CMPT 419 – Assignment 1

1 Satellite Orbiting Earth

a)

$$
\begin{aligned}
&x_1 = r\\
&x_2 = \dot{r}\\
&x_3 = \theta\\
&x_4 = \dot{\theta}\\
&therefor th,\\
&\dot{x_1} = x_2\\
&\dot{x_2} = x_1 x_4^2 - \frac{k}{x_1^2} + u_1\\
&\dot{x_3} = x_4\\
&\dot{x_4} = -\frac{2x_2x_4}{\mathrm{x}_1} + \frac{1}{\mathrm{x}_1}\mathrm{u}_2
\end{aligned}
$$

b)

$$
\begin{aligned}
&let\ \mathrm{u}_1 = \mathrm{u}_2 = 0\\
&\dot{x_1} = x_2 = 0\\
&\dot{x_2} = x_1 x_4^2 - \frac{k}{x_1^2} = 0\\
&\dot{x_3} = x_4 = 0\\
&\dot{x_4} = -\frac{2x_2x_4}{\mathrm{x}_1} = 0\\
&k = x_1^3 x_4^2\\
&\dot{x_2} = 0 - \frac{\mathrm{k}}{\mathrm{x}_1^2}\\
&\text{Equilibrium point is when } \dot{x}_{1,2,3,4} = 0\\
&\dot{x_2}\text{is not 0. We can't reach any Equilibrium}
\end{aligned}
$$

Being at an equilibrium point implies we are at $\dot{x}_{1,2,3,4}$=0. All $\dot{x}$ and $x_{2,4}$ are 0. x_3 is constant, as x_4 is 0 (recall that $x`_3 = x_4$). Once again, this is not the case as $x`_2$ is positive, because both k and r, are positives. <u>This contradicts, we are not at equilibrium.</u>

Being at an equilibrium point, that's is, that we are at $\dot{x}_{1,2,3,4}$=0, means the two objects are at a fixed distance from each other. $x`_2 = -k/x_1$ We need a rocket will need to be firing at the perfect amount to counter gravity. We need a *control*, like in the next question.

c)

$$
let\ \mathrm{u}_1 = \frac{\mathrm{k}}{\mathrm{r}^2};\ \mathrm{u}_2 = 0
$$

$$
\dot{x_1} = x_2
$$

$$\dot{x_2} = x_1 x_4^2 - \frac{k}{x_1^2} + \frac{k}{x_1^2}$$
$$\dot{x_3} = x_4$$
$$\dot{x_4} = -\frac{2x_2x_4}{\mathrm{x}_1}$$

Equilibrium means $\dot{x}_{1,2,3,4}$=0. We have $x_{2,4}$ = 0 = $\dot{x}_{1,3}$, which results in $\dot{x}_{2,4}$ =0.

To my understanding, the system is at equilibrium, the effect of gravity is mitigated, due to u_1 the satellite can be at any distance x_1 from earth, at some x_3, the phase of orbit. The control is present and sufficient to establish and maintain equilibrium.

d)

Linearize the model with respect to a reference orbit given by $r(t) \equiv \rho$; $\theta(t)$ = $omg*t$; $u_1 = u_2 = 0$.

$$x_1 = r$$
$$x_2 = \dot{r}$$
$$x_3 = \theta$$
$$x_4 = \dot{\theta}$$
$$therefforth,$$
$$f_1 = \dot{x_1} = x_2$$
$$f_2 = \dot{x_2} = x_1 x_4^2 - \frac{k}{x_1^2} + u_1$$
$$f_3 = \dot{x_3} = x_4$$
$$f_4 = \dot{x_4} = -\frac{2x_2x_4}{\mathrm{x}_1} + \frac{u_2}{x_1}$$

please ignore the potential incorrect matrix nesting. I keep U1 & u2 for this step.

$$\frac{\partial f}{\partial x} = mat\left(\{0,1,0,0\}\ ,\left\{\frac{2\mathrm{k}}{\mathrm{x}_1^3} + \mathrm{x}_4^2, 0, 0, 2\mathrm{x}_3\mathrm{x}_1\right\}, \{0,0,0,1\}, \left\{\frac{2\mathrm{x}_2x_4}{x_1^2} - \frac{u_2}{x_1^2}, \frac{2\mathrm{x}_4}{x_1}, \frac{2x_2}{x_1}, 0\right\}\right)$$
$$\frac{\partial f}{\partial u} = mat\left(\{0,0\}, \{1,0\}, \{0,0\}, \left\{0, \frac{1}{x_!}\right\}\right)$$

