# Problem Solving and Search

Chapter 3

# Outline

- Problem-solving agents
- Problem formulation
- Example problems
- Basic search algorithms

# Problem-Solving Agents

In the *simplest* case, an agent will:

- formulate (or be given) a goal and a problem;
- search for a sequence of actions that solves the problem;
- then execute the actions.

When done it may formulate another goal and start over.

- In this case the performance measure is simply whether or not the goal is attained.

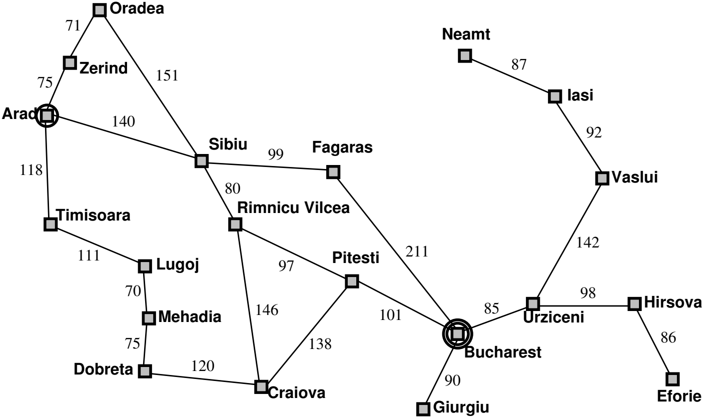
This is *offline* problem solving, executed "eyes closed."

- Requires complete knowledge about the domain
- *Online* problem solving involves acting without necessarily having complete knowledge.

## Example: Romania

- On holiday in Romania; currently in Arad.
  - Flight leaves tomorrow from Bucharest
- Formulate *goal*
  - Be in Bucharest
- Formulate *problem*
  - *states*: various cities
  - *actions*: drive between cities
- Find *solution*
  - Sequence of cities, e.g., Arad, Sibiu, Fagaras, Bucharest

# Example: Romania



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   - I.e. $RESULT(s, a) =$ state resulting from doing a in s.
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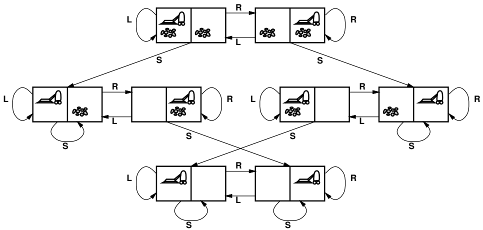
A *solution* is a sequence of actions from initial state to a goal state.

# Selecting a State Space

- The real world is highly complex and contains lots of irrelevant information.
  $\Rightarrow$ state space must be *abstracted* for problem solving
- (Abstract) state will have irrelevant detail removed.
- Similarly, actions must be at the right level of astraction
  - e.g., "Go(Zerind)" omits things like starting the car, steering, etc.
- (Abstract) solution =
  set of paths that are solutions in the real world

# Example: Vacuum World State Space Graph

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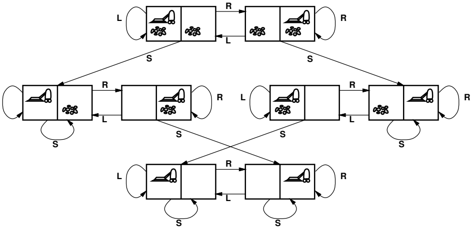


states: dirt and robot locations (so $2 \times 2^2$ possible states)

actions:

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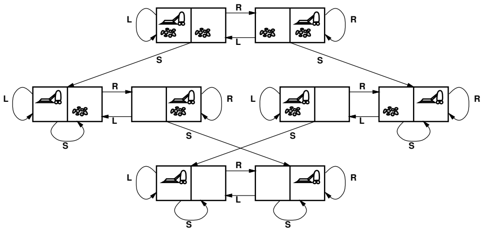


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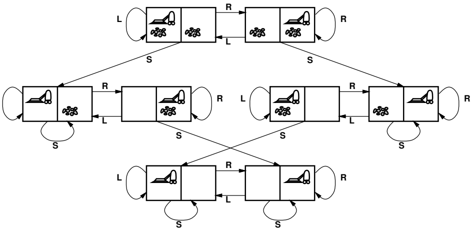


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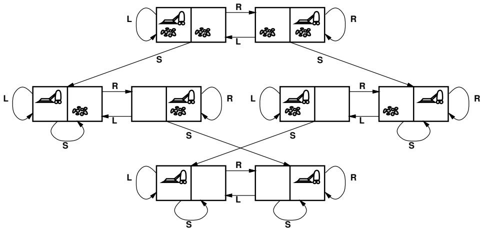


