Math 1171 Practice Final

Total 110 marks

(10 marks) 0. Find the following derivatives. Do not simplify your answers.

(2 marks) (a) $\left[\dfrac{2^x}{1+x^2}\right]'$

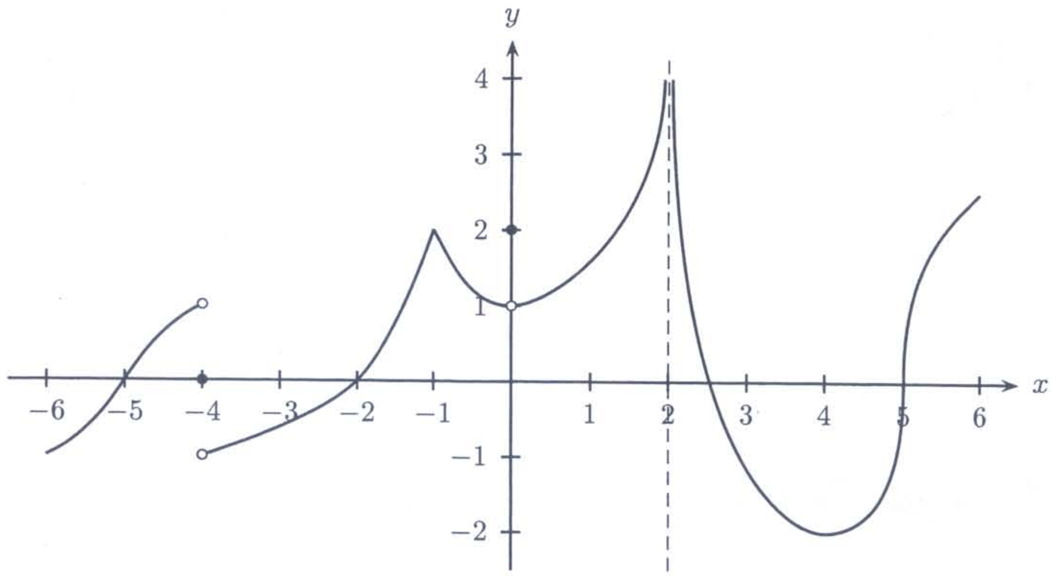
(2 marks) (b) $\dfrac{d}{dx}\left(\cos\left(\tan\sqrt{1+x^3}\right)\right)$

(3 marks) (c) $\left[\sin^{-1}(2x)\right]''$

(3 marks) (d) $y'(0)$ for $y=(1-x)^{(x-1)}$

(10 marks) 1. The graph of a function $f(x)$ is shown below. Answer the following questions based on the graph.

Be sure to justify each answer using one or two short statements.



(a) What is $\lim_{x \to 0} f(x)$?

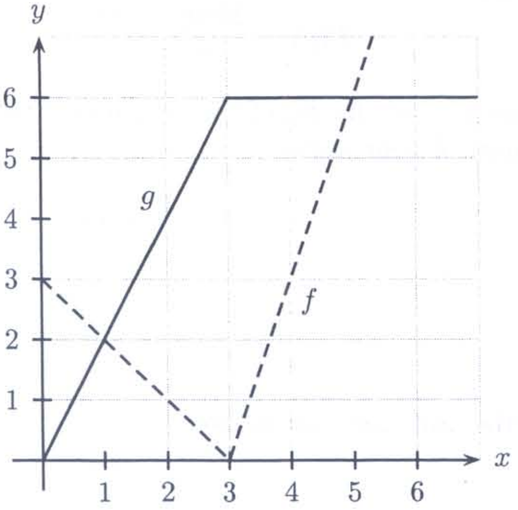
(b) What is $\lim_{x \to -4} f(x)$?

(c) What is $f'(4)$?

(d) For which values of x is f **not** continuous?

(e) For which values of x is f continuous but **not** differentiable?

(8 marks) 2. Two functions $f(x)$ and $g(x)$ are shown in the graph below. Use these graphs to answer the following questions.



(a) Let $P(x) = f(x)g(x)$. Find $P'(1)$.

(b) Let $Q(x) = \dfrac{f(x)}{g(x)}$. Find $Q'(5)$.

