Distributed Rigidity-Based Active Localization for Heterogeneous Robotic Teams

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Abstract. This work considers a heterogeneous robotic team structured with a leader and a set of children. Each agent has limited sensing and mapping capabilities. Specifically, the whole team is provided with range sensors for localization purposes and generic feature-detection technology to map the surrounding unknown environment. Our goal is to localize the entire team with respect to the leader reference frame while maximizing the area mapped by the agents. We address these goals by combining Active localization with Rigidity Theory: the former consists of a decision-making process in which each agent chooses its future control actions to keep accurate localization, while the latter consists of a mathematical framework used to find the conditions for unique localizability. We first exploit the Rigidity Theory to develop a distributed motion policy so that all the agents can recover their position with respect to the leader. Secondly, we propose a geometrical interpretation of rigidity, maximizing the team's exploration capabilities, and eventually addressing both localization and mapping from the same mathematical perspective. We test the performance of our proposed solutions through a batch of Monte Carlo simulations.

Maybe here is not fundamental to say why the leader is different. The leader-follower structure is quite standard.

1 Introduction

Recently, increasing interest has been drawn to collaborative robotics, specifically in heterogeneous multi-agent systems (MAS) [1]; these are characterized by agents differing in cognitive or physical properties, typically operating in unknown environments [2,3]. In these scenarios, collaborative exploration becomes a fundamental task, mainly addressed through *Simultaneous Localization And Mapping* (SLAM) [4–7], Bayesian filtering-based, and distributed localization solutions [8–11]. Sensors usually consider bearing and distance measurements, both in *Line of Sight* (LOS) and *Non-Line of Sight* (NLOS) conditions [12–15].

In this context, Active localization arises as a valid methodology [16, 17] to localize MAS. Active localization consists of a decision-making process in which each agent chooses its future control actions, balancing exploring new areas and exploiting those already seen to improve localization. This approach dates back to the 90s, with the *Cooperative Positioning System* (CPS) proposed by Hirose and Kurazume [18–22].

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Among the proposed Active localization approaches, in this work, we focus on those exploiting *Rigidity Theory*, which consists of a mathematical framework hinged on *Graph Theory* that can be used to find the conditions for unique localizability of a network whose agents are provided with distances from their neighbors [23–26]. Extensions have been developed exploiting bearing measurements [27–30]. We recall the fundamentals of Rigidity Theory in sec. 3.

1.1 Related work and contributions

This work considers a heterogeneous team with a leader and a set of children[1], as in [18], and focuses on two main aspects: network localization and environment mapping coverage [31–34]. We outline the case study considered in this work in sec. 2. We now describe previous works that inspired our research and outline our main contributions.

Network localization Rigidity Theory provides conditions to unambiguously locate all nodes in a positioned network, depending on the connectivity properties of its communication graph. In Active localization, agents not satisfying such conditions are provided with a motion policy that increases their chances of recovering them. Shames et al. [35], considers a team of agents exploring an unknown environment scattered with fixed landmarks with unknown locations. Each agent detects a set of landmarks and shares them with any neighbor. Rigidity Theory is exploited to find the conditions under which agents can merge their local maps, embedding them as constraints in the exploring actions. In the first part of this work (sec. 4), we develop a distributed rigidity-based localization algorithm, relaxing the assumptions from [35]; we localize the entire team with respect to the leader, starting from a non-rigid and simply connected network. Most importantly, the proposed algorithm is modular, allowing us to expand the goals of the motion policy to the team mapping coverage.

Mapping coverage The second part of this work interprets rigidity in terms of the agents' spatial disposition and, in particular, dispersion, connectivity, and symmetry (sec. 5). The goal is to define a rigidity-based cost function to embed into our previous algorithm, allowing the team to localize and best explore the surroundings. In this sense, we were inspired by two works, one by Zhu and Hu [36] and the other by Zelazo et al. [37], which we integrate linking rigidity to coverage. Conclusions and future research directions are drawn in sec. 6.

1.2 Notation

Vector quantities are described by bold lowercase letters. Symbols (μ, σ) refer to mean and standard deviation. $\|\cdot\|$ defines the 2-norm operator. $\dot{x}$ refers to the time derivative of x. The matrix operator $\sigma(\mathbf{M})$ of a generic square matrix $\mathbf{M}$ returns its set of eigenvalues, i.e., its spectrum. We refer to a ball centered in $\boldsymbol{x}_0 \in \mathbb{R}^n$ with radius $\epsilon > 0$ as the set $\mathcal{B}_\epsilon(\boldsymbol{x}_0) \triangleq \{\boldsymbol{x} \in \mathbb{R}^n \text{ s.t. } \|\boldsymbol{x} - \boldsymbol{x}_0\| < \epsilon\}$. Graphs will be referred to with the calligraphic letter $\mathcal{G}$. The set of nodes and edges in

[1] We borrow the terminology leader-children from CPS in [18].

$\mathcal{G}$ are $\mathcal{V}$ and $\mathcal{E}$, respectively, and $|\mathcal{V}| = n$ and $|\mathcal{E}| = m$ are their cardinalities. In this work, we consider undirected graphs, i.e., $(i, j) \in \mathcal{E} \iff (j, i) \in \mathcal{E}$. Thus, for a generic agent A_i, the set of neighbors is $\mathcal{N}_i \triangleq \{j \in \mathcal{V} \text{ s.t. } (i, j) \in \mathcal{E}\}$. The $\sim$ before a logical condition describes its negation.