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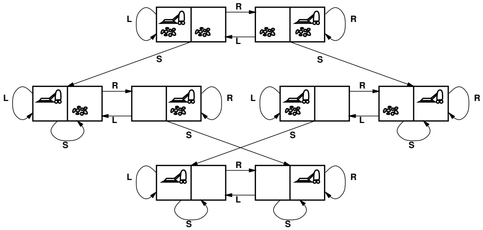


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path cost: 1 per action (0 for *NoOp*)

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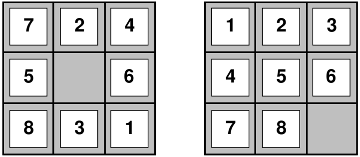


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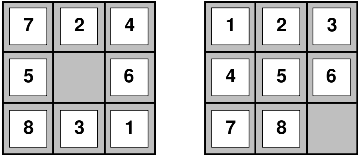


Start State Goal State

states: (integer) locations of tiles.

☞ Ignore intermediate positions

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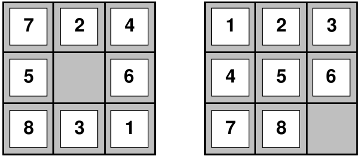

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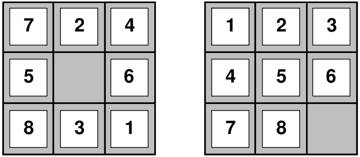


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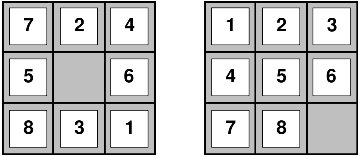


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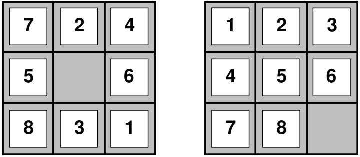

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path cost: 1 per move

[Aside: optimal solution of $n$-Puzzle family is NP-hard]

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*goal test:* At the final destination?

*path cost:* Depends on total cost, time, waiting time, seat type, type of plane, etc.

# Others Examples

How about:

- Crosswords?
- n-Queens?
- Propositional Satisfiability?
- Coffee and Mail Delivering Robot?
- Others?

# Tree Search Algorithms

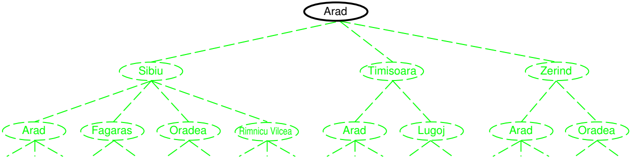
Basic idea:

- Offline exploration of the state space
  - So, exploring a *directed graph*
  - Result of exploration is a *tree*
- Generate successors of already-explored states (a.k.a. *expanding* states)

$\Rightarrow$ The set of nodes available for expansion is the *fringe* or *frontier*.

- Key issue: Which node should be expanded next?

# Tree search example



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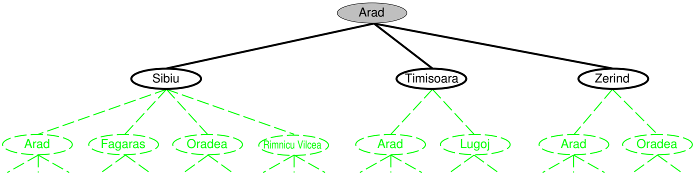

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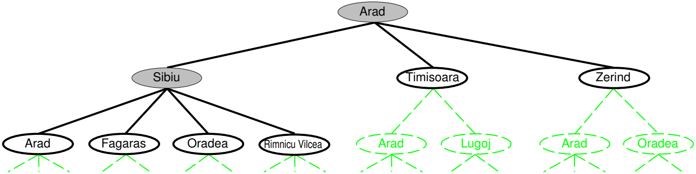


# Implementation: General Tree Search

In outline:

Function Tree-Search(problem) returns a solution or failure
  Initialize the search tree by the initial state of problem
  loop do {
    if there are no candidates for expansion then return failure
    choose a leaf node for expansion (according to some *strategy*)
      - remove the leaf node from the frontier
    if the node satisfies the goal state then return the solution
    expand the node and add the resulting nodes to the search tree
  }

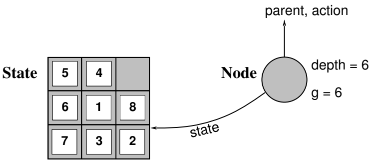
Aside: *Strategy* will most often be implicit in the resulting function.

