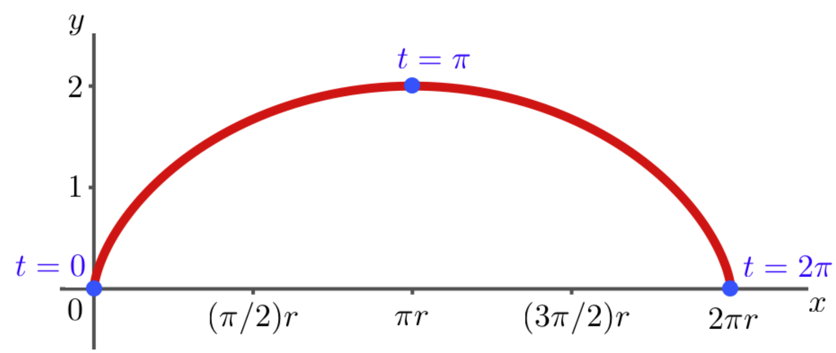
Calculus with Parametric Curves

1. **Quote.** "If you have a procedure with 10 parameters, you probably missed some." (Alan Perlis, American computer scientist and mathematician, 1922-1990)

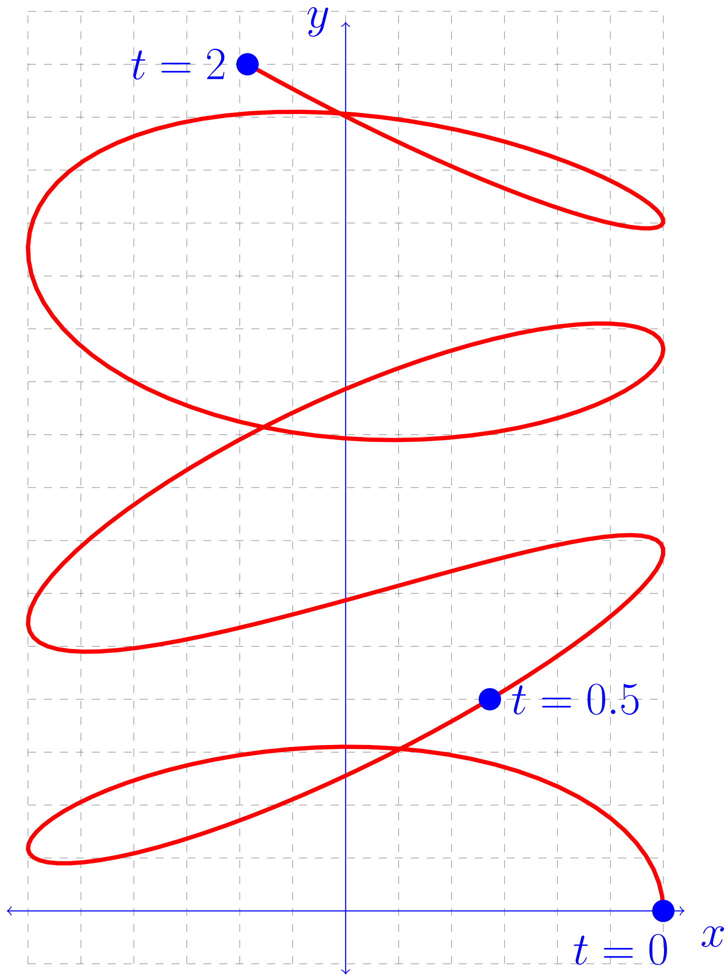
2. **Problem.** Find the arc length of one arch of the cycloid

$$x = r(t - \sin t), y = r(1 - \cos t), 0 \leq t \leq 2\pi.$$

Also find the area under this arch.



3. **Parametric Curves**



Parametrization gives us a curve (path), but at the same time describes the position of a particle as a function of time: $(x(t), y(t))$.

Velocity is $(x'(t), y'(t))$;

Acceleration is $(x''(t), y''(t))$.

Clearly, the curve to the left cannot be the graph of a function; parametrization allows path crossings, and closed curves.

A particle could trace part or all of its path more than once.

You will study curves like these in 2D and 3D in more detail in Math 251.

4. **Some previous results in a new context.**

Calculus for Parametric Curves.

Suppose the function $y(x)$, for $x \in [a, b]$, is defined by the parametric equations

$$x = f(t) \quad \text{and} \quad y = g(t) \quad \text{for } t \in [\alpha, \beta]$$

and let C be the corresponding parametric curve.

(We assume that f and g satisfy all conditions that will guarantee that the function $y(x)$ has the necessary properties that allow for the existence of all listed integrals.)