Use the following points:

$$x_1 = r(t) = p$$
$$x_2 = 0$$
$$x_3 = \theta(t) = \omega t$$
$$x_4 = \omega$$
$$u_1 = u_2 = 0$$
$$\frac{\partial f}{\partial x} = mat\left(\{0,1,0,0\}\ ,\left\{\frac{3\mathrm{k}}{p^3}, 0, 0, 2\omega\mathrm{p}\right\}, \{0,0,0,1\}, \left\{0, \frac{2\omega}{p}, 0, 0\right\}\right)$$
$$\dot{x_1} = 0$$
$$\dot{x_2} = p\omega^2 - \frac{k}{p^2} = 0$$
$$\dot{x_3} = \omega = 0$$
$$\dot{x_4} = 0$$
$$k = \omega^2 p^3;\ \ \omega^2 = \frac{\mathrm{k}}{\mathrm{p}^3}$$

Substitute values & make system linear:

$$\tilde{x}` = \ mat\left(\{0,1,0,0\}\ ,\{3\omega^2,0,0,2\omega p\},\{0,0,0,1\},\left\{0,\frac{2\omega}{p},0,0\right\}\right)*\tilde{x}+\ mat\left(\{0,0\},\{1,0\},\{0,0\},\left\{0,\frac{1}{p}\right\}\right)*\tilde{u}$$

Where ~x and ~u are:

$$\tilde{x} = mat(\{x_1-p\},\{x_2\},\{x_3-\omega t\},\{x_4-\ \omega\})$$
$$\tilde{u} = mat(\{u_1\},\{u_2\})$$

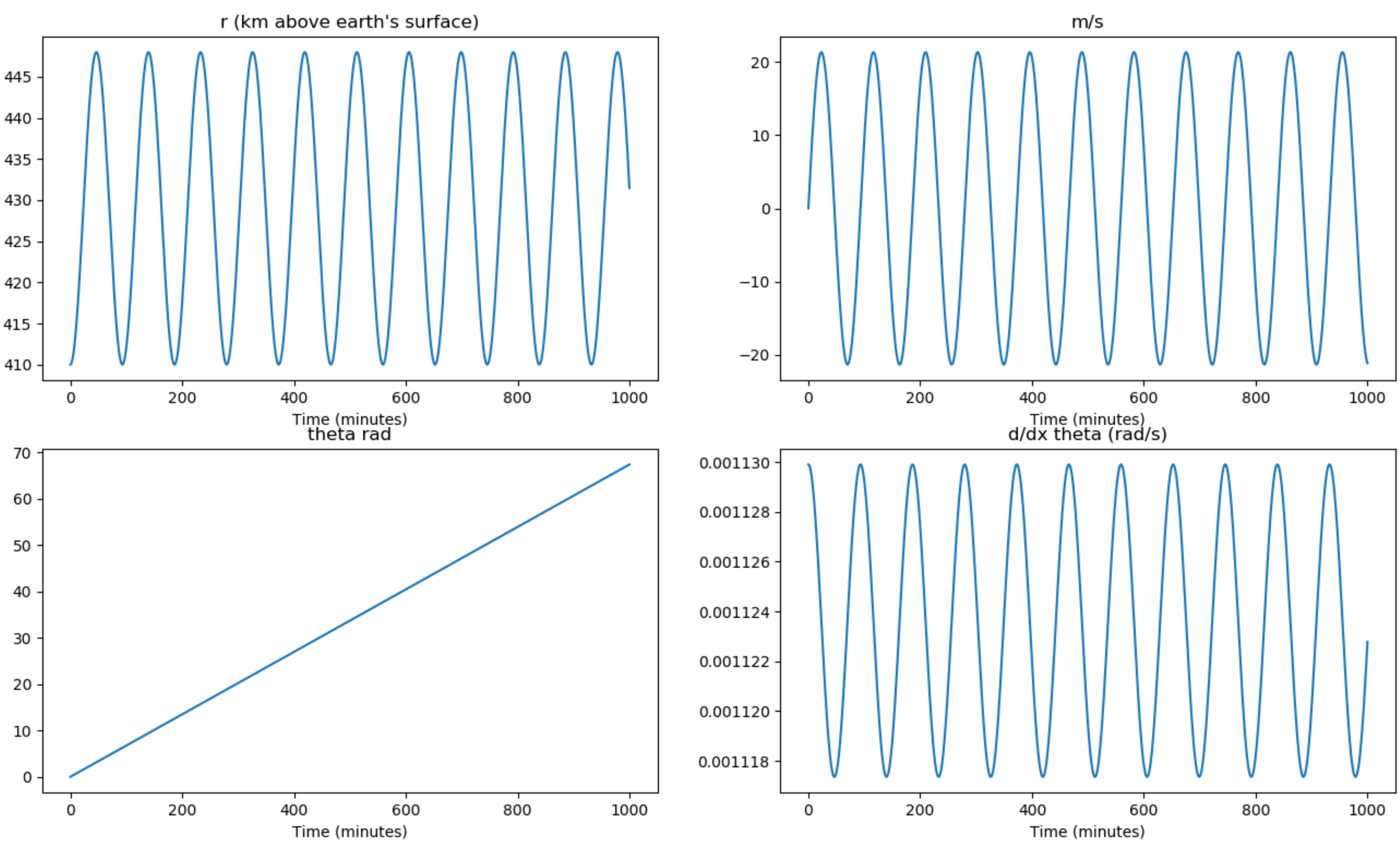
Which is:

$$\tilde{x}` = \ mat\left(\{0,1,0,0\}\ ,\{3\omega^2,0,0,2\omega p\},\{0,0,0,1\},\left\{0,\frac{2\omega}{p},0,0\right\}\right)*mat(\{x_1-p\},\{x_2\},\{x_3-\omega t\},\{x_4-\ \omega\})$$
$$+\ mat\left(\ \{0,0\},\{1,0\},\{0,0\},\left\{0,\frac{1}{p}\right\}\right)*mat(\{u_1\},\{u_2\})$$

which is:

$$\tilde{x}` = \ mat(\{0,0,0,0\}\ ,\{3\omega^2*(x_1-p),0,0,2\omega p*(x_4-\ \omega)\},\{0,0,0,x_4-\ \omega\},\{0,0,0,0\})$$
$$+\ mat\left(\ \{0,0\},\{u_1,0\},\{0,0\},\left\{0,\frac{u_2}{p}\right\}\right)$$
$$note\ that\ x_2 = 0$$

e)



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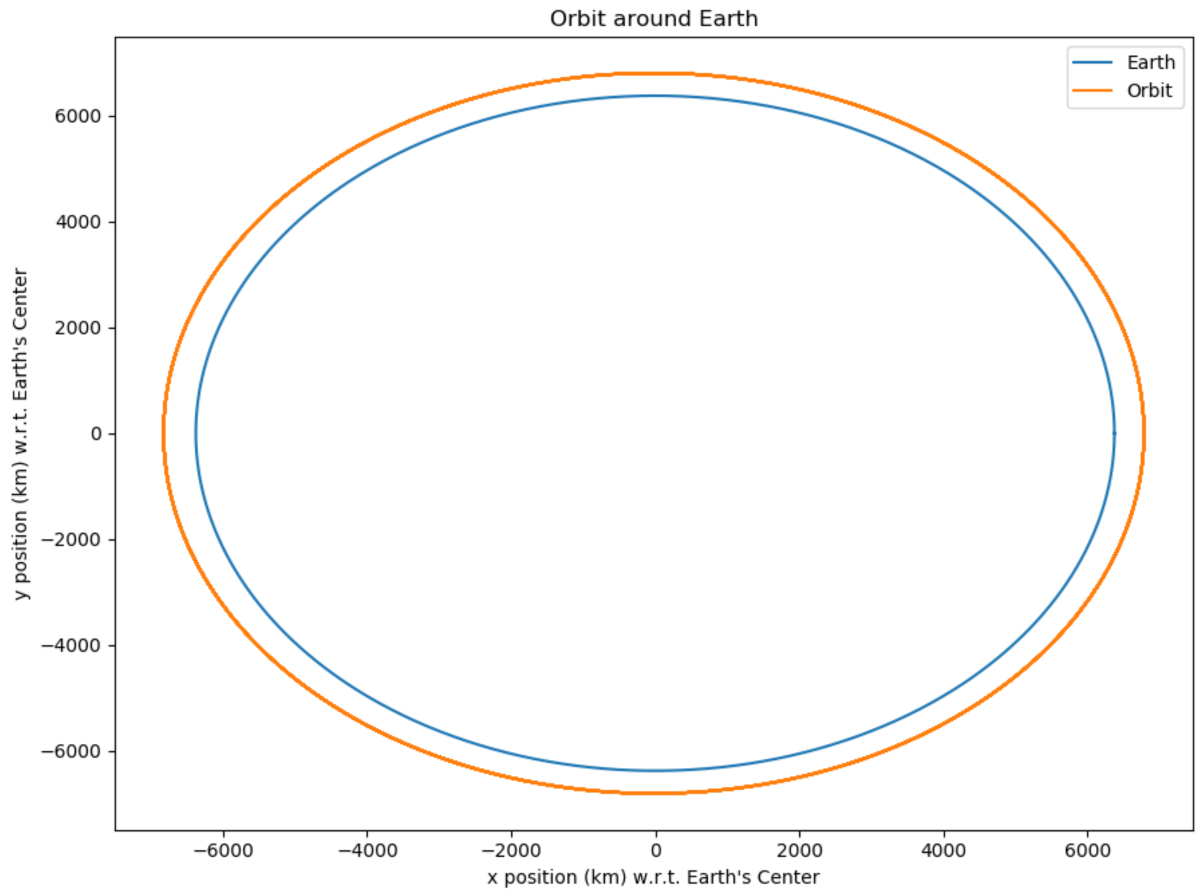


Orbit around Earth
Earth
Orbit
y position (km) w.r.t. Earth's Center
x position (km) w.r.t. Earth's Center
6000
4000
2000
0
−2000
−4000
−6000

2 The Lotka-Volterra Predator-Prey Model

Predator-prey model: x is number of preys, y is number of predators.