# Implementation: States vs. Nodes

It is important to distinguish the *state space* and the *search tree*.

- A *state* represents a configuration in the problem space.
- A *node* is part of a search tree.
  - has attributes *parent*, *children*, *depth*, *path cost* $g(x)$.

States do not have parents, children, depth, or path cost (though one state may be reachable from another).




An EXPAND function creates new nodes, filling in the various fields and using a SUCCESSORFN of the problem to create the corresponding states.

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- Time and space complexity are measured in terms of
  - $b$ – maximum *branching factor* of the search tree
  - $d$ – depth of the least-cost solution
  - $m$ – maximum depth of the state space (may be $\infty$)

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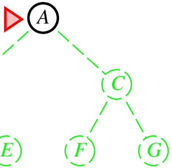
- Examples:
  - Breadth-first search
  - Uniform-cost search
  - Depth-first search
  - Depth-limited search
  - Iterative deepening search

# Breadth-first search

Expand the shallowest unexpanded node

*Implementation*

*fringe* is a FIFO queue, i.e., new successors go at end



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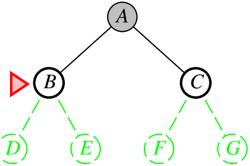

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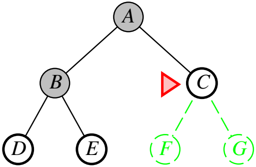

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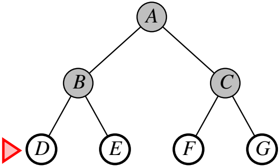

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Optimal: Yes (if cost $= 1$ per step); not optimal in general

*Space* is the big problem; can easily generate nodes at 100MB/sec.
So 24hrs $=$ 8640GB.

## Uniform-Cost Search

- Expand the least-cost unexpanded node
- *Implementation*

  *fringe* = queue ordered by path cost, lowest first
- Equivalent to breadth-first if step costs all equal
- For the travel-in-Romania example, expand the node on the fringe for that city closest in distance to the city at the root (Arad).

# Uniform-Cost Search

**Complete:** Yes, if step cost $\geq \epsilon$, for $\epsilon$ some small positive constant.

- So *NoOps* of cost 0 can be a problem.

**Time:** $O(b^{\lceil C^*/\epsilon \rceil})$, where $C^*$ is the cost of the optimal solution

**Space:** $O(b^{\lceil C^*/\epsilon \rceil})$

- Time and space complexity can be worse than $b^d$.

**Optimal:** Yes

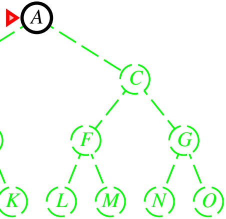
- Nodes expanded in increasing order of $g(n)$ where $g(n)$ is the cost to get to node $n$.

## Depth-First Search

Expand the deepest unexpanded node

*Implementation*

*fringe* = LIFO queue, i.e., put successors at front



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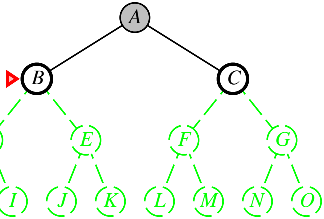

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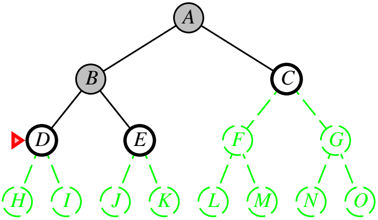

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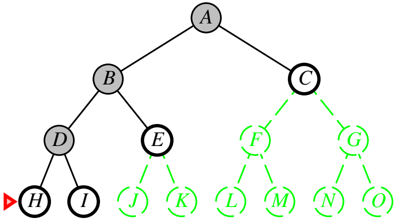

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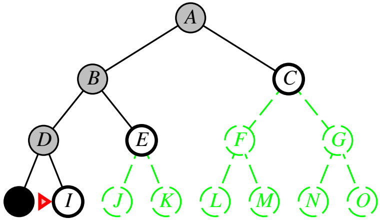

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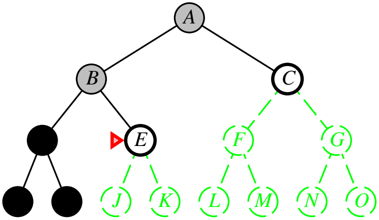

