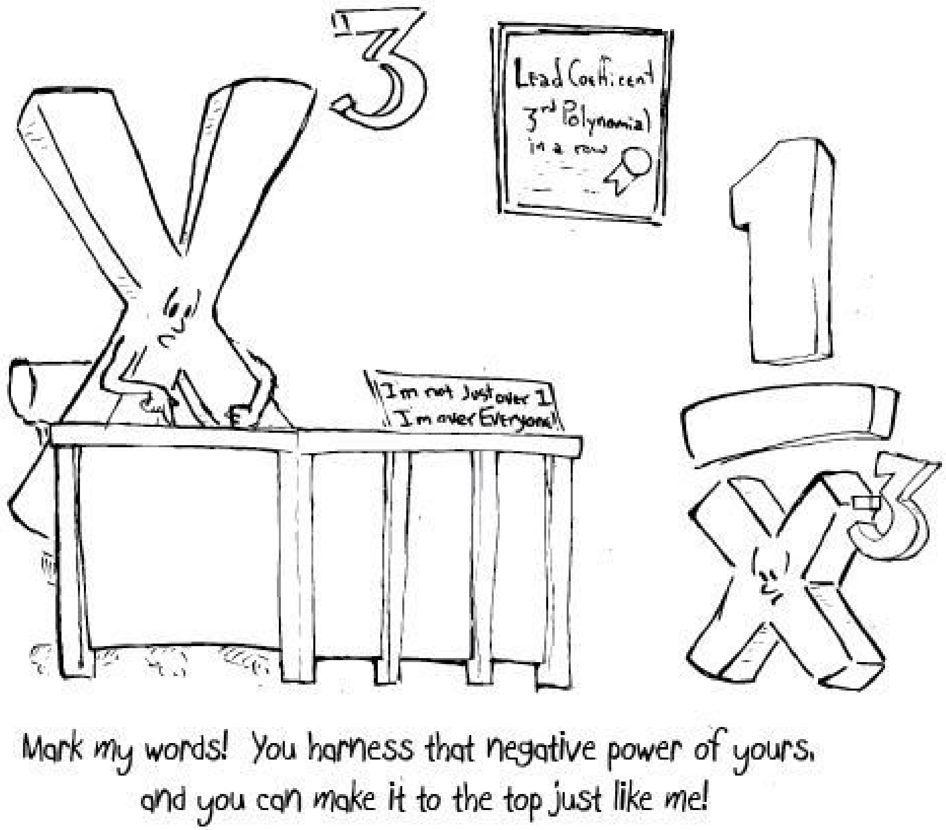
Integration of Rational Functions by Partial Fractions



1. **Problem.** Evaluate

$$\int \frac{x-1}{x^2-5x+6}dx.$$

2. **General Problem (Integrating Rational Functions).**

   Problem. Evaluate $\int \frac{P(x)}{Q(x)}dx$, where P and Q are polynomials.

   If $\deg P \geq \deg Q$ then (by long division) there are polynomials $q(x)$ and $r(x)$ such that

   $$\frac{P(x)}{Q(x)} = q(x) + \frac{r(x)}{Q(x)}$$

   and either $r(x)$ is identically 0 or $\deg r < \deg Q$. The polynomial q is the quotient and r the remainder produced by the long division process.

   If $r(x) = 0$, then $\frac{P(x)}{Q(x)}$ is really just a polynomial, so we can ignore that case here.

   Now $\int \frac{P(x)}{Q(x)}dx = \int q(x)dx + \int \frac{r(x)}{Q(x)}dx$.

   We can easily integrate the polynomial q, so the general problem reduces to the problem of integrating a rational function $\frac{r(x)}{Q(x)}$ with $\deg r < \deg Q$.

3. So, for the purposes of investigating how to integrate a rational function we can suppose $f(x) = \frac{P(x)}{Q(x)}$ with $\deg P(x) < \deg Q(x)$.

4. **Fact About Every Polynomial Q.**

   Q can be factored as a product of linear factors (i.e. of the form $ax + b$)

   and / or

   irreducible quadratic forms (i.e. of the form $ax^2 + bx + c$, where $b^2 - 4ac < 0$).

   Our strategy to integrate the rational function $f(x)$ is as follows:

   - Factor $Q(x)$ into linear and irreducible quadratic factors
   - Write $f(x)$ as a sum of **partial fractions**, where each fraction is of the form

   $$\frac{K}{(ax+b)^s} \quad \text{or} \quad \frac{Lx+M}{(ax^2+bx+c)^t}.$$

   - Integrate each partial fraction in the sum.

5. **Question.** How do we find K, L, and M?

   Let's look at some examples.

