0. **A Piecewise Example** left over from our previous lecture. Let

$$f(x) = \begin{cases} 0 & \textbf{if } x < 0 \\ x & \textbf{if } 0 \le x \le 1 \\ 2-x & \textbf{if } 1 < x \le 2 \\ 0 & \textbf{if } x > 2 \end{cases}$$

and let $g(x) = \displaystyle\int_0^x f(t)dt$.

(a) Find an expression for $g(x)$ similar to the one for $f(x)$.

(b) Sketch the graphs of f and g.

(c) Where is f differentiable? Where is g differentiable?

$$g(x) = \begin{cases} 0 & \textbf{if } x < 0 \\ x^2/2 & \textbf{if } 0 \le x \le 1 \\ 1-(2-x)^2/2 & \textbf{if } 1 < x \le 2 \\ 1 & \textbf{if } x > 2 \end{cases}$$

$0 \le x \le 1$:

$g(x) = \int_0^x t\,dt = \left[\frac{t^2}{2}\right]_0^x = x^2/2.$

$1 \le x \le 2$: $g(x) = g(1) + \int_1^x (2-t)dt =$

$= \frac{1}{2} - \left[2t - \frac{t^2}{2}\right]_1^x = \ldots = -\frac{x^2}{2} + 2x - 1.$

Indefinite Integrals

1. **Reminder. The Fundamental Theorem of Calculus, Part 2:.**
   If f is continuous on $[a, b]$, then

   $$\int_a^b f(x)dx = F(b) - F(a) = [F(x)]_a^b$$

   where F is any antiderivative of f, that is a function such that $F' = f$.

   Note: Your textbook uses the notation $F(b) - F(a) = F(x)]_a^b$;
   I prefer brackets around the expression to be evaluated.

5.3

2. **Problem.** So, to be able to evaluate an integral, we need a way to find any antiderivative F of the given function f . How do we find antiderivatives?

3. A new name for an old idea...

Definition (Indefinite Integral). The symbol $\int f(x)dx$ is called an **indefinite integral**, and it represents an antiderivative of f. That is,

$$\int f(x)dx = F(x) \textbf{ means } F'(x) = f(x)$$

4. **Warning!** It could be confusing: The notation $\int f(x)dx$ is used to represent

   - the **set** of all antiderivatives of f

     $$\int f(x)dx = \{F : F' = f\}$$

   - a single **function** that is an antiderivative of f.

5. Integrals you should know:

$$\int cf(x)dx = c\int f(x)dx \qquad \int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$\int k dx = kx + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{dx}{x} = \ln|x| + C$$

$$\int e^x dx = e^x + C \qquad \int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x dx = -\cos x + C \qquad \int \cos x dx = \sin x + C$$

$$\int \sec^2 x dx = \tan x + C \qquad \int \csc^2 x dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C \qquad \int \csc x \cot x dx = -\csc x + C$$

$$\int \frac{dx}{x^2+1} = \tan^{-1} x + C \qquad \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

6. **Examples.** Find the following indefinite integrals:

(a) $\int x^{-2/3} dx$

(b) $\int t^2(3 - 4t^5) dt$

(c) $\int (u - 1)(u^2 + 3) du$

(d) $\int (4e^v - \sec^2 v) dv$

(e) $\int \frac{\cos z}{1 - \cos^2 z} dz$

7. **The Net Change Theorem.** The integral of a rate of change is the net change:

$$\int_a^b F'(x)dx = F(b) - F(a)$$

8. **Example.** If a swimming pool is filled at the rate of $w(t)$ litres per minute, what does $\int_0^{90} w(t)dt$ represent?

9. **Example.** (Linear Motion of a Particle)
A particle is moving along a line with the acceleration (in m/s^2) $a(t) = 2t + 3$ and the initial velocity $v(0) = -4$ m/s with $0 \leq t \leq 3$. Find

(a) the velocity at time t,

(b) the distance traveled during the given time interval.