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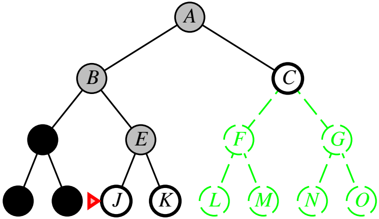

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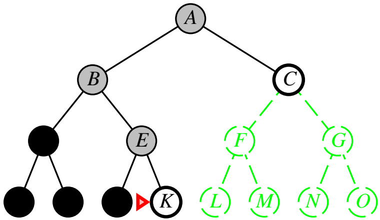

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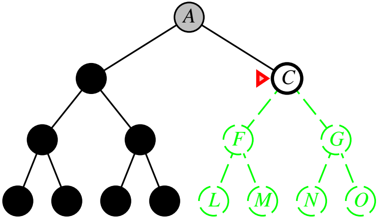
*fringe* = LIFO queue, i.e., put successors at front



# Depth-first search

Expand the deepest unexpanded node

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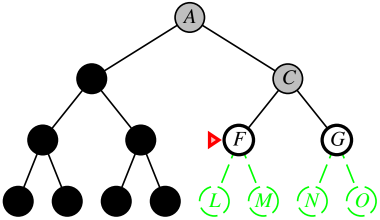

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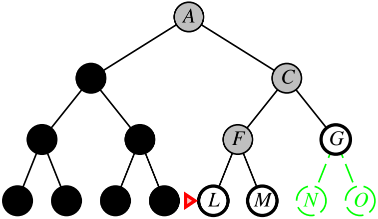

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*Implementation*
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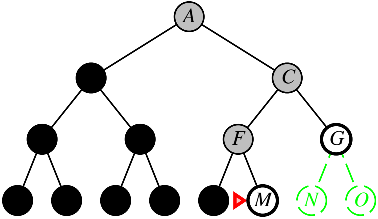

# Properties of depth-first search

Complete: ??

## Properties of depth-first search

Complete: No: fails in infinite-depth spaces, spaces with loops
$\Rightarrow$ Modify to avoid repeated states along path
$\Rightarrow$ Complete in finite spaces

Time: ??

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- But if solutions are dense, may be much faster than breadth-first

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Optimal: ??

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Space: $O(bm)$, i.e., linear space!

Optimal: No

# Depth-Limited Search

Depth-limited search = depth-first search with depth limit $l$,

- i.e., nodes at depth $l$ have no successors

## Recursive implementation:

The implementation simply calls a "helper" function (described on the next slide):

Function Depth-Limited-Search(problem,limit)
    returns soln/fail/cutoff
  Recursive-DLS(Make-Node(Initial-State[problem]),
    problem,limit)

# Depth-Limited Search

## Recursive implementation:

```
Function Recursive-DLS(node,problem,limit) returns soln/fail/cutoff
    cutoff-occurred? ←false
    if Goal-Test(problem,State[node]) then return node
    else if Depth[node] = limit then return cutoff
    else for each successor in Expand(node,problem) do
        result ←Recursive-DLS(successor,problem,limit-1)
        if result = cutoff then cutoff-occurred? ←true
        else if result ≠ failure then return result
    if cutoff-occurred? then return cutoff else return failure
```

- Note: second edition has a bug in the recursive call!

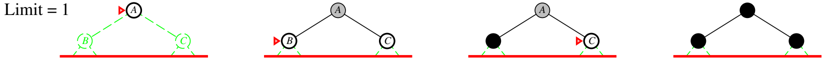
# Iterative Deepening Search

```
Function Iterative-Deepening-Search(problem) returns a solution
    inputs: problem a problem
    for depth ← 0 to ∞ do
        result ←Depth-Limited-Search(problem,depth)
        if result ≠ cutoff then return result
    end
```