2 Problem statement

Consider a team of agents on a 2D map; the team is split into a leader A_L and the remaining children. Every agent has a range sensor with a sensing limit $\delta = 8$m, which is a distance comparable to *Ultra Wide Band* (UWB) antennas' range limit. UWB is a low-cost ranging technology widely used in robotics and sensor networks. The team leader is additionally provided with a vision system with a *field of view* (FOV) and a maximum distance of $\delta_C = 3$m; each agent falling within the camera's FOV is localized in the leader's reference frame. The agents are also provided with environment mapping sensors (e.g., wide-angle cameras, air quality sensors, etc.) with maximum range $\delta_M = 3$m. A graphic representation of the model is depicted in Fig. 1[2].

In this work, we consider a noise-free scenario. However, this model aims to describe a heterogeneous team where only the leader is provided with accurate (and more expensive) sensing technology, while all the other agents can only rely on higher-noise off-the-shelf sensors. Given this setup, we now define the two problems addressed in this work: the network localization problem and the environment coverage problem.

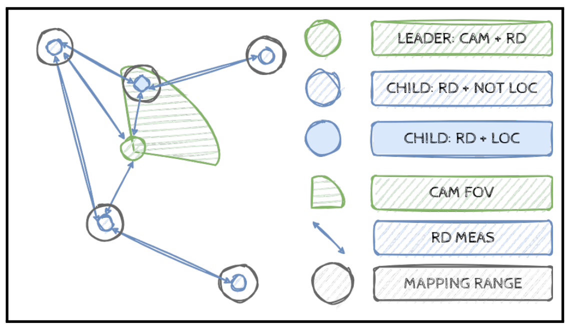

Fig. 1: Graphic representation of our case study. RD stands for *Relative Distance*, LOC for *Localized*.

The network localization problem As introduced in sec. 1, we aim to localize all the children with respect to the leader. This problem can be modeled through a framework, i.e., a positioned graph. For a team with agents' positions $\boldsymbol{p} = \{\boldsymbol{p}_i\}$ and communication graph $\mathcal{G}$ defined by the ranging sensors, the related framework is the pair $(\mathcal{G}, \boldsymbol{p})$. Every localized agent is called a *beacon* node. Beacons are a subset of the agents' set, i.e., $n_b < n$. With this notation, we define the following problem:

Problem 1 (Network localization problem). The network localization problem with distance information consists in unambiguously determining the locations $\boldsymbol{p}_i \in \boldsymbol{p}$ of framework $(\mathcal{G}, \boldsymbol{p})$, given the positions of the beacon nodes, and the distances among neighbors.

[2]The distances in the figure are not correctly scaled.

The environment coverage problem We define the mapping area of agent i as $M_i = \pi\delta_M^2$. Indeed, when $\|\boldsymbol{p}_i - \boldsymbol{p}_j\| < 2\delta_M$, the two mapping areas have an overlap I_{ij}. With this notation, we define the following problem:

Problem 2 (Environment coverage problem). The environment coverage problem for a localized heterogeneous team described by a generic framework $(\mathcal{G}, \boldsymbol{p})$ consists of maximizing the static team coverage, which is defined as

$$\mathcal{C} \triangleq \frac{\sum_{i=1}^{n} \left(M_i - \sum_{j \in \mathcal{N}_i} I_{ij} \right)}{A_{\text{env}}}, \tag{1}$$

where A_{env} represents the total area of the environment.

3 Rigidity Theory

This section recalls the main concepts and results of the Rigidity Theory for distance-based network localization [23–25,37]. As introduced in sec. 1, Rigidity Theory deals with agents' positions and connectivity; thus, it is well-described through frameworks. The two framework properties fundamental to rigidity are *equivalence* and *congruence*. Two frameworks $(\mathcal{G}, \boldsymbol{p}_0)$ and $(\mathcal{G}, \boldsymbol{p}_1)$ are *equivalent* when all their edges' lengths are equal, while they are *congruent* when this is true for all the node's distances. Consider Fig. 2a for an example. For a graph $\mathcal{G}$, all the frameworks equivalent up to isometric transformations are called *realizations*. When only one framework realization exists, the network localization problem can be *unambiguously* solved. We now introduce a first definition of rigidity:

Definition 1 ((Infinitesimal) Rigidity). *A framework $(\mathcal{G}, \boldsymbol{p}_0)$ is rigid there exists an $\epsilon > 0$ such that every framework $(\mathcal{G}, \boldsymbol{p}_1)$ which is equivalent to $(\mathcal{G}, \boldsymbol{p}_0)$ and satisfies $\|\boldsymbol{p}_{0,i} - \boldsymbol{p}_{1,i}\| < \epsilon \ \forall i \in \mathcal{V}$, is congruent to $(\mathcal{G}, \boldsymbol{p}_0)$.*

Rigidity can be seen as the capability of a framework to maintain congruence when equivalence-keeping transformations are applied with displacements limited to a ball $\mathcal{B}_\epsilon(\boldsymbol{p}_i)$ for each node (i.e., local behavior). If we consider any edge $(i, j) \in \mathcal{E}$, the following holds for a generic framework $(\mathcal{G}, \boldsymbol{p})$:

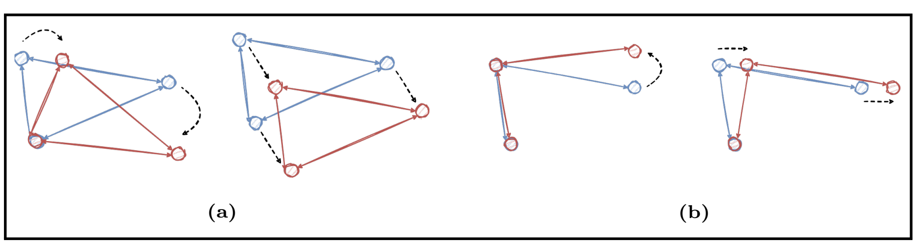

Fig. 2: (a) the blue framework can be transformed through rotations and translations (i.e., isometric transformations) without affecting the edges' length or the nodes' distances. (b) this is not true here, where equivalence is maintained but not congruence.

$$(\boldsymbol{p}_i - \boldsymbol{p}_j)^T(\boldsymbol{p}_i - \boldsymbol{p}_j) = d_{ij}^2, \tag{2}$$

with d_{ij} the distance between nodes. To investigate the infinitesimal motions of a framework, we study the local behavior of the solutions to (2) through its tangent space:

$$(\boldsymbol{p}_i - \boldsymbol{p}_j)^T(\dot{\boldsymbol{p}}_i - \dot{\boldsymbol{p}}_j) = 0 \implies \mathcal{R}(\mathcal{G}, \boldsymbol{p})\dot{\boldsymbol{p}} = 0, \tag{3}$$

where $\mathcal{R} \in \mathbb{R}^{m \times 2n}$ is called *rigidity matrix* [24]. Each row of $\mathcal{R}$ represents an edge, and it is structured as follows:

$$\mathcal{R}(\mathcal{G}, \boldsymbol{p})_{ij,i<j} = \Big[\underbrace{\mathbf{0}}_{1\ldots2(i-1)} \; (\boldsymbol{p}_i - \boldsymbol{p}_j)^T \; \underbrace{\mathbf{0}}_{2(i+1)\ldots2(j-1)} \; (\boldsymbol{p}_j - \boldsymbol{p}_i)^T \; \underbrace{\mathbf{0}}_{2(j+1)\ldots2n} \Big]. \tag{4}$$

The rigidity matrix is of great help in assessing infinitesimal rigidity [24]:

Lemma 1 (Rigidity matrix rank). *A framework $(\mathcal{G}, \boldsymbol{p})$ in $\mathbb{R}^2$ infinitesimally rigid if and only if $\textbf{\textit{rank}}(\mathcal{R}(\mathcal{G}, \boldsymbol{p})) = 2n - 3$.*

Moore [38] states that "non-rigid graphs can be continuously deformed to produce an infinite number of realizations." Thus, rigid graphs can't be *continuously* deformed, but still have *discontinuous* deformations that can prevent a realization from being unique. Consequently, infinitesimal rigidity can't be used as a sufficient condition for the unicity of a framework realization. A more strict condition is needed, i.e., global rigidity:

Definition 2 (Global rigidity). *A framework $(\mathcal{G}, \boldsymbol{p}_0)$ is globally rigid if every framework which is equivalent to $(\mathcal{G}, \boldsymbol{p}_0)$, is also congruent to $(\mathcal{G}, \boldsymbol{p}_0)$.*

Global rigidity is related to frameworks' unique realizations ([24] [Theorem 1]):

Theorem 1. *Consider a framework $(\mathcal{G}, \boldsymbol{p})$ in $\mathbb{R}^2$, with $n_b \geq 3$ beacon nodes placed at $p_b = \{\boldsymbol{p}_1 \; \ldots \; \boldsymbol{p}_{n_b}\} \subset \boldsymbol{p}$ and $n - n_b$ ordinary nodes placed at $p_o = \{\boldsymbol{p}_{n_b+1} \; \ldots \; \boldsymbol{p}_n\} \subset \boldsymbol{p}$. We also assume that at least three beacons are not aligned. Then, the network localization problem is solvable if and only if the framework $(\mathcal{G}, \boldsymbol{p})$ is globally rigid.*

The global rigidity and beacon alignment conditions in Theorem 1 address the network's spatial disposition and connectivity, characterizing the framework's two constructive entities, i.e., the graph $\mathcal{G}$ and the positions $\boldsymbol{p}$. Constructive conditions to check global rigidity on the plane are provided in [24] [Theorem 6]:

Theorem 2. *A graph $\mathcal{G}$ with $n \geq 4$ vertices is generically globally rigid in $\mathbb{R}^2$ if and only if it is 3-connected and redundantly rigid in $\mathbb{R}^2$.*

Here, *generic global rigidity* refers to globally rigid frameworks robust to small perturbations. Moreover, *3-connectivity* refers to graphs that remain connected after removing any pair of nodes. *Redundant rigidity* refers to frameworks that remain infinitesimally rigid after removing any single edge. We are left with three constructive conditions to check whether the localization problem can be unambiguously solved: beacon alignment, 3-connectivity, and redundant rigidity. We will use these in our distributed Active localization algorithm, which will be presented in the next section.

4 Distributed Active localization

In this section, we address the network localization problem stated in sec. 2, proposing a distributed heuristic algorithm based on the results from sec. 3.

