### FUNDAMENTAL GROUPS OF LOG CALABI-YAU SURFACES

#### CÉCILE GACHET, ZHINING LIU, AND JOAQUÍN MORAGA

ABSTRACT. In this article, we study the orbifold fundamental group  $\pi_1^{\rm reg}(X,\Delta)$  of a log Calabi–Yau pair  $(X,\Delta)$ . We conjecture that the orbifold fundamental group  $\pi_1^{\rm orb}(X,\Delta)$  of a n-dimensional log Calabi–Yau pair admits a normal solvable subgroup of rank at most 2n and index at most c(n). We prove this conjecture in the case that n=2. More precisely, for a log Calabi–Yau surface pair  $(X,\Delta)$  we show that  $\pi_1^{\rm orb}(X,\Delta)$  is the extension of a nilpotent group of length 2 and rank at most 4 by a finite group of order at most 7200. In the previous setting, we show that the group  $\pi_1^{\rm orb}(X,\Delta)$  may not be virtually abelian. Further, the rank and the order stated above are optimal. Finally, we show some geometric features of log Calabi–Yau surfaces  $(X,\Delta)$  for which  $\pi_1^{\rm orb}(X,\Delta)$  is infinite.

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#### 1. Introduction

1.1. **Motivation.** The topology of complex algebraic varieties is a long-standing topic with numerous ramifications; from the classical Riemann uniformization theorem (see [34] for a historical survey) to a plethora of more recent results and open questions. There are some recent notable results about the most central notions of fundamental groups and universal covers of varieties. To just cite a few striking results in varied situations, in chronological order of publication, see [3, 17, 60, 78, 4, 18, 25, 61, 26, 1, 7, 30, 29, 19, 36, 21, 8, 33, 23, 6, 24, 49, 5, 28, 14, 2, 32, 31]. One of the simplest ideas governing that realm of questions is that the fundamental group of a complex algebraic variety should be easiest to control when the variety's curvature is the most positive. For instance, any smooth Fano variety is simply connected [59, 16, 64], the fundamental group of a smooth Calabi–Yau variety<sup>1</sup> is virtually abelian [50, 9], and the fundamental groups of smooth canonically polarized varieties are still not quite fully understood (see, e.g., [4] for partial results).

Allowing singularities in the picture unleashes many more unruly topological phenomena (see, for example, [77, Theorem 12.1] and [57, Theorems 1 and 2]). However, the few classes of singularities appearing in the

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<sup>&</sup>lt;sup>1</sup>Here, we refer to a variety as Calabi–Yau if it has a numerically trivial canonical divisor.

minimal model program [66, Definition 2.34] exhibit more reasonable topological behaviors, at least locally. For example, Braun shows in [14] that Kawamata log terminal (klt) singularities have finite local fundamental groups. Log canonical (lc) singularities have virtually solvable local fundamental groups in dimension 2, but can have larger local fundamental groups in dimension 3, and even have any free group as local fundamental group in dimension 4 by [40]. There are global counterparts to these local results: In [14], it also is proven that the fundamental group of the smooth locus of a klt Fano variety is finite, whereas [23] shows that the fundamental group of the smooth locus of a klt Calabi–Yau surface is virtually abelian. In higher dimensions, the study of fundamental groups of the smooth locus of klt Calabi–Yau varieties is yet to be completed (for some partial results, see [48, 38, 55, 22]).

Note that the previously cited papers [14, 23] have definitive results about klt Fano and Calabi–Yau surface pairs. This more familiar notion has appeared in the past, for example in [58, Theorem 1.4], where it is shown that the orbifold fundamental group of a plt Fano surface pair is virtually cyclic.

1.2. Main results. In this paper, we propose a detailed description of the orbifold fundamental groups of log Calabi–Yau surface pairs. Our description encompasses detailed results on the orbifold fundamental groups of log canonical Fano surface pairs as well. We then establish our main result.

**Theorem 1.** Let  $(X, \Delta)$  be a log canonical Calabi–Yau surface pair. Then the orbifold fundamental group  $\pi_1^{\text{orb}}(X, \Delta)$  admits a normal subgroup that is nilpotent of length at most 2, of rank at most 4, and of index at most 7200.

Concretely, that normal subgroup can most often be taken to be abelian. In fact, we provide one example, Example 7.2, where the group  $\pi_1^{\text{orb}}(X,\Delta)$  is not virtually abelian, and show that this example is *grosso modo* the only obstruction to virtual abelianity. The constant 7200 and the rank 4 are optimal here. Note that Theorem 1 also holds for log canonical Fano surface pairs  $(X,\Delta)$  as they admit log Calabi–Yau structures  $(X,\Delta+\varepsilon\Gamma)$  with  $\varepsilon$  arbitrarily small.

**Theorem 2.** Let  $(X, \Delta)$  be a log canonical Calabi-Yau surface pair. Assume that the group  $\pi_1^{\text{orb}}(X, \Delta)$  is not virtually abelian. Then there exists a finite cover of the pair  $(X, \Delta)$  such that, after a birational transformation, the underlying surface is isomorphic to a projectivized bundle  $\mathbb{P}(\mathcal{O}_E \oplus L)$ , where E is an elliptic curve, and L an ample line bundle on E, and the boundary is the sum of two disjoint sections.

In the case that we consider klt surfaces, we obtain an effective version of a theorem due to Campana and Claudon [23].

**Theorem 3.** Let  $(X, \Delta)$  be a klt Calabi–Yau surface pair with standard coefficients. Then, the fundamental  $\pi_1^{\text{orb}}(X, \Delta)$  admits a normal abelian subgroup of rank at most 4 and of index at most 3840.

Example 7.2 and Example 7.3 show that Theorem 3 fails if we either drop the klt condition or the standard coefficients condition. The rank in the previous theorem is optimal. Indeed, the fundamental group of an abelian surface is isomorphic to  $\mathbb{Z}^4$ . We emphasize that Theorem 3 is independent of the previous theorems (see Proposition 4.1). Finally, we characterize log Calabi–Yau surfaces whose orbifold fundamental groups are not finite.

**Theorem 4.** Let  $(X, \Delta)$  be a log Calabi–Yau surface. If  $\pi_1^{\text{orb}}(X, \Delta)$  is infinite, then one of the following conditions is satisfied:

- (i) there is a birational transformation  $(\tilde{X}, \tilde{\Delta})$  of  $(X, \Delta)$  that, up to a finite cover, admits a fibration onto an elliptic curve, or
- (ii) there is a birational transformation  $(\tilde{X}, \tilde{\Delta})$  of  $(X, \Delta)$  that, up to a finite cover, admits a  $\mathbb{G}_m$ -action.

Note that a log Calabi–Yau surface pair  $(X, \Delta)$  whose underlying surface satisfies the geometric assumption (i) or (ii) does not necessarily have infinite orbifold fundamental group  $\pi_1^{\text{orb}}(X, \Delta)$ . The previous theorem states that whenever  $\pi_1^{\text{orb}}(X, \Delta)$  is infinite, it is due to the existence of a complexification of a product of circles.

1.3. **Residually finite groups.** We present the following theorem regarding fundamental groups of dlt Fano varieties.

**Theorem 5.** Let n be a positive integer. There exists a constant J(n), only depending on n, satisfying the following. Let  $(X, \Delta)$  be a n-dimensional dlt Fano pair. If the group  $\pi_1^{\text{orb}}(X, \Delta)$  is residually finite, then it admits an abelian normal subgroup of index at most J(n).

The proof of Theorem 5 is given in Section 6 and is independent of all the other results in the article. Theorem 5 is very useful, as it reduces the task of proving that  $\pi_1^{\text{orb}}(X,\Delta)$  is effectively virtually abelian to showing that it is residually finite.

Conversely, it follows a posteriori from Theorem 1 that for any log Calabi–Yau surface pair  $(X, \Delta)$ , the group  $\pi_1^{\mathrm{orb}}(X, \Delta)$  is residually finite. In particular, it embeds in its profinite completion. Thus, we have the following corollary.

Corollary 1. Let  $(X, \Delta)$  be a log Calabi–Yau pair of dimension at most 2. Then, the group  $\pi_1^{\text{orb}}(X, \Delta)$  is residually finite. In particular, the group homomorphism given by the profinite completion

$$\pi_1^{\mathrm{orb}}(X,\Delta) \to \pi_1^{\mathrm{alg}}(X,\Delta)$$

is injective.

Some examples of non-residually finite fundamental groups of smooth projective varieties are given in [78]. This corollary furthers the folklore expectation that positivity of  $K_X$  relates to a large fundamental group  $\pi_1(X)$ , whereas positivity of  $-K_X$  relates to a small enough fundamental group  $\pi_1(X)$ . In [61], Kollár shows that a smooth projective variety X with large algebraic fundamental group  $\pi_1^{\text{alg}}(X)$  satisfies that  $2K_X$  is linearly equivalent to an effective divisor. In the previous statement, by a large algebraic fundamental group, we mean that the image of  $\pi_1(Z) \to \pi_1(X)$  has infinite image for any non-constant morphism  $Z \to X$ . On the contrary, whenever  $-K_X$  is positive, for instance globally generated, we expect that  $\pi_1(X)$  is residually finite. This expectation is consolidated in a particular case by Corollary 1.

1.4. Some conjectures. Finally, we introduce two conjectures that encompass our expectations for higher-dimensional pairs. The first predicts that the fundamental group of n-dimensional lc Fano pairs satisfy the Jordan property.

**Conjecture 1.** Let n be a positive integer. There exists a constant c(n) satisfying the following. Let  $(X, \Delta)$  be a log canonical Fano pair. Then, there is a short exact sequence

$$1 \to A \to \pi_1^{\mathrm{orb}}(X, \Delta) \to N \to 1,$$

where A is an abelian group of rank at most n and N is a finite group of order at most c(n).

The second conjecture predicts that the fundamental group of a n-dimensional log canonical log Calabi–Yau pair is solvable of rank at most 2n. This conjecture is motivated by [20, Conjecture 13.10.(2)] and [71, Conjecture 4.46].

Conjecture 2. Let n be a positive integer. There is a constant k(n) satisfying the following. Let  $(X, \Delta)$  be an n-dimensional lc log Calabi-Yau pair. Then, there is a short exact sequence

$$1 \to S \to \pi_1^{\mathrm{orb}}(X, \Delta) \to N \to 1,$$

where S is a solvable group of rank at most 2n and N is a finite group of order at most k(n).

In the case that  $(X, \Delta)$  is a klt Calabi–Yau pair, it is expected that  $\pi_1^{\text{orb}}(X, \Delta)$  is virtually abelian. In Example 7.1, we show that  $\pi_1^{\text{orb}}(X, \Delta)$  may not be virtually abelian if  $(X, \Delta)$  has some non–klt, log canonical singularities.

1.5. **Sketch of the proof.** In this subsection, we show the ideas that lead to the proof of Theorem 1. The statement of the theorem is preserved under surjective homomorphisms. In Lemma 2.20 and Lemma 2.21, we show that if we take a dlt modification of  $(X, \Delta)$  and run a  $K_X$ -MMP, then it suffices to prove the statement for the log Calabi-Yau pair induced on the outcome of the MMP. Thus, we may replace X with the surface obtained by the MMP and  $\Delta$  with the push-forward on this surface.

Case 1: The MMP terminates with a klt Calabi–Yau variety X.

This case is covered by Proposition 4.1, which is proved more generally for a klt Calabi–Yau pair  $(X, \Delta)$  with standard coefficients, using [23], the classification of smooth Calabi–Yau varieties, and the description of their automorphism groups due to Fujiki [44] and Mukai [73]. This is carried out in Subsection 4.1.

<u>Case 2:</u> The MMP terminates with a Mori fiber space to a curve C.

This is carried out in Section 3. In this case, Nori's trick (see Lemma 2.36) yields:

$$\pi_1(F, \Delta|_F) \to \pi_1^{\text{orb}}(X, \Delta) \to \pi_1(C, \Delta_C) \to 1,$$

where F is the general fiber and  $(C, \Delta_C)$  is the log pair induced on the base of the fibration (see Definition 2.33). In Definition 2.38, we introduce a trichotomy for curve pairs  $(C, \Delta_C)$  with  $\deg(K_C + \Delta_C) \leq 0$ . This trichotomy depends on the abelianization morphism of  $(C, \Delta_C)$ , i.e., the smallest cover  $(C', \Gamma') \to (C, \Delta_C)$  for which  $\pi_1(C', \Gamma')$  is abelian. We say that  $(C, \Delta_C)$  is of elliptic type (resp. toric type, sporadic type) if C' is an elliptic curve (resp.  $(C', \Gamma')$  is a toric pair,  $\pi_1(C', \Gamma')$  is trivial). We first study  $\pi_1^{\text{orb}}(X, \Delta)$  under the assumption that the base  $(C, \Delta_C)$  is of elliptic type. In this case, the group  $\pi_1(C, \Delta_C)$  is the largest possible, i.e., contains a copy of  $\mathbb{Z}^2$ . Then, we study  $\pi_1^{\text{orb}}(X, \Delta)$  under the assumption that the group  $(F, \Delta|_F)$  is of elliptic type. Finally, the case when neither of the two pairs  $(C, \Delta_C)$  or  $(F, \Delta|_F)$  is of elliptic type, is derived as an application of Lemma 2.50. This lemma is a variation of Theorem 5.

Case 3: The MMP terminates with a klt Fano surface of rank 1.

This step proceeds in three different cases depending on the coregularity of  $(X, \Delta^{\text{st}})$ . Here,  $\Delta^{\text{st}}$  is the standard approximation of  $\Delta$ . The coregularity of the pair is an invariant that measures the singularities of its complements (see Definition 2.6).

Case 3.1: The coregularity of  $(X, \Delta^{st})$  is zero.

In this case, we know that  $(X, \Delta^{\rm st})$  admits a 2-complement  $(X, \Gamma)$ , i.e., the pair  $(X, \Gamma)$  is lc,  $2(K_X + \Gamma) \sim 0$ , and  $\Gamma \geq \Delta^{\rm st}$  (see e.g., [39]). By Lemma 2.19, it suffices to show the statement for  $(X, \Gamma)$ . We take the index one cover of  $K_X + \Gamma$  and a dlt modification. By doing so, we may assume that  $(X, \Gamma)$  is dlt and  $K_X + \Gamma \sim 0$ . Hence, X is a Gorenstein canonical surface. Proceeding as before, we run a  $K_X$ -MMP. If it terminates with a Mori fiber space to a curve, then we are in the setting of Case 2. We may assume that the MMP terminates with a Gorenstein del Pezzo of rank one. Thus, it suffices to prove the statement for dlt pairs  $(X, \Delta)$  where  $K_X + \Delta \sim 0$  and X is a Gorenstein del Pezzo of rank 1. These surfaces have been classified by Miyanishi and Zhang. The anti-canonical systems of these surfaces have been studied extensively (see, e.g. [80]). For instance, all these surfaces can be transformed birationally into  $\Sigma_2$ . This transformation is quite explicit (see, e.g., [68]). Using the toric fibration of  $\Sigma_2$  we conclude by mimicking the second case.

<u>Case 3.2:</u> The coregularity of  $(X, \Delta^{st})$  is one.

If  $\Delta^{\text{st}} = \Delta$ , then we take the index one cover of  $K_X + \Delta$  and a dlt modification, leading to a dlt pair of index one. Afterward, we proceed similarly to Case 3.1.

Now we can assume that  $\Delta^{\rm st} \neq \Delta$ . We can reduce to the situation in which  $(X, \Delta^{\rm st})$  is a plt Fano surface of rank one and  $\lfloor \Delta^{\rm st} \rfloor$  has a unique prime component S. If  $\Delta^{\rm st}$  is reduced, then the statement follows from the work of Keel and McKernan [65, Theorem 1.4] and Theorem 5. Indeed, a virtually cyclic group is residually abelian. Thus, we may assume that  $\Delta^{\rm st}$  has at least two components. In particular, the sum of the coefficients of  $\Delta^{\rm st}$  is at least  $\frac{3}{2}$  so the complexity of  $(X, \Delta^{\rm st})$  is at most  $\frac{3}{2}$ . From the perspective of the complexity the log surface  $(X, \Delta^{\rm st})$  is close to being toric (see Definition 2.14 and Lemma 2.15). In Subsection 5.3, we analyze the singularities of X along S. In most cases, using the theory of complements and the complexity we show that (X, S) is a toric pair. In Subsection 5.1, we develop lemmata related to toric fibrations to prove that  $\pi_1^{\rm orb}(X, \Delta^{\rm st})$  is residually finite when (X, S) is a toric pair. There are only two configurations of singularities of X along S for which we are not able to prove that (X, S) is toric. These two special cases are treated in Subsection 5.2, where we show that  $\pi_1^{\rm orb}(X, \Delta^{\rm st})$  is residually finite nevertheless. Hence, for any log pair  $(X, \Delta)$  satisfying the assumptions of this case, the group  $\pi_1^{\rm orb}(X, \Delta^{\rm st})$  is residually finite. We conclude applying Theorem 5.

<u>Case 3.3:</u> The coregularity of  $(X, \Delta^{st})$  is two.

As in the previous case, if  $\Delta^{\text{st}} = \Delta$ , then we take the index one cover of  $K_X + \Delta$  and a dlt modification. So we can proceed as in Case 3.1.

Otherwise,  $\Delta^{\text{st}} \neq \Delta$  and we can reduce to the situation in which  $(X, \Delta^{\text{st}})$  is a klt Fano surface. By [14],  $\pi_1^{\text{orb}}(X, \Delta^{\text{st}})$  is a finite group. By taking the universal cover of  $(X, \Delta^{\text{st}})$ , we observe that  $\pi_1^{\text{orb}}(X, \Delta^{\text{st}})$  is a finite subgroup of the plane Cremona group  $\text{Bir}(\mathbb{P}^2)$ . Thus, the statement follows from the work of Dolgachev and Iskovskikh [37] (see also [79]).

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## 2. Preliminaries

In this section, we recall some concepts that will be used throughout the article and prove some preliminary results. We work over the field  $\mathbb{C}$  of complex numbers. For a group G the rank, denoted by rank(G), is the least number of generators of G. For a group G and elements  $g_1, \ldots, g_r \in G$ , we write  $\langle g_1, \ldots, g_r \rangle_n$  for the smallest normal subgroup of G containing the elements  $g_1, \ldots, g_r$ .

In Subsection 2.1, we introduce results regarding Fano and Calabi–Yau pairs. Then, in Subsections 2.2-2.8 we prove several results related to orbifold fundamental groups. Finally, in Subsection 2.9 we prove lemmas about residually finite groups.

2.1. Singularities of pairs, Fano pairs, and Calabi–Yau pairs. In this subsection, we recall several results regarding the singularities of pairs and the geometry of Fano and Calabi–Yau pairs. More precisely: the adjunction formula, dual complexes, coregularity, theory of complements, and complexity. For the basic concepts of singularities of pairs, we refer the reader to [63].

First, we recall the formula for the coefficients of the boundary divisor under adjunction. The following can be found in [76, Proposition 3.9].

**Lemma 2.1.** Let (X, S+B) be a plt pair, where S is a reduced irreducible Weil divisor. Let  $B = \sum_{j=1}^{k} b_j B_j$  be the decomposition into prime components, and define the following effective  $\mathbb{Q}$ -divisor on S:

$$B_S := \sum_{P \subset S} \left( 1 - \frac{1}{m_P} + \sum_{j=1}^k \frac{m_{j,P} b_j}{m_P} \right) P,$$

where the sum runs over all prime divisors of S,  $m_P$  denotes the orbifold index of the germ (X; P), and  $m_{j,P}$  denotes the multiplicity of  $B_j$  at P. Then we have an adjunction formula:

$$(K_X + S + B)|_S \sim K_S + B_S.$$

**Remark 2.2.** Note that in Lemma 2.1, the component S is normal due to the plt assumption, see [76, Lemma 3.6].

**Definition 2.3.** A pair  $(X, \Delta)$  is said to be  $\log \operatorname{Calabi-Yau}$  if  $(X, \Delta)$  has lc singularities and  $K_X + \Delta \sim_{\mathbb{Q}} 0$ . A pair  $(X, \Delta)$  is said to be of  $\operatorname{Fano}$  type if there exists  $\Gamma \geq \Delta$  for which  $(X, \Gamma)$  is klt and  $-(K_X + \Gamma)$  is a nef and big divisor. If  $(X, \Delta)$  is a Fano type pair, then X is a Mori dream space. Furthermore,  $(X, \Delta)$  being Fano type is equivalent to the existence of  $\Gamma \geq 0$  for which  $(X, \Delta + \Gamma)$  is klt,  $\Delta + \Gamma$  is big, and  $K_X + \Delta + \Gamma \sim_{\mathbb{Q}} 0$ .

A Fano pair is a pair  $(X, \Delta)$  for which  $-(K_X + \Delta)$  is ample. We say that a pair  $(X, \Delta)$  is lc Fano (resp. plt Fano, klt Fano) if the pair  $(X, \Delta)$  is a Fano pair and it has lc singularities (resp. plt singularities, klt singularities).

We turn to recall the concepts of dual complexes and coregularity.

**Definition 2.4.** Let E be a simple normal crossing divisor on a smooth variety X. The dual complex  $\mathcal{D}(E)$  of E is the CW complex whose k-simplices correspond to irreducible components of intersections of k+1 prime components of E.

**Definition 2.5.** Let  $(X, \Delta)$  be a log canonical pair. Let  $p: Y \to X$  be a log resolution of  $(X, \Delta)$  and  $p^*(K_X + \Delta) = K_Y + \Delta_Y$ . Let E denote the reduced sum of all components of  $\Delta_Y$  that have coefficient one. The dual complex of  $(X, \Delta)$ , denoted by  $\mathcal{D}(X, \Delta)$ , is the dual complex  $\mathcal{D}(E)$  obtained from the incidence geometry of the components of E.

By [43, Theorem 1.6], the dual complex of a log Calabi-Yau surface is an equidimensional topological manifold, possibly with boundary. It has dimension at most one by [72, Theorem 2]. Still by [43, Theorem 1.6], it is empty, or homeomorphic to [0,1],  $S^1$ ,  $\{0,1\}$ , or  $\{0\}$ .

The *coregularity* of the pair  $(X, \Delta)$  is the integer

$$\operatorname{coreg}(X, \Delta) := \dim X - \dim \mathcal{D}(X, \Delta) - 1.$$

(We use as a convention that the empty set has dimension -1). In particular, if  $(X, \Delta)$  is a log Calabi–Yau surface, the coregularity is in the set  $\{0, 1, 2\}$  and is 2 if and only if the log Calabi–Yau surface is klt.

**Definition 2.6.** Let  $(X, \Delta)$  be a log Fano pair. The *coregularity* of  $(X, \Delta)$  is the minimum of the values  $coreg(X, \Delta + B)$ , when  $(X, \Delta + B)$  ranges over all possible log Calabi-Yau pairs of that form.

We recall the definition of complement and some results regarding complements for surfaces.

**Definition 2.7.** Let X be a normal  $\mathbb{Q}$ -Gorenstein projective variety, and  $\Gamma$  be an effective  $\mathbb{Q}$ -Weil divisor on X. For an integer  $N \geq 1$  we say that the pair  $(X, \Gamma)$  is an N-complement of (X, 0) if it is log canonical, the effective divisor  $N\Gamma$  is Weil, and  $N(K_X + \Gamma) \sim 0$ .

**Definition 2.8.** Let  $(X, \Delta)$  be an lc pair. We say that it admits an N-complement for an integer  $N \geq 1$  if there is an effective  $\mathbb{Q}$ -divisor  $\Gamma \geq \Delta$  such that  $(X, \Gamma)$  is an N-complement. In the previous context, we also say that  $\Gamma$  is a N-complement of  $(X, \Delta)$ .

The following lemma is a special case of [41, Theorem 1] and [39, Theorem 2].

**Lemma 2.9.** Let  $(X, \Delta)$  be a log Calabi–Yau surface with standard coefficients. If  $\operatorname{coreg}(X, \Delta) = 1$ , then  $N(K_X + \Delta) \sim 0$  for some  $N \in \{1, 2, 3, 4, 6\}$ . If  $\operatorname{coreg}(X, \Delta) = 0$ , then  $2(K_X + \Delta) \sim 0$ . In particular, if  $(X, \Delta)$  is not klt, then its index is at most 6.

The following lemma is a consequence of Kawamata-Viehweg vanishing Theorem. It can be found for instance in [11, Proposition 6.7].

**Lemma 2.10.** Let  $(X, \Delta)$  be a plt Fano pair of dimension two. Assume that  $\Delta$  has standard coefficients. Let S be a component of  $\Delta$  with coefficient one. Let  $(S, \Delta_S)$  be the pair obtained by adjunction of  $(X, \Delta)$  to S. If  $(S, \Delta_S)$  admits an N-complement  $\Gamma_S$ , then  $(X, \Delta)$  admits an N-complement  $\Gamma$  that for which  $K_X + \Gamma|_S \sim_{\mathbb{Q}} K_S + \Gamma_S$ .

The following lemma is a special case of [39, Theorem 4 and Theorem 5].

**Lemma 2.11.** If  $(X, \Delta)$  is a log Fano surface with standard coefficients and coregularity zero, then it admits a 2-complement. If  $(X, \Delta)$  is a log Fano surface with standard coefficients and coregularity 1, then it admits an N-complement for some  $N \in \{1, 2, 3, 4, 6\}$ .

The following lemmas are related to the singularities and the MMP for dlt Calabi–Yau surfaces of index one.

**Lemma 2.12.** Let  $(X, \Delta)$  be a dlt pair of dimension two with  $K_X + \Delta \sim 0$ . Then  $\Delta$  is contained in  $X_{\text{reg}}$ .

*Proof.* Since  $K_X$  is a Weil divisor, so is  $\Delta$ . If  $\Delta$  is empty, the result is clear. If any component of  $\Delta$  has numerically trivial canonical class, then by adjunction (see Lemma 2.1), such component is contained in  $X_{\text{reg}}$ .

Using the condition  $K_X + \Delta \sim 0$ , we argue that  $\mathcal{D}(X, \Delta)$  is either a point, two points, or homeomorphic to a circle. Indeed, if  $\mathcal{D}(X, \Delta)$  is homeomorphic to an interval, then we may perform adjunction of  $(X, \Delta)$  to a rational curve C that corresponds to an endpoint of the interval. In this case, we obtain a pair  $(C, \Delta_C)$  with  $\Delta_C$  an effective reduced Weil divisor. If  $\Delta_C$  has zero or one component, then  $(C, \Delta_C)$  is not Calabi–Yau, leading to a contradiction. On the other hand, if  $\Delta_C$  has two or more components, then  $\Delta$  has two or more components of coefficient one intersecting C, which contradicts the fact that C is an endpoint of the interval  $\mathcal{D}(X, \Delta)$ .

Assume that  $\mathcal{D}(X,\Delta)$  is one or two points. Let S be a component of  $\Delta$ . We perform adjunction  $(K_X + \Delta)|_S \sim K_S + \Delta_S$  and obtain a log Calabi–Yau pair  $(S,\Delta_S)$ . Since  $K_X + \Delta \sim 0$ , we have  $K_S + \Delta_S \sim 0$ , so  $\Delta_S$  is a reduced Weil divisor. Assume by contradiction that  $\Delta_S \neq 0$ , then it contains a point p with coefficient one. By Lemma 2.1, there must be a component of  $\Delta$  other than S containing p, and by our assumption on  $\mathcal{D}(X,\Delta)$ , that component does not have coefficient one. But  $\Delta$  is a reduced Weil divisor, a contradiction. Hence,  $\Delta_S = 0$  and S is a smooth elliptic curve. So  $\Delta$  is contained in  $X_{\text{reg}}$ .

Assume that  $\mathcal{D}(X, \Delta)$  is homeomorphic to the circle. By Lemma 2.1, each component of  $\Delta$  is a rational curve. Let  $E_1, \ldots, E_r$  be the components of  $\Delta$ . If r = 1, then the rational curve  $E_1$  has a single node. The surface X is smooth away from the node by adjunction (Lemma 2.1) and at the node by the dlt condition. Otherwise,  $r \geq 2$ , and for any i, taking the indices modulo r, we have

$$(K_X + \Delta)|_{E_i} \sim K_{E_i} + E_i \cap E_{i-1} + E_i \cap E_{i+1}.$$

The surface X is thus smooth along  $E_i$  except possibly at the two points at  $E_i \cap E_{i-1}$  and  $E_i \cap E_{i+1}$ , and it is also smooth at these two points by the dlt condition.

