# Linear Algebra

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Lecture Note 1: Introduction to Vector Spaces - This lecture note was taken while reading **Linear Algebra** by Friedberg, Insel, and Spence.

## 1 Vector Spaces

**Definition 1.** A **vector space** (or **linear space**) $V$ over a field $\mathbb{F}$ consists of a set with 2 operations (*addition* and *scalar multiplication*) defined so that for all $x, y \in V$, there is a unique element $(x + y) \in V$, and for all $\alpha \in \mathbb{F}$ and $x \in V$ there is a unique element $\alpha x \in V$ such that the following conditions hold:

[(VS 1)]

(1) For all $x, y \in V$ we have $x + y = y + x$.

(2) For all $x, y, z \in V$ we have $(x + y) + z = x + (y + z)$.

(3) There exists an element (unique) $0 \in V$ such that $x + 0 = x$ for all $x \in V$.

(4) For all $x \in V$ there exists an element (unique) $y \in V$ such that $x + y = 0$.

(5) For all $x \in V$ we have that $1 \cdot x = x$.

(6) For all $a, b \in \mathbb{F}$ and $x \in V$ we have $(ab)x = a(bx)$.

(7) For all $a \in \mathbb{F}$ and $x, y \in V$ we have $a(x + y) = az + ay$.

(8) For all $a, b \in \mathbb{F}$ and $x \in V$ we have that $(a + b)x = ax + bx$.

The elements of $\mathbb{F}$ are called **scalars** while the elements of $V$ are are called **vectors**.

**Definition 2.** An object of the form $(a_1, a_2, \ldots, a_n)$ for $a_i \in \mathbb{F}$ is called an $n -$ **tuple** with entries from $\mathbb{F}$. Two $n$-tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ with entries from $\mathbb{F}$ are called **equal** if $a_i = b_i$ for $i = 1, 2, \ldots, n$.

**Example 3.** The set of all $n$-tuples from a field $\mathbb{F}$ is a vector space (over $\mathbb{F}$ itself), which we denote by $\mathbb{F}^n$, under the operations of coordinatewise addition and scalar multiplication, that is, if $u = (a_1, a_2, \ldots, a_n) \in \mathbb{F}^n$, $v = (b_1, b_2, \ldots, b_n) \in \mathbb{F}^n$, and $c \in \mathbb{F}$, then

$$u + v = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \quad \text{and} \quad cu = (ca_1, ca_2, \ldots, ca_n).$$

For example, in $\mathbb{R}^4$ :

$$(3, -2, 0, 5) + (-1, 1, 4, 2) = (2, -1, 4, 7) \quad \text{and} \quad -5(1, -2, 0, 3) = (-5, 10, 0, 15).$$

**Notation 4.** *Elements from $\mathbb{F}^n$ may be written as* ***column vectors*** *:*

$$\begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix}$$

Since $\mathbb{F}^1$ are all the 1-tuples we write $\mathbb{F}$ instead of $\mathbb{F}^1$.

**Definition 5.** An $m \times n$ **matrix** with entries from a field $\mathbb{F}$ is a rectangular array of the form

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

where each entry $a_{ij}$ is an element of $\mathbb{F}$. We call the elements $a_{ij}$ where $i = j$ the **diagonal entries** of the matrix.

The rows of the $m \times n$ matrix $A$ previously sketched with entries $a_{ij}$, are elements of $\mathbb{F}^n$ while the rows of $A$ are elements of $\mathbb{F}^m$. We say that two matrices $A$ and $B$ are *equal* if $a_{ij} = b_{ij}$ for all $i \in [\![1, m]\!]$ and $j \in [\![1, n]\!]$.

**Notation 6.** *The $m \times n$ matrix such that $a_{ij} = 0$ for all $i, j$ is called the* ***zero matrix*** *and it is denoted by $O$.*

**Notation 7.** *If $A$ is an $m \times n$ matrix, then $A_{ij} = a_{ij}$.*

**Example 8.** The set of all $m \times n$ matrices with entries from a field $\mathbb{F}$ is a vector space denoted by $M_{m \times n}(\mathbb{F})$ in which

$$(A + B)_{ij} = A_{ij} + B_{ij} \quad \text{and} \quad (cA)_{ij} = cA_{ij}.$$

**Example 9.** Let $S$ be a nonempty set and $\mathbb{F}$ a field, and let $\mathcal{F}(S, \mathbb{F})$ denote the set of all the functions from $S$ to $\mathbb{F}$. Two elements $f$ and $g$ in $\mathcal{F}(S, \mathbb{F})$ are called equal if $f(s) = g(s)$ for all $s \in S$. The set $\mathcal{F}(S, \mathbb{F})$ is a vector space under the operations of addition and scalar multiplication defined for $f, g \in \mathcal{F}(S, \mathbb{F})$ and $c \in \mathbb{F}$ by

$$(f + g)(s) = f(s) + g(s) \quad \text{and} \quad (cf)(s) = c[f(s)] \quad \text{for all } s \in S.$$

**Definition 10.** A **polynomial** with coefficients from a field $\mathbb{F}$ is an expression of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $n \in \mathbb{N}$ and $a_k \in \mathbb{F}$ for all $k \in [\![0, n]\!]$. If $f(x) = 0$, that is, if $a_n = a_{n-1} = \cdots = a_0 = 0$, then $f(x)$ is called the **zero polynomial** and it is convenient to set its degree to be $-1$. Note that the polynomials of degree 0 are of the form $f(x) = c$ for some nonzero scalar $c$.