4.1 The algorithm

We focused on two goals: keeping the assumptions on the initial network configuration as general as possible and the distributed structure of the algorithm. Thus, we structured the algorithm in three actions, which are asynchronously repeated by every agent A_i:

- `GetNeighbors`: A_i collects all the measurements in a `LOStable`. Ranging sensors provide each agent with the ID and distance of other agents within δ, defining $\mathcal{N}_{i,RD}$. When A_i is the leader, the camera provides children's ID, distance, and bearing (i.e., position) within δ_C, defining $\mathcal{N}_{L,CAM}$.
- `GetPosition`: the information from `LOStable` is combined with Theorem 1 and Theorem 2, checking the node's localizability.
- `MovePolicy`: if `GetPosition` couldn't positively assess the localizability of A_i, the agent moves accordingly to a heuristic policy aiming to increase the chances to become so in the next iteration.

Algorithm 1 `GetPosition`

function GetPosition(A_i)
    if $A_i = A_L \mid A_i \in \mathcal{N}_{L,CAM}$ **then** A_i.`localized` $\leftarrow$ `true`
        return
    end if
    if $|\overline{\mathcal{N}}_{i,RD}| < 3$ **then** A_i.`localized` $\leftarrow$ `false`
        return
    end if
    $\mathcal{F} \leftarrow A_i$.`BuildFrameWork`($\overline{\mathcal{N}}_{i,RD}$)
    if Is3Connected($\mathcal{F}$) & IsRedundant($\mathcal{F}$) **then** A_i.`localized` $\leftarrow$ `true`
    else
        do $(\mathcal{F}^*, \overline{\mathcal{N}}^*_{i,RD}) \leftarrow A_i$.`ReduceFrameWork`($\mathcal{F}$)
        while ($\sim$Is3Connected($\mathcal{F}^*$) & $\sim$IsRedundant($\mathcal{F}^*$)) | ($|\overline{\mathcal{N}}^*_{i,RD}| < 3$)
        if Is3Connected($\mathcal{F}^*$) & IsRedundant($\mathcal{F}^*$) & ($|\overline{\mathcal{N}}^*_{i,RD}| \geq 3$) **then**
            A_i.`localized` $\leftarrow$ `true`
        else
            A_i.`localized` $\leftarrow$ `false`
        end if
    end if
end function

Let's look at Algorithm 1 for more details on `GetPosition`; we define $\overline{\mathcal{N}}_{i,RD}$ as the subset of localized agents from $\mathcal{N}_{i,RD}$ with at least three non-aligned elements. After checking the most trivial conditions as being localized by the leader or not having the minimum number of localized neighbors (CFR. Theorem 1), A_i selects only the `LOStable` entries referring to $\overline{\mathcal{N}}_{i,RD}$, creating a framework $\mathcal{F}$ (CFR. A_i.`BuildFrameWork`). Theorem 2 conditions are checked on $\mathcal{F}$; if they are not satisfied, agents are removed from $\mathcal{F}$ until the largest globally rigid sub-framework $\mathcal{F}^*$ is found (CFR. A_i.`ReduceFrameWork`). When a globally rigid framework is found, the agent is localized by solving a trilateration problem.

If A_i can't localize itself through `GetPosition`, as long as $\mathcal{N}_{i,RD} \neq \emptyset$, it can move towards its neighbors; by doing so, the chances of either increasing $|\mathcal{N}_{i,RD}|$ or falling within $\mathcal{N}_{L,CAM}$ get higher. The idea behind the heuristic `MovePolicy` (CFR. Algorithm 2) is to numerically compute the gradient of the measured distances' average by moving A_i of ϵ on the four points $[\ \pm\epsilon,\ \pm\epsilon\ ]$ (CFR. A_i.`DistGrad`). If A_i has localized neighbors, they are the only ones used to compute the gradient. Otherwise, all the neighbors are considered. The point with the lowest measured distances' average is selected as the next A_i's motion $\boldsymbol{v}$. Lastly, A_i actuates $\boldsymbol{v}$ (CFR. A_i.`Move`). In this heuristic, every agent's movement is done in feed-forward, as no localization is available yet.

Algorithm 2 `MovePolicy`

1: **function** MovePolicy(A_i) `Assumption:` $\mathcal{G}$ `connected at time` t_0
2:     **if** $\mathcal{N}_{i,RD} \neq \emptyset$ **then**
3:         **if** $\overline{\mathcal{N}}_{i,RD} \neq \emptyset$ **then** $\boldsymbol{v} \leftarrow A_i$.`DistGrad`($\epsilon, \overline{\mathcal{N}}_{i,RD}$)
4:         **else** $\boldsymbol{v} \leftarrow A_i$.`DistGrad`($\epsilon, \mathcal{N}_{i,RD}$)
5:         **end if**
6:     **else** $\boldsymbol{v} \leftarrow A_i$.`RandSelect`($\mathcal{B}_\epsilon$)
7:     **end if**
8:     A_i.`Move`($\boldsymbol{v}$)
9: **end function**

