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Journal of the Franklin Institute 354 (2017) 7983–7995

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Distributed extended Kalman filter with nonlinear consensus estimate[☆]

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Received 20 November 2016; received in revised form 20 June 2017; accepted 30 September 2017
Available online 13 October 2017

Abstract

This paper is concerned with the distributed filtering problem for discrete-time nonlinear systems over a sensor network. In contrast with the distributed filters with linear consensus estimate, a distributed extended Kalman filter (EKF) is developed with nonlinear consensus estimate. Specifically, a new nonlinear consensus protocol with polynomial form is proposed to generate the consensus estimate. By using the variance-constrained approach, the Kalman gain matrix is determined for each node to guarantee an optimized upper bound on the state estimation error covariance despite consensus terms and linearization errors. It is shown that the Kalman gain matrix can be derived by solving two Riccati-like difference equations. The effectiveness of the proposed filter is evaluated on an indoor localization of a mobile robot with visual tracking systems.

☆ This work was supported by the NSFC (61573031, 61327807, 61532006, 61320106006, 61520106010) and (9140C590201150C59005, 4162070).

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https://doi.org/10.1016/j.jfranklin.2017.09.033
0016-0032/

1. Introduction

In the past decade, consensus-based distributed filters have received increasing attention in the signal processing and control communities, and different kinds of distributed filtering algorithms have been proposed, according to the communication scheme, the sensor node links and the available information. In the consensus-based algorithms, each node in a network shares its information with its immediate neighbors and corrects its own state using the information sent by its neighbors. A large number of works on consensus-based distributed filtering have been reported since they can drastically reduce the utilization of communication resources [1,2].

Among various distributed filters, Kalman consensus filter has been considered as one of the most popular algorithms for estimating the state of a dynamical system [3–13]. The Kalman consensus filter was implemented by adding a consensus term of predicted estimates in the standard Kalman filter [6]. The main disadvantage of the Kalman consensus filter is that the covariance information is not used in a distributed way. Specifically, the covariance update algorithm of the Kalman consensus filter is the same as a standard Kalman filter, although a consensus term has been introduced in a standard Kalman filter. In [14], the Kalman consensus filter has been extended to the scenario with stochastic sensor activation to reduce the sensor energy consumption in communications. The result in [14] has been further extended to address the distributed filtering with power constraint in [15]. It should be pointed out that the algorithms in [14,15] are not scalable since the edge-covariance matrices are required to be computed among nodes. In [16], the Kalman consensus filter has been used to develop optimal and suboptimal distributed Kalman filters with intermittent measurements. However, the communication packet dropout among sensors is not addressed. In [17], the Kalman consensus filter has been used to develop a distributed filter with an event-triggered communication protocol. By using the Lyapunov-based approach, a sufficient condition has been established for ensuring the stochastic stability of the Kalman consensus filter. Recently, consensus-based distributed extended Kalman filter (EKF) have been proposed for discrete-time nonlinear systems in [18–20]. Following the lines of the distributed Kalman filter [13], the measurement and the estimates are exchanged between neighboring nodes in [18,19] and it has been shown that the estimation error is bounded under certain conditions. However, the linearization errors of the EKF are not addressed in the distributed EKF. To address this problem, the linearization errors have been handled by using the variance-constrained approach in [20], where the Kalman gain matrix is determined for each node by minimizing the trace of the upper bound matrix.

Most of the aforementioned works focused on developing distributed filters with linear consensus estimate, i.e., the consensus estimate is a linear combination of the difference between the predicted estimates between neighboring nodes. To the best of our knowledge, very few studies are devoted to developing distributed filters with nonlinear consensus estimates. In the field of consensus control for multi-agent systems, it is well known that the nonlinear consensus protocol may result in high-speed convergence [21–23]. Inspired by this fact, nonlinear consensus estimate can be expected to be used in distributed filters to improve estimation accuracy.

In this paper, we attempt to develop a distributed EKF for discrete-time nonlinear systems over a sensor network. The proposed filter is derived for each node by applying nonlinear consensus estimate with the EKF. It should be pointed out that the proposed nonlinear consensus protocol is not adopted in literature for multi-agent systems [21–23]. This is due to the fact that the existing nonlinear protocols are mainly designed for scalars not vectors. In this paper, a new polynomial form nonlinear protocol is proposed to generate consensus estimate with the EKF. By using the variance-constrained approach, the Kalman gain matrix is determined for each node to guarantee an optimized upper bound on the state estimation error covariance despite consensus terms and linearization errors. It is shown that the Kalman gain matrix can be derived by solving two Riccati-like difference equations. Moreover, the edge-covariances have been bounded by their individual covariances so that the proposed distributed filter is scalable. A numerical example is provided to verify the effectiveness of the proposed filter.

The rest of this paper is organized as follows. In Section 2, the distributed filtering problem for discrete-time nonlinear systems is formulated. The state estimation error covariances are derived and the corresponding upper bound matrices are provided to design the gain matrices in Section 3. A numerical example involving an indoor localization of a mobile robot is shown in Section 4. Conclusions and future work are given in Section 5.

2. Problem statement

In this paper, we are interested in estimating the target state using a undirected sensor network with communication topology $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, N\}$ and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ denote the nodes and the edges, respectively. Two nodes are said to be connected if they can communicate directly with each other. The set of nodes connected with a certain node i is called the neighborhood of node i and is denoted by $\mathcal{N}_i$. The number of neighbors of node i is called its degree and is denoted by $d_i = |\mathcal{N}_i|$. The target state and the sensor measurement are described by the following discrete-time nonlinear system

$$x_{k+1} = f(x_k) + w_k \tag{1}$$

$$z_{i,k} = h_i(x_k) + v_{i,k}, \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the state vector and $z_{i,k} \in \mathbb{R}^p$ is the measurement vector of the ith sensor. $f(\cdot)$ and $h_i(\cdot)$ are known nonlinear functions that are assumed to be continuously differentiable. The process noise w_k and the measurement noise $v_{i,k}$ are assumed to be mutually uncorrelated zero-mean white Gaussian with covariances Q_k and $R_{i,k}$, respectively.