**Lemma 2.13.** Let X be a klt surface that admits a 1-complement  $\Gamma$ . Let  $X \to X_0$  be a step of the  $K_X$ -MMP contracting an irreducible curve C to a point  $x_0$ . Then either C is a component of  $\Gamma$ , or  $\Gamma$  and C have exactly one intersection point p and at most two components of  $\Gamma$  contain p.

Furthermore, if the variety X is smooth at a point  $q \in C$ , then at most one component of  $\Gamma$  (besides possibly C itself) contains q.

*Proof.* Assume that C is not a component of  $\Gamma$ . Since  $-K_X \cdot C > 0$ , the curve C intersects  $\Gamma$  in at least one point. Let  $\Gamma_0$  be the push-forward of  $\Gamma$  in  $X_0$ . The pair  $(X_0, \Gamma_0)$  is log canonical.

Since  $X_0$  is a klt surface, it is Q-factorial, and its local class group at any point is thus finite, see [72, Remark 3.29]. By [72, Theorem 1.2], at most two components of  $\Gamma_0$  contain the point  $x_0$ , i.e., either one or two components of  $\Gamma$  intersect C.

By [43, Theorem 1.1], and since the contraction  $X \to X_0$  is birational, the intersection of C with the non-klt locus of  $(X, \Gamma)$  is connected. It consists of finitely many points, hence of exactly one point p. In particular,  $\Gamma \cap C = \{p\}$ , as wished.

Now, assume that the variety X is smooth at  $q \in C$  and that there are two components  $\Gamma_1$  and  $\Gamma_2$  of  $\Gamma$  that are different from C and contain q. Let  $\Gamma_{1,0}$  and  $\Gamma_{2,0}$  be the images of  $\Gamma_1$  and  $\Gamma_2$  in  $X_0$ . By [72, Theorem 1 and 2], the pairs  $(X, \Gamma_1 + \Gamma_2; q)$  and  $(X_0, \Gamma_{1,0} + \Gamma_{2,0}; x_0)$  are formally toric surface singularities. Let  $Y_0 \to X_0$  be a toric resolution of singularities. Write  $(Y_0, \Gamma_{Y_0,0} + \Gamma_{Y_0,1} + E_{0,1} + \cdots + E_{0,r})$  for the log pullback of  $(X_0, \Gamma_{1,0} + \Gamma_{2,0})$  to  $Y_0$ . By performing further toric blow-ups on  $Y_0$ , we may assume that the center of C on  $Y_0$  is not a toric strata. Thus, we may extract C by blowing-up  $E_{0,j}$  at a smooth point not contained in another  $E_{0,i}$ . We write  $Y \to Y_0$  for the smooth blow-up extracting C. We write  $(Y, \Gamma_{Y,0} + \Gamma_{Y,1} + E_1 + \cdots + E_r)$  for the log pullback of  $(X, \Gamma_1 + \Gamma_2)$  to Y. By construction, we have  $E_j^2 = E_{j,0}^2 - 1$  and  $E_i^2 = E_{i,0}^2$  for every  $i \neq j$ . As Y and X are smooth and  $Y \to X$  is a projective birational morphism, the variety X can be obtained from Y by performing a sequence of smooth blow-downs, i.e., contracting a sequence of (-1)-curves. By blowing down (-1)-curves simultaneously on Y and  $Y_0$  over X and  $X_0$  respectively, we arise to a model on which the strict transform of  $E_{j,0}$  has self-intersection zero. This leads to a contradiction.

To conclude this subsection, we recall the characterization of toric varieties using the complexity (see e.g., [15, 72]).

**Definition 2.14.** Let X be a Q-factorial variety and Let  $(X, \Delta)$  be a log Calabi–Yau pair. The *complexity* of  $(X, \Delta)$  is

$$c(X, \Delta) := \rho(X) + \dim X - |\Delta|,$$

where  $|\Delta|$  denotes the sum of the coefficients of  $\Delta$ .

The following is a special case of the main theorem of [15].

**Lemma 2.15.** Let  $(X, \Delta)$  be a log Calabi–Yau surface. Then, we have that  $c(X, \Delta) \geq 0$ . Furthermore, if  $c(X, \Delta) < 1$ , then  $(X, |\Delta|)$  is a toric surface.

2.2. **Orbifold fundamental groups.** In this subsection, we recall the definition of orbifold fundamental group and prove some statements about it.

**Definition 2.16.** Let X be a normal quasi-projective variety, and D be a non-trivial prime effective Weil divisor on X. We say that an analytic open subset U of X is a trimmed neighborhood of D in X if there are a Zariski open set  $U_{\text{Zar}} \subset X$  intersecting D, and an (analytic) tubular neighborhood  $U_{\text{tub}}$  for  $D \cap (X_{\text{reg}} \setminus D_{\text{sing}})$  in  $X_{\text{reg}} \setminus D_{\text{sing}}$ , such that  $U = U_{\text{Zar}} \cap U_{\text{tub}}$ . If U is a trimmed neighborhood of D, we denote by  $U^*$  the topological manifold  $U \setminus D \cap U$ . It can be thought of as a pointed neighborhood of D.

Of course, the notion of trimmed neighborhood for a fixed divisor D is stable by finite intersection and arbitrary union.

An inclusion of trimmed neighborhoods  $V \subseteq U$  induces inclusions  $V_{\text{tub}} \subseteq U_{\text{tub}}$  and  $V_{\text{Zar}} \subseteq U_{\text{Zar}}$ , thus a surjection

$$\pi_1(V^*) \simeq \pi_1(U_{\text{tub}}^* \cap V_{\text{Zar}}) \twoheadrightarrow \pi_1(U^*),$$

where we use that there is a homeomorphism between  $V_{\text{tub}}^*$  and  $U_{\text{tub}}^*$  that preserves  $V_{\text{Zar}}$  (since its complement has finitely many components, it suffices to preserve them).

**Definition 2.17.** Let X be a normal quasi-projective variety, and D be a non-trivial prime effective Weil divisor on X. We define the loop around D as the data, for any trimmed neighborhood U of D, of the class  $\gamma_D|_U$  of the positive oriented loop in the fiber of the normal circle bundle of  $D \cap U$ , in the fundamental group  $\pi_1(U^*)$ . Since the aforementioned surjection  $\pi_1(V^*) \to \pi_1(U^*)$  induced by an inclusion  $V \subset U$  sends  $\gamma_D|_V$  to  $\gamma_D|_U$ , we might omit to precise the trimmed neighborhood used and simply write  $\gamma_D$  whenever possible.

These definitions can be found in various places, but to the authors' knowledge, appear first in [27, Definition 4.4] in the global case, and [14] in the local case.

**Definition 2.18.** Let  $(X, \Delta)$  be a pair with standard coefficients. Let  $P_1, \ldots, P_r$  be the prime components of  $\Delta$ . Let  $m_i$  be the positive integer for which  $\operatorname{coeff}_{P_i}(\Delta) = 1 - \frac{1}{m_i}$ . For each  $i \in \{1, \ldots, r\}$ , we let  $\gamma_i := \gamma_{P_i}$  be a loop around  $P_i$ . The fundamental group of  $(X, \Delta)$  is

$$\pi_1(X,\Delta) := \pi_1(X \setminus \operatorname{supp}(\Delta)) / \langle \gamma_i^{m_i} \rangle_n,$$

where  $\langle \gamma_i^{m_i} \rangle_n$  is the smallest normal group containing each  $\gamma_i^{m_i}$ .

For any real number  $\alpha \in [0,1]$ , we define

$$\alpha^{\text{st}} := \max \left\{ 1 - \frac{1}{m} \le \alpha | m \in \mathbb{N} \cup \infty \right\}.$$

Let  $\Delta = \sum b_i D_i$  be the prime decomposition of  $\Delta$ . We define  $\Delta^{\text{st}} := \sum b_i^{\text{st}} D_i$ . We call  $\Delta^{\text{st}}$  the standard approximation of  $\Delta$ .

We define the *orbifold fundamental group* of  $(X, \Delta)$  to be

$$\pi_1^{\mathrm{orb}}(X,\Delta) := \pi_1(X_{\mathrm{reg}}, \Delta^{\mathrm{st}}|_{X_{\mathrm{reg}}}),$$

where  $X_{\text{reg}}$  denotes the smooth locus of X.

The regional fundamental group of  $(X, \Delta)$  at a closed point  $x \in X$ , denoted by  $\pi_1^{\text{reg}}(X, \Delta; x)$ , is the inverse limit of  $\pi_1^{\text{orb}}(U, \Delta|_U)$ , where the limit runs over all the analytic neighborhoods U of x in X.

**Lemma 2.19.** Let  $(X, \Delta)$  and  $(X, \Gamma)$  be two log pairs with standard coefficients. Assume that for every prime divisor  $P \subset X$ , the orbifold index of  $\Delta$  at P divides the orbifold index of  $\Gamma$  at p. Then, we have a surjective homomorphism  $\pi_1^{\text{orb}}(X, \Gamma) \to \pi_1^{\text{orb}}(X, \Delta)$ .

Proof. We may add components to  $\Delta$  with orbifold index 1, i.e., add components with coefficient 0. By doing so, we may assume that  $E := \operatorname{supp} \Delta = \operatorname{supp} \Gamma$ . For each prime divisor P in X we let  $n_P$  be its orbifold index in  $\Delta$  and  $m_P$  its orbifold index in  $\Gamma$ . So,  $n_P \mid m_P$  for each P. We have two surjective homomorphisms  $\pi_1(X^{\operatorname{reg}} \setminus E) \to \pi_1^{\operatorname{orb}}(X, \Delta)$  and  $\pi_1(X^{\operatorname{reg}} \setminus E) \to \pi_1^{\operatorname{orb}}(X, \Gamma)$ . For each prime divisor P, we let  $\gamma_P$  be a loop around P. The kernel of the former is  $N_1$  the smallest normal subgroup that contains  $\gamma_P^{n_P}$  for every P. Note that  $\gamma_P^{m_P}$  is a power of  $\gamma_P^{n_P}$  for every P. So  $N_1$  contains  $N_2$ . The statement follows.

**Lemma 2.20.** Let  $(X, \Delta)$  be a log Calabi-Yau surface. Let  $(Y, \Delta_Y)$  be a dlt modification of  $(X, \Delta)$ . Then, we have an isomorphism

$$\pi_1^{\mathrm{orb}}(X,\Delta) \simeq \pi_1^{\mathrm{orb}}(Y,\Delta_Y).$$

*Proof.* Let  $E \subseteq \Delta_Y$  be the reduced exceptional locus of  $p: Y \to X$ . Note that p(E) is a finite union of smooth and singular points of X. So, we have  $\pi_1^{\text{orb}}(X \setminus p(E), \Delta) \simeq \pi_1^{\text{orb}}(Y \setminus E, \Delta_Y) \simeq \pi_1^{\text{orb}}(Y, \Delta_Y)$ . The last isomorphism follows as  $E \leq \Delta_Y$ .

**Lemma 2.21.** Let  $(X, \Delta)$  be a log pair. Let  $X \to Y$  be a birational contraction and  $\Delta_Y$  be the push-forward of  $\Delta$  in Y. Then, there is a surjective homomorphism

$$\pi_1^{\operatorname{orb}}(Y, \Delta_Y) \twoheadrightarrow \pi_1^{\operatorname{orb}}(X, \Delta).$$

Proof. We may replace  $\Delta$  with its standard approximation and assume it has standard coefficients. Let B be the strict transform of  $\Delta_Y$  in X and E be the reduced exceptional of  $p\colon X\to Y$ . Note that p(E) is a finite union of smooth and singular points of Y. So, we have  $\pi_1^{\mathrm{orb}}(Y,\Delta_Y)\simeq\pi_1^{\mathrm{orb}}(X\setminus E,B)\simeq\pi_1^{\mathrm{orb}}(X,B+E)$ . Note that both B+E and  $\Delta$  have standard coefficients and for every prime divisor  $P\subset X$  the orbifold index of  $\Delta$  divides such of B+E. By Lemma 2.19, we conclude that there is a surjective homomorphism  $\pi_1^{\mathrm{orb}}(Y,\Delta_Y)\to\pi_1^{\mathrm{orb}}(X,\Delta)$ .

The following lemma can be found in [65, Lemma 7.3].

**Lemma 2.22.** Let X be a normal surface and  $p: X \to Y$  be a birational contraction. Then, there is a surjective homomorphism  $\pi_1^{\text{orb}}(Y) \twoheadrightarrow \pi_1^{\text{orb}}(X)$ . If the image of Ex(p) in Y lies in the smooth locus, then the previous homomorphism is an isomorphism.

2.3. The loop around a ramification divisor. In this subsection, we study the behavior of the loop around a ramification divisor under morphisms.

**Definition 2.23.** Let X and X' be normal projective varieties, together with a proper surjective map  $p: X' \to X$ . We say that p can pullback if Im(p) is a smooth curve, or p is finite. In this case, there is a well-defined pullback map  $p^*: Cl(Im(p)) \to Cl(X')$  at the level of Weil divisors (cf. [51, 21.10.1]).

Let D' be a non-trivial prime effective Weil divisor on X'. We say that p ramifies at D' with degree m if D := p(D') has codimension one in Im(p), and D' appear as a component of  $p^*D$  with coefficient m.

We define the ramification divisor of p as

$$Ram(p) := \sum_{(m,D) \in R} (m-1)D,$$

where R is the set of pairs (m, D) such that  $m \geq 2$  is an integer and p ramifies at D with degree m. Note that the index set R is finite.

The following result is an elementary consequence of a computation in local coordinates, and explains how ramification affects fundamental groups.

**Lemma 2.24.** Let X and Y be normal projective varieties, together with a proper map  $p: X \to Y$  of image  $Y_0$  in Y. Assume that p can pullback. Let D be an irreducible effective Weil divisor in X, along which p ramifies with degree m. Let  $D_0 = p_0(D)$  in  $Y_0$ . Then we have

$$p_{0*}(\gamma_D) = \gamma_{D_0}^{m}$$
.

*Proof.* In local coordinates, we want to pushforward the positively oriented loop  $\gamma$  generating  $\pi_1(\mathbb{C}^*)$  by the map  $f: z \in \mathbb{C}^* \mapsto z^m \in \mathbb{C}^*$ . Clearly,  $f_*(\gamma) = \gamma^m$ , as wished.

2.4. A pushforward map for orbifold fundamental groups. In this subsection, we construct a functorial pushforward map for orbifold fundamental groups of pairs.

**Definition 2.25.** Let  $p: X' \to X$  be a proper surjective map of normal projective varieties that can pullback. Let  $\Delta'$  and  $\Delta$  be effective  $\mathbb{Q}$ -divisors on X' and X, whose components appear with coefficients at most one each. We say that p is *compatible* with the pairs  $(X', \Delta')$  and  $(X, \Delta)$  if

$$p^*\Delta = \operatorname{Ram}(p) + \Delta',$$

and the following conditions are satisfied:

- (1) the divisor  $\Delta'$  is an effective  $\mathbb{Q}$ -divisor on X' whose irreducible components appear with coefficient at most one each.
- (2) for any component D' appearing in  $\Delta'$  with coefficient a > 0 and in Ram(p) with coefficient m-1 > 0, there is a component D appearing in  $\Delta$  with coefficient b > 0 such that D' appears in  $p^*D$  and we have:

$$m(1 - b^{\text{st}}) = 1 - a^{\text{st}}.$$

For simplicity, we may say that  $p: (X', \Delta') \to (X, \Delta)$  is compatible.

**Remark 2.26.** One may note that for a compatible morphism  $p:(X',\Delta')\to (X,\Delta)$ , it holds

$$p^*(\Delta^{\mathrm{st}}) = \mathrm{Ram}(p) + (\Delta')^{\mathrm{st}}.$$

**Remark 2.27.** If  $p: X' \to X$  is a finite quasi-étale cover, and  $\Delta$  is an effective  $\mathbb{Q}$ -divisor on X whose components appear with coefficient at most one each, then setting  $\Delta' := p^*\Delta$ , and we note that the map p is compatible with  $(X', \Delta')$  and  $(X, \Delta)$ .

An example of a compatible map that is not a finite cover, but a Mori fiber space onto a curve, is given in Remark 2.35.

**Construction 2.28.** Let  $(X', \Delta')$  and  $(X, \Delta)$  be pairs, and  $p: (X', \Delta') \to (X, \Delta)$  be a compatible map. We construct a pushforward group homomorphism:

$$p_{\bullet}: \pi_1^{\mathrm{orb}}(X', \Delta') \to \pi_1^{\mathrm{orb}}(X, \Delta).$$

The construction goes as follows. Let  $B := X_{\text{sing}} \cup \text{Supp}(\Delta^{\text{st}})$ . We can restrict and corestrict p to obtain a proper surjective map

$$\overline{p}: X' \setminus p^{-1}(B) \to X \setminus B.$$

We claim that  $p^{-1}(B)$  is contained in  $X'_{\text{sing}} \cup \text{Supp}(\Delta'^{\text{st}} + \text{Ram}(p)) \cup Z$ , for some Zariski closed subset Z in  $X'_{\text{reg}}$  of codimension at least two. Indeed, it is clear that  $p^{-1}(\text{Supp}\,\Delta^{\text{st}}) = \text{Supp}\,\Delta'^{\text{st}} + \text{Ram}(p)$ . If X is a smooth curve, that is enough. Otherwise,  $p: X' \to X$  is a finite cover, and since X is normal,  $p^{-1}(X_{\text{sing}})$  has codimension at least two in X'.

Therefore, we have an open immersion

$$i: X'_{\text{reg}} \setminus \text{Supp}(\Delta'^{\text{st}} + \text{Ram}(p)) \cup Z \hookrightarrow X' \setminus p^{-1}(B)$$

and we define  $p_0 := \overline{p} \circ i$ . Since Z is a Zariski closed subset of codimension at least two in  $X'_{\text{reg}}$ , it does not affect fundamental groups. So we obtain the following diagram

$$\pi_{1}(X'_{\text{reg}} \setminus \text{Supp}(\Delta'^{\text{st}} + \text{Ram}(p))) \xrightarrow{p_{0*}} \pi_{1}(X_{\text{reg}} \setminus \text{Supp}(\Delta^{\text{st}}))$$

$$\downarrow s \qquad \qquad \downarrow s$$

Here, we claim that there exists a unique group homomorphism  $p_{\bullet}$  that makes this diagram commute. Provided it exists, its unicity is clear from the fact that s' and t' are surjective.

First, we prove that  $s \circ p_{0*}$  factors through s'. Applying Van Kampen's theorem to add trimmed neighborhoods of components one by one, we see that

$$\ker(s') = \langle \gamma_R \mid R \text{ component of Supp Ram}(p) \setminus \operatorname{Supp} \Delta'^{\operatorname{st}} \rangle_n.$$

Fixing such a component R of ramification order  $m \geq 2$ , we see by Definition 2.25 that D := p(R) appears in  $\Delta^{\text{st}}$  with coefficient  $1 - \frac{1}{m}$ . So we have  $s(p_{0*}\gamma_R) = s(\gamma_D^m) = 1$ . That proves the existence of a group homomorphism  $p_{\text{int}} : \pi_1(X'_{\text{reg}} \setminus \text{Supp}(\Delta'^{\text{st}})) \to \pi_1^{\text{orb}}(X, \Delta)$  making the diagram commutative.

Second, we prove that  $p_{\rm int}$  factors through t'. Recall that

$$\ker(t') = \left\langle \gamma_{D'}^{k} \mid D' \text{ component of } \Delta'^{\text{st}} \text{ with coefficient } 1 - \frac{1}{k} \right\rangle_{n}.$$

Fixing such a pair (D', k) and denoting by  $m \ge 1$  the ramification order of p along D', we see by Definition 2.25 that D := p(D') appears in  $\Delta^{\text{st}}$  with coefficient  $1 - \frac{1}{km}$ . So we have

$$p_{\text{int}}(\gamma_{D'}{}^{k}) = s(p_{0*}\gamma_{D'}{}^{k}) = s(\gamma_{D}{}^{km}) = 1.$$

This shows that  $p_{\text{int}}$  factors through t', hence the existence of the group homomorphism  $p_{\bullet}$  as wished.

We show that our construction is, in some sense, functorial.

**Proposition 2.29.** Let  $(X'', \Delta'')$ ,  $(X', \Delta')$ , and  $(X, \Delta)$  be pairs. Let  $q: (X'', \Delta'') \to (X', \Delta')$  and  $p: (X', \Delta') \to (X, \Delta)$  be compatible maps, and assume that p is a finite cover. Then the composition  $p \circ q: (X'', \Delta'') \to (X, \Delta)$  is a compatible map.

Proof of Proposition 2.29. Clearly, it suffices to show that

$$Ram(p \circ q) = q^*Ram(p) + Ram(q).$$

Fix an irreducible effective divisor D'' in X'' and see whether it appears with the same coefficient on the left handside and on the right handside. If q(D'') = X', then D'' does not appear in  $\operatorname{Ram}(p \circ q)$  or in  $\operatorname{Ram}(q)$ . It does not appear in  $q^*\operatorname{Ram}(p)$  either, since  $q(q^*\operatorname{Ram}(p)) \neq X'$ . So D'' appears neither on the left, nor on the right handside.

Otherwise, D' := q(D'') is an irreducible effective Weil divisor in X'. Since p is a finite cover, D := p(D') is an irreducible effective divisor in X. Let  $m', m \ge 1$  be the multiplicities of D'' and D' in  $q^*D'$  and  $p^*D$ , respectively. The multiplicity of D'' in  $q^*p^*D$  is then exactly mn, and of course mn-1=m(n-1)+(m-1) as wished.

**Proposition 2.30.** Let  $(X'', \Delta'')$ ,  $(X', \Delta')$  and  $(X, \Delta)$  be pairs. Let  $q:(X'', \Delta'') \to (X', \Delta')$  and  $p:(X', \Delta') \to (X, \Delta)$  be compatible maps, and assume that  $p \circ q$  is also compatible with the corresponding pairs. Then we have  $(p \circ q)_{\bullet} = p_{\bullet} \circ q_{\bullet}$ .

Proof of Proposition 2.30. Note that the compatible map q can be restricted and corestricted into a map

$$q_{0,p}: X''_{\text{reg}} \setminus \operatorname{Supp}(\Delta''^{\text{st}} + \operatorname{Ram}(p \circ q)) \cup Z' \to X'_{\text{reg}} \setminus \operatorname{Supp}(\Delta'^{\text{st}} + \operatorname{Ram}(p))$$

where Z' is a Zariski closed subset of  $X''_{\text{reg}}$  of codimension at least 2. Clearly, the two group homomorphisms  $(p \circ q)_{0_*}$  and  $p_{0_*} \circ q_{0,p_*}$  then coincide. Factoring through the natural surjections of fundamental groups induced by the relevant open immersions, we see that  $q_{0,p_*}$  and  $q_{0_*}$  both induce the same unique group homomorphism  $q_{\bullet}$  as in Construction 2.28. The fact that  $(p \circ q)_{\bullet} = p_{\bullet} \circ q_{\bullet}$  then follows from the fact that  $(p \circ q)_{0_*} = p_{0_*} \circ q_{0,p_*}$ .

2.5. A Galois correspondence for orbifold fundamental groups. In this subsection, we prove a Galois correspondence for orbifold fundamental groups. Recall that a finite cover  $p: X' \to X$  is called *Galois* if there is a finite subgroup  $\operatorname{Gal}(p)$  of  $\operatorname{Aut}(X')$  such that p is isomorphic to the natural quotient map of  $X' \to X'/\operatorname{Gal}(p)$ . Recall also that a finite cover is called *cyclic* if it is a Galois cover with a cyclic Galois group.

**Proposition 2.31.** Let  $(X', \Delta')$  and  $(X, \Delta)$  be pairs and  $p : (X', \Delta') \to (X, \Delta)$  be a compatible map. If p is a finite Galois cover, then  $p_{\bullet}$  is injective and  $p_{\bullet}\pi_1^{\mathrm{orb}}(X', \Delta')$  is a normal subgroup of finite index in  $\pi_1^{\mathrm{orb}}(X, \Delta)$ , with quotient isomorphic to the Galois group  $\mathrm{Gal}(p)$ .

*Proof.* By Zariski's purity of the branch locus, the map  $\bar{p}$  defined at the beginning of Construction 2.28 is a finite Galois étale cover. Moreover,  $p^{-1}(X_{\text{sing}} \cup \text{Supp}\,\Delta^{\text{st}})$  coincides exactly with  $X'_{\text{sing}} \cup \text{Supp}({\Delta'}^{\text{st}} + \text{Ram}(p)) \cup Z$ , where Z is a Zariski closed subset of  $X_{\text{reg}}$  of codimension at least 2, so that the pushforward  $i_*$  by the open inclusion i is an isomorphism of groups.

Using the usual Galois correspondence for fundamental groups, the pushforward homomorphism  $\overline{p}_*$  yields an exact sequence

$$1 \to \pi_1(X' \setminus p^{-1}(B)) \xrightarrow{\overline{p}_*} \pi_1(X \setminus B) \to \operatorname{Gal}(p) \to 1.$$

Moreover, an easy computation shows that the image of  $\overline{p}_*$  contains the kernel of s. Hence, by the first isomorphism theorem and since s is surjective, the image of  $s \circ \overline{p}_*$  is a normal subgroup of  $\pi_1^{\rm orb}(X,\Delta)$  with corresponding quotient group isomorphic to  ${\rm Gal}(p)$ . Since  $i_*$  is an isomorphism, the image of  $s \circ \overline{p}_*$ . Since s' and t' are surjective, this is again the same as the image of  $p_{\bullet}$ . Hence, the image of  $p_{\bullet}$  is a normal subgroup of finite index in  $\pi_1^{\rm orb}(X,\Delta)$ , with quotient isomorphic to the Galois group  ${\rm Gal}(p)$ , as wished.

From here on, assume that  $p(X'_{\text{sing}}) \subseteq X_{\text{sing}} \cup p(\text{Ram}(p))$ , and let us show that  $p_{\bullet}$  is injective. By the usual Galois correspondence, the group homomorphism  $\bar{p}_{*}$  is injective. Since the pushforward map  $i_{*}$  is an isomorphism,  $p_{0_{*}}$  is injective too.

We now want to prove that, with the notation of Construction 2.28,

$$p_{0*} \ker(t' \circ s') = \ker(s).$$

Consider the subgroups

$$K' := \langle \gamma_Z^k \mid Z \text{ is a component of } \Delta'^{\text{st}} + \text{Ram}(p), \ 1 - k^{-1} \text{ is the coefficient of } Z \text{ in } \Delta'^{\text{st}} \rangle$$
 inside  $G' := \pi_1(X'_{\text{reg}} \setminus \text{Supp}(\Delta'^{\text{st}} + \text{Ram}(p))),$  and

$$K := \langle \gamma_D^n \mid D \text{ is a component of } \Delta^{\text{st}} \text{ with coefficient } 1 - n^{-1} \rangle,$$

inside  $G := \pi_1(X_{\text{reg}} \setminus \text{Supp}\,\Delta^{\text{st}})$ . Let us fix a base point  $x' \in X'_{\text{reg}} \setminus \text{Supp}(\Delta'^{\text{st}} + \text{Ram}(p))$  and let x = p(x') be fixed as our base point in  $X_{\text{reg}} \setminus \text{Supp}\,\Delta^{\text{st}}$ .

We want to show that  $p_{0*}K'_n = K_n$ , where  $K_n$  and  $K'_n$  are smallest normal subgroups containing K and K' respectively. We already proved in Construction 2.28 that  $p_{0*}K' = K$ . Clearly,  $(p_{0*}K'_n)_n = (p_{0*}K')_n = K_n$ , so it suffices to show that  $p_{0*}K'_n$  is a normal subgroup of G to conclude.

Note that the action of G by conjugation on  $p_{0*}G'$  induces a group homomorphism from  $\operatorname{Gal}(p) \simeq G/p_{0*}G'$  to  $\operatorname{Aut}(G')/\operatorname{Inn}(G')$ . This action can also be more concretely viewed as follows: Fix for each  $h \in \operatorname{Gal}(p)$  a path  $c_h$  from h(x') to x'. For every  $h \in \operatorname{Gal}(p)$ , we can consider the automorphism of G' sending a loop  $\gamma'$  based at x' to another loop  $c_h h(\gamma') c_h^{-1}$  based at x'. Its class [h] in the group  $\operatorname{Aut}(G')/\operatorname{Inn}(G')$  does not depend on the choice of the path  $c_h$ , and we obtain in this way our natural group homomorphism  $\operatorname{Gal}(p) \to \operatorname{Aut}(G')/\operatorname{Inn}(G')$ . In other words,  $\operatorname{Gal}(p)$  acts on the set of G'-conjugacy classes in G'.

