Phil 320
Chapter 9: A precis of 1st-order logic

I. Syntax (part 1)

1) Logical symbols (compulsory, except for =):

Official	***Unofficial***
Connectives: ~ & $\vee$	$\rightarrow$ and $\leftrightarrow$ (abbreviations)
Variables: v_0, v_1, …	*x*, *y*, *z*
Quantifiers: $\forall v_0$, $\exists v_1$, …	$\forall x$, $\exists y$,…
Parentheses: ( )	Omit these when obvious (e.g., *F*&*G*&*H*)
Comma: ,	
Identity: =	Write $x \neq y$ for ~($x=y$)

Note that =, while a 2-place predicate, is still considered a logical symbol.

2) Nonlogical symbols

	Official	***Unofficial***
Constants/ individual symbols	f^0_0, f^0_1, f^0_2,…	c_0, c_1, c_2 or *a*, *b*, *c*,…
Predicates/ relation symbols	A^n_k (n = # places; k = position on list)	*P*, *Q*, *R*
Function symbols:	f^n_k (n = # places; k = position on list)	*f*, *g*, *h*,…

Def: A *language* $\mathcal{L}$ is an enumerable set of nonlogical symbols.

Examples:

1) The *empty language*: no nonlogical symbols.
2) The language of arithmetic, *L**:
   - One constant, **0** (officially: f^0_0)
   - One two-place predicate, **<** (A^2_0)
   - One one-place function symbol, ′ (f^1_0)
   - Two two-place function symbols, **+** (addition) (f^2_0) and · (multiplication) (f^2_1)

Write $x < y$, x', $x+y$ and $x \cdot y$ instead of $<(x, y)$, $'(x)$ and so forth. Use **1**, **2**, … for **0**′, **0**′′,…

3) Formulas and terms

Informal definition. Formulas are sequences of symbols that correspond to grammatically well-formed English sentences. *Closed formulas* (sentences) have no free variables; *open formulas* are those with one or more free variables.

Examples. *Closed*: $\forall x \forall y(Rxy \vee Ryx)$, R**00**, $\forall$x (x=x) *Open*: R*xy*, R**0***x*, $\forall y Rxy$

Formal definition.

Case 1: $\mathcal{L}$ has no identity and no function symbols.

Atomic formulas: R(t_1, …, t_n) [R is an n-place predicate and each t_i is a variable or constant]
Non-atomic formulas:
(1) If *F* is a formula, so is ~*F*.
(2) If *F* and *G* are formulas, so are (*F* & *G*) and ($F \vee G$)
(3) If *F* is a formula and *x* is a variable, then $\forall x\, F$ and $\exists x\, F$ are formulas.

Nothing else is a formula: every formula is built up from atomic formulas by finitely many applications of (1) – (3).

Case 2: $\mathcal{L}$ has identity (but no function symbols).
Additional atomic formula: $t_1 = t_2$ [t_1 and t_2 are atomic terms (variables or constants)]

Case 3: $\mathcal{L}$ has identity and function symbols.

Informal definition of terms. Terms are sequences of symbols that correspond to grammatically well-formed singular-noun phrases. *Closed terms* contain no variables (just constants and function symbols); *open terms* include variables.

Examples: *Closed*: $\mathbf{0}$, $\mathbf{0}+\mathbf{0}'$ *Open*: x', $x \cdot y$

Formal definition of terms.

Atomic terms: variables and constants

Non-atomic terms: $f(t_1,\ldots, t_n)$ [f is an n-place function symbol and $t_1, \ldots, t_n$ are terms]

Nothing else is a term: every term is built up from atomic terms by finitely many applications of function symbols.

The definition of formulas differs only for the clause about atomic formulas:

Atomic formulas:

(1) $R(t_1,\ldots,t_n)$ [R is an n-place predicate and each t_i is a term]

(2) $t_1=t_2$ [t_1 and t_2 are terms]

4) Official vs. unofficial formulas.

$\forall v_0 A^2_0(f^0_0,\ f^2_0(v_0, f^1_0(f^0_0)))$ *becomes* $\forall x\,(0 < x + 1)$. [Advantages of the unofficial language are obvious!]

Note: The formulas and terms of a language are just those whose nonlogical symbols all belong to the language. There may be no closed terms, but the set of formulas is always enumerably infinite, even for the empty language.

II. Interpretations

Recall from Phil 220: we use *interpretations* to show that certain arguments are invalid.

