Phil 320
Chapter 1: Sets, Functions and Enumerability

I. Sets

Informally: a *set* is a collection of objects. The objects are called *members* or *elements* of the set.

a) Use *capital letters* to stand for sets (A, B, C, S, T, X, Y are common choices).

b) *Special sets* (with reserved letters):

> P: the positive integers 1,2,3, …
> N: the natural numbers 0,1,2,3,…
> E: the even integers 2, 4, 6,…
> O: the odd integers 1, 3, 5, …
> Z: the integers 0, 1, -1, 2, -2, …
> Q: the rational numbers
> R: the real numbers
> ϕ: the empty set (has no members)

c) More examples of sets (everything inside the curly parentheses):

> S = {1, 2, 3}
> S = {1, 2, 3, …, *n*}
> S = {1} [a *singleton* set: just one element]
> S = {P, {1}, ϕ} [a set of sets]

d) Power set: P* = the set of all subsets of P [again, a set of sets]

e) Two basic relations: $\subseteq$ (subset), $\in$ (member).
> A $\subseteq$ B means A is a subset of B: every element of A is an element of B.
> $a \in$ A means *a* is an element of A. [Typically use lower-case for *elements* of a set.]

f) *Size of sets*.

> *Finite sets.* All the examples in part c) are finite sets. The size of a finite set is the number of elements it contains. F is bigger than E if F has more elements than E.
>
> *Infinite sets.* All the examples in part b), apart from the empty set, are infinite sets. Any infinite set is bigger than any finite set. But how do we compare two infinite sets?
> - Sometimes one is bigger (Ex: R is bigger than P)
> - Sometimes both have the same size (Ex: P and E)
>
> To make these relationships precise, we need the idea of a *function*.

II. Functions

a) Total functions

Informal: A function $f:X \to Y$ is a rule that assigns one member of Y to each member of X.

Examples:

1) $f(n) = 2n$ is a function from $P \to P$ (or from $P \to E$)
2) $f(1) = 2$, $f(2) = 1$, $f(3) = 1$ is a function $f:\{1, 2, 3\} \to \{1, 2\}$

As example 2) illustrates, it is often possible to define a function using a *table of values*. (You don't always need to give a formula for $f(n)$ in terms of n.)

X is called the *domain* of f; the members of X are *arguments* of f.
The members of Y are the *values* of f.
The set of all values that f actually assigns to one or more arguments is the *range* of f. (In example 1, the range of f is E; in example 2, the range of f is {1, 2}.)

A function $f:X \to Y$ is 1-1 (*one-to-one*) if whenever $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$.
A function $f:X \to Y$ is *onto* Y if for each $y \in Y$, there is at least one $x \in X$ such that $y = f(x)$.

b) Partial functions

A partial function $f:X \to Y$ is a rule that assigns one member of Y to *some* (but not all) members of X (the members to which f assigns a value constitute its domain). The function is left undefined for some members of X.

Example: $f(n) = \begin{cases} 2n, & n \text{ even} \\ \text{undefined}, & n \text{ odd} \end{cases}$

is a partial function $P \to P$ (or $P \to E$).

Application to sizes of sets: Two sets X and Y have the *same size* if there is a function $f:X \to Y$ that is total, 1-1 and onto.

Example: E and P have the same size. For $f(n) = 2n$ is 1-1 and onto.

III. Enumerable sets

Sets like P and E are the ‘smallest’ kind of infinite sets. They are called *enumerably infinite* sets. (Synonymous terms: *countably infinite*, *denumerable*.)

Examples: P, E, O

Informal definition: a set S is *enumerable* if its members can be *put in a list*, so that each member of the set appears some finite number of steps from the start of the list.

Formal definition: S is *enumerable* if there is a function f: P $\rightarrow S$ whose range is all of S.

Notes: 1) *Enumerable* includes both *finite* and *enumerably infinite* sets.
2) Any two enumerably infinite sets have the same size.
3) The function f need not be 1-1, but it must be onto S.
4) The function f may be partial.

This gives us two techniques for showing enumerability:

- Informal: show you can put them in a list.

  Need only make it clear that every member of the set will eventually appear as the n’th member of the list, for some finite n.
  [Note: 1, 3, 5, 7, …, 2, 4, 6, 8, …. does not count as a list.]

- Formal: state a function f:P $\rightarrow S$ whose range is all of S.

Examples:

a) O (odd numbers) is enumerable. Use $f(n) = 2n-1$.

b) {1, 2, 3} is enumerable. (But not enumerably infinite.)

c) Any subset S of P is enumerable.

Use $f(n) = n$, $n \in S$; undefined otherwise.

d) ϕ is enumerable.

We can use the partial function e, whose domain is empty: undefined everywhere! Then the range of e is ϕ.

e) The set of all people on earth is enumerable.

Why? It’s finite. And *any* finite non-empty set, by definition, can be written as $S = \{s_1, \ldots, s_n\}$ for some positive number n. But this is just to say that there is a function from P to S, namely, $f(k) = s_k$ if $k \leq n$, and undefined otherwise.

f) Z (the integers) is enumerable. Here is one way to list its members: 0, 1, -1, 2, -2, … . [What function f enumerates this list? Use a table of values to see the pattern.]

g) Ordered pairs of positive integers: P^2 = {(1, 1), (1, 2), …, (2,1), (2,2) , …}. This is enumerable.

