Absolute Convergence and the Ratio and Root Tests

1. **Quote.** ”Power tends to corrupt, and absolute power corrupts absolutely. Great men are almost always bad men.”

   (John Emerich Edward Dalberg Acton (Lord Acton), British histortian and moralist, 1834-1902)

2. **Quote.** “Power corrupts; PowerPoint corrupts absolutely.” (computer geek version)

3. **Summary - Absolute convergence**
   If a series converges absolutely ($\sum |a_n|$ is convergent), then working with this series is “safe”.
   Absolute convergence is the important concept!

4. **Key concept - comparison to geometric series.** Recall:

   Comparison test. Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with $0 \le a_n \le b_n$ for all $n \ge M$.

   (a) If $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

   (b) If $\sum_{n=1}^{\infty} a_n$ is divergent, then $\sum_{n=1}^{\infty} b_n$ is also divergent.

   Use this with $a_n = r^n$ $(r > 1)$ to prove divergence of $\sum b_n$,
   or $b_n = r^n$ $(r < 1)$ to prove convergence of $\sum a_n$.

5. **Problem.** Test if $\displaystyle\sum_{n=1}^{\infty} \frac{n2^n}{n!}$ is convergent or divergent.

6. **Definition.**

A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

For example, the series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \ldots = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is absolutely convergent since $\sum \frac{1}{n^2}$ is convergent.

7. **Definition.**

A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

We have already seen that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent, but we know that $\sum \frac{1}{n}$ (the harmonic series) is divergent.

8. **Theorem. (absolute convergence vs. convergence).**
If a series $\sum a_n$ is absolutely convergent then it is convergent.

(a) $\sum |a_n| < \infty$

(b) $b_n = a_n + |a_n|$

(c) $\sum b_n \leq$

(d) Since $a_n = \ldots\ldots$

$\sum a_n = \ldots\ldots$

9. **Example.** Determine if the series

$$\sum_{n=0}^{\infty} \frac{\sin n}{n^2 + 2n + 1}$$

is convergent or divergent.

10. **Test for absolute convergence (part 1): The Ratio Test.**

(a) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(b) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(c) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L = 1$, the Ratio Test is inconclusive; that is no conclusion can be drawn about the convergence or divergence of $\sum_{n=1}^{\infty} a_n$.

11. **Examples.** Test for convergence, using the ratio test.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n2^n}{n!}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n^2}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

12. **Test for absolute convergence (part 2): The Root Test.**

(a) **If** $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, **then the series** $\sum_{n=1}^{\infty} a_n$ **is absolutely convergent.**

(b) **If** $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ **or** $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, **then the series** $\sum_{n=1}^{\infty} a_n$ **is divergent.**

(c) **If** $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$, **the Root Test is inconclusive; no conclusion can be drawn about the convergence or divergence of** $\sum_{n=1}^{\infty} a_n$.

13. **Examples.** Test for convergence, using the root test.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5n-3n^3}{7n^3+2}\right)^n$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

Notes.