Now, let  $h \in \operatorname{Gal}(p)$  and  $\gamma_Z^k$  a generator of K' as in the beginning of this proof, based at x'. Then it is easy to check that h sends the conjugacy class of  $\gamma_Z^k$  to the conjugacy class of  $\gamma_{h(Z)}^k$ , still based at x', which is an element of K' as well. In particular, applying h to the normal subgroup  $K'_n$  of G', we obtain  $K'_n$  again. Hence,  $p_{0*}K'_n$  is indeed normal in G, as wished.

**Lemma 2.32.** Let  $(X, \Delta)$  be a pair, and let H be a normal subgroup of finite index in  $\pi_1^{\text{orb}}(X, \Delta)$ . Then there exists a pair  $(X', \Delta')$  and a compatible map  $p: (X', \Delta') \to (X, \Delta)$  that is a finite Galois cover, such that  $p_{\bullet}\pi_1^{\text{orb}}(X', \Delta')$  coincides with H as a subgroup of  $\pi_1^{\text{orb}}(X, \Delta)$ .

*Proof.* Let  $H_0$  be the pre-image of H in  $\pi_1(X_{\text{reg}} \setminus \text{Supp } \Delta^{\text{st}})$ . It is again a normal subgroup of finite index, and by the usual Galois correspondence, we have a finite Galois unramified cover in the analytic topology

$$p_0: U \to X_{\text{reg}} \setminus \text{Supp}(\Delta^{\text{st}})$$

such that  $p_{0*}\pi_1(U)$  coincides with  $H_0$  as a normal subgroup of  $\pi_1(X_{\text{reg}} \setminus \text{Supp}(\Delta^{\text{st}}))$ .

By [35, Theorems 3.4 and 3.5], there is an algebraic finite cover  $p: X' \to X$  extending  $p_0$ . Since X is normal, p is a finite Galois cover and  $\operatorname{Gal}(p) = \operatorname{Gal}(p_0)$ , which identifies with the quotient of  $\pi_1^{\operatorname{orb}}(X, \Delta)$  by the normal subgroup H. We set  $\Delta' := p^*\Delta - \operatorname{Ram}(p)$ , and want to show that  $(X', \Delta')$  is a pair that makes p a compatible map.

For that, it suffices to track the coefficients in  $p^*\Delta$ ,  $p^*\Delta^{\rm st}$ ,  $\Delta'$ ,  $\Delta'^{\rm st}$ , and  ${\rm Ram}(p)$  of any fixed irreducible effective divisor in X'. Fix Z such a divisor. If Z is not contained in  ${\rm Ram}(p)$ , then it appears with the same coefficient in  $\Delta'$  as in  $p^*\Delta$  and with the same coefficient in their standard approximations, respectively. That is enough. Assume now that Z appears in  ${\rm Ram}(p)$ . Let m be the ramification order of p along Z. Since  $p_0$  is étale, the irreducible effective divisor D := p(Z) appears as a component of  $\Delta^{\rm st}$  with coefficient  $b^{\rm st}$ . If  $b^{\rm st} = 1$ , then Z appears with coefficient 1 in each of the divisors  $p^*\Delta$ ,  $p^*\Delta^{\rm st}$ ,  $\Delta'$ ,  $\Delta'^{\rm st}$ , and that's enough.

Otherwise, we have  $b^{\rm st}=1-\frac{1}{n}$  for some  $n\geq 2$ . Let a the (possibly zero) coefficient of Z in  $\Delta'$ . Then clearly,  $a=mb-(m-1)\in [0,1]$ . From there on, it suffices to check that n is divisible by m to conclude the proof. Let  $\gamma_Z$  be the loop around Z, and  $\gamma_D$  be the loop around D. Note that  $\gamma_D{}^n$  belongs to the kernel of the defining surjection  $\pi_1(X_{\rm reg}\setminus {\rm Supp}\,\Delta^{\rm st}) \twoheadrightarrow \pi_1^{\rm orb}(X,\Delta)$ , and in particular to the normal subgroup  $H_0$ . By definition of the finite Galois cover  $p_0$ , we can find  $\gamma\in\pi_1(U)$  such that  $p_{0*}\gamma=\gamma_D{}^n$ . Take  $U_Z$  and  $U_D$  to be trimmed neighborhoods of Z in X' and D in X. Noting that the subgroup of  ${\rm Gal}(p)$  whose elements fix every single point of Z is cyclic of order m (because Z has codimension one), we have a commutative diagram

$$1 \longrightarrow \pi_1(U_Z \setminus Z) \longrightarrow \pi_1(U_D \setminus D) \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 1$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$1 \longrightarrow \pi_1(U) \longrightarrow \pi_1(X_{\text{reg}} \setminus \text{Supp}(\Delta^{\text{st}})) \longrightarrow \text{Gal}(p) \longrightarrow 1$$

Here  $\gamma_D$  in  $\pi_1(U_D \setminus D)$  is sent to a generator of  $\mathbb{Z}/m\mathbb{Z}$ . Hence, the order of  $\gamma_D$  in  $\mathrm{Gal}(p_0)$  equals m. We already showed that the image of  $\gamma_D^n$  is trivial in the quotient  $\mathrm{Gal}(p_0)$ , so n is divisible by m, as wished.

To conclude, we note that  $p_{\bullet}\pi_1^{\text{orb}}(X', \Delta')$  is the image of  $p_{0*}\pi_1(U) = H_0$  by the defining surjection  $\pi_1(X_{\text{reg}} \setminus \text{Supp } \Delta^{\text{st}}) \twoheadrightarrow \pi_1^{\text{orb}}(X, \Delta)$ , which is exactly the initial normal subgroup H.

2.6. Fundamental groups and fibrations. In this subsection, we recall and prove some statements regarding the fundamental groups of fibrations. First, we define a log pair structure on the base of a fibration.

**Definition 2.33.** Let X be a normal projective surface, and  $\Delta$  be an effective  $\mathbb{R}$ -divisor on X, whose irreducible reduced components each have coefficient at most 1. Let C be a smooth curve, and let  $f: X \to C$  be a surjective map with connected fibers. Assume that for every point  $p \in C$ , there are an integer  $m_p \geq 1$  and a standard rational number  $0 \leq a_p^{\text{st}} = a_p \leq 1$  (both necessarily unique) such that  $f^*p = m_p(f^*p)_{\text{red}}$  and the  $\mathbb{Q}$ -divisor  $\Delta^{\text{st}} - a_p(f^*p)_{\text{red}}$  is effective and has no common component with  $f^*p$ .

For any closed point  $p \in C$ , we set

$$\delta(p) = 1 - \frac{1 - a_p}{m_p}.$$

We define the effective divisor  $\Delta_C = \sum_{p \in C} \delta(p)p$ , and note that it has standard coefficients. We say that  $(C, \Delta_C)$  is the pair induced by the fibration f or the log pair encoding multiple fibers. In the previous setting, we say that the fibration  $f: (X, \Delta) \to (C, \Delta_C)$  is equimultiple.

**Remark 2.34.** Note that if f is a Mori fiber space, its fibers are all (possibly non-reduced and) irreducible, so the assumptions of Definition 2.33 are automatically satisfied. Thus, if  $f: X \to Y$  is a Mori fiber space,  $(X, \Delta)$  a log pair with standard coefficients, and  $(C, \Delta_C)$  is the pair induced by the fibration, then  $f: (X, \Delta) \to (C, \Delta_C)$  is an equimultiple Mori fiber space.

**Remark 2.35.** Let  $f:(X,\Delta)\to (C,\Delta_C)$  be an equimultiple fibration. We have

$$f^*\Delta_C = \Delta_{\text{vert}}^{\text{st}} + \text{Ram}(f),$$

and the map f is compatible with the pairs  $(X, \Delta^{\rm st}_{\rm vert})$  and  $(C, \Delta_C)$ . Using Construction 2.28 and Lemma 2.19, we obtain a group homomorphism  $\pi_1^{\rm orb}(X,\Delta) \to \pi_1(C,\Delta_C)$ . The inclusion of the general fiber  $i: F \hookrightarrow X$  also induces a group homomorphism  $\pi_1(F,\Delta|_F) \to \pi_1^{\rm orb}(X,\Delta)$ . These two maps play an important role in the next lemma.

A key ingredient in the proof of Proposition 3.2 is the so-called Nori's trick. To its core, it can be found in [74, Lemma 1.5.C]. The version that we use here is closer to [40, Lemma 3.13]. Let us state a variation of it that best fits our needs.

**Lemma 2.36.** Let  $f:(X,\Delta) \to (C,\Delta_C)$  be an equimultiple fibration. Fix F a general fiber of f. The following sequence is exact:

$$\pi_1(F, \Delta|_F) \to \pi_1^{\text{orb}}(X, \Delta) \to \pi_1(C, \Delta_C) \to 1.$$

Moreover, if  $(X, \Delta)$  is a Calabi-Yau pair, then  $K_C + \Delta_C$  has non-positive degree, and if  $(X, \Delta)$  is a Fano pair,  $K_C + \Delta_C$  has negative degree.

The following lemma is an important consequence of Lemma 2.36. Under particular assumptions, it describes a *large* subgroup of the orbifold fundamental group as a quotient of a local orbifold fundamental group. We will use it to prove the residual finiteness of certain orbifold fundamental groups.

**Lemma 2.37.** Let  $f:(X,\Delta) \to (C,\Delta_C)$  be an equimultiple fibration. Let  $\Delta_{hor}$  be the horizontal part of  $\Delta$  with respect to the fibration f. Let  $x \in X$  a point. Assume that for any small enough analytic open ball V containing  $f(x) \in C$ , the point x is contained in every connected component of  $\Delta_{hor} \cap f^{-1}(V)$ , except for

possibly one. Moreover, denote by  $F_x$  the fiber of f containing x. Assume that  $f_{\bullet}(\gamma_{F_x})$  generates a subgroup of finite index N in  $\pi_1(C, \Delta_C)$ . Then, the image of the group homomorphism (induced by inclusion)

$$\pi_1^{\text{reg}}(X,\Delta;x) \to \pi_1^{\text{orb}}(X,\Delta)$$

is a subgroup of finite index at most N in  $\pi_1^{\text{orb}}(X,\Delta)$ .

*Proof.* Let U be a small enough connected simply-connected analytic neighborhood of  $x \in X$  such that  $\pi_1^{\text{reg}}(X, \Delta; x) = \pi_1^{\text{orb}}(U, \Delta|_U)$ . Let  $p \in U$  be a general point and  $F_p$  be the fiber of f containing p. Let  $\Delta|_{F_p}$  be the restriction of  $\Delta$  to  $F_p$ . By Lemma 2.36, we have the following exact sequence:

$$\pi_1^{\mathrm{orb}}(F_p,\Delta|_{F_p},p) \xrightarrow{p_1} \pi_1^{\mathrm{orb}}(X,\Delta,p) \xrightarrow{f_{\bullet}} \pi_1^{\mathrm{orb}}(C,\Delta_C,f(p)) \longrightarrow 1.$$

where the points p and f(p) are precised as chosen base points for these fundamental groups. Let  $U_p$  be the analytic subset  $F_p \cap U$  in  $F_p$ . Considering the inclusions  $U_p \hookrightarrow U \hookrightarrow X$  and  $U_p \hookrightarrow F_p \hookrightarrow X$ , we get a commutative diagram:

$$\begin{split} \pi_1^{\mathrm{orb}}(U_p, \Delta|_{U_p}, p) & \longrightarrow \pi_1^{\mathrm{orb}}(U, \Delta|_U, p) \\ & \downarrow^{\pi_1} & \downarrow^{\pi_2} \\ \pi_1^{\mathrm{orb}}(F_p, \Delta|_{F_p}, p) & \stackrel{p_1}{\longrightarrow} \pi_1^{\mathrm{orb}}\left(X, \Delta, p\right) & \stackrel{f_{\bullet}}{\longrightarrow} \pi_1^{\mathrm{orb}}\left(C, \Delta_C, f(p)\right) & \longrightarrow 1. \end{split}$$

We claim that  $\pi_1$  is surjective: Indeed, the intersection  $F_p \cap \Delta_{\text{hor}}$  contains finitely many, say k, distinct points, each corresponding to a distinct local branch of  $\Delta_{\text{hor}}$ . Since  $F_p$  is a smooth rational curve, the loops  $\gamma_1, \ldots, \gamma_{k-1}, \gamma_k$  around the points of  $F_p \cap \Delta_{\text{hor}}$  satisfy

$$\langle \gamma_1, \dots, \gamma_k \mid \gamma_1 \cdots \gamma_k = 1 \rangle \twoheadrightarrow \pi_1(F_p, \Delta|_{F_p}).$$

Since the point x is contained in all local branches of  $\Delta_{\text{hor}}$  except possibly one, the open subset  $U_p$  in  $F_p$  contains all of the k points  $F_p \cap \Delta_{\text{hor}}$  except possibly one. Hence, all the loops  $\gamma_1, \ldots, \gamma_k$ , except possibly one, are in the image of  $\pi_1$ . But any k-1 of these loops already generate the whole group  $\pi_1(F_p, \Delta|_{F_p})$ . So the group homomorphism  $\pi_1$  is surjective.

Since  $\pi_1$  is surjective, the image of  $\pi_2$  contains the image of  $p_1$ , that is the kernel of  $f_{\bullet}$ . Moreover, the loop  $\gamma_{F_x}$  appears in the image of  $\pi_2$ , thus and by our finite index assumption, the subgroup  $f_{\bullet}\text{Im}(\pi_2)$  has finite index at most N in  $\pi_1^{\text{orb}}(C, \Delta_C)$ . The set of left  $\text{Im}(\pi_2)$ -cosets in  $\pi_1^{\text{orb}}(X, \Delta)$  is thus in bijection by  $f_{\bullet}$  with the set of left  $f_{\bullet}\text{Im}(\pi_2)$ -cosets in  $\pi_1^{\text{orb}}(C, \Delta_C)$ , which is finite of cardinal at most N.

2.7. Three types of Calabi–Yau pairs on curves. In this subsection, we classify pairs  $(C, \Gamma)$ , where C is a smooth curve,  $\Gamma$  has standard coefficients and the divisor  $K_C + \Gamma$  has non–positive degree, into three types, which we subsequently study.

**Definition 2.38.** Let  $(C,\Gamma)$  be a pair, where C is a smooth curve,  $\Gamma$  has standard coefficients, and the divisor  $K_C + \Gamma$  has non-positive degree. We say that  $(C,\Gamma)$  is of:

- (1) toric type if there is an N-complement  $(C, \Gamma_+)$  of  $(C, \Gamma)$  such that  $\Gamma_+$  has a point with coefficient one:
- (2) elliptic type if  $(C, \Gamma)$  is a Calabi–Yau pair and  $\Gamma$  has no point with coefficients 1;
- (3) sporadic type if  $(C,\Gamma)$  is not a Calabi–Yau pair, and it admits no N-complement  $(C,\Gamma_+)$  with  $\Gamma_+$  having a point with coefficient 1.

These three types partition all possibilities for one-dimensional log Calabi–Yau pairs with standard coefficients and non-positive degree. The purpose of this trichotomy is the succession of the following three lemmata.

**Lemma 2.39.** Let  $(C,\Gamma)$  be a pair of toric type. Then  $\pi_1(C,\Gamma)$  is a cyclic or a dihedral group. It is infinite if and only if  $(C, \Gamma)$  is a Calabi-Yau pair.

*Proof.* Clearly,  $C = \mathbb{P}^1$  and  $\Gamma$  has degree at most 2 and is supported at zero, one, two points, or at three points with twice the coefficient  $\frac{1}{2}$ .

If  $\Gamma$  is supported at two or less points,  $(C,\Gamma)$  has in fact a 1-complement of fundamental group  $\pi_1(\mathbb{P}^1 \setminus$  $\{0,\infty\}$ ) =  $\mathbb{Z}$ . By Lemma 2.19 the group  $\pi_1(C,\Gamma)$  is a quotient of that, thus cyclic.

If  $\Gamma$  is supported at three points, then  $\Gamma$  has in fact a 2-complement of fundamental group

$$\pi_1\left(\mathbb{P}^1, \frac{1}{2}\{0\} + \frac{1}{2}\{1\} + \{\infty\}\right) = \langle a, b \mid a^2 = b^2 = 1\rangle,$$

which is the infinite dihedral group (with a unique normal subgroup  $\langle ab \rangle \simeq \mathbb{Z}$  of index two). By Lemma 2.19 the group  $\pi_1(C,\Gamma)$  is a quotient of that, thus cyclic or dihedral.

**Lemma 2.40.** Let  $(C,\Gamma)$  be a pair of sporadic type. Then  $\pi_1(C,\Gamma)$  is one of the groups  $\mathfrak{A}_4,\mathfrak{S}_4,$  or  $\mathfrak{A}_5.$ 

*Proof.* Clearly,  $C = \mathbb{P}^1$ . Since  $\Gamma$  has standard coefficients and degree strictly below 2, it is supported on at most three points. Since  $(C,\Gamma)$  has no N-complement with any coefficient 1, it is not possible that  $\Gamma$  is supported at two points, or supported at three points with twice the coefficient  $\frac{1}{2}$ . So  $(C,\Gamma)=(\mathbb{P}^1,\frac{1}{2}\{0\}+1)$  $\frac{2}{3}\{1\} + \frac{n-1}{n}\{\infty\}$ ) for some  $n \geq 3$ . Since  $\Gamma$  has degree strictly below 2, we have  $n \in \{3,4,5\}$ . The three values of n yield the three possible groups, using that these three groups do each have faithful actions on  $\mathbb{P}^1$ with the right numbers and stabilizers of fixed points and using the Galois correspondence as in Proposition 2.31. 

**Lemma 2.41.** Let  $(C,\Gamma)$  be a pair of elliptic type. Then  $\pi_1(C,\Gamma)$  has a normal subgroup of index at most 6 that is isomorphic to  $\mathbb{Z}^2$ .

*Proof.* If C is an elliptic curve, it is clear. Assume now that  $C \simeq \mathbb{P}^1$ . Since  $\Gamma$  has standard coefficients strictly smaller than 1 and degree two, there are only a few possibilities. Let  $E_j$  and  $E_i$  be the elliptic curves with complex multiplication by  $j = e^{2i\pi/3}$  and  $i = e^{i\pi/2}$ .

- If  $\Gamma$  is supported at four distinct points with coefficient  $\frac{1}{2}$  each, then Proposition 2.31 applies to the double cover  $p:(E,0)\to (\mathbb{P}^1,\Gamma)$  that ramifies at those exact four points. The fact that E is an elliptic curve concludes.
- If Γ is supported at three distinct points with coefficient <sup>2</sup>/<sub>3</sub> each, then Proposition 2.31 applied to the finite cyclic cover of degree three p: (E<sub>j</sub>,0) → (P¹ ≃ E<sub>j</sub>/⟨j⟩, Γ) concludes.
  If Γ is supported at three distinct points with coefficients <sup>1</sup>/<sub>2</sub>, <sup>2</sup>/<sub>3</sub>, <sup>5</sup>/<sub>6</sub> each, then Proposition 2.31 applies to the finite cyclic cover of degree six p: (E<sub>j</sub>,0) → (P¹ ≃ E<sub>j</sub>/⟨-j⟩, Γ).
  Finally, if Γ is supported at three distinct points with coefficients <sup>1</sup>/<sub>2</sub>, <sup>3</sup>/<sub>4</sub>, <sup>3</sup>/<sub>4</sub> each, then Proposition 2.31
- applies to the finite cyclic cover of degree four  $p:(E_i,0)\to (\mathbb{P}^1\simeq E_i/\langle i\rangle,\Gamma)$ .

This concludes. 

**Definition 2.42.** Let  $(C,\Gamma)$  be a curve pair, where  $\Gamma$  has standard coefficients and  $K_C + \Gamma$  has non-positive degree. We define the abelianization map of  $(C,\Gamma)$  as the compatible finite Galois cover  $p:(C',\Gamma'_{ab})\to(C,\Gamma)$ corresponding to the maximal normal abelian subgroup of finite index in  $\pi_1(C,\Gamma)$ .

Remark 2.43. Note that there is one normal abelian subgroup of finite index that is maximal for this property in  $\pi_1(C,\Gamma)$ , and that it is unique by Lemma 2.39, Lemma 2.40, and Lemma 2.41. By the same lemmata, we note that  $\Gamma' = 0$  if we start with a pair  $(C, \Gamma)$  of sporadic type or of elliptic type. We also note that if  $(C,\Gamma)$  is of toric type, then p either is an isomorphism or a double cover, and it is a double cover if and only if  $\Gamma$  is supported at three points, with coefficient  $\frac{1}{2}$  exactly twice.

2.8. Base change by the abelianization map. In this subsection, we prove a main lemma on the Cartesian square induced by a Mori fiber space  $f:(X,\Delta)\to(C,\Delta_C)$ , and the abelianization map  $p:(C',\Delta_{C',ab})\to(C,\Delta_C)$ . Before that, we prove a simple result.

**Lemma 2.44.** Let  $f: X \to C$  be a Mori fiber space from a klt surface X to a smooth projective curve C, and let  $p: C' \to C$  be a finite cover of smooth curves. Denote by X' the normalization of  $X \times_C C'$  with its projections  $f': X' \to C'$  and  $q: X' \to X$ . Then f' is a Mori fiber space too.

Proof. It is clear that  $X' \to C'$  is a fibration with (geometrically) irreducible fibers, and that its general fiber is isomorphic to the general fiber of  $X \to C$ , which is a smooth rational curve. We want to prove that  $\rho(X') = 2$  to conclude. Fix a Cartier divisor D on X', and let us show that it is numerically equivalent to a linear combination of  $K_{X'}$  and of the fiber F of f'. Consider the Cartier divisor  $L = 2mD + (D \cdot F)mK_{X'}$  for some m large enough. Its restriction to any fiber of f' is numerically trivial, and since  $\operatorname{Pic}^0(\mathbb{P}^1)$  is trivial, its restriction to the general fiber is trivial as a line bundle.

Define V in C' as the smooth locus of f', and U as its preimage in X', and apply [53, Exercise III.12.4] to the smooth fibration  $U \to V$  to show that  $L|_U$  is a multiple of the general fiber F. Finally, note that the fibers of  $X' \setminus U \to C' \setminus V$  are irreducible, and apply the excision exact sequence [46, Proposition 1.8] to conclude that L is numerically still a multiple of F on X'.

**Lemma 2.45.** Let  $(X, \Delta)$  be a log Calabi–Yau pair of dimension two. Let  $f: (X, \Delta) \to (C, \Delta_C)$  be an equimultiple Mori fiber space onto a curve C. Let  $p: (C', \Delta_{C',ab}) \to (C, \Delta_C)$  be the abelianization map. Denote by X' the normalization of  $X \times_C C'$  with its projections  $f': X' \to C'$  and  $q: X' \to X$ . Then:

- (1) there is a divisor  $\Delta'$  on X' such that  $(X', \Delta')$  is a log Calabi–Yau pair and  $q: (X', \Delta') \to (X, \Delta)$  is a compatible finite Galois cover, and that  $q: (X', \Delta'_{\text{vert}}) \to (X, \Delta_{\text{vert}})$  is compatible too;
- (2) the fibration f' is a Mori fiber space, and the divisor  $\Delta'_{C'}$  that f' induces on C' coincides with  $\Delta_{C',ab}$ .

*Proof.* We first claim that there is an effective  $\mathbb{Q}$ -divisor  $\Delta'$  on X' such that the finite Galois cover q compatible with both  $(X', \Delta')$  and  $(X, \Delta)$ , and with  $(X', \Delta'_{\text{vert}})$  and  $(X, \Delta_{\text{vert}})$ . Those compatibilities will clearly imply that  $(X', \Delta')$  is a log Calabi–Yau pair.

To prove that such a divisor  $\Delta'$  exists, we establish the following stronger result: Any irreducible component of the branching divisor of q with branching order m is a component of  $\Delta^{\rm st}_{\rm vert}$ , with coefficient either 1 or  $1 - \frac{1}{m}$ . Fix an irreducible component B of the branching divisor of q with branching order m.

Since being a finite étale morphism is preserved by base change, f sends the generic point of B to a branching point of p, hence a point in  $\Delta_C$ . So  $B = f^{-1}(x)$  for some component x of  $\Delta_C$ . If x has coefficient 1 in  $\Delta_C$  then by Remark 2.35, B has coefficient 1 in  $\Delta_{\text{vert}}^{\text{st}}$ , as wished. Assume otherwise that x has coefficient  $1 - \frac{1}{kd}$  in  $\Delta_C$ , where k is the multiplicity of the fiber of f above x and  $1 - \frac{1}{d}$  is the coefficient of  $B = f^{-1}(x)$  in  $\Delta^{\text{st}}$ . We want to prove that d = m.

We distinguish two cases: First, we assume that x does not appear in  $p(\operatorname{Supp} \Delta_{C',\operatorname{ab}})$ . In that case, since p is compatible, it must ramify with order exactly kd above x, and that is of course divisible by k. So the fiber of f' above any  $x' \in p^{-1}(x)$  is now reduced. In particular, the non-reduced scheme  $f'^*p^*x = q^*f^*x$  has uniform multiplicity kd = km, and so d = m as wished. Second, we assume that there is a point  $x' \in C'$  that appears in  $\Delta_{C',\operatorname{ab}}$  and such that p(x') = x. That can only happen if  $\Delta_{C',\operatorname{ab}}$  is not empty, in which case the pair  $(C, \Delta_C)$  is of toric type by Remark 2.43, so that p is an isomorphism or a double cover. If p is an isomorphism, then q is an isomorphism and has no branching divisor, a contradiction. Otherwise, still by Remark 2.43, the double cover p ramifies at exactly two points that have coefficient  $\frac{1}{2}$  in  $\Delta_C$ , and x is the third and last point of  $\Delta_C$ , and p is étale in a neighborhood of x. Since being étale is preserved by base change, we then note that  $B = f^{-1}(x)$  is not contained in the branching divisor of p, a contradiction too. This discussion proves the first item in the lemma. Set  $\Delta' := p^*(\Delta) - \operatorname{Ram}(q)$ .

We just showed that  $\operatorname{Ram}(p)$  and  $\Delta_{C',ab}$  are supported on disjoint sets, that  $f'(\operatorname{Ram}(q))$  and  $\Delta_{C',ab}$  are supported on disjoint sets as well, and that  $f'(\operatorname{Ram}(f'))$  is supported in the support of  $\Delta_{C',ab}$ . We can now prove the second item. Note that f' is a Mori fiber space by Lemma 2.44. Now, it suffices to prove the equality

$$f'^*\Delta_{C',ab} = \Delta'_{vert}^{st} + Ram(f').$$

Let x' be a point in C', let  $R = f'^{-1}(x')$  be the fiber above x', and m be its multiplicity.

First assume that x' appears in  $\Delta_{C',ab}$  with a coefficient  $a^{st} > 0$ . Then p is étale at x and so  $B = q(R) = f^{-1}(p(x'))$  is a fiber of f of multiplicity m as well. Since p is compatible, the coefficient of x = p(x') in  $\Delta_C$  is  $a^{st}$  too. By Remark 2.35 for f, the coefficient of B in  $\Delta_{\text{vert}}^{st}$  is thus  $ma^{st} - m + 1$ . Again, since by assumption x' appears in  $\Delta_{C',ab}$ , the component R does not appear in the ramification divisor of q. But we proved that q is compatible, so R has the same coefficient as B, namely  $ma^{st} - m + 1$ , in  $\Delta'_{\text{vert}}^{st}$ . Finally, the coefficient of R in  $f'^*\Delta_{C',ab} - \text{Ram}(f')$  is  $ma^{st} - m + 1$  too, as wished.

Now assume that x' does not appear in  $\Delta_{C',ab}$ . Then R is in fact a reduced fiber of f', and in particular, does not appear in  $\operatorname{Ram}(f')$ . First, if the image x = p(x') is not in the support of  $\Delta_C$ , then B = q(R) is not contained in  $\Delta^{\operatorname{st}}_{\operatorname{vert}}$ , and so since q is compatible, R does not appear in  $\Delta'_{\operatorname{vert}}$ , as wished. Second, assume otherwise that x appears in  $\Delta_C$ . Since x' does not appear in  $\Delta_{C',ab}$ , the coefficient of x in  $\Delta_C$  is of the form  $1 - \frac{1}{kd}$ , where k is the multiplicity of the fiber  $B = f^{-1}(x)$  for f, where  $1 - \frac{1}{d}$  is the coefficient of B in  $\Delta^{\operatorname{st}}_{\operatorname{vert}}$ , and where kd is also the branching order of p at x. Clearly, q then branches along B with order d, and since  $q:(X',\Delta'_{\operatorname{vert}}) \to (X,\Delta^{\operatorname{st}}_{\operatorname{vert}})$  is compatible, the component R does not appear in  $\Delta'_{\operatorname{vert}}$ , which concludes the proof.