- a represents the birth rate of prey,
- d represents the death rate of predators,
- b is the prey's susceptibility to predators, and
- c is the ability of predators to hunt prey

x' = *ax - bxy*
y' = -dy + cxy

a) Find the (non-trivial) equilibrium of the system.

x' = x(a-by)

y' = y(-d+cx)

trivial points,

x' = 0 => y=0

y' = 0 => 0 = *-dy + cxy* // because y=0

non-trivial equilibrium point,

$$y = \frac{a}{b}, x = \frac{d}{c}$$

non-trivial equilibrium point = $\left(\frac{d}{c}, \frac{a}{b}\right)$

b) Using Lyapunov Stability – slide 9 of non-linear systems part II lecture.

Let $V(x;\ y) = \ y^a\, e^{-by}\, x^d\, e^{-cx}$. Show that $\dot{V}(x;\ y) = \ 0$

$\dot{V}(x;\ y) = \ y^a\, e^{-by}\, x^d\, e^{-cx}$

$0 = \ x^d\, e^{-cx} y^a\, e^{-by}$

$0 = \ y^a e^{-by}\left(\mathrm{d}\mathrm{x}^{\mathrm{d}-1} e^{-cx} - x^d \mathrm{ce}^{-\mathrm{cx}}\right)\mathrm{x}` \ + \ x^d\, e^{-cx}\left(\mathrm{ay}^{\mathrm{a}-1}\mathrm{e}^{-\mathrm{by}} - \mathrm{y}^{\mathrm{a}}\mathrm{be}^{-\mathrm{by}}\right)\mathrm{y}`$

$0 = \mathrm{y}^{\mathrm{a}}\mathrm{x}^{\mathrm{d}}\mathrm{e}^{-\mathrm{by}}\mathrm{e}^{-\mathrm{cx}}(\mathrm{da} - \mathrm{cax} - \mathrm{bdy} + \mathrm{cbxy}) + \mathrm{y}^{\mathrm{a}}\mathrm{e}^{-\mathrm{by}}(-\mathrm{ed} + \mathrm{cby} + \mathrm{dex} - \mathrm{cbxy})$

Cancel out likes terms

$0 = \left(\mathrm{y}^{\mathrm{a}}\mathrm{x}^{\mathrm{d}} e^{-cx}\mathrm{e}^{-\mathrm{by}}\right)(0) \rightarrow \dot{V}(\mathrm{x}, \mathrm{y}) = 0$

{Used wolfram alpha}

c) Prove that the system is stable around the equilibrium point.

Hint: Find the maximum of *V* (*x; y*), by using the fact that log *V* (*x; y*) has the same maximum.
In addition, consider the convexity of log *V* (*x; y*)

$$V(x;\ y) = \ y^a\ e^{-by}\ x^d\ e^{-cx}$$

$$\ln\left(y^a\ e^{-by}\ x^d\ e^{-cx}\right)$$

Use product rule & power rule

$$\ln\left(y^a\right) + \ln\left(e^{-by}\right) + \ln\left(\ x^d\right) + \ln\ \left(e^{-cx}\right)$$
$$a * ln(y) - by + d * \ln(x) - cx$$

Want: ∇ ln($V(x, y)$) = 0. Function is concave, so we set to 0.

$$\frac{\partial \ln\left(V(x,y)\right)}{\partial x} = \frac{d}{x} - c = 0$$
$$x = \frac{d}{c}; as\ above$$

$$\frac{\partial \ln\left(V(x,y)\right)}{\partial y} = \frac{a}{y} - b = 0$$
$$y = \frac{a}{b}; as\ above$$

(d/c, a/b) is the maximum and equilibrium of V(x,y). While V\`(x,y) is 0, for any x, y.

3 Stabilization via Linear Feedback

$$\dot{x} = Ax + Bu$$

$$A = \begin{matrix} 1 & 0 \\ 1 & -2 \end{matrix}$$

$$B = \begin{matrix} 1 \\ 1 \end{matrix}$$

$$(A - \lambda I)x = 0$$

$$Det\begin{pmatrix} 1-\lambda & 0 \\ 1 & -2-\lambda \end{pmatrix} = (1-\lambda) * (-2-\lambda) - 0$$

$$Det\begin{pmatrix} \lambda-1 & 0 \\ 1 & \lambda+2 \end{pmatrix} = (\lambda-1) * (\lambda+2) - 0$$

$$Det(\lambda I - A) = 0 = \lambda^2 + \lambda - 2$$
$$\lambda = -2, 1$$

Instable system

Choose K so that $u = -Kx$ stabilizes the system.