The distributed filter is developed for the ith sensor by introducing nonlinear consensus estimates into the EKF

$$\bar{x}_{i,k+1} = f(\hat{x}_{i,k}) \tag{3}$$

$$\hat{x}_{i,k+1} = \bar{x}_{i,k+1} + K_{i,k+1}[z_{i,k+1} - h(\bar{x}_{i,k+1})] + \sum_{j \in \mathcal{N}_i} a_{ij} g(\bar{x}_{j,k+1} - \bar{x}_{i,k+1}), \tag{4}$$

where $\bar{x}_{i,k+1}$ and $\hat{x}_{i,k+1}$ denote the predicted and the updated estimates at time instant $k+1$, respectively. $K_{i,k+1}$ is the Kalman gain matrix to be determined. a_{ij} denote the predefined weights between neighboring nodes. $g(\cdot)$ is a continuously differentiable nonlinear function satisfying $g(x) = 0$ if and only if $x = 0$. It is also known as the nonlinear consensus protocol in multi-agent systems [21–23]. Similar to the linear consensus estimate in the existing literature, the third term on the right hand side of Eq. (4) is called as the nonlinear consensus estimate in this paper.

The updated estimation error and the corresponding covariance are defined as

$$e_{i,k+1} = x_{k+1} - \hat{x}_{i,k+1} \tag{5}$$

$$P_{i,k+1} = \mathbb{E}\{e_{i,k+1}e_{i,k+1}^T\} \tag{6}$$

By using the variance-constrained criterion for nonlinear filters [28–30], the aim of this paper is to design distributed filters described by Eqs. (3) and (4), such that there exists a sequence of positive-definite matrices $\Phi_{i,k+1}$ satisfying

$$P_{i,k+1} \leq \Phi_{i,k+1} \tag{7}$$

The gain matrix $K_{i,k+1}$ is determined by minimizing the trace of the upper bound matrix $\Phi_{i,k+1}$ at each time instant.

Remark 1. In the existing literature [6], distributed filters have been developed with linear consensus estimates, i.e., $g(\bar{x}_{j,k+1} - \bar{x}_{i,k+1}) = C_k(\bar{x}_{j,k+1} - \bar{x}_{i,k+1})$ is used as the consensus estimate in Eq. (4) where C_k is the consensus gain matrix. To our knowledge, this is the first attempt to develop distributed filters with nonlinear consensus estimates. As the nonlinear consensus estimate is used to update the state estimate for each node, the gain matrix $K_{i,k+1}$ is determined with respect to an upper bound matrix of the covariance matrix by using the variance-constrained approach.

3. Main results

In this section, the covariance matrices of the predicted and the updated estimation errors are firstly derived. Subsequently, an upper bound matrix is obtained and the Kalman gain matrix is determined by minimizing the trace of the upper bound matrix.

3.1. Distributed EKF

The distributed EKF is derived based on the following lemmas.

Lemma 1 ([24]). *Given matrices A, B, C and D with appropriate dimensions such that $CC^T \leq I$. Let U be a symmetric positive definite matrix and $a > 0$ be an arbitrary positive constant such that $a^{-1}I - DUD^T > 0$. Then the following matrix inequality holds:*

$$(A+BCD)U(A+BCD)^T \leq A(U^{-1} - aD^TD)^{-1}A^T + a^{-1}BB^T \tag{8}$$

Lemma 2 ([25]). *For $0 \leq k < n$, suppose that $A = A^T > 0$. Let $\varphi_k(\cdot)$ and $\psi_k(\cdot)$ be two sequences of matrix functions such that*

$$\varphi_k(A) = \varphi_k(A^T), \quad \psi_k(A) = \psi_k(A^T) \tag{9}$$

If there exists a matrix $B = B^T > A$ such that

$$\varphi_k(B) \geq \varphi_k(A), \quad \psi_k(B) \geq \varphi_k(B) \tag{10}$$

then the solutions X_k and Y_k to the following difference equations:

$$X_k = \varphi_k(X_{k-1}), \quad Y_k = \psi_k(Y_{k-1}), \quad X_0 = Y_0 > 0 \tag{11}$$

satisfy $X_k \leq Y_k$.

Define the predicted estimation error and the corresponding covariance matrix as

$$\bar{e}_{i,k+1} = x_{k+1} - \bar{x}_{i,k+1} \tag{12}$$

$$\bar{P}_{i,k+1} = \mathbb{E}\{\bar{e}_{i,k+1}\bar{e}_{i,k+1}^T\} \tag{13}$$

The predicted estimation error can be derived by subtracting Eq. (3)from Eq. (1)

$$\bar{e}_{i,k+1} = f(x_k) - f(\hat{x}_{i,k}) + w_k \tag{14}$$

As shown in [26–30], the system transition function can be represented by a Taylor series about $\hat{x}_{i,k}$

$$f(x_{i,k}) = f(\hat{x}_{i,k}) + F_{i,k}e_{i,k} + U_{i,k}\Omega_{i,k}e_{i,k}, \tag{15}$$

where $F_{i,k} = \partial f(x)/\partial x|_{x=\hat{x}_{i,k}}$ denotes the Jacobian matrix, $U_{i,k}$ is a problem-dependent scaling matrix and $\Omega_{i,k}$ is a time-varying matrix accounting for the linearization error satisfying $\Omega_{i,k}\Omega_{i,k}^T \le I$. As shown in Appendix C of [26], the matrix $U_{i,k}$ can be chosen as a diagonal matrix $U_{i,k} = \frac{\sqrt{n}}{2}\mathrm{diag}\{m_{i,1}^{(1)} \ldots, m_{i,n}^{(1)}\}$, where $m_{i,s}^{(1)}$ $(s = 1, 2, \ldots, n)$ is a constant such that the Hessian matrix of the sth element of $f(x)$ is $||\tilde{F}_s(x)|| \le m_{i,s}^{(1)}$.

