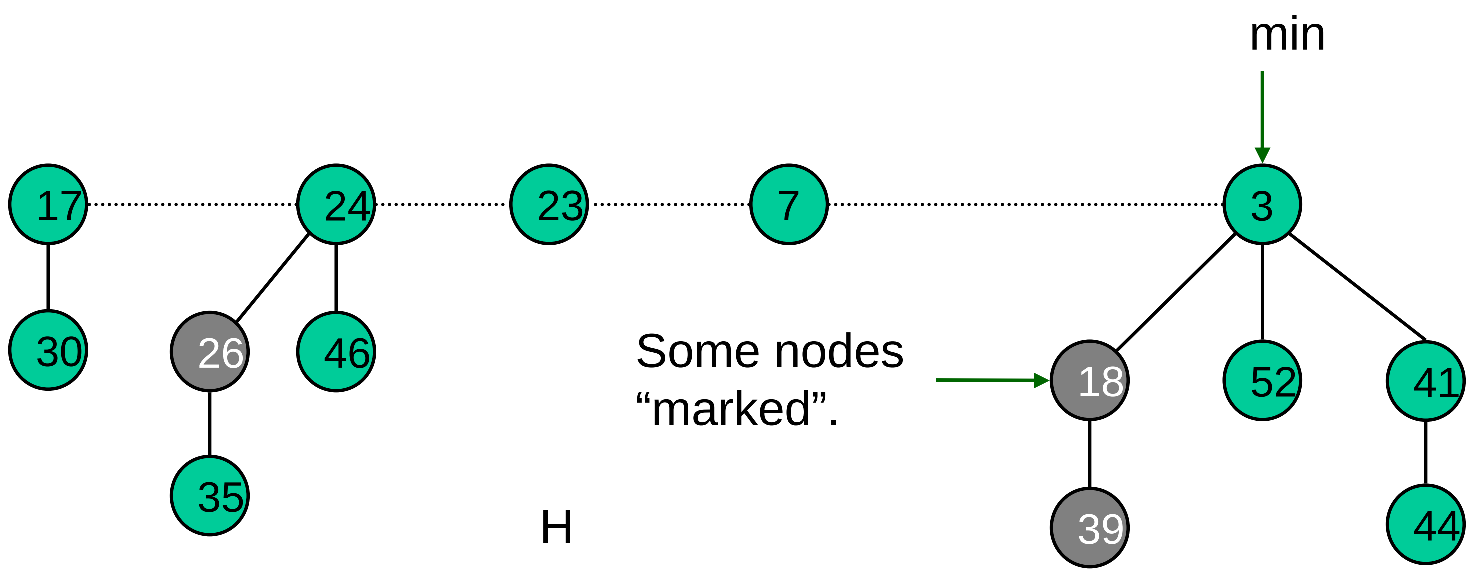
Fibonacci Heaps

Data Structures and Algorithms

Andrei Bulatov

Structure

Set of min-heap ordered trees



Implementation

Each node has 4 pointers: parent, 1st child, next & previous siblings.

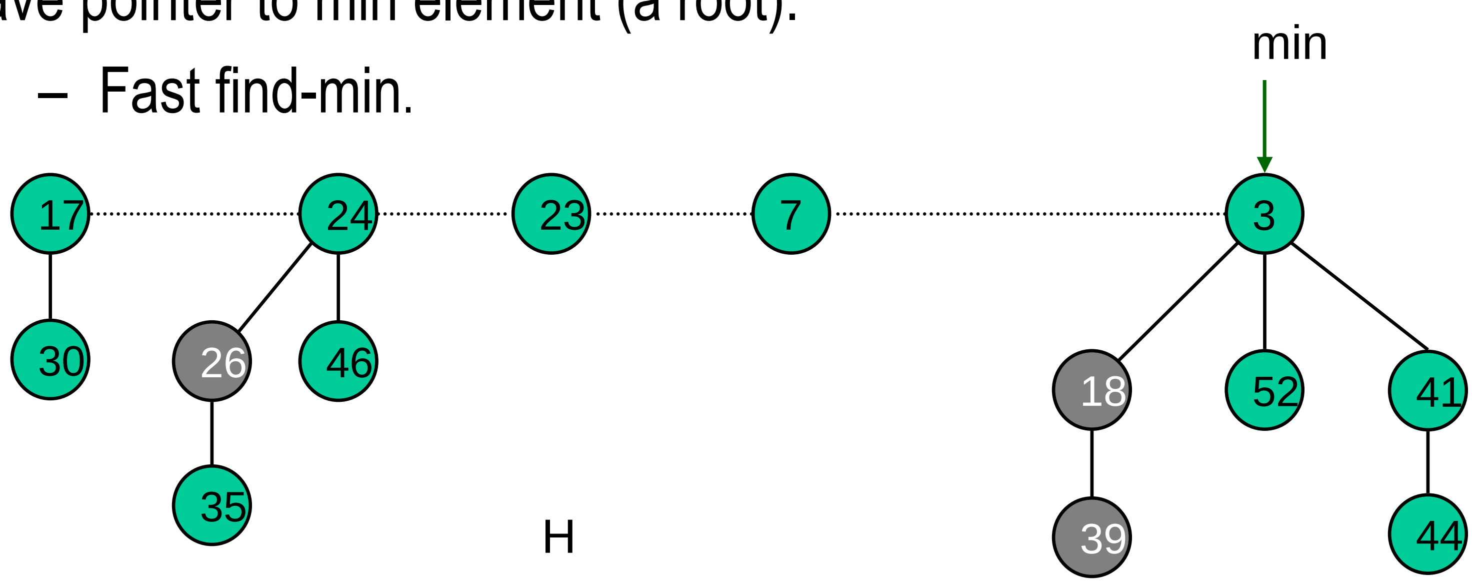
- Sibling pointers form circular, doubly-linked list.
- Can quickly splice off subtrees.

Roots in circular, doubly-linked list.

- Fast union.

Have pointer to min element (a root).

- Fast find-min.



Implementation

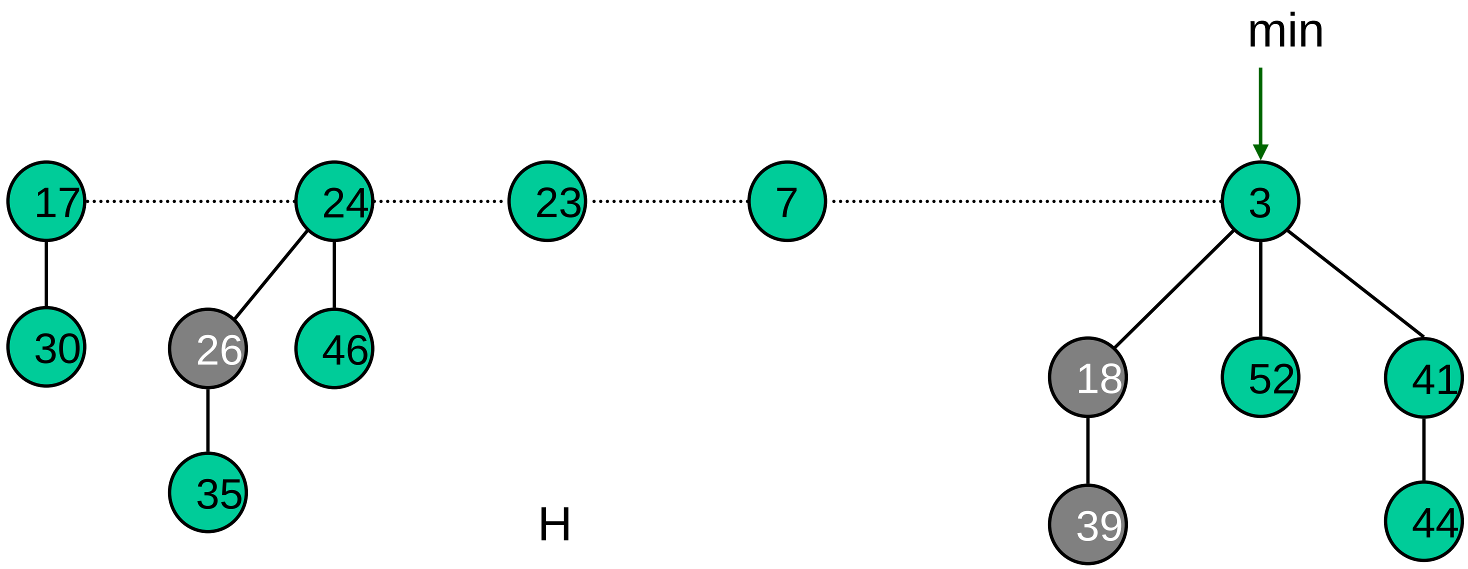
Key quantities:

- degree(x) = degree of node x.
- mark(x) = is node x marked?
- t(H) = # trees.
- m(H) = # marked nodes.
- Φ(H) = t(H) + 2m(H)

t(H) = ? 5

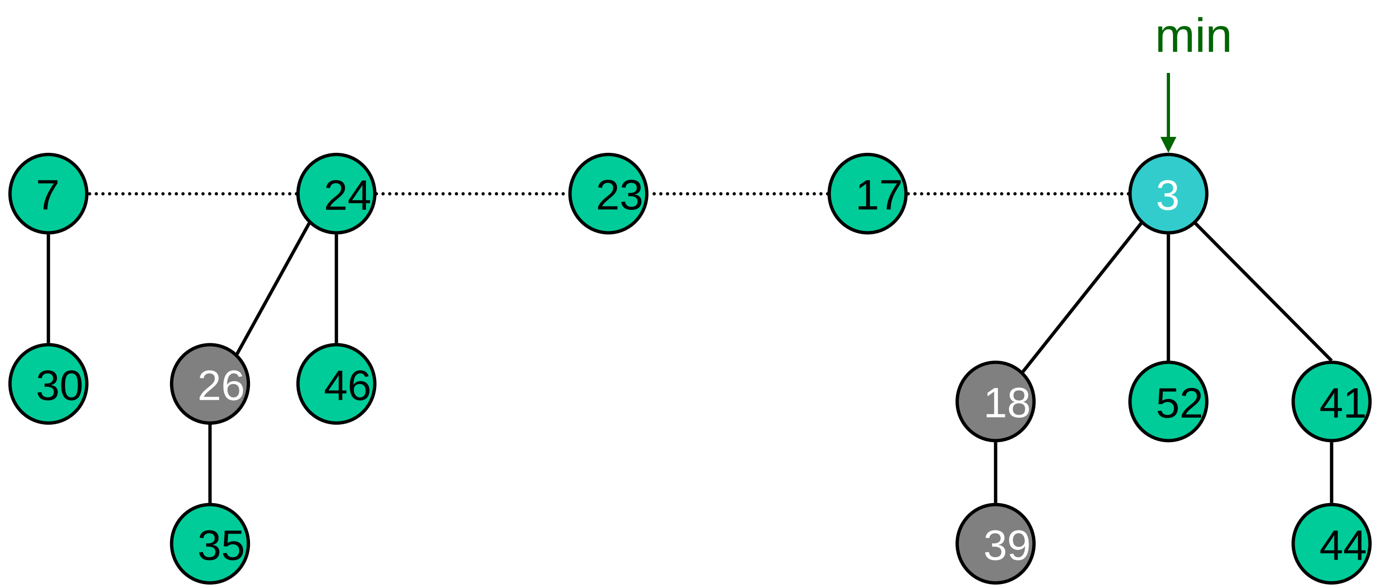
m(H) = ? 3

Φ(H) = ? 11



Delete Min

1. Delete min, concatenate its children into root list.
2. Consolidate trees so that no two roots have same degree, and finding new min.



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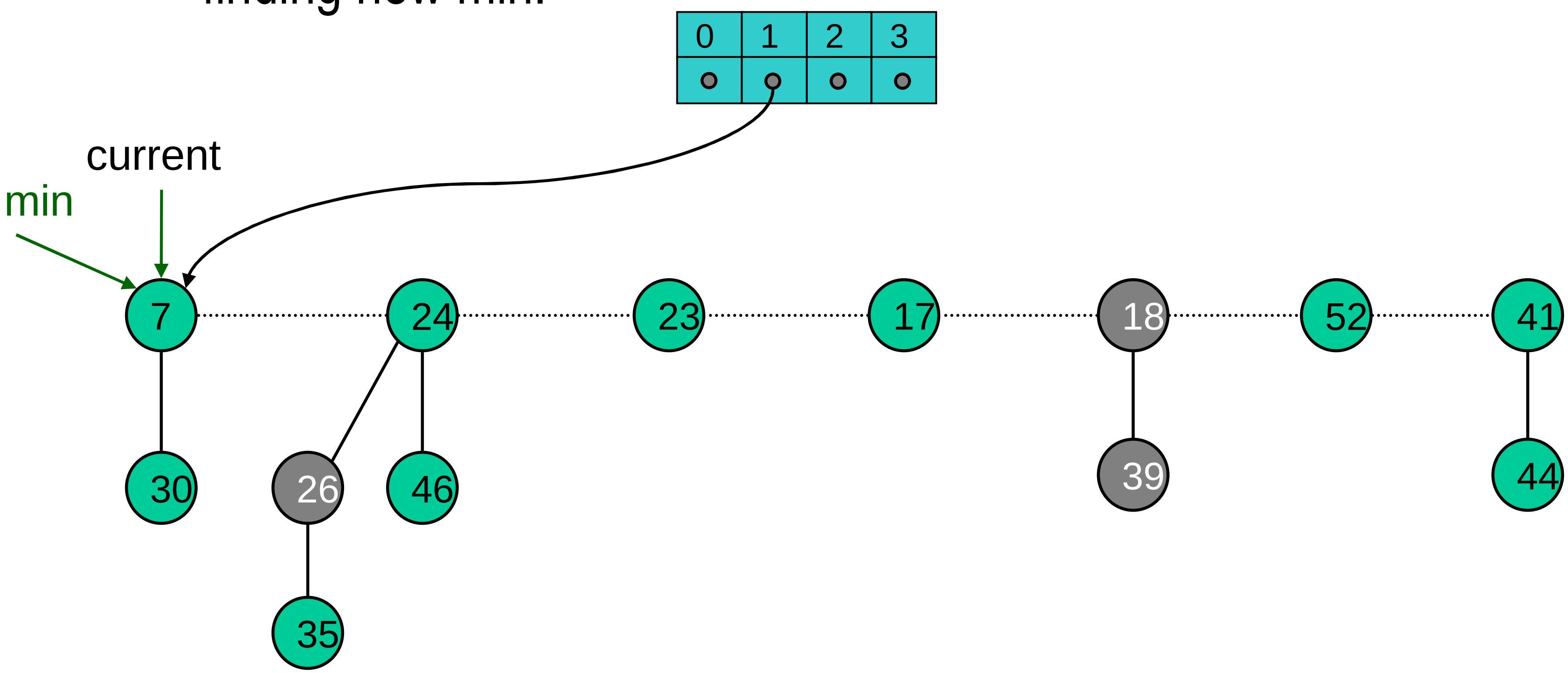

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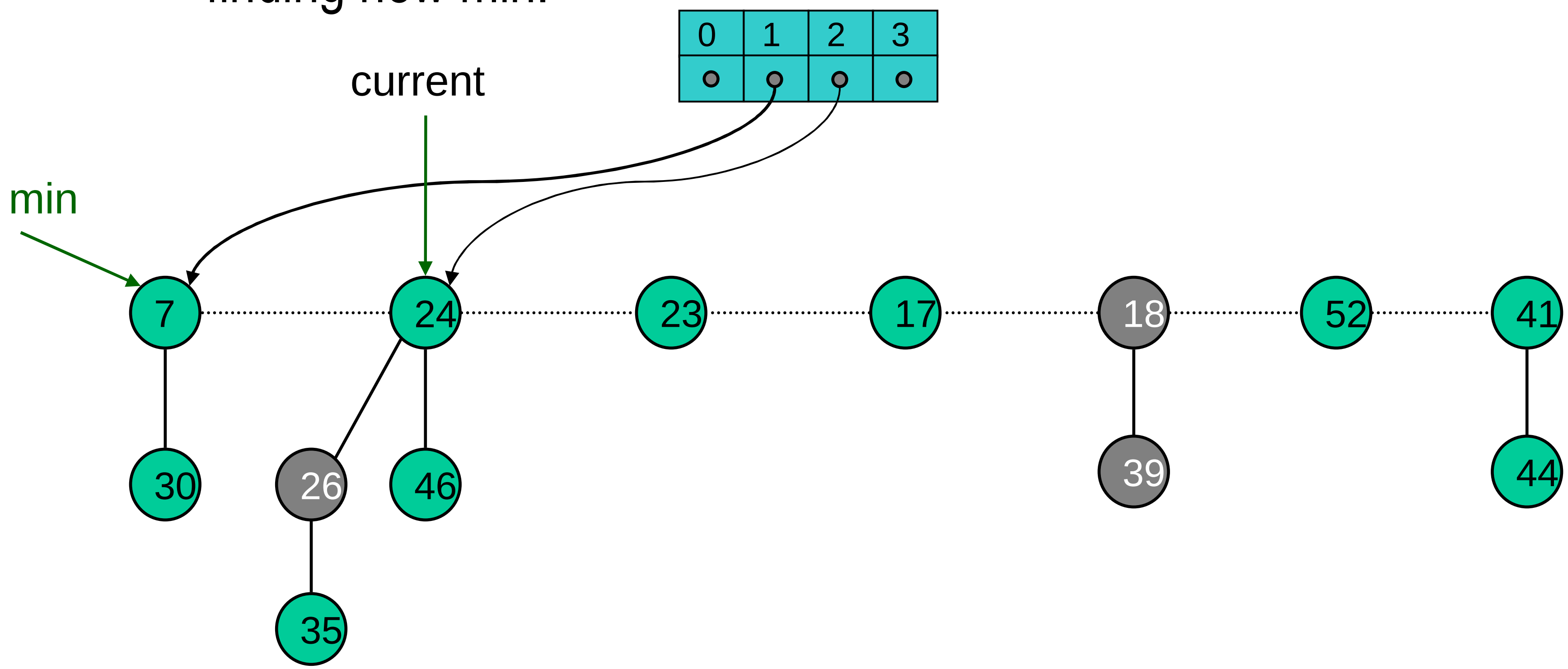