(c) Let $C(x) = f(g(x))$. Find $C'(2)$.

(d) Let $D(x) = g(f(x))$. Find $D'(4)$.

(8 marks) 3. An area of the forest was logged and then re-planted. The number of trees T growing in the area y years later is given by

$$T(y) = \frac{2000(1+y)}{2+5y} \quad .$$

(a) What is the average rate of change of the number of trees growing during the first two years? Give the units of your answer.

(b) Find $T'(2)$. Give the units for your answer.

(c) Write a sentence interpreting your answer in (b) as a rate of change.

(d) Find $\lim_{y\to\infty} T(y)$.

(e) What will happen to the number of trees after a long time? Justify your answer.

(6 marks) 4. Evaluate the following limits. Be sure to indicate where you use l'Hôpital's Rule, if necessary.

(a) $\displaystyle\lim_{x\to 1} \frac{\ln x}{x^2 - 1}$

(b) $\displaystyle\lim_{x\to\infty} x \sin\left(\frac{1}{x}\right)$

(8 marks) 5. Suppose a certain function y that depends on x satisfies the equation

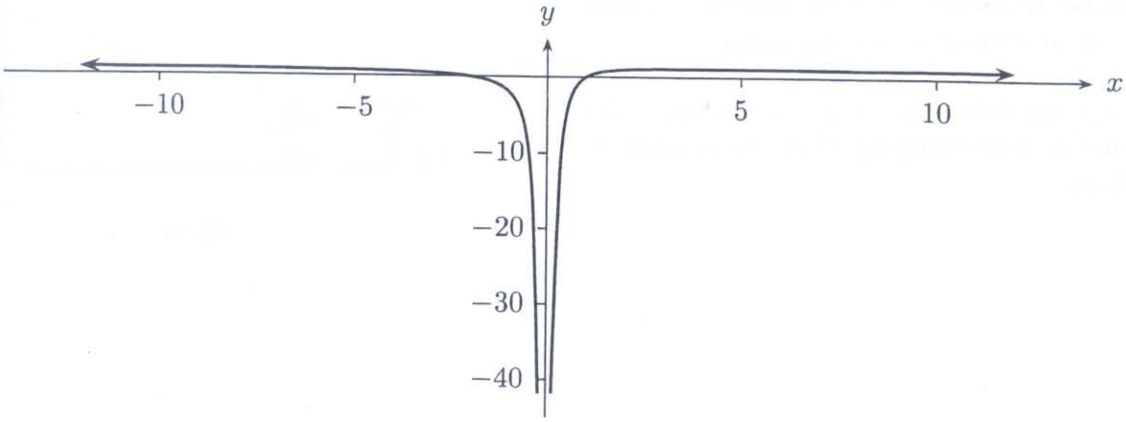
$$x^2 + xy + y^3 = 11$$

so that $y(1) = 2$.

(a) Find $y'(1)$.

(b) Use a linearization (linear approximation) to estimate $y(1.03)$.

(10 marks) 6. The graph of the function $f(x) = 1 + \frac{1}{x} - \frac{2}{x^2}$ fo is shown below:

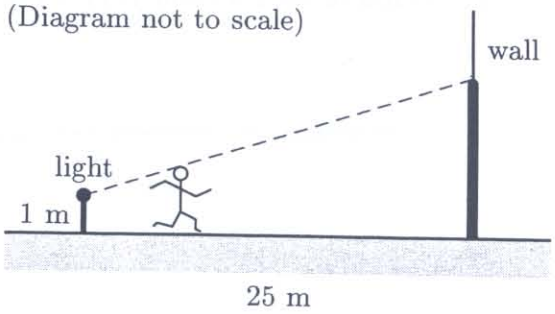


Use derivatives, limits, and algebra to determine the exact location (i.e., do not just estimate from the graph) of each intercept, asymptote, local minimum, absolute minimum, local maximum, absolute maximum, and inflection point.

(8 marks) 7. A spotlight 1 m off the ground and 25 m from a wall shines on the wall. A man 1.8 m tall walks from the spotlight directly towards the wall at 0.75 m/s (see diagram).

At what rate is the height of the man's shadow on the wall changing when the shadow is 5 m high?

