Stochastic Finance (FIN 519)
Midterm Exam

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BM stands for Brownian motion. Assume that B_t is a standard **BM**. **RN** and **RV** stand for random number and random variable respectively. The PDF and CDF of the standard normal distribution are denoted by $n(z)$ and $N(z)$ respectively. You can use $n(z)$ and $N(z)$ in your answers without further evaluation.

1. (5 points) [**Lognormal distribution**] A lognormal random variables with parameters (μ, σ) is given by

$$Y = \mu \exp(\sigma Z - \sigma^2/2) \quad \text{for a standard normal variable} \quad Z.$$

(a) (2 points) Obtain the mean and variance of Y.

(b) (3 points) Suppose that two lognormal random variables are give by

$$Y_1 = \mu_1 \exp(\sigma_1 Z_1 - \sigma_1^2/2) \quad \text{and} \qquad Y_2 = \mu_2 \exp(\sigma_2 Z_2 - \sigma_2^2/2),$$

and that the two standard normals, Z_1 and Z_2, are correlated by ρ (i.e., $E(Z_1 Z_2) = \rho$). Obtain the covariance and correlation between Y_1 and Y_2.

Solution:

(a) The mean and variance are given by μ and $\mu^2(e^{\sigma^2} - 1)$ respectively.

$$E(Y) = \mu E(\exp(\sigma Z - \sigma^2/2)) = \mu.$$
$$E(Y^2) = \mu^2 E(\exp(2\sigma Z - \sigma^2)) = \mu^2 \exp(\sigma^2) E(\exp(2\sigma Z - (2\sigma)^2/2)) = \mu^2 \exp(\sigma^2).$$
$$\mathrm{Var}(Y) = E(Y^2) - E(Y)^2 = \mu^2 \left(\exp(\sigma^2) - 1\right).$$

(b) Using $\sigma_1 Z_1 + \sigma_2 Z_2 \sim N(0, \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)$,

$$\begin{aligned}
E(Y_1 Y_2) &= \mu_1\mu_2 E(\exp(\sigma_1 Z_1 + \sigma_2 Z_2 - (\sigma_1^2 + \sigma_2^2)/2)) \\
&= \mu_1\mu_2 \exp(\rho\sigma_1\sigma_2) E(\exp(\sigma_1 Z_1 + \sigma_2 Z_2 - (\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)/2)) \\
&= \mu_1\mu_2 \exp(\rho\sigma_1\sigma_2) \\
\mathrm{Cov}(Y_1, Y_2) &= E(Y_1 Y_2) - E(Y_1)E(Y_2) = \mu_1\mu_2 \left(\exp(\rho\sigma_1\sigma_2) - 1\right) \\
\mathrm{Corr}(Y_1, Y_2) &= \frac{\mathrm{Cov}(Y_1, Y_2)}{\sqrt{\mathrm{Var}(Y_1)\mathrm{Var}(Y_2)}} = \frac{\exp(\rho\sigma_1\sigma_2) - 1}{\sqrt{(\exp(\sigma_1^2) - 1)(\exp(\sigma_2^2) - 1)}}.
\end{aligned}$$

2. (2×4 points) [**Martingale related to BM**] If B_t is a standard BM, determine whether the following is a martingale or not. Give a brief reason.
   (a) $Y_t = B_{\lambda t}^2 - \lambda^2 t$
   (b) $Y_t = \exp(-B_{at} - a^2 t/2)$
   (c) $Y_t = \begin{cases} 2A - B_t & \text{if} \quad t \le \tau \\ B_t & \text{if} \quad t > \tau \end{cases}$, where τ is the first time B_t hits A ($\tau = \min\{t : B_t = A\}$)
   (d) $S_t = S_0 + \sigma B_{t\wedge\tau}$ where $\tau = \min\{t : S_{t+1} - S_t < -A\}$ for some $A > 0$.

Solution:
(a) No. $Y_t = B_{\lambda t}^2 - \lambda t$ is a martingale.
(b) No. $Y_t = \exp(-B_{at} - at/2)$ is a martingale.
(c) Yes. Y_t is a BM starting with $2A$ from the reflection principle. Although Y_t is not a standard BM, it is still a martingale.
(d) No. τ is not a proper stopping time because it is based on forward-looking information (i.e., S_{t+1}).

3. (3 points) [**Forward-starting option**] A forward-starting option with expiry T is an option whose strike price is set relative to the stock price at time $t = T'$ ($< T$) not at time $t = 0$. Suppose that the strike will be set as $K = S_{T'} + \Delta$ at $t = T'$. Therefore, the payout of the forward-starting call option at expiry T is given by

$$\text{Payout} = \max(S_T - K, 0) = \max(S_T - S_{T'} - \Delta, 0).$$

Assume that the underlying stock price follows a BM, $S_t = S_0 + \sigma B_t$ (and $r = q = 0$). Derive the price of the forward-starting call option. You may use the Bachelier model option formula without proof.