2.9. **Residually finite groups.** In this subsection, we prove a lemma regarding residually finite fundamental groups. First, we recall the following statement about finite subgroups of the plane Cremona (see, e.g., [79, Theorem 1.9]).

**Theorem 2.46.** Let G be a finite subgroup of the plane Cremona group  $Bir(\mathbb{P}^2)$ . Then G admits a normal abelian subgroup of rank at most 2 and index at most 7200.

**Remark 2.47.** This bound is in fact sharp, as the group  $(\mathfrak{A}_5 \times \mathfrak{A}_5) \rtimes \mathbb{Z}/2\mathbb{Z}$  acts faithfully on  $\mathbb{P}^1 \times \mathbb{P}^1$ , and has no proper normal abelian subgroup, see Example 7.6.

Recall that a group G is called *residually finite* if it admits a set of normal subgroups of finite index  $(H_i)_{i\in I}$  such that  $\bigcap_{i\in I} H_i = \{1\}$ . If a group G is finitely presented, it only admits finitely many normal subgroups of a given finite index, and thus countably many normal subgroups of finite index. In that case, it thus suffices to check residual finiteness for decreasing sequences of normal subgroups of finite index (indexed by  $I = \mathbb{N}$ ) in G.

We adopt the following definitions: A normal variety X is called rationally connected if for any two general points x, y in X, there is a rational curve on X passing through x and y. This property is invariant by birational equivalence, and equivalent to rationality in dimension 1 and 2. A normal variety X is called rationally chain connected if for any two general points x, y in X, there is a finite chain of rational curves  $C_1, \ldots, C_n$  on X such that  $x \in C_1, y \in C_n$ , and  $C_i \cap C_{i+1} \neq \emptyset$  for  $1 \le i \le n-1$ . Note that the notions of rational connectedness, and of rational chain connectedness coincide for complex projective varieties that are smooth, or that have klt singularities as well (see, e.g., [62, Theorem IV.3.10]).

**Lemma 2.48.** Let n be a positive integer. Then there exists a constant J(n) such that the following holds: Let  $(X, \Delta)$  be a log Calabi–Yau pair of dimension n such that:

- (1) the group  $\pi_1^{\text{orb}}(X, \Delta)$  is residually finite,
- (2) and for every compatible finite Galois cover  $p:(X',\Delta')\to (X,\Delta)$ , the normal variety X' is rationally connected.

Then the group  $\pi_1^{\text{orb}}(X,\Delta)$  admits a normal abelian subgroup of index at most J(n).

*Proof.* By [75, Theorem 1.8], we can consider the minimal constant J(n) such that any finite group Q acting on a rationally connected variety of dimension n admits a normal subgroup that is abelian of index at most J(n).

Take a log Calabi–Yau pair  $(X, \Delta)$  as in the lemma's statement, and denote the group  $\pi_1^{\operatorname{orb}}(X, \Delta)$  by G. By residual finiteness, we have  $(H_i)_{i\in I}$  a nested sequence of normal subgroups of finite index in G such that  $\bigcap_{i\in I} H_i = \{1\}$ . By Lemma 2.32, we have corresponding compatible finite Galois covers  $p_i: (Y_i, \Delta_i) \to (X, \Delta)$  such that  $\operatorname{Gal}(p_i) \simeq G/H_i$  acts on  $Y_i$ . Since  $Y_i$  is rationally connected and by definition of J(n), there exists a normal abelian subgroup  $A_i \leqslant G/H_i$  that has index at most J(n) in  $G/H_i$  Let  $G_i$  be the preimage of  $A_i$  in G: It is a normal subgroup of G of index at most J(n). Since G is a finitely presented group, there are only finitely many subgroups of G of index at most G0. Hence, there exists G1 a normal subgroup of index at most G2 a normal subgroup of index at most G3. Note that for each G4 we have an exact sequence

$$(2.1) 1 \to H_m \to G_k \to A_m \to 1,$$

where  $A_m$  is a finite abelian group. We argue that  $G_k$  is an abelian group. Let  $x, y \in G_k$ . The image of  $z := xyx^{-1}y^{-1}$  is the identity in  $A_m$  for every  $m \ge k$ , so  $z \in H_m$  for every  $m \ge k$ . Hence  $z \in \bigcap_{m \ge k} H_m = \{1\}$ , by assumption. So x and y commute. This finishes the proof.

The following lemma is standard.

**Lemma 2.49.** Let G be a finitely generated abelian group. Assume there exists a nested sequence of normal subgroups  $(H_i)_{i\in I}$  for which  $G/H_i$  has rank at most n and  $\cap_{i\in I}H_i=\{1\}$ . Then G has rank at most n.

Proof. Write  $G \simeq \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_s} \oplus \mathbb{Z}^k$  where  $m_1 \mid \cdots \mid m_s$ . Then, the rank of G is precisely s+k. Consider the subgroup  $H := \{0\} \oplus \cdots \oplus \{0\} \oplus m_{s+1}\mathbb{Z} \oplus \cdots \oplus m_{s+k}\mathbb{Z}$  of G where the  $m_i$ 's are such that  $m_1 \mid \cdots \mid m_s \mid m_{s+1} \mid \cdots \mid m_{s+k}$ . Since the subgroups  $H_i$  are nested and  $\bigcap_{i \in I} H_i = \{1\}$ , then we have  $H_i \leq H$  for some i. This implies that there is a surjective homomorphism

$$G/H_i \to \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_{s+k}}.$$

By assumption, the left hand side has rank at most n. On the other hand, the right hand side has rank s+k. We conclude that  $s+k \le n$  and so the rank of G is at most n.

The following lemma is an improvement of Lemma 2.48 for surfaces. In its proof, we use the notation of the proof of Lemma 2.48.

**Lemma 2.50.** Let  $(X, \Delta)$  be a log Calabi–Yau surface such that:

- (1) the group  $\pi_1^{\text{orb}}(X,\Delta)$  is residually finite,
- (2) for every compatible finite Galois cover  $p:(X',\Delta')\to (X,\Delta)$ , the normal variety X' is rationally connected

Then the group  $\pi_1^{\text{orb}}(X,\Delta)$  admits a normal abelian subgroup of rank at most 2 and index at most 7200.

*Proof.* In dimension 2, every rationally connected surface is rational. Hence, the constant J(2) in Lemma 2.48 can be taken as in Theorem 2.46 and equals 7200. Further, by Theorem 2.46, the groups  $A_m$  in the short exact sequence (2.1) are generated by 2 elements. Thus, by Lemma 2.49, we conclude that  $G_k$  is an abelian normal subgroup of  $\pi_1^{\text{orb}}(X, \Delta)$  of rank at most 2 and index at most 7200.

We conclude this section with a group-theoretic result, and an application to fundamental groups of log Calabi–Yau pairs admitting Mori fiber spaces.

**Lemma 2.51.** Let G be a finitely generated group. Assume that there is an exact sequence

$$1 \to K \to G \to Q \to 1$$
,

where K is residually finite and Q is virtually abelian. Then the group G is residually finite.

*Proof.* By the classification of finitely generated abelian groups, it suffices to prove the lemma for  $Q = \mathbb{Z}^r$  for some  $r \geq 0$ . If r = 0, there is nothing to prove. If we can prove the lemma for r = 1, then we deduce it for any r by induction, using the result for r = 1 to prove that the kernel of a surjective morphism  $G \to Q = \mathbb{Z}^r \to \mathbb{Z}^{r-1}$  remains residually finite.

From here on,  $Q = \mathbb{Z}$ . Let  $a \in G$  be a pre-image of  $1 \in \mathbb{Z}$ . Let  $(K_n)_{n \in \mathbb{N}}$  be a nested sequence of normal subgroups of finite index in K, with trivial intersection. Consider the subgroups  $H_n := \langle a^{n!}, K_n \rangle$  in G. They form a decreasing sequence of subgroups of G, and clearly have trivial intersection. Now,  $H_n$  is a subgroup of finite index in  $\langle a^{n!}, K \rangle$ , which is itself a normal subgroup of finite index in G. This concludes the proof.  $\square$ 

Corollary 2.52. Let  $(X, \Delta)$  be a log Calabi–Yau surface pair with a Mori fiber space onto a curve. Then the group  $\pi_1^{\text{orb}}(X, \Delta)$  is residually finite.

*Proof.* By Lemma 2.36, we have an exact sequence

$$\pi_1(F, \Delta_F) \to \pi_1^{\mathrm{orb}}(X, \Delta) \to \pi_1(C, \Delta_C) \to 1.$$

By Lemma 2.39, Lemma 2.40, and Lemma 2.41, we conclude that  $\pi_1(C, \Delta_C)$  is a virtually abelian group. On the other hand, by the same lemmata, we conclude that  $\pi_1(F, \Delta|_F)$  is a finitely generated virtually abelian group. Thus, the group  $\pi_1(F, \Delta|_F)$  is residually finite. Lemma 2.51 then implies that  $\pi_1^{\text{orb}}(X, \Delta)$  is a residually finite group.

### 3. Log Calabi-Yau surfaces admitting Mori fiber spaces to curves

In this section, we prove the main theorem of the paper under the assumption that the log Calabi–Yau surface admits a Mori fiber space to a curve.

**Definition 3.1.** We define the discrete Heisenberg-style groups for  $k \in \mathbb{Z}_{>0}$  as

$$H_k := \langle a, b, c \mid [a, b] = [a, c] = 1, [b, c] = a^k \rangle.$$

They are nilpotent groups of length two. Note that  $H_k$  is virtually abelian if and only if k = 0. Further, we have an exact sequence

$$1 \to \mathbb{Z} \to H_k \to \mathbb{Z}^2 \to 1$$
,

where  $\mathbb{Z}$  is generated by a and  $\mathbb{Z}^2$  is generated by the images of b and c. In particular,  $H_k$  is a metabelian group of rank at most 3.

**Proposition 3.2.** Let X be a surface with klt singularities, and let  $(X, \Delta)$  be a log Calabi–Yau surface that is a Mori fiber space onto a curve. Then the group  $\pi_1^{\mathrm{orb}}(X, \Delta)$  admits a normal subgroup of index at most 7200 that is abelian of rank at most 4, or a quotient of the nilpotent group  $H_k$  for some  $k \geq 1$ .

In Example 7.1, it is explained how the group  $H_k$  naturally appears in the set-up of the proposition.

Throughout this section, we denote by  $(X, \Delta)$  a log Calabi-Yau surface and by  $f: (X, \Delta) \to (C, \Delta_C)$  an equimultiple Mori fiber space onto a smooth curve. We denote by F the general fiber of f. We consider the

commutative diagram induced by the abelianization of the base

$$\begin{array}{c|c} (X', \Delta') & \xrightarrow{q} & (X, \Delta) \\ f' \downarrow & & \downarrow f \\ (C', \Delta'_{C'}) & \xrightarrow{p} & (C, \Delta_C) \end{array}$$

as introduced and analyzed in Definition 2.42 and Lemma 2.45. The general fiber of f' is isomorphic to a general fiber of f, so we denote it by F. The proof then proceeds with a careful study of the possible cases for the log pairs induced on the base and general fibers.

3.1. The base is of elliptic type. In this subsection, the pair  $(C, \Delta_C)$  is assumed to be of elliptic type, as in Lemma 2.41. In particular,  $\Delta'_{C'} = 0$  and so f' has no multiple fibers. Since f' is a Mori fiber space, by [62, Theorem II.2.8] and by Tsen's theorem, the surface X' is smooth and of the form  $\mathbb{P}_{C'}(V)$  for some vector bundle V of rank 2 on C'.

This first lemma is a consequence of standard facts on the Mori cone of surfaces of the form  $\mathbb{P}_{C'}(V)$ .

**Lemma 3.3.** In the previous set-up, exactly one of the two following possibilities occur:

- (1) The Mori cone of X' satisfies  $\overline{\text{NE}}(X') = \mathbb{R}_{\geq 0}[-K_{X'}] + \mathbb{R}_{\geq 0}[F]$ . The components of  $\Delta'$  are all proportional to  $[-K_{X'}]$ , in particular, they are disjoint.
- (2) The Mori cone of X' satisfies  $\overline{\text{NE}}(X') = \mathbb{R}_{\geq 0}[B] + \mathbb{R}_{\geq 0}[F]$  for some irreducible curve B such that  $B \cdot F = 1$  and  $e := B^2 < 0$ . The curve B appears as a component of  $\Delta'$  with coefficient 1, and the remaining components of  $\Delta'$  are all proportional to [B + eF], in particular, they do not intersect B.

*Proof.* Recall that X' has Picard rank 2. By [10, Proposition III.18], we have  $-K_{X'}^2 = 0$ ,  $-K_{X'} \cdot F = 2$ , and of course  $F^2 = 0$ .

Let e be the integer and B be the section defined by [53, Proposition V.2.8]. Since C' is a smooth elliptic curve, by [53, Theorems V.2.12 and V.2.15], we have  $e \ge -1$ , and  $B^2 = -e$ , and we can write  $-K_{X'} \equiv 2B + eF$ . If e = -1, then [53, Proposition V.2.21] shows that the nef cone of X' is spanned by F and  $2B + eF = -K_{X'}$ , as wished. If e = 0, then  $-K_{X'} = 2B$ , and again spans the nef cone of X' together with F. The consequences on the components of  $\Delta'$  are clear from the fact that  $[\Delta'] = [-K_{X'}]$  generates an extremal ray of  $\overline{\text{NE}}(X')$  and that  $(-K_{X'})^2 = 0$ .

Assume now that  $e \ge 1$ . Then [53, Proposition V.2.20] yields the promised description of  $\overline{\text{NE}}(X')$ . It also shows that a class [aB+bF] can represent a reduced and irreducible curve if and only if a=1 or  $b \ge ea$ . Since  $[\Delta'] = [B] + [B+eF]$ , this shows that B appears in  $\Delta'$  with coefficient 1 and all remaining components of  $\Delta'$  are proportional to [B+eF], as wished.

Corollary 3.4. The divisor  $\Delta'$  is purely f'-horizontal.

*Proof.* By Lemma 3.3, the fiber F does not appear in the list of possible components of  $\Delta'$ .

**Lemma 3.5.** Assume that  $\Delta'|_F$  has no component of coefficient 1. Then the pair  $(X', \Delta')$  is klt. Further, the components of  $\Delta'$  are pairwise disjoint smooth elliptic curves.

Proof. By Corollary 3.4, the divisor  $\Delta'$  has no component of coefficient 1. Hence, by Lemma 3.3, we are in a situation where  $-K_{X'}$  is pseudoeffective, and the components of  $\Delta'$  are pairwise disjoint and proportional to the anticanonical class. Moreover, each component D' of  $\Delta'$  is a smooth elliptic curve since it has arithmetic genus  $p_a(D') = 1$  (from the proportionality to  $[-K_{X'}]$ ) and geometric genus  $p_g(D') \geq 1$  (from the finite surjective map to C'). This proves that  $\Delta'$  is an snc divisor on the smooth surface X'. In particular, the pair  $(X', \Delta')$  has klt singularities.

**Lemma 3.6.** Assume that  $(F, \Delta'|_F)$  is of elliptic type. Then the group  $\pi_1^{\text{orb}}(X, \Delta)$  admits a normal abelian subgroup of rank at most 4 and of index at most 3840.

*Proof.* Since  $(F, \Delta'|_F)$  is of elliptic type, the divisor  $\Delta'|_F$  has degree 2 and standard coefficients strictly smaller than 1. By Corollary 3.4, this means that  $\Delta'$  has standard coefficients strictly smaller than 1 too. Back on X, the same clearly holds for  $\Delta$ . By Lemma 3.5 and [66, Proposition 5.20(4)], the pair  $(X, \Delta)$  is a klt Calabi–Yau pair with standard coefficients, and so Theorem 3 concludes.

**Lemma 3.7.** Assume that the pair  $(F, \Delta'^{\text{st}}|_F)$  is of sporadic type. Then the group  $\pi_1^{\text{orb}}(X, \Delta)$  admits a normal abelian subgroup of rank 2 and of index at most 360.

*Proof.* By the classification of pairs of sporadic type in Lemma 2.40, we note that the pair  $(F, \Delta'^{\text{st}}|_F)$  is isomorphic to  $(\mathbb{P}^1, \frac{1}{2}\{0\} + \frac{2}{3}\{1\} + \frac{n-1}{n}\{\infty\})$  for  $n \in \{3,4,5\}$ .

Assume first that the divisor  $\Delta'^{\text{st}}$  has exactly three components  $s_0, s_1, s_\infty$ , which are sections of the Mori

Assume first that the divisor  $\Delta'^{\text{st}}$  has exactly three components  $s_0, s_1, s_\infty$ , which are sections of the Mori fiber space f'. These sections are pairwise disjoint by Lemma 3.5, so the surface X' (with the fibration f') is isomorphic to  $\mathbb{P}^1 \times C'$  as a C'-scheme. Via this isomorphism, the divisor  $\Delta'^{\text{st}}$  on X' identifies with  $(\frac{1}{2}\{0\}+\frac{2}{3}\{1\}+\frac{n-1}{n}\{\infty\})\times C'$  for the appropriate integer n. Finally, considering the action of  $\operatorname{Gal}(q)$  on X' and the diagonal action of  $\operatorname{Gal}(p)$  on  $\mathbb{P}^1 \times C'$ , they both preserve  $\Delta'$ , hence make this isomorphism equivariant. Quotienting yields an identification

$$(X, \Delta^{\mathrm{st}}) \simeq (F, \Delta^{\mathrm{st}}|_F) \times (C, \Delta_C).$$

This concludes the proof in this case.

Assume now that n=3 and that  $\Delta'^{\text{st}}=\frac{1}{2}s_0+\frac{2}{3}b$ , where  $s_0$  is a section of f' and b is a bisection of f'. These two curves are disjoint by Lemma 3.5, and numerically equivalent to  $-\frac{1}{2}K_{X'}$  and  $-K_{X'}$  respectively. So the linear system  $|-K_{X'}|$  is a basepoint-free pencil on X', yielding an elliptic fibration  $\phi:(X',\Delta')\to (\mathbb{P}^1,\frac{3}{4}\{0\}+\frac{2}{3}\{\infty\})$  that is equimultiple in the sense of Definition 2.33. By Lemma 2.36, the group  $\pi_1^{\text{orb}}(X',\Delta')$  is thus a quotient of  $\mathbb{Z}^2$ . It is a normal subgroup of index at most 6 in  $\pi_1^{\text{orb}}(X,\Delta)$ , which concludes.

**Lemma 3.8.** Assume that  $(F, \Delta'^{\text{st}}|_F)$  is of toric type. Then the group  $\pi_1^{\text{orb}}(X, \Delta)$  admits a normal subgroup of index at most 864 that is isomorphic to a quotient of a discrete Heisenberg-style group  $H_k$  for some  $k \geq 0$ .

Proof. By the classification in Lemma 2.39, the coefficients of the components of  $\Delta_{\text{hor}}^{\text{st}}$  can be increased to obtain a divisor  $\Gamma \geq \Delta^{\text{st}}$  such that  $\Gamma_{\text{vert}} = \Delta_{\text{vert}}$  and the pair  $(F, \Gamma|_F)$  is of the form  $(\mathbb{P}^1, \{0\} + \{\infty\})$ ,  $(\mathbb{P}^1, \frac{1}{2}\{0\} + \frac{1}{2}\{1\} + \frac{1}{2}\{\infty\})$ , or  $(\mathbb{P}^1, \frac{1}{2}\{0\} + \frac{1}{2}\{1\} + \{\infty\})$ . The usual total order coincides with the divisibility order on the set  $\{1, 2, \infty\}$  of orbifold indices. Hence, by Lemma 2.19, we are left proving that  $\pi_1^{\text{orb}}(X, \Gamma)$  admits a normal subgroup of index at most 864 that is isomorphic to a quotient of a discrete Heisenberg–style group. We denote by  $\Gamma'$  the corresponding divisor on X', which still satisfies  $K_{X'} + \Gamma' = q^*(K_X + \Gamma)$ .

Since C' is an elliptic curve, the higher homotopy group  $\pi_2(C')$  is trivial, which yields, together with Lemma 2.36, a commutative diagram

$$1 \longrightarrow \pi_{1}(F, \Gamma'|_{F}) \longrightarrow \pi_{1}^{\text{orb}}(X', \Gamma') \longrightarrow \pi_{1}(C') \longrightarrow 1$$

$$\simeq \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$1 \longrightarrow \pi_{1}(F, \Gamma|_{F}) \longrightarrow \pi_{1}^{\text{orb}}(X, \Gamma) \longrightarrow \pi_{1}(C, \Delta_{C}) \longrightarrow 1$$

Note that  $\pi_1(F,\Gamma|_F)$  is isomorphic to  $\mathbb{Z}$ , to  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , or to the infinite dihedral group  $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , whereas  $\pi_1(C,\Delta_C)$  is isomorphic to  $\mathbb{Z}^2 \times \mathbb{Z}/d\mathbb{Z}$  for  $d \in \{1,2,3,4,6\}$ . From here on, we conclude with a group theoretic argument explained in the next three lemmata: Lemma 3.9, Lemma 3.10, and Lemma 3.11.

**Lemma 3.9.** Consider an exact sequence of groups

$$1 \to \mathbb{Z} \to G \to Q \to 1$$
,

and assume that Q contains a normal subgroup of finite index N isomorphic to  $\mathbb{Z}^2$ . Then G contains a normal subgroup of index 4N that is isomorphic to a discrete Heisenberg group  $H_k$  for some  $k \geq 0$ .

Proof. Consider the subgroup H of G that is the pre-image of the normal subgroup  $(2\mathbb{Z})^2$  of Q by the surjection  $G \to Q$ . Clearly, the subgroup H is normal and has index 4N in G. Moreover, the action of H by conjugation on the normal subgroup  $\mathbb{Z}$  of G is trivial. Indeed,  $\mathbb{Z}$  is abelian itself, every element of H is of the form  $zg^2$  for  $z \in \mathbb{Z}$  and  $g \in G$ , and the action of G by conjugation yields a homomorphism  $G \to \operatorname{Aut}(\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ . Hence, we have a presentation of H as

$$\langle a, b, c \mid [a, b] = [a, c] = 1, [b, c] = a^k \rangle$$

for some  $k \in \mathbb{Z}_{\geq 0}$ . This concludes.

**Lemma 3.10.** Consider an exact sequence of groups

$$1 \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to G \to Q \to 1$$
,

and assume that Q contains a normal subgroup of finite index N isomorphic to  $\mathbb{Z}^2$ . Then G contains a normal subgroup of index 144N that is abelian of rank 4.

*Proof.* Consider the subgroups K and H of G that are the pre-images of the normal subgroups  $(6\mathbb{Z})^2$  and  $(12\mathbb{Z})^2$  of Q by the surjection  $G \to Q$ . Clearly, the subgroups K and H are normal in G, we have  $H \leq K$ , and H has index 144N in G. Note that the action of K by conjugation on  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  is trivial: Indeed, every element of K is of the form  $zg^6$  for  $z \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  and  $g \in G$ , and the order of the group  $\operatorname{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$  is G. So we have a presentation of G as

$$\langle s, t, b, c \mid s^2 = t^2 = [s, t] = [b, s] = [b, t] = [c, s] = [c, t] = 1, [b, c] = s \rangle,$$

and we see that H is isomorphic to the subgroup generated by  $s, t, b^2, c^2$  in K. It is now easy to check that H is abelian of rank 4, as wished.

Lemma 3.11. Consider an exact sequence of groups

$$1 \to \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} \to G \to Q \to 1,$$

and assume that Q contains a normal subgroup of finite index N isomorphic to  $\mathbb{Z}^2$ . Then G contains a normal subgroup of index 8N that is isomorphic to a discrete Heisenberg group  $H_k$  for some  $k \geq 0$ .

*Proof.* Since  $\mathbb{Z}$  is a characteristic subgroup in  $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , its isomorphic image in G is a normal subgroup of G. Hence, we have an exact sequence

$$1 \to \mathbb{Z} \to G \to R \to 1$$
,

where the quotient R itself lies in an exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to R \to Q \to 1.$$

Consider the subgroup S of R that is the pre-image of the normal subgroup  $(2\mathbb{Z})^2$  of Q. It is easy to check that S is isomorphic to  $\langle s,b,c \mid s^2=1, [b,s]=[c,s]=[b,c]=1 \rangle \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^2$ . Now, consider the subgroup H of G that is the pre-image of the characteristic subgroup  $\mathbb{Z}^2$  of S, viewed as a normal subgroup of R. It is easy to check (as in the proof of Lemma 3.9) that H is a discrete Heisenberg–style group. It also has index SN, as wished.

We conclude with a summary of this subsection.

**Proposition 3.12.** Let X be a klt surface. Let  $(X, \Delta)$  be a log Calabi-Yau pair. Assume that there is a Mori fiber space  $f:(X, \Delta) \to (C, \Delta_C)$  such that the pair  $(C, \Delta_C)$  is of elliptic type. Then  $\pi_1^{\text{orb}}(X, \Delta)$  admits a normal subgroup of index at most 3840 that is abelian of rank at most 4, or a quotient of a discrete Heisenberg-style group  $H_k$ .

*Proof.* The statement follows from Lemma 3.6, Lemma 3.7, or Lemma 3.8 depending on the type of the pair  $(F, \Delta^{\text{st}}|_F)$ .

3.2. The fiber is of elliptic type. In this subsection, we prove the following result.

**Proposition 3.13.** Let X be a klt surface. Let  $(X, \Delta)$  be a log Calabi-Yau pair. Assume that there is an equimultiple Mori fiber space  $f:(X, \Delta) \to (C, \Delta_C)$  with general fiber F such that the pair  $(F, \Delta|_F)$  is of elliptic type. Then either  $\pi_1^{\text{orb}}(X, \Delta)$  admits a normal subgroup of index at most 7200 that is abelian of rank at most 4, or it is the quotient of a discrete Heisenberg-style group  $H_k$ .

We first prove that result under the stronger assumption that the pair  $(X, \Delta)$  is dlt.

**Lemma 3.14.** Let  $(X, \Delta)$  be a log Calabi–Yau pair of dimension two. Assume that any log canonical center of  $(X, \Delta)$  lies in the support of  $\lfloor \Delta \rfloor$ . Assume that there is an equimultiple Mori fiber space  $f: (X, \Delta) \to (C, \Delta_C)$  with general fiber F such that the pair  $(F, \Delta|_F)$  is of elliptic type. Then either  $\pi_1^{\text{orb}}(X, \Delta)$  admits a normal subgroup of index at most 7200 that is abelian of rank at most 4, or it is the quotient of a discrete Heisenberg-style group  $H_k$ .

*Proof.* Note that if every compatible finite Galois cover of  $(X, \Delta)$  is rationally connected, Corollary 2.52 and Lemma 2.48 conclude. Let us assume now that  $q:(Y, \Delta_Y) \to (X, \Delta)$  is a compatible finite Galois cover such that the surface Y is not rationally connected, and prove that the group  $\pi_1^{\text{orb}}(X, \Delta)$  nevertheless admits a normal subgroup of index at most 7200, that is abelian of rank at most 4, or a quotient of a discrete Heisenberg-style group  $H_k$ .

Since  $q:(Y,0) \to (X,\operatorname{Branch}(q))$  is still a compatible finite Galois cover, and since the pair  $(X,\operatorname{Branch}(q))$  has no log canonical center, i.e., is klt, the surface Y has klt singularities by [66, Proposition 5.20(4)].

As a first case, assume that the klt surface Y is not uniruled. Then, for a minimal resolution  $\tilde{Y}$  of Y, the canonical class  $K_{\tilde{Y}}$  is pseudoeffective by [13, Corollary 0.3]. Yet, by the log Calabi–Yau condition on the pair  $(Y, \Delta_Y)$ , the anticanonical classes  $-K_Y$ , and a fortiori  $-K_{\tilde{Y}}$ , are numerically effective. We conclude that  $K_{\tilde{Y}}$  is numerically trivial, and so is  $K_Y$ . In particular, we see that  $\Delta = \operatorname{Branch}(q)$  has standard coefficients, and that the pair  $(X, \Delta)$  is thus a klt Calabi–Yau pair with standard coefficients. Theorem 3 concludes in this case.