6. **Example.** Integrate

(a) $\displaystyle\int \frac{x-1}{x^2-5x+6}dx.$

(b) $\displaystyle\int \frac{4x^2 - 3x - 4}{x^3 + x^2 - 2x} dx$

$$x^3 + x^2 - 2x = x(x^2 + x - 2) = x(x+2)(x-1)$$

$$\frac{x-1}{x^2 - 5x + 6} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-1}$$

(c) $\displaystyle\int \frac{x^3-4x-1}{x(x-1)^3}dx$

$$\begin{aligned}\frac{x^3-4x-1}{x(x-1)^3} &= \frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}+\frac{D}{(x-1)^3}\\ x^3-4x-1 &= A(x-1)^3+Bx(x-1)^2+Cx(x-1)+Dx\\ &= A(x^3-3x^2+3x-1)+B(x^3-2x^2+x)+C(x^2-x)+Dx\end{aligned}$$

(d) $\displaystyle\int \frac{5x^3 - 3x^2 + 2x - 1}{x^4 + x^2} dx$

(e) $\displaystyle\int \frac{1}{x(x^2+1)^2}dx$

(f) $\displaystyle\int \frac{1}{(x^2+1)^2}dx$

(g) $\displaystyle\int \frac{1}{(x^2+x+1)}dx$

(h) $\int \sec(x)\, dx = \int \cos^{-1} x\, dx$.

That does not look like a rational function - does it? Rationalize!

Recall the transformation in Section 7.2 that seemed to come from nowhere:

$v = \sec x + \tan x = \frac{1+\sin x}{\cos x}$.

We have an integral involving powers (here negative) of $\cos$ **and** $\sin$**, with an odd power. Hence, we follow the instructions from 7.2, and set**

$u = \sin x, \qquad du = \cos x\, dx, \qquad \cos^2(x) = 1 - \sin^2(x) = 1 - u^2.$

$$\int \frac{1}{\cos x} dx = \int \frac{\cos x}{\cos^2 x} dx = \int \frac{1}{1-u^2}\, du \qquad \leftarrow \textbf{rational!}$$

$$\frac{1}{1-u^2} = \frac{\frac{1}{2}}{1+u} + \frac{\frac{1}{2}}{1-u}$$

$$\int \frac{1}{1-u^2}\, du = \frac{1}{2}\left(\ln(1+u) - \ln(1-u)\right) + C = \ln\sqrt{\frac{1+u}{1-u}} + C.$$

Now substitute $u = \sin x$**:**

$$\frac{1+u}{1-u} = \frac{1+\sin x}{1-\sin x} = \frac{(1+\sin x)^2}{1-\sin^2 x} =$$

7. The steps to integrate a rational function f: a technical look

Suppose $f(x) = \frac{P(x)}{Q(x)}$ with $\deg P < \deg Q$.

- **Step 1:** First factor $Q(x)$ into its linear and irreducible quadratic pieces. If there are n distinct linear factors and m distinct quadratic factors, then

$$Q(x) = (a_1 x + b_1)^{r_1} \dots (a_n x + b_n)^{r_n} (c_1 x^2 + d_1 x + e_1)^{s_1} \dots (c_m x^2 + d_m x + e_m)^{s_m}$$

- **Step 2:** The $f(x)$ can be written as a sum of **partial fractions** as follows

$$\begin{aligned}
\frac{P(x)}{Q(x)} &= \frac{A_{1,1}}{a_1 x + b_1} + \frac{A_{1,2}}{(a_1 x + b_1)^2} + \dots + \frac{A_{1,r_1}}{(a_1 x + b_1)^{r_1}} + \\
&\vdots \\
&+ \frac{A_{n,1}}{a_n x + b_n} + \frac{A_{n,2}}{(a_n x + b_n)^2} + \dots + \frac{A_{n,r_n}}{(a_n x + b_n)^{r_n}} + \\
&+ \frac{B_{1,1} x + C_{1,1}}{c_1 x^2 + d_1 x + e_1} + \frac{B_{1,2} x + C_{1,2}}{(c_1 x^2 + d_1 x + e_1)^2} + \dots + \frac{B_{1,s_1} x + C_{1,s_1}}{(c_1 x^2 + d_1 x + e_1)^{s_1}} + \\
&\vdots \\
&+ \frac{B_{m,1} x + C_{m,1}}{c_m x^2 + d_m x + e_m} + \frac{B_{m,2} x + C_{m,2}}{(c_m x^2 + d_m x + e_m)^2} + \dots + \frac{B_{m,s_m} x + C_{m,s_m}}{(c_m x^2 + d_m x + e_m)^{s_m}}
\end{aligned}$$

- **Step 3:** Integrate each partial fraction in the sum.

Notes.