Modeling with Differential Equations, Direction Fields 9.1,9.2

1. **Quote.** Once you learn the concept of a differential equation, you see differential equations all over, no matter what you do. If you want to apply mathematics, you have to live the life of differential equations. When you live this life, you can then go back to molecular biology with a new set of eyes that will see things you could not otherwise see.

   (Gian-Carlo Rota in 'A Mathematician's Gossip', Indiscrete Thoughts (2008), 213)

2. **Velocity of a Falling Object (considering air friction).**

   Imagine a sky-diver in free fall after jumping out of a plane. There are two forces acting on the falling mass: gravity ($F_g = mg$) and air friction ($F_a = -\gamma v$). Assume air friction is proportional to the velocity of the object.
   The physical law that governs the motion of objects is Newton's second law, which states $F = ma$, where m is the mass of the object, a its acceleration, and F the net force on the object (in our case this is $F_g - F_a$).
   Show that the velocity of the sky-diver satisfies this *differential equation*

   $$m\frac{dv}{dt} = mg - \gamma v$$

   where g and γ are constants.

3. **Examples of differential equations.**

type	form
antidifferentiation $\int \cdots dx$	$\dfrac{dy}{dx} = x + \sin x$
natural growth	$\dfrac{dP}{dx} = kP$
Newton's Law of Cooling/Heating	$\dfrac{dT}{dt} = k(T - M)$
logistic growth	$\dfrac{dT}{dt} = P(1 - P)$
object falling under force of gravity (ignore friction)	$\dfrac{d^2s}{dt^2} = -g$
harmonic oscillator	$\dfrac{d^2\phi}{dt^2} + \omega^2\phi = 0$

4. **Terminology for Differential Equations.**

A **differential equation** is an equation that contains one or more derivatives of an unknown function.

An **ordinary** differential equation (ODE) deals with unknown functions of **one** variable.

The **order** of a differential equation is the order of the highest derivative that occurs in the equation.

The differential equation is called **linear**, if the unknown function and its derivatives only occur linearly (i.e., no terms like y^2, $\sqrt{y'}$, $\sin y$).

A function f is called a **solution** of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation.

To **solve** a differential equation means to find all possible functions that satisfy the equation.

An **initial value problem** (IVP) is a differential equation together with an **initial condition**, which specifies a value that the function must take at given t_0. An initial condition is presented in the form $y(t_0) = y_0$; a solution to an (IVP) is function y which satisfies the differential equation and has value y_0 at $t = t_0$.

A **boundary value problem** (BVP) is a differential equation together with boundary conditions, for example $y(a) = \alpha$ and $y(b) = \beta$. Since there are at least two conditions, the differential equation must have order ≥ 2.

"Most" differential equations need to be solved numerically.

5. **Example.** Show that $\dfrac{dy}{dx} = \dfrac{x}{y}$ has solutions $y = \sqrt{x^2 + c}$.

6. **Example.** (a) Show that $y = e^{2t}$ is a solution to the second-order differential equation

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = 0.$$

(b) Show that $y = e^{-3t}$ is another solution.

7. **Example.** Find the value of c so that $y = \sqrt{x^2 + c}$ is a solution to the *initial value problem*

