each member of S behaves like a set of *satisfiable* sentences, and that each such member can be *expanded* in ways that would preserve satisfiability).

(S0) If Γ is in S and Γ_0 is a subset of Γ, then Γ_0 is in S.
(S1) If Γ is in S, then for no A are both A and $\sim A$ in Γ.
(S2) If Γ is in S and $\sim\sim B$ is in Γ, then $\Gamma \cup \{B\}$ is in S.
(S3) If Γ is in S and $(B \vee C)$ is in Γ, then either $\Gamma \cup \{B\}$ or $\Gamma \cup \{C\}$ is in S.
(S4) If Γ is in S and $\sim(B \vee C)$ is in Γ, then $\Gamma \cup \{\sim B\}$ and $\Gamma \cup \{\sim C\}$ are in S.
(S5) If Γ is in S and $\exists x\, B(x)$ is in Γ, and c does not occur in Γ or in $\exists x\, B(x)$, then $\Gamma \cup B(c)$ is in S.
(S6) If Γ is in S and $\sim\exists x\, B(x)$ is in Γ, then for every closed term t, $\Gamma \cup \{\sim B(t)\}$ is in S.
(S7) If Γ is in S, then $\Gamma \cup \{t = t\}$ is in S for any closed term t in the language of Γ.
(S8) If Γ is in S and $B(s)$ and $s = t$ are in Γ, then $\Gamma \cup \{B(t)\}$ is in S.

If S is the family of all sets that are satisfiable, then S has all of these properties. In fact, the properties also hold for S^*, the family of all sets <u>whose every finite subset</u> is satisfiable. This is basically the point of the **Finite Character Lemma**.

iii) The closure properties. The *closure properties* for a set Γ^* of sentences in the language $\mathcal{L}+$ say, roughly, that Γ^* contains no contradictions and Γ^* contains 'enough' of its consequences: 'most' consequences of Γ^* are already in Γ^*. (C5) gives us in addition a concrete 'witness sentence' in Γ^* for each existential sentence.

(C1) For no A are both A and $\sim A$ in Γ^*.
(C2) If $\sim\sim B$ is in Γ^*, then B is in Γ^*.
(C3) If $B \vee C$ is in Γ^*, then B is in Γ^* or C is in Γ^*.
(C4) If $\sim(B \vee C)$ is in Γ^*, then $\sim B$ and $\sim C$ are in Γ^*.
(C5) If $\exists x\, B(x)$ is in Γ^*, then for some closed term t of $\mathcal{L}+$, $B(t)$ is in Γ^*.
(C6) If $\sim\exists x\, B(x)$ is in Γ^*, then for every closed term t of $\mathcal{L}+$, $\sim B(t)$ is in Γ^*.
(C7) For each closed term t of $\mathcal{L}+$, $t = t$ is in Γ^*.
(C8) If $B(s)$ and $s=t$ are in Γ^*, then $B(t)$ is in Γ^*.

One easy result (not actually needed in our proof of compactness):

Proposition 13.4: Let $\mathcal{L}+$ be a language and $\mathcal{M}$ an interpretation in which each element of the domain is the denotation of a closed term. Then the set Γ^* of sentences true in $\mathcal{M}$ has the closure properties.

Proof: Most follow from the definition of an interpretation, and the fact that if $\mathcal{M} \models A$ and A logically implies B, then $\mathcal{M} \models B$. Only (C5) is special, and it follows from the fact that $\mathcal{M} \models B[m]$ for some m in the domain; since we can find a closed term t with $t^{\mathcal{M}} = m$, it follows that $\mathcal{M} \models B(t)$.

<u>**Step 2: Γ is extendable (Finite Character Lemma, §13.1)**</u>

Finite Character Lemma 13.2:
If S is a family of sets of sentences with the satisfaction properties, then the family S^* of all sets of sentences **whose every finite subset** is in S also has the satisfaction properties.

***Proof*:** We have to verify (S0) **-** (S8) for S^*. We know they hold for S.

(S0): If Γ in S^* and Γ_0 is a subset of Γ, then every finite subset of Γ is in S and hence every finite subset of Γ_0 is in S. So Γ_0 is in S^*.

(S1): If Γ is in S^* and both A, $\sim A$ are in Γ, then $\{A, \sim A\}$ is in S, contradicting (S1) for S.

(S2): Suppose Γ is in S^* and $\sim\sim B$ is in Γ. Any finite subset of $\Gamma \cup \{B\}$ is either a finite subset of Γ, and hence in S, or of form $\Gamma_0 \cup \{B\}$ where Γ_0 is a finite subset of Γ. But in the second case, $\Gamma_0 \cup \{\sim\sim B\}$ is still a finite subset of Γ and hence in S, so by (S2) for S, we have $\Gamma_0 \cup \{\sim\sim B\} \cup \{B\}$ in S and thus $\Gamma_0 \cup \{B\}$ in S by (S0). This shows $\Gamma \cup \{B\}$ is in S^*.