As a second case, assume that the klt surface Y is klt and uniruled, yet not rationally connected. We have an MRC fibration  $\phi: Y \to B$ , where B is a smooth, non-rational curve (see, e.g. [64]). By the canonical bundle formula for the log canonical Calabi–Yau pair  $(Y, \Delta_Y)$ , the curve B is elliptic, the divisor  $\Delta_Y^{\text{st}}$  has no vertical component, and  $\phi$  has no multiple fiber, so  $\phi: (Y, \Delta_Y) \to (B, 0)$  is an equimultiple fibration. The action of Gal(q) on Y descends equivariantly to an action on B, with quotient map  $p: B \to C_0$ . We thus obtain a compatible, equimultiple fibration  $f: (X, \Delta) \to (C_0, \text{Branch}(p))$  whose general fiber is a smooth rational curve. Since  $X \to C$  is a Mori fiber space, the variety X has Picard rank 2, and so the fibration f is a Mori fiber space. The pair  $(C_0, \text{Branch}(p))$  is of elliptic type, so Proposition 3.12 concludes.

We now prove Proposition 3.13 in full generality using some birational geometry.

Proof of Proposition 3.13. We apply [12, Corollary 1.4.3] to a finite set  $\mathcal{F}$  of divisorial valuations of  $(X, \Delta)$  of discrepancy -1. We choose  $\mathcal{F}$  so that it contains a valuation above each fiber  $f^{-1}(p_i)$  of f that does not have coefficient 1 in  $\Delta$ , yet such that  $f^{-1}(p_i)$  contains a log canonical center of the pair  $(X, \Delta)$ . It yields a birational morphism  $\mu: Z \to X$  with an exceptional divisor  $\sum_{1 \le i \le k} E_i$  such that  $f(\mu(E_i)) = p_i$  are distinct

points. Denoting by  $\Delta_{Z,0}$  the strict transform of  $\Delta$  by  $\mu$ , we have  $K_Z + \Delta_{Z,0} + E \equiv 0$ , and denoting by  $F_i$  the strict transform of  $f^{-1}(p_i)$  by  $\mu$ , we note that the  $F_i$  are disjoint, have negative square, and negative intersection with  $K_Z + \Delta_{Z,0}$ . By the cone theorem, we can contract them all by a birational morphism  $\mu': (Z, \Delta_{Z,0}) \to (\tilde{X}, \mu'_* \Delta_{Z,0})$  over C.

Setting  $\tilde{\Delta} := \mu'_* \Delta_{Z,0} + \sum_{1 \leq i \leq k} \mu'_* E_i$ , we obtain an equimultiple Mori fiber space  $\tilde{f} : (\tilde{X}, \tilde{\Delta}) \to (C, \tilde{\Delta}_C)$ . By construction, any log canonical center of  $(\tilde{X}, \tilde{\Delta})$  is now contained in  $\lfloor \tilde{\Delta} \rfloor$ . Furthermore, the general fiber of  $(\tilde{X}, \tilde{\Delta}) \to C$  is of elliptic type. By Lemma 3.14, we conclude that either  $\pi_1^{\text{orb}}(\tilde{X}, \tilde{\Delta})$  admits a normal subgroup of index at most 7200 that is abelian of rank at most 4, or it is a quotient of a discrete Heisenberg-style group  $H_k$ . Note that  $\pi_1^{\text{orb}}(\tilde{X}, \tilde{\Delta})$  surjects onto  $\pi_1^{\text{orb}}(X, \Delta)$  by Lemma 2.20. Hence, the statement holds for the group  $\pi_1^{\text{orb}}(X, \Delta)$ .

## 3.3. The general case. In this subsection, we prove Proposition 3.2.

Proof of Proposition 3.2. Let  $(X, \Delta)$  be a log Calabi–Yau surface pair, where X is a surface with klt singularities that admits an equimultiple Mori fiber space  $f: (X, \Delta) \to (C, \Delta_C)$  onto a curve C. By Corollary 2.52, the group  $\pi_1^{\text{orb}}(X, \Delta)$  is residually finite. In particular, if every compatible finite Galois cover of  $(X, \Delta)$  is rationally connected, Lemma 2.50 concludes.

Assume that there exists a compatible finite Galois cover  $q:(Y,\Delta_Y)\to (X,\Delta)$  such that Y is not rationally connected, and let us prove the proposition anyways. Denote by  $\phi:Y\to \tilde{C}$  the fibration and by  $p:\tilde{C}\to C$  the finite Galois cover obtained by the Stein factorization of  $f\circ q$ . Considering the ramification of p, it is easy to check that there exists an effective divisor  $\tilde{\Delta}$  with standard coefficients such that  $p:(\tilde{C},\tilde{\Delta})\to (C,\Delta)$  is a compatible finite Galois cover. In particular, if  $\tilde{C}$  is an elliptic curve, the pair  $(C,\Delta_C)$  is of elliptic type. Proposition 3.12 shows that  $\pi_1^{\text{orb}}(X,\Delta)$  admits a normal subgroup of index at most 3840 that is abelian of rank at most 4 or a quotient of a discrete Heisenberg–style group, as wished.

Assume that  $\hat{C}$  is a smooth rational curve. If the general fiber of  $\phi$  is a rational curve too, we can apply [47, Corollary 1.3] to show that Y is rationally connected, a contradiction. Hence, the general fiber of  $\phi$  is a smooth elliptic curve, which means that the fiber pair  $(F, \Delta|_F)$  on the initial variety X is of elliptic type. Proposition 3.13 then concludes.

## 4. Log canonical Calabi-Yau surfaces with standard coefficients

In this section, we study the fundamental group of lc Calabi–Yau surfaces. In the first subsection, we deal with the log terminal case, and in the second subsection, we deal with the non-klt case. The main result of this section is the following:

4.1. Log terminal Calabi-Yau surface pairs with standard coefficients. In this section, we study the fundamental group  $\pi_1^{\text{orb}}(X,\Delta)$  of klt Calabi-Yau surfaces with standard coefficients. The following statement implies Theorem 3.

**Proposition 4.1.** Let  $(X, \Delta)$  be a klt Calabi–Yau surface with  $\Delta^{\text{st}} = \Delta$ . Then  $\pi_1^{\text{orb}}(X, \Delta)$  admits a normal abelian subgroup of rank at most 4 and index at most 3840.

*Proof.* By [31, Lemma 2.7] we know that for any projective klt pair (Y, D) with standard coefficients, there exists an open  $Y^0 \subset Y$  such that  $\operatorname{codim}_{Y \setminus Y^0}(Y) \geq 3$  and  $(Y^0, D|_{Y^0})$  is an orbifold. In particular, the klt pair  $(X, \Delta)$  is in fact a Calabi–Yau orbifold. Hence [23, Theorem 4.2] implies the following exact sequence

$$1 \to A' \to \pi_1^{\mathrm{orb}}(X, \Delta) \to G' \to 1,$$

where A' is an abelian group of rank at most 4 and G' is a finite group.

By Lemma 2.32, we have a finite orbifold cover  $\pi: (X', \Delta') \to (X, \Delta)$  corresponding to the normal group A. The covering  $\pi$  is Galois, has Galois group  $\operatorname{Gal}(\pi) = G$  and  $(X', \Delta')/G \cong (X, \Delta)$ . Since  $\pi$  is an orbifold covering, we have  $\pi^*(K_X + \Delta) = K_{X'} + \Delta' \sim_{\mathbb{Q}} 0$ .

Suppose that  $m \in \mathbb{N}$  is the smallest natural number such that  $m(K_{X'} + \Delta') \sim \mathcal{O}_{X'}$ . The statement [31, Proposition 2.5] shows that there exists an orbifold covering  $X'' \to X'$  such that  $\mathcal{O}_{X''}(K_{X''}) \cong \mathcal{O}_{X''}$ . Let s be an isomorphism  $\mathcal{O}_{X'} \cong \mathcal{O}_{X'}(m(K_{X'} + \Delta'))$ . Then the proof of [31, Proposition 2.5] in fact shows that  $X'' \to X'$  is the ramified  $\mathbb{Z}/m\mathbb{Z}$ -cover determined by s.

Thus we get a Galois covering of orbifolds  $(X'',0) \to (X,\Delta)$ . Let H be its Galois group. Then we have a short exact sequence

$$1 \to \mathbb{Z}/m\mathbb{Z} \to H \to G' \to 1.$$

Take the minimal resolution  $r: X''' \to X''$ . The statement [66, Theorem 4.5] implies that X''' is Calabi-Yau. Consider  $H \ni h: X \to X$ . Then  $h \circ r: X''' \to X''$  is another resolution. Hence we have a unique automorphism  $\tilde{h}: X''' \to X'''$  such that  $h \circ r = r \circ \tilde{h}$ . Thus we have that H acts on X''' by automorphisms and r is H-equivariant. Note that X''' is either a K3 surface or an abelian surface.

As X'' (being a cyclic cover) has canonical singularities, the minimal resolution has rational curves as exceptional divisors. If X''' is an abelian surface, the resolution r has no exceptional divisor. Hence  $X''' \simeq X''$ . Let  $T \subset H$  be the subgroup of H acting on X'' by translation. Then T is a normal subgroup of H. We thus get another abelian surface S := X''/T, such that  $S \to (X, \Delta)$  is a Galois orbifold covering with Galois group isomorphic to H/T. The group action  $H/T \curvearrowright S$  fixes the neutral element of S, hence it is an automorphism group of the Lie group S. Now [44, Lemma 3.2 and 3.3] implies  $|H/T| \le 96$ . Set G := H/T and we reach the following short exact sequence

$$1 \to \mathbb{Z}^4 \to \pi_1^{\mathrm{orb}}(X, \Delta) \to G \to 1.$$

When X''' is a K3 surface, [67, Theorem 1] implies that  $|H| \leq 3840$ .

4.2. Non-klt Calabi–Yau surface pairs with standard coefficients. In this section, we study the fundamental group of non-klt log Calabi–Yau surfaces  $(X, \Delta)$  with  $\Delta^{\text{st}} = \Delta$ . The main result of this section, worth comparing to the main result of Section 4.1, is the following.

**Proposition 4.2.** Let  $(X, \Delta)$  be a non-klt log Calabi-Yau surface with  $\Delta^{\text{st}} = \Delta$ . Then there is a normal subgroup of  $\pi_1^{\text{orb}}(X, \Delta)$  of index at most 7200, that is nilpotent of length at most 2 and of rank at most 3.

We first describe what happens under the additional assumptions that  $\Delta$  is not empty, that  $(X, \Delta)$  is a 1-complement, and that the group  $\pi_1(X_{\text{reg}})$  is trivial. Lemma 4.3 and Lemma 4.4 deal with rather particular cases of low Picard number.

**Lemma 4.3.** Let  $(X, \Delta)$  be a dlt pair of dimension 2 such that  $K_X + \Delta \sim 0$  and  $\Delta \neq 0$ . Assume that the fundamental group  $\pi_1^{\text{orb}}(X)$  is trivial, and that X has Picard number one. Then the group  $\pi_1^{\text{orb}}(X, \Delta)$  is abelian of rank at most 2.

**Lemma 4.4.** Let  $(X, \Delta)$  be a dlt pair of dimension 2 such that  $K_X + \Delta \sim 0$  and  $\Delta \neq 0$ . Assume that the fundamental group  $\pi_1^{\mathrm{orb}}(X)$  is trivial, and that there is a Mori fiber space  $f: X \to C$ , where C is a curve. Then the group  $\pi_1^{\mathrm{orb}}(X, \Delta)$  is abelian of rank at most 2.

Let us first prove these two lemmas.

Proof of Lemma 4.3. By Lemma 2.12, we know that  $\Delta$  is contained in the smooth locus of X. By assumption, the surface X is a normal Gorenstein Fano surface with quotient singularities, with  $\rho(X) = 1$ , and with  $\pi_1(X_{\text{reg}}) = \{1\}$ . By the work of Miyanishi-Zhang (see, e.g., [68, Lemma 6, Table I]), the singularities of

X are one of the following:  $A_1, A_1 + A_2, A_4, D_5, E_6, E_7$ , and  $E_8$ . By [80, Theorem 1.2], the singularities in this list entirely determine X, except in the case of  $E_8$ , which can appear on exactly two non-isomorphic Gorenstein del Pezzo surfaces of Picard number one. By [68, Lemma 6, Table I] again, we have that either  $-K_X$  generates  $\operatorname{Pic}(X) \simeq \mathbb{Z}$ , or X is isomorphic to  $\mathbb{P}(1,1,2)$ . By Lemma 2.12, every component of  $\Delta$  is in the smooth locus of X, hence a Cartier divisor. So either  $\Delta$  has one component, or it has two components and  $X \simeq \mathbb{P}(1,1,2)$ .

If  $\Delta$  has one component, then by adjunction (Lemma 2.1) and by [43, Theorem 1.6],  $\Delta$  is either a smooth elliptic curve or a rational curve with a single node. If  $\Delta$  has two components and  $X \simeq \mathbb{P}(1,1,2)$ , then both components belong to the linear system  $|\mathcal{O}_{\mathbb{P}}(2)|$  they are smooth rational curves and they intersect at two smooth points of X.

Let  $\psi \colon V \to X$  be a minimal resolution of X, with reduced exceptional divisor  $E_V$ . Let  $\Delta_V$  be the strict transform of  $\Delta$  by  $\psi$ , and note that  $E_V$  and  $\Delta_V$  are disjoint. By Lemma 2.20, we have an isomorphism:

$$\pi_1^{\mathrm{orb}}(X,\Delta) \simeq \pi_1^{\mathrm{orb}}(V,\Delta_V + E_V)$$

Note that the pair  $(V, \Delta_V + E_V)$  is dlt, and that the pair  $(V, \Delta_V)$  is a 1-complement. By [68, Lemma 3], there is a  $K_V$ -MMP that terminates with the second Hirzebruch surface  $\Sigma_2$  and its Mori fiber space structure  $f_2 \colon \Sigma_2 \to \mathbb{P}^1$ . By [80, Appendix, Figure 1-6], the image  $E_2$  of  $E_V$  in  $\Sigma_2$  is contained in the union of a fiber  $F_2$  of  $f_2$  and the section S of negative square of  $f_2$ . Let  $\Delta_2$  be the image of  $\Delta$  in  $\Sigma_2$ . By Lemma 2.21, it suffices to prove that the fundamental group  $\pi_1^{\text{orb}}(\Sigma_2, \Delta_2 + E_2)$  is abelian of rank at most 2.

If we started with  $\Delta$  having two components intersecting at two smooth points and  $X \simeq \mathbb{P}(1,1,2)$ , then  $V = \Sigma_2$ ,  $E_2 = S$ , and  $\Delta_2$  has two components that are sections of  $f_2$  and intersect at exactly two points in  $X_{\text{reg}} \setminus S$ . Let p be such an intersection point, then  $\pi_1^{\text{orb}}(\Sigma_2, \Delta_2 + E_2 p) \simeq \mathbb{Z}^2$ , and by Lemma 2.37, the group  $\pi_1^{\text{orb}}(\Sigma_2, \Delta_2 + E_2)$  is abelian of rank at most two.

Otherwise,  $\Delta_2$  has only one component, which is horizontal for  $f_2$ . Since  $(\Sigma_2, \Delta_2)$  is a 1-complement, the fibration  $f_2$  restricts to a double cover  $\Delta_2 \to \mathbb{P}^1$ . By Lemma 2.13, the  $K_V$ -MMP has only been contracting curves intersecting  $\Delta_V$  transversally at a single point. So the curves  $\Delta_V$  and  $\Delta_2$  are isomorphic, and  $\Delta_2$  must be an elliptic curve or a rational curve with a single node. Take p to be either the node of  $\Delta_2$  (if it has one) or a ramification point of the double cover induced by  $f_2$  that is not contained in  $F_2$  (if  $\Delta_2$  is an elliptic curve). Then  $\pi_1^{\text{reg}}(\Sigma_2, \Delta_2 + E_2; p)$  is abelian of rank at most two, and by Lemma 2.37, the group  $\pi_1^{\text{orb}}(\Sigma_2, \Delta_2 + E_2)$  is abelian of rank at most two.

Proof of Lemma 4.4. Since the general fiber F of  $f: X \to C$  is a rational curve, and since  $\Delta$  is a reduced Weil divisor, the pair  $(F, \Delta|_F)$  is isomorphic to  $(\mathbb{P}^1, \{0\} + \{\infty\})$ .

By Lemma 2.36 applied to the pair (X,0), and since by assumption  $\pi_1(X_{\text{reg}})$  is trivial, the curve C is rational, and the Mori fiber space f has at most one multiple fiber. By Lemma 2.36 applied to the pair  $(X,\Delta)$ , since  $\Delta$  is a Weil divisor, the divisor  $\Delta_C$  is supported at most two points. Moreover, the horizontal part  $\Delta_{\text{hor}}$  consists of either two sections of f or one bisection of f. By Lemma 2.12, those components are Cartier divisors, and so any multiple fiber of f has multiplicity 2.

By Lemma 2.37, if  $\Delta_C$  is supported at zero or one point, the group  $\pi_1^{\mathrm{orb}}(X,\Delta)$  is cyclic, which concludes this case. Otherwise,  $\Delta_C$  is supported at two points. The pair  $(C,\Delta_C)$  is then of the form  $(\mathbb{P}^1,\frac{1}{2}\{0\}+\{\infty\})$  or  $(\mathbb{P}^1,\{0\}+\{\infty\})$ . In either case, the fiber  $F_{\infty}$  is a component of  $\Delta$ , and by [72, Theorem 1], the fibration  $f:X\to C$  is formally toric over a neighborhood of  $\infty\in C$ . Then there is a point  $p\in F_{\infty}$  contained in a branch of  $\Delta_{\mathrm{hor}}$ , at which  $F_{\infty}$  and the branch of  $\Delta_{\mathrm{hor}}$  intersect transversally. The group  $\pi_1^{\mathrm{reg}}(X,\Delta;x)$  is abelian of rank 2, and Lemma 2.37 concludes again.

We also prove a technical lemma about running a  $K_X$ -MMP.

**Lemma 4.5.** Let  $(X, \Delta)$  be a dlt pair of dimension 2 such that  $K_X + \Delta \sim 0$  and  $\Delta \neq 0$ . Then, if we run a  $K_X$ -MMP on X, the last surface  $X_k$  that we obtain has canonical singularities and satisfies

$$\pi_1(X_{\text{reg}}) \simeq \pi_1(X_{k,\text{reg}}).$$

*Proof.* Note that X is both a klt and a Gorenstein surface. Hence, it has canonical singularities. We run a  $K_X$ -MMP: Let

$$X =: X_0 \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_2} \dots \xrightarrow{\pi_k} X_k \downarrow \phi$$

and note that  $\phi: X_k \to Y$  is a Mori fiber space. We show by induction that  $X_k$  has canonical singularities. For each  $i \ge 1$ , the map  $\pi_i$  contracts a unique curve  $C_i$  of negative square and negative intersection with  $K_{X_{i-1}}$ , so that

$$\pi_i^* K_{X_i} + a_i C_i = K_{X_{i-1}},$$

for some  $a_i > 0$ . We claim that the point  $\pi_i(C_i)$  is in the smooth locus of  $X_i$ . Since  $X_i$  is a surface, it is enough to show that it is in the terminal locus. Resolving minimally the canonical singularities of  $X_{i-1}$  that are in  $C_i$  by a proper birational map  $\varepsilon : \tilde{X}_{i-1} \to X_{i-1}$ , we see that the exceptional locus of  $\pi_i \circ \varepsilon$  coincides with the support of  $\varepsilon^*C$ . Since

$$\varepsilon^* \pi_i^* K_{X_i} + a_i \varepsilon^* C_i = K_{\tilde{X}_{i-1}},$$

and  $a_i > 0$ , this concludes that  $\pi_i(C_i)$  is in the terminal, hence smooth, locus of  $X_i$ . So  $X_i$  has canonical singularities, and by Lemma 2.22, we also have

$$\pi_1(X_{i-1,\text{reg}}) = \pi_1(X_{i,\text{reg}}).$$

This concludes our proof by induction.

The following proposition completes the full picture under the assumptions that  $\Delta$  is not empty, that  $(X, \Delta)$  is a 1-complement, and that the group  $\pi_1(X_{reg})$  is trivial.

**Proposition 4.6.** Let  $(X, \Delta)$  be a dlt pair of dimension 2 such that  $K_X + \Delta \sim 0$  and  $\Delta \neq 0$ . Assume that the group  $\pi_1(X_{\text{reg}})$  is trivial. Then  $\pi_1^{\text{orb}}(X, \Delta)$  is an abelian group of rank at most 2.

*Proof.* We run a  $K_X$ -MMP:

$$X =: X_0 \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_2} \dots \xrightarrow{\pi_k} X_k \bigvee_{V}^{\phi}$$

and note that  $\phi: X_k \to Y$  is a Mori fiber space. By Lemma 4.5,  $X_k$  has canonical singularities, and the group  $\pi_1(X_{k,\text{reg}})$  is trivial. Let  $\Delta_k$  be the push-forward of  $\Delta$  to  $X_k$ . By Lemma 2.21, we have a surjective homomorphism

$$\pi_1^{\mathrm{orb}}(X_k, \Delta_k) \twoheadrightarrow \pi_1^{\mathrm{orb}}(X, \Delta),$$

and it suffices to show that  $\pi_1^{\text{orb}}(X_k, \Delta_k)$  is an abelian subgroup of rank at most 2.

Since  $X_k$  is a canonical Gorenstein surface and by [66, Theorem 2.44], we still have that  $(X_k, \Delta_k)$  is a dlt pair and a 1-complement. Depending on whether the dimension of Y is 0 or 1, Lemma 4.3 or Lemma 4.4 concludes the proof of the proposition.

In the following lemma, we characterize the possible fundamental groups of the smooth locus of a surface X when  $(X, \Delta)$  is a dlt pair of dimension 2 such that  $K_X + \Delta \sim 0$  and  $\Delta \neq 0$ .

**Lemma 4.7.** Let  $(X, \Delta)$  be a dlt pair of dimension 2 such that  $K_X + \Delta \sim 0$  and  $\Delta \neq 0$ . Then, one of the following statements holds:

- the fundamental group  $\pi_1(X_{reg})$  is finite of order at most 9;
- the fundamental group is  $\pi_1(X_{reg})$  is isomorphic to  $\mathbb{Z}^2$ ;
- the fundamental group is  $\pi_1(X_{reg})$  an extension of  $\mathbb{Z}^2$  by  $\mathbb{Z}/2\mathbb{Z}$ .

*Proof.* We run a  $K_X$ -MMP that terminates with a Mori fiber space  $\phi: X_k \to C$ . By Lemma 4.5, the surface  $X_k$  is canonical (hence Gorenstein), and we have an isomorphism  $\pi_1(X_{kreg}) \simeq \pi_1(X_{reg})$ . Let  $\Delta_k$  be the push-forward of  $\Delta$  to  $X_k$ . Since  $X_k$  is a canonical Gorenstein surface, we still have  $K_{X_k} + \Delta_k \sim 0$ . By [66, Theorem 2.44], the pair  $(X_k, \Delta_k)$  is dlt too.

If C is a point, then  $X_k$  is a Gorenstein del Pezzo surface of Picard rank one. By [68, Lemma 6, Table I], we conclude that the group  $\pi_1^{\text{orb}}(X_k)$  is finite, of order at most 9.

Now we assume that C is a curve. By Lemma 2.36 for the pair  $(X, \Gamma := 0)$ , we have an isomorphism:

$$\pi_1(X_{k,\text{reg}}) \simeq \pi_1(C,\Gamma_C).$$

By Lemma 2.12, the boundary  $\Delta_k$  is contained in the smooth locus of  $X_k$ , in particular every component of  $\Delta_k$  is a Cartier divisor. Moreover, by adjunction on the general fiber of  $\phi$ , we note that  $\Delta_{k\text{hor}}$  consists of two sections or of a single bisection of  $\phi$ . Hence, any multiple fiber of  $\phi$  has multiplicity 2.

Hence, the divisor  $\Gamma_C$  only has components of coefficient  $\frac{1}{2}$ , and the divisor  $K_C + \Gamma_C$  has non-positive degree. There are only a few cases:

- If C is an elliptic curve, and  $\Gamma_C = 0$ , then  $\pi_1(C, \Gamma_C) \simeq \mathbb{Z}^2$ ;
- If C is a rational curve, and  $\Gamma_C$  is supported at at most three points, then  $\pi_1(C, \Gamma_C)$  is finite, of order at most 4;
- If C is a rational curve, and  $\Gamma_C$  is supported at four points, then  $\pi_1(C, \Gamma_C)$  has a normal subgroup of index 2 isomorphic to  $\mathbb{Z}^2$ .

This concludes the proof.

Proof of Proposition 4.2. We assume that  $(X, \Delta)$  is log canonical but not klt. Hence, Lemma 2.9 applies and by [66, Definition 5.19], there is an index one cyclic cover  $p: Y \to X$  of degree at most 6 such that  $p^*(K_X + \Delta) \sim \mathcal{O}_Y$ . Let  $\Delta_Y = p^*\Delta - \operatorname{Ram}(p)$ , we have  $K_Y + \Delta_Y \sim 0$ , and the pair  $(Y, \Delta_Y)$  remains log canonical. Since  $\Delta$  has standard coefficients, the divisor  $\Delta_Y$  is a reduced effective divisor.

Let G be the cyclic group acting on  $(Y, \Delta_Y)$  with quotient  $(X, \Delta)$ . By [72, Proposition 2.16], we can take a G-equivariant dlt modification  $(Y', \Delta_{Y'})$  of  $(Y, \Delta_Y)$ , with quotient  $(X', \Delta')$ . Since  $(X, \Delta)$  is not klt, the  $\mathbb{Q}$ -divisor  $\Delta'$  has a component of coefficient one, and so does  $\Delta_{Y'}$ . We have a commutative diagram as follows:

$$(Y, \Delta_Y) \stackrel{\pi_Y}{\longleftarrow} (Y', \Delta_{Y'})$$

$$/G \downarrow \qquad \qquad \downarrow /G$$

$$(X, \Delta) \stackrel{\pi}{\longleftarrow} (X', \Delta').$$

where  $\pi$  and  $\pi_Y$  are projective birational maps. Note that  $\pi_1^{\text{orb}}(Y', \Delta_{Y'})$  embeds as a normal subgroup of  $\pi_1^{\text{orb}}(X', \Delta')$  of index at most 6. Meanwhile, since  $\pi_Y$  is a dlt modification, every exceptional prime divisor of  $\pi$  appears in  $\Delta'$  (which is a reduced Weil divisor), so we have an isomorphism

$$\pi_1^{\mathrm{orb}}(X', \Delta') \simeq \pi_1^{\mathrm{orb}}(X, \Delta).$$

We proceed in two cases, depending on the rationally connectedness of finite Galois covers of Y'.

<u>Case 1:</u> We assume that there is a finite Galois cover  $p: Y'' \to (Y', \Delta_{Y'})$  with p compatible, such that Y'' is not rationally connected.

Using the orbifold Galois correspondence (see, e.g., Proposition 2.31 and Lemma 2.32), and the fact that any finite-index subgroup in a given group contains a finite-index subgroup that is normal in the whole group, we provide a finite Galois compatible cover  $q:(X'',\Delta'')\to (X',\Delta')$  that factorizes through the finite cover  $Y''\to (Y',\Delta_{Y'})\to (X',\Delta')$ . In particular, note that Y'' dominates X'', hence X'' is not rationally connected. Note that X'' is a klt surface by [66, Proposition 5.20(4)]. We run a  $\mathrm{Gal}(q)$ -equivariant  $K_{X''}$ -MMP. Since X'' is not rationally connected, this must terminate with a  $\mathrm{Gal}(q)$ -equivariant Mori fiber space to a curve (since klt Fano surfaces are rationally connected). Quotienting by  $\mathrm{Gal}(q)$  yields a  $K_{X'}$ -MMP that terminates with a Mori fiber space to a curve, and Lemma 2.21 and Proposition 3.2 conclude.

<u>Case 2:</u> We assume that for every finite Galois cover  $p:Y''\to (Y',\Delta_{Y'})$  with p compatible, we have that Y'' is rationally connected.

By Lemma 2.50, it suffices to show that  $\pi_1^{\text{orb}}(X', \Delta')$  is residually finite to conclude. In fact, it suffices to show that  $\pi_1^{\text{orb}}(Y', \Delta_{Y'})$  is residually finite.

