HW2

1. $\ell = -\log p_c$

$$= \log \sum_{j=1}^{c} e^{z_j} - z_i$$

where $z_i = w_i^T x + b_i$

thus $$\frac{\partial \ell}{\partial w_i} = \frac{\partial \ell}{\partial z_i} \cdot \frac{\partial z_i}{\partial w_i} = \left(\frac{e^{z_i}}{\sum_{j=1}^{c} e^{z_j}} - \frac{\partial z_j}{\partial z_i} \right) \cdot x$$

$$= \begin{cases} (p_i - 1) \cdot x & i = j \\ p_j \, x & i \neq j \end{cases}$$

$$\frac{\partial \ell}{\partial b_i} = \frac{\partial \ell}{\partial z_i} \cdot \frac{\partial z_i}{\partial b_i} = \left(\frac{e^{z_i}}{\sum_{j=1}^{c} e^{z_j}} - \frac{\partial z_j}{\partial z_i} \right) \cdot 1$$

$$= \begin{cases} p_i - 1 & i = j \\ p_j & i \neq j \end{cases}$$

2. $\tilde{\ell} = -\log \tilde{p}_c$

$$= \log\left(e^{z_i} + \frac{1}{K}\sum_{j\in S} q_j e^{z_j}\right) - z_i$$

$$\frac{\partial \tilde{\ell}}{\partial w_i} = \frac{\partial \hat{\ell}}{\partial z_j} \cdot \frac{\partial z_j}{\partial w_j} = \left(\frac{e^{z_i} + \frac{1}{K} q_i e^{z_i}}{e^{z_i} + \frac{1}{K}\sum_{j\in S} q_j e^{z_j}} - 1 \right) \cdot x$$

$$= \begin{cases} \left(\frac{1+K}{K}\tilde{p}_i - 1\right) x & i = j \\ \frac{1+K}{K}\tilde{p}_i \cdot x & i \neq j \end{cases}$$

Similarly, $$\frac{\partial \tilde{\ell}}{\partial b_i} = \begin{cases} \frac{1+K}{K}\tilde{p}_i - 1 & i = j \\ \frac{1+K}{K}\tilde{p}_i & i \neq j \end{cases}$$