Phil 320

Chapter 12: Models

Let Γ be a set of sentences. A *model* of Γ is an interpretation that makes each sentence in Γ true. We are interested in knowing what kinds of models a given set of sentences has.

Initial examples:

(i) Γ is a set of valid sentences. Any interpretation at all is a model.
(ii) Γ is $\{\exists x\, \forall y\, y=x\}$. Any interpretation whose domain has one object is a model.
(iii) Γ is the set of sentences of arithmetic true in the standard interpretation. There are many distinct models: N, non-negative rationals, non-negative reals.

12.1 Size and Number of Models; Isomorphisms

The *size* of a model M is the size of its domain $|M|$. For the examples above, (i) has models of all sizes; (ii) has only models of size 1; (iii) has only infinite models.

Example 1 (at least size *n*): For each n, let I_n be

$$\forall x_1\, \forall x_2 \ldots \forall x_{n-1}\, \exists x_n\, (x_n \neq x_1\ \&\ x_n \neq x_2\ \&\ \ldots\ \&\ x_n \neq x_{n-1}).$$

This sentence is true in M if and only if there are at least n objects in the domain of M.

Example 2 (at most *n*): The sentence $J_n = {\sim}I_{n+1}$ asserts that there are at most n objects.

Example 3 (exactly *n*): $I_n\ \&\ J_n$ is true iff the domain contains exactly n objects.

Example 4 (only infinite models):

a) (Infinite set Γ) $\Gamma = \{I_1, I_2, \ldots\}$, i.e., all the I_n's. If M is a model, then $|M|$ has at least n objects for each n, so it must be infinite.

b) (Finite set Γ) Let R be a 2-place predicate. Let Γ be:

(1) $\forall x\, \exists y\, R(x, y)$
(2) $\forall x\, \forall y\, {\sim}(R(x, y)\ \&\ R(y, x))$
(3) $\forall x\, \forall y\, \forall z((R(x, y)\ \&\ R(y, z)) \rightarrow R(x, z))$.

Γ has an infinite model: N, with R interpreted as <. Γ has no finite model.

Isomorphisms

Example 1: $L = \{<\}$ – a single 2-place predicate.

Interpretation M_1:	Interpretation M_2:
Domain $|M_1| = \mathrm{N}$	Domain $|M_2| = -\mathrm{N}$, i.e., $\{0, -1, -2, \ldots\}$
$<^{M1}$ is less-than	$<^{M2}$ is greater-than

These two interpretations of L are *isomorphic* because: a) there is a *correspondence* between the domains: a 1-1 function j from the domain $|M_1|$ onto the domain $|M_2|$, namely, $j(n) = -n$. Also, b) $m <^{M1} n$ iff $m < n$ iff $-m > -n$ iff $j(m) <^{M2} j(n)$, so j 'preserves' the relation < between interpretations.

Definition (Isomorphism):

Two interpretations P and Q of a language L are *isomorphic* if there is a 1-1 function j from the domain $|P|$ onto the domain $|Q|$ that satisfies the following three conditions for all predicates, constants and function symbols in L:

(I1) For every n-place predicate R and all $p_1,\ldots,p_n$ in $|P|$,

$$R^P(p_1,\ldots,p_n) \text{ iff } R^Q(j(p_1),\ldots,j(p_n))$$

(I2) For every constant c,

$$j(c^P) = c^Q$$

(I3) For every n-place function symbol f and all $p_1,\ldots, p_n$ in $|P|$,

$$j([f^P(p_1,\ldots,p_n)]) = f^Q(j(p_1),\ldots,j(p_n)).$$

Method for proving two interpretations are isomorphic:

1) Clearly indicate the language and the two interpretations, P and Q.
2) Write down the function j with domain $|P|$ and range $|Q|$, and explain why j is 1-1 and onto.
3) Show that clauses (I1), (I2) and (I3) are correct for each nonlogical symbol.

Example 2: L is $\{\mathbf{0}, \equiv\}$, where $\mathbf{0}$ is a constant and $\equiv$ is a two-place relation.

1) P: domain $|P|$ is N; $m \equiv^P n$ iff $n - m$ is even; $\mathbf{0}^P = 0$
Q: domain $|Q|$ is $\{0,2,4,6,\ldots\}$; $m \equiv^Q n$ iff $n - m$ is evenly divisible by 4; $\mathbf{0}^Q=0$.

2) $j(n) = 2n$ is 1-1 and onto.

3) $m \equiv^P n$ iff $n\text{-}m = 2k$ for some k iff $2(n\text{-}m) = 4k$ for some k
iff $2n \equiv^Q 2m$ iff $j(n) \equiv^Q j(m)$.

$j(\mathbf{0}^P) = j(0) = 2 \cdot 0 = 0 = \mathbf{0}^Q$.

Proposition 12.4 (interchangeable domains): Suppose P and Q are sets, and there is a 1-1 onto function j from P to Q. Suppose Q is any interpretation with domain Q. Then there is an interpretation P with domain P such that P is isomorphic to Q.

***Corollary*:** For any intepretation with a domain having n elements, there is an isomorphic interpretation whose domain is $\{0, 1, \ldots, n\text{-}1\}$. For any interpretation with an enumerably infinite domain, there is an isomorphic interpretation whose domain is N.

Proposition 12.5 (Isomorphism Lemma): If P and Q are isomorphic interpretations of the same language L, then for every sentence A of L, we have

(*) $P \models A$ iff $Q \models A$.

Upshot: Two isomorphic interpretations are the same in every important way. We say that two models have the same *isomorphism type* if they are isomorphic.