<u>Case 2.1:</u> We first assume that  $\pi_1^{\text{orb}}(Y')$  is finite.

Note that  $(Y', \Delta_{Y'})$  is dlt and  $K_{Y'} + \Delta_{Y'} \sim 0$ . Let N be the kernel of the surjective homomorphism  $\pi_1^{\text{orb}}(Y', \Delta_{Y'}) \twoheadrightarrow \pi_1^{\text{orb}}(Y')$  given by Lemma 2.19. Let  $p: (Y'', \Delta_{Y''}) \to (Y', \Delta')$  be the finite Galois compatible cover associated to N by Lemma 2.32. Observe that the following conditions are satisfied:

- we have a surjective homomorphism  $\pi_1^{\text{orb}}(Y'', \Delta_{Y''}) \twoheadrightarrow N$  induced by the finite Galois compatible cover p;
- the pair  $(Y'', \Delta_{Y''})$  is dlt;
- we have  $K_{Y''} + \Delta_{Y''} \sim 0$ ;
- we have  $\Delta_{Y''} \neq 0$ ; and
- the fundamental group  $\pi_1^{\text{orb}}(Y'')$  is trivial.

The last statement follows by construction. Indeed, the fundamental group of  $p^{-1}(Y'_{reg})$  is trivial, so does the fundamental group of  $Y''_{reg}$ . By Lemma 4.4, we conclude that  $\pi_1^{\text{orb}}(Y'', \Delta_{Y''})$  is an abelian group of rank at most 2, and thus N is residually finite. Hence, the group  $\pi_1^{\text{orb}}(Y', \Delta_{Y'})$  is residually finite as wished.

<u>Case 2.2:</u> We assume that  $\pi_1^{\text{orb}}(Y')$  is infinite.

We run a G-equivariant  $K_{Y'}$ -MMP. It terminates with a G-equivariant Mori fiber space  $Y'' \to C$ . By Lemma 2.22, we have a surjection  $\pi_1^{\text{orb}}(Y'') \to \pi_1^{\text{orb}}(Y')$ . So, the group  $\pi_1^{\text{orb}}(Y'')$  is infinite. In particular, by [14, Theorem 2], we see that C is a curve. Quotienting by G, we obtain a  $K_{X'}$ -MMP that terminates with a Mori fiber space to a curve. We conclude by Lemma 2.21 and Proposition 3.2.

# 5. Log canonical Fano surfaces

In this section, we prove the statement of the main theorem for log Fano surfaces.

5.1. **Toric fibrations.** In this subsection, we study the fundamental group of log Fano pairs  $(X, \Delta)$  where  $(X, \lfloor \Delta \rfloor)$  is a toric pair. However, we do not assume that the components of  $\Delta \setminus \lfloor \Delta \rfloor$  are toric.

**Lemma 5.1.** Let  $(X, S_1 + S_2)$  be a toric pair with  $S_1$  and  $S_2$  reduced and  $\rho(X) = 1$ . Let C be a curve that intersects  $S_1$  transversally at a unique smooth point different from  $S_1 \cap S_2$ . Then  $\pi_1^{\text{orb}}(X, S_1 + S_2 + C)$  the group is abelian of rank at most two.

Proof. Let  $\Sigma$  be the fan corresponding to X. Let  $\Sigma(1) = \{v_1, v_2, v_3\}$  where  $v_1$  and  $v_2$  correspond to  $S_1$  and  $S_2$ , respectively. Let  $Y \to X$  be the toric blow-up corresponding to adding the ray  $-v_3$  to the fan. In other words,  $Y \to X$  is a toric blow-up supported at the point  $S_1 \cap S_2$ . Let E be the exceptional divisor,  $S_{Y,i}$  be the strict transform of  $S_i$ , and  $S_i$ , and  $S_i$  be the strict transform of  $S_i$ . From the fan of the toric surface  $S_i$ , it is clear that  $(S_{Y,i})^2 = 0$ ,  $S_{Y,1} \cdot S_{Y,2} = 0$ , and  $S_{Y,i} \cdot S_{Y,i} > 0$ . Since  $S_i = 0$ , this shows that  $S_i = 0$ , and  $S_i = 0$ , are numerically proportional, and span an extremal ray of both the nef and the Mori cone of  $S_i = 0$ . In particular, there is a Mori fiber space structure  $S_i = 0$ , and  $S_i = 0$ , we have  $S_i = 0$ , where  $S_i = 0$ , we have  $S_i = 0$ , we have  $S_i = 0$ , where  $S_i = 0$ , we have  $S_i = 0$ , we have  $S_i = 0$ , where  $S_i = 0$ , we have  $S_i = 0$ , where  $S_i = 0$ , we have  $S_i = 0$ , where  $S_i = 0$ , we have  $S_i = 0$ , where  $S_i = 0$ , we have  $S_i = 0$ , where  $S_i = 0$ , we have  $S_i = 0$ , where  $S_i = 0$ , we have  $S_i = 0$ , where  $S_$ 

By Lemma 2.20, it suffices for us to show that  $\pi_1^{\text{orb}}(Y, S_{Y,1} + S_{Y,2} + C_Y + E)$  is abelian of rank at most two. Let x be the unique point of (transversal) intersection of  $C_Y$  and  $S_{Y,1}$ , which lies in the smooth locus of Y. The local fundamental group  $\pi_1^{\text{reg}}(Y, S_{Y_1} + S_{Y,2} + C_Y + E; x)$  is isomorphic to  $\mathbb{Z}^2$ . Moreover, since  $\pi$  is a toric fibration above  $\mathbb{P}^1$ , its multiple fibers are contained in its two torus-invariant fibers, namely  $S_{Y,1}$  and  $S_{Y,2}$ . Hence, the group  $\pi_1^{\text{orb}}(B, \Delta_B)$  is cyclic and generated by the class  $\pi_{\bullet}(\gamma_{S_{Y,1}})$ . The point x hence satisfies the assumptions of Lemma 2.37, whence there is surjective group homomorphism

$$\pi_1^{\text{reg}}(Y, S_{Y_1} + S_{Y,2} + C_Y + E; x) \simeq \mathbb{Z}^2 \twoheadrightarrow \pi_1^{\text{orb}}(Y, S_{Y,1} + S_{Y,2} + C_Y + E).$$

**Lemma 5.2.** Let  $(X, S + \Delta)$  be a plt Fano pair with standard coefficients, and with S reduced irreducible. Assume that (X, S) is a toric pair,  $\rho(X) = 1$ , and either S contains exactly one singular point of X, or S contains no singular point of X and  $\Delta$  has exactly three components. Then the group  $\pi_1^{\text{orb}}(X, S + \Delta)$  is finite.

*Proof.* Since X is a toric surface and  $\rho(X) = 1$ , the corresponding fan  $\Sigma$  is generated by exactly three vectors, one of which  $v_1$  corresponds to the torus-invariant divisor S. Hence, the quasi-projective variety  $X \setminus S$  is isomorphic to  $U_{\sigma}$ , where  $\sigma$  is the only cone of the fan  $\Sigma$  that does not contain  $v_1$ . Hence,  $X \setminus S$  is an affine toric surface.

By [45, Section 2.2, Page 32-33], there is a finite toric cyclic cover from the affine plane to  $X \setminus S$ , and it ramifies above the singular locus of  $X \setminus S$ , which consists in at most one point. This toric cover corresponds to the linear endomorphism of  $\mathbb{Z}^2$  sending (1,0) to (m,-k) for some m,k coprime, and preserving (0,1). By [35, Theorem 3.4 and Theorem 3.5], we can extend it uniquely to a finite cyclic cover  $q: X' \to X$ , that ramifies at most along S and a single point of  $X \setminus S$ . In particular, X' and the map q are toric. The divisor  $q^*S - \operatorname{Ram}(q)$  is a reduced effective Weil divisor, which we can denote by S'. Note that X' is toric and its torus-invariant divisors are every component of S' and the closure of the two coordinate lines in the affine plane contained in X'. In particular, if S' has more than one component, looking at the fan of X, we see that S' has a component that does not intersect the closure of one of the coordinate lines. But S intersects both of the two other torus-invariant divisors in X, which by the projection formula is a contradiction. So S' has only one component,  $\rho(X') = 1$ , and the pair (X', S') is toric.

Note that the fan corresponding to X' is generated by the three vectors (1,0), (0,1), (a,b) in the lattice  $\mathbb{Z}^2$ , and thus the fan of X is generated by (m,-k), (0,1), (ma,b-ak). Since S contains at most one singular point of X, ma, or both b and m, must equal  $\pm 1$ . In any case, m=1 and X' contains at most one singular point, which then belongs to S'. It is now clear from its fan that X' is isomorphic to  $\mathbb{P}(1,1,n)$  for an integer n > 1.

Let  $\Delta' := q^* \Delta$ , so that  $q^*(K_X + S + \Delta) = K_{X'} + S' + \Delta'$ . By Proposition 2.31, we get an exact sequence  $\pi_1^{\text{orb}}(X', S' + \Delta') \xrightarrow{q} \pi_1^{\text{orb}}(X, S + \Delta) \rightarrow \text{Gal}(q) \rightarrow 1$ ,

and we are left to show that the group  $\pi_1^{\text{orb}}(X', S' + \Delta')$  is finite.

First, let us assume that X' is singular, i.e.,  $n \geq 2$ . Let  $r: Y \to X'$  be the minimal resolution; clearly, Y is a Hirzebruch surface with a section E of negative square -n. Note that  $r^*S' = S_Y + \frac{1}{n}E$ , where  $S_Y$  is the strict transform of S' by r, and let  $\Delta_Y$  be the strict transform of  $\Delta'$ . Since  $(X', \Delta')$  is plt, we have  $r^*(K_{X'} + S' + \Delta') = K_Y + S_Y + \Delta_Y + \left(1 + \frac{\Delta_Y \cdot E - 1}{n}\right)E$ . Since  $(X', \Delta')$  is a plt pair and  $\Delta_Y$  has standard coefficients, there is at most one component  $\Delta_1$  of  $\Delta_Y$  such that  $\Delta_1 \cdot E > 0$ . Moreover, if there is such a component, if satisfies  $\Delta_1 \cdot E = 1$  and appears with a standard coefficient strictly smaller than one in  $\Delta_Y$ , say  $1 - \frac{1}{k_1}$ . All other components  $\Delta_i$  of  $\Delta_Y$  are numerically proportional to  $E + nS_Y$ , and appear with standard coefficients strictly smaller than one (because they intersect  $S_Y$ , and the pair  $(Y, S + \Delta_Y)$  is plt), say  $1 - \frac{1}{k_2}$ . Intersecting with  $S_Y$  in the equality

$$r^*(K_{X'} + S' + \Delta') = K_Y + S_Y + \Delta_Y + \left(1 - \frac{1}{k_1 n}\right) E,$$

we obtain  $\sum_{i \in I} (1 - \frac{1}{k_i}) \Delta_i \cdot S_Y < 1 + \frac{1}{k_1 n} \leq \frac{3}{2}$ . This bounds the number of possibilities: We list them exhaustively. Recall that  $f: Y \to \mathbb{P}^1$  is the Mori fiber space of that Hirzebruch surface.

- First,  $k_1 = 1$ , i.e.,  $\Delta_1$  does not appear in  $\Delta_Y$ . Then  $\Delta_Y$  can be a single bisection with coefficient  $\frac{1}{2}$  or  $\frac{2}{3}$ , the sum of two sections with standard coefficients strictly smaller than one, or a single section with a standard coefficient strictly smaller than one.
- Second,  $k_1 \geq 2$  and  $\Delta_1 \equiv S_Y$ , and  $\Delta_Y \Delta_1$  is one of the few cases listed for  $\Delta_Y$  in the previous item.
- Third,  $k_1 \geq 2$  and  $\Delta_1 \not\equiv S_Y$ , and  $\Delta_Y$  can be a single bisection with coefficient  $\frac{1}{2}$ , the sum of two sections with standard coefficients strictly smaller than one, or a single section with a standard coefficient strictly smaller than one.

By Lemma 2.20, it suffices to show that the group  $\pi_1^{\text{orb}}(Y, S_Y + \Delta_Y + E)$  is finite to conclude. If  $\Delta_{Y,\text{hor}}$  is supported on a single section of f, then Lemma 2.36 applied to f shows that  $\pi_1^{\text{orb}}(Y, S_Y + \Delta_Y + E)$  is an extension of a finite cyclic group by another finite cyclic group, hence finite itself.

If  $\Delta_{Y,\text{hor}}$  is supported on two distinct sections of f, then they must intersect at a point x. Moreover, x is in a  $\Delta_i$  for  $i \neq 1$ , hence not in E, and x is not in  $S_Y$  either by the plt condition. So the pair  $(Y, S_Y + \Delta_Y + E)$  is locally klt at the point x, hence  $\pi_1^{\text{reg}}(Y, S_Y + \Delta_Y + E; x)$  is finite [14, Theorem 1]. By Lemma 2.37, we conclude that  $\pi_1^{\text{orb}}(Y, S_Y + \Delta_Y + E)$  is the extension of a finite group by a finite cyclic group, hence finite itself.

Otherwise,  $\Delta_{Y \text{hor}}$  is supported on a single bisection D of f. Since the base of f is  $\mathbb{P}^1$ , that bisection maps to it with at least two ramification points. Let x be a ramification point of the bisection. Assume that  $x \in S_Y$ . Then, since  $D \cdot S_Y = 2$ , D appears with coefficient at least  $\frac{1}{2}$  in  $\Delta_Y$ , and x is a smooth point of Y, blowing up Y at x contradicts the plt condition on the pair  $(Y, S_Y + \Delta_Y)$ . Hence, x is not in  $S_Y$ . Moreover, D intersects E in at most one point, so we can choose x to be not in E. In that way, the pair  $(Y, S_Y + \Delta_Y + E)$  is locally klt at x, hence  $\pi_1^{\text{reg}}(Y, S_Y + \Delta_Y + E; x)$  is finite [14, Theorem 1]. By Lemma 2.37, we conclude that  $\pi_1^{\text{orb}}(Y, S_Y + \Delta_Y + E)$  is the extension of a finite group by a finite cyclic group, hence finite itself.

Now, let us assume that X' is smooth, i.e., n=1 and (a,b)=1,  $X'\simeq \mathbb{P}^2$ , S' is a line, and S contains no singular point of X, so by assumption  $\Delta'$  has exactly three components. Since  $(X', S' + \Delta')$  is a plt pair, the components of  $\Delta'$  all have coefficients strictly smaller than one. These coefficients being standard, hence at least  $\frac{1}{2}$ , the Fano condition yields that every component of  $\Delta'$  is a line. Let  $r: Y \to X'$  be the blow up of X' at the intersection point of S' with one component of  $\Delta'$ , let E be the exceptional divisor,  $S_Y, \Delta_1, \Delta_2, \Delta_3$  the strict transforms of S and of the components of  $\Delta'$ , where  $\Delta_1$  is the only one that intersects E. Let  $\Delta_Y$ 

be the strict transform of  $\Delta'$ . Looking at the smooth fibration  $f: Y \to \mathbb{P}^1$  that contracts both  $S_Y$  and  $\Delta_1$ , we see that  $E, \Delta_2, \Delta_3$  are sections of f. Let x be the intersection point of  $\Delta_2$  and  $\Delta_3$ , clearly x is not in  $S_Y$  or E. So  $(Y, S_Y + \Delta_Y + E)$  is locally klt near x, hence  $\pi_1^{\text{reg}}(Y, S_Y + \Delta_Y + E; x)$  is finite by [14, Theorem 1]. By Lemma 2.37, we conclude that  $\pi_1^{\text{orb}}(Y, S_Y + \Delta_Y + E)$  is the extension of a finite group by a finite cyclic group, hence finite itself.

We combine the previous lemmas with a characterization of toric pairs, to prove the following result.

**Lemma 5.3.** Let  $(X, S + \Delta)$  be a plt Fano surface where S is a reduced irreducible Weil divisor and  $\rho(X) = 1$ . Assume that  $\Delta$  has standard coefficients, and at least three irreducible components. Then the group  $\pi_1^{\text{orb}}(X, S + \Delta)$  is residually finite.

*Proof.* Let n be the number of components of  $\Delta$ . Since every component of  $\Delta$  appears with coefficient at least  $\frac{1}{2}$ , and since  $(X, \Delta)$  is a Fano pair, the complexity of  $(X, S + \Delta)$  as defined in Lemma 2.15 is strictly smaller than  $2 - \frac{n}{2}$ . In particular, by Lemma 2.15, n = 3 and the pair  $(X, \lfloor S + \Delta \rfloor)$  is toric, in particular the pair (X, S) is toric too.

Since S appears with coefficient one, the coregularity of  $(X, S + \Delta)$  as in Definition 2.6 is either 0 or 1. If it is zero, then by Lemma 2.11, there is an effective  $\mathbb{Q}$ -divisor  $\Gamma \geq S + \Delta$  such that the pair  $(X, \Gamma)$  is a 2-complement. In particular, every component in  $\Gamma$  has coefficient  $\frac{1}{2}$  or 1, and so for every component P of  $\Delta$ , the orbifold index of P in  $\Delta$  divides the orbifold index of P in  $\Gamma$ . By Lemma 2.19, it suffices to show that  $\pi_1^{\text{orb}}(X,\Gamma)$  is residually finite, and that follows from Proposition 4.2 and the fact that supersolvable groups, in particular finitely generated nilpotent groups, are residually finite (see, e.g., [54]).

Assume now that the coregularity of the pair  $(X, S + \Delta)$  is one. Let  $(S, \Delta_S)$  be the pair obtained by adjunction form  $(X, S + \Delta)$  to S. If it has an N-complement  $(S, \Gamma_S)$  for some N, then by Lemma 2.10 there is an N-complement  $(X, S + \Delta + \Gamma)$  which recovers  $(S, \Gamma_S)$  by adjunction. Since  $(X, S + \Delta)$  has coregularity one,  $(X, S + \Delta + \Gamma)$  has coregularity one, so  $(S, \Gamma_S)$  has coregularity one, by inversion of adjunction [52, Theorem 0.1]. So  $\Gamma_S$  has no component of coefficient one.

This allows to classify the possible pairs klt Fano pairs  $(S, \Delta_S)$ : They are of the form  $(\mathbb{P}^1, \frac{1}{2} + \frac{2}{3} + \frac{n-1}{n})$  for  $n \in \{3, 4, 5\}$ . Since all three components of  $\Delta$  intersect S, contributing positively to the divisor  $\Delta_S$  (see Lemma 2.1), it must be that either X is smooth along S, or n = 4 and S passes through a unique singular point of X, which has orbifold index 2. From there, Lemma 5.2 concludes.

**Lemma 5.4.** Let  $(X, S + \frac{1}{2}C)$  be a plt Fano pair with S and C two reduced irreducible Weil divisors such that (X, S) is toric. Assume that  $\rho(X) = 1$ , and X has two singular points  $x_2$  and  $x_3$  contained in S, of respective orbifold index 2 and 3. Assume that the curve C intersects S at exactly two points, namely at  $x_2$  with multiplicity one, and transversally at another point  $x \in X_{reg}$ . Then the group  $\pi_1^{orb}(X, S + \frac{1}{2}C)$  is virtually cyclic.

Proof. As in the proof of Lemma 5.2, we have a finite toric cyclic cover  $q: X' \to X$  that is étale above  $X_{\text{reg}} \setminus S$ , with  $\rho(X') = 1$ , where  $S' := q^*S - \text{Ram}(q)$  is a reduced irreducible torus-invariant Weil divisor, and  $X' \setminus S'$  is isomorphic to the affine plane. Denote  $C' := q^*C$ , and note that  $q^*(K_X + S + \frac{1}{2}C) = K_{X'} + S' + \frac{1}{2}C'$ . By [76, Lemma 3.6], the curve S is normal, hence smooth, and by Lemma 2.1 since it passes through some singular points of X, it is a rational curve. Similarly, S' is smooth and toric, hence rational.

Let  $(S, \Delta_S)$  be the log pair obtained from the pair (X, S) by adjunction to S, and  $(S', \Delta_{S'})$  be the log pair obtained from the pair (X', S') by adjunction to S'. We have that  $\Delta_S = \frac{1}{2}\{x_2\} + \frac{2}{3}\{x_3\}$  and that  $\Delta_{S'}$  has standard coefficients by Lemma 2.1, and that  $(q|_{S'})^*\Delta_S = \Delta_{S'} + \text{Ram}(q|_{S'})$  since adjunction is well-behaved with respect to finite Galois covers.

From that, we see that  $q|_{S'}$  can ramify up to order 2 above  $x_2$  and up to order 3 above  $x_3$ . Since  $\Delta_{S'}$  has standard coefficients, it cannot ramify with order 2 above  $x_3$ . That is enough to see that  $q|_{S'}$ 

is a compatible cyclic cover from  $(S', \Delta_{S'})$  to  $(S, \Delta_S)$ . By Proposition 2.31, by Lemma 2.32, and since  $\pi_1^{\text{orb}}(S, \Delta_S) = \langle a, b \mid a^2 = b^3 = ab = 1 \rangle = \{1\}$ , we see that  $q|_{S'}$  is in fact an isomorphism. In particular, q is totally ramified along S', and using the projection formula for an irreducible component of C' and S' near their smooth intersection point, we see that C' is in fact irreducible.

From its fan spanned by (1,0), (0,1), (-2,-3), we see that X' is isomorphic to the weighted projective space  $\mathbb{P}(1,2,3)$  (with coordinates [x:y:z]), and S' is the line  $\{x=0\}$ . Let Y be the toric surface of Picard number two with fan spanned by (1,0), (0,1), (-2,-3), (0,-1), with a birational map  $\mu:Y\to X$ . The surface Y has two singular points of type  $A_1$ , one of which lies at the intersection of the prime exceptional divisor E and the strict transform  $S_Y$  of S'. Let  $C_Y$  be the strict transform of C'. From the fan of Y, we see that  $(S_Y)^2 = 0$  and  $-K_Y \cdot S_Y > 0$ , so we have a Mori fiber space  $f: Y \to \mathbb{P}^1$  contracting  $S_Y$ . Its fibers are in the pencil spanned by  $2S_Y$  and by the strict transform  $\ell_Y$  of the line  $\{y=0\}$ , whose members are all smooth, reduced and irreducible, except for  $2S_Y$  itself. Let F be the general fiber.

The scheme-theoretic intersection of  $C_Y$  and  $S_Y$  hasn't changed, so  $C_Y \cdot F = 2 C_Y \cdot S_Y = 3$ , i.e.,  $C_Y$  is a trisection of f. On the other hand, E and  $S_Y$  intersect at a unique singular point of type  $A_1$ , both with multiplicity one, so E is a section of f. It suffices to show that  $\pi_1^{\text{orb}}(Y, S_Y + E_Y + \frac{1}{2}C_Y)$  is virtually cyclic. Let  $\Delta_Y := S_Y + E_Y + \frac{1}{2}C_Y$ . Note that  $C_Y$  intersects  $S_Y$  at two points: transversally at a smooth point p and at a  $A_1$ -singularity with multiplicity 1. Locally, the point p is contained in two branches of  $C_Y$  over  $\mathbb{P}^1$ .

Let  $U_p$  be an analytic neighborhood of p in Y, biholomorphic to a ball. Since  $C_Y$  and  $S_Y$  intersect transversally at p, the group  $\pi_1^{\mathrm{orb}}(U_p, \Delta_Y|_{U_p})$  is isomorphic to  $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .

Let q be a general point in  $f(U_p)$ , and  $F_q := f^{-1}(q)$ , so that  $F_q \cap U_p \cap C_Y$  consists in two distinct points. The group  $\pi_1^{\text{orb}}(F_q \cap U_p, \Delta_Y|_{F_q \cap U_p})$  is then isomorphic to the orbifold fundamental group of  $(\mathbb{C}, \frac{1}{2}\{0\} + \frac{1}{2}\{1\})$ , hence generated by two generators x, y of order two. Note that the homomorphism

$$p_1: \pi_1^{\mathrm{orb}}(F_q \cap U_p, \Delta_Y|_{F_q \cap U_p}) \to \pi_1^{\mathrm{orb}}(U_p, \Delta_Y|_{U_p})$$

sends x, y, or xy to the neutral element, since the second group only has one element of order two.

Note that, by Lemma 2.36, we have a surjection  $\pi_1^{\text{orb}}(F_q, \Delta_Y|_{F_q}) \to \pi_1^{\text{orb}}(Y, \Delta_Y)$ . Note also that this first group is generated by three generators x, y, a of order two, where the first two generators x and y come from  $\pi_1^{\text{orb}}(F_q \cap U_p, \Delta_Y|_{F_q \cap U_p})$ . The natural inclusions induce a commutative diagram:

$$\begin{split} \pi_1^{\mathrm{orb}}(F_q \cap U_p, \Delta_Y|_{F_q \cap U_p}) & \xrightarrow{p_1} \pi_1^{\mathrm{orb}}(U_p, \Delta_Y|_{U_p}) \\ & \downarrow^{\pi_1} & \downarrow^{\pi_2} \\ & \pi_1^{\mathrm{orb}}(F_q, \Delta_Y|_{F_q}) & \xrightarrow{p_2} & \pi_1^{\mathrm{orb}}(Y, \Delta_Y). \end{split}$$

So  $\pi_1^{\text{orb}}(Y, \Delta_Y)$  is generated by three generators  $p_2(x), p_2(y), p_2(a)$ , each of order at most two. But note that  $p_2(x), p_2(y)$ , or  $p_2(xy)$  is trivial (because of  $p_1$ ). Hence, the group  $\pi_1^{\text{orb}}(Y, \Delta_Y)$  can be in fact generated by two elements of order at most two, i.e., it is a quotient of

$$\langle u,v\mid u^2=v^2=1\rangle,$$

which itself is an extension of  $\langle uv \rangle \simeq \mathbb{Z}$  by  $\mathbb{Z}/2\mathbb{Z}$ . Hence, the group  $\pi_1^{\text{orb}}(Y, \Delta_Y)$  is virtually cyclic, as wished.

5.2. **Purely log terminal Fano surfaces.** In this section, we study some special plt Fano surfaces. In this subsection, the underlying klt Fano surfaces that we consider are not necessarily toric.

**Lemma 5.5.** Let  $(X, S + \frac{1}{2}L_0)$  be a plt Fano surface such that S and  $L_0$  are reduced irreducible Weil divisors,  $\rho(X) = 1$ , S contains exactly two singular points of X, which both have orbifold index 3, and  $L_0$  intersects

S transversally at a single smooth point of X. Then the fundamental group  $\pi_1^{\text{orb}}(X, S + \frac{1}{2}L_0)$  is residually finite.

*Proof.* Note that the pair  $(S, \Delta_S)$  obtained by adjunction from (X, S) with respect to S is isomorphic to  $(\mathbb{P}^1, \frac{2}{3}\{0\} + \frac{2}{3}\{\infty\})$ , hence it admits a 1-complement  $(\mathbb{P}^1, \{0\} + \{\infty\})$ .

By Lemma 2.10, it extends to a 1-complement  $(X, S + \Gamma)$  for (X, S), with  $(S, \Gamma_S) \simeq (\mathbb{P}^1, \{0\} + \{\infty\})$ . In particular, note that  $\Gamma$  is a Weil divisor.

By Lemma 2.15,  $\Gamma$  either has one component, or it has two components and the pair  $(X, S + \Gamma)$  is toric. In the second case, let  $\Gamma_1$  be one of those two components; applying Lemma 5.1 to the pair  $(X, S + \Gamma_1 + L_0)$  together with Lemma 2.19 concludes.

We now assume that  $\Gamma$  consists of a single component. By Lemma 2.1,  $\Gamma$  contains the two singular points that lie on S. Since the pair  $(X,\Gamma)$  is also plt,  $\Gamma$  is a smooth curve. Applying Lemma 2.1 to perform adjunction of the log Calabi-Yau pair  $(X,S+\Gamma)$  with respect to  $\Gamma$ , we see that  $\Gamma$  is a smooth rational curve and that X is smooth along  $\Gamma$  outside of the two intersection points of S and  $\Gamma$ .

Since X is a klt surface, these two singularities are quotient singularities, and since  $\mathbb{Z}/3\mathbb{Z}$  is the sole group of order 3, they are cyclic quotient singularities, locally isomorphic to the Duval singularity  $A_2$  or to the singularity  $\mathbb{C}^2/\langle \operatorname{diag}(\zeta_3,\zeta_3)\rangle$ , which we denote by  $C_3$ . Since  $K_X+S+\Gamma\sim 0$ , every point in  $X\setminus S\cup \Gamma$  is a canonical singularity.