Solution: Since $S_T - S_{T'} = \sigma(B_T - B_{T'}) \sim \sigma\sqrt{T - T'}\, Z$ for a standard normal Z, we can use the Bachelier option price formula with time-to-expiry $T - T'$, spot price 0, and strike price Δ:

$$C = -\Delta N(d_{\text{N}}) + \sigma\sqrt{T - T'}\, n(d_{\text{N}}), \quad d_{\text{N}} = -\frac{\Delta}{\sigma\sqrt{T - T'}}.$$

4. (7 points) [**Stochastic integral**] Based on the highschool calculus, $\int_0^x e^{-x} dx = 1 - e^{-x}$, I make the statement on the following stochastic integral:

$$\int_0^T e^{-B_t} dB_t = 1 - e^{-B_T}.$$

We will check if this is true or false.

(a) (3 points) What is the mean and variance of the left-hand side (LHS)?

(b) (3 points) What is the mean and variance of the right-hand side (RHS)?

(c) (1 point) Is the statement true or false?

Solution:

(a) The mean of LHS is zero by symmetry. Using the Itô's isometry,

$$\mathrm{Var}(\mathrm{LHS}) = E\left(\int_0^T e^{-2B_t}dt\right) = \int_0^T E(e^{-2B_t})dt = \int_0^T e^{2t}dt = \frac{e^{2T}-1}{2}$$

(b) Regarding RHS,

$$\begin{aligned} E(\mathrm{RHS}) &= E(1-e^{-B_T}) = 1 - E(e^{-B_T}) = 1 - e^{T/2}, \\ \mathrm{Var}(\mathrm{RHS}) &= E\left((e^{-B_T} - e^{T/2})^2\right) = E(e^{-2B_T}) - 2e^{T/2}E(e^{-B_T}) + e^T \\ &= e^{2T} - 2e^{T/2}\cdot e^{T/2} + e^T = e^T(e^T-1). \end{aligned}$$

(c) Because the means and variance are not same, the statement is false.

5. (3 points) [**Geometric BM**] Assume that a stock price follows a geometric BM with volatility σ and initial price S_0,

$$S_t = S_0 \exp\left(\sigma B_t - \frac{\sigma^2 t}{2}\right)$$

When you observe the stock price ever year ($t = 1, 2, \cdots$), find the probability that the annual stock return is positive for the following 3 years,

$$P(S_1 > S_0 \text{ and } S_2 > S_1 \text{ and } S_3 > S_2).$$

Express the answer with $n(\cdot)$ or $N(\cdot)$.

Solution: For $t = n$,

$$\begin{aligned} P(S_{n+1} > S_n) &= P(S_{n+1}/S_n > 1) = P(\sigma(B_{n+1} - B_n) - \sigma^2/2 > 0) \\ &= P(\sigma Z - \sigma^2/2 > 0) = P(Z > \sigma/2) \quad \text{for a standard normal} \quad Z \\ &= 1 - N(\sigma/2) \quad \text{or} \quad N(-\sigma/2) \end{aligned}$$

Thanks to the independent increments of BM, $S_1 > S_0$, $S_2 > S_1$, and $S_3 > S_2$ are independent events. Therefore,

$$P(S_1 > S_0 \text{ and } S_2 > S_1 \text{ and } S_3 > S_2) = (1 - N(\sigma/2))^3$$

For example, if $\sigma = 20\%$ ($= 0.2$), the probability is $(1 - N(0.1))^3 \approx (0.460)^3 = 9.74\%$.

6. (4 points) [**Last hitting time of BM**] In class, we know that the **first** hitting time of BM to δ, $\tau = \min\{t : B_t = \delta\}$, has the following CDF:

$$P(\tau \le t) = 2 - 2N(\delta/\sqrt{t}).$$

Instead, let us consider the **last** hitting time τ' as

$$\tau' = \max\{t : B_t = \delta\}.$$

Derive the CDF (i.e., $P(\tau' \le t)$) and PDF of τ'. **Hint**: $Y_t = t\,B_{1/t}$ ($Y_0 = 0$) is also a standard BM.

Solution: If τ is the first hitting time, $1/t$ is the last exit time of BM. Using that $Y_t = t\,B_{1/t}$ is also a standard BM,

$$\begin{aligned}\tau' &= \max\{t : Y_t = \delta\} = \max\left\{t : tB_{1/t} = \delta\right\} = \max\left\{t : B_{1/t} = \delta/t\right\}\\ &= \max\{1/t : B_t = \delta t\} = 1/\min\{t : B_t = \delta t\} = 1/\tau.\end{aligned}$$

Therefore, the CDF and PDF are respectively given by

$$\begin{aligned}P(\tau' \le t) = P(\tau \ge 1/t) = 2N(\delta\sqrt{t}) - 1,\\ f_{\tau'}(t) = \frac{1}{dt}P(\tau' \le t) = \frac{\delta}{\sqrt{t}}\,n(\delta\sqrt{t}).\end{aligned}$$

Note that the PDF is similar to, but different from, the PDF of the first hitting time τ:

$$f_\tau(t) = \frac{\delta}{\sqrt{t^3}}\,n\left(\frac{\delta}{\sqrt{t}}\right).$$

This question was inspired by a question on Math StackExcahnge.