Informed Search Algorithms

Chapter 4

Outline

Informed Search and Heuristic Functions

- For informed search, we use *problem-specific* knowledge to guide the search.

Topics:

- Best-first search
- A^* search
- Heuristics

Recall: General Tree Search

```
function Tree-Search(problem) returns a solution or failure
  initialize the search tree by the initial state of problem
  loop do {
    if there are no candidates for expansion then return failure
    choose a leaf node for expansion (according to some strategy)
      - remove the leaf node from the frontier
    if the node satisfies the goal state then return the solution
    expand the node and add the resulting nodes to the search tree
  }
```

Informed (Heuristic) Search

- *Idea*: use an *evaluation function* for each node
  - estimate of "desirability" or proximity to a goal.
- Expand the most desirable unexpanded node

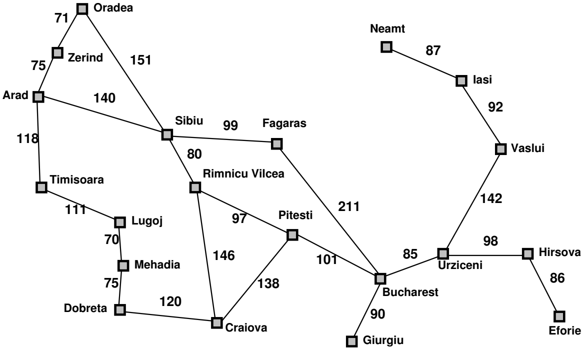
Informed (Heuristic) Search

- *Idea*: use an *evaluation function* for each node
  - estimate of "desirability" or proximity to a goal.
- Expand the most desirable unexpanded node
- Most generally we have:
  - Evaluation function: $f(n) = g(n) + h(n)$
  - $g(n)$ = cost from root to node n
  - $h(n)$ = estimated cost from node n to the goal
    $h(n)$ – *heuristic function*
  - $f(n)$ = estimated total cost of path through n to goal
- Thus for uniform-cost search $f(n) = g(n)$.

Greedy Best-First Search

- Evaluation function $f(n) = h(n)$
  $= \text{estimate of cost from } n \text{ to the closest goal}$
- So, $g(n) = 0$
  - I.e. the cost from the root to n is not considered.
- E.g., $h_{\mathrm{SLD}}(n) = $ straight-line distance from n to Bucharest
- Greedy search expands the node that *appears* to be closest to goal

Example: Romania with step costs in km



Straight−line distance to Bucharest

Arad	366
Bucharest	0
Craiova	160
Dobreta	242
Eforie	161
Fagaras	178
Giurgiu	77
Hirsova	151
Iasi	226
Lugoj	244
Mehadia	241
Neamt	234
Oradea	380
Pitesti	98
Rimnicu Vilcea	193
Sibiu	253
Timisoara	329
Urziceni	80
Vaslui	199
Zerind	374

Greedy search example

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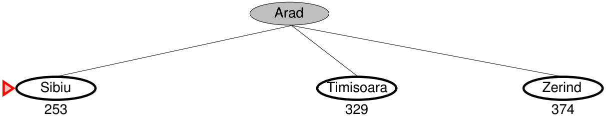

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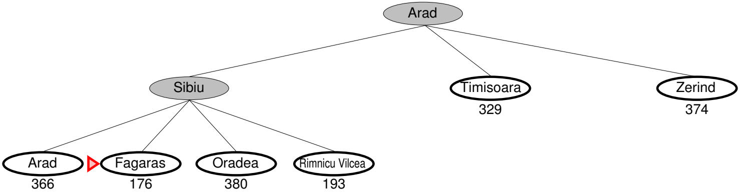

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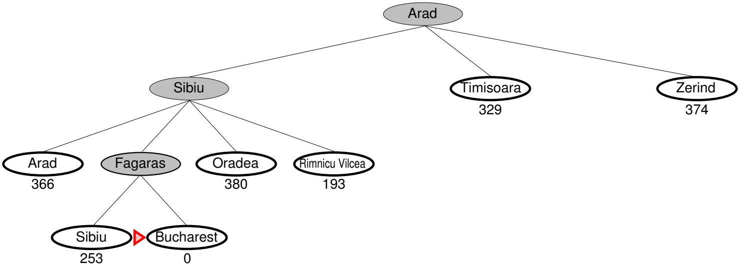

Other examples

- Games (i.e. as a search technique in *adversarial search*)
- Others?

Properties of greedy search

Complete: ??

Properties of greedy search

Complete: No – can get stuck in loops,

- E.g., with Oradea as goal,
  Iasi $\rightarrow$ Neamt $\rightarrow$ Iasi $\rightarrow$ Neamt $\rightarrow$

☞ Complete in finite space with repeated-state checking

Time: ??