(Diagram not to scale)



(8 marks) 8. A cylindrical cup with a bottom but **no top** is constructed from 75π cm^2 of material.

(a) Find the dimensions of the cup with the largest volume.

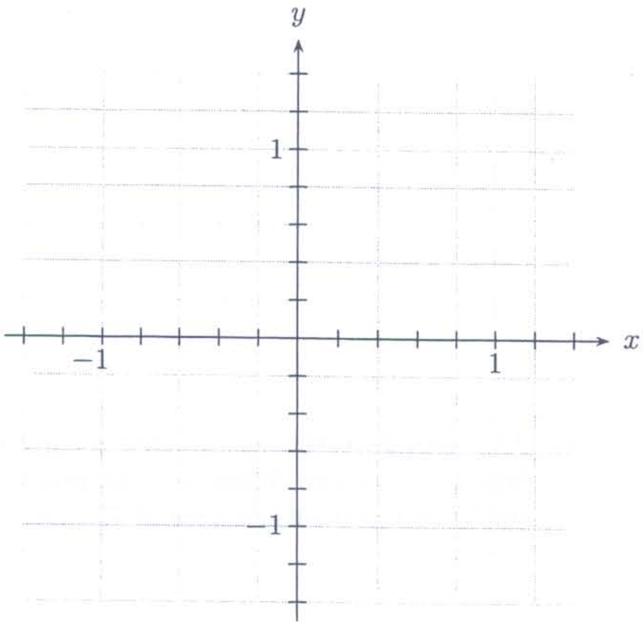
(b) The manufacturer decides that a child will not be able to hold the cup if the radius is larger than 3 cm. What are the dimensions of the **child's** cup with the largest volume that still has a surface area of 75π cm^2?

(8 marks) 9. Consider the curve described by the parametric equations

$$x(t) = \frac{1-t^2}{1+t^2}, \quad y(t) = \frac{2t}{1+t^2} .$$

(a) Show that $[x(t)]^2 + [y(t)]^2 = 1$ for all t.

(b) Plot the parametric curve for all values of $t \geq 0$ on the axes below. Label several points with the corresponding values of t.



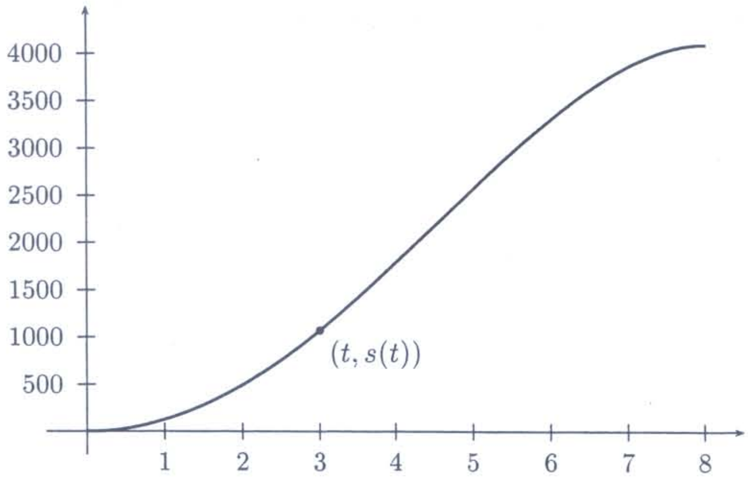
(c) Find the slope of the curve at the point $\left(-\frac{4}{5}, \frac{3}{5}\right)$.

(6 marks) 10. You are driving down the highway at 90 km/h when you see a bear at the side of the road. You want to stop and take a picture so you apply the brakes and decelerate at 2 m/s^2 (that is, accelerate at -2 m/s^2.)

(a) How many seconds does it take to stop the car? (Carefully note the units given in the question.)

(b) If the bear is 50 m in front of the car when you first apply the brakes, how far past the bear is the car when you finally stop?

(9 marks) 11. A particle travels a distance $s(t) = 128t^2 - t^4$ over the interval $0 \le t \le 8$:



(3 marks) (a) Find the time at which the particle's speed is a maximum.

