The Substitution Rule

1. **Problem.** Find
$$\int -2xe^{-x^2}\,dx$$

2. **Hint.** What if we think of the "dx" above as a differential? If $u = e^{-x^2}$, what is the differential du?

3. **The Substitution Rule.**

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

4. **Notes:**

(a) This rule can be proved using the Chain Rule for differentiation. In this sense, it is a reversal of the Chain Rule.

(b) The substitution rule says that we can work with "dx" and "du" that appear after the $\int$ symbols as if they were differentials.

5. **Examples.** Find the following indefinite integrals:

(a) $\displaystyle\int x^2(x^3+5)^9 dx$

(b) $\displaystyle\int \frac{dt}{\sqrt{3-5t}}$

(c) $\displaystyle\int \sin 3t \; dt$

(d) $\displaystyle\int \frac{du}{u(\ln u)^2}$

(e) $\displaystyle\int \frac{\sin(\pi/v)}{v^2} dv$

(f) $\displaystyle\int \frac{z^2}{\sqrt{1-z}}\, dz$

6. Computers are ideal for computing integrals, and Wolfram|Alpha (www.wolframalpha.com) gives you easy access to this computing power. Use it as a tool to help you study. **But be warned:** you still have to understand how to do these computations yourself, since Wolfram|Alpha won't be with you for quizzes and exams.

WolframAlpha computational knowledge engine

integrate x^2(x^3+5)^9

Indefinite integrals:

Hide steps

$$\int x^2 (x^3+5)^9 \, dx =$$

$$\frac{x^{30}}{30} + \frac{5 x^{27}}{3} + \frac{75 x^{24}}{2} + 500 x^{21} + 4375 x^{18} + 26\,250 x^{15} + 109\,375 x^{12} + 312\,500 x^9 + \frac{1\,171\,875 x^6}{2} + \frac{1\,953\,125 x^3}{3} + \text{constant}$$

Possible intermediate steps:

$$\int x^2 (5+x^3)^9 \, dx$$

For the integrand $x^2 (x^3+5)^9$, substitute $u = x^3+5$ and $du = 3x^2 dx$:

$$= \frac{1}{3} \int u^9 \, du$$

The integral of u^9 is $\frac{u^{10}}{10}$:

$$= \frac{u^{10}}{30} + \text{constant}$$

Substitute back for $u = x^3+5$:

$$= \frac{1}{30} (x^3+5)^{10} + \text{constant}$$

7. **Substitution Rule for Definite Integrals.**

If g' is continuous on $[a, b]$ and if f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

8. **Notes:**

(a) When we make the substitution $u = g(x)$, then the interval $[a, b]$ on the x-axis becomes the interval $[g(a), g(b)]$ on the u-axis.

(b) Writing

$$\int_a^b f(g(x))g'(x)\,dx \;=\; \int_{a}^{b} f(u)\,du \;=\; \int_{g(a)}^{g(b)} f(u)\,du$$

would **NOT** be right.

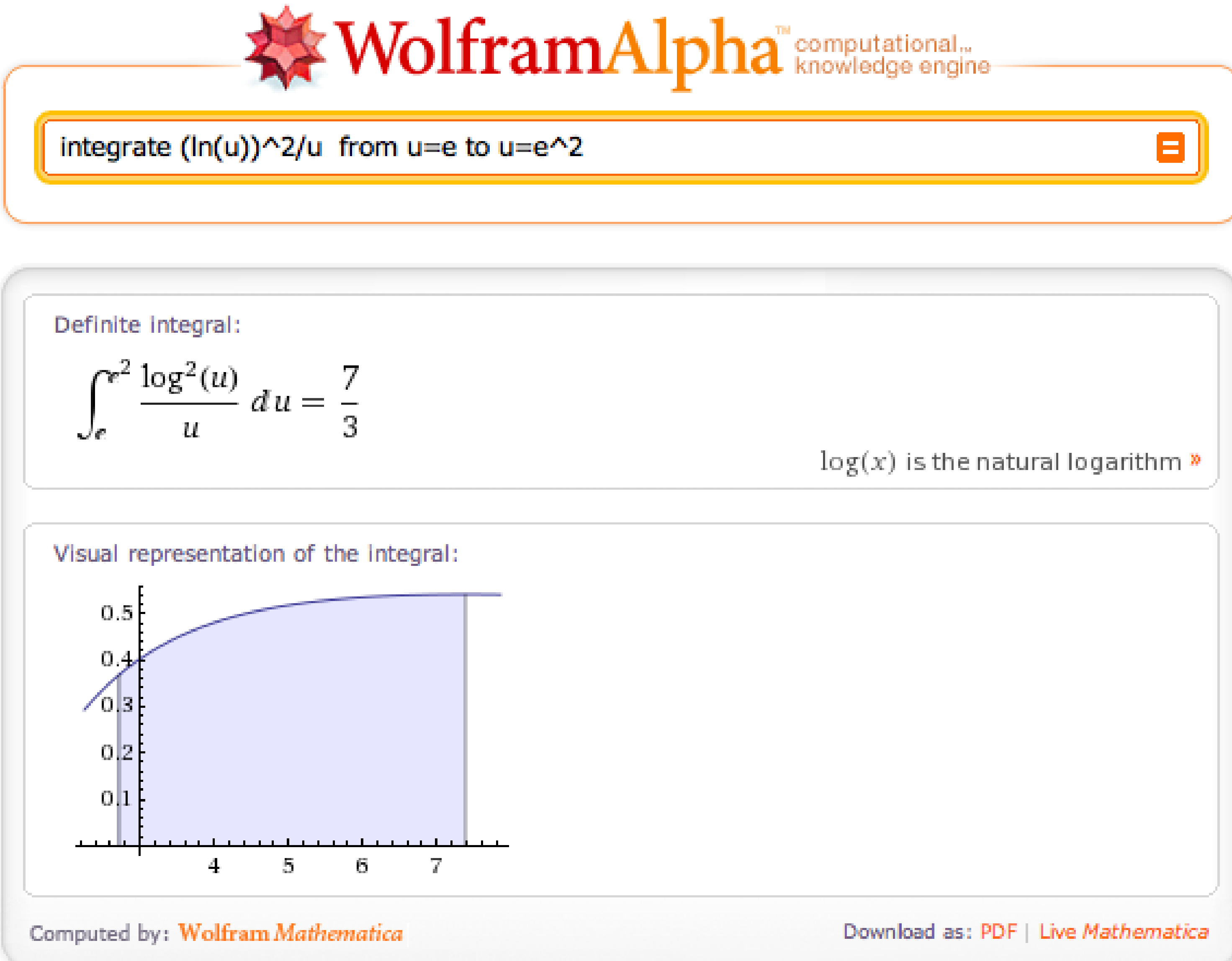
Make the substitution **AND** change the limits of integration at the same time!

