Sequences

1. **Quote.** "Life is a full circle, widening until it joins the circle motions of the infinite." (Anaïs Nin, French-American diarist, essayist, novelist, 1903-1977)

2. **Quote.**"I don't want to belong to any club that would have me as a member."

   (Julius Henry "**Groucho**" Marx, American comedian, writer, stage, film, radio, and television star, 1890-1977)

3. **Mensa Puzzle.** What number comes next in this sequence?

   $$1 \quad 3 \quad 8 \quad 19 \quad 42 \quad ?$$

   What is the 100th number in the sequence?

   Let's call the elements of this sequence a_n, $n = 1, 2, 3, \ldots$.

   Can you see a pattern?

$a_{100} = 3 \cdot 2^{99} - 101 = 2^{100} + 2^{99} - 101.$

4. **Sequence.**

A **sequence** is a function whose domain is the set $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ of positive integers.

If the function is $s : \mathbb{Z}^+ \to \mathbb{R}$, then the output $s(n)$ is usually written as s_n, we also write the whole sequence as $s = \{s_n\}$.

Note: Sometimes the domain of a sequence is may be taken as $\mathbb{N} = \mathbb{Z}^+ \cup \{0\}$, in which case we write $\{s_n\}_{n=0}^{\infty}$.

5. **Examples.**

(a) Write out the first few terms of the sequence

$$\{\cos n\pi\}_{n=2}^{\infty}.$$

Is it possible to write this sequence in a different form?

(b) Graph the sequence $\left\{1 + \frac{(-1)^n}{n}\right\}$.

6. **Definition: Limit of a sequence.**
(Informal definition)

A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \text{ or } a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.

If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or it is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**). Or even simpler, we say the sequence does not converge.

7. **Definition: Limit of a sequence.**
(Formal or mathematically rigorous definition, called the "ϵ-N definition")

A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \text{ or } a_n \to L \text{ as } n \to \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$|a_n - L| < \varepsilon \text{ whenever } n > N.$$

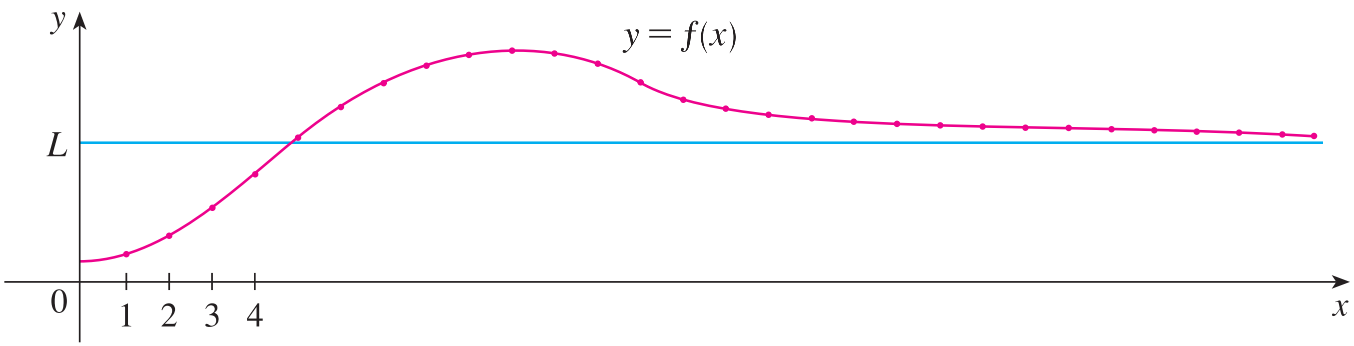
8. **Example.** Is the sequence $\left\{\dfrac{2n}{n+3}\right\}$ convergent or divergent?

9. **Example.** Is the sequence $1000\left\{(1+0.03)^n\right\}$ convergent or divergent?

   Note: This describes the amount of money in your bank account after year n, if you start with \$1000, never deposit or withdraw anything, and the bank pays you 3% interest. Or, how much you owe after n years if you borrow \$1000 at 3% interest, and don't make any payments.

