Hw3, Immersed Boundary Method

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1 Introduction

The notation exactly follows Prof. Peskin's notes and I will refer to equations there directly. The codes I used are online in my github repository. Except pictures shown in this report, I also made movies of simulations available on my webpage.

2 Straight Line

For the straight line pivoted at the center $[L/2, L/2]$ of the square $[L, L]$, we denote the length of straight line as L_0, the length of one end from the pivoted point is L_p, the angle of the straight line is θ. A uniform force density in the x direction, i.e. $\mathbf{f^0} = [f_0, 0]$ is applied on the fluid. So the PDEs to solve is eq. (1).

$$\begin{gathered}\rho\left(\frac{\partial u}{\partial t} + \underline{U}\cdot\nabla\underline{u}\right) + \nabla p = \mu\Delta\underline{u} + \underline{f}^0 + \underline{f}^{IB} \\ \nabla\cdot\underline{u} = 0\end{gathered} \tag{1}$$

The temporal integrator shown in the course is used here. In brief, explicitly forward a half-time step and then do a fluid solver to get the velocity at half time step. Then, forward one time step using the velocity at half time step, which is a multistage implicate method. As for the fluid solver, we use the FFT method with periodic boundary condition. Also, a spring force penalty with magnitude K is used to bond the target points between fluid target points and boundary target points.

2.1 Positive excess mass, i.e. $m_0 \neq 0$

The task is to observe the behavior for different position of the pivot, i.e. different L_p, and different mass density of the straight line, i.e. m_0. The code is in the 'straight_line' folder.

Configuration setup

Length of the periodic box $L = 1$, straight line width $L_0/L = 1/4$, the number of grid on each dimension is $N = 64$. Spatial step of the boundary line, i.e. the straight line, is $\Delta Z = h/2$, where $h = L/N$ is spatial step size of the regular grid. Viscousity $\mu = 0.01$ so that we get Reynold number around 100 for velocity $u = 1$. The uniform force on the x direction is $f_0 = 0.1$. The density of the fluid $\rho = 1$.

The spring force magnitude K is chosen such that the target points are bond well, which we can see from the movie. And we need small enough Δt so that the temporal integrator is stable, which we can tell from the bond of the target points as well as the CFL number.

2.1.1 Non-symmetric pivoting

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$L_p/L_0 = 1/4, \theta_0 = \pi/4, m_0 = 4.$

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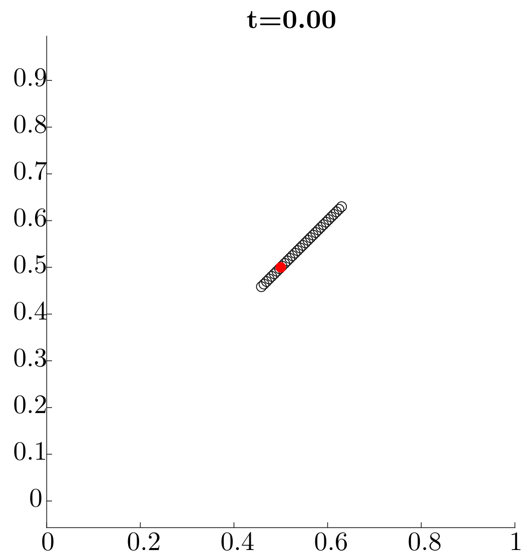

Figure 1: Initial configuration, $\theta_0 = \pi/4$, the red point is the pivot and the black circles are the target boundary points.

Let us first try an easy case where the straight line is not symmetric about the pivot and it start with $\theta_0 = \pi/4$. Intuitively, the stable point is where the longer part fall downstream, i.e. horizontally to the right. And the numerical result is consistent with this figs. 1 and 2. The flow velocity is plotted with quiver in equi-spaced grids. See the movie for more details. It accelerates toward the balanced state and oscillate around there with decaying amplitude since it has inertia. So, the only stable balanced state is when $\theta = 0$.

2.1.2 Symmetric pivoting.

If we change $L_p/L0 = 1/2$, the straight line is symmetric about the pivot. This is a quite tricky and interesting case due to the symmetry. And typically the torque will be very minor due to the symmetry and the uniform external force, so this system will be very sensitive to perturbation. As a result of symmetry, we could argue that $\theta = \pi/2, 0$ are the balanced state. I started with $\theta_0 = \pi/2$, in which case the straight line is vertical at first and is supposed to balanced without knowing it's stable or not.

However, one thing I find important is that the discretization of boundary points plays a vital role here. In the notes, we discrete it as left-most grids points, i.e. $[0, ..., Nb - 1] * \Delta Z$, which break the symmetry since it make the mass distribution uneven artificially. This effect can be verified by changing θ_0 to $\theta_0 + \pi$, which should be no difference in reality but reverse the rotation direction in simulation because of the discretization. And I observe that it will always rotate towards one direction if $\theta_0 = \pi/2$, which is un-physical. So one thing we should put efforts into is to use evenly distributed points in this case, e.g. mid-points, as well as the later S-shape case.