(1 marks) (b) The particle's average speed up until time t is given by $s(t)/t$. Draw a line on the graph through the origin whose slope represents $s(t)/t$.

(2 marks) (c) Clearly label the point on the graph at which you think the average speed $s(t)/t$ is maximum.

(3 marks) (d) Show that $s(t)/t$ will be a maximum at a time when $s'(t) = s(t)/t$, and find this value of t.

(7 marks) 12. (a) Prove that the equation $x = 1 + \frac{1}{2}\sin x$ has *exactly* one solution.
[*Hint*: Consider *f(x)* and its derivative for a suitable function *f(x)*.]

(b) Use Newton's method to find the value of x to one decimal place.

(4 marks) 13. If $\$A$ is invested at 8% p.a. continuously compounded, then the amount after t years will be $y = Ae^{0.08t}$. *Use differentials* to estimate the percentage increase in y:
(i) over any one-year period;
(ii) over any 18-month period.

Answers:

0. $(a)\ \dfrac{(1+x^2)2^x \ln 2 - 2^x.2x}{(1+x^2)^2}$; $(b)\ -\sin(\tan\sqrt{1+x^3}).\sec^2(\sqrt{1+x^3}).\dfrac{1}{2\sqrt{1+x^3}}.3x^2$;

$(c)\ 8x(1-4x^2)^{-\frac{3}{2}}$; $(d)\ y' = (1-x)^{x-1}(1+\ln(1-x))$; $y'(0)=1$.

1. (a) 1; (b) DNE; (c) 0; (d) $-4,\ 0,\ 2$; (e) $-1,\ 5$.

2. (a) 2; (b) $\frac{1}{2}$; (c) 6; (d) DNE.

3. (a) -250 *trees/ yr*; (b) $-41\frac{2}{3}$ *trees/ yr*; (c) After 2 years, the number of trees growing is declining at the rate of $41\frac{2}{3}$ *trees/ yr*; (d) 400 *trees*, (e) *approach* 400.

4. (a) $\frac{1}{2}$; (b) 1.

5. (a) $-\frac{4}{13}$; (b) 1.99.

6. *Intercepts*: $x=-2,\ 1$; *VA*: $x=0$, *HA*: $y=1$; *loc. and abs.* max $\frac{9}{8}$ *at* $x=4$; IP $(6,\frac{10}{9})$.

7. $-0.6\,m/s$.

8. (a) *radius* 5*cm*, *height* 5*cm*; (b) *radius* 3*cm*, *height* 11*cm*.

9. (a) You're on the unit circle! (b) Semicircle traced counterclockwise starting at (1,0), reaching top at $t=1$, and on towards the limit (-1,0) as t goes to infinity. (c) 4/3 (note $y' = -x/y$ from (a)).

10. (a) 12.5 sec. (b) 106.25 m.

11. (a) $\dfrac{8}{\sqrt{3}} \approx 4.62$; (b) Line through origin and point $(t,s(t))$; (c) Point of tangency of tangent line to graph through origin (near top of curve); (d) slope of line drawn in (c) is both $\dfrac{s(t)}{t}$ *and* $s'(t)$ for $t=\sqrt{\dfrac{128}{3}} \approx 6.53$. (Note also $\left(\dfrac{s(t)}{t}\right)' = \dfrac{ts'(t)-s(t)}{t^2} = 0 \Leftrightarrow s'(t) = \dfrac{s(t)}{t}$).

12. (a) $f(x) = x - 1 - \frac{1}{2}\sin x$; $f(1) = -\frac{1}{2}\sin 1 < 0$, $f(\frac{\pi}{2}) = \dfrac{\pi}{2} - \dfrac{3}{2} > 0 \Rightarrow f(x)$ has a zero near $\dfrac{\pi}{2}$ in $(1,\dfrac{\pi}{2})$. Also $f'(x) = 1 - \frac{1}{2}\cos x \geq \dfrac{1}{2} > 0 \Rightarrow f(x)$ *increases on* $(-\infty,\infty)$, so there can be only one zero.
(b) 1.5

13. $dy = 0.08Ae^{+0.08t}dt \Rightarrow \dfrac{dy}{y} = 0.08dt$ (i) 8% (dt=1) (ii) 12% (dt=1.5).