$$\overline{A} = A - BK$$

$$K = \{k_1 k_2\}$$

$$\overline{A} = \begin{matrix} 1 - k_1 & 0 - k_2 \\ 1 - k_1 & -2 - k_2 \end{matrix}$$

$$\det(\overline{A} - \lambda I) = (1 - k_1 - \lambda)(-2 - k_2 - \lambda) - (-k_2)(1 - k_1)$$

Choose $k1$, $k2$ such that det(A_bar – λI) = 0 gives λ in the open left half plane

$$0 = \det(\overline{A} - \lambda I) = -2 + 2k_1 + 2\lambda - k_2 - \lambda + 2k_1 K_2 + \lambda k_1 + 2\lambda k_2 + \lambda^2 - [-k_2 + k_1 k_2]$$

$$0 = \det(\overline{A} - \lambda I) = -2 + \lambda + \lambda^2 + \lambda k_1 + 2\lambda k_2 + 1k_2 \ + k_1 k_2$$
$$0 = \lambda^2 + k_1 \lambda + k_2 \lambda + \lambda + 2k_1 - 2$$
$$0 = \lambda^2 + \lambda(k_1 + k_2 + 1) + 2k_1 - 2$$

Slides say there is one choice for lambda, lecture says we can choose…

$$I\ wanted\ to\ use\ (\lambda + 2)(\lambda - 1), but\ that\ didnt\ work$$

$$So\ use\ (\lambda + 2)(\lambda + 1) = \lambda^2 + 3\lambda + 2$$

let's do "match coefficients" way from lecture.

Given,

$$2 = 2k_1 - 2; 4 = 2k_1; k_1 = 2$$

$$3 = k_1 + k_2 + 1; 2 = 2 + k_2; k_2 = 0$$

From simplifying each equation independently, we got k_1 =2, and K_2 = 0.

$u = [2 \quad 0]..therefor th, 2x_1 0x_3$ $stabilizes\ the\ system$

$u = -Kx => \dot{x} = (A - BK)x$

4 Numerical Solutions to ODEs

a) Determine conditions on the time step size that must be satisfied for the forward Euler method to be stable.

I heavy referenced slide 20/42 of the ODE lecture.

The eigenvalues are: [-1., -500.]

And just for fun, the normed eigenvectors are [[0.707, -0.002], [-0.707, 0.999]]))

- Forward Euler: $y^{k+1} = y^k + hf(y^k)$
  $= y^k + hAy^k$
  $= (I + hA)y^k$

Where y is the X matrix and h is time step.

Eigenvalues for hA: -h, -500h

Eigenvalues for I + hA: 1 - h, 1 -500h

So, we need |1-h| < 1 and |1-500h| < 1.

h = 0.02/5 = 0.004. Because 1 - 500*0.004 = -1. Note that sqrt(-1 *-1) is 1

so, for forwards Euler's method to be stable <u>we need h=0.004</u>. note the other value is h=2, but of course, the smaller h is the minimum step size.

b) Repeat the above two steps for the backward Euler method.

Backward Euler

- Our system: $\dot{x} = Ax$
- Backward Euler:
  - $y^{k+1} = y^k + hf(y^{k+1})$
  - $y^{k+1} = y^k + hAy^{k+1}$
  - $(I - hA)y^{k+1} = y^k$
  - $y^{k+1} = (I - hA)^{-1}y^k$
    - Eigenvalues of $(I - hA)^{-1}$ are $\left(1 - h\sigma(A)\right)^{-1}$
    - No restrictions on h if eigenvalues of A have negative real part

Where y is the X matrix and h is time step.

eig = -1, -500

we need |(I – hA)^-1| < 1, which is the same as (I – h*eig)^-1 < 1 , for all eigenvalues

which, to my understanding is the same as |I – h*eig| >= 1

We want:

|I + h*1| >= 1, which is satisfied for all h > 0

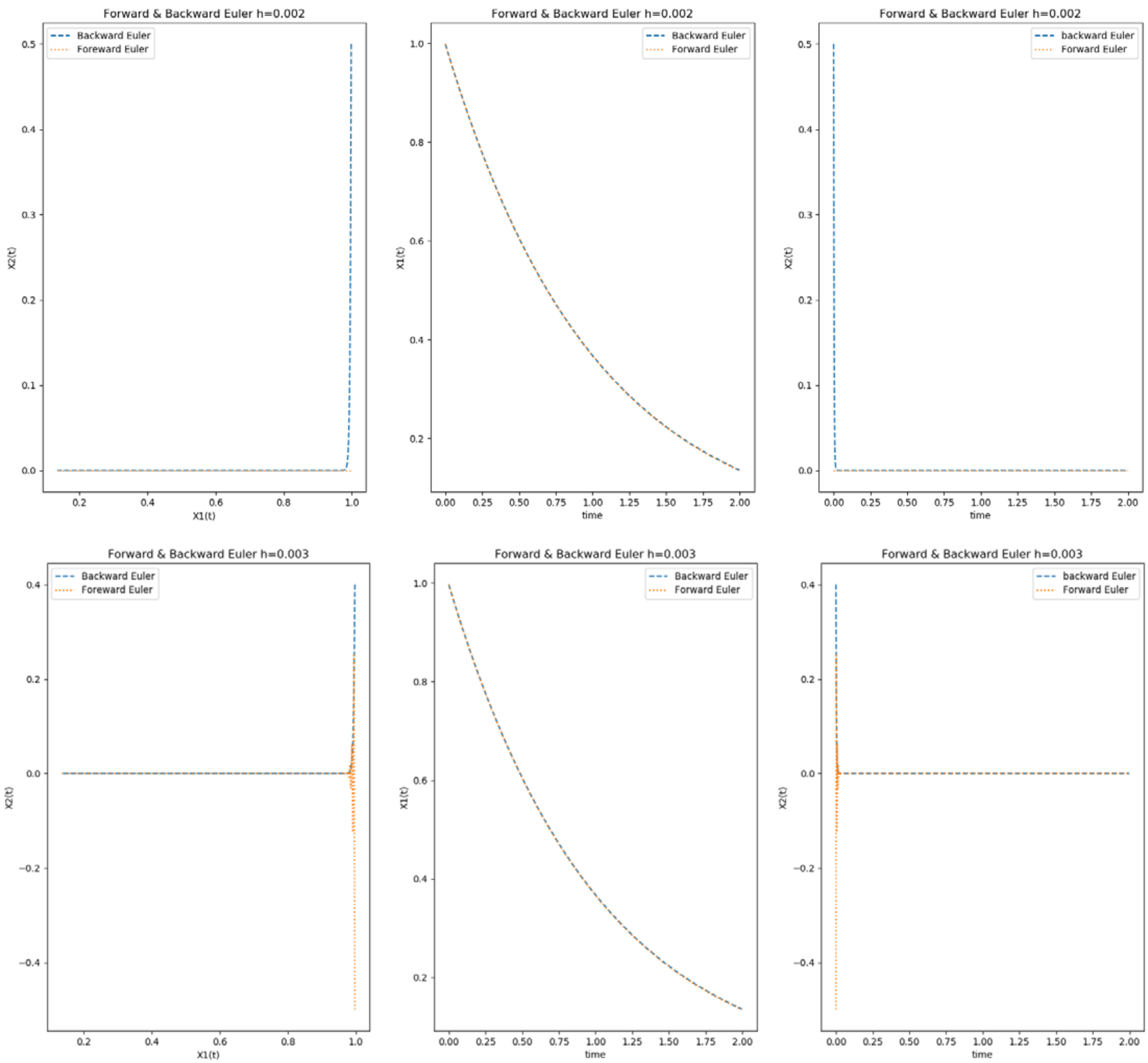
|I + h*500| >= 1, which is satisfied for all h > 0