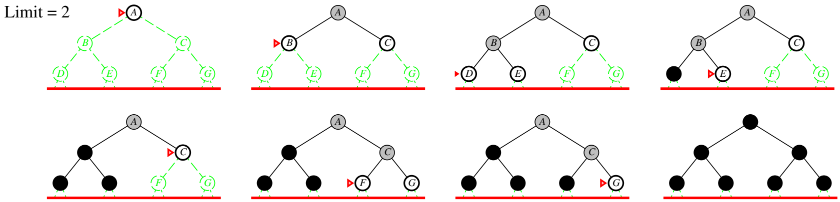
# Iterative deepening search $l = 0$

Limit = 0

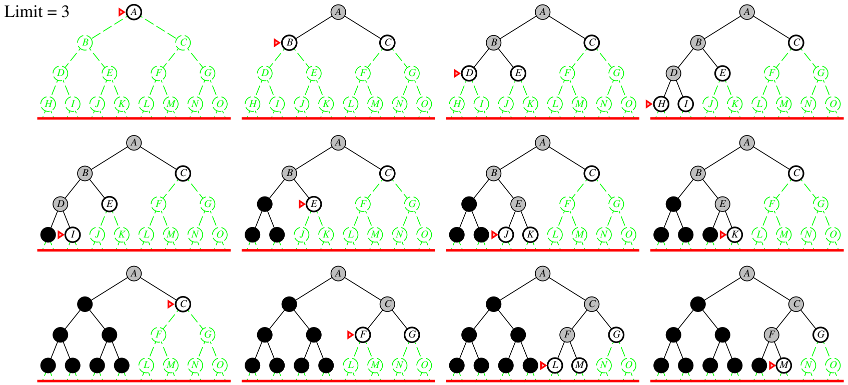

# Iterative deepening search $l = 1$



# Iterative deepening search $l = 2$



# Iterative deepening search $l = 3$



# Properties of iterative deepening search

Complete: ??

# Properties of iterative deepening search

Complete: Yes

Time: ??

## Properties of iterative deepening search

Complete: Yes

Time: $(d+1)b^0 + db^1 + (d-1)b^2 + \ldots + b^d = O(b^d)$

Space: ??

## Properties of iterative deepening search

Complete: Yes

Time: $(d+1)b^0 + db^1 + (d-1)b^2 + \ldots + b^d = O(b^d)$

Space: $O(bd)$

Optimal:

# Properties of iterative deepening search

Complete: Yes

Time: $(d+1)b^0 + db^1 + (d-1)b^2 + \ldots + b^d = O(b^d)$

Space: $O(bd)$

Optimal: Yes, if step cost $= 1$

## Properties of iterative deepening search

- Comparison for $b = 10$ and $d = 5$, solution at far right leaf:

$$\begin{aligned} N(\mathsf{IDS}) &= 50+400+3,000+20,000+100,000 = 123,450 \\ N(\mathsf{BFS}) &= 10+100+1,000+10,000+100,000 \\ &\quad +999,990 = 111,100 \end{aligned}$$

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- For BFS, we have the following ratio of IDS to BFS:

| b | Ratio |
|---|---|
| 2 | 3 |
| 3 | 2 |
| 5 | 1.5 |
| 10 | 1.2 |

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| 10 | 1.2 |

- Can be modified to explore uniform-cost tree

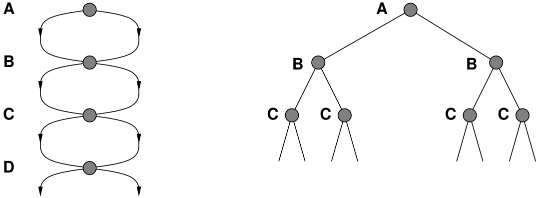
# Summary of algorithms

| Criterion | Breadth-First | Uniform-Cost | Depth-First | Depth-Limited | Iterative Deepening |
|---|---|---|---|---|---|
| Complete? | Yes* | Yes* | No | Yes if $l \geq d$ | Yes |
| Time | $b^{d+1}$ | $b^{\lceil C^*/\epsilon \rceil}$ | $b^m$ | $b^l$ | $b^d$ |
| Space | $b^{d+1}$ | $b^{\lceil C^*/\epsilon \rceil}$ | $bm$ | $bl$ | $bd$ |
| Optimal? | Yes* | Yes | No | No | Yes* |

*: If $b$ is finite.

# Repeated states

- Failure to detect repeated states can turn a linear problem into an exponential one!




- If we detect repeated states, then our search algorithm amounts to searching a graph rather than a tree.
  - Keep a list of encountered nodes, called the *closed* list.

# Graph search

```
Function Graph-Search(problem,fringe) returns a solution, or failure
    closed ←an empty set
    fringe ←Insert(Make-Node(Initial-State[problem]),fringe)
    loop do
        if fringe is empty then return failure
        node ←Remove-Front(fringe)
        if Goal-Test(problem,State[node]) then return node
        if State[node] is not in closed then
            add State[node] to closed
            fringe ←InsertAll(Expand(node,problem),fringe)
    end
```

# Summary

- Problem formulation usually requires abstracting from real-world details to define a state space that can feasibly be explored
- Variety of uninformed search strategies
- Iterative deepening search uses only linear space and not much more time than other uninformed algorithms
- Graph search can be exponentially more efficient than tree search