First-order logic

Chapter 8

Outline

◇ Why FOL?

◇ Syntax and semantics of FOL

◇ Fun with sentences

◇ Wumpus world in FOL

Pros and cons of propositional logic

🙂 Propositional logic is **declarative**: pieces of syntax correspond to facts

🙂 Propositional logic allows partial/disjunctive/negated information
(unlike most data structures and databases)

🙂 Propositional logic is **compositional**:
meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$

🙂 Meaning in propositional logic is **context-independent**
(unlike natural language, where meaning depends on context)

🙁 Propositional logic has very limited expressive power
(unlike natural language)
E.g., cannot say "pits cause breezes in adjacent squares"
except by writing one sentence for each square

First-order logic

Whereas propositional logic assumes world contains **facts**,
first-order logic (like natural language) assumes the world contains

- Objects: people, houses, numbers, theories, Ronald McDonald, colors, baseball games, wars, centuries ...

- Relations: red, round, bogus, prime, multistoried ...,
  brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, ...

- Functions: father of, best friend, third inning of, one more than, end of ...

Logics in general

Language	Ontological Commitment	Epistemological Commitment
Propositional logic	facts	true/false/unknown
First-order logic	facts, objects, relations	true/false/unknown
Temporal logic	facts, objects, relations, times	true/false/unknown
Probability theory	facts	degree of belief
Fuzzy logic	facts + degree of truth	known interval value

Syntax of FOL: Basic elements

Constants $KingJohn, \ 2, \ SFU, \dots$

Predicates $Brother, \ >, \dots$

Functions $Sqrt, \ LeftLegOf, \dots$

Variables $x, \ y, \ a, \ b, \dots$

Connectives $\land \ \lor \ \lnot \ \Rightarrow \ \ \Leftrightarrow$

Equality $=$

Quantifiers $\forall \ \exists$

FOL Sentences

$\text{Sentence} = AtomicSentence$

$\qquad\qquad \textbf{or}\ (Sentence\ Connective\ Sentence)$

$\qquad\qquad \textbf{or}\ Quantifier\ Variable, \ldots\ Sentence$

$\qquad\qquad \textbf{or}\ \neg Sentence$

Atomic sentences

Atomic sentence $= predicate(term_1, \ldots, term_n)$
or $term_1 = term_2$

Term $= function(term_1, \ldots, term_n)$
or $constant$ or $variable$

E.g., $Brother(KingJohn, RichardTheLionheart)$

$> (Length(LeftLegOf(Richard)), Length(LeftLegOf(KingJohn)))$

Complex sentences

Complex sentences are made from atomic sentences using connectives

$$\neg S, \quad S_1 \wedge S_2, \quad S_1 \vee S_2, \quad S_1 \Rightarrow S_2, \quad S_1 \Leftrightarrow S_2$$

E.g. $Sibling(KingJohn, Richard) \Rightarrow Sibling(Richard, KingJohn)$
$>(1,2) \vee \leq(1,2)$
$>(1,2) \wedge \neg >(1,2)$

