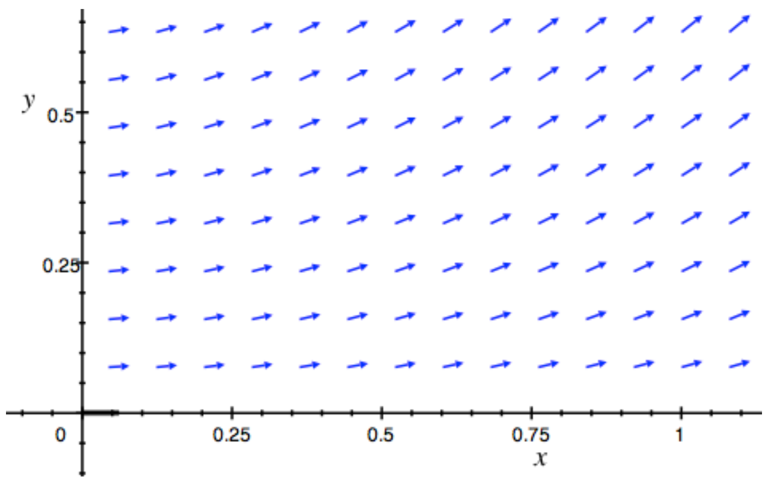
Separable Equations

1. **Quote.** "Ideologies separate us. Dreams and anguish bring us together."

   (Eugene Ionesco, Romanian born French dramatist, 1909-1994)

2. **Problem.** Solve the differential equation

$$\frac{dy}{dx} = \sqrt{xy}, \ x > 0, \ y > 0.$$



3. **Separable Equation.**

   A **separable equation** is a first-order differential equation in which the expression for dy/dx can be factored as a product of a function of x and a function of y:

$$\frac{dy}{dx} = f(x) \cdot g(y).$$

4. **Example.** Solve the Initial Value Problem $\dfrac{dy}{dt} = \alpha y(t)$, with $y(0) = y_0$.

Separable equations lead us right back to integration.

