Assignment 2: Optimization and Optimal Control

Due Mar. 5 at 23:59

This assignment is to be done individually.

Important Note: The university policy on academic dishonesty (cheating) will be taken very seriously in this course. You may not provide or use any solution, in whole or in part, to or by another student.

You are encouraged to discuss the concepts involved in the questions with other students. If you are in doubt as to what constitutes acceptable discussion, please ask! Further, please take advantage of office hours offered by the instructor and the TA if you are having difficulties with this assignment.

DO NOT:

- Give/receive code or proofs to/from other students
- Use Google to find solutions for assignment

DO:

- Meet with other students to discuss assignment (it is best not to take any notes during such meetings, and to re-work assignment on your own)
- Use online resources (e.g. Wikipedia) to understand the concepts needed to solve the assignment.

Submitting Your Assignment

The assignment must be submitted on Canvas. You must submit one zip file (student_number_a2.zip) containing:

1. An assignment report in **PDF format**, named `student_number_a2.pdf`. This report should contain your solutions to questions 1-4.
2. Your code for question 1, named `a2_q1.py` or `a2_q1.m`.
3. Your code for question 2, named `a2_q2.py` or `a2_q2.m`.
4. Your code for question 3, named `a2_q3.py` or `a2_q3.m`.
5. Your code for question 4, named `a2_q4.py`.

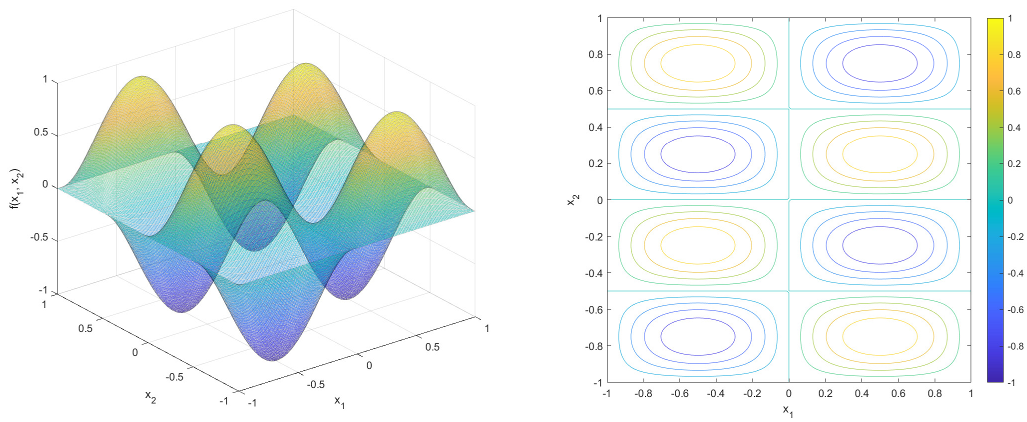

Figure 1: Blank plot for Question 1.

1 Sequential Quadratic Programming

Use Sequential Quadratic Programming, as introduced in class, to solve the NLP below. Solve the quadratic subproblems using `cvx`.

$$\begin{aligned} \text{minimize} \quad & \sin(\pi x_1)\sin(2\pi x_2) && (1)\\ \text{subject to} \quad & -1 \le x_1 \le 1 && (2)\\ & -1 \le x_2 \le 1 && (3) \end{aligned}$$

Notation:

- The decision variable is $x = (x_1, x_2)$.
- Subscripts denote vector components, and superscripts denote the iteration number.

a) Construct a quadratic subproblem that minimizes the quadratic approximation of the objective, subject to linearization of the constraints, centred around a given iterate x^k.

b) Write code to solve the above quadratic subproblem using cvx.

c) Write code to solve the entire NLP, starting from several different initial points x^0.

Make surface and contour plots showing several sequences of $\{x^k\}$ starting these initial points. The plots should follow the same axis limits as those in Figure 1.