Truth in first-order logic

Sentences are true with respect to a model and an interpretation

Model contains ≥ 1 objects (domain elements) and relations among them
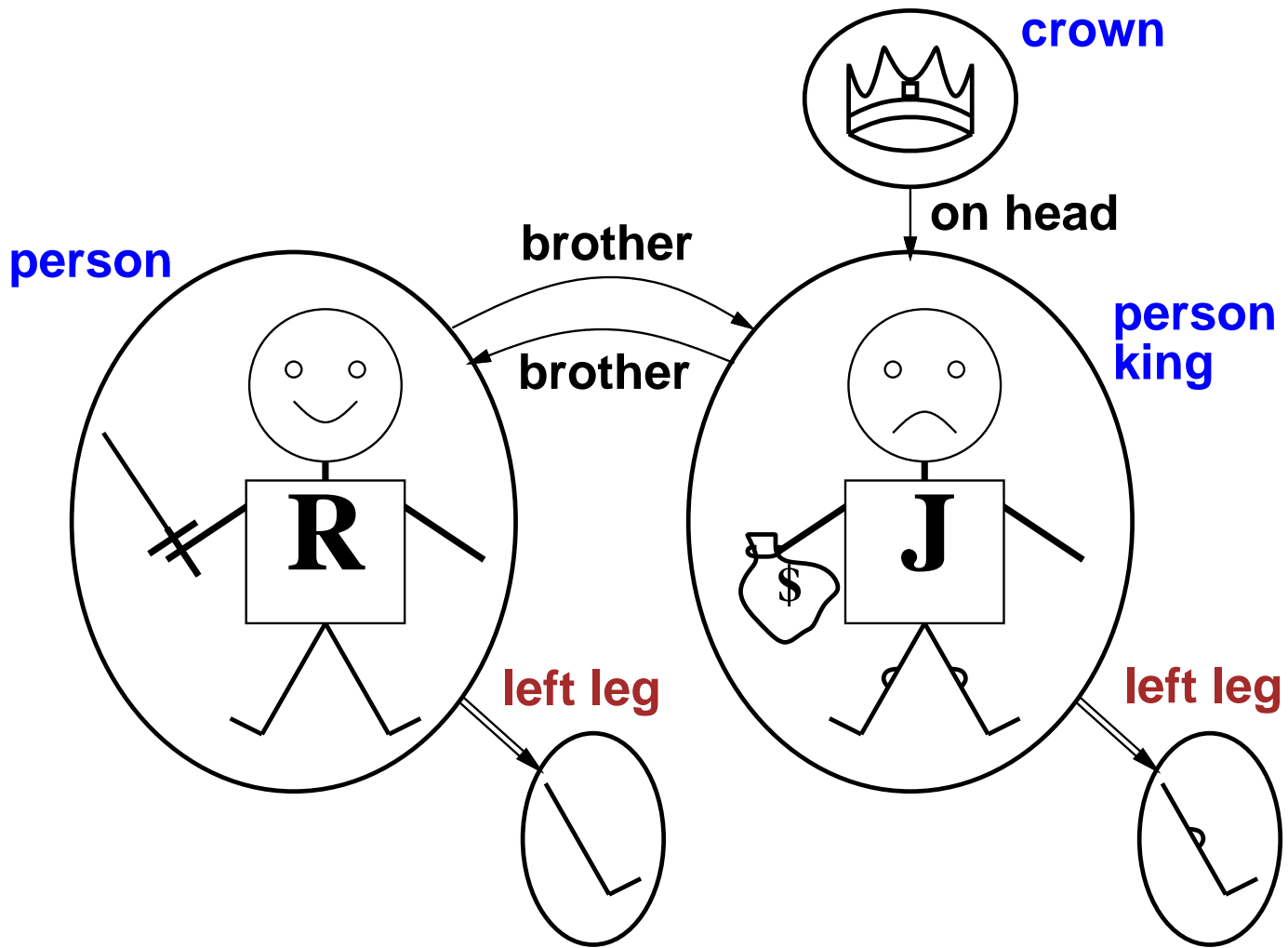
Interpretation specifies referents for
    constant symbols $\rightarrow$ objects
    predicate symbols $\rightarrow$ relations
    function symbols $\rightarrow$ functional relations

An atomic sentence $predicate(term_1, \ldots, term_n)$ is true
iff the objects referred to by $term_1, \ldots, term_n$
are in the relation referred to by $predicate$

Models for FOL: Example

Truth example

Consider the interpretation in which
$Richard \rightarrow$ Richard the Lionheart (man with R on chest)
$John \rightarrow$ the evil King John (man with J on chest)
$Brother \rightarrow$ the brotherhood relation

Under this interpretation, $Brother(Richard, John)$ is true
just in the case that Richard the Lionheart and the evil King John
are in the brotherhood relation in the model

Models for FOL: Lots!

Entailment in propositional logic can be computed by enumerating models

We **can** enumerate the FOL models for a given KB vocabulary:

For each number of domain elements n from 1 to ∞
    For each k-ary predicate P_k in the vocabulary
        For each possible k-ary relation on n objects
            For each constant symbol C in the vocabulary
                For each choice of referent for C from n objects . . .

Computing entailment by enumerating FOL models is not easy!