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- An (online) depth-first agent could perform in constant space using via *local* search (later).

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Optimal: No

A* search

Idea:

- Try to avoid expanding paths that look to be expensive
  - Evaluation function $f(n) = g(n) + h(n)$
  - $g(n)$ = cost so far to reach n
  - $h(n)$ = estimated cost to the goal from n
  - $f(n)$ = estimated total cost of path through n to goal
- Expand the node where the cost so far, plus the estimated cost, is minimal.
- Note that $f(n)$ is a heurisitic function. It may not give the best value.
- A good choice of a heurisitic function is crucial for good performance.

A^* search

A^* search (ideally) uses an *admissible* heuristic

- Let $h^*(n)$ be the *true* (unknown) cost from n to the goal.
- A heuristic function $h(n)$ is admissable just if:

$$h(n) \leq h^*(n)$$

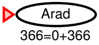
☞ So $h(n)$ never overestimates the cost.

- Also require $h(n) \geq 0$, so $h(G) = 0$ for any goal G.

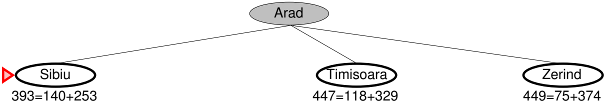
E.g., $h_{\mathrm{SLD}}(n)$ never overestimates the actual road distance

Theorem: A^* search is optimal
Corollary: Uniform cost search is optimal (why?)

A* search example



A* search example



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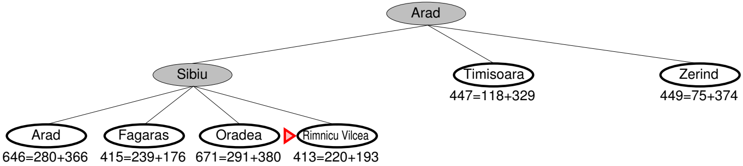

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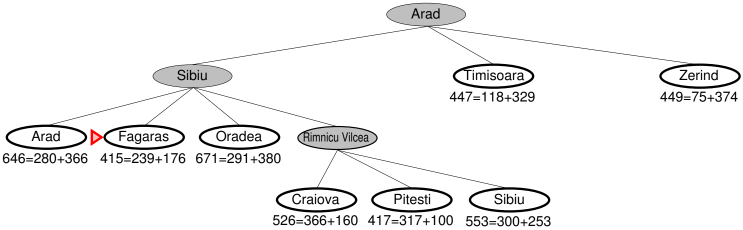

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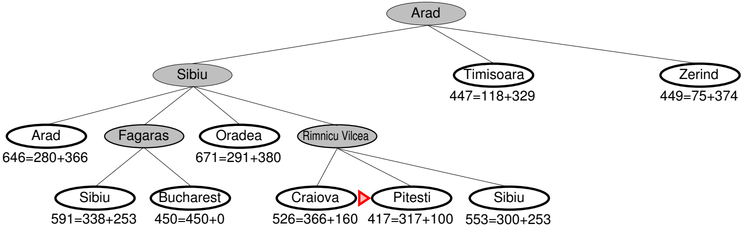

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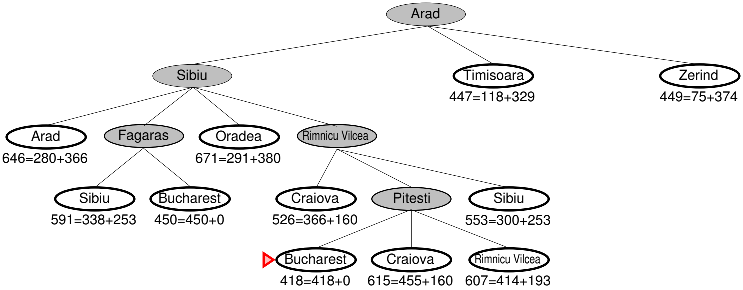
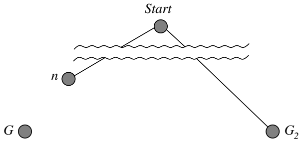


Optimality of A* (standard proof)

- Suppose G_2 is a suboptimal goal.
- Let n be an unexpanded node on a shortest path to an optimal goal G:



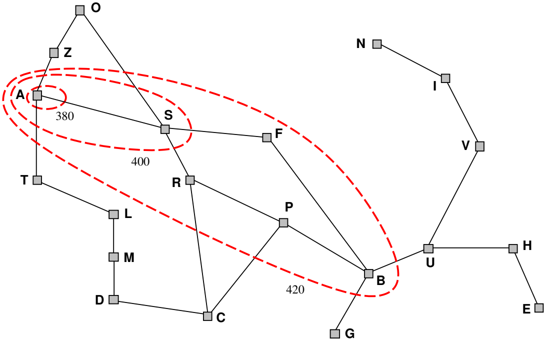
- Then:

$$
\begin{aligned}
f(G_2) &= g(G_2) && \text{since } h(G_2) = 0 \\
&> g(G) && \text{since } G_2 \text{ is suboptimal} \\
&\geq f(n) && \text{since } h \text{ is admissible}
\end{aligned}
$$

