STAT 485/685
Stationary Series and trends

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Purposes of These Notes

- Today two topics. First:
- Define stationary processes.
- Show some examples of stationary processes.
- Working on 2.3, in the text.
- I will not discuss the random cosine process on p 18.

Stationarity

- Goal: find assumptions on a discrete time process which will permit us to make reasonable estimates of the parameters.
- Intuition: need some notion of replication.
- Time series $Y_t; t = 0, \pm 1, \ldots$ is *stationary* if joint distribution of $Y_{t+1}, \cdots, Y_{t+k}$ is same as joint distribution of $Y_1, \cdots, Y_k$ for all t and all k.
- More precise terminology: *strictly* stationary.
- Time series $Y_t; t = 0, \pm 1, \ldots$ is *weakly* (or *second order*) stationary if

$$\mathrm{E}(Y_t) \equiv \mu$$

for all t (that is the mean does not depend on t) and

$$\mathrm{Cov}(Y_t, Y_{t+h}) = \mathrm{Cov}(Y_0, Y_h) \equiv \gamma_h$$

is a function of h only (and does not depend on t).

Relationships, Gaussian processes

- X finite variance, strictly stationary implies X weakly stationary.
- X second order stationary and Gaussian implies X strictly stationary.
- **Def'n**: Y is *Gaussian* if for each $t_1, \ldots, t_k$ the vector $(X_{t_1}, \ldots, X_{t_k})'$ has a Multivariate Normal Distribution
- MVN distribution defined by a mean vector $\boldsymbol{\mu}$ and a variance covariance matrix $\boldsymbol{\Sigma}$.
- If $Z = (Z_1, \ldots, Z_k)$ is column vector with MVN$(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ dist then
- $\mu_i = \mathrm{E}(Z_i)$ and $\Sigma_{ij} = \mathrm{Cov}(Z_i, Z_j)$.

Stationary Gaussian Time Series

- **Def'n**: The process Y has **stationary** covariance if:

$$\begin{aligned}\mathrm{Cov}(Y_t, Y_s) &= \mathrm{Cov}(Y_{t+1}, Y_{s+1}) \\ &= \mathrm{Cov}(Y_{t+2}, Y_{s+2}) = \cdots\end{aligned}$$

- If so then for all t and h we find

$$\begin{aligned}\mathrm{Cov}(Y_t, Y_{t+h}) &= \mathrm{Cov}(Y_0, Y_h) \\ &\equiv \gamma_h\end{aligned}$$

- Call γ_h *autocovariance* function of Y.

Autocovariance and Covariance Matrices

- Notice: $\boldsymbol{\Sigma}$ for $Y_1, \ldots, Y_T$ has

  γ_0 down the diagonal
  γ_1 down the first sub and super diagonals
  γ_2 down the next sub and super diagonals and so on.

- Such a matrix is called a Toeplitz matrix.
- For $T = 3$:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 \\ \gamma_1 & \gamma_0 & \gamma_1 \\ \gamma_2 & \gamma_1 & \gamma_0 \end{bmatrix}$$

Computing Example

- I will show plots of
- A weakly correlated Moving Average:

$$Y_t = \epsilon_t + \epsilon_{t-1}/10$$

- A more strongly correlated Moving Average:

$$Y_t = \epsilon_t + \epsilon_{t-1}$$

- A random walk

$$Y_t = \epsilon_1 + \cdots + \epsilon_t$$

- The *sample autocorrelation* functions of these.
- The *sample cross-correlation* between the two MA series.

Purposes of These Notes

- Define a *trend*
- Discuss some specific trends: seasonal, linear.
- Estimating a constant mean.
- Sections 3.1 and 3.2 in text.

Trend

- Some series have a mean which is quite predictable.
- Three common structures: constant, periodic, and linear.
- Constant mean: for all t

$$\mu_t = \mu.$$

- Periodic with period S: for all t

$$\mu_{t+s} = \mu_t.$$

- Linear trend: for all t

$$\mu_t = \beta_0 + \beta_1 t.$$

Estimating a constant mean

- Since each Y_t has the same expected value use

$$\hat{\mu} = \bar{Y} = \frac{1}{T}\sum_{t=1}^{T} Y_t.$$

- How accurate is that?
- Measure its *standard error* (SE):

$$\mathrm{SE} = \sqrt{\mathrm{Var}(\bar{Y})}.$$

- Use variance formulas:

$$\mathrm{Var}(\bar{Y}) = \frac{1}{T^2}\sum_{ij}\mathrm{Cov}(Y_i, Y_j). = \frac{1}{T^2}\sum_{ij}\gamma_{i,j}.$$

- So just add up all the covariances and divide by T^2.

For a stationary series

- For a stationary series go back to the covariance matrix I showed.
- Total of T diagonal terms all $= \gamma_0$.
- Total of $2(T-1)$ terms γ_1.
- And so on.
- Resulting SE is

$$\frac{1}{\sqrt{T}}\sqrt{\gamma_0 + 2\sum_{j=1}^{T-1}\left(1-\frac{j}{T}\right))\gamma_j} = \frac{\sqrt{\gamma_0}}{\sqrt{T}}\sqrt{1 + 2\sum_{j=1}^{T-1}\left(1-\frac{j}{T}\right))\rho_j}$$

- How big is that?

Special cases

- Proportional to SD of Y so make comparison with same value of γ_0.
- For white noise get usual iid sampling formula

$$\mathrm{SE} = \frac{\sqrt{\gamma_0}}{\sqrt{T}} = \frac{\sigma}{\sqrt{T}}.$$

  where $\sigma^2 = \mathrm{Var}(\epsilon_t)$ is the noise variance.
- For MA(1) $Y_t = \epsilon_t + a\epsilon_{t-1}$ we have

$$\gamma_0 = (1 + a^2)\sigma^2 \text{ and } \gamma_1 = a\sigma^2$$

  and

$$\mathrm{SE} = \frac{\sqrt{\gamma_0}}{\sqrt{T}}\sqrt{1 + 2(T-1)\rho_1/T}$$

- So larger, compared to SD of single Y, than white noise if $a > 0$.

Random Walk, non-stationary series

- Suppose $Y_t = \mu + \epsilon_1 + \cdots + \epsilon_t$.
- Then $Y_1 + \cdots + Y_T$ is

$$T\mu + \epsilon_T + 2\epsilon_{T-1} + \cdots + T\epsilon_1$$

  because every Y_t contains ϵ_1, all but Y_1 contain ϵ_2 and so on.
- So $\mathrm{Var}(Y_1 + \cdots + Y_T)$ is

$$\sigma^2 + 4\sigma^2 + \cdots + T^2\sigma^2$$

  and

$$\mathrm{Var}(\bar{Y}) = \sigma^2 \left(\frac{1}{T^2} + \frac{4}{T^2} + \cdots \frac{T^2}{T^2} \right).$$

- Algebraic tricks permit us to compute

$$\mathrm{Var}(\bar{Y}) = \sigma^2(2T+1)\frac{T+1}{6T}$$

  which gets *bigger* at T gets bigger.