(S4) - (S8) are just like (S2).

(S3): Suppose Γ is in S^* and $(B \vee C)$ is in Γ. We must show either $\Gamma \cup \{B\}$ in S^* or $\Gamma \cup \{C\}$ in S^*. Any finite subset of $\Gamma \cup \{B\}$ is either a finite subset of Γ and hence in S (done), or of form $\Gamma_0 \cup \{B\}$, for Γ_0 a finite subset of Γ. If for all finite subsets Γ_0 we have $\Gamma_0 \cup \{B\}$ in S, we are done. If not, there is some Γ_0 such that $\Gamma_0 \cup \{B\}$ not in S. But then any finite subset of $\Gamma \cup \{C\}$ will be in S. This is obvious for finite subsets that omit C (since they are subsets of Γ); consider a finite subset $\Gamma_1 \cup \{C\}$. Then $\Gamma_0 \cup \Gamma_1 \cup \{(B \vee C)\}$ is a finite subset of Γ and hence in S, so by (S3) for S, $\Gamma_0 \cup \Gamma_1 \cup \{B\}$ or $\Gamma_0 \cup \Gamma_1 \cup \{C\}$ is in S. But not the first, because $\Gamma_0 \cup \{B\}$ is not in S and we can appeal to (S0). But then $\Gamma_0 \cup \Gamma_1 \cup \{C\}$ is in S and by (S0) again, $\Gamma_1 \cup \{C\}$ is in S.

Step 3: Extend Γ to Γ* (Closure Lemma, §13.4)

Lemma 13.6 (Closure lemma): Let $\mathcal{L}$ be a language and $\mathcal{L}+$ obtained by adding infinitely many new constants to $\mathcal{L}$. If S^* is a set of sets of sentences of $\mathcal{L}+$ having the satisfaction properties, then every set Γ of sentences of $\mathcal{L}$ in S^* can be extended to a set Γ^* of sentences in $\mathcal{L}+$ having the closure properties.

$\mathcal{L}+$ is obtained from $\mathcal{L}$ by adding infinitely many constants, S^* is a set of sets of sentences of $\mathcal{L}+$ having the satisfaction properties (S0)-(S8), and Γ is a set of sentences of $\mathcal{L}$ in S^*.

To show: Γ can be extended to a set Γ^* of sentences of $\mathcal{L}+$ with closure properties (C1) - (C8).

Idea of Proof: Obtain Γ^* by defining a sequence of progressively larger sets:

$\Gamma_0, \Gamma_1, \Gamma_2, \ldots$

where Γ_0 is just Γ, and each Γ_{n+1} is of the form $\Gamma_n \cup \{B_n\}$ – i.e., we add just one sentence at each stage. Then Γ_n is a subset of all Γ_k for $k>n$. We ensure that each Γ_n is still in S^*. Finally, Γ^* is the union of all the Γ_n. We make sure we've added enough so that Γ^* has the closure properties.

In more detail:
Each closure property other than (C1) raises *requirements* for what we need to add. For instance, if Γ_n contains $\sim\sim B$ but not B, then we need to add B at some stage. If Γ_n contains $(B \vee C)$, then we need to add either B or C at some stage. If Γ_n contains $\exists x B(x)$, then we need to add $B(t)$ for some closed term t at some stage. And so on for each of (C2) - (C8), as summarized on p. 161. Note that we need not fulfill these requirements at the very next stage. It is enough that they be met eventually; then the needed sentences will be included in Γ^*.

The properties (S2) - (S8) tell us that we can <u>start granting</u> each demand raised by (C2) - (C8) by adding a sentence, and the expanded set will remain in S^*. This is summarized on p. 161 as well. So all we need is an orderly way to add sentences to Γ to ensure that each original demand, and each new demand ever raised, is eventually met, and that nothing is added except to meet some demand. Then Γ^* will have the closure properties. Γ^* will also have the property (C1), because if A and $\sim A$ are in Γ^* they must also belong to some Γ_n, contradicting the fact that Γ_n is in S^* and so has property (S1).

So here is how to make sure every demand raised is eventually met:

(i) Write a complete enumeration of all sentences of the language:

$A_1, A_2, \ldots$

(ii) For each stage n: omit all sentences on this list except those for which a demand is raised, yielding a reduced (but still probably infinite) list of sentences demanded at stage n.

(iii) The ith sentence demanded at stage n (even for n=0) is precisely defined (using these lists).

