CMPT 308 - Computability and Complexity: Homework 3 **(Due: Oct 27)**

1. Describe two different Turing machines, M and N, such that, when started on any inputs, M outputs $\langle N \rangle$ and N outputs $\langle M \rangle$.

2. Using the Recursion Theorem, complete the following, alternative proof that A_{TM} is undecidable.

   > Suppose A_{TM} is decidable by a decider TM H. Define a new TM M="On input w, get own description $\langle M \rangle$; simulate H on $\langle M, w \rangle$; if H accepts, then reject; if H rejects, then accept."

   Derive a contradiction by analyzing the question: Does M accept the empty string ϵ?

3. **Kolmogorov complexity**
   (a) Define $L = \{x \mid K(x) \geq |x|\}$, where $K(x)$ is the Kolmogorov complexity of the binary string x. Prove that L is undecidable.
   (b) Show that the set $\{x \mid K(x) \geq |x|\}$ of incompressible strings contains no infinite subset that is semi-decidable.

4. For each $m > 1$ let $Z_m = \{0, 1, \ldots, m-1\}$. Consider arithmetic formulas over Z_m where addition and multiplication operations are interpreted as addition modulo m and multiplication modulo m, respectively, and where the variables are assumed to take values from Z_m. Argue that, for each $m > 1$, the language of true arithmetic sentences over Z_m is decidable. That is, argue that for each fixed $m > 1$, there is an algorithm for deciding if a given arithmetic sentence over Z_m is true or false.

5. Recall that a proof in a proof system P is a sequence of formulas such that each formula is either an axiom of P or follows from some earlier formulas in the sequence by inference rules of P. Suppose that a proof system P has finitely many inference rules, but *infinitely* many axioms. However, the axioms are *enumerable* by a TM A: the TM A when run on the empty string ϵ will be outputting axioms a_i so that every axiom of P is eventually output. Define the language

   $$\mathit{Provable} = \{\langle \phi \rangle \mid P \text{ proves } \phi\}.$$

   Argue that the language *Provable* is semi-decidable.