2 Differential Flatness

Consider the following simple model of the car (that is slightly more complex than the one introduced in class):

$$\dot{x} = v\cos\theta \tag{4}$$
$$\dot{y} = v\sin\theta \tag{5}$$
$$\dot{\theta} = \omega \tag{6}$$
$$\dot{v} = a \tag{7}$$

The state consist of the position (x, y), the heading θ, and the longitudinal speed v. The controls are the turn rate ω and the longitudinal acceleration a.

a) Show that the system is differentially flat by letting $z = (x, y)$, and deriving the functions β and γ such that

$$(x, y, \theta, v) = \beta(z, \dot{z}, \ldots, z^{(q)}) \tag{8}$$
$$(\omega, a) = \gamma(z, \dot{z}, \ldots, z^{(q)}) \tag{9}$$

b) Consider a maneuver given by the following initial condition at $t = 0$ and final condition at $t = T = 10$:

$$\begin{bmatrix} x(0) \\ y(0) \\ \theta(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} x(T) \\ y(T) \\ \theta(T) \\ v(T) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{\pi}{2} \\ 1 \end{bmatrix} \tag{10}$$

Using the basis functions $\psi_0(t) = 1, \psi_1(t) = t, \psi_2(t) = t^2, \psi_3(t) = t^3$, we parameterize the flat outputs as follows:

$$x(t) = \sum_{i=0}^{3} b_{0i}\psi_i(t) \tag{11}$$
$$y(t) = \sum_{i=0}^{3} b_{1i}\psi_i(t) \tag{12}$$

Write a system of equations in terms of the coefficients $\{b_{0i}\}$ and $\{b_{1i}\}$ such that the initial and final conditions are satisfied.

c) Solve the above systems of equations using a software of your choice to obtain and plot the state and control trajectories. Use the format provided in the blank plot in Figure 2 to show the path of the car in (x,y) space, the trajectory of all four state variables over time, and the trajectory of all control variables over time.

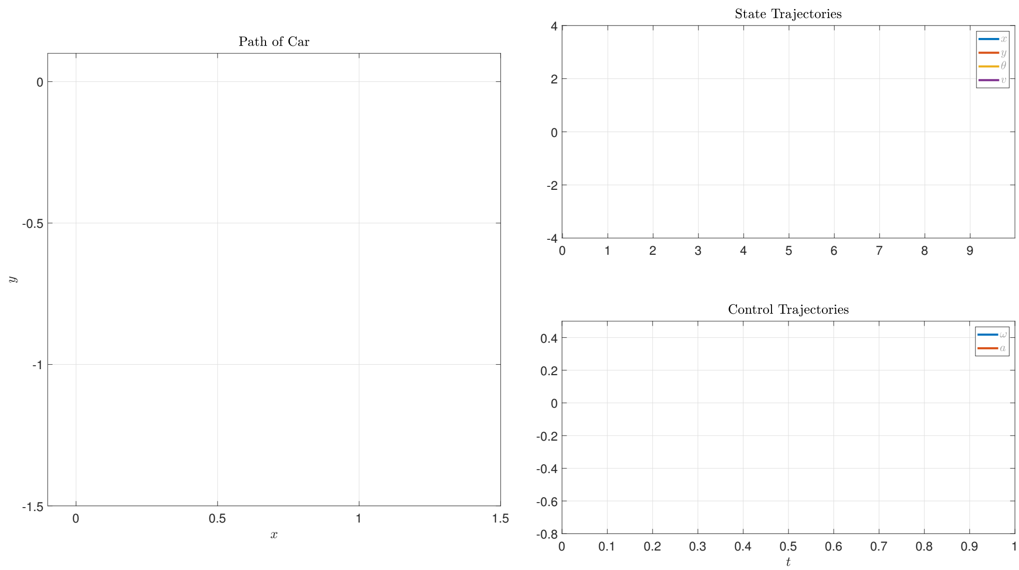

Figure 2: Blank plot for Question 2.

3 Multiple Shooting with Casadi

Consider a system involving an inverted pendulum on a cart. For this system, the state is (x, v, θ, ω), representing the position of the cart, speed of the cart, angle of the pendulum, and angular speed of the pendulum, respectively. The initial state at time $t = 0$ is $(0, 0, 0, 0)$.

The problem parameters are as follows: $M = 0.5$ kg is the mass of the cart, $m = 0.2$ kg is the mass of the pendulum, $l = 0.3$ m is the half-length of the pendulum, $I = 0.006\text{kg} \cdot \text{m}^2$ is the moment of inertial of the pendulum.