Substituting Eq. (15) into Eq. (14) yields

$$\bar{e}_{i,k+1} = (F_{i,k} + U_{i,k}\Omega_{i,k})e_{i,k} + w_k \tag{16}$$

The covariance matrix of the predicted estimation error can be obtained with respect to Eq. (16)

$$\bar{P}_{i,k+1} = (F_{i,k} + U_{i,k}\Omega_{i,k})P_{i,k}(F_{i,k} + U_{i,k}\Omega_{i,k})^T + Q_k \tag{17}$$

Similarly, the measurement function and the nonlinear consensus function can be represented by Taylor series as follows:

$$h(x_{i,k+1}) = h(\bar{x}_{i,k+1}) + H_{i,k+1}\bar{e}_{i,k+1} + V_{i,k+1}\Theta_{i,k+1}\bar{e}_{i,k+1} \tag{18}$$

$$g(\bar{e}_{j,k+1} - \bar{e}_{i,k+1}) = G_{ij,k+1}(\bar{e}_{j,k+1} - \bar{e}_{i,k+1}) + W_{ij,k+1}\Psi_{ij,k+1}(\bar{e}_{j,k+1} - \bar{e}_{i,k+1}), \tag{19}$$

where $H_{i,k+1} = \partial h(x)/\partial x|_{x=\bar{x}_{i,k+1}}$ and $G_{ij,k+1} = \partial g(x)/\partial x|_{x=\bar{x}_{j,k+1}-\bar{x}_{i,k+1}}$. $V_{i,k+1}$ and $W_{ij,k+1}$ are problem-dependent scaling matrices. $\Theta_{i,k+1}$ and $\Psi_{ij,k+1}$ are time-varying matrices accounting for the linearization error satisfying $\Theta_{i,k+1}\Theta_{i,k+1}^T \le I$ and $\Psi_{ij,k+1}\Psi_{ij,k+1}^T \le I$, respectively. As shown in Appendix C of [26], the matrices $V_{i,k+1}$ and $W_{ij,k+1}$ can be chosen as diagonal matrices $V_{i,k+1} = \frac{\sqrt{p}}{2}\mathrm{diag}\{m_{i,1}^{(2)} \ldots, m_{i,p}^{(2)}\}$ and $W_{ij,k+1} = \frac{\sqrt{p}}{2}\mathrm{diag}\{m_{i,1}^{(3)} \ldots, m_{i,p}^{(3)}\}$ where $m_{i,s}^{(2)}$ $(s = 1, 2, \ldots, p)$ is a constant such that the Hessian matrix of the sth element of $h(x)$ is $||\tilde{H}_s(x)|| \le m_{i,s}^{(2)}$ and $m_{i,s}^{(3)}$ $(s = 1, 2, \ldots, p)$ is a constant such that the Hessian matrix of the sth element of $g(x)$ is $||\tilde{G}_s(x)|| \le m_{i,s}^{(3)}$.

Then, the updated estimation error can be derived by substituting Eqs. (2)–(4) and (18)–(19) into Eq. (5)

$$\begin{aligned} e_{i,k+1} &= \bar{e}_{i,k+1} - K_{i,k+1}(H_{i,k+1} + V_{i,k+1}\Theta_{i,k+1})\bar{e}_{i,k+1} - \sum_{j\in\mathcal{N}_i} a_{ij}g(\bar{e}_{i,k+1} - \bar{e}_{j,k+1}) - K_{i,k+1}v_{i,k+1} \\ &= [I - K_{i,k+1}(H_{i,k+1} + V_{i,k+1}\Theta_{i,k+1})]\bar{e}_{i,k+1} \\ &\quad + \sum_{j\in\mathcal{N}_i} a_{ij}(G_{ij,k+1} + W_{ij,k+1}\Psi_{ij,k+1})(\bar{e}_{j,k+1} - \bar{e}_{i,k+1}) - K_{i,k+1}v_{i,k+1} \end{aligned} \tag{20}$$

Then the covariance of the updated estimation error can be obtained with respect to Eq. (20)

$$
\begin{aligned}
P_{i,k+1} &= [I - K_{i,k+1}(H_{i,k+1} + V_{i,k+1}\Theta_{i,k+1})]\bar{P}_{i,k+1}[I - K_{i,k+1}(H_{i,k+1} + V_{i,k+1}\Theta_{i,k+1})]^T \\
&\quad + \sum_{j\in\mathcal{N}_i} a_{ij}\mathbb{E}\{[I - K_{i,k+1}(H_{i,k+1} + V_{i,k+1}\Theta_{i,k+1})]\bar{e}_{i,k+1}(\bar{e}_{j,k+1} - \bar{e}_{i,k+1})^T \\
&\quad \times (G_{ij,k+1} + W_{ij,k+1}\Psi_{ij,k+1})^T\} + \sum_{j\in\mathcal{N}_i} a_{ij}\mathbb{E}\{(G_{ij,k+1} + W_{ij,k+1}\Psi_{ij,k+1})(\bar{e}_{j,k+1} - \bar{e}_{i,k+1})\bar{e}_{i,k+1}^T \\
&\quad \times [I - K_{i,k+1}(H_{i,k+1} + V_{i,k+1}\Theta_{i,k+1})]^T\} + K_{i,k+1}R_{i,k+1}K_{i,k+1}^T \\
&\quad + \sum_{j\in\mathcal{N}_i}\sum_{l\in\mathcal{N}_i} a_{ij}a_{il}\mathbb{E}\{(G_{ij,k+1} + W_{ij,k+1}\Psi_{ij,k+1})(\bar{e}_{i,k+1} - \bar{e}_{j,k+1}) \\
&\quad \times (\bar{e}_{i,k+1} - \bar{e}_{l,k+1})^T(G_{il,k+1} + W_{il,k+1}\Psi_{il,k+1})^T\}
\end{aligned} \tag{21}
$$