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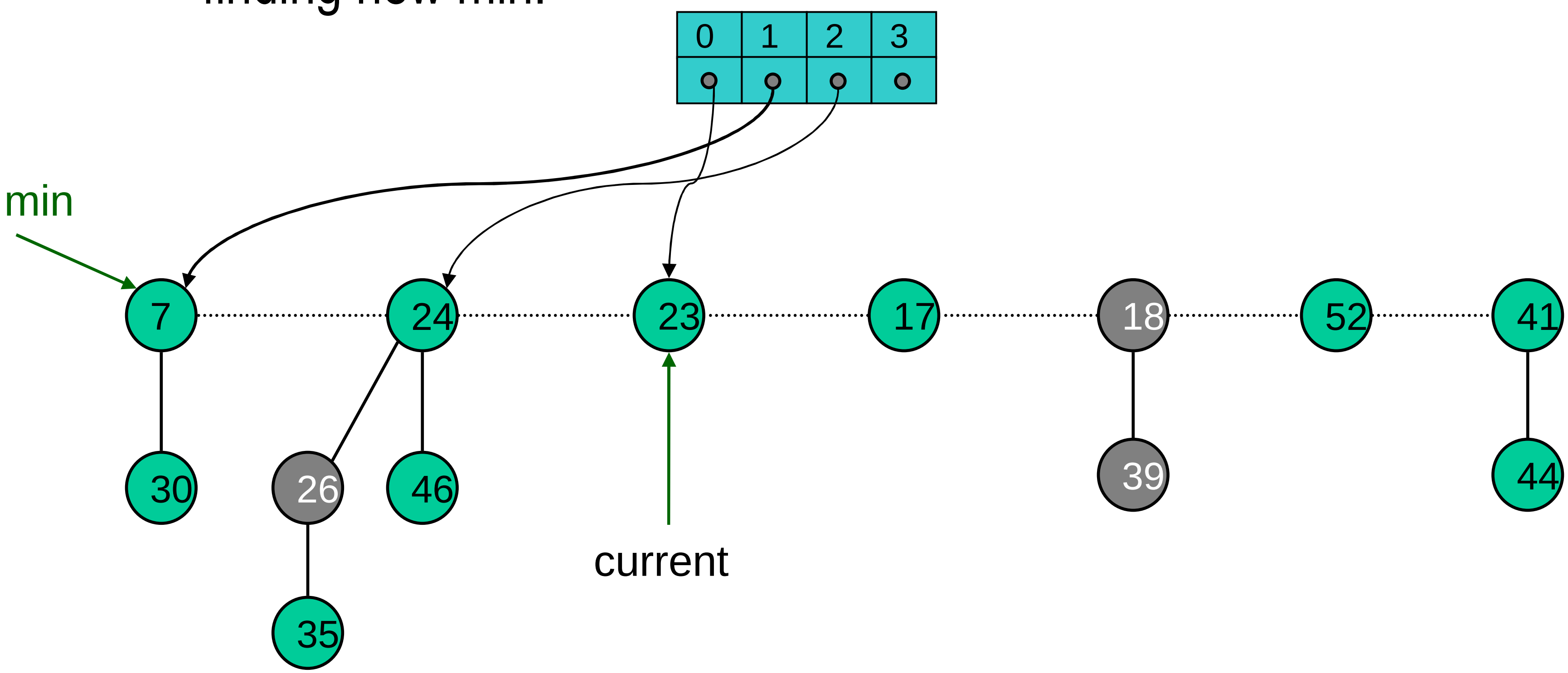

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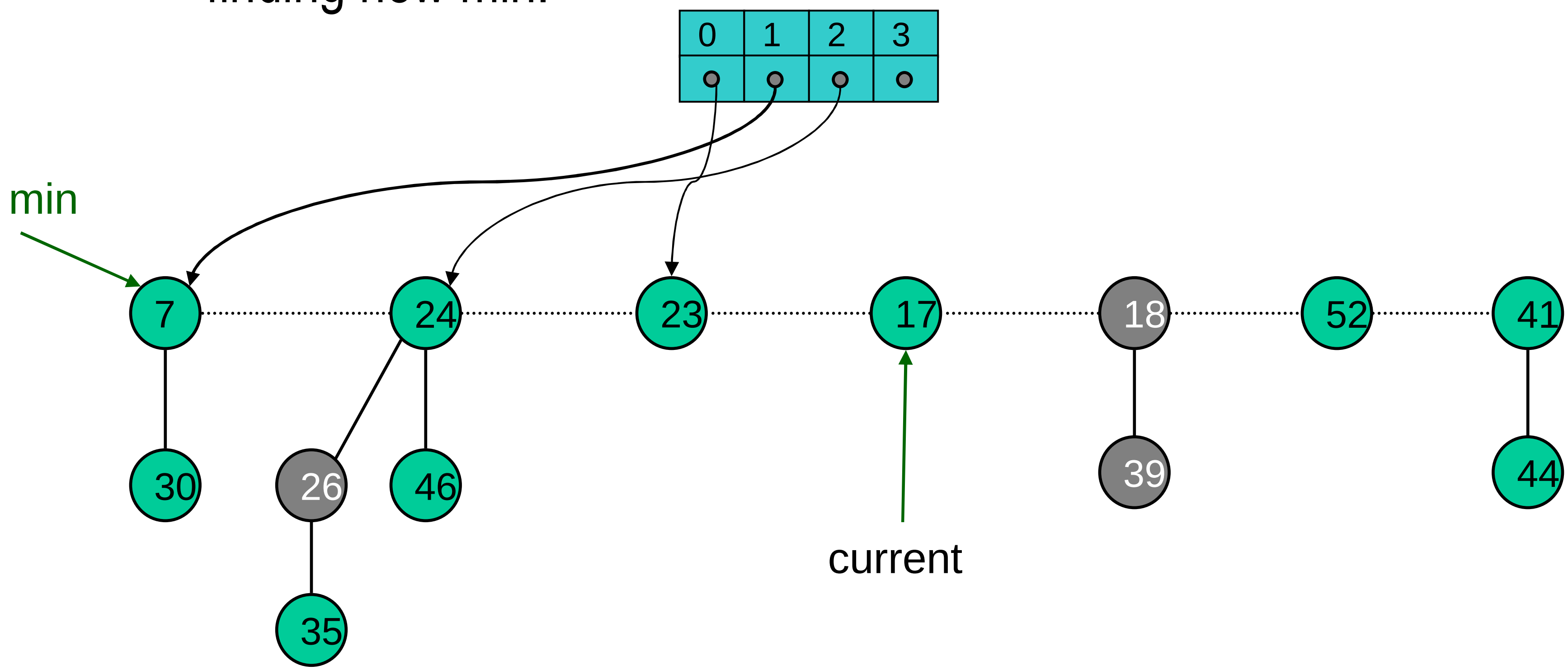

Merge 17 & 23 trees.

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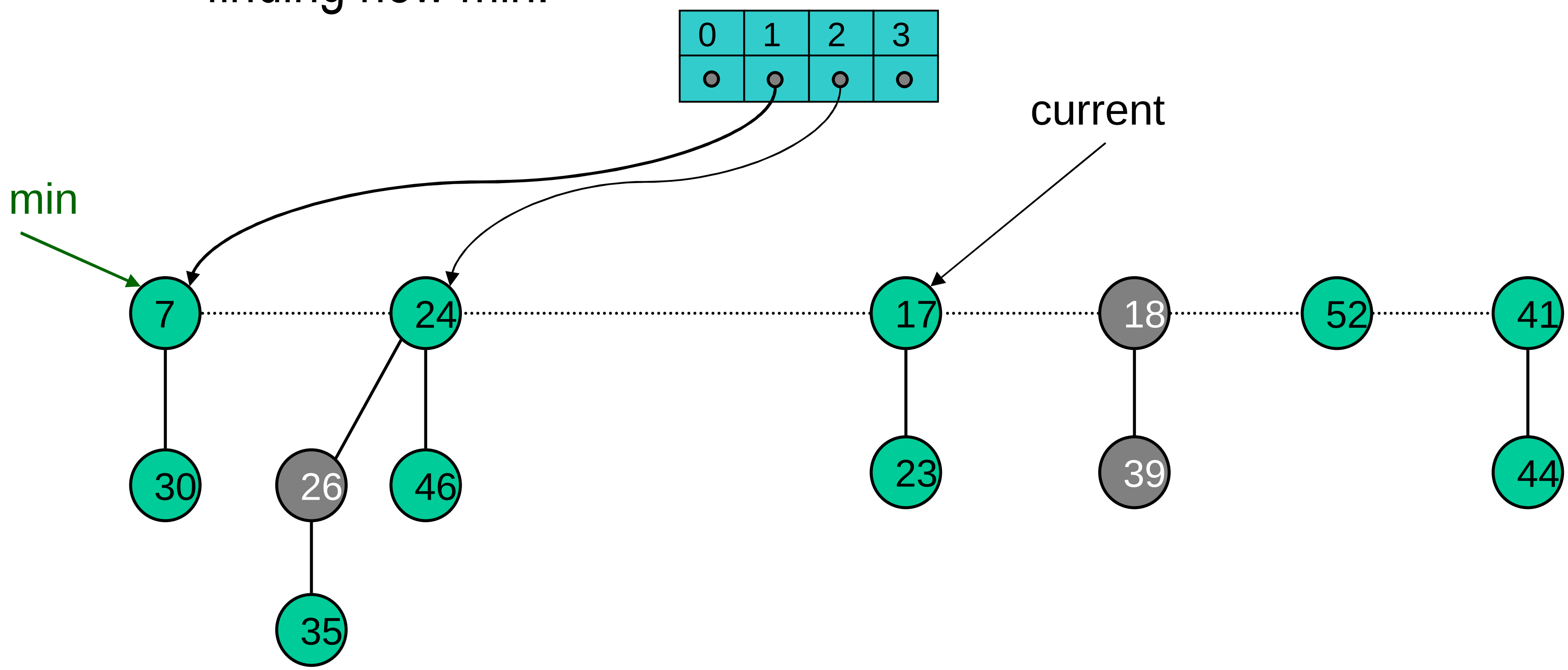

Merge 17 & 7 trees.