Comments From Algorithm 1 and Algorithm 2, we see that the proposed procedure only considers the presence of neighbors for A_i; this is the only assumption we make, i.e., for $\mathcal{G}$ to be connected from the beginning. No rigidity assumption is required, differently from [35]. Connectivity is maintained during the exploration by discarding any movement $\boldsymbol{v}$ resulting in a neighbor loss. If this results in no feasible movement $\boldsymbol{v}$, A_i randomly chooses a feasible motion $\boldsymbol{v}$ in $\mathcal{B}_\epsilon$, similar to [35] (CFR. A_i.`RandSelect`). Specifically, $\boldsymbol{v}$ is feasible if a minimum distance $d_{\min}$ is kept between A_i and its neighbors. All these conditions are checked in A_i.`DistGrad`. Lastly, A_i.`ReduceFramework` complexity scales up with $|\overline{\mathcal{N}}^*_{i,RD}|$; thus, Theorem 2 has been checked only on $\mathcal{F}^*$ such that $|\overline{\mathcal{N}}^*_{i,RD}| = |\overline{\mathcal{N}}_{i,RD}| - 1$.

4.2 Simulation results

To show the effectiveness of our solution, we consider a batch of 100 randomly generated formations with $n = 8$ agents and the setup from sec. 2, where, for simplicity, the camera FOV is set to 360°. A single algorithm iteration consists of every agent A_i performing the three actions from sec. 4.1. The agents' sequence is randomly selected at every iteration. Figure 3 shows four frames of the algorithm execution on one of the formations. The algorithm succeeds if all the agents are localized after at most 200 iterations. Table 1 shows the success rate $\mathtt{S}_{\%}$, the average number of iterations $\#_{\text{it}}$, and the coverage change $\mathtt{C} = \overline{C_0}/\overline{C_{\text{end}}}$ depending on $d_{\min}$ and ϵ. $\overline{C}$ is the average formation coverage. From the results,

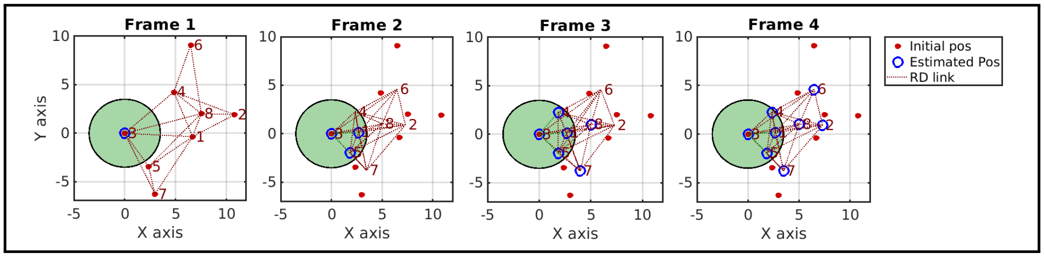

Fig. 3: Evolution of the solution to the network localization problem according to sec. 4.1. The green area around A_3 is the camera FOV.

ϵ \ $d_{\min}$	0.5m	1.0m	2.0m
0.25m	$\mathtt{S}_{\%} = 99, \#_{\text{it}} = 29.2, \mathtt{C} = 0.74$	$\mathtt{S}_{\%} = 96, \#_{\text{it}} = 30.7, \mathtt{C} = 0.75$	$\mathtt{S}_{\%} = 91, \#_{\text{it}} = 27.7, \mathtt{C} = 0.80$
0.50m	$\mathtt{S}_{\%} = 99, \#_{\text{it}} = 15.7, \mathtt{C} = 0.74$	$\mathtt{S}_{\%} = 97, \#_{\text{it}} = 15.9, \mathtt{C} = 0.77$	$\mathtt{S}_{\%} = 86, \#_{\text{it}} = 17.6, \mathtt{C} = 0.77$
0.75m	$\mathtt{S}_{\%} = 100, \#_{\text{it}} = 11.4, \mathtt{C} = 0.71$	$\mathtt{S}_{\%} = 98, \#_{\text{it}} = 10.7, \mathtt{C} = 0.75$	$\mathtt{S}_{\%} = 95, \#_{\text{it}} = 11.7, \mathtt{C} = 0.79$

Table 1: Algorithm performance on 100 formations depending on $d_{\min}$ and ϵ.

we see that increasing ϵ and reducing $d_{\min}$ results in higher performance and faster convergence due to the increased accuracy in the gradient computation and the higher number of feasible motions. The proposed algorithm successfully solves the network localization problem from sec. 2, with a coverage reduction of 20/25% as a side effect. We aim to address the coverage problem by reducing the formation contraction through the motion policy of the localized agents, which is currently not considered.

5 Rigidity spatial interpretation

This section studies how to exploit rigidity to address the environment coverage problem in the Active localization algorithm from sec. 4. From (1), reducing the overlapping areas I_{ij} increases the coverage. Generally speaking, to do so, increasing the team dispersion and symmetry is a reasonable policy. Thus, we analyze rigidity structural properties to understand whether exploiting it for coverage maximization is possible.

