Supplementary Material 2. Matrix Algebra

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1 Differentiation

1. $\mathbf{a} = (a_1, \ldots, a_n)'$ and $\mathbf{x} = (x_1, \ldots, x_n)^{'}$, let $y = \mathbf{a}'\mathbf{x} = \sum_{i=1}^{n} a_i x_i$. Then

$$\frac{\partial\left(\mathbf{a}'\mathbf{x}\right)}{\mathbf{x}} = \mathbf{a}.$$

2. $A = \begin{bmatrix} \mathbf{a_1^{'}} \\ \vdots \\ \mathbf{a_k^{'}} \end{bmatrix}$, and $\mathbf{x} = (x_1, \ldots, x_n)^{'}$, let $\mathbf{y} = \mathbf{Ax}$, then

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_k \end{bmatrix} = \begin{bmatrix} \mathbf{a_1^{'}x} \\ \vdots \\ \mathbf{a_k^{'}x} \end{bmatrix}$$

and

$$\frac{\partial \mathbf{Ax}}{\partial \mathbf{x}'} = \begin{bmatrix} \partial y_1/\partial \mathbf{x}' \\ \vdots \\ \partial y_k/\partial \mathbf{x}' \end{bmatrix} = \begin{bmatrix} \mathbf{a_1^{'}} \\ \vdots \\ \mathbf{a_k^{'}} \end{bmatrix} = A.$$

Also,

$$\frac{\partial \mathbf{Ax}}{\partial \mathbf{x}} = A'$$

3. $\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}'} = \left(\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}}\right)^{'}$

4. $\mathbf{x}'A\mathbf{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$, then
(1) $\frac{\partial(\mathbf{x}'A\mathbf{x})}{\partial \mathbf{x}} = (A + A')\,\mathbf{x}$. And when A is symmetric, $\frac{\partial(\mathbf{x}'A\mathbf{x})}{\partial \mathbf{x}} = 2A\mathbf{x}$;
(2) $\frac{\partial(\mathbf{x}'A\mathbf{x})}{\partial A} = \mathbf{xx}'$, especially, $\frac{\partial(\mathbf{x}'A\mathbf{x})}{\partial a_{ij}} = x_i x_j$.

2 Kronecker Product

1. $A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1K}B \\ a_{21}B & a_{22}B & \cdots & a_{2K}B \\ \\ a_{n1}B & a_{n2}B & \cdots & a_{nK}B \end{bmatrix}$
2. $(A \otimes B)' = A' \otimes B'$
3. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$
4. $(A \otimes B)(C \otimes D) = AC \otimes BD$

3 Eigenvalues and Matrix Decomposition

1. Definition: we say λ is the ***eigenvalue*** (or characteristic root) of A and $\mathbf{x}$ is the ***eigenvector*** (or characteristic vector) of A if
$$A\mathbf{c} = \lambda \mathbf{c}$$
And $\mathbf{x}$is often normalized such that c'c = 1.
2. Suppose that A has k non-zero eigenvalues, then $rank(A) = k$, $tr(A) = \sum_{i=1}^{k} \lambda_i$ and $det(A) = \prod_{i=1}^{k} \lambda_i$.
3. Suppose A is a $k \times k$ real symmetric matrix, it has $k$distinct eigenvalues, $\lambda_1, \cdots, \lambda_k$, $c_1, \cdots, c_k$ are the corresponding eigenvectors, and they are ***orthonormal***. Let $C = [c_1 \cdots, c]$ and $\Lambda = diag(\lambda_1, \cdots, \lambda_k)$. Then $AC = C\mathit{\Lambda}$.
   (a) The ***diagonalization*** of A is $C'AC = C'C\mathit{\Lambda} = \Lambda$.
   (b) The ***spectral decomposition*** of A is $A = C\mathit{\Lambda}C' = \sum_{i=1}^{k} \lambda_k c_k c_k^{'}$.
4. Let A be an $m \times n$ matrix of rank $r > 1$. Then there exists orthogonal matrices $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$. The ***singular value decomposition (SVD)*** of A is
$$A = U\mathit{\Lambda}V^{'},$$
where $\mathit{\Lambda} \in \mathbb{R}^{m \times n}$ has $\mathit{\Lambda}_{ij} = 0$ for $i \neq j$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > \lambda_{r+1} = \cdots = \lambda_{min\{m,n\}} = 0$, where $\lambda_i := \Lambda_{ii}$ is the $i - th$ singular value of A. And
$$A = \sum_{i=1}^{r} \lambda_i u_i v_i^{'},$$
where $u_i$is the $i - th$ column of U and v_i is the $i - th$ column of V.