$$\frac{dy}{dx} = f(x) \cdot g(y), \qquad y(x_0) = y_0.$$

Leads to

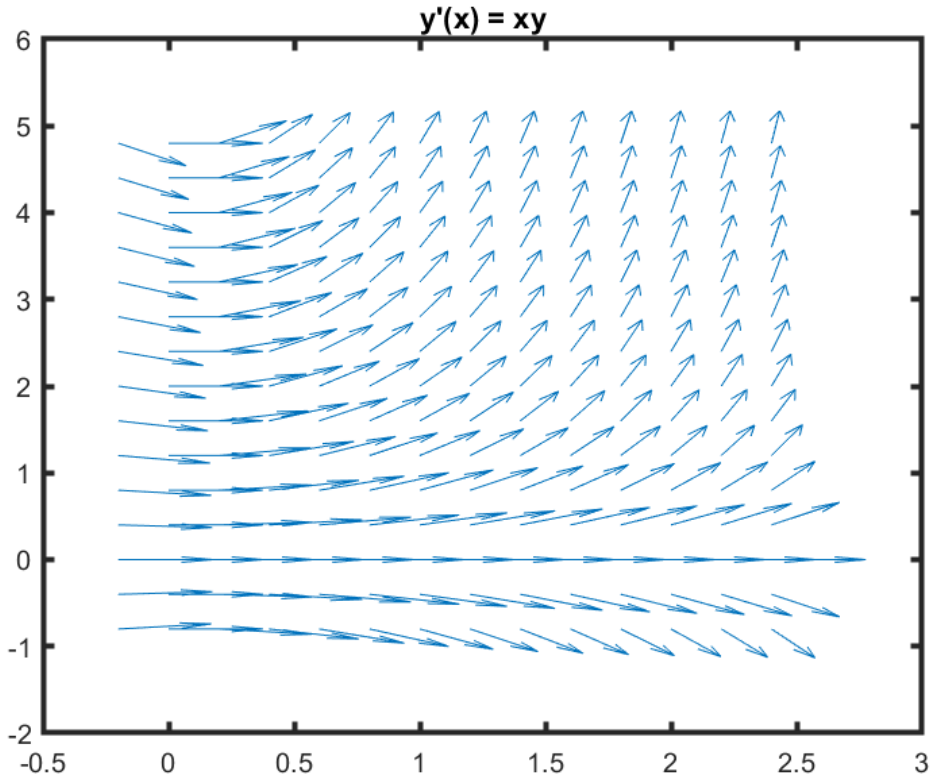
$$\frac{dy}{g(y)} = f(x)\ dx$$

$$\int \frac{dy}{g(y)} = \int f(x)\ dx \qquad \text{Note: there is an integration constant}$$

or

$$\int_{y_0}^{y} \frac{dy}{g(y)} = \int_{x_0}^{x} f(x)\ dx$$

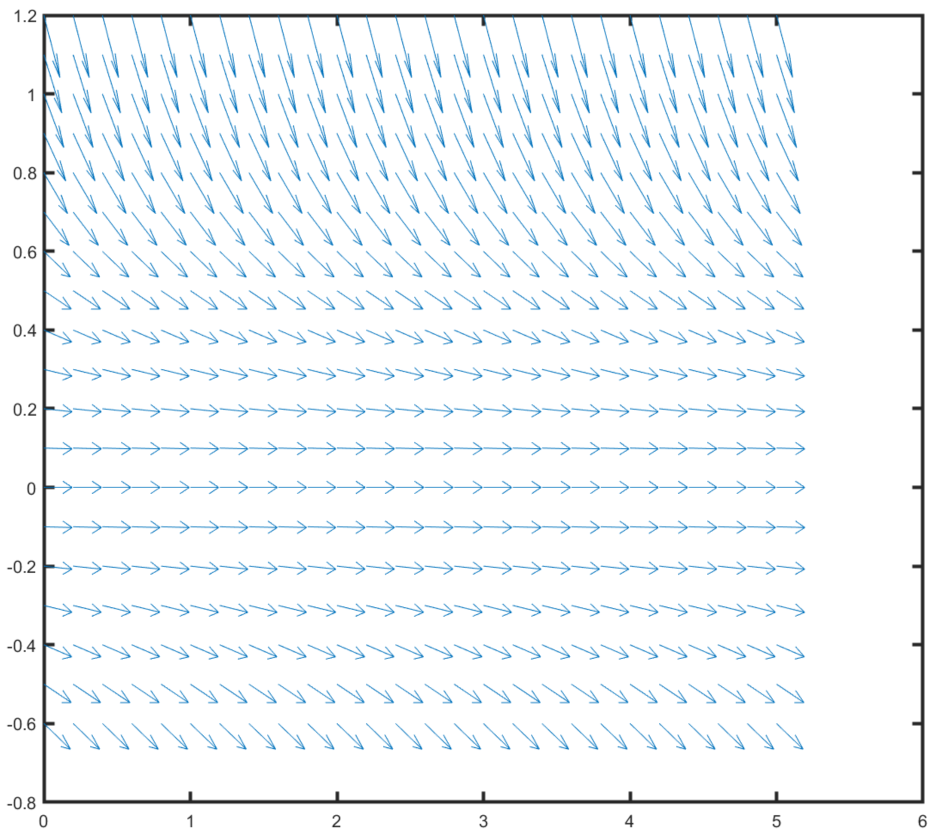
5. **Example.** Solve the differential equation $y'(x) = xy(x)$.