(iv) Actually, we list the demands for stages 0,1,2,… in an infinite two-dimensional array **as each stage is defined** (i.e., as we add sentence B_n). Use slanting diagonals to put all of the demanded sentences into a single list. We add the nth member of this list (i.e., B_n) at stage n: i.e., add it to Γ_{n-1} to get Γ_n. In this way, we eventually add every sentence ever demanded to get Γ^* which has the closure properties.

Step 4: Construct a model for Γ*

13.2 First stage: Term Models Lemma, no identity or function symbols

Lemma 13.5 (Term models lemma): Let Γ^* be a set of sentences with the closure properties. Then there is an interpretation $\mathcal{M}$ in which every element of the domain is the denotation of some closed term, and every sentence in Γ^* is true in $\mathcal{M}$.

[**Note:** It's actually easier to find a model for the bigger "closed set" Γ^* than for the original Γ.]

Idea: Γ^* has the closure properties (C1) - (C8). They let us construct a 'simple' model $\mathcal{M}$ of Γ^* in which every element of the domain is the denotation of some constant in the language of Γ^*.

1) The domain $|\mathcal{M}|$

Pick a distinct object $c^{\mathcal{M}}$ for each distinct constant c in the language of Γ^*. This can always be done with some set of numbers (finite or all of N). The set of all such objects is the domain. Each element of the domain is the denotation of some constant in the language of Γ^*.

2) Denotations

Constants. We've already defined $c^{\mathcal{M}}$ for each c.
Predicates. Define $R^{\mathcal{M}}$ on n-tuples in the domain by:

$$R^{\mathcal{M}}(c_1^{\mathcal{M}},\ldots,c_n^{\mathcal{M}}) \leftrightarrow R(c_1,\ldots,c_n) \text{ is in } \Gamma^*.$$

3) $\mathcal{M}$ is a model of Γ^*

We prove this by induction on complexity. It's easiest to prove by induction that the following two statements hold for all sentences A (even though (1) is all we care about):

(1) For all A, if A is in Γ^*, then $\mathcal{M} \models A$.

(2) For all A, if $\sim A$ is in Γ^*, then $\mathcal{M} \models \sim A$.

Atomic case. A is $R(c_1,\ldots,c_n)$. Then $\mathcal{M} \models A$ iff A is in Γ^* by definition. So (1) and (2) both hold.

Non-atomic case.

Negation. Suppose (1) and (2) hold for A. Then (1) holds for $\sim A$: if $\sim A$ in Γ^*, then $\mathcal{M} \models \sim A$ is just what we mean by (2) holds for A. Also (2) holds for $\sim A$: if $\sim\sim A$ is in Γ^*, then by (C2), A is in Γ^*, and since (1) holds for A, we have $\mathcal{M} \models A$ and hence $\mathcal{M} \models \sim\sim A$.

Disjunction. Suppose (1) and (2) hold for each of A and B. Then (1) holds for $A \vee B$: if $A \vee B$ is in Γ^*, then either A or B is in Γ^* by (C3), so either $\mathcal{M} \models A$ or $\mathcal{M} \models B$, and in either case, we get $\mathcal{M} \models (A \vee B)$. Also (2) holds for $A \vee B$: if $\sim(A \vee B)$ is in Γ^*, then by (C4), both $\sim A$ and $\sim B$ are in Γ^*, so $\mathcal{M} \models \sim A$ and $\mathcal{M} \models \sim B$. Hence $\mathcal{M} \models \sim(A \vee B)$, as required.

Existential quantification. Assume (1) and (2) hold for $A(t)$, each closed term t (in this case, each constant). Then (1) holds for $\exists x A(x)$: if $\exists x A(x)$ is in Γ^*, then by (C5), there is some closed term t such that $A(t)$ is in Γ^*, and hence $\mathcal{M} \models A(t)$. But then $\mathcal{M} \models \exists x A(x)$. Also (2) holds for $\exists x A(x)$: if $\sim\exists x A(x)$ is in Γ^*, then by (C6), for every closed term t, $\sim A(t)$ is in Γ^*. But then $\mathcal{M} \models \sim A(t)$ for each t, which ((C5) again) implies $\mathcal{M} \models \sim\exists x A(x)$.

13.3 Second stage: identity and function symbols are present

(a) We first define an **equivalence relation** between closed terms. It will be used to specify the domain of the interpretation.

Lemma 13.7: Let Γ^* be a set of sentence with the closure properties (C1) - (C8). If t and s are closed terms, put

$t \equiv s$ iff $t = s$ is in Γ^*.

