Power Series

1. **Quote.** "Knowledge is power." (Francis Bacon, English Philosopher, 1561-1626)

2. **Quote.** "When the power of love overcomes the love of power the world will know peace." (Jimi Hendrix, American rock guitarist, singer, and songwriter, 1942-1970)

3. **Power Series - Motivation.**

   We know a lot about polynomials, $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$; polynomials are easy to evaluate!
   If we look at values of p and its derivatives at $x = 0$ we find that

   $p(0) = c_0$; $p'(0) = c_1$, $p''(0) = 2c_2$, $p'''(0) = 3 \cdot 2c_3$, ..., $p^{(n)}(0) = n!c_n$, and $p^{(k)}(x) \equiv 0$, for $k > n$.

   We've just learned about sequences and series; we can apply this knowledge to taking limits of polynomials in the sense of considering the infinite series

   $$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots$$

   One of the simplest and most important examples is our familiar geometric series

   $$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots = \frac{1}{1-x} \text{ if } |x| < 1.$$

4. **Power Series.**

A polynomial is a function of the form

$$p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

where n is a nonnegative integer, and the numbers $c_0, c_1, \ldots, c_n$ are constants called coefficients of the polynomials.

We define a **power series** to be an infinite series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

where x is a variable and the numbers $c_0, c_1, \ldots$ are constants called the **coefficients** of the series.

Note. For each value of x we have an "ordinary" infinite series. For those x for which the series converges, it defines a function of x.

A **power series in** x – a, or a **power series centered at** a, has the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots.$$

5. **Example.** For what values of $x \in \mathbb{R}$ is the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{n \cdot 4^n}$$

convergent?

Back to the geometric series.

(a)

$$\sum_{n=0}^{\infty} cx^n = c + cx + cx^2 + cx^3 + cx^4 + \ldots = \frac{c}{1-x} \text{ if } |x| < 1.$$

For example, take $c = 5$, $x = 1/10$, and you get $\frac{5}{9} = 5.55555\ldots$

(b) Derivative:

$$g(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$$

$$g'(x) = \sum_{n=0}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \ldots = \underline{\hspace{3cm}}.$$

[In a §11.2 i>clicker question we used this to compute $\sum_{n=0}^{\infty} n\left(\frac{1}{2}\right)^{n+1} = 1$.]

(c)

$$g'(x) - g(x) = \sum_{n=0}^{\infty} nx^{n-1} - \sum_{n=0}^{\infty} x^n = 1 + 2x + 3x^2 + 4x^3 + \ldots - (1 + x + x^2 + x^3 + \ldots)$$

$$= x + 2x^2 + 3x^3 + \ldots = \frac{1}{(1-x)^2} - \frac{1}{1-x} = \frac{x}{(1-x)^2}.$$

We get the same series by multiplying the derivative $g'(x)$ by x.

(d) Multiply the geometric series by itself:

$$\begin{aligned}\frac{1}{(1-x)^2} = [g(x)]^2 =& (1+x+x^2+x^3+\dots)(1+x+x^2+x^3+\dots)\\ =& 1+2x+3x^2+4x^3+\dots = g'(x).\end{aligned}$$

So we have a solution to the differential equation

(e) We know how to integrate $\frac{1}{1-x}$ as well as the monomials x^n:

$$\begin{aligned}\int_0^x (1+t+t^2+t^3+\dots)dt =& x+\frac{x^2}{2}+\frac{x^3}{3}+\dots = \int_0^x \frac{1}{1-t}\,dt =\\ \int_0^x (1-t+t^2-t^3+\dots)dt =& x-\frac{x^2}{2}+\frac{x^3}{3}+\dots = \int_0^x \frac{1}{1+t}\,dt =\end{aligned}$$

(f) We can replace x by x^2 or $-x^2$ in the geometric series:

$$\sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + x^6 + x^8 + \cdots = \frac{1}{1-x^2}$$

$$\sum_{n=0}^{\infty} (-x)^{2n} = 1 - x^2 + x^4 - x^6 + x^8 + \cdots = \frac{1}{1+x^2}$$

If we integrate the second equation we get

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} = \int_0^x \frac{1}{1+t^2}\, dt =$$

and,

6. **Example.** The function J_1 defined by

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} \left(\frac{x}{2}\right)^{2n+1}$$

is called the *Bessel function of the first kind*. [It is a solution to the differential equation $x^2y''(x) + xy'(x) + (x^2 - 1)y(x) = 0$.] What is the domain of J_1. (Note that $0!$ is by definition equal to 1 – see Assignment 6.)

7. **Where does a power series converge?**

Theorem.

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

(a) The series converges only when $x = a$.

(b) The series converges for all $x \in \mathbb{R}$.

(c) There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

Terminology.

- R - the **radius of convergence**
- the **interval of convergence** - the interval that consists of all values of x for which the series converges

8. **Example.** Find the interval of convergence of the following series.

(a) $\displaystyle\sum_{n=1}^{\infty} n^n x^n$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n \cdot 3^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{(2n)!}$

(d) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

Notes.