9. **Examples.** Evaluate the following definite integrals:

(a) $\displaystyle\int_{\pi}^{2\pi} \cos 3t \; dt$

(b) $\displaystyle\int_{e}^{e^2} \frac{(\ln u)^2 du}{u}$

10. Again, you may use Wolfram|Alpha to check your answer.



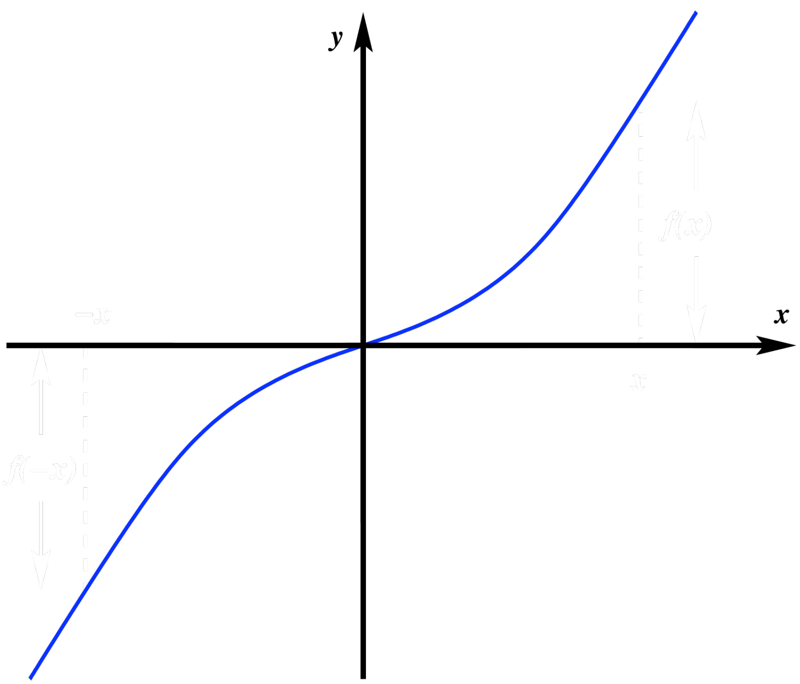
11. **Even or Odd?** Let $a > 0$ and let f be continuous on $[-a, a]$.

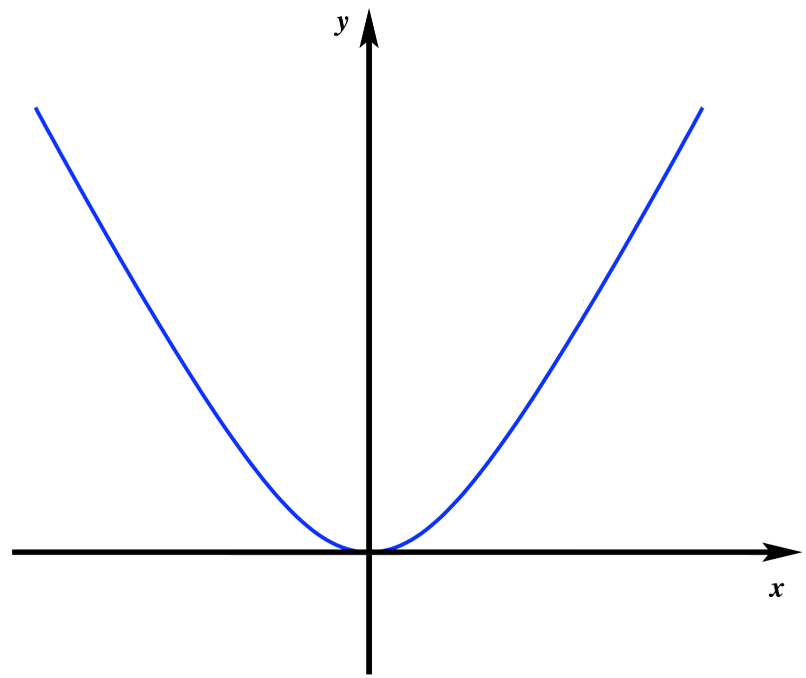
- If f is **odd** then

$$\int_{-a}^{a} f(x)dx = 0$$

- If f is **even** then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$





12. **Examples.** Evaluate the following definite integrals:

(a) $\displaystyle\int_{-3}^{3} (2x^4 + 3x^2 + 4)dx$

(b) $\displaystyle\int_{-e}^{e} \frac{e^{-u^2} \sin u \, du}{u^2 + 10}$