<u>Case 1:</u> Assume that both singularities contained in S are  $C_3$  singularities.

Let  $\mu: Y \to X$  be a resolution of the two toric singular points contained in S. Let  $E_1$  and  $E_2$  be the exceptional divisors, and  $S_Y, \Gamma_Y, L_Y$  the strict transforms of  $S, \Gamma, L_0$ . We have

(5.1) 
$$\mu^* \left( K_X + S + \frac{1}{2} L_0 \right) = K_Y + S_Y + \frac{2}{3} E_1 + \frac{2}{3} E_2 + \frac{1}{2} L_Y,$$

and

(5.2) 
$$\mu^* (K_X + S + \Gamma) = K_Y + S_Y + \Gamma_Y + E_1 + E_2,$$

Note that Y is a klt Gorenstein (hence canonical) surface of Picard rank three, and it is smooth along the cycle of rational curves  $S_Y + \Gamma_Y + E_1 + E_2$ .

Of course,  $K_Y$  is anti-effective, hence not pseudoeffective, and we can start running a  $K_Y$ -MMP which will terminate with a Mori fiber space. Let us prove that it terminates with a Mori fiber space to a curve. Assume by contradiction that it successively contracts two curves of negative square.

- If neither of the two contracted curves are in  $S_Y + \Gamma_Y + E_1 + E_2$ , we can push-forward the pair  $(Y, S_Y + \Gamma_Y + E_1 + E_2)$  to a log canonical pair on the minimal model, which has complexity -1 there, a contradiction with Lemma 2.15.
- Assume that both contracted curves are in  $S_Y + \Gamma_Y + E_1 + E_2$ . Since Y is smooth along that divisor, the contracted curves are both (-1)-curves in that case. So the first step must contract either  $S_Y$  or  $\Gamma_Y$ , and turn  $E_1$  and  $E_2$  into two curves of square -2. The second step must then contract the remaining curve from  $S_Y \cup \Gamma_Y$ , and turn  $E_1$  and  $E_2$  into two curves of square -1. But that cannot occur on a surface of Picard number one, contradiction.
- Assume that the first contracted curve is in  $S_Y + \Gamma_Y + E_1 + E_2$ , but not the second one. Then the first contracted curve is either  $S_Y$  or  $\Gamma_Y$ , and we still have a 1-complement after the first step, supported on the image of  $S_Y + \Gamma_Y + E_1 + E_2$ . The second contracted curve, say C, is not contained in that image, so by Lemma 2.13 it intersects it in exactly one point p, which is a smooth point of the surface, so by Lemma 2.13 again C is disjoint from one of the two divisors  $E_1$  and  $E_2$ . After

contracting C, we are left again with a curve of negative self-intersection on a surface of Picard number one, a contradiction.

• We can finally assume that the first curve C contracted is not in  $S_Y + \Gamma_Y + E_1 + E_2$ . By Lemma 2.13 again, C is disjoint from one of the two curves  $E_1$  and  $E_2$ . The discrepancy for the exceptional divisor C equals  $\frac{K_Y \cdot C}{C^2} > 0$ , so C is contracted to a terminal singularity, i.e., a smooth point of the surface. So the image  $S_Z + \Gamma_Z + E_{1,Z} + E_{2,Z}$  of  $S_Y + \Gamma_Y + E_1 + E_2$  after that first contraction stays in the smooth locus of the surface Z. Thus, the second contraction (which contracts a curve in the image of  $S_Z + \Gamma_Z + E_{1,Z} + E_{2,Z}$ ) contracts a (-1)-curve of Z. Since  $E_{1,Z}$  and  $E_{2,Z}$  are disjoint, contracting one of them would still yield a curve of square -3 on a surface of Picard number one, which is not possible. Otherwise, if  $S_Z$  or  $\gamma_Z$  is a (-1)-curve, since  $(Z, S_Z + \Gamma_Z + E_{1,Z} + E_{2,Z})$  is still a 1-complement, and  $S_Z \cdot E_{i,Z} \geq 1$  and  $\Gamma_Z \cdot E_{i,Z} \geq 1$ , we get  $S_Z \cdot E_{i,Z} = \Gamma_Z \cdot E_{i,Z} = 1$ , and after the second contraction, we still have a curve of square -2 on a surface of Picard number one, a contradiction.

We conclude that the  $K_Y$ -MMP must indeed contract a single curve of negative square, and then induce a Mori fiber space. Let  $\varepsilon: Y \to Z$  be the contraction and  $f: Z \to \mathbb{P}^1$  be the Mori fiber space.

If  $\varepsilon$  contracts a curve in  $S_Y + \Gamma_Y + E_1 + E_2$ , it contracts a (-1)-curve, which must be  $S_Y$  or  $\Gamma_Y$ . Hence the images of  $E_1, E_2$ , denoted by  $E_{1,Z}, E_{2,Z}$  both have square -2, hence are distinct sections of f. They thus represent distinct extremal rays of the Mori cone of Z; the general fiber of f is another extremal ray of it. But  $\rho(Z) = 2$ , so there cannot be three extremal rays, a contradiction.

So  $\varepsilon$  contracts a curve C that is not contained in the support of  $S_Y + \Gamma_Y + E_1 + E_2$ . Computing the discrepancy, we note that C is contracted onto a terminal singularity (hence a smooth point) of Z. Let  $S_Z, \Gamma_Z, L_Z, E_{i,Z}$  be the images of  $S_Y, \Gamma_Y, L_Y, E_{i,Y}$  in Z. By Lemma 2.13, the curves  $E_{1,Z}$  and  $E_{2,Z}$  are thus disjoint, and at least one of them has self-intersection -3, say it is  $E_{1,Z}$ . Hence, the Mori cone of Z is generated by  $E_{1,Z}$  and the general fiber F of f, neither  $E_{1,Z}$  nor  $E_{2,Z}$  is contracted by f, and  $E_{2,Z}$  has positive self-intersection. Since  $E_2$  has negative square, it means that C intersects  $E_2$ , hence by Lemma 2.13, C is disjoint from  $S_Y \cup \Gamma_Y$ . In particular,  $L_Z$  and  $S_Z$  still intersect transversally at exactly one smooth point of Z.

Using our 1-complement  $(Z, S_Z + \Gamma_Z + E_{1,Z} + E_{2,Z})$ , adjunction on the general fiber F, and the fact that  $E_{i,Z} \cdot F \geq 1$ , we see that  $E_{1,Z}$  and  $E_{2,Z}$  are both sections of f, while  $S_Z$  and  $\Gamma_Z$  are contained in fibers of f. Since  $E_{1,Z} \cdot S_Z = E_{1,Z} \cdot \Gamma_Z = 1$ , they are both reduced irreducible fibers of f. In particular,  $L_Z$  is a section of f too. Since  $\rho(Z) = 2$  and  $E_{1,Z}$  is disjoint from both  $E_{2,Z}$  and  $E_{2,Z}$  and  $E_{2,Z}$  and  $E_{2,Z}$  and  $E_{2,Z}$  and  $E_{2,Z}$  and  $E_{2,Z}$  are disjoint,  $E_{2,Z} = E_{2,Z} \cdot E_{2,Z} = E_{2$ 

<u>Case 2:</u> The singular points contained in S are an  $A_2$  singularity and a  $C_3$  singularity.

As in Case 1, we denote by  $\mu: Y \to X$  a resolution of the  $C_3$  singular point. We use the same notation as above for  $S_Y, \Gamma_Y, L_Y$ , and denote by E the prime exceptional divisor. Then, we can write

(5.3) 
$$\mu^* \left( K_X + S + \frac{1}{2} L_0 \right) = K_Y + S_Y + \frac{2}{3} E + \frac{1}{2} L_Y,$$

and

(5.4) 
$$\mu^*(K_X + S + \Gamma) = K_Y + S_Y + \Gamma_Y + E.$$

Note that  $\rho(Y) = 2$ , Y is Gorenstein and has canonical singularities, and Y is smooth along E. Since  $K_Y$  is anti-effective, we run a  $K_Y$ -MMP, which terminates in one step.

First, we claim that  $\varepsilon: Y \to Z$  cannot be a birational contraction contracting a curve in the support of  $S_Y + \Gamma_Y + E$ . It clearly cannot contract E because it is in the smooth locus and has square -3. If  $S_Y$  or  $\Gamma_Y$  is contracted to a point, then computing the discrepancy shows that it is contracted to a terminal, hence a smooth point. Let us denote by  $C_Y$  the curve (of  $S_Y$  and  $\Gamma_Y$ ) that is contracted. We thus have that  $K_Y \cdot C_Y = (C_Y)^2$ , in particular  $(C_Y)^2$  is an integer. Let  $(C_Y, \Delta_{C_Y})$  be the pair obtained by adjunction from the pair  $(Y, C_Y)$  with respect to  $C_Y$ , following Lemma 2.1. We obtain that the degree of  $\Delta_{C_Y}$  is an integer. But  $C_Y$  contained exactly one singular point of Y, of orbifold index 3, a contradiction.

Now, assume that  $\varepsilon\colon Y\to Z$  is a birational contraction, contracting a curve C not contained in the support of  $S_Y+\Gamma_Y+E$ . Since  $\rho(Z)=1$ , the image of E in Z must have positive self-intersection, hence C and E must intersect. Since Y is smooth along E, by Lemma 2.13, C does not intersect  $S_Y$  or  $\Gamma_Y$ , and intersects E at exactly one point. Let  $S_Z, \Gamma_Z, E_Z, L_Z$  be the images of  $S_Y, \Gamma_Y, E, L_Y$  in Z. By Lemma 2.15, the pair  $(Z, S_Z + \Gamma_Z + E_Z)$  is toric. Since C and  $S_Y$  are disjoint, the curves  $S_Z$  and  $L_Z$  still intersect transversally at a single smooth point of Z. That point is not on  $E_Z$  because S and  $S_Y$  are disjoint, the curves  $S_Z$  and  $S_Z$  are disjoint, the curves  $S_Z$  are disjoint, the curves  $S_Z$  and  $S_Z$  are disjoint, the curves  $S_Z$  and  $S_Z$  are disjoint, the curves  $S_Z$  are disjoint, the curves  $S_Z$  and  $S_Z$  are disjoint, the curves  $S_Z$  are disjoint  $S_Z$  are di

Finally, assume that  $f: Y \to B$  is a Mori fiber space onto a smooth curve B. Note that  $L_Y$  and E have non-zero self-intersection, so they must be horizontal over B. let F be the general fiber of f. Since the divisor (5.4) is numerically trivial, intersecting with F we notice that there are two possibilities: Either  $E \cdot F = 2$ , and  $S_Y$  and  $\Gamma_Y$  are contracted by f, or E and one of  $S_Y$ ,  $\Gamma_Y$  are sections of f, and the remaining curve in  $S_Y, \Gamma_Y$  is contracted by f. If  $S_Y$  and  $\Gamma_Y$  are both contracted by f, since f is a Mori fiber space they are numerically proportional to F, hence disjoint, even though they are supposed to intersect at an  $A_2$  singular point of Y, a contradiction. Hence, we see that E is a section of f. Intersecting the anti-nef divisor (5.3) with the Cartier divisor F, we see that  $S_Y \cdot F = 0$ , hence  $\Gamma_Y$  is a section of f and  $S_Y$  is contracted by f. Let x be the intersection point of  $S_Y$  and  $L_Y$ , which is clearly not in E. Since it is a smooth point of transversal intersection, we have  $\pi_1^{\text{reg}}(Y, S_Y + E + \frac{1}{2}L_Y; x) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$ , and since  $L_Y$  and  $S_Y$  do not intersect anywhere else, x is contained in all the branches of  $L_Y$ , and only fails to be contained in E, which is a section of f. Finally, the canonical bundle formula for the pair  $(Y, S_Y + \Gamma_Y + E)$  shows that the base curve B of f is a smooth rational curve, and that besides possible  $S_Y$ , the fibration f must have either zero or one multiple fiber, or two multiple fibers both with multiplicity two. In either case, since x belongs to the fiber  $S_Y$ , we can apply Lemma 2.37 and see that  $\pi_1^{\text{orb}}(Y, S_Y + E + \frac{1}{2}L_Y)$  admits a finite index (not necessarily normal) subgroup isomorphic to a quotient of  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$ . So it is residually finite.

<u>Case 3:</u> Both singular points contained in S are  $A_2$  singularities.

In that last case, X is a Gorenstein del Pezzo surface of Picard rank 1. By [68, Lemma 6], there is a finite quasi-étale Galois cover  $q: \mathbb{P}^2 \to X$ . Let  $S' := q^*S$  and  $L' := q^*L_0$ , so that

$$q^* \left( K_X + S + \frac{1}{2} L_0 \right) = K_{\mathbb{P}^2} + S' + \frac{1}{2} L'.$$

Then, the pair  $(\mathbb{P}^2, S' + \frac{1}{2}L')$  is still a plt Fano pair, so S' is smooth. Since  $\rho(\mathbb{P}^2) = 1$ , the divisor S' is in fact an irreducible curve, and it must be a line or a smooth quadric. If S' is a quadric then L' must be a line, and the pair  $(\mathbb{P}^2, S' + L')$  is a 1-complement, so Lemma 2.19 and Proposition 4.2 conclude. Otherwise, S' is a line, and we can take  $\Gamma$  to be one, two or three general lines so that  $(\mathbb{P}^2, S' + \frac{1}{2}L' + \frac{1}{2}\Gamma)$  is a 2-complement, and Lemma 2.19 and Proposition 4.2 conclude again.

**Lemma 5.6.** Let  $(X, S + \frac{2}{3}L_0)$  be a plt Fano surface such that such that S and  $L_0$  are reduced irreducible Weil divisors,  $\rho(X) = 1$ , S contains exactly two singular points of X, which have respective orbifold indices 2 and 4, and  $L_0$  intersects S transversally at a single smooth point of X. Then the group  $\pi_1^{\text{orb}}(X, S + \frac{2}{3}L_0)$  is residually finite.

*Proof.* The first four paragraphs in the proof of Lemma 5.5 still hold, and yield a 1-complement  $(X, S + \Gamma)$  of (X, S), such that  $\Gamma$  has a single component which passes through the two singular points of X contained in S, with multiplicity one at each, and X is smooth along  $\Gamma$  outside of these two points.

Again,  $K_X + S + \Gamma \sim 0$ , so the klt surface X is Gorenstein outside of  $S \cup \Gamma$ . It has two singular points contained in S: The one of orbifold index 2 is an  $A_1$  singularity, and the one of index 4 is a quotient singularity by a group G of order 4 acting faithfully and freely in codimension 1. In particular, any element of order two in G must be  $\operatorname{diag}(-1, -1)$ , so G is cyclic and the singular point is either an  $A_3$  singularity, or modelled after  $\mathbb{C}^2/\langle \operatorname{diag}(\zeta_4, \zeta_4) \rangle$ , which we call a  $C_4$  singularity.

If the singular points of X contained in S are an  $A_1$  and a  $C_4$  singularities, then the proof of Case 2 in Lemma 5.5 applies *verbatim*.

Otherwise, the surface X has canonical Gorenstein singularities, including an  $A_1$  and an  $A_3$  singularities. Since X is also a del Pezzo surface of Picard number one, by [68, Lemma 3] the number and type of the singular points of X is prescribed as  $2A_1 + A_3$ ,  $A_1 + 2A_3$ , or  $2A_1 + 2A_3$ .

If the singularities of X are either  $A_1 + 2A_3$  or  $2A_1 + 2A_3$ , then by [68, Lemma 6, Table I], there is a finite Galois quasi-étale cover  $q: \mathbb{P}^1 \times \mathbb{P}^1 \to X$ . Taking  $S' := q^*S$  and  $L' := q^*L_0$ , by Proposition 2.31, it suffices to show that  $\pi_1^{\text{orb}}(\mathbb{P}^1 \times \mathbb{P}^1, S' + \frac{2}{3}L')$  is residually finite. But that pair is still a plt Fano pair, and Corollary 2.52 shows the required residual finiteness.

Otherwise, the singularities of X are  $2A_1 + A_3$ , and by [68, Lemma 6, Table I], there is a finite quasi-étale double cover  $q: X' \to X$  such that  $\rho(X') = 1$  and X' has a single  $A_1$  singularity. Let  $S' := q^*S$  and  $L' := q^*L_0$ , so that

$$q^* \left( K_X + S + \frac{2}{3} L_0 \right) = K_{X'} + S' + \frac{2}{3} L'.$$

Then,  $(X', S' + \frac{2}{3}L')$  is still a plt Fano surface, and the unique singular point of X' is contained in S' and outside of L'. Moreover, by We run a  $(K_{X'} + S' + \frac{2}{3}L'_0)$ -MMP. If it terminates with a Mori fiber space, then Proposition 3.2 concludes. Hence, we may assume it terminates with a Fano variety of Picard rank one. Let  $X' \to X''$  be the MMP and  $(X'', S'' + \frac{2}{3}L''_0)$  be the associated pair on X''. By construction, S'' has at most one singular point. By Lemma 2.10, the pair (X', S' + L') admits a 1-complement, and by Lemma 2.15, it is toric. In particular,  $X' \setminus S'$  is isomorphic to the (smooth) complex affine plane. Thus, the group  $\pi_1^{\text{orb}}(X', S' + L')$  is cyclic, and so is the group  $\pi_1^{\text{orb}}(X, S + \frac{2}{3}L_0)$ , as wished.

5.3. **Log canonical Fano surfaces.** In this subsection, we study the fundamental groups of log canonical Fano surfaces.

**Proposition 5.7.** Let  $(X, \Delta)$  be a log canonical Fano pair of dimension 2. Then there is a normal subgroup of  $\pi_1^{\text{orb}}(X, \Delta)$  of index at most 7200, that is nilpotent of length at most 2 and of rank at most 3.

*Proof.* In most cases, we will show that  $\pi_1^{\text{orb}}(X,\Delta)$  is residually finite and conclude by Theorem 5.

If  $\operatorname{coreg}(X, \Delta^{\operatorname{st}}) = 2$ , then the pair  $(X, \Delta^{\operatorname{st}})$  is klt, so  $\pi_1^{\operatorname{orb}}(X, \Delta)$  is finite by [14, Theorem 2]. In particular it is indeed residually finite.

If  $\operatorname{coreg}(X, \Delta^{\operatorname{st}}) = 0$ , then by Lemma 2.11, we know that  $(X, \Delta^{\operatorname{st}})$  admits a 2-complement  $(X, \Gamma)$ . Each component of  $\Gamma$  has coefficient  $\frac{1}{2}$  if and only if it appears with coefficient 0 or  $\frac{1}{2}$  as a component of  $\Delta^{\operatorname{st}}$ , and coefficient 1 otherwise; so the Lemma 2.19 and Proposition 4.2 conclude.

From now on, we assume that  $\operatorname{coreg}(X, \Delta^{\operatorname{st}}) = 1$ . By Lemma 2.11, the pair  $(X, \Delta^{\operatorname{st}})$  admits an Ncomplement  $(X,\Gamma)$  for some  $N \in \{1,2,3,4,6\}$ . Let  $(Y,\Gamma_Y) \to (X,\Gamma)$  be a dlt modification of the pair  $(X,\Gamma)$ , let E be its reduced exceptional divisor, and denote by  $\Delta_Y$  be the strict transform of  $\Delta$ . We have  $\pi_1^{\text{orb}}(Y, \Delta_Y + E) \simeq \pi_1^{\text{orb}}(X, \Delta)$ , and the pair  $(Y, \Delta_Y + E)$  is still a log Calabi-Yau pair. By [43, Theorem 1.1] and since by assumption  $\operatorname{coreg}(Y, \Delta_Y + E) = 1$ , we know that the dual complex  $\mathcal{D}(E)$  is either one or two points.

If E contains two components  $S_0$  and  $S_1$ , then we can run a  $(K_Y + \Delta_Y + E - S_0)$ -MMP. Since  $K_Y + \Delta_Y +$  $E-S_0$  is not pseudoeffective, this minimal model program terminates with a Mori fiber space, and since  $\operatorname{coreg}(Y, E) = 1$ , the two curves  $S_0$  and  $S_1$  don't intersect in any birational model. So this MMP yields a Mori fiber space  $Z \to C$ , where C is a smooth curve. Furthermore, the push-forward of both  $S_0$  and  $S_1$  to Z are sections over C. Let  $\Delta_Z + E_Z$  be the push-forward of  $\Delta_Y + E$  to Z. We have a surjective homomorphism  $\pi_1^{\text{orb}}(Z, \Delta_Z + E_Z) \to \pi_1^{\text{orb}}(Y, \Delta_Y + E)$  by Lemma 2.21. Note that  $(Z, \Delta_Z + E_Z)$  is of log Calabi–Yau type and it admits a Mori fiber space. The statement of the proposition holds for  $\pi_1^{\text{orb}}(Z, \Delta_Z + E_Z)$  by Proposition 3.2.

From now on, we assume that E is irreducible. In particular,  $\rho(Y/X) \leq 1$ . We run a  $(K_Y + \Delta_Y + E)$ -MMP. Note that the pair  $(Y, \Delta_Y + E)$  is plt. If it terminates with a Mori fiber space onto a curve, then we argue as in the previous paragraph. We now assume that it terminates with a Fano surface Z of Picard rank one. Let  $\Delta_Z$  and  $E_Z$  be the images of  $\Delta_Y$  and E on Z. The pair  $(Z, \Delta_Z + E_Z)$  is a plt Fano pair with Z of Picard rank 1, and we are left to prove that the group  $\pi_1^{\text{orb}}(Z, \Delta_Z + E_Z)$  is residually finite.

We write  $(\Delta_Z + E_Z)^{\text{st}}$  as a sum S + L, where S is a reduced irreducible Weil divisor, and L has standard coefficients smaller than one. Since  $\operatorname{coreg}(Z, S+L)=1$ , the pair (Z, S+L) is a Fano pair, and by Lemma 2.10, the pair  $(S, \Delta_S)$  obtained by adjunction of (X, S + L) to S is one of the following three log pairs:

- $\begin{array}{l} (1) \ \ {\rm the \ pair \ } (\mathbb{P}^1,\frac{1}{2}\{0\}+\frac{2}{3}\{1\}+\frac{2}{3}\{\infty\}),\\ (2) \ \ {\rm the \ pair \ } (\mathbb{P}^1,\frac{1}{2}\{0\}+\frac{2}{3}\{1\}+\frac{3}{4}\{\infty\}), \ {\rm or \ } \\ (3) \ \ {\rm the \ pair \ } (\mathbb{P}^1,\frac{1}{2}\{0\}+\frac{2}{3}\{1\}+\frac{4}{5}\{\infty\}). \end{array}$

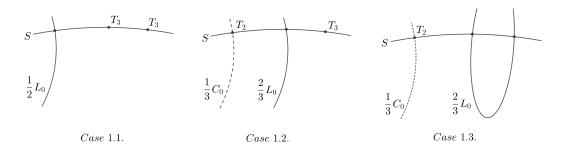
We say that  $(S, \Delta_S)$  has configuration (2,3,3), (2,3,4), or (2,3,5), respectively in these cases. If L=0, then by [65, Corollary 1.6] it follows that  $\pi_1^{\text{orb}}(X,\Delta)$  is virtually abelian, hence residually finite. We may now assume that  $L \neq 0$ . If L has at least three components, then Lemma 5.3 concludes.

If L has exactly two components  $L_0$  and  $L_1$ , since (Z, S + L) is a Fano pair, the pair (Z, S) is toric due to Lemma 2.15. One of these components, say  $L_0$ , must intersect S transversally at a unique smooth point. The pair  $(Z, S + L_1)$  is toric as it is le Fano and has complexity 1. Hence, the statement follows from Lemma 5.1.

Finally, we assume that  $\Delta^{\rm st} = S + (1 - \frac{1}{n})L_0$  where both S and  $L_0$  are prime divisors. We proceed in three cases depending on the configuration of  $(S, \Delta_S)$ .

<u>Case 1:</u> The pair  $(S, \Delta_S)$  has configuration (2,3,3).

The following figure describes the possible configurations, where each  $T_i$  denotes a singular point of orbifold index i.

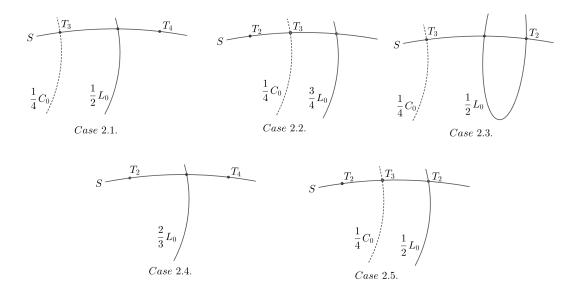


In Case 1.1, Lemma 5.5 shows that  $\pi_1^{\text{orb}}(Z, S + \frac{1}{2}L_0)$  is residually finite, as wished.

In Case 1.2. and Case 1.3., the pair  $(S, \Delta_S)$  admits a 3-complement (increase the coefficient  $\frac{1}{2}$  to  $\frac{2}{3}$ ), which by Lemma 2.10 lifts to a 3-complement  $(Z, S + \frac{m}{3}C_0 + \frac{2}{3}L_0)$  for some positive integer m and a curve  $C_0$  intersecting S at the point  $T_2$ . By Lemma 2.1, m=1 and  $C_0$  passes through  $T_2$  with multiplicity one. So  $(Z, S + C_0)$  is also a log canonical pair. Since  $L_0 - C_0$  has positive intersection with S and  $\rho(Z) = 1$ , the divisor  $L_0 - C_0$  is ample, and so the pair  $(Z, S + C_0)$  is also a Fano pair. By Lemma 2.15, we conclude that  $(Z, S + C_0)$  is toric and Lemma 5.1 concludes.

<u>Case 2:</u> The pair  $(S, \Delta_S)$  has configuration (2, 3, 4).

The following figure describes the possible configurations:

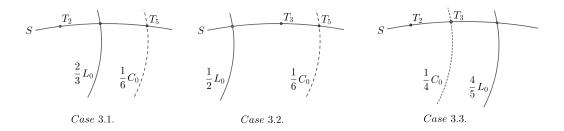


Noting that the pair  $(S, \Delta_S)$  admits a 4-complement (increasing the coefficient  $\frac{2}{3}$  to  $\frac{3}{4}$ ), we can find in Case 2.1, Case 2.2, and Case 2.3 a curve  $C_0$  passing through  $T_3$  with multiplicity one, such that  $(Z, S + \frac{1}{4}C_0 + \frac{1}{2}L_0)$  is 4-complement, and  $(Z, S + C_0)$  is again a log Fano pair, hence a toric pair by Lemma 2.15. We conclude by Lemma 5.1 in Case 2.1 and Case 2.2, and by Lemma 5.4 in Case 2.3. In Case 2.4, Lemma 5.6 concludes.

In Case 2.5, the same approach as in Case 2.1, Case 2.2, and Case 2.3 provides a curve  $C_0$  that intersects S with multiplicity one at  $T_3$ . But in that case, the pair  $(X, S + C_0)$  is a 4-complement. We let  $q: Y \to X$  be the index one cover associated to  $K_X + S + C_0$ , which is quasi-étale, and denote  $S_Y = q^*S$  and  $L_Y = q^*L_0$ , so that  $q^*(K_X + S + \frac{1}{2}L_0) = K_Y + S_Y + \frac{1}{2}L_Y$ . The curve  $L_Y$  intersects  $S_Y$  at a smooth point and  $S_Y$  has two  $T_3$  singularities. By running a  $(K_Y + S_Y + \frac{1}{2}L_Y)$ -MMP we either induce a Mori fiber space onto a curve, obtain a plt Fano pair of coregularity 0, or reduce to Case 1.1. In any case, we conclude that  $\pi_1^{\text{orb}}(Y, S_Y + \frac{1}{2}L_Y)$  and hence  $\pi_1^{\text{orb}}(X, S + \frac{1}{2}L_0)$  is residually finite.

<u>Case 3:</u> The pair  $(S, \Delta_S)$  has configuration (2, 3, 5).

The following figure describes the possible configurations:



Cases 3.1 and 3.2 proceed similarly as before: The pair  $(S, \Delta_S)$  admits a 6-complement (increasing the coefficient  $\frac{4}{5}$  to  $\frac{5}{6}$ ). This 6-complement lifts to X as  $(X, S + \frac{1}{6}C_0 + \frac{2}{3}L_0)$  in Case 3.1 and  $(X, S + \frac{1}{6}C_0 + \frac{1}{2}L_0)$  in Case 3.2, where  $C_0$  is a curve that intersects S at the  $T_5$  singularity with multiplicity 1. Hence, the pair  $(X, S + C_0)$  is a log Fano pair, thus a toric pair by Lemma 2.15. The residual finiteness then follows from Lemma 5.1.

