Hw1, Immersed Boundary Method

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1 Problem 1: The kernel function

1.1 Plot the 4 point delta function

I use Matlab symbolic package to do the calculation and plotting. See fig.1 for $\phi(r), r \in [-3, 3]$, fig.2 for $\phi'(r), r \in [-3, 3]$. So, this delta function is C^1

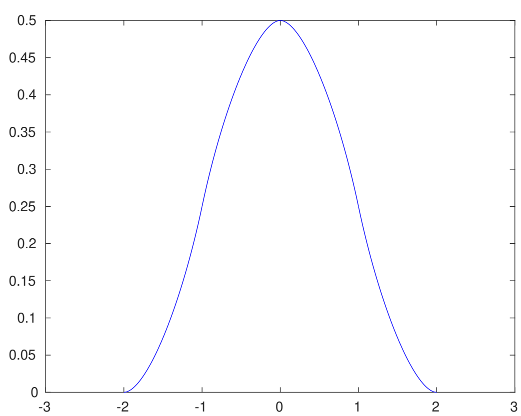

Figure 1: $\phi(r)$

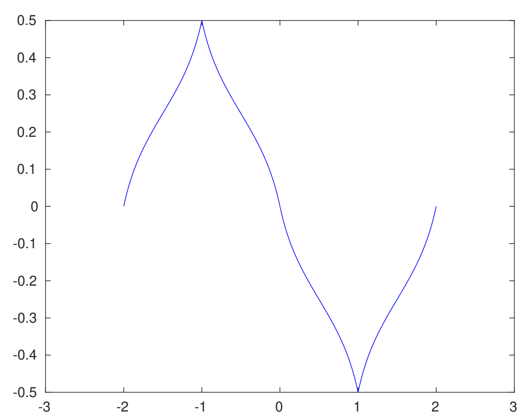

Figure 2: $\phi'(r)$

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2.1 Similarly, fix $0 \le r \le 1$, we have

$$\begin{cases} \varphi_{-3} + \varphi_{-1} + \varphi_{1} = \varphi_{-2} + \varphi + \varphi_{2} = \frac{1}{2} & ①, ② \\ (r-3)^{\alpha}\varphi_{-3} + (r-2)^{\alpha}\varphi_{-2} + (r-1)^{\alpha}\varphi_{-1} + r^{\alpha}\varphi + (r+1)^{\alpha}\varphi_{1} + (r+2)^{\alpha}\varphi_{2} = 0 & ③, ④, ⑤ \\ \varphi_{-3}^2 + \varphi_{-2}^2 + \varphi_{-1}^2 + \varphi^2 + \varphi_1^2 + \varphi_2^2 = C & ⑥ \end{cases}$$

, where $\alpha = 1, 2, 3$. And we denote $\varphi_{\alpha} = \varphi(r+\alpha)$

To decide C, take $r = 0$,

$$\begin{cases} \varphi(-1) + \varphi(1) = \varphi(-2) + \varphi(0) + \varphi(2) = \frac{1}{2} \\ -2\varphi(-2) - \varphi(-1) + \varphi(1) + 2\varphi(2) = 0 \\ 4\varphi(-2) + \varphi(-1) + \varphi(1) + 4\varphi(2) = 0 \\ -8\varphi(-2) - \varphi(-1) + \varphi(1) + 8\varphi(2) = 0 \end{cases}$$

It's five linear Eqns for five unknowns, we get

$\varphi(-2) = -1/16$, $\varphi(-1) = 1/4$, $\varphi(0) = 5/8$, $\varphi(1) = 1/4$, $\varphi(2) = -1/16$

Thus, $C = \sum_{\alpha} \varphi_{\alpha}^2 = 67/128$

Since ① + ②, for ③, we have

$$r\left(\sum_{\alpha} \varphi_{\alpha}\right) = 3\varphi_{-3} + 2\varphi_{-2} + \varphi_{-1} - \varphi_{1} - 2\varphi_{2} = r$$

Then, for ④, we have

$$r^2 - 2r \cdot r + 9\varphi_{-3} + 4\varphi_{-2} + \varphi_{-1} + \varphi_{1} + 4\varphi_{2} = 0$$

For ⑤, we have,

$$r^3 - 3r^2 \cdot r + 3r \cdot r^2 - 27\varphi_{-3} - 8\varphi_{-2} - \varphi_{-1} + \varphi_{1} + 8\varphi_{2} = 0$$

Solve $\varphi_{-3}, \varphi_{-2}, \varphi_{-1}, \varphi_{1}, \varphi_{2}$ in terms of φ,

$$\Rightarrow, \quad \varphi_{-3} = r^2/8 + \frac{\varphi(r)}{2} - \frac{5}{16}$$

$$\varphi_{-2} = r^3/12 - r^2/4 - r/12 - \frac{3}{2}\varphi(r) + \frac{7}{8}$$

$$\varphi_{-1} = \frac{-r^3}{6} + \frac{r^2}{4} + \frac{2r}{3} + \varphi(r) - \frac{3}{8}$$

$$\varphi_1 = \frac{r^3}{6} - \frac{3}{8}r^2 - \frac{2}{3}r - \frac{3}{2}\varphi(r) + \frac{19}{16}$$

$$\varphi_2 = -\frac{r^3}{12} + \frac{r^2}{4} + \frac{r}{12} + \frac{1}{2}\varphi(r) - \frac{3}{8}$$

plug into ⑥,

$$\Rightarrow \varphi(r) = -\frac{7}{12}r^6 + \frac{7}{4}r^5 + \frac{125}{48}r^4 - \frac{65}{8}r^3 - \frac{187}{48}r^2 + \frac{33}{4}r + \frac{81}{64})^{1/2}/42 - \frac{11}{56}r^2 + \frac{1}{12}r^3 + \frac{61}{112}$$

(It's solved by Matlab & I drop the unreasonable solution)

2.2 Plot the 6 point delta function

Similarly, I plotted the 6 point delta function. See fig.3 for $\phi(r), r \in [-4,4]$, fig.4 for $\phi'(r), r \in [-4,4]$. So, this delta function is C^1

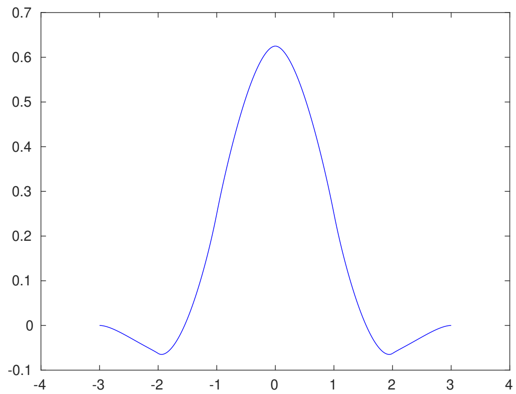

Figure 3: $\phi(r)$

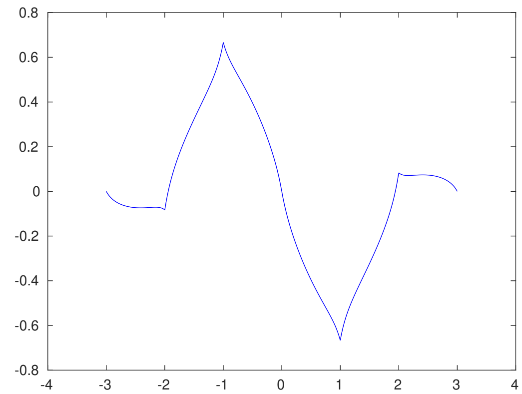

Figure 4: $\phi'(r)$