Phil 320
Chapter 2: Diagonalization

Some sets are so big that they are *not* enumerable. Chapter 2 presents a technique called *diagonalization*. Given a list of sets (functions, numbers), we show how to construct a new set (function, number) that's distinct from each one on the list. We do this by making sure it's different in at least one place, "along the diagonal".

1. Diagonalization with sets

P* = the set of all subsets of P
= {ϕ, {1}, {1, 2}, …, E, O, …, P}
{2}, {4, 7}, …

Theorem 2.1: P* is not enumerable.

***Proof (by reductio)*:** Suppose P* is enumerable. Then it can be written in a list. Let the list

L: $S_1, S_2, \ldots,$

be a complete enumeration of all the members of P*: the list includes every subset of P.

We'll show that in fact there is a set S of positive integers that does not appear in this list, contradicting the claim that L is a complete enumeration.

We define S as follows:

Put 1 in S if $1 \notin S_1$; don't put 1 in S if $1 \in S_1$ → so S is different from S_1
$2 \in S$ iff $2 \notin S_2$ → so S is different from S_2
…
$n \in S$ iff $n \notin S_n$ → so S is different from S_n

For every n, S differs from S_n and therefore S does not appear on the list L. But this is a contradiction, since the list was supposed to include every subset of P. //

Note 1: In general, if L is any (countable) list of sets $S_1, S_2, \ldots,$ of positive integers, and we define the *diagonal* set $\Delta(L)$ by:

$n \in \Delta(L) \leftrightarrow n \notin S_n$,

then $\Delta(L)$ is not on the list L. So this technique always produces a set not on the list.

Note 2: A very similar proof shows that the set of subsets of *any* enumerably infinite set is not enumerable.

Objections:

1. What if S turns out to be the empty set because $n \in S_n$ for each n?
   Then ϕ could not have been on the original list.
2. Why not just add $\Delta(L)$ to the start of the list to get a new list L′?

You could run Δ again to get a set $\Delta(L')$ that is not on L'. That is, your list would still be missing something. You can't win! You can't get a full list.

Also: this is a logical mistake. The argument is a *reductio*: we assume L is complete and derive a contradiction.

3. The set of all FINITE subsets of P *is* enumerable, but diagonalization seems to show that it is not enumerable.

If L is a complete list of all the finite subsets of P, you can indeed apply Δ and you get a subset of P that is not on the list. But it's not a finite subset of P, so there is no reason why it should be on the list! [The key to diagonalization arguments is that you have to generate a *contradiction*: the new object generated is supposed to be on the original list, but isn't.]

2. Diagonalization with functions

Consider a sequence of functions whose domain is P:

$$f_1, f_2, f_3, \ldots$$

Problem: Find a function f that is different from each f_n, whose domain is also P.
Solution: just make sure to define f so that $f(n)$ differs from $f_n(n)$.

Ex. 1: Let each $f_n: P \to \{0, 1\}$; assume each f_n assigns each positive integer the value 0 or 1. Here is a function f that is different from each f_n:

$$f(n) = \begin{cases} 1, \text{ if } f_n(n) = 0 \\ 0, \text{ if } f_n(n) = 1 \end{cases}$$

Ex. 2: Let each $f_n: P \to P$.

Case 1: Each f_n is total. Put $f(n) = f_n(n) + 1$. Then f is different from each f_n.

Case 2: f_n may be partial. Put $f(n) = \begin{cases} f_n(n) + 1, \text{ if } f_n(n) \text{ is defined} \\ 1, \text{ if } f_n(n) \text{ is undefined.} \end{cases}$

Characteristic Functions

Definition: If S is any subset of P, the *characteristic function* $c_S: P \to \{0, 1\}$ is defined by

$$c_S(x) = \begin{cases} 1, x \in S \\ 0, x \notin S \end{cases}$$

Sets and their characteristic functions are inter-definable: if we have a set, then we have a characteristic function; and if we have a characteristic function, we can single out the set of elements where its value is 1.

Another way to understand the proof of **Theorem 2.1** is to introduce c_n (rather than the more cumbersome c_{S_n}), the characteristic function for S_n, and write out the values of these functions in a row.

	1	2	3	4	5	…
c_1	$c_1(1)$	$c_1(2)$	$c_1(3)$	$c_1(4)$	$c_1(5)$	…
c_2	$c_2(1)$	$c_2(2)$	$c_2(3)$	$c_2(4)$	$c_2(5)$	…
…						

Row n has "1" under each integer that is in S_n and "0" under each integer not in S_n.

Now define a new characteristic function c by setting

$$c(n) = \begin{cases} 1, & \text{if } c_n(n) = 0 \\ 0, & \text{if } c_n(n) = 1. \end{cases}$$ [Just like Ex. 1 on page 2 above.]

More succinctly: $c(n) = 1 - c_n(n)$. It's clear that c is different from each c_n, so that the set *S* of **all numbers n such that c(n) = 1** is distinct from each S_n. [The text calls c the *antidiagonal* function.] Actually, the set *S* is just the 'diagonal set' from our earlier proof of **Theorem 2.1**. (This is just another way to prove that there is no function from P to P* whose range is all of P*, i.e., that P* is not enumerable.)

3. Diagonalization with Numbers

Theorem: the set X of all real numbers between 0 and 1 (including 1 but not 0) is not enumerable.

Proof: Suppose X is enumerable. Write the complete list as

$r_1, r_2, \ldots$

Now select for each number on the list its non-terminating decimal expansion (i.e., if it's a finite decimal, choose the equivalent ending in an infinite string of 9's). Every positive real number has a unique non-terminating decimal expansion.

Define r by the decimal expansion whose n'th term is $9 - r_n(n)$ if $r_n(n) \neq 9$, and 1 if $r_n(n) = 9$; then r is not equal to any $r_n(n)$.

4. Summary of Techniques

Use diagonalization to show that a set S of sets, functions or numbers is not enumerable. In each case, **step 1** is to suppose that the set S is enumerable, i.e., that all of its members can be written down as a list L:

L = $S_1, S_2, \ldots$ (if S is a set of sets)

L = $f_1, f_2, \ldots$ (if S is a set of functions)

L = $r_1, r_2, \ldots$ (if S is a set of numbers)

Step 2 is to employ the diagonalization technique to generate a new set, function or number $\Delta(L)$ that both is not on the list (because it differs from each item on the list) and is on the list (because it still has the property that puts it in S). This contradiction then shows that our supposition that S was enumerable must be false.