5.1 Analytical analysis

For this analysis, we were first inspired by [36, 37], where the rigidity matrix properties are studied, proving a lemma equivalent to Lemma 1:

Lemma 2 (Rigidity eigenvalue). *Consider a framework $(\mathcal{G}, \boldsymbol{p})$ in $\mathbb{R}^2$ with rigidity matrix $\mathcal{R} = \mathcal{R}(\mathcal{G}, \boldsymbol{p})$ from (3). Consider also the spectrum of the symmetric (positive semidefinite) rigidity matrix $\overline{\mathcal{R}} \triangleq \mathcal{R}^T\mathcal{R} \in \mathbb{R}^{2n\times 2n}$, i.e., the set of eigenvalues $0 \le \lambda_1 \le \dots \lambda_{2n}$. If $(\mathcal{G}, \boldsymbol{p})$ is infinitesimally rigid, then $\lambda_i = 0\ \forall i \le 3$.*

The first nonzero eigenvalue λ_4 is called *Worst-case Rigidity Index* (WRI) in [36], and *rigidity eigenvalue* in [37]. In both these papers, it is clearly shown that the greater λ_4, the higher the rigidity of the formation. However, [36] goes further, providing the following geometric interpretations:

1. Increasing the connectivity between any pair of agents will always increase the formation's rigidity.
2. In general, the symmetry of the formation may positively impact the rigidity increase.

To better understand these intuitions, we consider the symmetric rigidity matrix $\overline{\mathcal{R}}$ for a framework $(\mathcal{G}, \boldsymbol{p})$, that can be written as

$$\overline{\mathcal{R}} = \begin{bmatrix} \sum\limits_{j\neq 1} a_1^j \mathrm{cov}(\boldsymbol{p}_1, \boldsymbol{p}_j) & -a_1^2 \mathrm{cov}(\boldsymbol{p}_1, \boldsymbol{p}_2) & \dots & -a_1^n \mathrm{cov}(\boldsymbol{p}_1, \boldsymbol{p}_n) \\ -a_2^1 \mathrm{cov}(\boldsymbol{p}_2, \boldsymbol{p}_1) & \sum\limits_{j\neq 2} a_2^j \mathrm{cov}(\boldsymbol{p}_2, \boldsymbol{p}_j) & \dots & -a_2^n \mathrm{cov}(\boldsymbol{p}_2, \boldsymbol{p}_n) \\ \vdots & \vdots & \vdots & \vdots \\ -a_n^1 \mathrm{cov}(\boldsymbol{p}_n, \boldsymbol{p}_1) & \dots & -a_n^{n-1} \mathrm{cov}(\boldsymbol{p}_n, \boldsymbol{p}_{n-1}) & \sum\limits_{j\neq n} a_n^j \mathrm{cov}(\boldsymbol{p}_n, \boldsymbol{p}_j) \end{bmatrix}, \tag{5}$$

where $\mathrm{cov}(\boldsymbol{p}_i, \boldsymbol{p}_j)$ is the covariance matrix[3] of a pair of agents. The scalar $a_i^j = a_j^i$ defines the presence or absence of a connection, i.e., $a_i^j \neq 0$, if and only if $j \in \mathcal{N}_i$. Thus, a framework described by a non-complete graph will have zero entries in $\overline{\mathcal{R}}$. If we further expand the structure of $\overline{\mathcal{R}}$, we find out that all the diagonal elements are nothing but the cumulative variances of all the neighbor sets $\mathcal{N}_i$, i.e., for a generic agent i, $\overline{\mathcal{R}}_{[2i-1,2i-1]} = \sum_{j\in\mathcal{N}_i} \sigma_x^2(\boldsymbol{p}_i, \boldsymbol{p}_j)$, and $\overline{\mathcal{R}}_{[2i,2i]} = \sum_{j\in\mathcal{N}_i} \sigma_y^2(\boldsymbol{p}_i, \boldsymbol{p}_j)$. If we consider the trace-eigenvalues property, we obtain:

$$\begin{aligned} \sum_i \lambda_i = \mathrm{tr}(\overline{\mathcal{R}}) &= \sum_i \sum_{j\in\mathcal{N}_i} \left(\sigma_x^2(\boldsymbol{p}_i, \boldsymbol{p}_j) + \sigma_y^2(\boldsymbol{p}_i, \boldsymbol{p}_j)\right) = \\ &= \sum_i \sum_{j\in\mathcal{N}_i} \left((p_{i,x} - p_{j,x})^2 + (p_{i,y} - p_{j,y})^2\right) \geq 0. \end{aligned} \tag{6}$$

From (6), we see that maximizing the inter-agent variances, namely spreading the formation, increases the eigenvalues of $\overline{\mathcal{R}}$. It is also clear that more connections generally yield a higher trace. However, (6) does not directly imply that a higher trace means higher λ_4. Furthermore, in [36], no maximum detection range is considered between agents; here, we consider a more realistic case where $a_i^j \neq 0$ if $\|\boldsymbol{p}_i - \boldsymbol{p}_j\| < \delta$, in accordance with our case study. In this situation, we might have different configurations maximizing the framework's eigenvalues, either with more connections or higher dispersion. We will numerically investigate these aspects in sec. 5.2.