Notice that the unknown terms $\Omega_{i,k}$, $\Theta_{i,k+1}$ and $\Psi_{ij,k+1}$ are introduced in order to address the linearization errors of $f(\cdot)$, $h(\cdot)$ and $g(\cdot)$. Hence, it is impossible to calculate the covariance matrices $\bar{P}_{i,k+1}$ and $P_{i,k+1}$ directly. On the other hand, the edge-covariance matrices $\mathbb{E}\{\bar{e}_{i,k+1}\bar{e}_{j,k+1}^T\}$ are present due to the nonlinear consensus estimates. As stated in [6], the distributed filter is not scalable if the edge-covariance matrices are computed explicitly. An alternative way is to find upper bound matrices for $\bar{P}_{i,k+1}$ and $P_{i,k+1}$ and then determine the gain matrix $K_{i,k+1}$ according to the upper bound matrix.

The main result of this paper is summarized as follows.

Theorem 1. *Consider the discrete-time nonlinear system described by Eqs. (1) and (2). Let α_k and β_{k+1} be positive scalars. If the following two Riccati-like difference equations:*

$$
\bar{\Phi}_{i,k+1} = F_{i,k}(\Phi_{i,k}^{-1} - \alpha_k I)^{-1}F_{i,k}^T + \alpha_k^{-1}U_{i,k}U_{i,k}^T + Q_k \tag{22}
$$

$$
\begin{aligned}
\Phi_{i,k+1} &= (1 + \bar{a}_i)[(I - K_{i,k+1}H_{i,k+1})(\bar{\Phi}_{i,k+1}^{-1} - \beta_{k+1}I)^{-1}(I - K_{i,k+1}H_{i,k+1})^T] \\
&\quad + K_{i,k+1}[(1 + \bar{a}_i)\beta_{k+1}^{-1}V_{i,k+1}V_{i,k+1}^T + R_{i,k+1}]K_{i,k+1}^T \\
&\quad + 2(1 + \bar{a}_i)\sum_{j\in\mathcal{N}_i} a_{ij}[G_{ij,k+1}(\bar{\Phi}_{i,k+1}^{-1} - \beta_{k+1}I)^{-1}G_{ij,k+1}^T \\
&\quad + G_{ij,k+1}(\bar{\Phi}_{j,k+1}^{-1} - \beta_{k+1}I)^{-1}G_{ij,k+1}^T + 2\beta_{k+1}^{-1}W_{ij,k+1}W_{ij,k+1}^T]
\end{aligned} \tag{23}
$$

have positive-define solutions $\bar{\Phi}_{i,k+1}$ and $\Phi_{i,k+1}$ with initial conditions $P_0 \leq \Phi_0$ such that the following inequalities:

$$
\Phi_{i,k}^{-1} > \alpha_k I \tag{24}
$$

$$
\bar{\Phi}_{i,k+1}^{-1} > \beta_{k+1} I \tag{25}
$$

hold for all $k \geq 0$, then the matrix $\Phi_{i,k+1}$ is an upper bound of $P_{i,k+1}$. Moreover, the Kalman gain matrix can be determined by minimizing the trace of the upper bound matrix $\Phi_{i,k+1}$ as follows:

$$
\begin{aligned}
K_{i,k+1} &= (\bar{\Phi}_{i,k+1}^{-1} - \beta_{k+1}I)^{-1}H_{i,k+1}^T\big[H_{i,k+1}(\bar{\Phi}_{i,k+1}^{-1} - \beta_{k+1}I)^{-1}H_{i,k+1}^T \\
&\quad + \beta_{k+1}^{-1}V_{i,k+1}V_{i,k+1}^T + (1 + \bar{a}_i)^{-1}R_{i,k+1}\big]^{-1},
\end{aligned} \tag{26}
$$

where $\bar{a}_i = \sum_{j\in\mathcal{N}_i} a_{ij}$.

Proof. We aim to provide upper bound matrices for the predicted and the updated covariances, respectively. To this end, an upper bound matrix can be derived for the predicted covariance according to Lemma 1

$$\bar{P}_{i,k+1} \leq F_{i,k}(P_{i,k}^{-1} - \alpha_k I)^{-1} F_{i,k}^T + \alpha_k^{-1} U_{i,k} U_{i,k}^T + Q_k \tag{27}$$

We are now ready to derive an upper bound matrix for the updated covariance. By using the elementary inequality $x^T y + xy^T \leq xx^T + yy^T$, the upper bound matrix can be obtained for the second and the third terms on the right hand side of Eq. (21)