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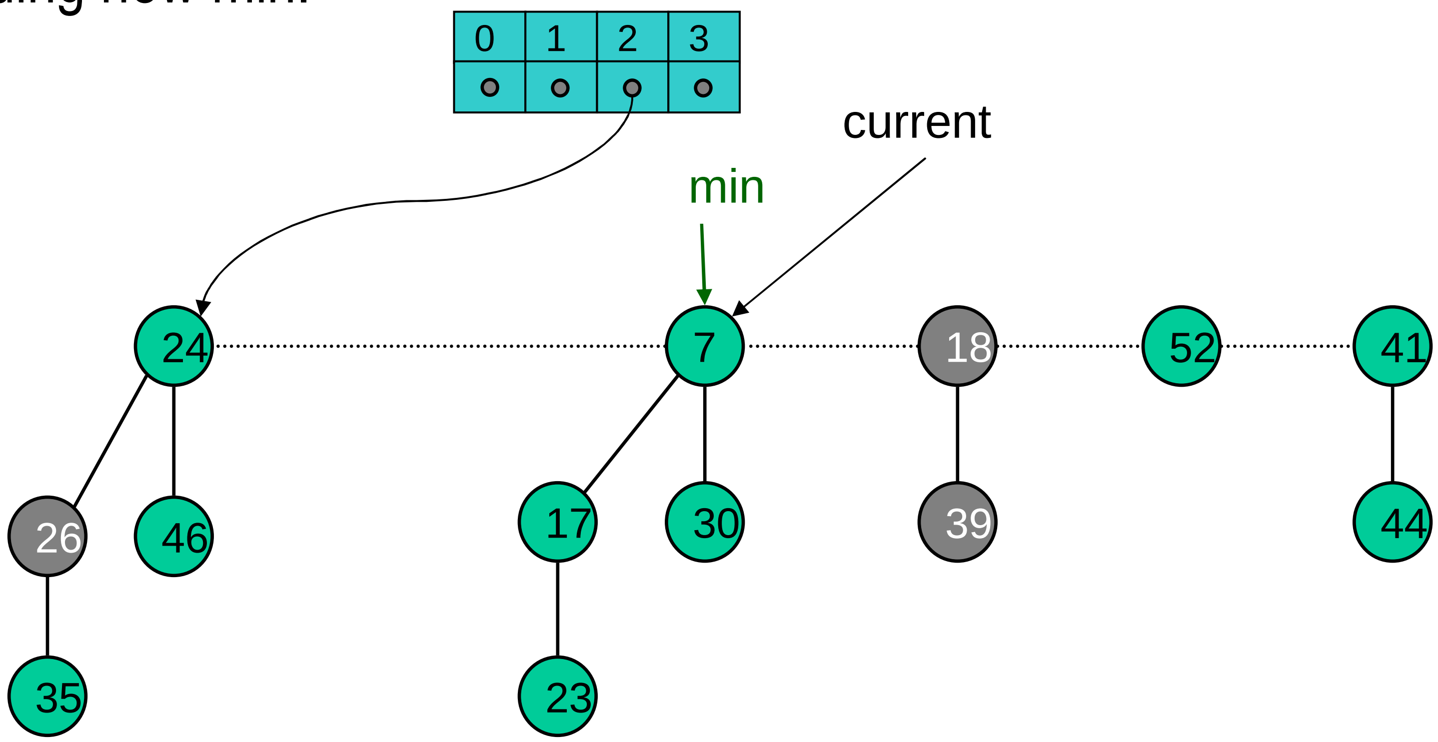

Merge 7 & 24 trees.

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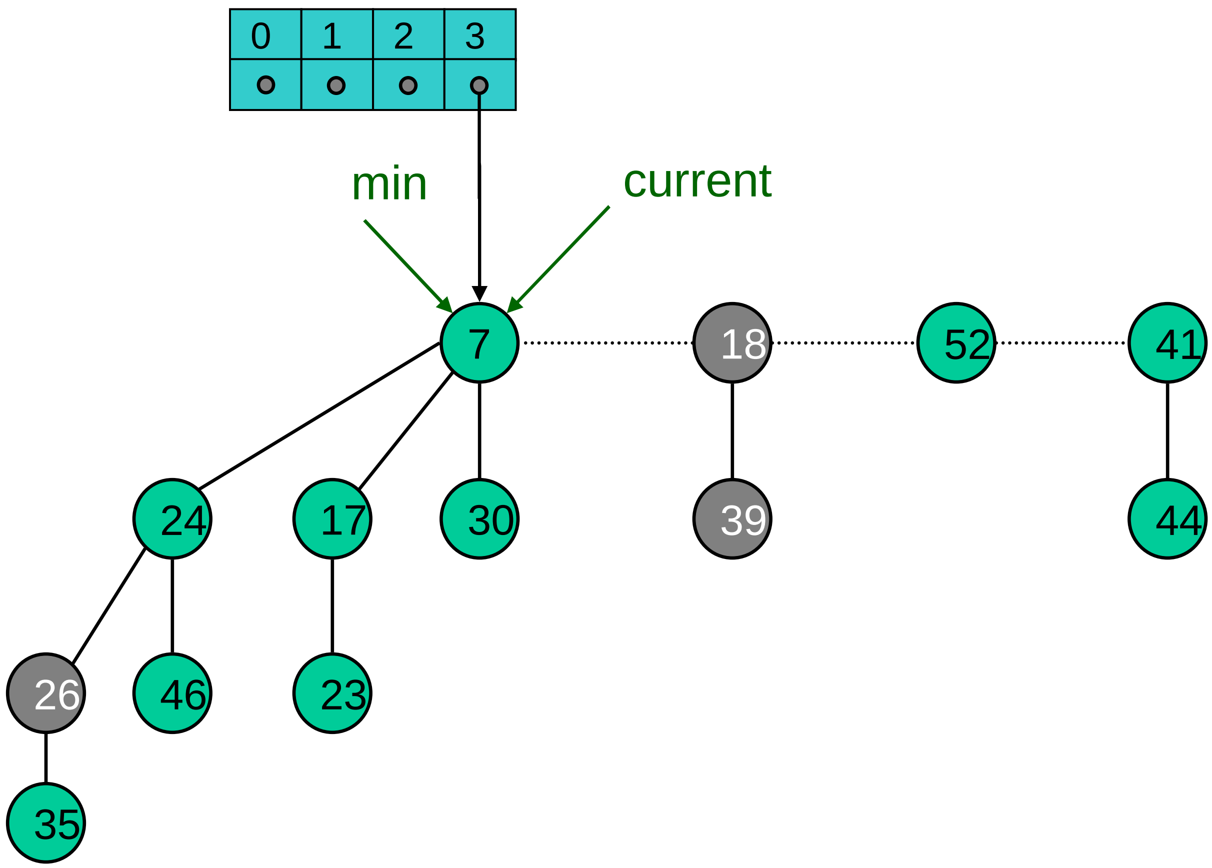