1. If f and g are differentiable with $f'(t) \neq 0$, then $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{g'(t)}{f'(t)}$.

2. If $y(x) \geq 0$, then the **area** under the curve C is given by

$$A = \int_a^b y(x)\, dx = \int_\alpha^\beta g(t) f'(t)\, dt$$

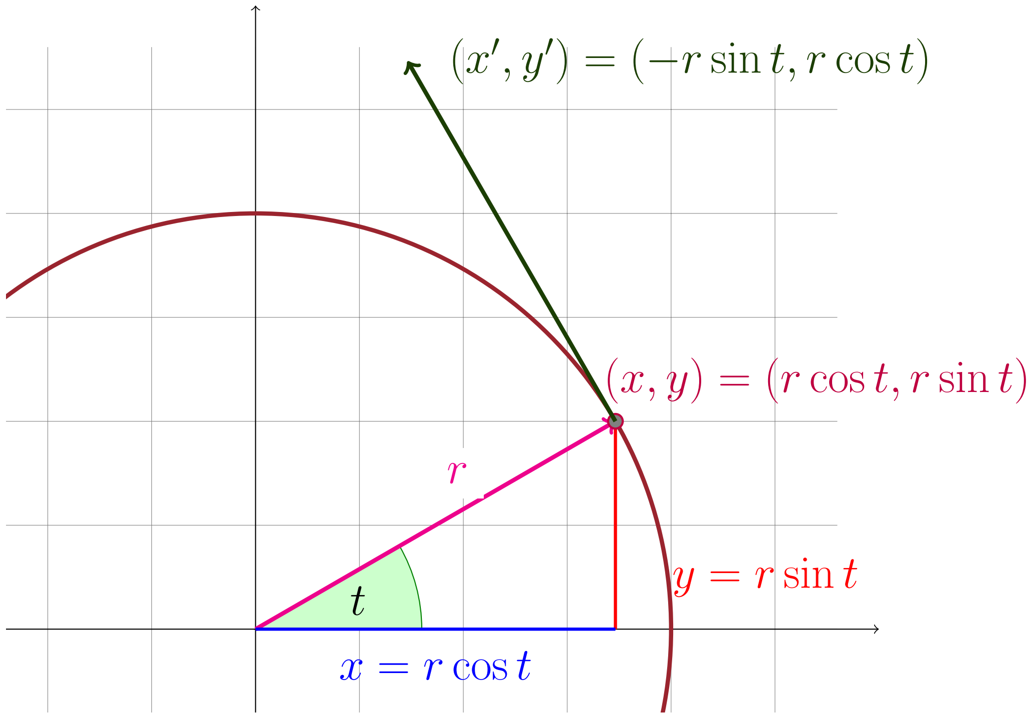
3. If f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the **length of the curve** C is given by

$$\boldsymbol{s} = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx = \int_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}} \sqrt{[\,\boldsymbol{f'(t)}]^2 + [\,\boldsymbol{g'(t)}]^2}\, \boldsymbol{dt}$$

4. If $g(t) \geq 0$ then the **area of the surface** obtained by rotating C about the x-axis is given by

$$\boldsymbol{S} = \int_a^b 2\pi y \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx = \int_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}} \boldsymbol{2\pi g(t)} \sqrt{[\,\boldsymbol{f'(t)}]^2 + [\,\boldsymbol{g'(t)}]^2}\, \boldsymbol{dt}$$

5. Going back to going around in circles



"Canonical" parametrization of the circle:
$(x, y) = (r\cos t, r\sin t)$.

Particle moves around circle with speed 1.

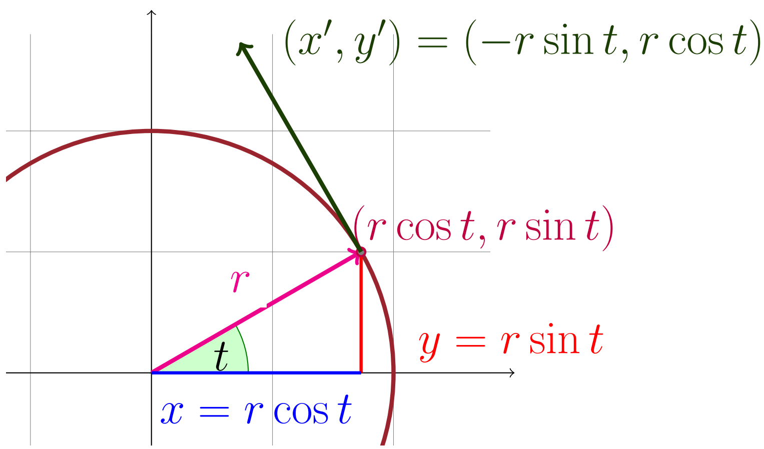
Note: $(x, y) = (r\cos(t^2), r\sin(t^2))$ traces the same curve, but particle moves with speed $2t$.
Velocity remains **tangent** to the circle.