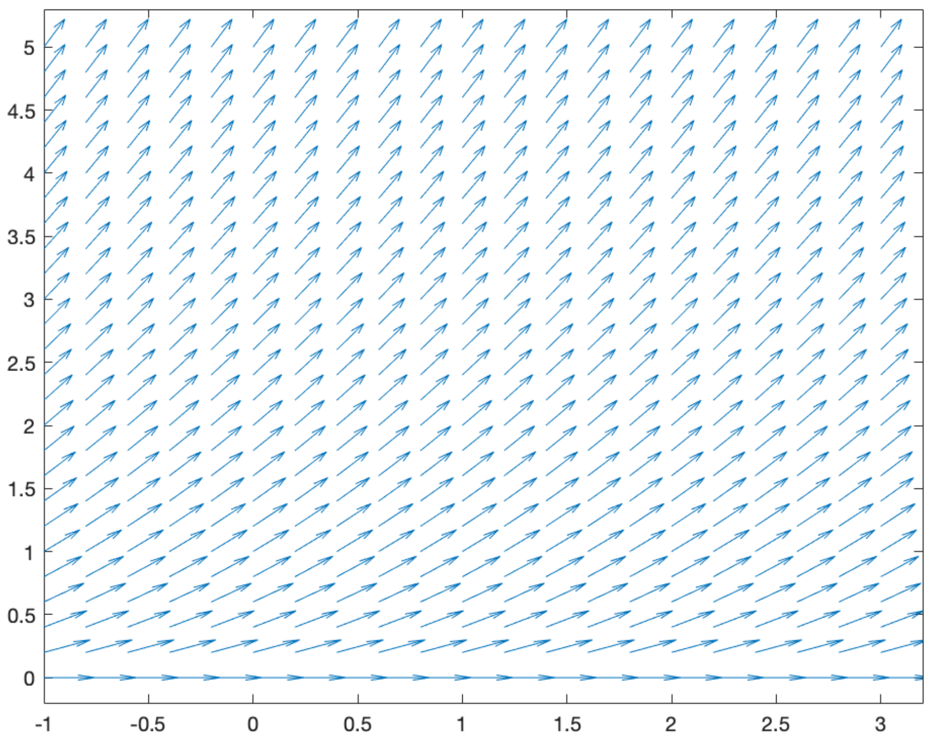
6. **Example.** Solve the initial value problems for $y' = -y^2$.