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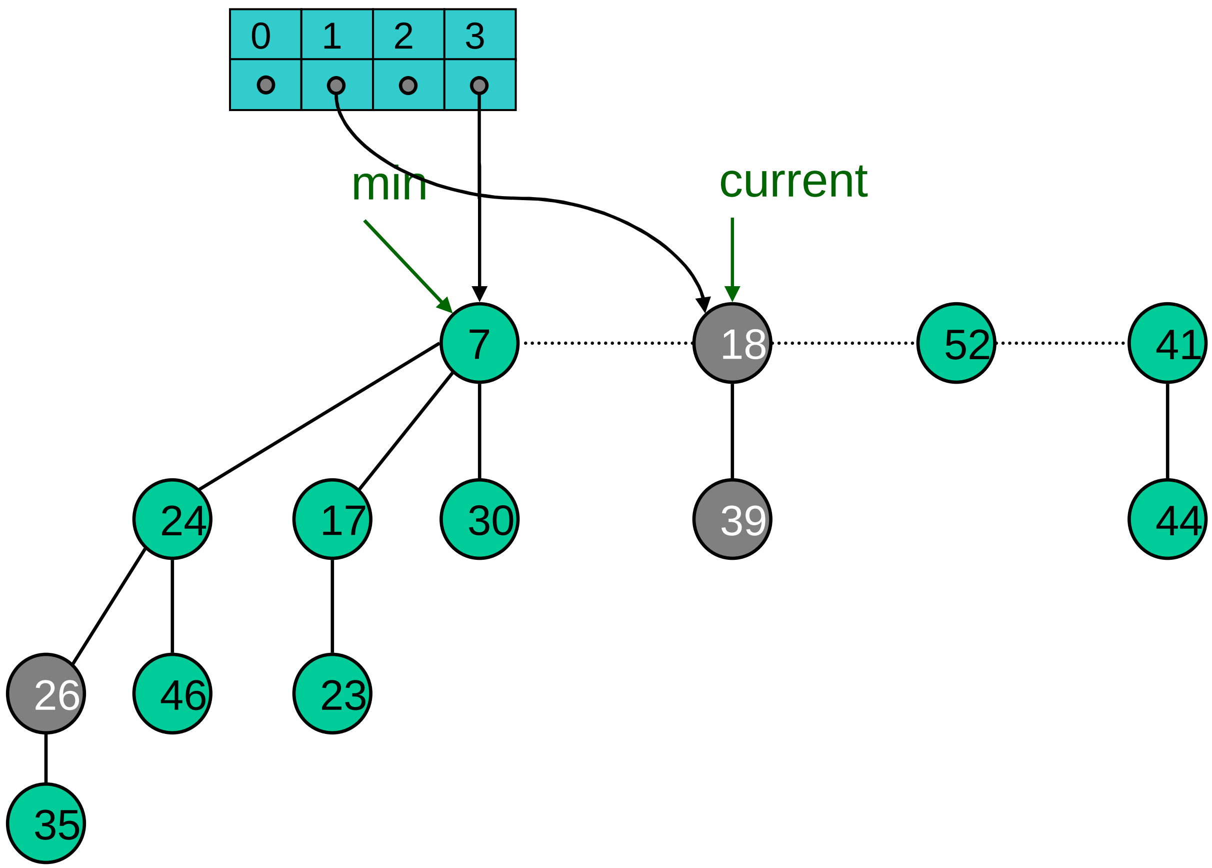

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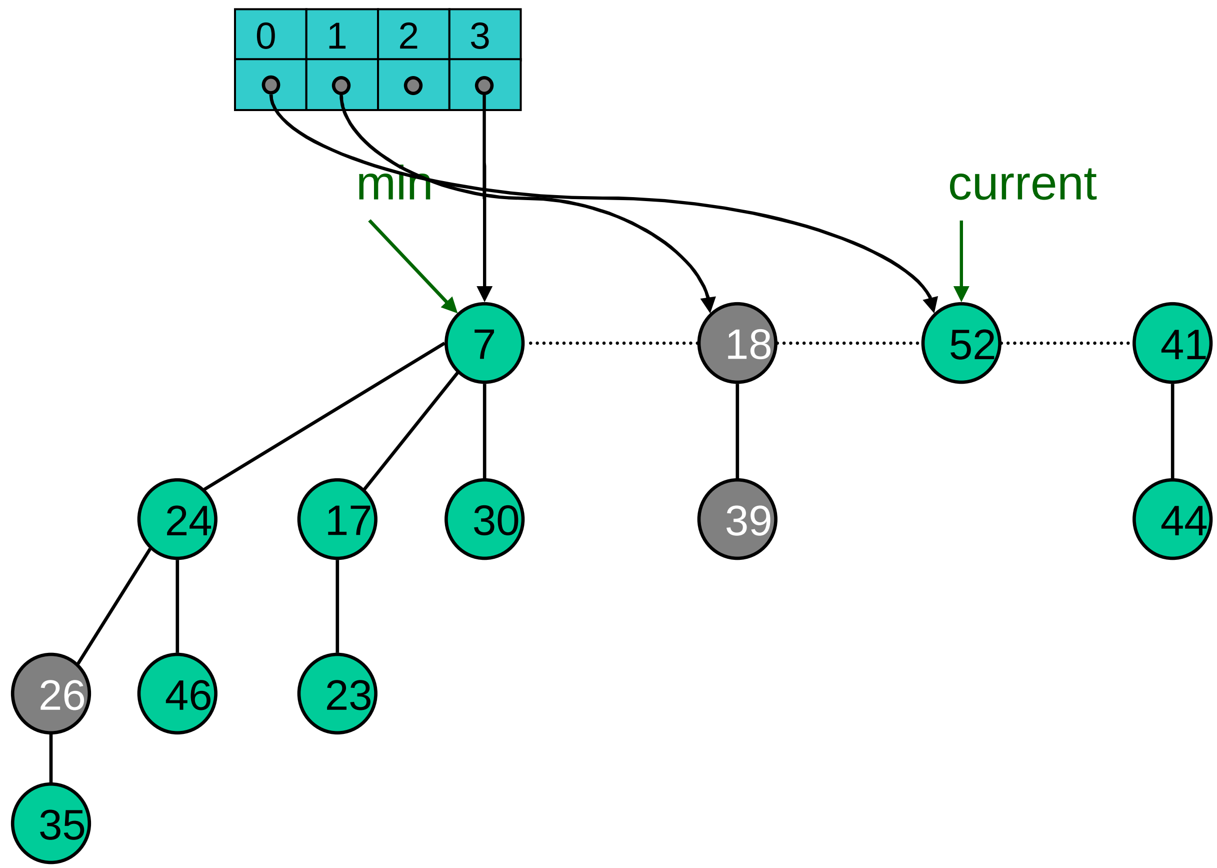

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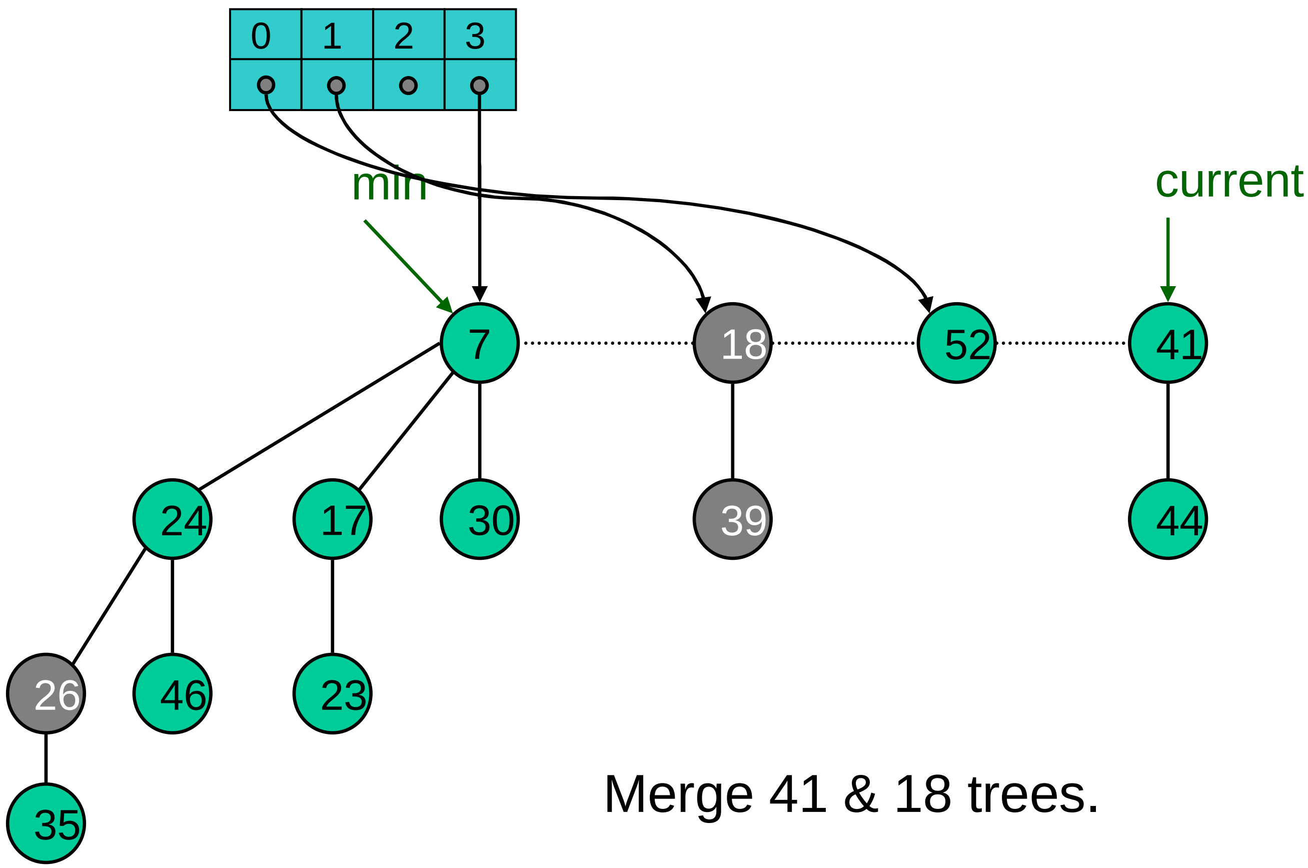

Merge 41 & 18 trees.