Velocity (x', y') is tangent to the circle, i.e., **orthogonal** to the *vector* (x, y).

This follows directly from $x(t)^2 + y(t)^2 = r^2$.

Acceleration is $(x'', y'') =$
Hence,

$$x'' + \qquad\qquad = 0, \qquad\qquad y'' + \qquad\qquad = 0.$$



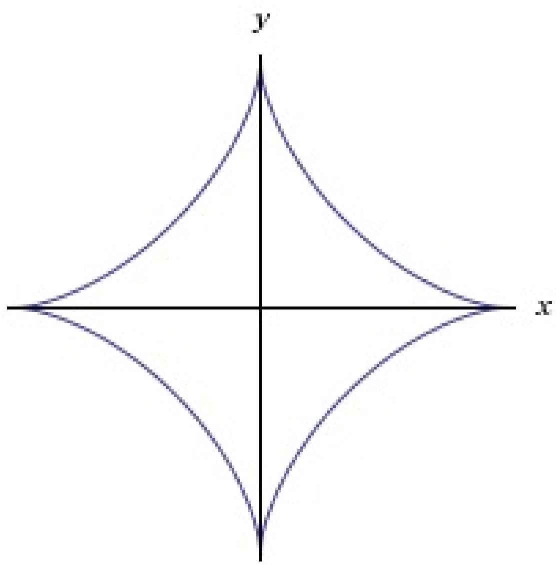
1. Slope at $(x(t), y(t))$:

2. Area under the quarter circle: $\int_0^r y(x)dx = -\int_0^{\pi/2} y(t)x'(t)dt =$

3. Arc length of semicircle: $\int_0^{\pi} \sqrt{[\,x'(t)]^2 + [\,y'(t)]^2}\,dt =$

4. Rotate upper semicircle about x axis, surface area: $\int_0^{\pi} 2\pi y(t)\sqrt{[\,x'(t)]^2 + [\,y'(t)]^2}\,dt =$

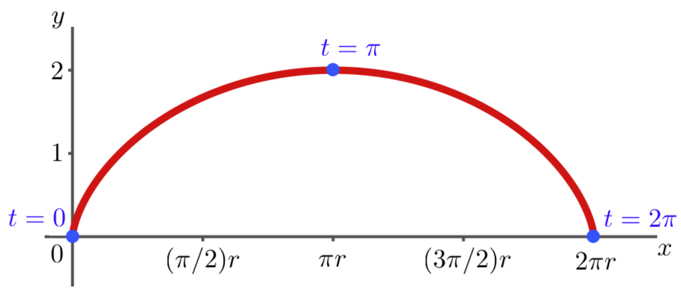
6. **Example.** Find the slope of the tangent to the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$ as a function of the parameter θ. At what points is the tangent horizontal. Vertical? At what points does that tangent have slope 1. What about slope -1?



7. **Example.** Find the area under one arch of the cycloid:

$$x = r(t - \sin t), y = r(1 - \cos t), 0 \le t \le 2\pi.$$

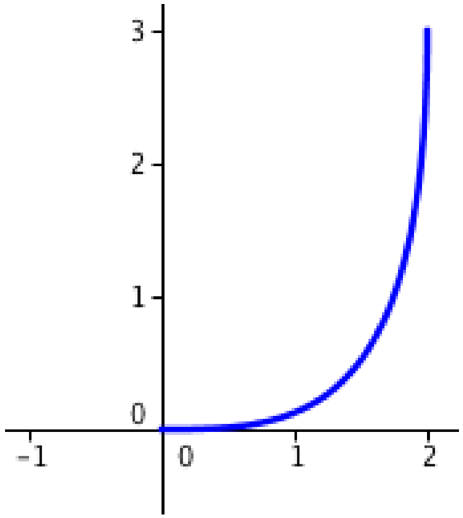
Also find the arc length of this arch.



8. **Example.** Find the area of the surface obtained by rotating the curve

$$x = 3t - t^3, \; y = 3t^2, \quad 0 \leq t \leq 1$$

about the x-axis.



9. **Polar Coordinates are just parametric equations, really...**
Find the arc length s of the cardioid with polar equation

$$r = 1 + \cos\theta.$$

Also, find also the surface area S generated by revolving the cardioid around the x-axis. This will look very much like an apple – without a stem.