We proceed by considering symmetry in the formation. We first note that (5) describes $\overline{\mathcal{R}}$ as a block matrix, the blocks being the inter-agent covariance matrices and their summation. Indeed, the eigenvalues of $\overline{\mathcal{R}}$ depend on its determinant, which nonlinearly depends too on $\det(\mathrm{cov}(\boldsymbol{p}_i, \boldsymbol{p}_j))$. We are not interested in this

[3] We normalized the terms, i.e., for a pair of agents, we multiplied by 2 the matrix.

exact relation but rather in the intuition that $\det(\overline{\mathcal{R}})$ will depend on a variance-related and a covariance-related term, similar to a generic covariance matrix determinant:

$$\det(\mathrm{cov}(\boldsymbol{p})) = (\sigma_x^2 + \sigma_y^2) - \sigma_{xy}^2 \implies \det(\overline{\mathcal{R}}) \propto \gamma_D(\sigma_x^2, \sigma_y^2) + \gamma_S(\sigma_{xy}), \tag{7}$$

with γ_D, γ_S continuous maps, and the variance and covariance terms defined as

$$\sigma_{x/y}^2 \triangleq \frac{1}{n}\sum_{i=1}^{n}\left(p_{i,x/y} - \frac{1}{|\mathcal{N}_i|}\sum_{j\in\mathcal{N}_i} a_i^j p_{j,x/y}\right)^2, \tag{8a}$$

$$\sigma_{xy} \triangleq \frac{1}{n}\sum_{i=1}^{n}\left(p_{i,x} - \frac{1}{|\mathcal{N}_i|}\sum_{j\in\mathcal{N}_i} a_i^j p_{j,x}\right)\left(p_{i,y} - \frac{1}{|\mathcal{N}_i|}\sum_{j\in\mathcal{N}_i} a_i^j p_{j,y}\right). \tag{8b}$$

From standard statistics, we know that $\sigma_{xy} = \rho\sigma_x\sigma_y$, where ρ is the correlation coefficient. For a pair of agents $\rho = \pm 1$, but for the whole formation, maximizing σ_{xy}^2 means placing the agents of a neighborhood $\mathcal{N}_i$ on a straight line ($\rho^2 \to 1$) while minimizing it means obtaining a symmetric formation ($\rho^2 \to 0$). We will further investigate this aspect through numerical simulations in sec. 5.2.

This analysis showed how the symmetric rigidity matrix $\overline{\mathcal{R}}$ and the inter-agent covariance matrices $\mathrm{cov}(\boldsymbol{p}_i, \boldsymbol{p}_j)$ share the same underlying structure. Exploiting this analogy, we analytically support the intuitions from [36] regarding the effects of connectivity and symmetry on a formation's rigidity. Furthermore, we also relate rigidity with the spatial dispersion of the framework. In the next section, we investigate these results through numerical simulations.

5.2 Simulation results

This section investigates the results from sec. 5.1 through numerical simulations, considering the setup from sec. 2. We also add a constraint on the minimum distance between agents to avoid collisions, mimicking the outcome of the motion policy proposed in sec. 4.

Dispersion and connectivity We study how the trace-eigenvalues relation from (6) affects the formation dispersion and connectivity and its relation to the rigidity eigenvalue λ_4. To do so, we consider the following optimization problems:

$$\begin{aligned} \min_{(\mathcal{G},\boldsymbol{p})} \quad & -\mathrm{tr}(\overline{\mathcal{R}}) \\ \text{s.t.} \quad & \lambda_4 > 0 \\ & d_{\min} \le d_{ij} \le d_{\max} \le \delta \end{aligned} \tag{9a}$$

$$\begin{aligned} \min_{(\mathcal{G},\boldsymbol{p})} \quad & -\lambda_4 \\ \text{s.t.} \quad & \lambda_4 > 0 \\ & d_{\min} \le d_{ij} \le d_{\max} \le \delta \end{aligned} \tag{9b}$$

where framework $(\mathcal{G}, \boldsymbol{p})$ is constrained the to be rigid. The collision safety condition is represented by the $d_{\min} \le d_{ij}$ constraint. The $d_{ij} \le d_{\max}$ condition manually limits the dispersion of the team. By solving (9a) and (9b), the question that we are addressing is: *among all the rigid frameworks with a minimum and a maximum inter-agent distance, does maximum rigidity coincide with either maximum connectivity or dispersion?*

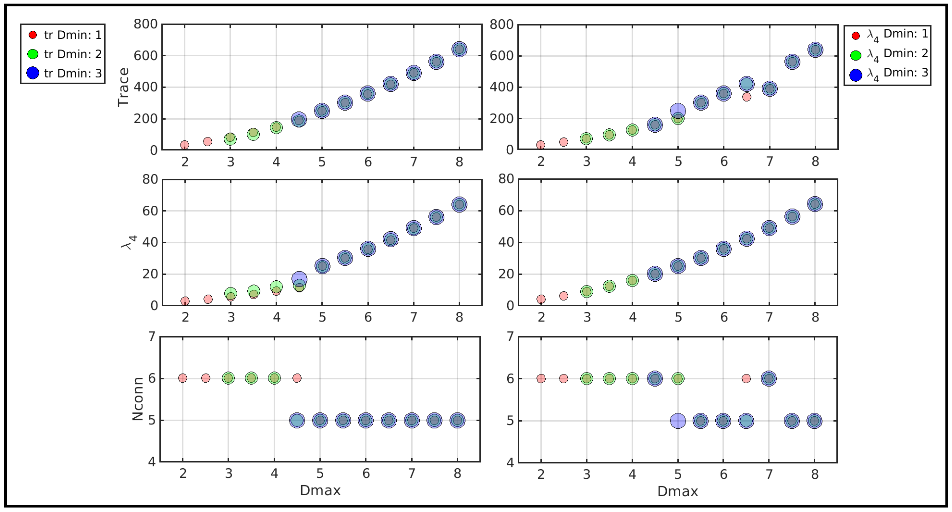

Fig. 4: Numerical solutions to (9a) and (9b) under different sets of constraints.