Universal quantification

$\forall \langle variables \rangle \ \langle sentence \rangle$

Everyone at SFU is smart:
$\forall x \ \ At(x, SFU) \ \Rightarrow \ Smart(x)$

$\forall x \ \ P \ \ $ is true in a model m iff P is true with x being **each** possible object in the model

Roughly speaking, equivalent to the conjunction of instantiations of P

$$(At(KingJohn, SFU) \ \Rightarrow \ Smart(KingJohn))$$
$$\wedge \ (At(Richard, SFU) \ \Rightarrow \ Smart(Richard))$$
$$\wedge \ (At(SFU, SFU) \ \Rightarrow \ Smart(SFU))$$
$$\wedge \ \ldots$$

A common mistake to avoid

Typically, $\Rightarrow$ is the main connective with $\forall$

Common mistake: using $\wedge$ as the main connective with $\forall$:

$$\forall\, x \quad At(x, SFU) \wedge Smart(x)$$

means "Everyone is at SFU and everyone is smart"

Existential quantification

$\exists \langle variables \rangle \ \langle sentence \rangle$

Someone at UBC is smart:
$\exists x \ \ At(x, UBC) \wedge Smart(x)$

$\exists x \ \ P \ \ $ is true in a model m iff P is true with x being
some possible object in the model

Roughly speaking, equivalent to the disjunction of instantiations of P

$$(At(KingJohn, UBC) \wedge Smart(KingJohn))$$
$$\vee \ (At(Richard, UBC) \wedge Smart(Richard))$$
$$\vee \ (At(UBC, UBC) \wedge Smart(UBC))$$
$$\vee \ \ldots$$

Another common mistake to avoid

Typically, $\wedge$ is the main connective with $\exists$

Common mistake: using $\Rightarrow$ as the main connective with $\exists$:

$$\exists x \; At(x, UBC) \; \Rightarrow \; Smart(x)$$

is true if there is anyone who is not at UBC!

Properties of quantifiers

$\forall x \ \forall y$ is the same as $\forall y \ \forall x$ (<u>why</u>??)

$\exists x \ \exists y$ is the same as $\exists y \ \exists x$ (<u>why</u>??)

$\exists x \ \forall y$ is **not** the same as $\forall y \ \exists x$

$\exists x \ \forall y \ Loves(x, y)$
$\forall y \ \exists x \ Loves(x, y)$

Properties of quantifiers

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$\exists\,x\ \ \exists\,y$ is the same as $\exists\,y\ \ \exists\,x$ (<u>why</u>??)

$\exists\,x\ \ \forall\,y$ is **not** the same as $\forall\,y\ \ \exists\,x$

$\exists\,x\ \ \forall\,y\ \ Loves(x,y)$
"There is a person who loves everyone in the world"

$\forall\,y\ \ \exists\,x\ \ Loves(x,y)$
"Everyone in the world is loved by at least one person"

Quantifier duality: each can be expressed using the other

$\forall\,x\ \ Likes(x, IceCream)$ $\neg\exists\,x\ \ \neg Likes(x, IceCream)$

$\exists\,x\ \ Likes(x, Broccoli)$ $\neg\forall\,x\ \ \neg Likes(x, Broccoli)$

Fun with sentences

Brothers are siblings

Fun with sentences

Brothers are siblings

$$\forall\, x, y \quad Brother(x, y) \;\Rightarrow\; Sibling(x, y).$$

"Sibling" is symmetric

Fun with sentences

Brothers are siblings

$\forall x, y \;\; Brother(x, y) \Rightarrow Sibling(x, y).$

"Sibling" is symmetric

$\forall x, y \;\; Sibling(x, y) \Leftrightarrow Sibling(y, x).$

One's mother is one's female parent

Fun with sentences

Brothers are siblings

$\forall\, x, y \;\; Brother(x, y) \;\Rightarrow\; Sibling(x, y)$.

"Sibling" is symmetric

$\forall\, x, y \;\; Sibling(x, y) \;\Leftrightarrow\; Sibling(y, x)$.

One's mother is one's female parent

$\forall\, x, y \;\; Mother(x, y) \;\Leftrightarrow\; (Female(x) \land Parent(x, y))$.