$$\begin{aligned}
&\sum_{j\in\mathcal{N}_i} a_{ij}\mathbb{E}\{[I - K_{i,k+1}(H_{i,k+1} + V_{i,k+1}\Theta_{i,k+1})]\bar{e}_{i,k+1}(\bar{e}_{j,k+1} - \bar{e}_{i,k+1})^T (G_{ij,k+1} + W_{ij,k+1}\Psi_{ij,k+1})^T\} \\
&\quad + \sum_{j\in\mathcal{N}_i} a_{ij}\mathbb{E}\{(G_{ij,k+1}+W_{ij,k+1}\Psi_{ij,k+1})(\bar{e}_{j,k+1} - \bar{e}_{i,k+1})\bar{e}_{i,k+1}^T[I-K_{i,k+1}(H_{i,k+1}+V_{i,k+1}\Theta_{i,k+1})]^T\} \\
&\leq \sum_{j\in\mathcal{N}_i} a_{ij}[I - K_{i,k+1}(H_{i,k+1} + V_{i,k+1}\Theta_{i,k+1})]\bar{P}_{i,k+1}[I - K_{i,k+1}(H_{i,k+1} + V_{i,k+1}\Theta_{i,k+1})]^T \\
&\quad + \sum_{j\in\mathcal{N}_i} a_{ij}(G_{ij,k+1}+W_{ij,k+1}\Psi_{ij,k+1})\mathbb{E}\{(\bar{e}_{j,k+1}-\bar{e}_{i,k+1})(\bar{e}_{j,k+1}-\bar{e}_{i,k+1})^T\}(G_{ij,k+1}+W_{ij,k+1}\Psi_{ij,k+1})^T \\
&\leq \bar{a}_i[I - K_{i,k+1}(H_{i,k+1} + V_{i,k+1}\Theta_{i,k+1})]\bar{P}_{i,k+1}[I - K_{i,k+1}(H_{i,k+1} + V_{i,k+1}\Theta_{i,k+1})]^T \\
&\quad + \sum_{j\in\mathcal{N}_i} a_{ij}(G_{ij,k+1}+W_{ij,k+1}\Psi_{ij,k+1})\mathbb{E}\{(\bar{e}_{j,k+1}-\bar{e}_{i,k+1})(\bar{e}_{j,k+1}-\bar{e}_{i,k+1})^T\}(G_{ij,k+1}+W_{ij,k+1}\Psi_{ij,k+1})^T
\end{aligned} \tag{28}$$

Similarly, the fourth term on the right hand side of Eq. (21) can be derived as follows:

$$\begin{aligned}
&\sum_{j\in\mathcal{N}_i}\sum_{l\in\mathcal{N}_i} a_{ij}a_{il}\mathbb{E}\{(G_{ij,k+1}+W_{ij,k+1}\Psi_{ij,k+1})(\bar{e}_{i,k+1}-\bar{e}_{j,k+1})(\bar{e}_{i,k+1}-\bar{e}_{l,k+1})^T (G_{il,k+1}+W_{il,k+1}\Psi_{il,k+1})^T\} \\
&\leq \sum_{j\in\mathcal{N}_i}\sum_{l\in\mathcal{N}_i} a_{ij}a_{il}\mathbb{E}\{(G_{ij,k+1}+W_{ij,k+1}\Psi_{ij,k+1})(\bar{e}_{i,k+1}-\bar{e}_{j,k+1})(\bar{e}_{i,k+1}-\bar{e}_{l,k+1})^T (G_{il,k+1}+W_{il,k+1}\Psi_{il,k+1})^T\} \\
&\leq \sum_{j\in\mathcal{N}_i}\sum_{l\in\mathcal{N}_i} a_{ij}a_{il}\frac{1}{2}\Big[\mathbb{E}\{(G_{ij,k+1} + W_{ij,k+1}\Psi_{ij,k+1})(\bar{e}_{i,k+1} - \bar{e}_{j,k+1})(\bar{e}_{i,k+1} - \bar{e}_{j,k+1})^T (G_{ij,k+1} + W_{ij,k+1}\Psi_{ij,k+1})^T\} \\
&\quad +\mathbb{E}\{(G_{ij,k+1} + W_{ij,k+1}\Psi_{ij,k+1})(\bar{e}_{i,k+1} - \bar{e}_{j,k+1})(\bar{e}_{i,k+1} - \bar{e}_{j,k+1})^T (G_{ij,k+1} + W_{ij,k+1}\Psi_{ij,k+1})^T\}\Big] \\
&\leq \bar{a}_i \sum_{j\in\mathcal{N}_i} a_{ij}\mathbb{E}\{(G_{ij,k+1} + W_{ij,k+1}\Psi_{ij,k+1})(\bar{e}_{i,k+1} - \bar{e}_{j,k+1})(\bar{e}_{i,k+1} - \bar{e}_{j,k+1})^T (G_{ij,k+1} + W_{ij,k+1}\Psi_{ij,k+1})^T\}
\end{aligned} \tag{29}$$

On the other hand, an upper bound matrix can be derived for the edge-covariance matrices in Eqs. (28) and (29)

$$\begin{aligned}
&(G_{ij,k+1} + W_{ij,k+1}\Psi_{ij,k+1})\mathbb{E}\{(\bar{e}_{i,k+1} - \bar{e}_{j,k+1})(\bar{e}_{i,k+1} - \bar{e}_{j,k+1})^T\}(G_{ij,k+1} + W_{ij,k+1}\Psi_{ij,k+1})^T \\
&\leq 2(G_{ij,k+1} + W_{ij,k+1}\Psi_{ij,k+1})(\bar{P}_{i,k+1} + \bar{P}_{j,k+1})(G_{ij,k+1} + W_{ij,k+1}\Psi_{ij,k+1})^T
\end{aligned} \tag{30}$$

Applying Eq. (30) into Eqs. (28)–(29) and substituting them into Eq. (21) leads to

