The Definite Integral

1. **Definition (The Definite Integral).** Suppose f is a continuous function defined on the closed interval $[a, b]$, we divide $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. Let

   $$x_0 = a,\ x_1,\ x_2,\ \ldots,\ x_n = b$$

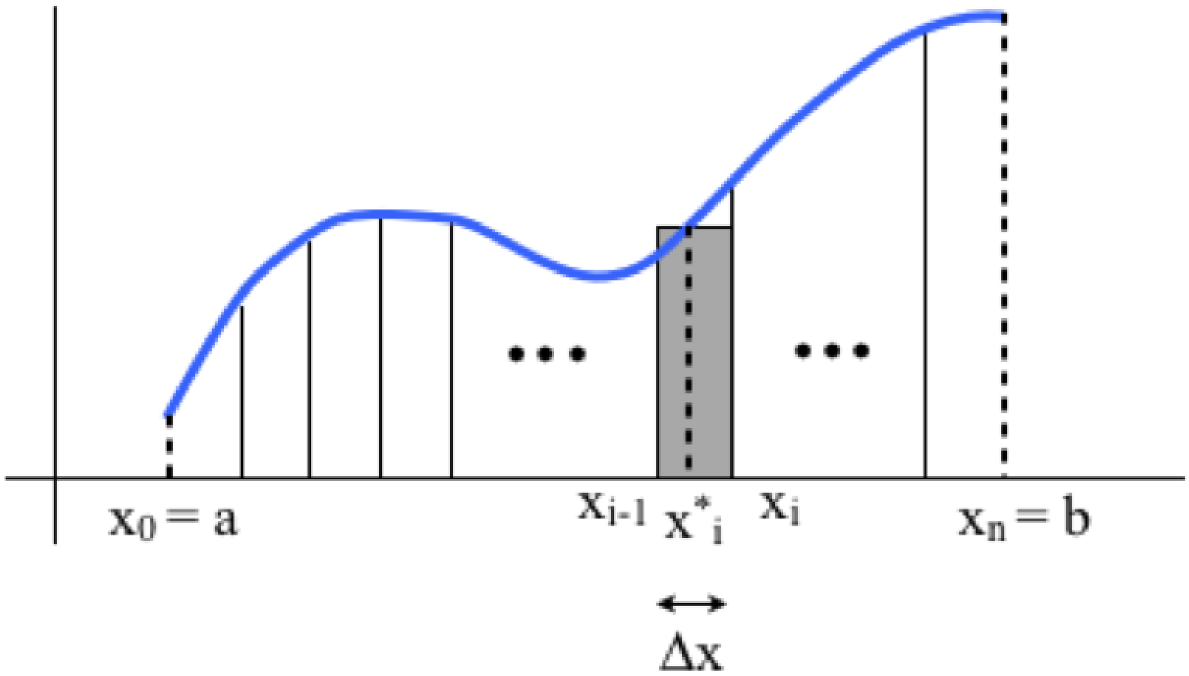
   be the end points of these subintervals. Let

   $$x_1^*, x_2^*, \ldots, x_n^*$$

   be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$.

   Then the **definite integral of** f **from** a **to** b is written as $\int_a^b f(x)dx$, and is defined as follows:

   $$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$



$x_0 = a$
x_{i-1}
x^*_i
x_i
$x_n = b$
Δx

2. The definite integral: some terminology

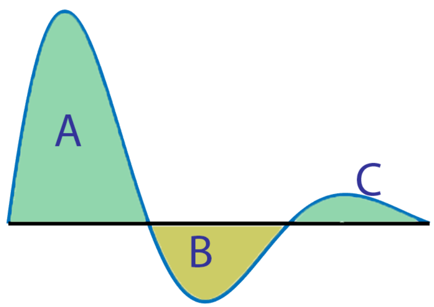
$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

- $\int$ is the *integral sign*
- $f(x)$ is the *integrand*
- a and b are the *limits of integration*
  - a - *lower limit*
  - b - *upper limit*
- The procedure of calculating an integral is called *integration*.
- $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a *Riemann sum*
  (named after the German mathematician Bernhard Riemann,1826-1866)

3. **Four Facts.**

(a) If $f(x) > 0$ on $[a, b]$ then $\int_a^b f(x)dx > 0$.