Example:	(1)	$\forall x(Fx \vee Gx)$		1. D = people
	(2)	$\exists x \sim Fx$	$/\therefore\ \forall x\, Gx$	2. $Fx \equiv x$ is female; $Gx \equiv x$ is male

We now make *interpretation* precise, and define truth on an interpretation (or model).

Definition. Let $\mathcal{L}$ be a language. An *interpretation* $\mathcal{M}$ of $\mathcal{L}$ is:

(a) A non-empty *domain* $|\mathcal{M}|$ of objects.

(b) A *denotation* or meaning for each nonlogical symbol of $\mathcal{L}$:

(i) *Constants*. For each constant c, an object $c^{\mathcal{M}}$ in $|\mathcal{M}|$.

(ii) *Predicates*. For each n-place predicate R, an n-place relation $R^{\mathcal{M}}$ on $|\mathcal{M}|$.

[Note that $=^{\mathcal{M}}$ must always be the identity relation on $|\mathcal{M}|$.]

(iii) *Function symbols*. For each n-place function symbol f, an n-place function $f^{\mathcal{M}}$ from $|\mathcal{M}|$ to $|\mathcal{M}|$.

Example 1: $\mathcal{L}$ is the language with the following nonlogical symbols

Constants: a, b; 2-place predicate: L; 1-place function: f

(a) $|\mathcal{M}|$ = people

(b) (i) $a^{\mathcal{M}}$ = Romeo; $b^{\mathcal{M}}$ = Juliet

(ii) $L^{\mathcal{M}}$ is the 2-place relation ‘likes’: $L^{\mathcal{M}}(d_1, d_2)$ iff d_1 likes d_2.

(ii) $f^{\mathcal{M}}(d)$ is the father of d.

Example 2 (The standard interpretation of arithmetic, N)* L* is the language $\{\mathbf{0}, <, ', +, \cdot\}$:

(a) $|N^*|$ = natural numbers $\{0, 1, 2, \ldots\}$

(b) (i) $\mathbf{0}^{N^*} = 0$ [the number 0]

(ii) $<^{N^*}$ is the usual less-than relation, <

(iii) $'^{N^*}$ is the successor function, $s(x) = x+1$

$+^{N^*}$ is the usual addition function, +

$\cdot^{N^*}$ is the usual multiplication function, $\cdot$

We use the interpretations supplied to the *symbols* to find the interpretation of each closed formula (sentence) and determine its truth value.

Example 1 (continued):

$L(f(a), b)$ amounts to "Romeo's father likes Juliet" (F)
$\forall x\, L(x, b)$ amounts to "Everybody likes Juliet" (F)
$\forall x\, \exists z\, (Lxz \,\&\, Lzx)$ amounts to "Everybody has somebody whom they like and who likes them." (F)
$\sim\exists x\, f(x) = x$ amounts to "Nobody is his/her own father" (T)
$L(y, f(y))$ amounts to *y* likes *y*'s father [Open: neither T nor F]

*Example 1** (alternative interpretation):

(a) $|\mathcal{M}| = N$
(b) i) $a^{\mathcal{M}} = 1$, $b^{\mathcal{M}} = 0$
ii) $L^{\mathcal{M}}$ is the relation $\geq$
iii) $f^{\mathcal{M}}$ is the successor function

$L(f(a), b)$	$2 \geq 0$	(T)
$\forall x\, L(x, b)$	Every number is ≥ 0	(T)
$\forall x\, \exists z\, (Lxz \,\&\, Lzx)$	For every x there is a z such that $x \geq z$ and $z \geq x$	(T)
$\sim\exists x\, f(x) = x$	No number is its own successor	(T)
$L(y, f(y))$	$y \geq y+1$	[Open, so neither T nor F]

Example 2 (continued):

$\forall x\, \forall y\, (x \cdot y = \mathbf{0} \rightarrow (x = \mathbf{0} \vee y = \mathbf{0}))$ For all x and y, if the product $x \cdot y$ is 0, then x=0 or y=0. (T)
$\forall x\, \exists y\, (x < y \,\&\, \sim\exists z\, (x < z \,\&\, z < y))$ For any x, there is a next largest number y (with nothing in between x and y). (T)
$\mathbf{0}''' + \mathbf{0}'''' = \mathbf{0}''''''$ $3 + 4 = 6$ (F)

*Example 2** (alternative interpretations of the language of arithmetic):

a) Q: non-negative rationals

$|Q|$ = non-negative rational numbers

$\mathbf{0}^{Q} = 0$, $<^{Q} = <$, $+^{Q} = +$, $\cdot^{Q} = \cdot$ and $'^{Q}$ is $f(x) = x+1$.