$$\frac{dy}{dx} = \frac{x}{y}, \qquad y(0) = 3$$

8. **Direction Fields (also known as Slope Fields)**

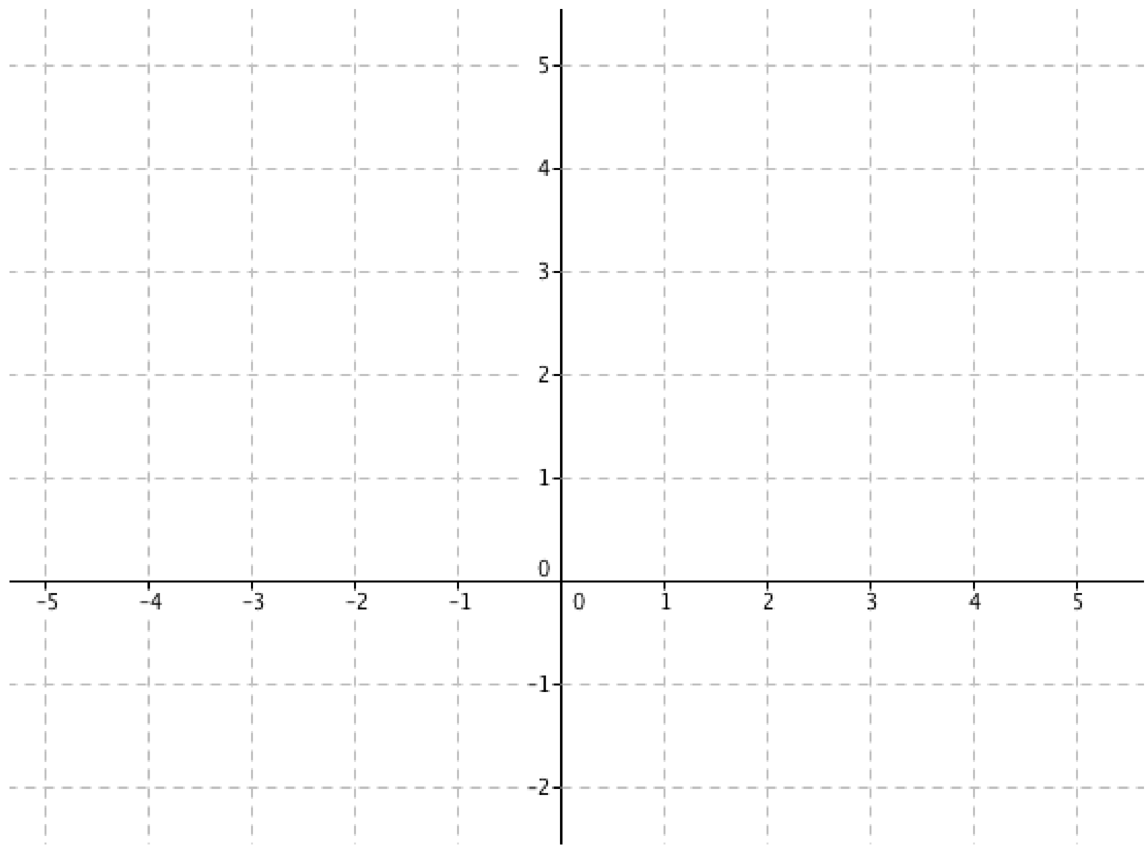
We now look at a visual approach for first-order differential equations.

Consider the differential equation

$$\frac{dy}{dt} = 3 - y.$$

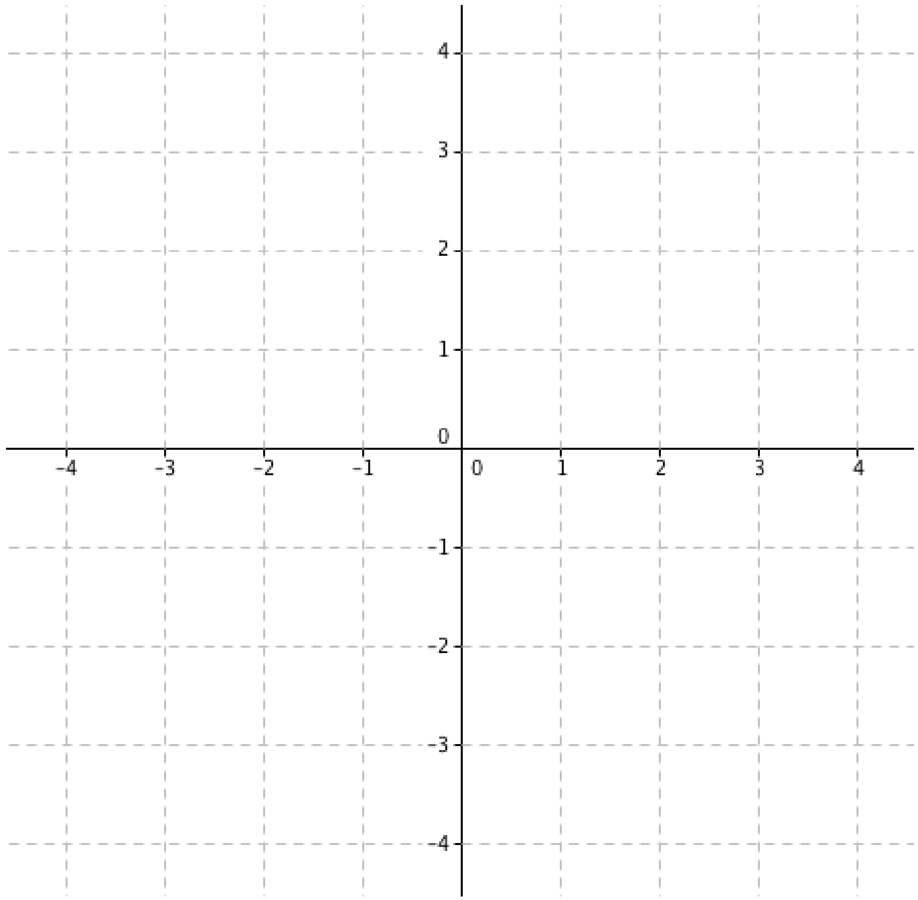
If $y = f(t)$ is a solution to this differential equation fill out values in the following table. Use this information to sketch a graph of f.