$$
\begin{aligned}
P_{i,k+1} \le\ & (1+\bar{a}_i)[I-K_{i,k+1}(H_{i,k+1}+V_{i,k+1}\Theta_{i,k+1})]\bar{P}_{i,k+1}[I-K_{i,k+1}(H_{i,k+1}+V_{i,k+1}\Theta_{i,k+1})]^T \\
& +2(1+\bar{a}_i)\sum_{j\in\mathcal{N}_i} a_{ij}(G_{ij,k+1}+W_{ij,k+1}\Psi_{ij,k+1})(\bar{P}_{i,k+1}+\bar{P}_{j,k+1}) \\
& \times(G_{ij,k+1}+W_{ij,k+1}\Psi_{ij,k+1})^T+K_{i,k+1}R_{i,k+1}K_{i,k+1}^T \\
\le\ & (1+\bar{a}_i)[(I-K_{i,k+1}H_{i,k+1})(\bar{P}_{i,k+1}^{-1}-\beta_{k+1}I)^{-1}(I-K_{i,k+1}H_{i,k+1})^T] \\
& +K_{i,k+1}[(1+\bar{a}_i)\beta_{k+1}^{-1}V_{i,k+1}V_{i,k+1}^T+R_{i,k+1}]K_{i,k+1}^T \\
& +2(1+\bar{a}_i)\sum_{j\in\mathcal{N}_i} a_{ij}[G_{ij,k+1}(\bar{P}_{i,k+1}^{-1}-\beta_{k+1}I)^{-1}G_{ij,k+1}^T \\
& +G_{ij,k+1}(\bar{P}_{j,k+1}^{-1}-\beta_{k+1}I)^{-1}G_{ij,k+1}^T+2\beta_{k+1}^{-1}W_{ij,k+1}W_{ij,k+1}^T],
\end{aligned}
\tag{31}
$$

where the second inequality is derived by applying Lemma 1.

Applying Lemma 2 to Eqs. (22), (23), (27) and (31), we can draw a conclusion that

$$
P_{i,k+1} \le \Phi_{i,k+1} \tag{32}
$$

Now the Kalman gain matrix can be derived by minimizing the trace of the matrix $\Phi_{i,k+1}$. Taking partial derivative of the trace of $\Phi_{i,k+1}$ with respect to the Kalman gain matrix $K_{i,k+1}$, we have

$$
\begin{aligned}
\frac{\partial \mathrm{tr}(\Phi_{i,k+1})}{\partial K_{i,k+1}} =\ & 2(1+\bar{a}_i)K_{i,k+1}H_{i,k+1}(\bar{\Phi}_{i,k+1}^{-1}-\beta_{k+1}I)^{-1}H_{i,k+1}^T \\
& -2(1+\bar{a}_i)(\bar{\Phi}_{i,k+1}^{-1}-\beta_{k+1}I)^{-1}H_{i,k+1}^T \\
& +2K_{i,k+1}[(1+\bar{a}_i)\beta_{k+1}^{-1}V_{i,k+1}V_{i,k+1}^T+R_{i,k+1}]
\end{aligned}
\tag{33}
$$

The Kalman gain matrix can be determined by setting $\frac{\partial \mathrm{tr}(\Phi_{i,k+1})}{\partial K_{i,k+1}}=0$ which leads to the solution as shown in Eq. (26). This completes the proof. □

For clarity, the proposed filter is listed in Algorithm 1.

As shown in the computations of the upper bound matrices in Eqs. (22) and (23), the edge-covariances are not required to be computed and therefore the proposed filter is scalable. On the other hand, the predicted state estimate $\bar{x}_{i,k+1}$ and its corresponding upper bound matrix $\bar{\Phi}_{i,k+1}$ should be exchanged between neighboring nodes. Compared with the distributed EKF with linear consensus estimates, the communication cost is not increased. It should be pointed out that the computational burden becomes higher due to the computations for Jacobian matrices of the nonlinear consensus functions.

3.2. Design of the nonlinear consensus protocol

In the field of consensus control for multi-agent systems, nonlinear consensus protocols have been proposed for finite-time convergence in [21–23]. However, they can only be used for scalar systems. Unlike the existing nonlinear consensus protocols for multi-agent systems,

Algorithm 1 Distributed EKF with nonlinear consensus estimate.

Given the state estimate $\hat{x}_{i,0}$ and the matrix $\Phi_{i,0}$ for node i at time $k=0$,the distributed EKF is implemented at time instant $k=1,2,\ldots$

Step 1. Compute the predicted state estimate and the upper bound matrix

$$\begin{aligned}
\bar{x}_{i,k+1} &= f(\hat{x}_{i,k}) \\
\bar{\Phi}_{i,k+1} &= F_{i,k}(\Phi_{i,k}^{-1}-\alpha_k I)^{-1}F_{i,k}^T+\alpha_k^{-1}U_{i,k}U_{i,k}^T+Q_k \\
F_{i,k} &= \frac{\partial f}{\partial x}(\hat{x}_{i,k}), \quad U_{i,k}=\frac{\sqrt{n}}{2}\mathrm{diag}\{m_{i,1}^{(1)}\ldots,m_{i,p}^{(1)}\}
\end{aligned}$$

Step 2. Broadcast the message $(\bar{x}_{i,k+1},\bar{\Phi}_{i,k+1})$ to neighbors