- Since $f(G_2) > f(n)$, A* will never select G_2 for expansion

Optimality of A* (another view)

- *Lemma*: A* expands nodes in order of increasing f value.
- Gradually adds "f-contours" of nodes
  - Cf.: breadth-first adds "layers"
- Contour i has all nodes with $f = f_i$, where $f_i < f_{i+1}$



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Optimal: Yes

- A* expands all nodes with $f(n) < C^*$, where
  C^* = cost of optimal solution
- A* expands some nodes with $f(n) = C^*$
- A* expands no nodes with $f(n) > C^*$

Admissible heuristics

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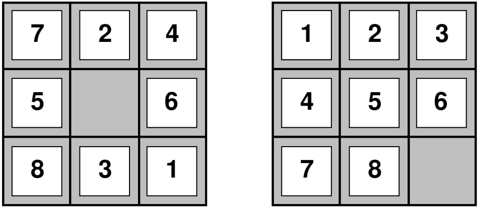

Start State **Goal State**

$h_1(S) = ??$

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Admissible heuristics

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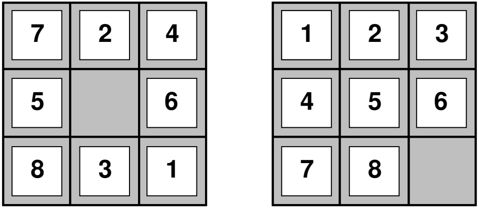

Start State **Goal State**

$h_1(S) = 6$

$h_2(S) = 4+0+3+3+1+0+2+1 = 14$

Dominance

- If $h_2(n) \geq h_1(n)$ for all n (both admissible) then h_2 *dominates* h_1, and is better for search

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d = 14 & \text{IDS} = 3{,}473{,}941 \text{ nodes} \\
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- For any admissible heuristics h_a, h_b,

$$h(n) = \max(h_a(n), h_b(n))$$

is also admissible and dominates h_a, h_b

Determining admissable heuristic functions

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- E.g.:
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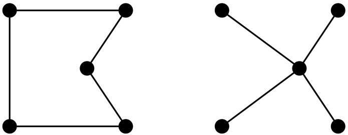
Key point:

The optimal solution cost of a relaxed problem is no greater than the optimal solution cost of the real problem

Relaxed problems contd.

- Well-known example: *travelling salesperson problem* (TSP)
- Find the shortest tour visiting all cities exactly once



- *Minimum spanning tree* can be computed in $O(n^2)$ and is a lower bound on the shortest (open) tour

Summary: Heuristic functions

- Heuristic functions estimate costs of shortest paths
- Good heuristics can dramatically reduce search cost
- Greedy best-first search expands lowest h
  - incomplete and not always optimal
- A^* search expands lowest $g + h$
  - complete and optimal
  - also optimally efficient (up to tie-breaks, for forward search)
- Admissible heuristics can be derived from exact solution of relaxed problems

Local Search: Outline

We consider next *local* search, where we maintain a single current state.

- Iterative improvement algorithms
- Hill-climbing
- Very briefly:
    - Simulated annealing
    - Local beam search

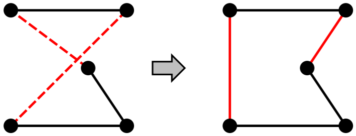
Iterative improvement algorithms

- Idea: In many optimization problems, the *path* to the goal is irrelevant.
  - The goal state itself is the solution
  - E.g. the n-queens problem
- So we may formulate a problem so that:

  state space = set of "complete" configurations
- Examples:
  - find *optimal* configuration, e.g., TSP
  - find configuration satisfying constraints, e.g., timetable
  - also, e.g. propositional satisfiability (SAT)
- In such cases, we can use *iterative improvement* algorithms
  - Keep a single "current" state; try to improve it
  - Uses constant space; suitable for online as well as offline search

Example: Travelling Salesperson Problem

- Start with any complete tour, perform pairwise exchanges



- Variants of this approach get within 1% of optimal very quickly with thousands of cities.

Example: n-queens

- Goal: Put n queens on an $n \times n$ board with no two queens on the same row, column, or diagonal.