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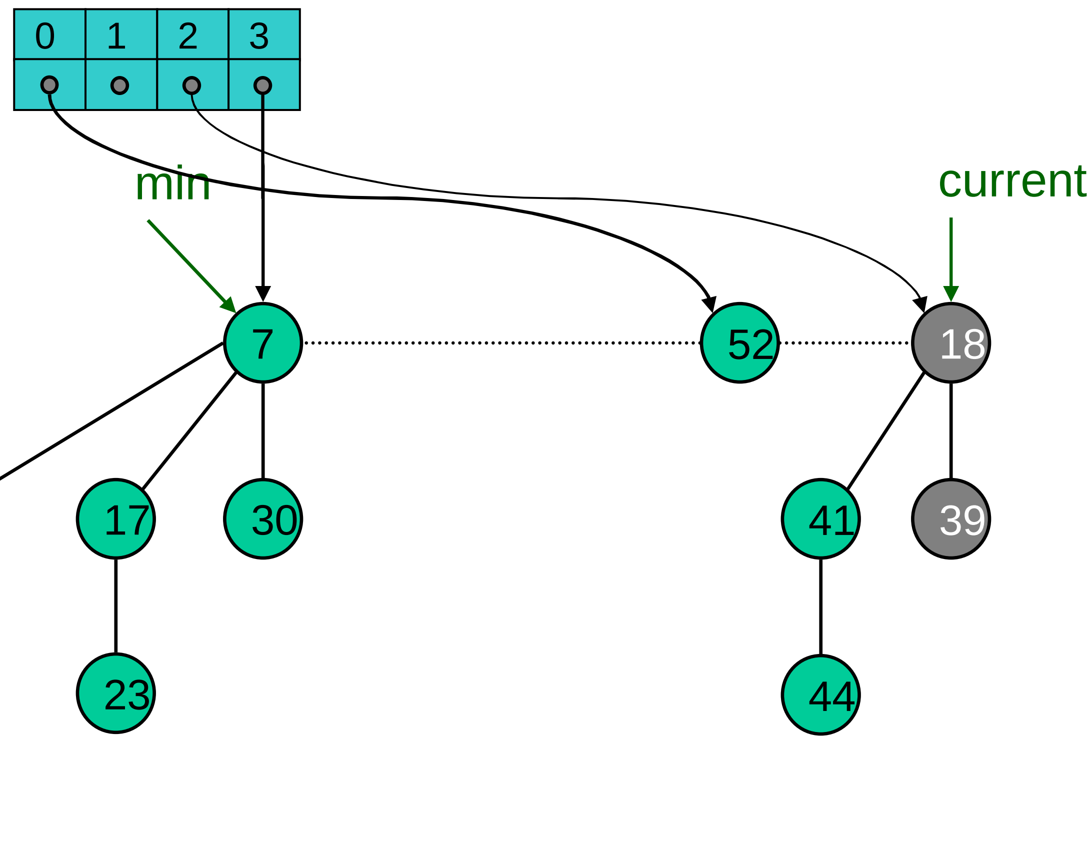


Delete-Min Analysis

$\Phi(H) = t(H) + 2m(H)$

$D(n) =$ max degree of any node in Fibonacci heap with n nodes

Actual cost = $O(D(n) + t(H))$

- $O(1)$ work adding min's children into root list & updating min.
- $O(D(n) + t(H))$ work consolidating trees.
  - At most $D(n)$ children of min node.
  - $\leq D(n) + t(H) - 1$ trees at beginning of consolidation.
  - #trees decreases by one after each merging

Amortized cost = $O(D(n) + t(H)) + \Delta\Phi(H) = O(D(n))$

- $t(H') \leq D(n) + 1$, since no two trees have same degree.
- $m(H') \leq m(H)$
- $\Delta\Phi(H) \leq D(n) + 1 - t(H)$

Delete-Min Analysis

Theorem

The Delete-Min operation can be implemented to run in $O(D(n) + t(n))$ actual time, and $O(D(n))$ amortized time

Is amortized cost of $O(D(n))$ good?

Yes, if only Insert, Union, & Delete-min supported.

- In this case, Fibonacci heap contains only binomial trees, since we only merge trees of equal root degree.
- $D(n) \leq \lfloor \log_2 N \rfloor$

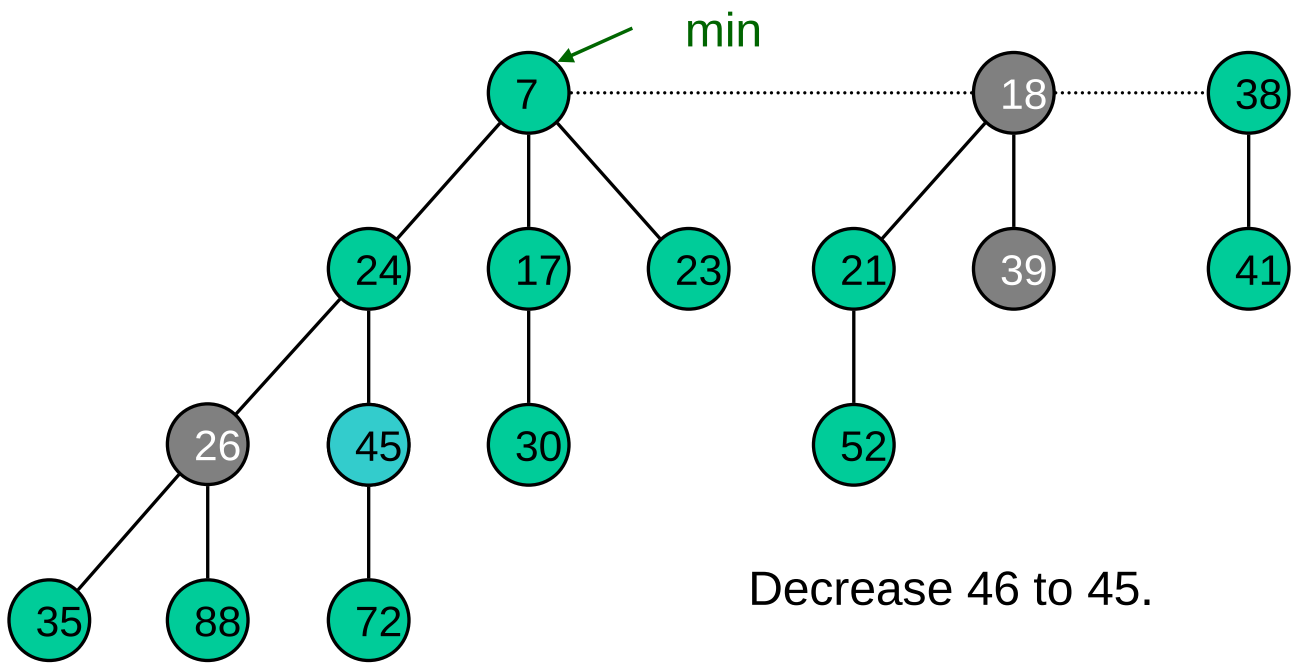
Yes, if we support Decrease-key cleverly.

- $D(n) \leq \lfloor \log_\phi N \rfloor$, where ϕ is golden ratio = 1.618…

Decrease Key

Case 0: min-heap property not violated.

1. Decrease key.
2. Change min pointer if necessary.

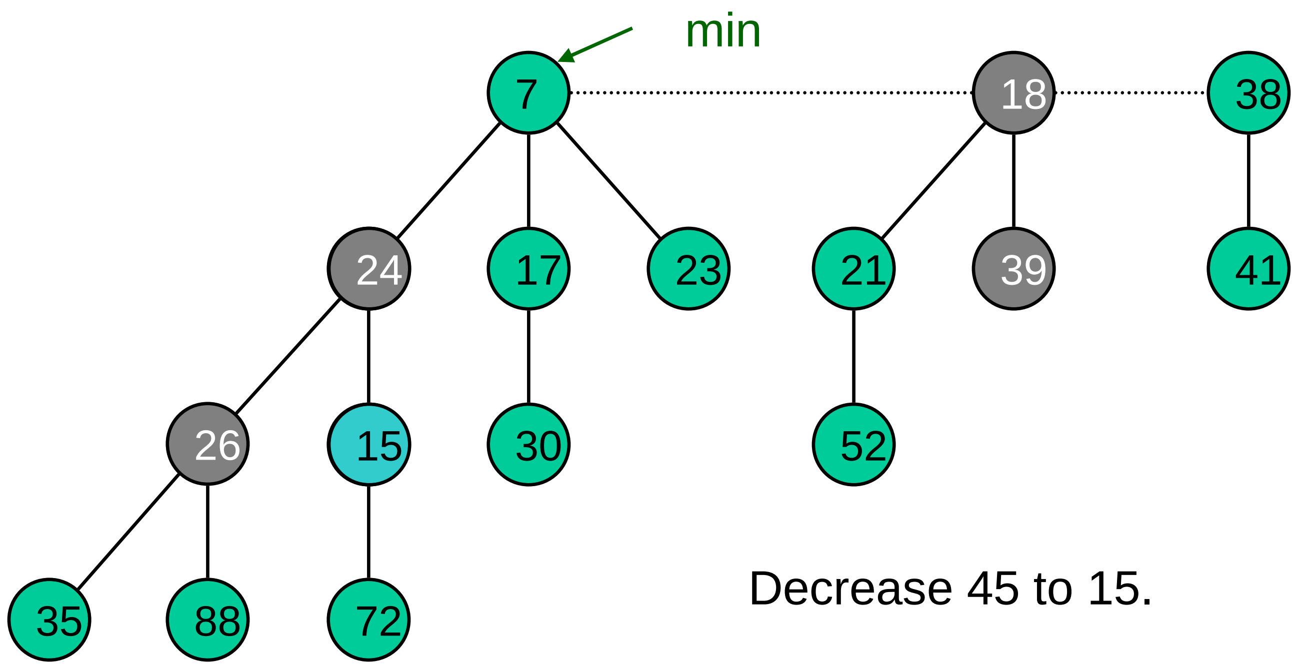


Decrease 46 to 45.