(a) $y(0) = 1$

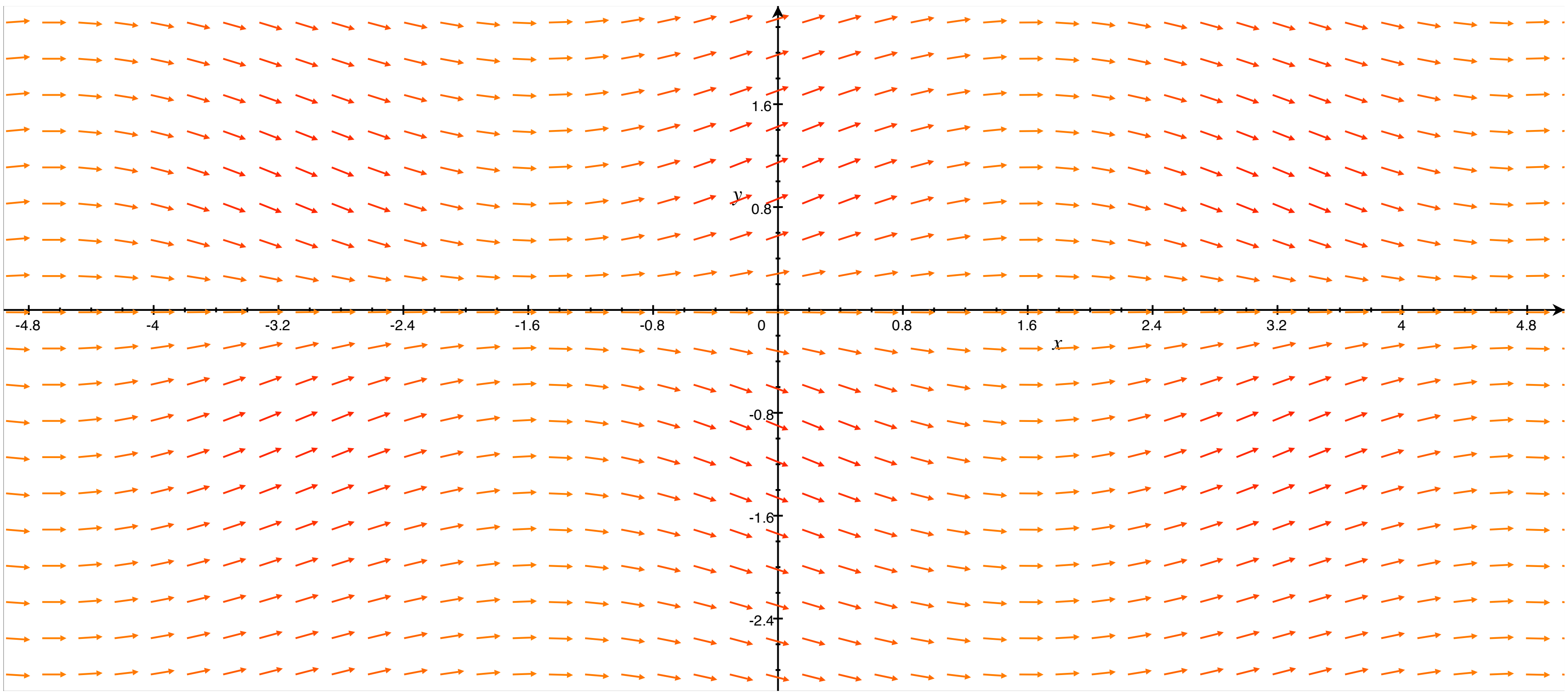
(b) $y(0) = -0.2$



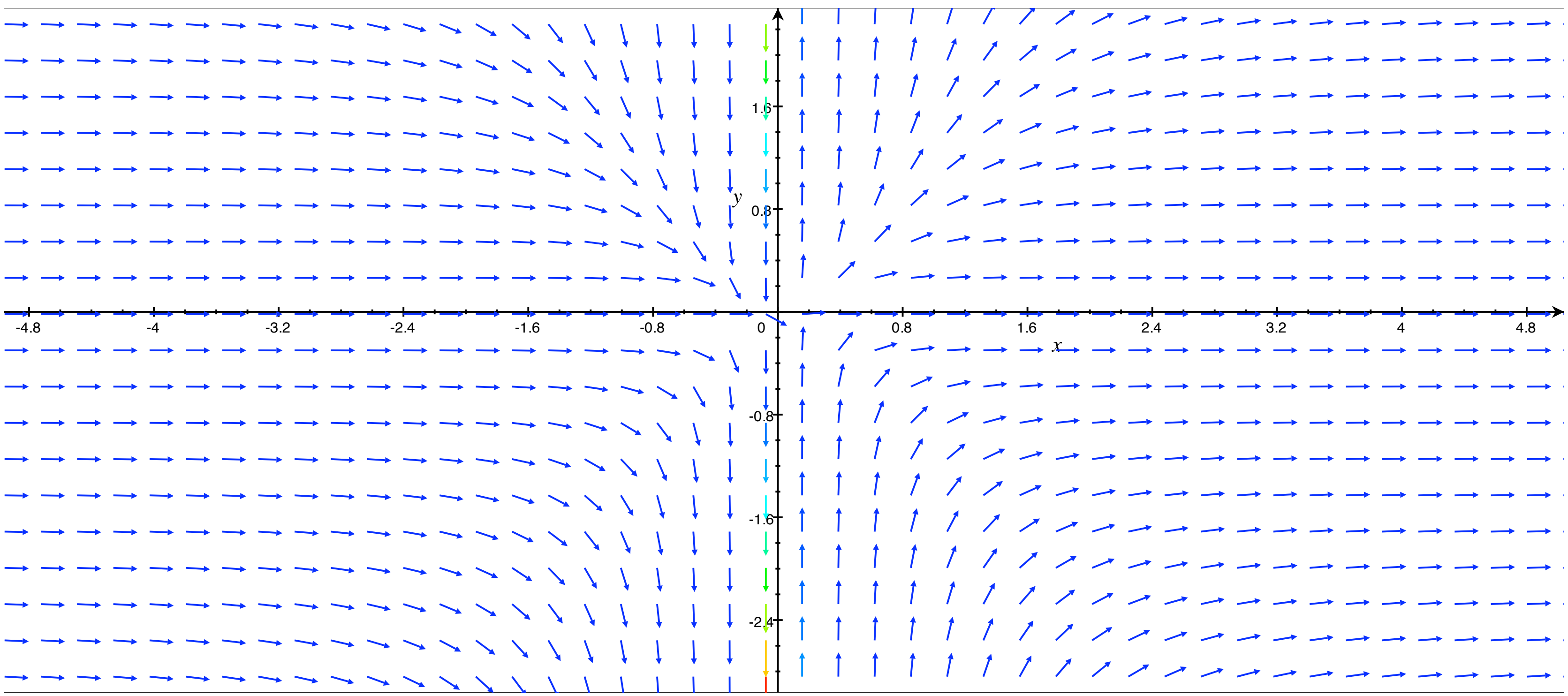
7. **Example.** Solve the initial value problem for $y' = \sqrt{y}$, with $y(0) = 0$.