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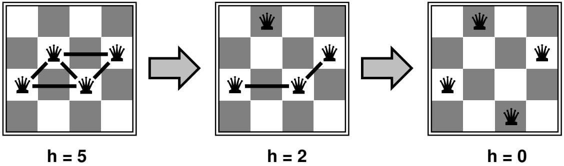
- Goal: Put n queens on an $n \times n$ board with no two queens on the same row, column, or diagonal.
- Move a queen to reduce number of conflicts.



- Almost always solves n-queens problems almost instantaneously for very large n, e.g., $n = 1,000,000$

Hill-climbing (or gradient ascent/descent)

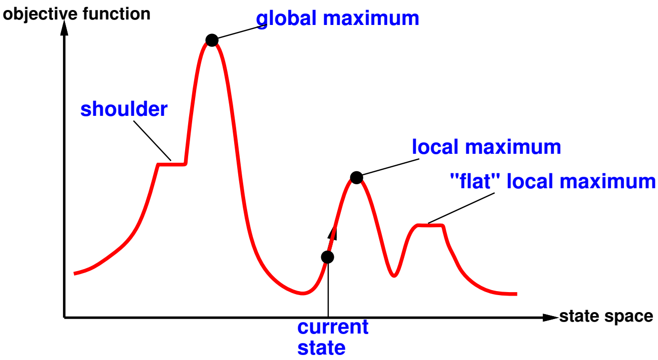
- Idea: Take the best move from a given position
- Aka *greedy local search*.
- "Like climbing a mountain in thick fog with amnesia"

Hill-climbing

```
Function Hill-Climbing(problem) returns a state that is a local
        maximum
  inputs: problem a problem
  local variables: current a node
      neighbor a node
  current ←Make-Node(Initial-State[problem])
  loop do
    neighbor ←a highest-valued successor of current
    if Value[neighbor] ≤ Value[current] then return State[current]
    current ←neighbor
  end
```

Hill-climbing contd.

Useful to consider *state-space landscape*



Hill-climbing contd.

- Hill climbing often gets stuck:

  Local Maxima: I.e. local "peaks". E.g. 8-queens gets stuck 86% of the time.

  Ridges: Essentially give a series of local maxima. Difficult for hill-climbing to navigate

  Plateaux: A plateau is a flat area in the search space. Search degenerates to exhaustive search, or may loop.

Hill-climbing: Strategies if stuck

- *Random-restart hill climbing*: Overcomes local maxima
    - Trivially complete *if* a goal is known to exist.
- *Random sideways moves*: Escape from shoulders but may loop on flat maxima
    - Can also define a hill-climbing version of depth-first search. (But then no longer a *local* search.)

Another Example: Propositional Satisfiabilty

- Goal: Find a *satisfying assignment* for a set of clauses in CNF.
- E.g.

$$(p \vee q \vee \neg r) \wedge (\neg p \vee r) \wedge (\neg p \vee \neg q)$$

is satisfied by setting: $p = \mathit{true}$, $q = \mathit{false}$, $r = \mathit{true}$.

Propositional Satisfiabilty

- Outline of an algorithm:

  Function Sat(problem) returns a solution or failure
    Assign truth values arbitrarily to the set of propositional variables
    loop do {
      if the truth assignment satisfies problem
          then return the assignment
      if timeout then return failure
      Find l such that $\bar{l}$ gives the largest increase in clauses satisfied
      Change the truth value of l to $\bar{l}$.
    }

☞ If l is p then $\bar{l}$ is $\neg p$;
if l is $\neg p$ then $\bar{l}$ is p.

Propositional Satisfiabilty

- This algorithm, when proposed in the 1990's, worked *very* well.
- The algorithm also featured random restarts. (I.e. after a while reassign all variable and start over).
  - It handily beat all previous algorithms (notably DPLL).
- Subsequent work in satisfiability has led to huge improvements over the naive greedy algorithm.

- Aside: Another thing that this work pointed out was the importance of choice of test instances.
  - DPLL (and other algorithms) appeared to work well because it turned out they were often tested on easy instances.

Simulated annealing

- Goal: Avoid local maxima
  - Local maxima is the biggest problem with local search.
- Idea: Take a step in a direction other than the best, from time to time.
  - Try to escape local maxima by allowing some "bad" moves *but gradually decrease their size and frequency*
  - These steps are designed to get the solver out of a possible local maximum
- The step size varies.
  - As time passes the step size and probabilty of a non-best step decreases.
- Simulated annealing has proven very effective in a wide range of problems, including VLSI layout, airline scheduling, etc.

Local beam search

Idea:

- Begin with k randomly-generated states.
- Keep k states instead of 1; choose top k of all their successors
- Not the same as k searches run in parallel!
- Searches that find good states recruit other searches to join them

Problem:

Quite often, all k states end up on same local hill

Variant: Stochastic beam search:

Choose k successors randomly, biased towards good ones

- Observe the analogy to natural selection!