Decrease Key

Case 1: parent of x is unmarked.

1. Decrease key.
2. Remove link to parent.
3. Mark parent.
4. Add x's tree to root list, updating heap min pointer.



Decrease 45 to 15.

Decrease Key

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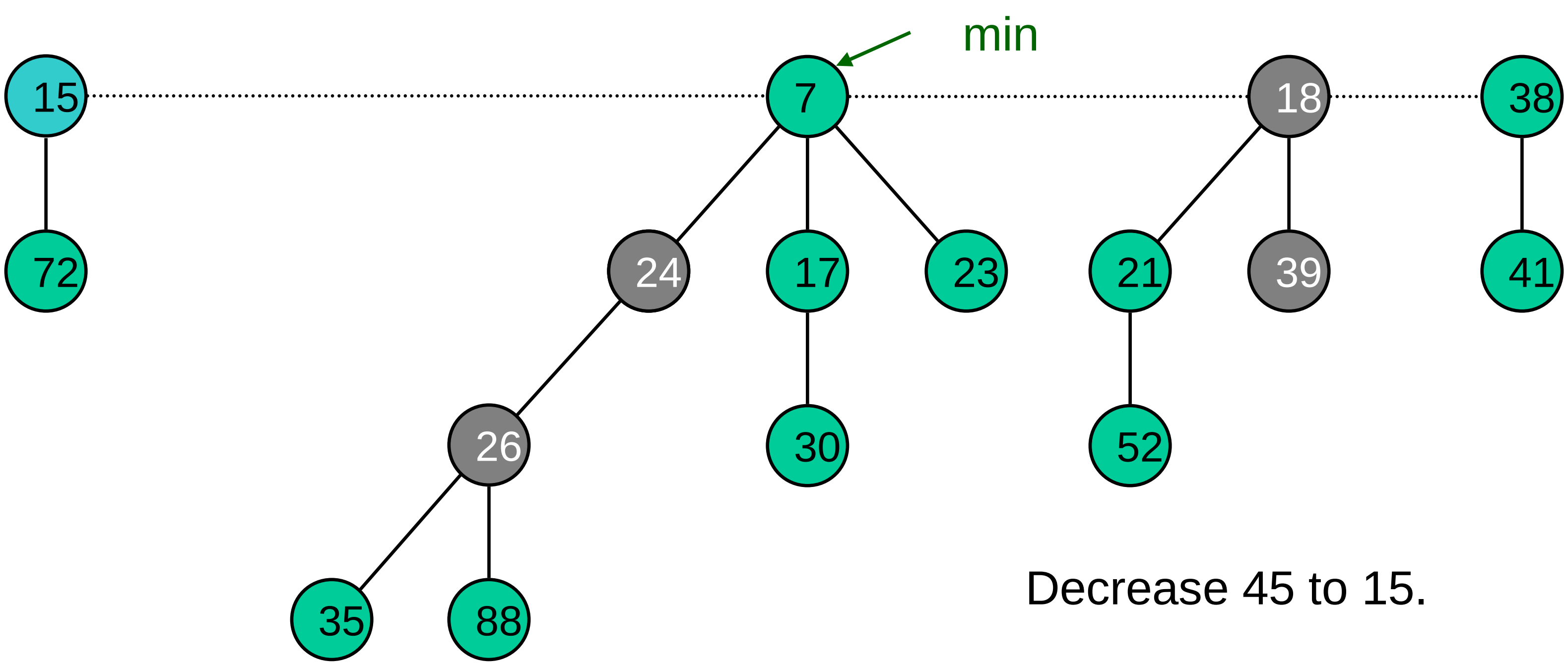
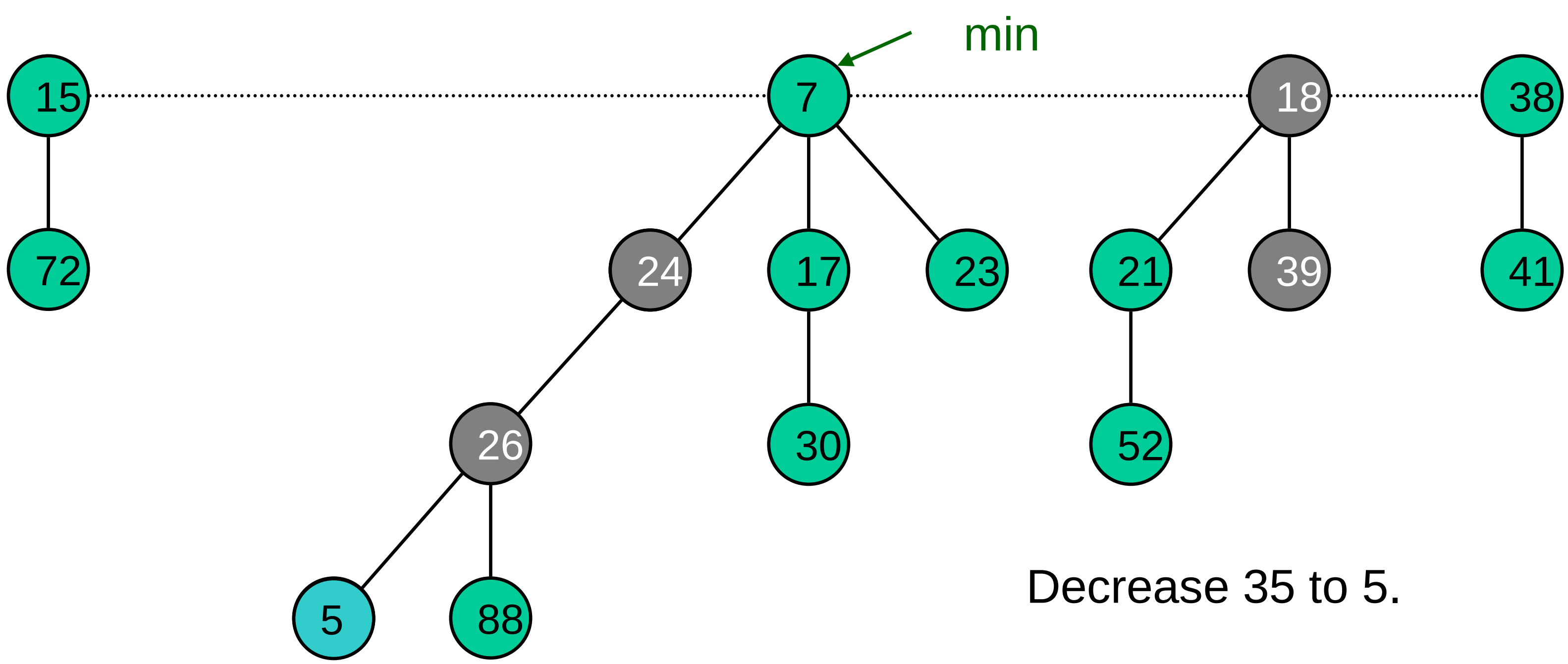


Decrease 45 to 15.

Decrease Key

Case 2: parent of x is marked.

1. Decrease key.
2. Move node to root list, updating heap min pointer.
3. Move chain of marked ancestors to root list, unmarking.
4. Mark first unmarked non-root ancestor.



Decrease 35 to 5.

Decrease Key

Case 2: parent of x is marked.