Corollary:

(a) Any set of sentences with a finite model has a model whose domain is $\{0, 1, \ldots, n\}$
(b) Any set of sentences with an infinite enumerable model has a model with domain N.

12.2 Equivalence Relations

Important for later work and a useful illustration of isomorphism types.

a) Basics

An *equivalence relation* $\equiv$ on a set is a two-place relation that is like "equality in some respect". For example:

1) People; $a \equiv b$ iff a and b have the same parents.
   [$a \equiv a$; if $a \equiv b$, then $b \equiv a$; if $a \equiv b$ and $b \equiv c$, then $a \equiv c$.]

2) Integers; $a \equiv b$ iff a-b is an even number.
   [$a \equiv a$; if $a \equiv b$, then $b \equiv a$; if $a \equiv b$ and $b \equiv c$, then $a \equiv c$.]

We want to look at interpretations for the language $\{\equiv\}$ which are models of the following three sentences (we refer their conjunction as the single sentence *Eq*):

(1) $\forall x\ x \equiv x$
(2) $\forall x\ \forall y\ (x \equiv y \to y \equiv x)$
(3) $\forall x\ \forall y\ \forall x\ ((x \equiv y\ \&\ y \equiv z) \to x \equiv z)$

We'll write E for the relation that is the denotation of $\equiv$. For any a, b and c, E will have to satisfy three properties corresponding to (1) - (3):

(1) $a\ E\ a$	(Reflexive)
(2) $a\ E\ b \to b\ E\ a$	(Symmetric)
(3) $(a\ E\ b\ \&\ b\ E\ c) \to a\ E\ c$	(Transitive)

Any relation satisfying these is called an *equivalence relation* on the domain X.

Def: If E is an equivalence relation on X, then the equivalence class of a, written [a], is the set of all b in X such that $b\ E\ a$.

Example 1: The equivalence class of a is all full siblings of a.

Example 2: The equivalence class of 0 is $\{0, \pm 2, \ldots\}$, i.e., the even numbers; the equivalence class of 1 is the odd numbers.

Definition: A *partition* of X is a set (family) of subsets of X satisfying two conditions:

(1) *Mutually exclusive*. No element of X belongs to more than one such subset.
(2) *Exhaustive*. Every element of X belongs to one such subset.

Proposition: Any partition of X gives rise to an associated equivalence relation E: aEb if and only if a and b belong to the same element of the partition. Any equivalence relation E on X gives rise to an associated partition.

b) Models

Focus on denumerable (= enumerably infinite) models of *Eq*, i.e., take N as the domain, and E as an equivalence relation on N that is the denotation of $\equiv$. We represent such a model by its *signature*: the number of equivalence classes with infinitely many elements, the number with 1 element, etc. List starting with the number of infinite classes:

(1, 0, 0, …)   Just one class with infinitely many elements.

$(0, \infty, 0, 0\ldots)$ Every number is a separate equivalence class.

Example 1: $\Gamma = \{Eq, \forall x\, \forall y\; x \equiv y\}$.

> $a\ E\ b$ for any two numbers a and b, so there is just one equivalence class consisting of all of N. The signature is (1,0,0,…).
>
> Any two infinite models of Γ are isomorphic: if the domain of M_1 is $\{a_1, a_2,\ldots\}$ and the domain of M_2 is $\{b_1, b_2,\ldots\}$, set $j(a_n) = b_n$ and j is an isomorphism.

Example 2: $\Gamma = \{Eq, \forall x\, \forall y\, (x \equiv y \leftrightarrow x = y)\}$

> Here, $a\ E\ b$ iff $a = b$, so each object is its own equivalence class. The signature is $\{0, \infty, 0, \ldots\}$. Again, any two infinite models of Γ are isomorphic, with $j(a_n) = b_n$.

Example 3: $\Gamma = \{Eq, \forall x\, \exists y\, (y \neq x\ \&\ x \equiv y\ \&\ \forall z\, (z \equiv x \rightarrow (z{=}x \vee z{=}y)))\}$

> Here, for each a there is exactly one b such that $b \neq a$ and $a \equiv b$. So each equivalence class has exactly two objects. Again, any two models are isomorphic.

12.3 Lowenheim-Skolem and Compactness Theorems

Lowenheim-Skolem Theorem: If a set of sentences Γ has a model, then it has an enumerable [finite or infinite] model.

Compactness Theorem: If every finite subset of a set of sentences Γ has a model, then Γ has a model.

Three consequences:

I. Corollary 12.16: If Γ has arbitrarily large finite models, then Γ has a denumerable (= enumerably infinite) model.

Proof: Let Γ* be all the sentences in Γ **plus** all of the sentences I_n: $\Gamma^* = \Gamma \cup \{I_1, I_2,\ldots\}$.

<u>Step 1</u>: Every finite subset of Γ* has a model.

<u>Step 2</u>: Γ* has a model.

<u>Step 3</u>: Γ has a denumerable model.

A set Γ of sentences is (*implicationally*) *complete* if for every *A* in the language, either *A* or ~*A* is a consequence of Γ.

A set Γ of sentences is *denumerably categorical* if any two denumerable models of Γ are isomorphic.

II. Corollary 12.17. If Γ is a denumerably categorical set of sentences having no finite models, then Γ is complete.

III. Corollary 12.18.

(a) If Γ has a model, then Γ has a model whose domain is $\{0,\ldots, n\}$ for some n, or else all of N.

(b) If Γ contains no function symbols or identity and has a model, then Γ has a model whose domain is all of N.