Step 3. Compute the updated state estimate and the upper bound matrix

$$\begin{aligned}
\hat{x}_{i,k+1} &= \bar{x}_{i,k+1}+K_{i,k+1}[z_{i,k+1}-h(\bar{x}_{i,k+1})]+\sum_{j\in\mathcal{N}_i}a_{ij}g(\bar{x}_{j,k+1}-\bar{x}_{i,k+1}) \\
\Phi_{i,k+1} &= (1+\bar{a}_i)[(I-K_{i,k+1}H_{i,k+1})(\bar{\Phi}_{i,k+1}^{-1}-\beta_{k+1}I)^{-1}(I-K_{i,k+1}H_{i,k+1})^T] \\
&\quad +K_{i,k+1}[(1+\bar{a}_i)\beta_{k+1}^{-1}V_{i,k+1}V_{i,k+1}^T+R_{i,k+1}]K_{i,k+1}^T \\
&\quad +2(1+\bar{a}_i)\sum_{j\in\mathcal{N}_i}a_{ij}[G_{ij,k+1}(\bar{\Phi}_{i,k+1}^{-1}-\beta_{k+1}I)^{-1}G_{ij,k+1}^T \\
&\quad +G_{ij,k+1}(\bar{\Phi}_{j,k+1}^{-1}-\beta_{k+1}I)^{-1}G_{ij,k+1}^T+2\beta_{k+1}^{-1}W_{ij,k+1}W_{ij,k+1}^T] \\
K_{i,k+1} &= (\bar{\Phi}_{i,k+1}^{-1}-\beta_{k+1}I)^{-1}H_{i,k+1}^T[H_{i,k+1}(\bar{\Phi}_{i,k+1}^{-1}-\beta_{k+1}I)^{-1}H_{i,k+1}^T \\
&\quad +\beta_{k+1}^{-1}V_{i,k+1}V_{i,k+1}^T+(1+\bar{a}_i)^{-1}R_{i,k+1}]^{-1} \\
H_{i,k+1} &= \frac{\partial h}{\partial x}(\bar{x}_{i,k+1}), \quad G_{ij,k+1}=\frac{\partial g}{\partial x}(\bar{x}_{j,k+1}-\bar{x}_{i,k+1}) \\
V_{i,k+1} &= \frac{\sqrt{p}}{2}\mathrm{diag}\{m_{i,1}^{(2)}\ldots,m_{i,p}^{(2)}\}, \quad W_{ij,k+1}=\frac{\sqrt{p}}{2}\mathrm{diag}\{m_{i,1}^{(3)}\ldots,m_{i,p}^{(3)}\}
\end{aligned}$$

we adopt the following polynomial form to generate nonlinear consensus estimates:

$$\begin{aligned}
g(\bar{x}_{j,k+1}-\bar{x}_{i,k+1}) &= (\bar{x}_{j,k+1}-\bar{x}_{i,k+1})+\gamma(\bar{x}_{j,k+1}-\bar{x}_{i,k+1})^T(\bar{x}_{j,k+1}-\bar{x}_{i,k+1})(\bar{x}_{j,k+1}-\bar{x}_{i,k+1}) \\
&= (\bar{x}_{j,k+1}-\bar{x}_{i,k+1})+\gamma||\bar{x}_{j,k+1}-\bar{x}_{i,k+1}||^2(\bar{x}_{j,k+1}-\bar{x}_{i,k+1}) \\
&= (1+\gamma||\bar{x}_{j,k+1}-\bar{x}_{i,k+1}||^2)(\bar{x}_{j,k+1}-\bar{x}_{i,k+1}),
\end{aligned} \tag{34}$$

where γ is a scalar representing the coefficient of the polynomial.

Notice that the proposed nonlinear consensus estimate reduces to the linear case if $\gamma=0$. Compared with the linear consensus estimate with constant weight, the term $(1+\gamma||\bar{x}_{j,k+1}-\bar{x}_{i,k+1}||^2)$ can be considered a time-varying weight related with the estimate difference between neighboring nodes. This is reasonable since larger estimate difference is expected to be given larger weight. It should be pointed out the proposed nonlinear consensus protocol only provides an alternative to derive more accurate estimate, more effective protocols are expected in the future work.

4. Numerical study

In this section, the performance of the proposed filter is evaluated via a numerical example involving an indoor localization of a mobile robot using visual tracking systems. Although a simple network is used in this paper, the proposed algorithm can be used for large scale networks since only local state estimates and their corresponding upper bound matrices are exchanged between neighbors. The kinematic model for the robot can be represented by [31]

$$x_{k+1} = x_k + \frac{s_k^R + s_k^L}{2} \cos\theta_k + w_k^x \tag{35}$$

$$y_{k+1} = y_k + \frac{s_k^R + s_k^L}{2} \sin\theta_k + w_k^y \tag{36}$$

$$\theta_{k+1} = \theta_k + \frac{s_k^R - s_k^L}{b} + w_k^\theta, \tag{37}$$

where $(x_k,\ y_k)$ and θ_k denote the position and the orientation, respectively. s_k^R and s_k^L denote the distances traveled by the right and left driving wheels during the time interval $[k, k+1)$, respectively. b is the distance between the right and left driving wheels. In the robot, the optical encoders are equipped to record the distances s_k^R and s_k^L. $w_k = (w_k^x, w_k^y, w_k^\theta)$ is zero mean white Gaussian noise with covariance Q_k.

In the visual tracking system, eight cameras are used to generate the measurement of the robot and the measurement model is given by [31]

$$p_{i,k} = \frac{\gamma_u}{z_f^c}[-(x_{i,f} - x_k)\sin\theta_k + (y_{i,f} - y_k)\cos\theta_k - d_2] + p_0 + v_{i,k}^p \tag{38}$$

$$q_{i,k} = \frac{\gamma_v}{z_f^c}[-(x_{i,f} - x_k)\cos\theta_k - (y_{i,f} - y_k)\sin\theta_k + d_1] + q_0 + v_{i,k}^q, \tag{39}$$

where $(p_{i,k},\ q_{i,k})$ denotes the coordinate of the feature in the image plane. $(d_1,\ d_2)$ is the coordinate in the robot frame. z_f^c is the distance from the optical center of the camera to the robot. γ_u and γ_v are pixel magnification factors. $(p_0,\ q_0)$ is the image coordinate of the camera's principal point and $(x_{i,f},\ y_{i,f})$ is the coordinate of the ith camera in the world frame. $v_{i,k} = (v_{i,k}^p, v_{i,k}^q)$ is zero mean white Gaussian noise with covariance $R_{i,k}$.

