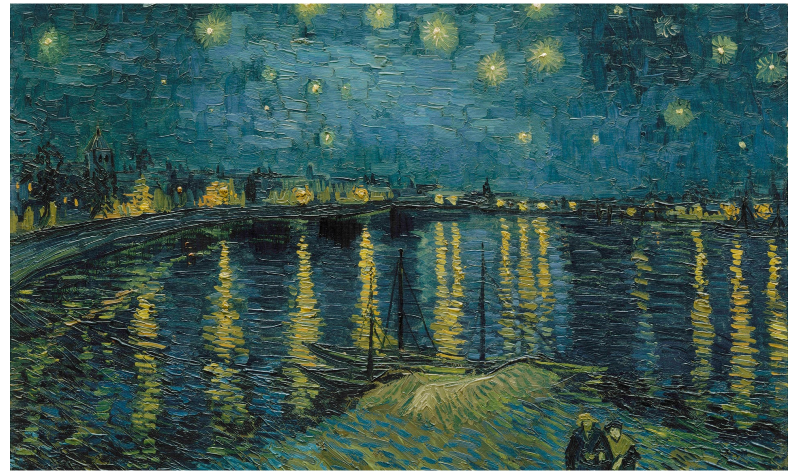
Taylor and Maclaurin Series

1. **Quote.** "Great things are done by a series of small things brought together."
   (Vincent Willem van Gogh, Dutch Post-Impressionist painter, 1853-1890)



2. **Problem.** Suppose the function f has a power series representation with radius of convergence R, that is,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad \text{for all } x \text{ such that } |x-a| < R,$$

   Can we express the coefficients c_n in terms of the function f?
   (Hint: We've already looked at this in the case of polynomials.
   What is the n^{th} derivative of f, evaluated at $x = a$? That is, calculate $f^{(n)}(a)$.)

3. **Theorem. (Power series representation is unique).**

If f has a power series representation at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad \text{for all } x \text{ such that } |x-a| < R,$$

then its coefficients are given by the formula $c_n = \dfrac{f^{(n)}(a)}{n!}$.

Here we adopt the convention that $f^{(0)}(x) = f(x)$; we already know that $0! = 1$.

So *if* a function f has a power series representation at a, then this representation *must be*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \tfrac{f'(a)}{1!}(x-a) + \tfrac{f''(a)}{2!}(x-a)^2 + \cdots$$

and this representation is called the **Taylor series of the function f at a.**

For the special case $a = 0$, the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) \ + \ \tfrac{f'(0)}{1!}x + \tfrac{f''(0)}{2!}x^2 + \cdots$$

and this is called the **Maclaurin series** of $f(x)$.

4. **Examples.** Find the Maclaurin series of the following functions.

(a) $f(x) = e^x$

(b) $f(x) = \cos x$

5. **Some Terminology.**

(a) $T_n(x) = \sum_{k=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^k$ is the nth-degree **Taylor polynomial** of f at a

That is, $T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

Notice that $\lim_{n\to\infty} T_n(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$, the Taylor series of f.

(b) The **remainder** of the Taylor series is defined as $R_n = f(x) - T_n(x)$.

6. **Theorem.**

Suppose $f(x) = T_n(x) + R_n(x)$, where T_n and R_n are as above. If

$$\lim_{n\to\infty} R_n(x) = 0, \quad \text{for } |x-a| < R,$$

then, on the interval $(a-R, a+R)$, the function f is equal to the sum of its Taylor series.

7. **Bounds on the size of the remainder.**

8. **Theorem.**

Suppose $f(x) = T_n(x)+R_n(x)$, where T_n and R_n are as above. If $f^{(n+1)}$, the $(n+1)^{st}$ derivative of f is continuous, then for every x there is a point t_x (note, it's not the same for all x) between a and x such that

$$R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(t_x)}{(n+1)!}(x-a)^{n+1}.$$

What does this theorem say for $n = 0$?

To show that any specific function f does have a power series representation, we must prove that $\lim_{n\to\infty} R_n(x) = 0$.

To do this, we usually use the following two facts.

Fact 1: Taylor's Inequality.

If

$$\left|f^{(n+1)}(x)\right| \leq M \textbf{ for } |x-a| \leq d$$

then the remainder of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \textbf{ for } |x-a| \leq d.$$

Fact 2. For every real number x, we have $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.

9. **Example.** Prove

(a) $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$, **for every real number** x

(b) $\cos x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, **for every real number** x

10. **Some important power series representations.**

These Maclaurin series can be derived just as in the previous examples.

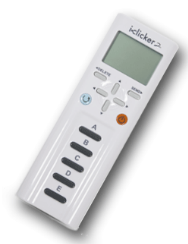
$$
\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n &&= 1 + x + x^2 + x^3 + \dots && (-1,1) \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} &&= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots && (-\infty,\infty) \\
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} &&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots && (-\infty,\infty) \\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} &&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots && (-\infty,\infty) \\
\arctan x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} &&= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots && (-1,1) \\
\ln(1+x) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} &&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots && (-1,1]
\end{aligned}
$$

11. **Example.** Find the Maclaurin series for the following functions.

(a) $f(x) = x^2 e^{-3x}$

(b) $g(x) = \sin(x^2)$

(c) $h(x) = \dfrac{x}{9 - x^2}$



12. **Example.** Find the sum of the series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots.$

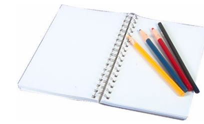
13. **Example.** Use series to evaluate

$$\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x}.$$

14. **Example.** Find the Taylor series for the following functions centered at the given value of a.

(a) $f(x) = e^{-x}$, $a = 1$

(b) $g(x) = \sin(2x)$, $a = \pi$



Notes.