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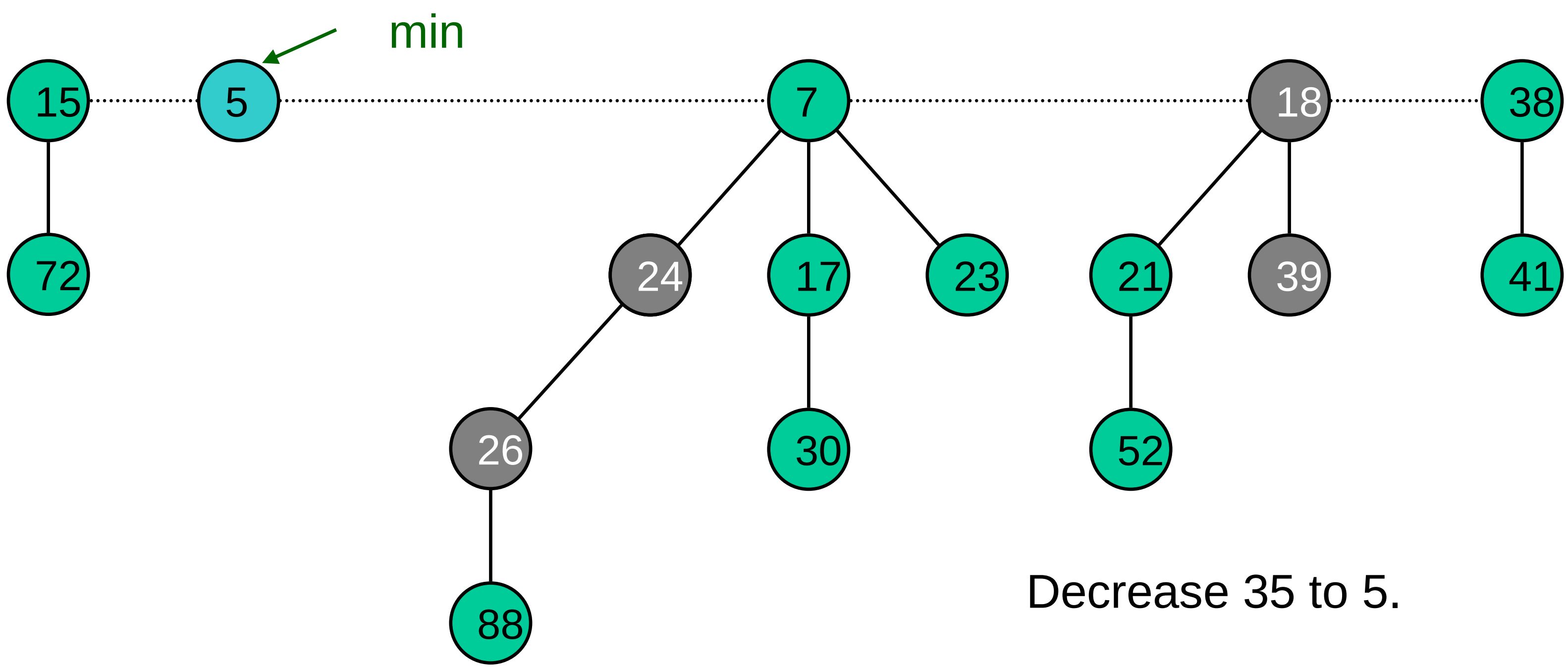

Decrease 35 to 5.

Decrease Key

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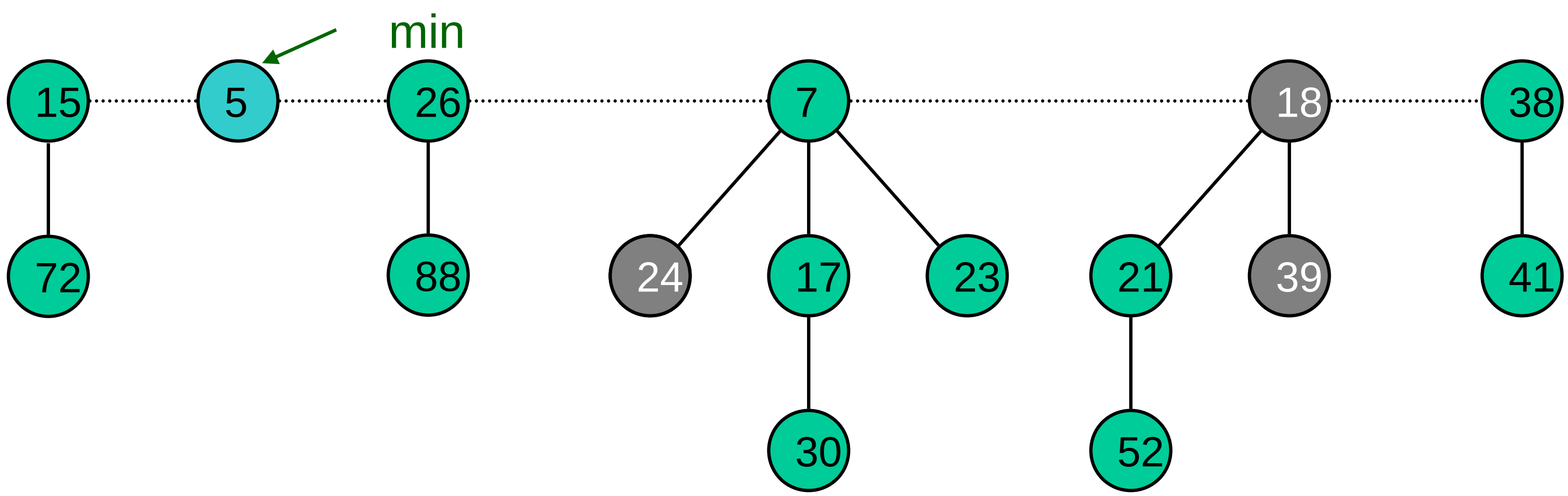

Decrease 35 to 5.

Decrease Key

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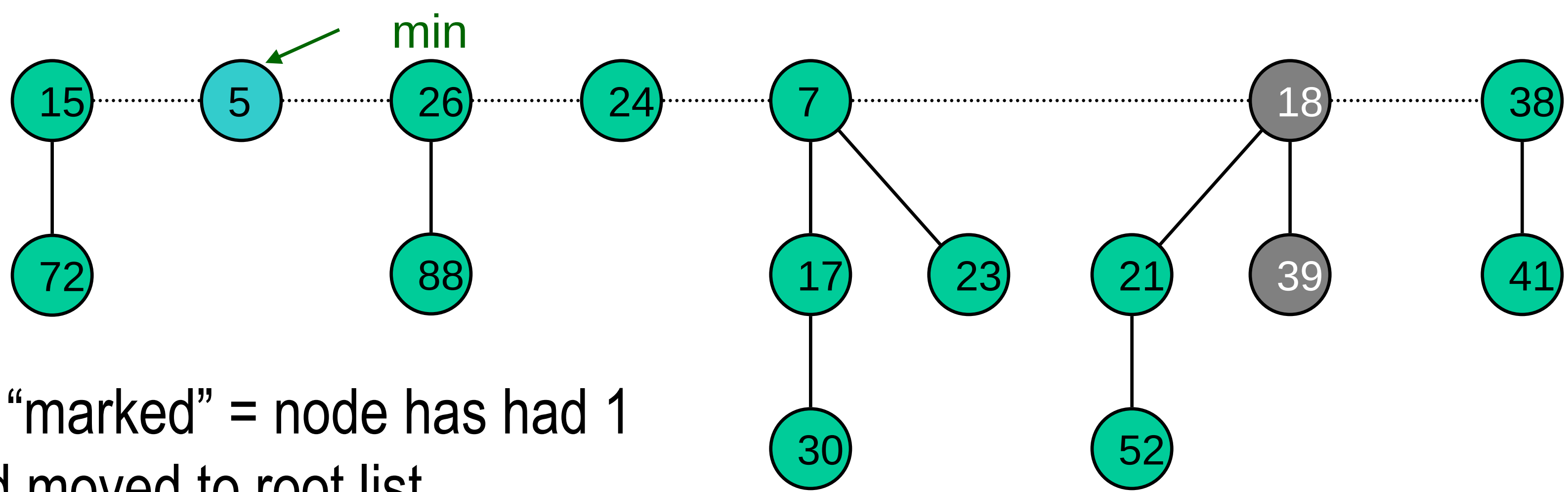

I.e., "marked" = node has had 1 child moved to root list.
Once we move a 2nd child of node, we also move the node.

Decrease 35 to 5.

Decrease-Key Analysis

- $t(H)$ = # trees in heap H.
- $m(H)$ = # marked nodes in heap H.
- $\Phi(H)$ = $t(H) + 2m(H)$.

Actual cost = O(c), where c = # of nodes "cut"

- O(1) time for decrease key.
- O(1) time **for each** cut.

Amortized cost = $O(c) + \Delta\Phi(H) = O(1)$

- $t(H') = t(H) + c$
- $m(H') \leq m(H) - c + 2$
  - Each cut unmarks a node.
  - Last cut could potentially mark a node.
- $\Delta\Phi(H) \leq c + 2(-c + 2) = 4 - c$.

Decrease-Key Analysis

Theorem

The Decrease-Key operation can be implemented to run in $O(c)$ actual time, where c is the number of nodes "cut", and in $O(1)$ amortized time

Delete

1. Decrease key of x to $-\infty$.
2. Delete min element in heap.

Amortized cost = O(D(n))

- O(1) for decrease-key.
- O(D(n)) for delete-min.
- D(n) = max degree of any node in Fibonacci heap.

Bounding D(n)

D(n) = max degree in Fibonacci heap with n nodes.

Theorem

$D(n) \le \log_\phi n$, where $\phi = (1 + \sqrt{5}) / 2$.

Thus, Delete & Delete-min take O(log n) amortized time.

Proof is somewhat tedious & explained well in [CLRS].

Key Lemma

Let size(x) = #nodes in the subtree rooted at x.

Then, $\phi^{\text{degree}(x)} \le \text{size}(x)$.