In the simulations, the parameters of the visual tracking system are adopted from the experiments in our lab [31]: $d_1 = -0.0668$, $d_2 = 0.0536$, $z_f^c = 2.1050$, $\gamma_u = 902.13283$, $\gamma_v = 902.50141$, $u_0 = 347.20436$, $v_0 = 284.34705$. The positions of the cameras are taken to be (0.6, 0.6), (0.6, 1.8), (1.2, 1.8), (1.8, 1.8), (2.4, 1.8), (2.4, 0.6), (1.8, 1.2), (1.2, 1.2). The network topology is shown in Fig. 1. The process noise covariance matrix is $Q_k = \mathrm{diag}\{1, 1, 1\}$ and the measurement noise covariance matrices are $R_{i,k} = \mathrm{diag}\{25^2, 25^2\}$ $(i = 1, \ldots, 8)$. The scalars in the calculation of the upper bound matrices are taken to be $\alpha_k = \beta_k = 0.1$ $(i = 1, \ldots, 8)$. In order to generate a fair comparison with the filter with linear consensus estimate, the consensus parameter ε and the weight a_{ij} is taken to be $\varepsilon = a_{ij} = 0.01$. The coefficient γ in the polynomial form nonlinear consensus protocol is taken to be $\gamma = 10$.

To illustrate the tracking performance of the proposed filter, the averaged root mean square errors (RMSE) over the network are used. They are derived over 100 Monte Carlo runs as shown in Fig. 2. For notational simplicity, the distributed EKF with linear consensus estimate

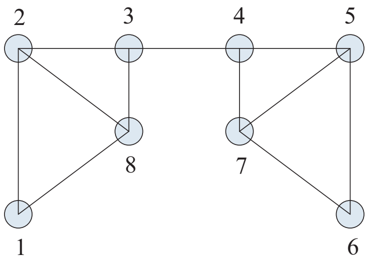

Fig. 1. Network topology.

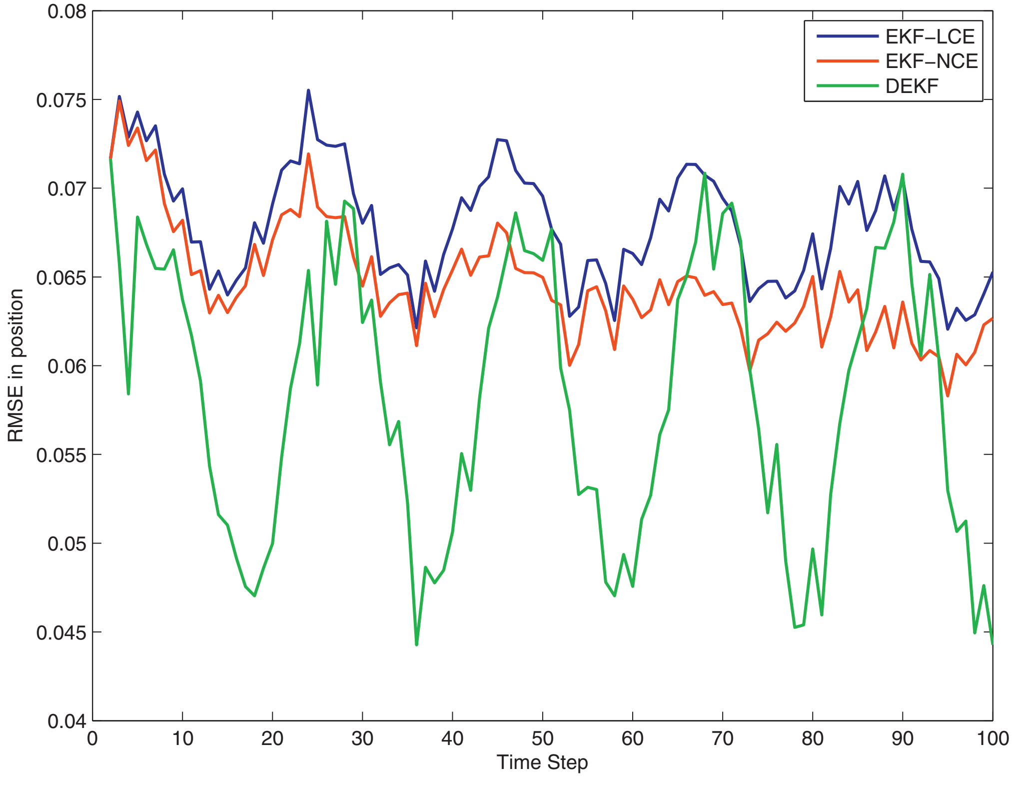

Fig. 2. RMSE in position versus time instants.

in [20] is shortly denoted by EKF-LCE and the proposed filter is denoted by EKF-NCE. The distributed EKF in [19] is shortly denoted by DEKF. Simulation results in Fig. 2 suggest that the proposed EKF-NCE performs better than the EKF-LCE. To be specific, the averaged RMSE for the EKF-NCE and the EKF—LCE are 0.63 and 0.67, respectively. This might be due to the fact that the more information can be extracted by using nonlinear consensus protocol. On the other hand, the averaged RMSE of the DEKF is 0.58 which indicates that it is better to fuse the information pairs than to fuse the predicted state estimates for designing the distributed filters.

To assess the computational requirements of the proposed filter, we computed the CPU time (averaged over 100 Monte Carlo runs) needed in MATLAB R2014a on a 2.5 GHz Intel(R) Core(TM) i7-6500 operating under Windows 10. The EKF–LCE consumes 0.08 s while the proposed EKF-NCE consumes 0.10 s. It can be seen that the computational cost is increased 25%. A modest conclusion can be drawn that the proposed filter provides a compromise between estimation accuracy and computational cost.

5. Conclusions and future work

In this paper, we have presented a distributed EKF for discrete-time nonlinear systems over a sensor network. A distinct feature of the proposed filter is that the nonlinear consensus estimate is adopted in the EKF. This is the main contribution of this paper with respect to the existing consensus-based distributed filters. The variance-constrained approach is used to determine the Kalman gain matrix for each node, and the edge-covariances have been bounded by their individual covariances so that the proposed distributed filter is scalable. Future work will focus on developing more effective nonlinear consensus protocols to improve estimation accuracy. The proposed distributed filter can be extended to the scenarios with network-based issues such as packet loss and event-triggered communication protocols.

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