On this interpretation, $\forall x\, \exists y (x < y \,\&\, \sim\exists z (x<z \,\&\, z<y))$ is false.

b) Z: Integers

$|Z| = \{0, \pm1, \pm2, \pm3, \ldots\}$

$\mathbf{0}^{T} = 0$, $<^{T} = <$, etc: all the usual denotations

On this interpretation, $\forall x\, \exists y\, (x+y=\mathbf{0})$ is true, but $\forall x\, (x=\mathbf{0} \vee \mathbf{0} < x)$ is false.

Semantic Notions

1. *Logical validity*. A sentence S is *valid* (logically valid) if no interpretation makes it false.
   Ex: $\forall x\, (x=x)$

2. *Logical implication*. A set Γ of sentences *logically implies* a sentence S if there is no interpretation that makes all the sentences in Γ true but S false.

Ex: $\forall x\, (xRx)$ logically implies $\forall x\, \exists y\, (xRy)$
Ex: $\forall x\, \forall y\, (xRy \vee yRx)$
$\sim aRb$ logically imply bRa

II. Syntax (part 2)

1. Induction on complexity

To prove that all formulas in a language have some property, here is the main technique.

Base case/Atomic formulas: Atomic formulas have the property. ($R(t_1,\ldots, t_n)$ and $t_1 = t_2$)
Induction step/Nonatomic formulas: Assume that F and G have the property, and prove:

(1) $\sim F$ has it
(2) $(F \,\&\, G)$ has it
(3) $(F \vee G)$ has it
(4) $\forall x\, F$ has it
(5) $\exists x\, F$ has it. [In fact, (3) and (5) aren't needed.]

Typically: we first do the proof for **Case 1** where = and function symbols are absent, then **Case 2** where identity is present, and finally **Case 3** where both identity and function symbols are present. In this last case, there is often a subsidiary induction on complexity to prove that all *terms* have some property:

Base case: Atomic terms (constants and variables) have the property
Induction step: If $t_1,\ldots,t_n$ have the property, then so does $f(t_1,\ldots,t_n)$.

First (trivial) example: Every formula is of finite length (# of symbols).

Textbook example (we'll just do part of it):

Lemma 9.4 (Parenthesis lemma):
a) Every formula ends in a right parenthesis (official form $=(t_1, t_2)$).
b) Every formula has the same number of left and right parentheses
c) If we divide any formula into a left and right part (nonempty), then there at least as many left as right parentheses in the left part, and more if that part contains at least one parenthesis.

2. Other syntactic notions

(i) *Subformulas* and *subterms*.

A consecutive string of symbols inside a formula is a *subformula* if it is itself a formula.
A consecutive string of symbols inside a term is a *subterm* if it is itself a term.

Lemma 9.5 (Unique readability):
(a) The only subformula of an atomic formula $R(t_1,\ldots,t_n)$ or $t_1=t_2$ is itself.
(b) The only subformulas of $\sim F$ are itself and subformulas of F.
(c) The only subformulas of $(F\&G)$ and $(F\vee G)$ are itself and subformulas of F and G.
(d) The only subformulas of $\forall x\, F$ and $\exists x\, F$ are itself and subformulas of F.

(ii) *Bound* and *free* occurrences of variables.

An occurrence of x in F is *bound* if it is part of a subformula beginning with $\forall x$ or $\exists x$; otherwise, *free*.

(iii) *Instances* of a formula.

Notation: $F(x)$ means a formula in which x is the only free variable; $F(x, y)$ means a formula in which x and y are the only free variables.
$F(c)$ means the result of substituting a constant c for each free occurrence of x in the formula F.

Ex: $F(x)$ is $\sim\forall y\,(y < x \vee y = x)$
$F(\mathbf{0})$ is $\sim\forall y\,(y < \mathbf{0} \vee y = \mathbf{0})$

An *instance* of the formula $F(x)$ is $F(t)$ where t is any closed term.

(iv) *Sentence*.

A formula is a *sentence* if it contains no free variables. A *subsentence* is a subformula that is a sentence.