If $f(x) < 0$ on $[a, b]$ then $\int_a^b f(x)dx < 0$.

(b) 

For a general function f,

$$\int_a^b f(x)dx = \text{(signed area of the region)}$$

$$= \text{(area above } x\text{-axis) - (area below } x\text{-axis)}$$

(c) For every $\varepsilon > 0$ there exists a number $n \in \mathbb{N}$ such that

$$\left| \int_a^b f(x)dx - \sum_{i=1}^{n} f(x_i^*)\Delta x \right| < \varepsilon$$

for every $n > N$ and every choice of $x_1^*, x_2^*, \ldots, x_n^*$. That's how limits $\lim_{n\to\infty}$ are defined ....

(d) Let f be continuous on $[a, b]$ and let $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ be any partition of $[a, b]$. Let $\Delta x_i = x_i - x_{i-1}$, and suppose $\max \Delta x_i$ approaches 0 as n tends to infinity. Then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

4. **Some formulas/facts you just have to know.**

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \tag{1}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \tag{2}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 \tag{3}$$

$$\sum_{i=1}^{n} c = cn \tag{4}$$

$$\sum_{i=1}^{n} (ca_i) = c\sum_{i=1}^{n} a_i \tag{5}$$

$$\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i \tag{6}$$

5. **Example.** Evaluate

$$\int_0^2 (x^2 - x)dx.$$

6. **Example.** Express the limit

$$\lim_{n\to\infty}\sum_{i=1}^{n}(1+x_i)\cos x_i\Delta x$$

as a definite integral on the interval $[\pi, 2\pi]$.

7. **Example.** Prove

$$\int_0^2 \sqrt{4-x^2}dx = \pi.$$

8. **Choosing a good sample point ...** (numerical quadrature)

Midpoint Rule. To approximate an integral it is usually better to choose x_i^* to be the midpoint $\overline{x}_i$ of the interval $[x_{i-1}, x_i]$:

$$\int_a^b f(x)dx \quad \approx \quad \sum_{i=1}^{n} f(\overline{x}_i)\Delta x \quad = \quad \Delta x \left[f(\overline{x}_1) + f(\overline{x}_2) + \ldots + f(\overline{x}_n) \right]$$

Recall the midpoint of an interval $[x_{i-1}, x_i]$ is given by $\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$.

9. **Example.** Use the Midpoint Rule with $n = 4$ to approximate the integral $\displaystyle\int_1^3 \frac{dx}{x^2}$ (exact value is $\frac{2}{3}$).

10. **Two Special Properties of the Integral.**

(a) If $a > b$ then

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

(b) If $a = b$ then

$$\int_a^b f(x)dx = \int_a^a f(x)dx = 0.$$

11. **Some More Properties of the Integral.**

(a) If c is a constant, then $\displaystyle\int_a^b c\,dx = c(b-a)$

(b) $\displaystyle\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

(c) If c is a constant, then $\displaystyle\int_a^b cf(x)dx = c\int_a^b f(x)dx$

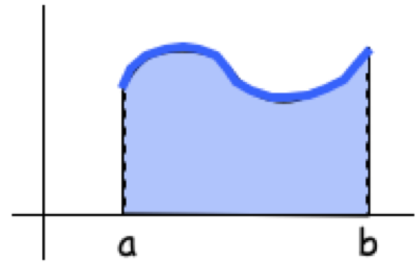
(d) $\displaystyle\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$

12. **Example.** Evaluate $\int_0^3 \left(2x - 3\sqrt{9 - x^2}\right) dx$.