In Fig. 4, we show the results of a three batch of simulations set for $n = 4$ agents: problems (9a) (left-side) and (9b) (right-side) were solved with $d_{\min} = \{1, 2, 3\}$ m, and with $d_{\max} \in [d_{\min} + 1,\ \delta]$. In the top plots, we show $\mathrm{tr}(\overline{\mathcal{R}})$, in the middle λ_4, and in the bottom the number of connections among agents. We first note that for every $d_{\min}$, the optimization objectives monotonically increase with $d_{\max}$ (CFR. top-left and mid-right). As for λ_4, these simulations show that dispersion does have a positive impact on rigidity. However, as expected, maximizing either connectivity or dispersion does not imply maximizing rigidity, as highlighted by the mid-left and top-right plots, which show different behaviors than the optimization objectives. Lastly, if we look at the number of connections (CFR. bottom plots), the general trend is to prefer dispersion over connectivity as $d_{\max}$ increases. However, this is not an absolute behavior, showing cases where connectivity helps rigidity more than dispersion [36]. It is clear, though, that, if combined, these two factors always help to increase λ_4.

Simmetry We now analyze the rigidity-symmetry relation provided in sec. 5.1 by considering again (9b) together with the following optimization problems:

$$\begin{aligned} \min_{(\mathcal{G},\boldsymbol{p})} \quad & -\lambda_4 + \rho^2 \\ \text{s.t.} \quad & \lambda_4 > 0 \\ & d_{\min} \le d_{ij} \le d_{\max} \le \delta \end{aligned} \tag{10a}$$

$$\begin{aligned} \min_{(\mathcal{G},\boldsymbol{p})} \quad & -\lambda_4 \\ \text{s.t.} \quad & \lambda_4 > 0 \\ & d_{\min} \le d_{ij} \le d_{\max} \le \delta \\ & |\rho| \ge 0.5 \end{aligned} \tag{10b}$$

Specifically, we address the following questions: *does rigidity maximization directly take into account symmetry maximization? Is it true that lower symmetry, in general, implies lower rigidity?*

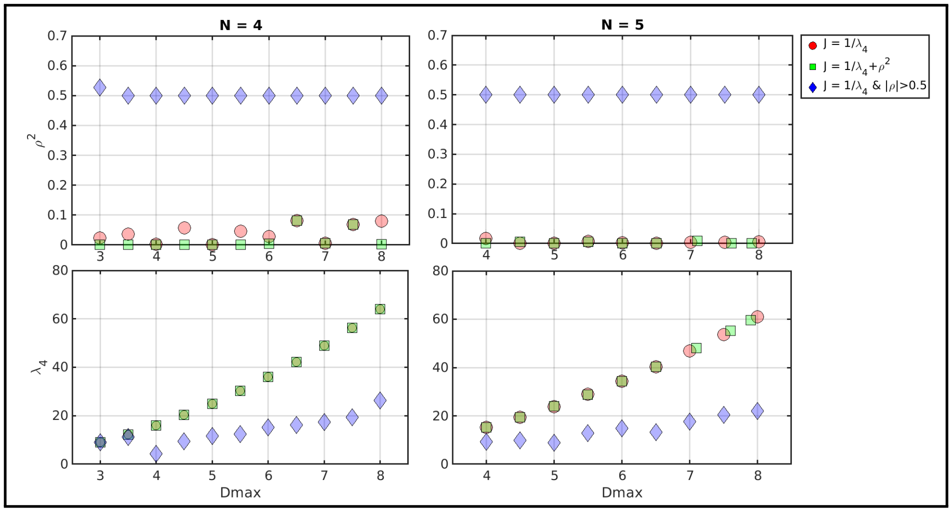

Fig. 5: Numerical solutions to (9b),(10a),(10b) under constraints and agents' number.

In Fig. 5, we show the solutions to (9b),(10a),(10b) for $n = 4$ (left-side), and $n = 5$ (right-side) agents. In the top plots, we show ρ^2, and in the bottom, λ_4. If we consider (9b) and (10a), we observe that the cost function values almost coincide; thus, optimizing on rigidity already comprehend maximizing symmetry. If we force a minimum $|\rho|$ value, it is pretty clear that rigidity worsens; thus, it is true that lower rigidity generally implies lower symmetry.

This section provided an analytical interpretation of rigidity; we showed through numerical simulations that increasing rigidity results in higher formation symmetry, dispersion, and hence higher coverage. Thus, the algorithm from sec. 4 can be extended by moving the localized agents to maximize λ_4, again maintaining connectivity. This will reduce the formation contraction, addressing the coverage problem as initially proposed.

6 Conclusions and future work

In this work we addressed the joint Active localization and exploration coverage problem for heterogenous teams organized with one leader and a set of children. In sec. 4, we distributedly solved the network localization through Rigidity Theory. In sec. 5 we analytically interpreted rigidity in terms of the team's dispersion, symmetry, and connectivity. Specifically, we showed that the team's coverage can be maximized by maximizing rigidity. As a consequence, both the localization and mapping problems can be addressed through the same matric, i.e., rigidity. We aim to extend the localization algorithm also to localized agents. Specifically, we intend to extend the motion policy so that each localized agent moves in order to maximize the neighborhoods' rigidity, and hence, the team's coverage.

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