A first cousin is a child of a parent's sibling

Fun with sentences

Brothers are siblings

$\forall x, y \;\; Brother(x, y) \;\Rightarrow\; Sibling(x, y)$.

"Sibling" is symmetric

$\forall x, y \;\; Sibling(x, y) \;\Leftrightarrow\; Sibling(y, x)$.

One's mother is one's female parent

$\forall x, y \;\; Mother(x, y) \;\Leftrightarrow\; (Female(x) \wedge Parent(x, y))$.

A first cousin is a child of a parent's sibling

$\forall x, y \;\; FirstCousin(x, y) \;\Leftrightarrow\; \exists p, ps \;\; Parent(p, x) \wedge Sibling(ps, p) \wedge Parent(ps, y)$

Equality

$term_1 = term_2$ is true under a given interpretation
if and only if $term_1$ and $term_2$ refer to the same object

E.g., $1 = 2$ and $\forall\, x \;\; \times(Sqrt(x), Sqrt(x)) = x$ are satisfiable
$\qquad 2 = 2$ is valid

E.g., definition of (full) $Sibling$ in terms of $Parent$:
$$\forall\, x, y \;\; Sibling(x, y) \;\Leftrightarrow\; [\neg(x = y) \wedge \exists\, m, f \;\; \neg(m = f) \wedge$$
$$Parent(m, x) \wedge Parent(f, x) \wedge Parent(m, y) \wedge Parent(f, y)]$$

Interacting with FOL KBs

Suppose a wumpus-world agent is using an FOL KB
and perceives a smell and a breeze (but no glitter) at $t = 5$:

$Tell(KB, Percept([Smell, Breeze, None], 5))$
$Ask(KB, \exists a \ Action(a, 5))$

I.e., does KB entail any particular actions at $t = 5$?

Answer: $Yes, \ \{a/Shoot\}$ $\quad \leftarrow$ substitution (binding list)

Given a sentence S and a substitution σ,
$S\sigma$ denotes the result of plugging σ into S; e.g.,
$S = Smarter(x, y)$
$\sigma = \{x/Hillary, y/Bill\}$
$S\sigma = Smarter(Hillary, Bill)$

$Ask(KB, S)$ returns some/all σ such that $KB \models S\sigma$

Knowledge base for the wumpus world

"Perception"
$\forall\, b, g, t \;\; Percept([Smell, b, g], t) \;\Rightarrow\; Smelt(t)$
$\forall\, s, b, t \;\; Percept([s, b, Glitter], t) \;\Rightarrow\; AtGold(t)$

Reflex: $\forall\, t \;\; AtGold(t) \;\Rightarrow\; Action(Grab, t)$

Reflex with internal state: do we have the gold already?
$\forall\, t \;\; AtGold(t) \wedge \neg Holding(Gold, t) \;\Rightarrow\; Action(Grab, t)$

$Holding(Gold, t)$ cannot be observed
$\qquad \Rightarrow$ keeping track of change is essential

Deducing hidden properties

Properties of locations:
$$\forall\, x, t \;\; At(Agent, x, t) \wedge Smelt(t) \;\Rightarrow\; Smelly(x)$$
$$\forall\, x, t \;\; At(Agent, x, t) \wedge Breeze(t) \;\Rightarrow\; Breezy(x)$$

Squares are breezy near a pit:

Diagnostic rule—infer cause from effect
$$\forall\, y \;\; Breezy(y) \;\Rightarrow\; \exists\, x \;\; Pit(x) \wedge Adjacent(x, y)$$

Causal rule—infer effect from cause
$$\forall\, x, y \;\; Pit(x) \wedge Adjacent(x, y) \;\Rightarrow\; Breezy(y)$$

Neither of these is complete—e.g., the causal rule doesn't say whether squares far away from pits can be breezy

Definition for the $Breezy$ predicate:
$$\forall\, y \;\; Breezy(y) \;\Leftrightarrow\; [\exists\, x \;\; Pit(x) \wedge Adjacent(x, y)]$$