Proof: First, list them in a two-dimensional array.

	1	2	3	4	5	6	…
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	…
2	(2,1)	(2,2)	(2,3)	…			
3	…						
4							
5							
6						(6,6)	
…							…

Now we can arrange them in a list in several ways.

Method 1: Slant down each constant-sum diagonal (moving NE to SW).
Method 2: Expanding square-fronts: traverse each backwards L (with corner (n, n)).
Method 3: Place each series in alternating slots that remain vacant (see text).

Write down the function G(m, n) that **encodes** (m, n) on methods 1 and 3.
[An *encoding function* $G:S \rightarrow P$ is a 1-1 function that assigns a positive integer to each member of S. A *decoding function* $g:P \rightarrow S$ is any function that enumerates S. If there is an encoding function G, then there will also be a decoding function, namely the inverse of G. Sometimes the best way to show a set is enumerable is to give an encoding function, rather than a decoding function.]

h) The set Q^+ of all positive rational numbers is enumerable.

Proof: List them in an array.

	1	2	3	4	…
1	1/1	½	1/3	¼	
2	2/1	2/2	2/3	…	
3	…				

Again, arrange them in a list in any of the three ways used in example g).

i) The set of ordered k-tuples of positive integers, for any k, is enumerable.

j) The set of all finite strings of letters of the alphabet is enumerable.

List: a, b, c, …, z; aa, ab, …, zz; aaa, aab, …, zzz; …

k) The set of all finite subsets of P is enumerable.

[Be clear on the difference: ordered k-tuples; subsets with k members; sequence of length k.]

l) The set of all finite sequences $<a_1,\ldots,a_n>$ in P.

Method 1. *Step 1*: For each n, the set of all sequences of length n is enumerable. *Step 2*: Form an infinite array, where row n contains all sequences of length n. *Step 3*: convert the array to a list.

Method 2. Use prime decomposition. Basic facts:

- Prime numbers: 2, 3, 5, 7, 11, … [evenly divisible only by itself and 1]
- There are infinitely many prime numbers.
- Prime factorization: $84 = 2^2 \cdot 3 \cdot 7$; $220 = 2^2 \cdot 5 \cdot 11$
  [every number can be written as the product of prime numbers]
- Uniqueness: this prime factorization is unique.

The encoding function G transforms each finite sequence into a single positive integer as follows:

	G	
<1, 22, 3, 5>	$\rightarrow$	$2^1 \cdot 3^{22} \cdot 5^3 \cdot 7^5$
<7, 63>	$\rightarrow$	$2^7 \cdot 3^{63}$
<4, 5, 9, 110, 37>	$\rightarrow$	$2^4 \cdot 3^5 \cdot 5^9 \cdot 7^{110} \cdot 11^{37}$

In general: $G(<a_1,\ldots,a_n>) = p_1^{a1} \cdot \ldots \cdot p_n^{an}$, where p_n is the nth prime number.
($p_1 = 2$, $p_2 = 3$, etc.)

Note: The decoding function is easy to apply. Just factor the number and use the exponents. *Example*: $24 = 2^3 \cdot 3^1$ stands for the sequence <3, 1>. Notice, though, that not every number encodes a sequence. For example, $20 = 2^2 \cdot 5$ does not encode any sequence.

Theorem 1.1 (subsets of an enumerable set are enumerable): If B is enumerable and $A \subseteq B$, then A is enumerable.

Proof: Informal – we can put the members of B in a list. Now just delete all members of the list that aren't in A, and we have a listing of A.

Formal: We are given the existence of $f : P \rightarrow B$ with the range of f being all of B. Define $g(n) = f(n)$ if $f(n) \in A$, and undefined otherwise.

Theorem 1.2 (an enumerable union of enumerable sets yields an enumerable set): If A_1, A_2, A_3, … are all enumerable, then so is the set A consisting of the union of all of them. We write this as:

$$A = \bigcup_{i=1}^{\infty} A_i$$

Proof: Same two-by-two array technique as for Q can be used to list all the members of A.

Summary: Three ways to show that a set *S* is enumerable

a) *Formal Method:* show directly that there is a function f : P $\rightarrow$ *S* whose range is all of *S*.

Example: the even numbers E.

f(n) = 2n is a function from P to E whose range is all of E.

OR

g(n) = n if n is even and undefined if n is odd

Variation: Show that there is an encoding function G:S $\rightarrow$ P that is 1-1.

b) *Informal Method:* show that it is possible to place all the members of *S* in a list in such a way that each member of *S* will eventually appear, a finite number of positions from the beginning of the list.

Example: the positive rational numbers. This uses the important technique of placing the entire set in a two-dimensional array, and then putting the members into list form by either moving along successive diagonals or expanding squares.

c) *By appealing to Theorem 1.2:* If A_1, A_2, … is an enumerable list of enumerable sets, then the set *A* consisting of the union of all the A_i (written as $A = \bigcup_{i=1}^{\infty} A_i$) is enumerable.

Example: the set *S* of all finite strings of symbols from an enumerable alphabet is enumerable. For if S_i is the set of finite strings of length *i*, then S_i is enumerable, and so by the Theorem, since $S = \bigcup_{i=1}^{\infty} S_i$, *S* is enumerable.