Then:

(1) $t \equiv t$. [by (C7)]

(2) If $s \equiv t$, then $t \equiv s$ [by (C8), letting $B(x)$ be $x=s$: since $B(s)$ and $s=t$ are in Γ^*, so is $B(t)$, i.e., so is $t=s$]

(3) If $s \equiv t$ and $t \equiv r$, then $s \equiv r$ [by (C8), letting $B(x)$ be $x=r$: since $B(t)$ and $t=s$ belong to Γ^*, so does $B(s)$, or $s=r$]

(4) If $t_1 \equiv s_1, \ldots, t_n \equiv s_n$, then for any predicate R, $R(t_1, \ldots, t_n)$ is in Γ^* iff $R(s_1, \ldots, s_n)$ is in Γ^*. [Repeated application of (C8) gives one direction; use of (2) and then (C8) gives the other direction.]

(5) If $t_1 \equiv s_1, \ldots, t_n \equiv s_n$, then for any function symbol f, $f(t_1, \ldots, t_n) = f(s_1, \ldots, s_n)$ is in Γ^*. [Apply (C8) first to $f(t_1, \ldots, t_n) = f(x_1, t_2, \ldots, t_n)$, then to $f(s_1, t_2, \ldots, t_n) = f(s_1, x_2, \ldots, t_n)$, and so forth. Both (C7) and (C8) are needed.]

Upshot: $\equiv$ is an equivalence relation on closed terms, and we can pick any member of an equivalence class to determine whether an atomic sentence belongs to Γ^*.

(b) Continuing the proof of the **Term Models Lemma**: first suppose identity is present but no function symbols. We want to specify a model for Γ^*.

1) The domain, $|\mathcal{M}|$

Pick a distinct C^* for each distinct *equivalence class* C of constants. So if $c_1 = c_2$ belongs to Γ^*, then $[c_1]^* = [c_2]^*$. The domain $|\mathcal{M}|$ is all the objects C^*.

2) Denotations

Constants. Define $c^{\mathcal{M}} = [c]^*$ for each c: the object associated with c's equivalence class.

Then $c^{\mathcal{M}} = d^{\mathcal{M}}$ iff $[c] = [d]$ iff $c \equiv d$, i.e., iff $c = d$ is in Γ^*. So this gives us

(*) $\mathcal{M} \models c{=}d$ iff $c{=}d$ is in Γ^*.

Predicates. Define $R^{\mathcal{M}}$ by:

$$R^{\mathcal{M}}(C_1^*,\ldots,C_n^*) \leftrightarrow R(c_1,\ldots,c_n) \text{ is in } \Gamma^* \text{ for any } c_i \text{ in } C_i. \text{ (Uses (4) of } \mathbf{13.7}.)$$

(**) Once again, $\mathcal{M} \models R(c_1,\ldots,c_n)$ iff $R(c_1,\ldots,c_n)$ is in Γ^*.

(*) and (**) give us the **two atomic cases** for the proof of "if A is in Γ^* then $\mathcal{M} \models A$", and "if $\sim A$ is in Γ^*, then $\mathcal{M} \models \sim A$". The rest of the proof is identical to the **first stage**.

(c) Finally, bring in function symbols. We have to construct a model for Γ^*.

1) The domain, $|\mathcal{M}|$

Pick a distinct object T^* for each distinct *equivalence class* T of closed terms. The domain consists of all the objects T^*.

2) Denotations

Constants. Again, $c^{\mathcal{M}} = [c]^*$ for each c: the object associated with c's equivalence class.

Predicates. No change.

Function symbols. Set $f^{\mathcal{M}}(T_1^*,\ldots,T_n^*) = T^*$ where $T = [f(t_1,\ldots,t_n)]$ for any t_i with $T_i = [t_i]$.

This is well-defined by (5) of Lemma 13.7. So we have

$$f^{\mathcal{M}}([t_1]^*,\ldots,[t_n]^*) = [f(t_1,\ldots,t_n)]^*.$$

To prove "A in $\Gamma^* \Rightarrow \mathcal{M} \models A$" and "$\sim A$ in $\Gamma^* \Rightarrow \mathcal{M} \models \sim A$", all we need to do is to show that for an arbitrary closed term, $t^{\mathcal{M}} = [t]^*$. The above proofs for atomic and non-atomic cases then apply.

Atomic case: constants. We still have $c^{\mathcal{M}} = [c]^*$.
Nonatomic case: Assume the result for $t_1,\ldots,t_n$. Then

$$\begin{aligned}(f(t_1,\ldots,t_n))^{\mathcal{M}} &= f^{\mathcal{M}}(t_1^{\mathcal{M}},\ldots,t_n^{\mathcal{M}}) \\ &= f^{\mathcal{M}}([t_1]^*,\ldots,[t_n]^*) \\ &= [f(t_1,\ldots,t_n)]^*, \quad \text{as required.}\end{aligned}$$

We are done! The **Model Existence Lemma** follows. From our original set Γ of sentences of $\mathcal{L}$, (i) extend to Γ^* in $\mathcal{L}+$ with the closure properties, and then (ii) find a model of Γ^* in which every element is denoted by some closed term. This will also be a model of Γ (with an enumerable domain).