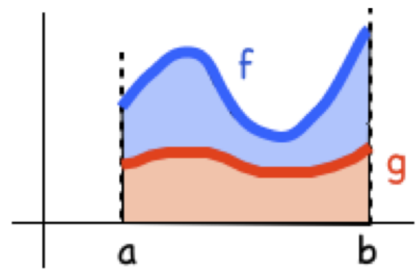
13. **Example.** Evaluate $\displaystyle\int_0^3 f(x)dx$ **if** $f(x) = \begin{cases} 1-x & \textbf{if } \; x \in [0,1] \\ -\sqrt{1-(x-2)^2} & \textbf{if } \; x \in (1,3] \end{cases}$

14. **More Properties of the definite integral.**

(a) **If** $f(x) \geq 0$ **for** $a \leq x \leq b$, **then** $\int_a^b f(x)dx \geq 0$.

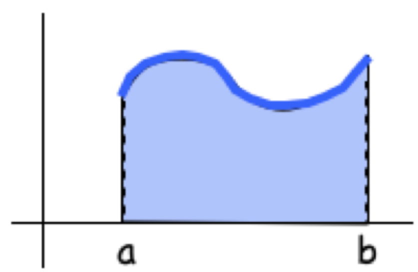


(b) **If** $f(x) \geq g(x)$ **for** $a \leq x \leq b$, **then** $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.



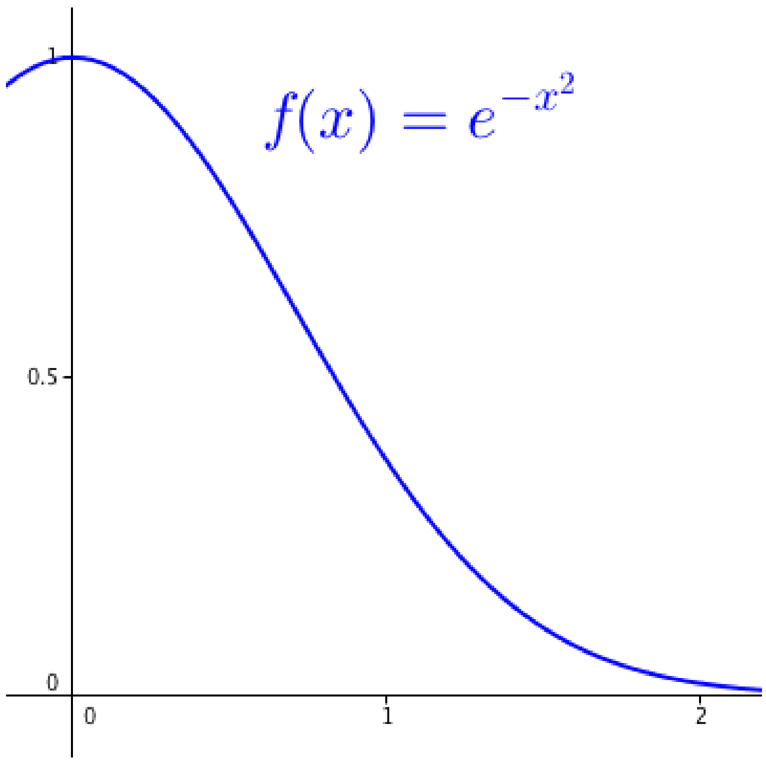
(c) **If** m **and** M **are constants, and** $m \leq f(x) \leq M$ **for** $a \leq x \leq b$, **then**

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$



15. **Example.** Prove

$$\frac{1}{e^4} \leq \int_1^2 e^{-x^2} dx \leq \frac{1}{e}$$



16. **Example.**

(a) If f is continuous on $[a,b]$, show that

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx.$$

(b) Show that if f is continuous on $[0,2\pi]$ then

$$\left|\int_0^{2\pi} f(x)\sin 2x dx\right| \leq \int_0^{2\pi} |f(x)|dx.$$