Hw2, Immersed Boundary Method

Zhe Chen*[1]

[1]Courant Institute of Mathematical Sciences, New York University

February 2021

1 Optional*: Energy identity of the discretized incompressible Navier-Stokes equations

WLOG, we assume it's 2d. Then, we write the discretized incompressible Navier-Stokes equations in this form,

$$\rho\left(D_t u + S(u)u\right) + Dp - f = \mu L u$$
$$D \cdot u = 0,$$

where $D = (D_1, D_2)$ is the centered difference in space and D_t is the discretized derivative in time. And $S(u)\varphi$ is the skew-symmetric operator $S(u)\phi = \frac{1}{2}u \cdot D\varphi + \frac{1}{2}D \cdot (u\varphi)$

Define the discrete inner product $(\cdot, \cdot)_h$ by

$$(\phi, \psi)_h = \sum_{\mathbf{x} \in g_h} \phi(\mathbf{x})\psi(\mathbf{x})h^2.$$

Hence, we can do the inner product on both side of the momentum equation. For the LHS, the first term is

$$(u, \rho D_t u)_h = \sum_{\mathbf{x} \in g_h} u_i \rho D_t u_i h^2 = D_t \left(\sum_{\mathbf{x} \in g_h} (\frac{1}{2}\rho u_i^2)h^2 \right) = D_t E_k,$$

where E_k is the total kinetic energy.

(Note that Einstein Notation is used in this homework.)

For the second term, due to its skew-symmetry, it should be zero,

$$(u, \rho S(u)u) = \sum_{\mathbf{x} \in g_h} \rho u_i \left(\frac{1}{2}u_j D_j u_i + \frac{1}{2}D_j(u_j u_i) \right) h^2$$
$$= \sum_{\mathbf{x} \in g_h} \rho u_i u_j D_i u_j h^2 = \sum_{\mathbf{x} \in g_h} -\rho u_j D_i (u_i u_j) h^2 = \sum_{\mathbf{x} \in g_h} -\rho u_j u_i D_i u_j h^2 = 0$$

Note that the partial integral law also hold for discrete form if we assume free boundary with vanishing value.

For the third term, the internal pressure would not apply work.

$$(u, Dp) = \sum_{\mathbf{x} \in g_h} u_i D_i p h^2 = \sum_{\mathbf{x} \in g_h} -p D_i u_i = 0$$

.

*zc1291@nyu.edu

For the fourth term, it's the work the potential force $F\Delta\theta = -D_X E_p$ apply, where we assume the force is given by potential of the boundary

$$(u, f) = \sum_{\mathbf{x}\in g_h} u_i f_i h^2 = \sum_{\mathbf{x}\in g_h} u_i F_{ki} \delta_h(x - X_k)\Delta\theta h^2 = F_{ki} D_t X_{ki} \Delta\theta = -D_t E_p.$$

For the RHS, it's the viscosity energy dissipation.

$$(u, \mu L u) = \sum_{\mathbf{x}\in g_h} \mu u_i D_2 u_i h^2 = \sum_{\mathbf{x}\in g_h} -\mu (D_i u_i)^2 h^2 \leq 0,$$

where we see monotone lose of energy.

Thus, we have the following energy identity

$$D_t(E_k + E_p) = \sum_{\mathbf{x}\in g_h} -\mu (D_i u_i)^2 h^2 \leq 0.$$

Note that this discrete identity is true only when the spreading and interpolation delta function is the same!

2 Preservation of total force and total torque during the force-spreading operation

This is an intrinsic property that comes from the fact that the spreading kernel adds no weight or first momentum.

Denote $\varphi(x) = \Pi_{i=1}^d \varphi(x_i)$, where $d = 2, 3$. And we know that $\delta_h(\underline{x}) = \frac{1}{h^2}\varphi(x/h)$. So, we have

$$\begin{aligned}
\sum_{x\in g_h} f(x) h^d &= \sum_{x\in g_l} \left(\sum_{k=1}^N F_k \delta_h\left(x - X_k\right) \Delta q \right) h^d \\
&= \sum_{x\in g_h} \sum_{k=1}^N F_k \varphi\left(\frac{x - X_k}{h}\right) \frac{\Delta q}{h^d} h^d \\
&= \Delta q \sum_{k=1}^N F_k \sum_i \varphi\left(\frac{x}{h} - i\right) \\
&= \sum_{k=1}^N F_k \Delta q,
\end{aligned}$$

where the last equality is because $\sum_i \varphi(r - i) = 1, \forall r$ and we can sum φ in each dimension in turn.

Also for the torque, we have

$$\begin{aligned}
\sum_{x\in g_h} x \times f(x) h^d &= \sum_{x\in g_h} x \times \left(\sum_{k=1}^N F_k \delta_h\left(x - X_k\right) \right) \Delta q \cdot h^d \\
&= \sum_{x\in g_h} \sum_{k=1}^N x \times F_k \varphi\left(\frac{x - X_k}{h}\right) \Delta q \\
&= \sum_{k=1}^N \left(\sum_{x\in g_h} x \varphi\left(\frac{x - X_k}{h}\right) \right) \times F_k \Delta q \\
&= \sum_{k=1}^N \left(\sum_{x\in g_h} X_k \varphi\left(\frac{x - X_k}{h}\right) + (\frac{x - X_k}{h}) \varphi\left(\frac{x - X_k}{h}\right) h \right) \times F_k \Delta q \\
&= \sum_{k=1}^N \left(X_k \times F_k\right) \Delta q,
\end{aligned}$$

where the last equality is because we know $\sum_i \varphi(r-i) = 1, \sum_i (r-i)\varphi(r-i) = 0, \forall r$.