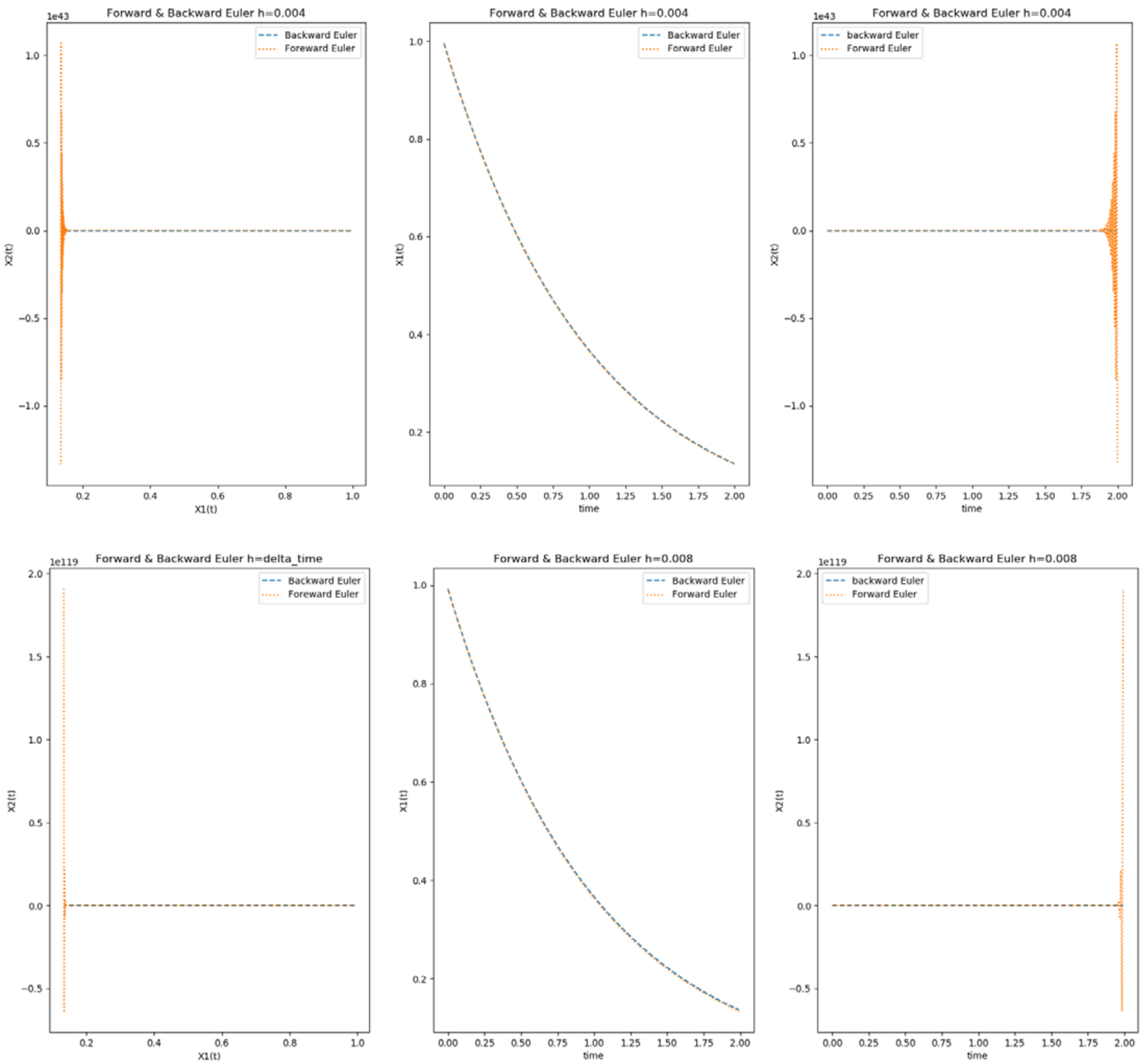
c)

The screen shots:

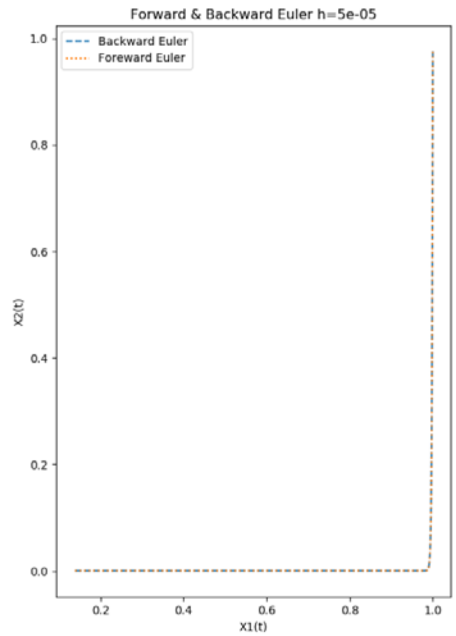
Note I got h = 0.004 in *4a*

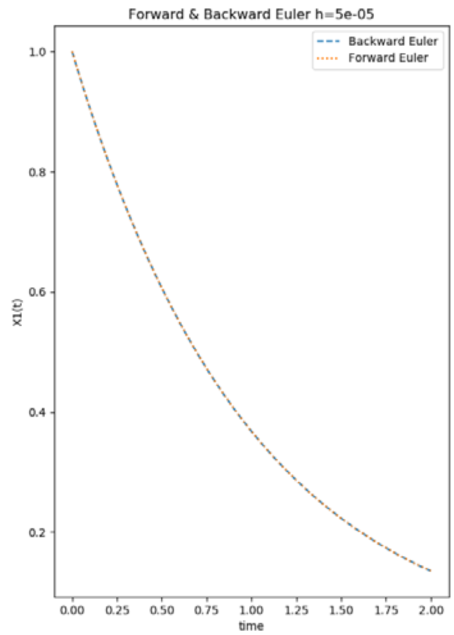
Note that my X_1 and X_2 maybe be **flipped**, compared to people who computed via the matrix, I used the eiganvalues method