We would like to apply a force $F(t)$ such that at time $t = 5$ s, the cart is stopped at the origin, with the pendulum being stationary at the upright position, represented by the state $(0, 0, \pi, 0)$. At the same time, we would like to minimize the control effort, given by the integral $\int_{t=0}^{5} F^2(t)dt$, while ensuring that the position of the cart stays within 1 m of the origin, $|x| \leq 1$. The control is also limited by $|F(t)| \leq 0.2$ at all times.

The equations of motion are given by

$$(M + m)\ddot{x} + b\dot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2 \sin\theta = F \tag{13}$$

$$(I + ml^2)\ddot{\theta} + mgl\sin\theta = -ml\ddot{x}\cos\theta \tag{14}$$

where $b = 0.1$ N/m/s is the coefficient of friction, and g = 9.81 m/s/s is the acceleration due to gravity on Earth. In terms of a first-order ODE, we have $\dot{x} = v, \dot{\theta} = \omega$. The rest of the system dynamics, for (v, ω), can be implicitly written as

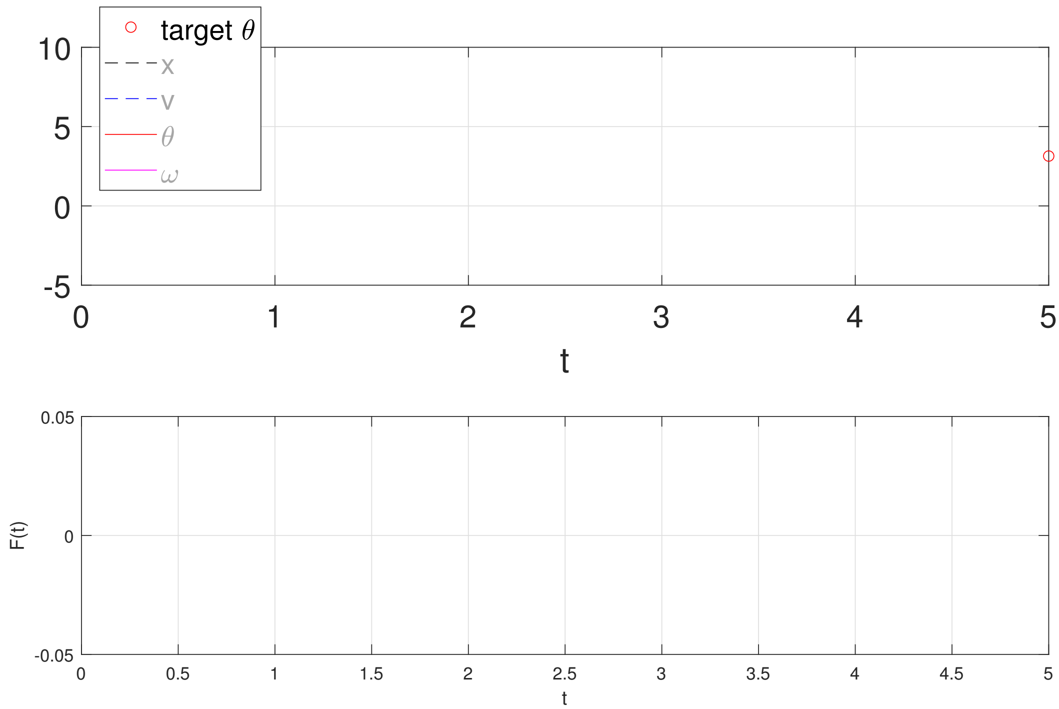

Figure 3: Blank plots for the cart pole problem.

$$\begin{bmatrix} M+m & ml\cos\theta \\ ml\cos\theta & I+ml^2 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} -bv + ml\omega^2\sin\theta + F \\ -mgl\sin\theta \end{bmatrix} \tag{15}$$

a) Write down the associated optimal control problem.

b) Discretize the optimal control problem using the multiple shooting method and write down the resulting nonlinear program. Use $N = 50$ time intervals, forward Euler integration, and left endpoint first order integration (the same schemes as presented in the lecture).

c) Using the Casadi toolbox, which can be found at `https://web.casadi.org/`, solve the nonlinear program, and plot $(x(t), v(t), \theta(t), \omega(t))$, and $F(t)$. A good starting point is the direct multiple shooting example in the example pack.