t	y	$\frac{dy}{dt} = f'(t)$
-3	1	2
-3	2	1
-3	3	
-3	4	
-2	1	2
-2	2	1



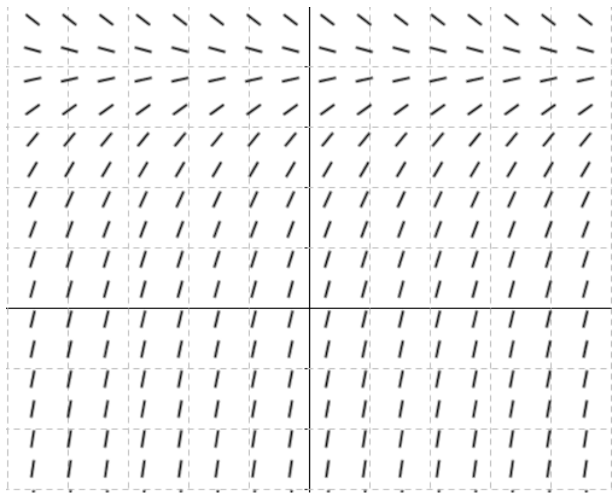
9. **Example.** Using the direction field, guess the form of the solution curves of the differential equation

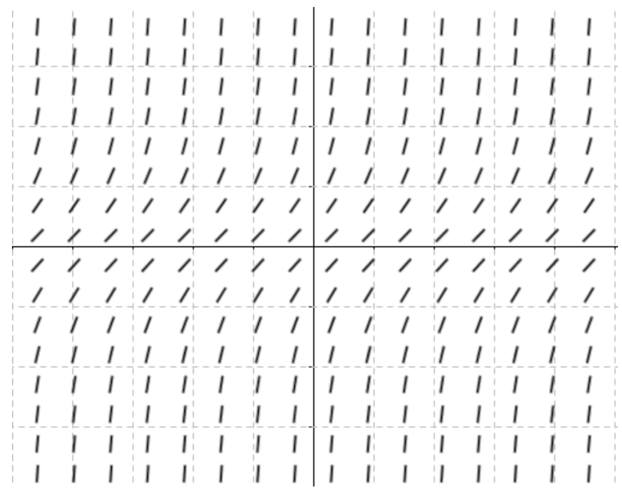
$$\frac{dy}{dx} = \frac{-x}{y}.$$

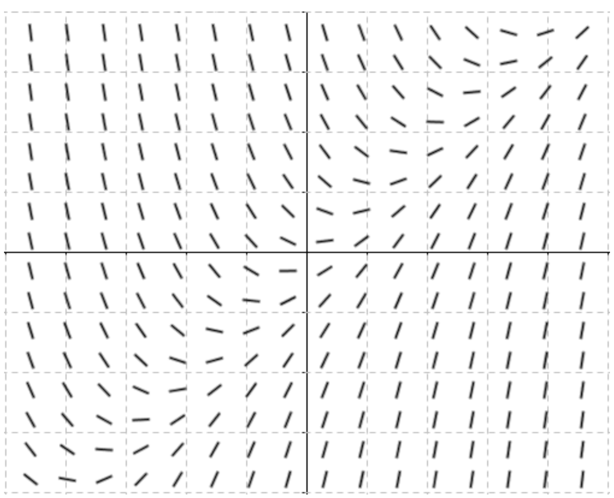


10. **Example.** Match the differential equation with its corresponding slope field.

(a) $y' = 1 + y^2$ (b) $y' = x - y$ (c) $y' = 4 - y$

(i) 

(ii) 

(iii) 

11. **Example.** The slope field for $\dfrac{dy}{dt} = -y(y-1)(y-3)$ is given below.

(a) Sketch the solution curves with initial conditions (A) $y(0) = 2$, (B) $y(0) = 0.5$.

(b) What are the equilibria, i.e., special solutions $y(t) =$ constant?
What is the long-time (large t) behaviour of $y(t)$?
For example does $\lim_{t\to\infty} y(t)$ exists? If so, what is its value?