Stochastic Finance (FIN 519)
Homework Solutions

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1. **HW 1-1** Consider the gambler's fortune with an unfair coin:

$$S_n = X_1 + X_2 + \cdots X_n \quad \text{where} \quad X_n = \begin{cases} 1 & \text{(probability } p) \\ -1 & \text{(probability } q) \end{cases}.$$

   (a) Prove that $M_n = (q/p)^{S_n}$ is a martingale.

   (b) If τ is the first time n that S_n hits A or $-B$, find $\text{Prob}(S_\tau = A)$ using the martingale property,

$$1 = M_0 = E(M_{n\wedge\tau}) \text{ for all } n = E(M_\tau).$$

Solution:

(a)

$$E(M_{n+1}|\mathcal{F}_n) = M_n E\left((q/p)^{X_{n+1}}\right) = \left(\frac{q}{p}p + \frac{p}{q}q\right) M_n = M_n$$

(b)

$$1 = E(M_\tau) = \text{Prob}(S_\tau = A)(q/p)^A + (1 - \text{Prob}(S_\tau = A))(q/p)^{-B}$$
$$\text{Prob}(S_\tau = A) = \frac{1 - (q/p)^{-B}}{(q/p)^A - (q/p)^{-B}} = \frac{(q/p)^B - 1}{(q/p)^{A+B} - 1}$$

2. **HW 1-2** Prove that, if B_t is a standard BM, the inverted process,

$$Y_0 = 0 \quad \text{and} \quad Y_t = t\, B_{1/t} \quad \text{for} \quad t > 0,$$

is also a standard BM.

Solution: Y_t satisfy the following requirements to be a standard BM:

(i) $Y_0 = 0$ by definition.

(ii) The increments of Y_t are independent because they are the (negative) increments of B_t. For example, $Y_{t_2} - Y_{t_1} = B_{1/t_2} - B_{1/t_1} = B_{t'_2} - B_{t'_1}$ with $t'_1 = 1/t_1$ and $t'_2 = 1/t_2$.

(iii) For $s \le t$,

$$Y_t - Y_s = tB_{1/t} - sB_{1/s} = (t-s)B_{1/t} + s(B_{1/t} - B_{1/s}).$$

Since

$$(t-s)B_{1/t} \sim N(0, (t-s)^2/t) \quad \text{and} \quad s(B_{1/t} - B_{1/s}) \sim N(0, s - s^2/t),$$

and $B_{1/t}$ and $B_{1/t} - B_{1/s}$ are independent from (iii), $Y_t - Y_s$ are normally distribute with variance

$$\mathrm{Var}(Y_t - Y_s) = (t-s)^2/t + s - s^2/t = t - s.$$

(iv) Y_t is continuous for $t > 0$. Y_t is also continuous at $t = 0$. $\lim_{t\to 0} Y_t \to Y_0 = 0$ because $E(Y_t) \to 0$ and $\mathrm{Var}(Y_t) = t^2/t = t \to 0$ as $t \to 0$.

3. **HW 2-1** The OU process is given by

$$dX_t = -\alpha X_t dt + \sigma dB_t \quad \text{for} \quad \alpha > 0.$$

(a) Find $\mathrm{Cov}(X_s, X_t)$ and $\mathrm{Corr}(X_s, X_t)$.

(b) When $\alpha \to 0$, the OU process converges to the BM with volatility σ. Therefore, show that

$$\lim_{\alpha\to 0} \mathrm{Cov}(X_s, X_t) \to \sigma^2 \min(s,t).$$

Solution:

(a) The OU process is represented as

$$X_t = X_0 e^{-\alpha t} + \sigma e^{-\alpha t}\int_0^t e^{\alpha u} dB_u = X_0 e^{-\alpha t} + \frac{\sigma e^{-\alpha t}}{\sqrt{2\alpha}} B'_{e^{2\alpha t}-1},$$

where B'_t is another BM independent from B_t. Therefore, the covariance can be obtained as

$$\begin{aligned}
\mathrm{Cov}(X_s, X_t) &= \frac{\sigma^2}{2\alpha} e^{-\alpha s} e^{-\alpha t}\, \mathrm{Cov}\left(B'_{e^{2\alpha s}-1}, B'_{e^{2\alpha t}-1}\right) \\
&= \frac{\sigma^2\, e^{-\alpha(s+t)}}{2\alpha} \min(e^{2\alpha s}-1, e^{2\alpha t}-1) = \frac{\sigma^2\, e^{-\alpha(s+t)}}{2\alpha}(e^{2\alpha \min(s,t)}-1) \\
&= \frac{\sigma^2}{2\alpha}\left(e^{-\alpha|t-s|} - e^{-\alpha(t+s)}\right)
\end{aligned}$$

(b) When $\alpha \to 0$, the covariance of the OU process converges to that of the BM with volatility σ.

$$\begin{aligned}
\lim_{\alpha\to 0} \mathrm{Cov}(X_s, X_t) &= \frac{\sigma^2}{2\alpha}\left(e^{-\alpha|t-s|} - e^{-\alpha(t+s)}\right) = \frac{\sigma^2}{2\alpha}\left(-\alpha|t-s| + \alpha(t+s)\right) \\
&= \frac{\sigma^2}{2}\left((t+s) - |t-s|\right) = \sigma^2 \min(s,t).
\end{aligned}$$

4. **HW 2-2** The inhomogeneous geometric brownian motion (IGBM) is the geometric BM with mean reversion. The SDE is given by

$$dX_t = \lambda(X_\infty - X_t)dt + \sigma X_t dB_t \quad \text{for} \quad \lambda,\ \sigma > 0.$$

Prove that the solution for X_t is given by

$$X_t = e^{-\left(\lambda+\frac{\sigma^2}{2}\right)t+\sigma B_t}\left(X_0 + \lambda X_\infty \int_0^t e^{\left(\lambda+\frac{\sigma^2}{2}\right)s-\sigma B_s} ds\right).$$

What happens if $\lambda = 0$? **Hint:** consider the stochastic derivative (i.e., dY_t) of

$$Y_t = X_t\, e^{\left(\lambda+\frac{\sigma^2}{2}\right)t-\sigma B_t}.$$

Solution: Following the hint, we take the stochastic derivative of Y_t:

$$\begin{aligned}
dY_t &= e^{\left(\lambda+\frac{\sigma^2}{2}\right)t-\sigma B_t}\left(dX_t + X_t\left((\lambda+\frac{\sigma^2}{2})dt - \sigma dB_t + \frac{\sigma^2}{2}(dB_t)^2\right) - \sigma(dB_t)(dX_t)\right) \\
&= e^{\left(\lambda+\frac{\sigma^2}{2}\right)t-\sigma B_t}\left(\lambda(X_\infty - X_t)dt + \sigma X_t dB_t + X_t\left(\left(\lambda+\sigma^2\right)dt - \sigma dB_t\right) - \sigma(dB_t)(\sigma X_t dB_t)\right) \\
&= \lambda X_\infty e^{\left(\lambda+\frac{\sigma^2}{2}\right)t-\sigma B_t} dt.
\end{aligned}$$

Here, you should not forget to include the second-order cross term (red). Integrating both sides, we obtain

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\begin{aligned}
Y_t &= Y_0 + \lambda X_\infty \int_0^t e^{\left(\lambda + \frac{\sigma^2}{2}\right)s - \sigma B_s} ds \\
X_t \, e^{\left(\lambda + \frac{\sigma^2}{2}\right)t - \sigma B_t} &= X_0 + \lambda X_\infty \int_0^t e^{\left(\lambda + \frac{\sigma^2}{2}\right)s - \sigma B_s} ds \\
X_t &= e^{-\left(\lambda + \frac{\sigma^2}{2}\right)t + \sigma B_t} \left(X_0 + \lambda X_\infty \int_0^t e^{\left(\lambda + \frac{\sigma^2}{2}\right)s - \sigma B_s} ds \right).
\end{aligned}
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