In Case 3.3., let us consider the pair  $(X, S + \frac{3}{4}L_0)$ . This pair admits a 4-complement (lifted from S) of the form  $(X, S + \frac{1}{4}C_0 + \frac{3}{4}L_0)$  where  $C_0$  is a curve intersecting S at the  $T_3$  singularity with multiplicity 1. The pair  $(X, S + C_0)$  is thus log Fano, in fact even toric by Lemma 2.15, and Lemma 5.1 concludes.

## 6. Proof of the theorems

In this section, we prove the main theorems of the article.

Proof of Theorem 5. Let  $(X, \Delta)$  be an n-dimensional dlt Fano pair. Assume that  $\pi_1^{\mathrm{orb}}(X, \Delta)$  is a residually finite group. Dlt pairs admit small  $\mathbb{Q}$ -factorializations. The orbifold fundamental group is unchanged by small  $\mathbb{Q}$ -factorializations. Hence, we may assume that X itself is  $\mathbb{Q}$ -factorial. Then, there are normal subgroups  $(H_i)_{i\in I}$  of finite index in  $\pi_1^{\mathrm{orb}}(X,\Delta)$  such that  $\cap_{i\in I}H_i=\{1\}$ . As the intersection of normal subgroups of finite indices are normal subgroups of finite indices, we may assume  $I=\mathbb{Z}_{\geq 0}$  and  $H_i\subsetneq H_{i-1}$  for every I. In particular, the intersection of infinitely many of the  $H_i$ 's is just the identity element. Now, if  $p_i\colon Y_i\to X$  is the finite cover associated to  $H_i$ . Since  $(X,\Delta)$  is dlt, the pair  $(X,\Delta-\epsilon\lfloor\Delta\rfloor)$  is klt and log Fano. Let  $n_i$  be

the maximum ramification index among the components of  $[\Delta]$ . Then, if we choose  $\epsilon < n_i^{-1}$ , the pullback

$$p_i^*(K_X + \Delta - \epsilon |\Delta|) = K_{Y_i} + B_{Y_i},$$

satisfies that  $B_{Y_i}$  is effective. In particular, we have that  $(Y_i, B_{Y_i})$  is a n-dimensional klt log Fano pair. Therefore,  $Y_i$  is a Fano type variety, so it is rationally connected. Then, Lemma 2.48 implies that  $\pi_1^{\text{orb}}(X, \Delta)$  admits a normal abelian subgroup of index at most J(n).

*Proof of Theorem 3.* This follows from Proposition 4.1.

Proof of Theorem 2. Let  $(X, \Delta)$  be a log Calabi–Yau surface for which  $\pi_1^{\text{orb}}(X, \Delta)$  is not virtually abelian. By the proof of Theorem 1, one of the two following conditions must be satisfied:

- (i) up to a birational transformation X admits a fibration onto an elliptic curve and  $\Delta$  consists of two sections, or
- (ii) up to a birational transformation  $(X, \Delta^{st})$  admits a complement with standard coefficients.

If we are in the setting of the first case, then we are done. Assume that we are in the second case. Let  $(X, \Gamma)$  be the complement of  $(X, \Delta^{\text{st}})$  with standard coefficients. Let  $(Y, \Delta_Y)$  be the index one cover of  $K_X + \Gamma$ . By the proof of Proposition 4.2, if  $\pi_1^{\text{orb}}(Y, \Delta_Y)$  is not virtually abelian, then a dlt modification of  $(Y, \Delta_Y)$  admits an MMP that terminates with a fibration onto an elliptic curve. This finishes the proof.

*Proof of Theorem 1.* Let  $(X, \Delta)$  be a log Calabi–Yau pair of dimension 2. By taking a dlt modification, we may assume that  $(X, \Delta)$  is dlt and in particular X is klt (see Lemma 2.20). We run a  $K_X$ -MMP. Let  $X \to Y$  be the sequence of divisorial contractions.

First, assume that Y is a minimal model. Then, the push-forward of  $\Delta$  to Y is trivial. Note that Y has klt singularities. By Lemma 2.21, it suffices to prove the statement for  $\pi_1^{\text{orb}}(Y)$ . This is proved in Proposition 4.1.

Now, assume that  $Y \to C$  is a Mori fiber space. By Lemma 2.21, it suffices to prove the statement for  $(Y, \Delta_Y)$ , where  $\Delta_Y$  is the push-forward of  $\Delta$  in Y. If dim C=1, then the statement follows from Proposition 3.2. On the other hand,  $\rho(Y)=1$  is a klt Fano surface. If  $\Delta_Y^{\rm st}=\Delta_Y$ , then the statement follows from Proposition 4.2. On the other hand, if  $\Delta_Y^{\rm st}\neq\Delta_Y$ , then we conclude that  $(Y,\Delta_Y^{\rm st})$  is a log canonical Fano pair. Thus, the statement follows from Proposition 5.7. This finishes the proof of the theorem.

Proof of Theorem 4. Let  $(X, \Delta)$  be a log Calabi–Yau surface for which  $\pi_1^{\mathrm{orb}}(X, \Delta)$  is infinite. Let  $\Gamma$  be a N-complement of  $(X, \Delta^{\mathrm{st}})$ . Here N only depends on the dimension (see [42, Theorem 1]). By replacing  $(X, \Delta)$  with  $(X, \Gamma)$ , we may assume that  $K_X + \Delta \sim_{\mathbb{Q}} 0$  and more precisely  $N(K_X + \Delta) \sim 0$ . By Theorem 1, up to a finite cover, we may assume that  $\pi_1^{\mathrm{orb}}(X, \Delta)$  surjects onto an infinite finitely generated abelian group. Hence, for each  $m \geq 0$ , we have a surjective homomorphism  $\pi_1^{\mathrm{orb}}(X, \Delta) \to \mathbb{Z}_m$ . Let  $(Y_m, \Delta_{Y_m}) \to (X, \Delta)$  be the cover associated to the previous surjective homomorphism. Hence,  $(Y_m, \Delta_{Y_m})$  is a log Calabi–Yau surface with a  $\mathbb{Z}_m$ -action. Let  $Y_m \to Y_m'$  be the outcome of a  $\mathbb{Z}_m$ -equivariant  $K_{Y_m}$ -MMP. Let  $Y_m' \to C_m$  be the corresponding  $\mathbb{Z}_m$ -equivariant Mori fiber space. We let  $\Delta_{Y_m'}$  be the push-forwrad of  $\Delta_{Y_m}$  on  $Y_m'$ . If  $C_m$  is an elliptic curve, then (i) holds and we are done. If  $C_m$  is a point then  $Y_m'$  is a Fano type variety. Hence, the statement holds by [69, Theorem 1].

Finally, assume that  $C_m$  is a rational curve. Note that  $N(K_{Y_m} + \Delta_{Y_m}) \sim 0$  and N is independent of m. Consider the short exact sequence

$$(6.1) 1 \to \mathbb{Z}_f \to \mathbb{Z}_m \to \mathbb{Z}_{C_m} \to 1,$$

where  $\mathbb{Z}_f$  is acting fiberwise on  $Y_m \to C_m$  and  $\mathbb{Z}_{C_m}$  is acting on  $C_m$ . By the equivariant canonical bundle formula, we have an induced  $\mathbb{Z}_{C_m}$ -equivariant log Calabi-Yau pair  $(C_m, \Gamma_{C_m})$  for which  $M(K_{C_m} + \Gamma_{C_m}) \sim 0$ . Here, M only depends on N (see, e.g. [70, Lemma 2.32]). By the short exact sequence (6.1), either  $|\mathbb{Z}_f|$  or  $|\mathbb{Z}_{C_m}|$  must be large compared to N. We proceed in two different cases.

Assume that  $|\mathbb{Z}_f|$  is large compared to N. By [69, Proposition 2.46], up to a  $\mathbb{Z}_m$ -equivariant transformation over  $C_m$ , we may assume that  $\lfloor \Delta_{Y'_m} \rfloor$  has two horizontal components over  $C_m$ . Note that [69, Proposition 2.46] assumes that the surface is of Fano type, however, the proof only uses the fact that the surface is of Fano type over the base, which holds in this case. Thus, up to a  $\mathbb{Z}_m$ -equivariant transformation over  $C_m$ , we may assume that  $\Delta_{Y'_m}$  has a vertical component over  $C_m$ . This implies that  $Y'_m$  is of Fano type. Indeed,  $\Delta_{Y'_m}$  supports an ample divisor on this model. Then, the statement follows from [69, Theorem 1].

Assume that  $\mathbb{Z}_{C_m}$  is large compared to N. Hence, we may assume that  $\Gamma_{C_m}$  is supported on  $\{0\}$  and  $\{\infty\}$ . In particular, the pair  $(Y'_m, \Delta_{Y'_m})$  has a log canonical center that maps to either  $\{0\}$  or  $\{\infty\}$  in  $C_m$ . Thus, up to a  $\mathbb{Z}_m$ -equivariant birational transformation, we may assume that  $\Delta_{Y'_m}$  has a vertical component with coefficient one. This implies that  $Y'_m$  is of Fano type. Indeed,  $\Delta_{Y'_m}$  supports an ample divisor on this model. Then, the statement follows from [69, Theorem 1].

## 7. Examples and Questions

In this section, we collect some interesting examples and questions related to the main results. The following two examples show that we can not improve the virtual nilpotency in the statement of Theorem 1.

**Example 7.1.** In this example, we compute the fundamental group of all the principal  $S^1$ -bundles over an elliptic curve E.

First note that for any smooth real manifold M, its classes of isomorphic principal  $S^1$ -bundles is  $H^2(M, \mathbb{Z})$ . In particular, the isomorphic class of principal  $S^1$ -bundles of an elliptic curve E is  $H^2(E, \mathbb{Z}) \cong \mathbb{Z}$ .

Topologically E is obtained by gluing a disk  $\mathbb{D}$  to  $S^1 \vee S^1$ . Set  $U \subset E$  to be the interior of the disc and  $V \subset E$  be an open neighborhood of  $S^1 \vee S^1$  such that V is homotopical to  $S^1 \vee S^1$  and there is a deformation retraction  $r: U \cap V \to S^1$ . Now for each  $k \in \mathbb{Z}$  we set  $B_k$  to be the  $S^1$ -bundle over E by the transition map  $t_k: (U \cap V) \times S^1 \to (U \cap V) \times S^1$ , defined by  $(x, \theta) \to (x, \theta + kr(x))$ , where  $S^1$  is regarded as an additive group.

We apply van Kampen theorem to the open cover  $\{U \times S^1, V \times S^1\}$  of  $B_k$ . We get that  $\pi_1(V \times S^1) = \pi_1((S^1 \vee S^1) \times S^1) = (\mathbb{Z} * \mathbb{Z}) \times \mathbb{Z}$  with generators a, b, c corresponding to the first, second and third factor,  $\pi_1(U \times S^1) = \mathbb{Z}$  with generator d, and  $\pi_1((U \cap V) \times S^1) = \pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$  with generators x, y corresponding to the first and second factor. We may further assume that r retracts  $U \cap V$  to a representative of x. Now regard  $(U \cap V) \times S^1$  as a subspace of  $V \times S^1$  being glued to  $U \times S^1$  by  $t_k$ . Then in  $(U \cap V) \times S^1$ , the inclusion morphism send x to the commutator [a, b] and y to c. Via  $t_k$ , the element x is mapped to kd and k is mapped to k. We thus get a presentation of  $\pi_1(B_k) = \langle a, b, c, d | [a, b] d^{-k}, [a, c], [b, c], cd^{-1} \rangle$  which can then be simplified to be

$$\pi_1(B_k) = \langle a, b, c | [a, b] c^{-k}, [a, c], [b, c] \rangle.$$

The long exact sequence of the fibration  $B_k \to E$  now has the form:

$$1 \to \langle c \rangle \to \langle a, b, c | [a, b] c^{-k}, [a, c], [b, c] \rangle \to \langle a, b | [a, b] \rangle \to 1,$$

where the second arrow maps c to the neutral element in  $\mathbb{Z}^2$ . Thus we see that for any  $u, v \in \pi_1(B_k)$  its commutator [u, v] lies in  $\langle c \rangle$ . Hence the derived length of  $\pi_1(B_k)$  is at most 2, i.e.,  $\pi_1(B_k)$  is metabelian.

If  $k \neq 0$ , then  $\pi_1(B_k)$  is not virtually abelian. Indeed, first note that

$$[a^m, b^m] = (a^m b a^{-m})^m b^{-m} = (c^{mk} b)^m b^{-m} = c^{km^2}$$

for any positive integer m. Suppose that  $H \leq \pi_1(B_k)$  is an abelian normal subgroup of finite index m, then its image H' in  $\langle a, b | [a, b] \rangle = \mathbb{Z}^2$  is also of finite index m. In particular  $(m, 0) \in H'$  and  $(0, m) \in H'$ . Thus for some integers u, v, the elements  $a^m c^u$  and  $b^m c^v$  belong to H. However their commutator is  $[a^m c^u, b^m c^v] = [a^m, b^m] = c^{km^2} \neq 1$ . Hence  $\pi_1(B_k)$  is not virtually abelian.

**Example 7.2.** Let E be an elliptic curve and L be a degree  $k \in \mathbb{Z}$  line bundle over E. The projective bundle  $X_k := \mathbb{P}(\mathcal{O} \oplus L) \to E$  has two disjoint sections  $s_0$  and  $s_\infty$  corresponding to the two factors of  $\mathcal{O} \oplus L$ . The pair  $(X, s_0 + s_\infty)$  is log Calabi–Yau.

First we claim that  $X_k \setminus (s_0 \cup s_\infty)$  is homotopical to the  $S^1$ -bundle  $B_k$  in Example 7.1. Indeed, we cover E by open subsets  $\{U_\alpha\}$  such that on each  $U_\alpha$  there is a trivialization  $\phi_\alpha: L|_{U_\alpha} \cong U_\alpha \times \mathbb{C}$ . We denote by  $\phi_{\alpha\beta} := \phi_\alpha \circ \phi_\beta^{-1}$  the transition function. We can also normalize the transition function. Let h be a Hermitian metric over L and denote by  $h_\alpha: U_\alpha \to \mathbb{R}^+$  its corresponding positive function on  $U_\alpha$ . Now consider the trivialization of L as a complex vector bundle  $\tilde{\phi}_\alpha(v_x) := h_\alpha(x)\phi_\alpha(v_x)$ . The transition functions become  $\tilde{\phi}_{\alpha\beta} = \phi_{\alpha\beta}|\phi_{\alpha\beta}^{-1}|$ . The transition function for the  $\mathbb{P}^1$ -bundle  $X_k = \mathbb{P}(\mathcal{O} \oplus L)$  is given by the class of

$$\begin{pmatrix} 1 & 0 \\ 0 & \phi_{\alpha\beta}^{-1} \end{pmatrix}$$
 in PGL(1,  $\mathbb{C}$ ).

The two sections  $s_0$  and  $s_\infty$  in each  $\mathbb{P}^1 \cong S^2$ -fiber correspond to the north and south poles. Hence after deformation retracting to an  $S^1$ -bundle B, the transition functions become  $\{\tilde{\phi}_{\alpha\beta}^{-1}\}$ . Consider now the following two exponential sequences:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{C}_{\mathbb{R}} \longrightarrow \mathcal{U} \longrightarrow 0,$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{C}_{\mathbb{C}} \longrightarrow \mathcal{C}_{\mathbb{C}}^* \longrightarrow 0$$

where  $\mathcal{U}$  is the sheaf of germs of complex-valued smooth functions of norm 1. Take the cohomology, we get the following diagram

$$H^{1}(E,\mathcal{U}) \longrightarrow H^{2}(E,\mathbb{Z}) \ .$$

$$\downarrow \qquad \qquad \qquad \downarrow_{\mathrm{id}}$$

$$H^{1}(E,\mathcal{C}_{\mathbb{C}}^{*}) \stackrel{\delta}{\longrightarrow} H^{2}(E,\mathbb{Z})$$

The upper row is an isomorphism that maps a cocycle of a  $S^1$ -bundle B to its characteristic class; the lower row maps a cocycle of a complex line bundle L to  $\delta(L) = -c_1(L)$  (cf. [56, Proposition 4.4.12]). In particular, beginning with the cocycle  $\{\tilde{\phi}_{\alpha\beta}^{-1}\}$ , we get  $\int_E -c_1(L^*) = k$ . Hence  $X_k \setminus (s_0 \cup s_\infty)$  is homotopic to  $B_k$ .

Thus  $\pi_1^{\text{orb}}(X, s_0 + s_\infty) = \langle a, b, c | [a, b] c^{-k}, [a, c], [b, c] \rangle$ , which is nilpotent of length 2, and virtually abelian if and only if k = 0.

The following example shows that even if the orbifold fundamental group of a klt log Calabi–Yau surface is virtually abelian, it is not effectively virtually abelian.

**Example 7.3.** Let  $X_k, s_0$  and  $s_\infty$  be the  $\mathbb{P}^1$ -bundle defined in Example 7.2. Note that  $s_\infty - K_{X_k} \sim s_0 + 2s_\infty$  is nef and big and  $s_\infty$  is nef and big. By base point free theorem there exists  $N \gg 0$  such that  $|Ns_\infty|$  has no base point. Let  $D \in |Ns_\infty|$  be a general member. Then  $K_X + s_0 + (1 - \frac{1}{m})s_\infty + \frac{1}{mN}D \sim 0$ . Hence  $(X_k, s_0 + (1 - \frac{1}{m})s_\infty + \frac{1}{mN}D)$  is log Calabi–Yau. Note that by definition

$$G_m := \pi_1^{\text{orb}} \left( X_k, s_0 + \left( 1 - \frac{1}{m} \right) s_\infty + \frac{1}{mN} D \right) = \pi_1^{\text{orb}} \left( X_k, s_0 + \left( 1 - \frac{1}{m} \right) s_\infty \right).$$

Let  $\gamma_{\infty}$  be a meridian loop  $s_{\infty}$ . Then  $G_m = \pi_1^{\mathrm{orb}}(X_k, s_0 + s_{\infty})/N$ , where N is the normal sub-group generated by  $\gamma_{\infty}^m$ . We represent  $\pi_1^{\mathrm{orb}}(X_k, s_0 + s_{\infty}) = \pi_1(B_k) = \langle a, b, c | [a, b] c^{-k}, [a, c], [b, c] \rangle$  as in Example 7.1. Note

that c comes from an  $S^1$ -fiber over E. Hence  $\gamma_{\infty}$  will have c as image in  $\pi_1(B_k)$ . Thus we get

$$\pi_1^{\text{orb}}\left(X_k, s_0 + \left(1 - \frac{1}{m}\right)s_\infty\right) = \langle a, b, c | [a, b]c^{-k}, [a, c], [b, c], c^m \rangle.$$

Consider the subgroup A generated by  $a^m$  and  $b^m$ . By Equation 7.1 one has  $[a^m, b^m] = c^{km^2} = 1$ . One easily checks that  $[a, b^m] = c^{-mk} = 1$  and  $[b, a^m] = c^{mk} = 1$ . Hence A is a normal abelian subgroup of  $G_m$  with finite index. Hence  $G_m$  is virtually abelian.

However, here for the family  $(G_m)_{m\in\mathbb{N}}$  we do not have effective virtual abelianity. In other words, we can not find a universal constant C such that each group  $G_m$  has a normal abelian subgroup of finite index at most C. Indeed, suppose that  $A' \leq G_m$  is the normal abelian subgroup of smallest index l. The image of A' in  $\mathbb{Z}^2$  is a subgroup of finite index l. The argument in the end of Example 7.1 shows that  $[a^l, b^l] = c^{kl^2}$ . Hence we have that  $l \geq \sqrt{\frac{m}{k}}$ .

The article [30] discusses the fundamental group of the smooth locus of a normal Calabi-Yau surface with canonical singularities. Inspired by a question from F. Catanese, we describe some finite abstract groups that can appear as such fundamental groups.

**Example 7.4.** Let G be a subgroup of one of the groups in the list by Mukai [73], acting faithfully and symplectically on a smooth algebraic K3 surface S. Consider  $p: S \to \Sigma := S/G$ . Since G acts symplectically, any element of G fixes finitely many points, in particular the finite morphism p is quasi-étale. The surface  $\Sigma$  is thus a normal Gorenstein Calabi-Yau surface, with quotient, hence klt, singularities. Hence,  $\Sigma$  has in fact canonical singularities. By the Galois correspondence (Lemma 2.32) and since S is simply connected, we have  $\pi_1(\Sigma_{\text{reg}}) = G$ .

The following classical example of the group  $\mathfrak{A}_6$  lies in Mukai's list. It allows us to realize  $\mathfrak{A}_6$  as the fundamental group of the smooth locus of a normal Calabi–Yau surface with canonical singularities.

**Example 7.5.** Consider the K3 surface  $S \subset \mathbb{P}^5 := \{\sum x_i = \sum x_i^2 = \sum x_i^3 = 0\}$ . The group  $\mathfrak{S}_6$  acts on  $\mathbb{P}^5$  by permuting the coordinates, and this action restricts to S. Every transposition is anti-symplectic, and fixes a locus of codimension 1 in S. This yields a symplectic action of  $\mathfrak{A}_6$ , to which Example 7.4 applies.

Note that the finite Galois cover  $p: S \to S/\mathfrak{S}_6$  ramifies in codimension 1. It is compatible with the klt Calabi–Yau pairs (S,0) and  $(S/\mathfrak{S}_6, \operatorname{Branch}(p))$ , so by Galois correspondence (Lemma 2.32), we have  $\pi_1^{\operatorname{orb}}(S/\mathfrak{S}_6, \operatorname{Branch}(p)) = \mathfrak{S}_6$ .

The following examples show that there exists log Calabi–Yau surface  $(X, \Delta)$  for which  $\pi_1^{\text{orb}}(X, \Delta)$  is finite and admits no metabelian subgroups of index less than 7200.

**Example 7.6.** We first construct a log Calabi–Yau surface  $(X, \Delta)$  with fundamental group  $\pi_1^{\text{orb}}(X, \Delta) = (\mathfrak{A}_5 \times \mathfrak{A}_5) \rtimes \mathbb{Z}/2\mathbb{Z}$ . Then we show that  $(\mathfrak{A}_5 \times \mathfrak{A}_5) \rtimes \mathbb{Z}/2\mathbb{Z}$  has no metabelian subgroups except  $\{1\}$ . Note that the order of  $(\mathfrak{A}_5 \times \mathfrak{A}_5) \rtimes \mathbb{Z}/2\mathbb{Z}$  is 7200 hence it has no metabelian subgroups of index less than 7200.

Recall that there is a faithful projective representation of  $\mathfrak{A}_5$  into  $\operatorname{PGL}(2,\mathbb{C})$ , yielding an action  $\mathbb{P}^1$ , and thus an action of  $(\mathfrak{A}_5 \times \mathfrak{A}_5) \rtimes \mathbb{Z}/2\mathbb{Z}$  on  $\mathbb{P}^1 \times \mathbb{P}^1$ . Now the finite subgroup of  $\operatorname{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$  generated by  $\mathfrak{A}_5 \times \mathfrak{A}_5$  and the involution  $\tau$  is  $(\mathfrak{A}_5 \times \mathfrak{A}_5) \rtimes \mathbb{Z}/2\mathbb{Z}$  with the  $\mathbb{Z}/2\mathbb{Z}$  acting on  $\mathfrak{A}_5 \times \mathfrak{A}_5$  by  $\tau.(g_1, g_2) = (g_2, g_1)$ . Set  $Y = \mathbb{P}^1 \times \mathbb{P}^1$  and X the quotient of Y by  $(\mathfrak{A}_5 \times \mathfrak{A}_5) \rtimes \mathbb{Z}/2\mathbb{Z}$ . We have

$$\pi: (Y,0) \to (X,\Delta),$$

with  $\pi^*(K_X + \Delta) \sim K_Y$  where  $\Delta$  is the branch divisor of  $\pi$ . In particular  $(X, \Delta)$  is a log Fano surface. By Galois correspondence, we have

$$1 \to \pi_1^{\mathrm{orb}}(Y,0) \to \pi_1^{\mathrm{orb}}(X,\Delta) \to (\mathfrak{A}_5 \times \mathfrak{A}_5) \rtimes \mathbb{Z}/2\mathbb{Z} \to 1.$$

As  $\pi_1(Y) = 1$ , we have thus

$$\pi_1^{\mathrm{orb}}(X,\Delta) = (\mathfrak{A}_5 \times \mathfrak{A}_5) \rtimes \mathbb{Z}/2\mathbb{Z}.$$

Now take one general curve  $C_1 \cong \mathbb{P}^1 \times \{\text{pt}\}$  and one general curve  $C_2 \cong \{\text{pt}\} \times \mathbb{P}^1$ . In particular the generic points of these curves lie outside the ramification locus of  $\pi$ . Set  $D' := \frac{1}{7200}(\pi_*C_1 + \pi_*C_2)$  then  $\pi^*(D') = C_1 + C_2$  and D' has no common components with  $\Delta$ . The pair  $(X, \Delta + D')$  is then log Calabi–Yau and  $(\Delta + D')^{\text{st}} = \Delta$ . Thus  $\pi_1^{\text{orb}}(X, \Delta + D') = (\mathfrak{A}_5 \times \mathfrak{A}_5) \rtimes \mathbb{Z}/2\mathbb{Z}$ 

We now show that  $G := \pi_1^{\text{orb}}(X, \Delta)$  has no metabelian normal subgroups other than  $\{1\}$ . Consider the short exact sequence

$$1 \to \mathfrak{A}_5 \times \mathfrak{A}_5 \to G \to \mathbb{Z}/2\mathbb{Z} \to 1.$$

First G is not metabelian. Indeed, its commutator subgroup [G,G] is mapped to 1 in  $\mathbb{Z}/2\mathbb{Z}$  hence  $[G,G]=\mathfrak{A}_5\times\mathfrak{A}_5$  which is not abelian. Now suppose that H is a proper normal subgroup of G then  $H\leqslant\mathfrak{A}_5\times\mathfrak{A}_5$ . In particular, the subgroup H can only be  $\mathfrak{A}_5\times\mathfrak{A}_5$ ,  $\mathfrak{A}_5\times\{1\}$ ,  $\{1\}\times\mathfrak{A}_5$  or  $\{1\}$ . Except for the trivial case, each of the first three cases has non-abelian commutator subgroup, hence is not metabelian.

We conclude with two questions of interest.

Question 7.7. Which are the finite groups that act on log Calabi-Yau surfaces?

The finite subgroups of the plane Cremona are precisely the finite groups that act on some smooth rational surface. These groups are classified by the work of Dolgachev and Iskovskikh [37]. It would be interesting to have a complete classification of all finite groups that act on a surface of log Calabi–Yau type. Every such subgroup is the homomorphic image of the fundamental group of a log Calabi–Yau surface. Thus, Theorem 1 will serve as a tool to give a classification of these finite groups. Note that every finite group acts on some canonically polarized surface. Hence, this classification is only interesting for finite actions on surfaces with non-negative curvature. Note that an answer to the previous question may shed light on group actions on log Calabi–Yau surfaces, whenever such a group has interesting finite subgroups.

**Question 7.8.** Let  $(X, \Delta)$  be a n-dimensional log Calabi–Yau pair with  $\pi_1^{\text{orb}}(X, \Delta)$  being infinite. How does this reflect on the geometry pair  $(X, \Delta)$ ?

Theorem 4 gives an answer to this in dimension 2. Up to a finite cover and a birational transformation, we either have a  $\mathbb{G}_m$ -action or a fibration onto an elliptic curve. In higher dimensions, it is natural to expect a similar behavior. For instance, in the context of Question 7.8, it is natural to expect that up to a finite cover and a birational transformation X admits a fibration to an abelian variety or a variety with a conic bundle structure. However, in higher dimensions, very little is known about the orbifold fundamental group of log Calabi–Yau pairs. Hence, it may be more reasonable to study the algebraic fundamental group  $\pi_1^{\text{alg}}(X, \Delta)$  first.

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Humboldt-Universität Zu Berlin, Institut für Mathematik, Unter den Linden 6, 10099 Berlin, Germany  $Email\ address$ : cecile.gachet@hu-berlin.de

Institute for Basic Science, Center for Complex Geometry, 34126 Daejeon, Republic of Korea Email address: zhining.liu@ibs.re.kr

UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555, USA  $Email\ address:$ jmoraga@math.ucla.edu