Keeping track of change

Facts hold in **situations**, rather than eternally
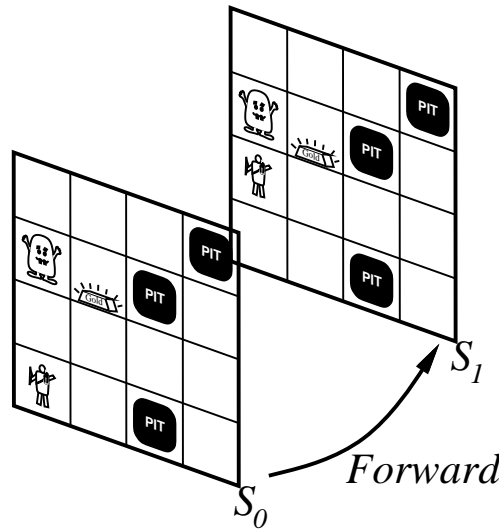E.g., $Holding(Gold, Now)$ rather than just $Holding(Gold)$

Situation calculus is one way to represent change in FOL:
       Adds a situation argument to each non-eternal predicate
       E.g., Now in $Holding(Gold, Now)$ denotes a situation

Situations are connected by the $Result$ function
$Result(a, s)$ is the situation that results from doing a in s

Describing actions I

"Effect" axiom—describe changes due to action

$$\forall s \; AtGold(s) \implies Holding(Gold, Result(Grab, s))$$

"Frame" axiom—describe **non-changes** due to action

$$\forall s \; HaveArrow(s) \implies HaveArrow(Result(Grab, s))$$

Frame problem: find an elegant way to handle non-change
       (a) representation—avoid frame axioms
       (b) inference—avoid repeated "copy-overs" to keep track of state

Qualification problem: true descriptions of real actions require endless caveats—
what if gold is slippery or nailed down or . . .

Ramification problem: real actions have many secondary consequences—
what about the dust on the gold, wear and tear on gloves, . . .

Describing actions II

Successor-state axioms solve the representational frame problem

Each axiom is "about" a **predicate** (not an action per se):

> P true afterwards $\Leftrightarrow$ [an action made P true
> $\vee$ P true already and no action made P false]

For holding the gold:
  ??

Describing actions II

Successor-state axioms solve the representational frame problem

Each axiom is "about" a **predicate** (not an action per se):

$$P \text{ true afterwards} \quad \Leftrightarrow \quad [\text{an action made P true}$$
$$\vee \quad \text{P true already and no action made P false}]$$

For holding the gold:
$$\forall\, a, s \;\; Holding(Gold, Result(a, s)) \;\Leftrightarrow\;$$
$$[(a = Grab \wedge AtGold(s))$$
$$\vee\, (Holding(Gold, s) \wedge a \neq Release)]$$

Making plans

Initial condition in KB:

$At(Agent, [1, 1], S_0)$
$At(Gold, [1, 2], S_0)$

Query: $Ask(KB, \exists s \; Holding(Gold, s))$
i.e., in what situation will I be holding the gold?

Answer: $\{s/Result(Grab, Result(Forward, S_0))\}$
i.e., go forward and then grab the gold

This assumes that the agent is interested in plans starting at S_0 and that S_0 is the only situation described in the KB

Making plans: A better way

Represent plans as action sequences $[a_1, a_2, \ldots, a_n]$

$PlanResult(p, s)$ is the result of executing p in s

Then the query $Ask(KB, \exists p\ Holding(Gold, PlanResult(p, S_0)))$
has the solution $\{p/[Forward, Grab]\}$

Definition of $PlanResult$ in terms of $Result$:
$\quad \forall s\ PlanResult([], s) = s$
$\quad \forall a, p, s\ PlanResult([a|p], s) = PlanResult(p, Result(a, s))$

Planning systems are special-purpose reasoners designed to do this type of inference more efficiently than a general-purpose reasoner

Summary

First-order logic:
  – objects and relations are semantic primitives
  – syntax: constants, functions, predicates, equality, quantifiers

Increased expressive power: sufficient to define wumpus world

Situation calculus:
  – conventions for describing actions and change in FOL
  – can formulate planning as inference on a situation calculus KB