PROJECT 2, PART # 1)

CONSIDER THE HEAT TRANSFER EQUATION $u_t - u_{xx} = f(x,t)$, $(x,t) \in (0,1) \times (0,1)$ WITH THE DIRICHLET BOUNDARY CONDITIONS $u(x,0) = \sin(\pi x)$, $u(0,t) = u(1,t) = 0$ AND FUNCTION $f(x,t) = (\pi^2 - 1)e^{-t}\sin(\pi x)$ THE ANALYTIC SOLUTION TO THIS PROBLEM IS $u(x,t) = e^{-t}\sin(\pi x)$

1. DEFINE THE TRIAL & TEST SPACES

   LET V BE THE FUNCTION SPACE IN WHICH SOLUTION u LIES. IN THE FINITE ELEMENT CONTEXT, V IS TYPICALLY CHOSEN AS A SPACE OF PIECEWISE POLYNOMIAL FUNCTIONS.

   FOR THIS PROBLEM, WE CAN USE THE SPACE OF CONTINUOUS PIECEWISE LINEAR FUNCTIONS ON THE INTERVAL OF $(0,1)$

   LET v BE THE FUNCTION IN THE SPACE V THAT SATISFIES THE DIRICHLET BOUNDARY CONDITIONS.

2. MULTIPLY BY A TEST FUNCTION AND INTEGRATE OVER THE DOMAIN

   MULTIPLY THE DIFFERENTIAL EQUATION BY A TEST FUNCTION v AND INTEGRATE OVER THE SPATIAL DOMAIN $(0,1)$ & THE TIME DOMAIN $(0,T)$

   WHERE T IS THE FINAL TIME:

$$\int_0^T \int_0^1 (u_t v - u_{xx} v)\, dx\, dt = \int_0^T \int_0^1 f(x,t)\, v\, dx\, dt$$

3. INTEGRATE BY PARTS IN SPACE

$$\int_0^T \int_0^1 u_t v\, dx\, dt + \int_0^T [u_x v]_0^1\, dt - \int_0^T \int_0^1 u_x v_x\, dx\, dt = \int_0^T \int_0^1 f(x,t)\, v\, dx\, dt$$

REPRESENTS THE BOUNDARY TERM, WHICH VANISHES DUE TO DIRICHLET BOUNDARY CONDITIONS

4. DISCRETIZE IN TIME

   NOW, WE CAN DISCRETIZE THE TIME DOMAIN. LET u^n DENOTE THE NUMERICAL APPROXIMATION OF u AT THE TIME LEVEL n, AND LET Δt BE THE TIME STEP.

   USING THE FORWARD EULER METHOD FOR TIME DISCRETIZATION, WE REPLACE u_t WITH $(u^{n+1} - u^n)/\Delta t$, WHERE n REPRESENTS THE CURRENT TIME STEP

   AND $n+1$ REPRESENTS THE NEXT TIME STEP:

$$\int_0^1 \frac{1}{\Delta t} \int_0^T (u^{n+1} v - u^n v)\, dx\, dt + \int_0^T \int_0^1 u_x v_x\, dx\, dt = \int_0^T \int_0^1 f(x,t)\, v\, dx\, dt$$

5. APPLY THE GALERKIN METHOD

   EXPAND THE SOLUTION u AND THE TEST FUNCTION v IN TERMS OF BASIS FUNCTIONS. LET u_h AND v_h BE THE FINITE ELEMENT APPROXIMATIONS:

$$u_h(x,t) = \sum_{i=1}^N u_i(t)\, \phi_i(x)$$

$$v_h(x,t) = \sum_{i=1}^N v_j(t)\, \phi_j(x)$$

   SUBSTITUTE THESE EXPRESSIONS INTO THE WEAK FORM:

$$\sum_{i=1}^N \left(\frac{1}{\Delta t} \int_0^T \int_0^1 (u^{n+1} \phi_i v_j - u_i^n \phi_i v_j)\, dx\, dt \right) + \sum_{i=1}^N \left(\int_0^T \int_0^1 u_{ix} v_{jx}\, dx\, dt \right) = \sum_{i=1}^N \left(\int_0^T \int_0^1 f \phi_j\, dx\, dt \right)$$