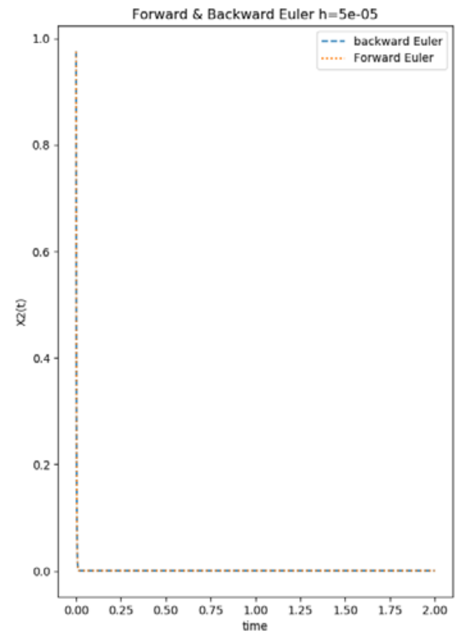


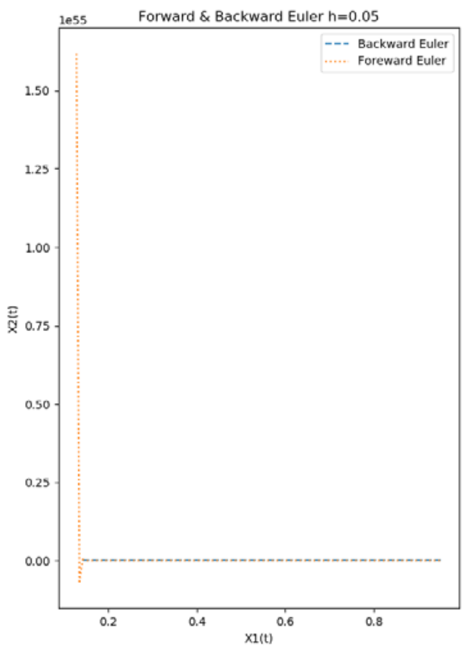

Forward & Backward Euler h=0.004
1e43
Backward Euler
Foreward Euler
X2(t)
X1(t)
Forward & Backward Euler h=0.004
Backward Euler
Forward Euler
X1(t)
time
Forward & Backward Euler h=0.004
1e43
backward Euler
Forward Euler
X2(t)
time
Forward & Backward Euler h=delta_time
1e119
Backward Euler
Foreward Euler
X2(t)
X1(t)
Forward & Backward Euler h=0.008
Backward Euler
Forward Euler
X1(t)
time
Forward & Backward Euler h=0.008
1e119
backward Euler
Forward Euler
X2(t)
time

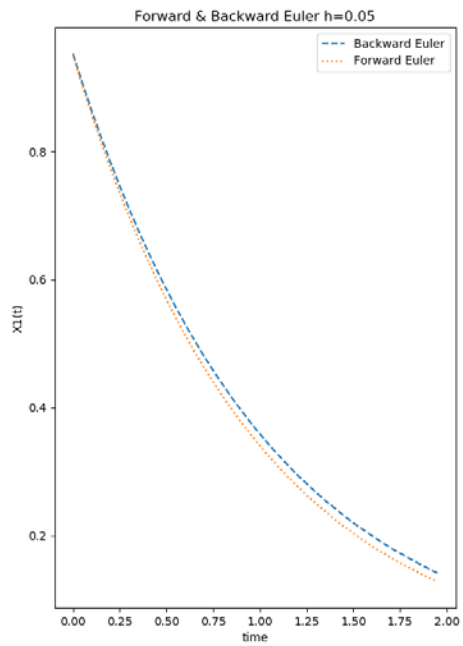
Bonus 2:

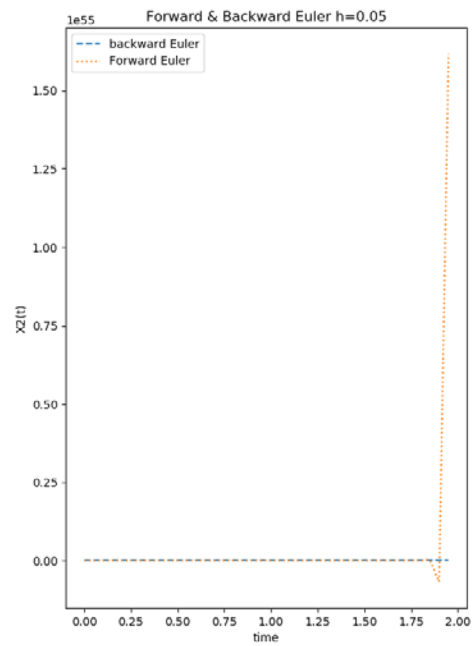

Forward & Backward Euler h=5e-05
Backward Euler
Foreward Euler
X2(t)
X1(t)


Forward & Backward Euler h=5e-05
Backward Euler
Forward Euler
X1(t)
time


Forward & Backward Euler h=5e-05
backward Euler
Forward Euler
X2(t)
time


Forward & Backward Euler h=0.05
1e55
Backward Euler
Foreward Euler
X2(t)
X1(t)


Forward & Backward Euler h=0.05
Backward Euler
Forward Euler
X1(t)
time


Forward & Backward Euler h=0.05
1e55
backward Euler
Forward Euler
X2(t)
time