8. Solve $\dfrac{dy}{dx} = \dfrac{y\cos x}{1+y^2}$



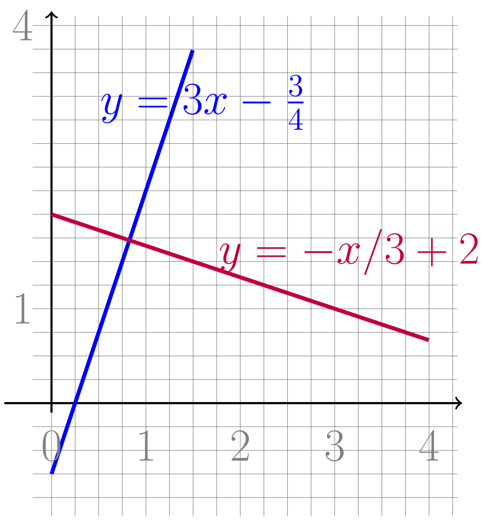
9. **Example.** Find an equation of the curve that passes through the point $(1, 1)$ and whose slope at (x, y) is y^2/x^3.

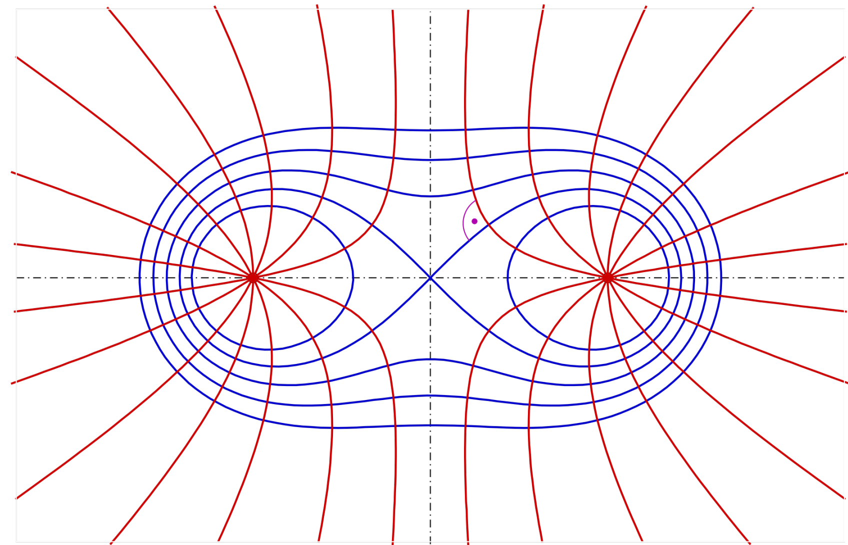


10. **Orthogonal Trajectories.**

An **orthogonal Trajectory** of a given family C of curves is a curve that intersects each member of the given curves at right angles. The family of all those orthogonal trajectories is denoted by $C^{\perp}$.

Recall: The lines $y = mx + b$ and $y = -\dfrac{1}{m}x + c$ are orthogonal.

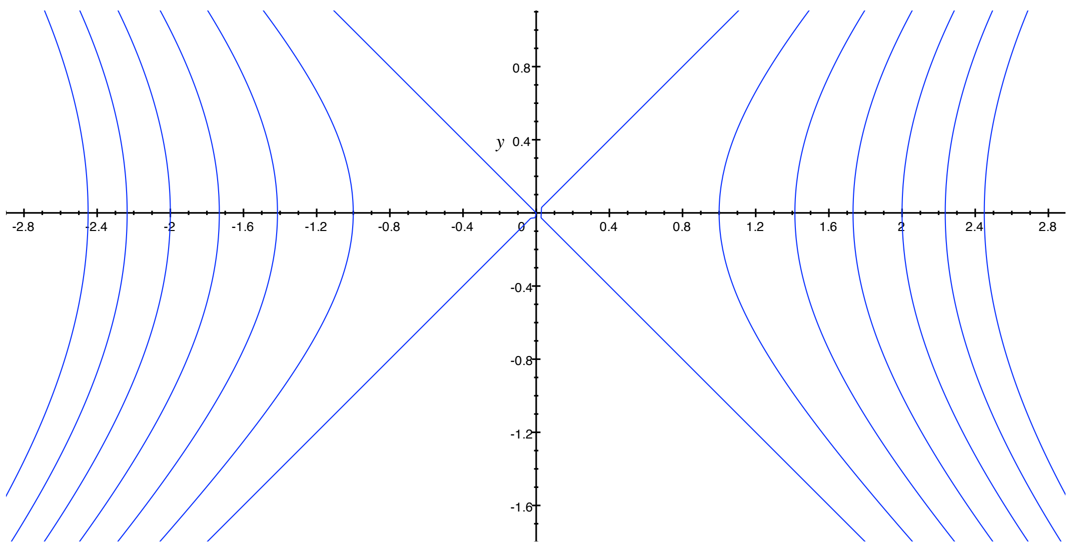




How are the slopes of the curves in C related to those in $C^{\perp}$?

11. **Example.** Find the orthogonal trajectories of the family of the curves

$$x^2 - y^2 = k.$$



12. **Example: A Mixing Problem**

A tank contains 1000L of pure water. Brine that contains 0.05 kg of salt per litre of water enters the tank at a rate of 5 L/min. Brine that contains 0.04 kg of salt per litre of water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at rate of 15 L/min. How much salt is in the tank (a) after t minutes and (b) after one hour?

Mixing Problem continued

13. **Example.** Mortgage payment. Let's use differential equations to estimate a mortgage or loan payment.

Amount outstanding at time t: $A(t)$
Amount borrowed: A_0 ($A_0 = \$100,000$)
Interest rate: α ($\alpha = 4\%$, i.e., $\alpha = .04$)
Amortization period: N ($N = 25$)
Yearly payment: yp (monthly payment $= yp/12$); set $\beta = -yp$

Note: actual mortgage payment calculation uses semi-annual compounding (or monthly compounding on a HELOC - home equity line of credit). This results in (slightly) different effective interest rates. This computation assumes continuous compounding.

Notes.