After changing the discretization of straight line to mid-points, we tried simulations. If $\theta_0 = 0, \pi/2$, that's the balanced state for sure. So we start somewhere in-between, say $\theta_0 = \pi/4$. The video shows that the straight line will oscillate around $\theta = \pi/2$ with increasing magnitude. It's surprising to me that it won't be attracted to the state $\theta = 0$. Maybe this is because we have the periodic boundary condition. And the non-zero uniform force is applied, so the fluid will keeps accelerating and the vertical state $\theta = \pi/2$ will block the flow.

2.1.3 Different excess mass m_0.

If we increase the excess mass density to $m_0 = 40$ and repeat the non-symmetric pivoting case with $\theta_0 = \pi/4$ in section 2.1.1, we see the similar motion scheme but with slower velocity since the rotational inertial is

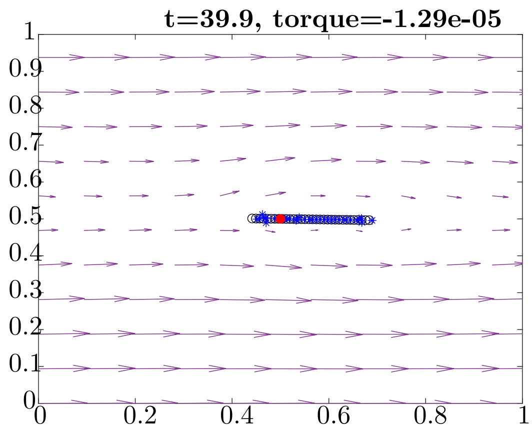

Figure 2: Stable balanced configuration, $\theta = 0$, the red point is the pivot, the black circles are the target boundary points and the blue asterisks are the target fluid points. The quiver vector field is plotted equispaced.

bigger now, see this movie.

2.2 Zero excess mass, i.e. neutrally buoyant

For neutral buoyant straight line, i.e. $m_0 = 0$, the previous method will not work since there will be no torque or more specifically, no rotational inertia. The angle of the straight line θ is decided by the torque-free condition. Following the notes, we know the formula is

$$(\cos\theta, \sin\theta) = \frac{\int_0^{L_0} \tilde{X}(s)\,(s - L_p)\,ds}{|\int_0^{L_0} \tilde{X}(s)\,(s - L_p)\,ds\,|},$$

where $\tilde{X} = X - [L/2, L/2]$. Then θ is easily computed by Matlab build-in function 'cart2pol'. Thus the force applied to the fluid, location of the target boundary points Z and velocity u is easily available. The code is in the 'straight_line_nuetral_buoyant' folder.

We tried the same simulation case as section 2.1.1 where $\theta_0 = \pi/4$ and $L_p/L_0 = 1/4$. The state $\theta = 0$ state is still the only stable balanced state. However, the difference is that the straight line will settle there directly without oscillation since there's no rotational inertial in this case (movie).

3 S-shape curve

The S shape curve is implemented as two semi-circle with radius r_0. The pivot is also in the center. It has only one degree of freedom, say θ, which is the angle of the line that connect the two end points. Then, the target boundary points can be directly written out via knowing θ. The same temporal integrator and spatial solver is used in this case, which is implemented in the 'curve' folder. Another quantity that we need to compute is the momentum of inertia, i.e. rotational inertia I_0,

$$I_0 = 2\int_0^{\pi} m_0 r_0^2((1-\cos(\theta))^2 + \sin^2(\theta))r_0 d\theta = 4\pi m_0 r_0^3.$$

Note this is same as section 2.1.2 that we need to make the discrete points in the boundary line evenly distributed about the center pivot point. We did the same simulation as in section 2.1.2 where $\theta_0 = \pi/4$ (movie). At the very beginning, the velocity of the flow is small, i.e. small Reynold number, the S-shape line rotate counterclock-wise a bit, which is against our intuition. However, as the Reynold number increases, it start to rotate clock-wise and faster and faster. Also, the instability also grows as we can see from the separation of target points since the velocity accelerates all the time due to the constant external force field. This suggests we should use adaptive time step.

3.1 Electricity generator, the most efficient torque resistance

If we apply a torque on the pivot in the opposite direction of its rotation, which is indeed what electricity generator does, the power of the generator is defined by the applied torque times the angular velocity. So we can determine the best external torque to decide the most efficient generator by simulations.

However, as suggested in the above section, the velocity keeps accelerating and instability grows. This model is not what reality is like the east river in nyc. I think we should implement the constant infinite boundary condition, i.e. flow velocity is constant is the infinity. Equivalently, we can make the fluid static in the infinity and the S-shape line pivot travel in a constant velocity and then compute the torque and power. This is free boundary condition in reality, but we can implement it as it is periodic boundary if we make $L >> r_0$. This is easily done by our numerical model here. The only thing we need to change is to make the pivot point travelling with constant velocity, which is feasible for target point method but I haven't done yet.