Alternating Series

1. **Quote.** ”Alternating periods of activity and rest is necessary to survive, let alone thrive. Capacity, interest, and mental endurance all wax and wane. Plan accordingly”

   (Tim Ferris, American podcaster, author, entrepreneur, 1977 –)

2. **Problem.** We have already seen that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

   But what happens if we alternately add and subtract the terms instead?

   Is the series
   $$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$
   convergent or divergent?

3. **Alternating series.**

If $\{b_n\}$ is a sequence of positive numbers then

$$b_1 - b_2 + b_3 - b_4 + \dots$$

is called an alternating series.

4. **The Alternating Series Test.**

If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \dots, (b_n > 0)$$

satisfies

(a) $b_{n+1} \leq b_n$ for all n

(b) $\lim_{n \to \infty} b_n = 0$

then the series is convergent.

5. **Example.** Test if

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

is convergent or divergent.

6. **Example.** Test for convergence

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{n^2+2n+1}$

(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(\frac{\pi}{2}-\arctan n\right)$

(c) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}$

7. **Alternating Series Estimation Theorem.**

If $s = \sum_{n=1}^{\infty} (-1)^n b_n$ is the sum of an alternating series which satisfies

$$\text{(i)} \quad 0 \le b_{n+1} \le b_n \quad \text{and} \quad \text{(ii)} \quad \lim_{n\to\infty} b_n = 0$$

then

$$|R_n| = |s - s_n| \le b_{n+1}.$$

8. **Example.** Use the fact that

$$\frac{1}{e} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \dots$$

to compute e^{-1} with an error of less than 10^{-2}.