Two polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{i=0}^{m} b_i x^i$ are called *equal* if $m = n$ and $a_i = b_i$ for all $i \in [\![0, n]\!]$.

**Notation 11.** *When $\mathbb{F}$ is a field containing infinite number of elements, we usually regard a polynimial with coefficients from $\mathbb{F}$ as a function from $\mathbb{F}$ to $\mathbb{F}$.*

**Definition 12.** Under the operations of addition and scalar multiplication, the set of all polynimials with coefficients from $\mathbb{F}$ is a vector space which we denote by $P(\mathbb{F})$.

**Example 13.** Let $\mathbb{F}$ be a field. A sequence in $\mathbb{F}$ is a function $\sigma$ from the positive integers into $\mathbb{F}$. We denote $\sigma(n) = a_n$ for all $n \in \mathbb{N}$ as $\{a_n\}$. Let $V$ be the set consisting of all the sequences $\{a_n\}$ in $\mathbb{F}$ that have only finite number of nonzero terms $a_n$. If $\{a_n\}, \{b_n\} \in V$, and $t \in \mathbb{F}$ define

$$\{a_n\} + \{b_n\} = \{a_n + b_n\} \quad \text{and} \quad t\{a_n\} = \{ta_n\}.$$

Then, $V$ is a vector space under these operations.

**Example 14.** Let $S = \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{R}\}$. For $(a_1, a_2), (b_1, b_2) \in S$ and $c \in \mathbb{R}$ define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, ca_2).$$

As $(a_1 + b_1, a_2 - b_2)$ may be different from $(b_1 + a_1, b_2 - a_2)$ we have that the addition is not commutative, and so $S$ is not a vector space.

**Proposition 15** (**Cancelative Law for Vector Addition**)**.** *If $x, y$ and $z$ are elements of the vector space $V$ such that $x + z = y + z$, then $x = y$.*

*Proof.* Take $v \in V$ such that $z + v = 0$. Then write

$$x = x + 0 = x + (z + v) = (x + z) + v = (y + z) + v = y + (z + v) = y + 0 = y.$$

☺

**Corollary.** *The vector $0$ is unique.*

**Corollary.** *The vector $y$ such that $x + y = 0$ where $x \in V$, is unique.*

**Definition 16.** The vector $0$ is called the **zero vector** and the vector $y$ such that $x+y = 0$ is called the **additive inverse** of $x$ and it is denoted by $-x$.

**Proposition 17.** *Let $V$ be a vector space. Then the following statement hold:*

*(1)* $0x = 0$ *for all* $x \in V$.

*(2)* $(-a)x = -(ax) = a(-x)$ *for all* $a \in \mathbb{F}$ *and* $x \in V$.

*(3)* $a0 = 0$ *for all* $a \in \mathbb{F}$.

*Proof.* (1) Write

$$0x + 0x = (0 + 0)x = 0x = 0x + 0..$$

Then $0x = 0$.

(2) Note that $-(ax)$ is the unique element of $V$ such that $ax + [-(ax)] = 0$. In addition, $ax + (-a)x = (a - a)x = 0x = 0$, and so $-(ax) = (-a)x$. In particular, $(-1)x = -x$, and it follows that

$$a(-x) = a[(-1)x] = [a(-1)]x = (-a)x = -(ax).$$

Therefore, $a(-x) = -(ax)$. ☺

**Definition 18.** The **zero vector space** is the vector space $V = \{0\}$ over $\mathbb{F}$ consisting of the single vector 0 and defined by $0 + 0 = 0$ and $c0 = 0$ for all $c \in \mathbb{F}$.

**Example 19.** Let $V$ be the set of all differentible real-valued functions defined on the real line. Then $V$ is a vector space. To prove this note first, that every differentiable function is continious, and second, that if $f, g \in V$, then $f(x)+g(x) = h(x)$ where $h$ is also differentiable, and so $h \in V$. In addition, take $k \in \mathbb{R}$, and so $kf(x) \in V$ as it is as well differentiable. This can be written as

$$\frac{d}{d_x}f(x) + \frac{d}{d_x}g(x) = \frac{d}{d_x}(f(x) + g(x)), \quad \text{and} \quad a\left(\frac{d}{dx}f(x)\right) = \frac{d}{dx}af(x).$$

**Example 20.** Both the set of real-valued functions and the set of continupus real-valued functions are vector spaces over $\mathbb{R}$. In addition, the set of all even real-valued functions is also a vector space over $\mathbb{R}$.

**Example 21.** Let $V$ be the set of all the sequences $\{a_n\}$ of real numbers. For $\{a_n\}$ and $\{b_n\}$ in $V$, and for all $t \in \mathbb{R}$, define

$$\{a_n\} + \{b_n\} = \{a_n + b_n\} \quad \text{and} \quad t\{a_n\} = \{ta_n\}.$$

Then, $V$ is a vector space over $\mathbb{R}$.

**Example 22.** Let $V$ and $W$ be vector spaces over a field $\mathbb{F}$, and let

$$Z = \{(v, w) \mid v \in V \text{ and } w \in W\}.$$

Then $Z$ is a vecot space over $\mathbb{F}$ under the following operations:

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \quad \text{and} \quad c(v_1, w_1) = (cv_1, cw_1).$$