Integration By Parts

1. **Thought of the Day.** "Do deaf mathematicians speak in sine language?"

2. **Integrals of products of functions**

   We know that

   $$\int p(x) + q(x)\,dx = \int p(x)\,dx + \int q(x)\,dx.$$

   What if we want to compute the integral of a product of functions?

   $$\begin{aligned} \int p(x)q(x)\,dx &\neq \int p(x)\,dx \cdot \int q(x)\,dx. \\ \int p(x)q(x)\,dx &= \;\; ? \end{aligned}$$

   Problem. For example, integrate

   $$\int xe^x dx.$$

   $$\int x dx = \frac{x^2}{2} + C_1, \qquad \int e^x dx = e^x + C_2 \qquad \Rightarrow \qquad \int xe^x dx = \; ?$$

3. **Integration By Parts.**

Let f and g be differentiable functions. Then

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx.$$

Here is an easier way to remember this: for $u = f(x)$ and $v = g(x)$

$$\int udv = uv - \int vdu.$$

4. **Example.** Integrate $\displaystyle\int xe^x dx$.

5. **Examples.** Integrate

(a) $\displaystyle\int \ln x dx$

(b) $\displaystyle\int \arcsin x dx$

(c) $\int x^2 e^{-x} dx$

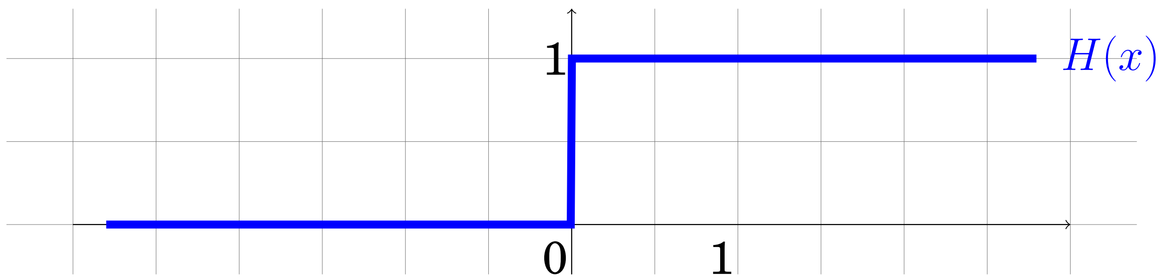
(d) $\int e^{2x} \cos 3x dx$

6. **Example.** What is

$$\int \cos^2 x \, dx \quad ?$$

7. **Example.** A rogue function - possibly even an impostor? The delta function.

The delta function $\delta(x)$ is the derivative of a step function $H(x)$.
The function H (the Heaviside function) is zero for all negative x, and then jumps to 1 at $x = 0$; it is equal to 1 for all positive x.



$$\delta(x) = \frac{d}{dx} H(x).$$

Of course, we know we are getting into trouble here, the derivative does not exist – we learned that lesson in Calculus I.

But, let's explore and be unconventional. What can go wrong?

The slope $\frac{d}{dx}H(x)$ of $H(x)$ is _____ everywhere except at $x = 0$, where it jumps – it must be infinite there, it is a thin and tall spike – *infinitely* thin and *infinitely* tall.

We cannot really give a value to $\delta(0)$, but we know its integral across the jump. For every $T > 0$,

$$\int_{-T}^{+T} \delta(x)\, dx = \int_{-T}^{+T} \frac{d}{dx} H(x)\, dx = [H(x)]_{-T}^{+T} = H(T) - H(-T) = 1.$$

"The area under the infinitely tall and infinitely thin spike δ equals 1.

8. **Integrals of the δ function.** Let $T > 0$, and let v be a continuous function.

$$\begin{array}{lrcl} (A) & \int_{-T}^{+T} \delta(x)\, dx & = & 1 \\[2ex] (B) & \int_{-T}^{+T} v(x)\delta(x)\, dx & = & v(0) \end{array}$$

9. Proof of (B) via integration by parts:

$$\begin{aligned} \int_{-T}^{+T} \underbrace{v(x)}_{\downarrow} \underbrace{\delta(x)}_{\uparrow}\, dx &= \\ &= \end{aligned}$$

Voilà!

10. **Examples.**

(a)

$$\int_{-5}^{5} e^{x}\delta(x)\ dx =$$

(b)

$$\int_{-T}^{T} (e^{x} + e^{-x})\delta(x)\ dx =$$

(c)

$$\int_{-1}^{1} \sin(x)\delta(x)\ dx =$$

(d)

$$\int_{-1}^{1} \left(\cos(x) + 2\sin(x + \frac{\pi}{2})\right)\ \delta(x)\ dx =$$

11. **Example.**

(a) Prove the reduction formula

$$\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx$$

(b) Use the reduction formula

$$\int \cos^n x dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx$$

from part (a) to evaluate $\int \cos^2 x dx$

(c) Use the reduction formula

$$\int \cos^n x dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx$$

from part (a) and part (b) to evaluate $\int \cos^4 x dx$

12. **Example.** Evaluate

$$\int_1^{\sqrt{3}} \arctan(1/x)dx.$$

Notes.