10. **Theorem.**

Consider the sequence $f(n) = a_n$ where n is an integer.
If $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} a_n = L$.



11. **Definition.**

$$\lim_{n\to\infty} a_n = \infty$$

means that for every positive number M there is an integer N such that

$$a_n > M \text{ whenever } n > N.$$

12. **Facts about sequences.**

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

(a) $\lim\limits_{n\to\infty}(a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$

(b) $\lim\limits_{n\to\infty}(ca_n) = c\lim\limits_{n\to\infty} a_n$ (in particular, this means that $\lim\limits_{n\to\infty} c = c$)

(c) $\lim\limits_{n\to\infty}(a_n b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$

(d) $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$, as long as $\lim\limits_{n\to\infty} b_n \neq 0$

(e) $\lim\limits_{n\to\infty}(a_n)^p = \left(\lim\limits_{n\to\infty} a_n\right)^p$ only for $p > 0$ and $a_n > 0$.

(f) If $\lim\limits_{n\to\infty}|a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

(g) If $a_n \leq c_n \leq b_n$ for all $n \geq N$, and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = L$, then $\lim\limits_{n\to\infty} c_n = L$.

(h) If $\lim\limits_{n\to\infty} a_n = L$ and a function f is continuous at L, then $\lim\limits_{n\to\infty} f(a_n) = f(L)$

13. **Examples.**

(a) Show that the sequence $\{\sqrt[n]{n}\}$ converges to 1.

(b) Is the sequence $a_n = \sin\left(\dfrac{n\pi}{2}\right)$ convergent or divergent?

14. **Examples.**

(a) Does the sequence $\left\{\frac{\cos(n\pi)}{n}\right\}$ converge or diverge?

(b) For what values of r is the sequence $\{r^n\}$ convergent?

15. **Definition.**

A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \ldots$.
It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$.
It is called **monotonic** if it is either increasing or decreasing.

Note: In many cases it is only important how the sequence behaves for large n. For example, we may call a sequence increasing, if $a_n < a_{n+1}$ for all $n \geq M$, where M is an integer.

16. **Examples.** Decide which of the following sequences is increasing, decreasing or neither.

(a) $a_n = 1 + \frac{1}{n}$

(b) $b_n = 1 - \frac{1}{n}$

(c) $c_n = 1 + \frac{(-1)^n}{n}$

(d) $d_n = \left(\frac{1}{2}\right)^n$

(e) $e_n = \dfrac{10^n}{n!}$

17. **Definition.**

A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leq M \textbf{ for all } n \geq 1.$$

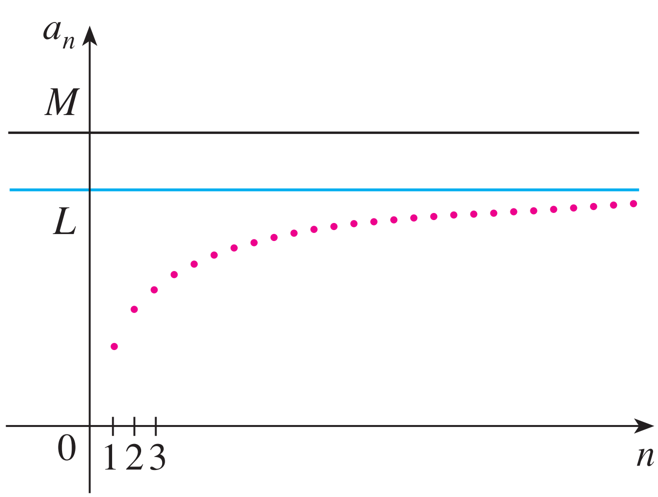
It is **bounded below** if there is a number m such that

$$m \leq a_n \textbf{ for all } n \geq 1.$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

18. **Monotonic Sequence Theorem.**

Every bounded, monotonic sequence is convergent.



19. **Example.** Investigate the sequence $\{a_n\}$ that is defined recursively by

$$a_1 = \sqrt{6},\ a_{n+1} = \sqrt{6 + a_n}\ , \quad \textbf{for } n \geq 1.$$

Notes.