

# Option Pricing under the Bachelier (Normal) Model Stochastic Finance (FIN 519)

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# Bachelier vs Black-Scholes-Merton model

- Let  $F_t$  be the  $T$ -forward price of stock price  $S_t$  observed at time  $t$ :

$$F_t = e^{(r-q)(T-t)} S_t \quad (F_T = S_T),$$

where  $r$  is interest rate,  $q$  is dividend rate and  $T$  is the time-to-expiry.

- Then,  $F_t$  is a martingale. (However, let us safely assume  $r = q = 0$ , so  $F_t = S_t$  for now.)
- Under the Bachelier model,  $S_T$  follows an arithmetic Brownian motion (BM) with volatility  $\sigma_N$ :

$$S_t = S_0 + \sigma_N B_t \quad (\text{SDE: } dS_t = \sigma_N dB_t).$$

- Under the Black-Scholes-Merton (BSM) model,  $S_T$  follows an geometric BM:

$$S_t = S_0 \exp \left( \sigma_{BS} B_t - \frac{1}{2} \sigma_{BS}^2 t \right) \quad \left( \text{SDE: } \frac{dS_t}{S_t} = \sigma_{BS} dB_t \right).$$

- The two models are approximately same if the two volatilities are related by

$$\sigma_N = S_0 \sigma_{BS}.$$

# Bachelier model

## Also known as

- Bachelier model (vs Black-Scholes-Merton model)
- Normal process (vs Log-normal process)
- Arithmetic BM (vs Geometric BM)

## Why Bachelier model?

- Bachelier model, once forgotten, has gained attention recently.
- Provides a model dynamics for some underlying assets. Daily changes are independent of the level of the price level (interest rate, inflation rate)
- Price can be indeed negative:
  - Negative (or near zero) interest rate after the 2008 financial crisis.
  - Negative oil futures due to the pandemic recession (April 2020); See [CME Model Switch](#).
- More intuitive than Black-Scholes-Merton

# Call Option Price

Underlying asset price at maturity  $T$ :

$$S_T = S_0 + \sigma_N B_T = S_0 + \sigma_N \sqrt{T} z, \quad \text{where } z \sim N(0, 1)$$

Payoff:

$$\begin{aligned} \max(S_T - K, 0) &= (S_T - K)^+ = (S_0 - K + \sigma_N \sqrt{T} z)^+ \\ S_T = K &\Rightarrow z = -d_N = \frac{K - S_0}{\sigma_N \sqrt{T}} \quad \left( d_N = \frac{S_0 - K}{\sigma_N \sqrt{T}} \right) \end{aligned}$$

Forward option value (undiscounted):

$$\begin{aligned} C(K) &= \int_{-d_N}^{\infty} (S_0 - K + \sigma_N \sqrt{T} z) n(z) dz \\ &= (S_0 - K)(1 - N(-d_N)) + \sigma_N \sqrt{T} n(-d_N) \\ &= (S_0 - K)N(d_N) + \sigma_N \sqrt{T} n(d_N) \end{aligned}$$

Here we used

$$\int z n(z) dz = \frac{z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = -n(z) + C.$$

Present option value (discounted):

$$C_0(K) = e^{-rT} C(K)$$

# Put Option Price

Payoff:

$$(K - S_T)^+ = (K - S_0 - \sigma_N \sqrt{T} z)^+$$

$$\text{The root of } S_T = K \Rightarrow z = -d_N = \frac{K - S_0}{\sigma_N \sqrt{T}} \quad \left( d_N = \frac{S_0 - K}{\sigma_N \sqrt{T}} \right)$$

Forward option value (undiscounted):

$$\begin{aligned} P(K) &= \int_{-\infty}^{-d_N} (K - S_0 - \sigma_N \sqrt{T} z) n(z) dz \\ &= (K - S_0) N(-d_N) - \sigma_N \sqrt{T} n(-d_N) \\ &= (K - S_0) N(-d_N) + \sigma_N \sqrt{T} n(d_N) \end{aligned}$$

Present option value (discounted):

$$P_0(K) = e^{-rT} P(K)$$

Put-Call parity holds!

$$\begin{aligned} C(K) - P(K) &= (S_0 - K) N(d_N) - (K - S_0) N(-d_N) \\ &= (S_0 - K) (N(d_N) + N(-d_N)) = S_0 - K \end{aligned}$$

# Option Price (At-The-Money)

If  $K = S_0$  (at-the-money),  $d_N = 0$  and the option prices are

$$C(K = S_0) = P(K = S_0) = \sigma_N \sqrt{T} n(0) = \frac{\sigma_N \sqrt{T}}{\sqrt{2\pi}} \approx 0.4 \sigma_N \sqrt{T}$$

$$\text{Straddle} = C + P \approx 0.8 \sigma_N \sqrt{T}$$

$$C_0(K = S_0) = P_0(K = S_0) = \frac{e^{-rT} \sigma_N \sqrt{T}}{\sqrt{2\pi}} \approx e^{-rT} 0.4 \sigma_N \sqrt{T}$$

Therefore, the option price is proportional to the *width* (or stdev) of the distribution of the future price,  $\sigma_N \sqrt{T}$ , which is consistent with the intuition. Before we derive Black-Scholes formula, let's keep this relation between the volatility and the option price in mind. Even without the Black-Scholes formula (which is somewhat complicated), this relation should give you a very good intuition.