   Show your results by plotting the state and control trajectories. Please use the format given by the blank plot in Figure 3, and mark the desired value of $\theta(T)$ as shown.

4 Robotic Safety via Reachability Analysis

Consider a simple model of a fixed-wing aircraft flying in a windy environment, given as follows:

$$\dot{x} = v\cos\theta + d_x \tag{16}$$
$$\dot{y} = v\sin\theta + d_y \tag{17}$$
$$\dot{\theta} = \omega \tag{18}$$
$$\dot{v} = a \tag{19}$$

The state (x, y, θ, v) consists of the x- and y-position, heading θ, and speed v. The aircraft controls its turn rate ω and acceleration a, with $|\omega| \leq 0.5$ rad/s and $|a| \leq 10$ m/s^2. d_x and d_y represent the wind disturbance. For parts a-e, assume $d_x = d_y = 0$; we explore disturbances in part f.

There is a no-fly zone with a radius of 1 km around a military base located at the origin.

a) What is the target set $\mathcal{T}$ representing the set of states inside the no-fly zone? Your answer should be in the form "$\mathcal{T} = \{(x, y, \theta, v) : \ldots\}$".

b) What is a suitable function $l(x, y, \theta, v)$ such that $l(x, y, \theta, v) \leq 0 \Leftrightarrow (x, y, \theta, v) \in \mathcal{T}$?

c) Let z denote the state and u denote the control: $z = (x, y, \theta, v), u = (\omega, a)$. The value function $V(t, z)$ representing the set of states from which the aircraft can reach the target set within t seconds is the solution to the HJI variational inequality

$$\min\left\{\frac{\partial V}{\partial t}(t,z) + \max_{u\in\mathcal{U}}\min_{d\in\mathcal{D}}\frac{\partial V}{\partial z}(t,z)^\top f(z,u,d), l(z) - V(t,z)\right\} = 0, \tag{20}$$

where $f(z, u, d)$ is given by the system dynamics in (16)-(19). Determine the optimal u and d optimizes the max-min expression. That is, find an analytic expression for u^* and d^*, where

$$u^*(t, x, y, \theta, v) = \arg\max_{u\in\mathcal{U}}\min_{d\in\mathcal{D}}\frac{\partial V}{\partial z}(t,z)^\top f(z,u,d), \tag{21}$$
$$d^*(t, x, y, \theta, v) = \arg\min_{d\in\mathcal{D}}\frac{\partial V}{\partial z}(t,z)^\top f(z,u^*,d), \tag{22}$$

and $u \in \mathcal{U}$ and $d \in \mathcal{D}$ represent the bounds on the control and disturbances. For bounds on the disturbances, use $|d_x|, |d_y| \leq 25$ m/s, which you will implement in part e.

d) Compute $\lim_{t\to-\infty} V(t, z)$ using the optimized_dp toolbox, which can be found at `https://github.com/SFU-MARS/optimized_dp`. You will need to first follow the quick start guide. A good starting point is `user_definer.py`.

For your computation, please use the following recommended number of grid points in each dimension. This ensures sufficient accuracy, and consistency in assignment submissions.

- x: 50 grid points
- y: 50 grid points

- v: 30 grid points
- θ: 35 grid points (periodic)

Requirements:

- Choose the time horizon to be $t \in [-T, 0]$ for a T large enough for $V(t, z)$ to converge.
- Choose a grid bound large enough to contain the set $\{z : V(-T, z) \leq 0$.
- Run the solver and plot your results using the following code (these should be the last two lines of code in your code submission):

```
po = PlotOptions("3d_plot", [0,1,3], [15])
HJSolver(my_car, g, Initial_value_f, tau, "minVWithV0", po)
```

  where `my_car` is the object representing the dynamical system.

e) Repeat part d with the additional constraint that the aircraft can only fly with a minimum speed of 80 m/s, up to a maximum speed of 300 m/s.

In one sentence, comment on the difference between the results of part d and e.

Hint: Modify the target set.

f) The effect of wind can be modeled by the disturbance $d = (d_x, d_y)$ with $|d_x|, |d_y| \leq 25$ m/s. Repeat part e with this additional consideration.

In one sentence, comment on the difference between the results of part e and f.