# Greeks (sensitivities of price)

## Delta: sensitivity on the underlying price

$$\frac{\partial C}{\partial S_0} = N(d_N), \quad \frac{\partial P}{\partial S_0} = -N(-d_N) \quad \left( d_N = \frac{S_0 - K}{\sigma_N \sqrt{T}} \right)$$
$$\left( \frac{\partial C}{\partial S_0} - \frac{\partial P}{\partial S_0} = 1 \right)$$

$N(d_N)$  measures how closely the call option price moves with the underlying stock, i.e., how much the option is in-the-money.

## Gamma: convexity on the underlying price

$$\frac{\partial^2 C}{\partial S_0^2} = \frac{\partial^2 P}{\partial S_0^2} = \frac{n(d_N)}{\sigma_N \sqrt{T}}$$

## Vega: sensitivity on the volatility

$$\frac{\partial C}{\partial \sigma_N} = \frac{\partial P}{\partial \sigma_N} = \sqrt{T} n(d_N)$$

# Comparison of the two models

Model	Bachelier (Normal)	BSM (Lognormal)
Reference	Bachelier [1900]	Black-Scholes, Merton [1973]
SDE	Arithmetic BM: $dS_t = \sigma_N dW_t$	Geometric BM: $dS_t/S_t = \sigma_{BS} dW_t$
Asset class	Interest rate, Inflation, Spread	Equity, FX
Moneyness	$d_N = \frac{S_0 - K}{\sigma_N \sqrt{T}}$	$d_{1,2} = \frac{\log(S_0/K)}{\sigma_{BS} \sqrt{T}} \pm \frac{1}{2} \sigma_{BS} \sqrt{T}$
Call option price	$(S_0 - K)N(d_N) + \sigma_N \sqrt{T} n(d_N)$	$S_0 N(d_1) - K N(d_2)$
Equivalent volatility	$\sigma_N \approx S_0 \sigma_{BS}$	
Digital, $P(S_t > K)$	$N(d_N)$	$N(d_2)$
Delta ( $\partial/\partial S_0$ )	$N(d_N)$	$N(d_1)$
Vega ( $\partial/\partial \sigma$ )	$\sqrt{T} n(d_N)$	$S_0 \sqrt{T} n(d_1)$
Gamma ( $\partial^2/\partial S_0^2$ )	$n(d_N)/\sigma_N \sqrt{T}$	$n(d_1)/S_0 \sigma_{BS} \sqrt{T}$
Theta ( $-\partial/\partial T$ )	$-\sigma_N n(d_N)/2\sqrt{T}$	$-S_0 \sigma_{BS} n(d_1)/2\sqrt{T}$



# Generalization

The price at maturity  $T$  has normal distribution with variance  $V_T$  (stdev  $\sqrt{V_T}$ ):

$$X_T = X_0 + \sqrt{V_T}z, \quad \text{where } z \sim N(0,1)$$

Then, for the payoff  $\max(\pm(X_T - K), 0)$ , the option prices are given by

$$\begin{cases} C(K) = (X_0 - K)N(d_N) + \sqrt{V_T} n(d_N) \\ P(K) = (K - X_0)N(-d_N) + \sqrt{V_T} n(d_N), \\ C(K = X_0) = P(K = X_0) = 0.4\sqrt{V_T}, \end{cases} \quad \text{where } d_N = \frac{X_0 - K}{\sqrt{V_T}}$$

- Spread/Basket option

$$X_t = X_0 + aW_t + bZ_t \text{ with } E(W_t Z_t) = \rho t \quad \Rightarrow \quad V_T = (a^2 + 2\rho ab + b^2)T$$

- Asian option

$$X_t = X_0 + \frac{\sigma}{N} \sum_{k=1}^N W_{kT/N} \quad \Rightarrow \quad V_T = \sigma^2 T \sqrt{\frac{15}{32}} \quad (N=4)$$

- Time-varying volatility

$$dS_t = f(t)dB_t \quad \Rightarrow \quad V_T = \int_0^T f^2(t)dt \quad (\text{Itô's isometry})$$

# Homework in the past years

More problems are available in **Problems and Solutions**.

- 1 Derive the (forward) price of the digital(binary) call/put option struck at  $K$  at maturity  $T$ . The digital(binary) call/put option pays \$1 if  $S_T$  is above/below the strike  $K$ , i.e.  $1_{S_T \geq K} / 1_{S_T \leq K}$ .
- 2 The payoff of the call option,  $\max(S_T - K, 0)$  can be decomposed into two parts,

$$S_T \cdot 1_{S_T \geq K} - K \cdot 1_{S_T \geq K}.$$

The first payout is the payout of the **asset-or-nothing** call option and the second payout if the binary call option multiplied with  $-K$ . What is the price of the asset-or-nothing call option?

- 3 Using the joint distribution of  $B_t$  and  $B_t^*$ , derive the price of the call option struck at  $K$  and knock-out at  $K_1$  ( $> K$ ). First, generalize the joint CDF function  $P(u < B_t, v < B_t^*)$  to  $\sigma_N B_t$ . Next, derive the PDF on  $u$  by taking derivative on  $u$ . Then, integrate the payoff  $(S_T - K)^+$  from  $K$  to  $K_1$ . (Assume that the risk-free rate is zero,  $r = 0$ , so that  $S_0 = F$ . Otherwise the problem is too complicated.)