Stochastic Finance (FIN 519) Final Exam

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BM stands for Brownian motion. **RN** and **RV** stand for random number and random variable respectively. Assume that B_t is a standard **BM**. The PDF and CDF for standard normal distribution is denoted by $n(z)$ and $N(z)$. You can use $n(z)$ and $N(z)$ in your answers without further evaluation.

1. (4=1x4 points) [**Stochastic calculus**] Give **True or False**. If false, give the correct answer.
   (a) $(dB_t)^2 = dt$
   (b) $d(B_t^2) = 2(B_t dB_t + dt)$
   (c) $d\sin(B_t) = \cos(B_t)dB_t + \frac{1}{2}\sin(B_t)dt$
   (d) $d\left(\dfrac{1}{1-B_t}\right) = \dfrac{dB_t}{(1-B_t)^2} + \dfrac{dt}{(1-B_t)^3}$

   Solution:
   (a) True.
   (b) False. $d(B_t^2) = 2B_t dB_t + dt$
   (c) False. $d\sin(B_t) = \cos(B_t)dB_t - \frac{1}{2}\sin(B_t)dt$ since $\sin'(x) = \cos(x)$ and $\sin''(x) = -\sin(x)$.
   (d) True.

2. (6=2x3 points) [**Stochastic Calculus**] Let us consider the two stochastic processes, X_t and Y_t:
$$X_t = \sinh(B_t) \quad \text{and} \quad Y_t = e^{B_t}\int_0^t e^{-B_s}dW_s,$$
   where B_t and W_t are two independent standard BMs. Reminded that $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$ and $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$.
   (a) What are the mean and variance of X_t and Y_t respectively?
   (b) Prove that $X_t = \sinh(B_t)$ satisfies the SDE:
$$dX_t = \sqrt{1+X_t^2}\,dB_t + \frac{1}{2}X_t dt.$$
   (c) Derive the SDE for Y_t. What can you say about the distributions of X_t and Y_t ?

   Solution:

(a) The means of X_t and Y_t are zero,

$$E(X_t) = E(Y_t) = 0$$

because $\sinh(-x) = -\sinh(x)$ and the symmetry of Y_t.

The variance of X_t is

$$\begin{aligned}\mathrm{Var}(X_t) &= E\left[\sinh^2(B_t)\right] = \frac{1}{4}E\left[\left(e^{2B_t} + e^{-2B_t} - 2\right)\right] \\ &= \frac{1}{4}\left(e^{2t} + e^{2t} - 2t\right) = \frac{1}{2}\left(e^{2t} - 1\right).\end{aligned}$$

The variance of Y_t is

$$\begin{aligned}\mathrm{Var}(Y_t) &= \int_0^t E[e^{2(B_t - B_s)}]ds = \int_0^t e^{2(t-s)}ds \\ &= e^{2t}\cdot\frac{1}{2}\left(1 - e^{-2t}\right) = \frac{1}{2}\left(e^{2t} - 1\right)\end{aligned}$$

where we used Itô's isometry and the property, $B_t - B_s \sim N(0, t-s)$. Therefore, X_t and Y_t have the same mean and variance.

(b)

$$\begin{aligned}dX_t &= \cosh(B_t)\,dB_t + \frac{1}{2}\sinh(B_t)dt \\ &= \sqrt{1 + X_t^2}\,dB_t + \frac{1}{2}X_t dt.\end{aligned}$$

Here, we used

$$1 + \sinh^2(x) = 1 + \frac{1}{4}(e^x - e^{-x})^2 = \frac{1}{4}(e^x + e^{-x})^2 = \cosh^2(x).$$

(c)

$$\begin{aligned}dY_t &= \left(e^{B_t}dB_t + \frac{1}{2}e^{B_t}dt\right)\int_0^t e^{-B_s}dW_s + e^{B_t}\cdot e^{-B_t}dW_t \\ &= Y_t dB_t + dW_t + \frac{1}{2}Y_t dt \\ &= \sqrt{1 + Y_t^2}\,dB_t' + \frac{1}{2}Y_t dt.\end{aligned}$$

In the last step, we combined $Y_t dB_t + dW_t$ into a new BM, B_t'. The two stochastic processes, X_t and Y_t, start from the same point, $X_0 = Y_0 = 0$, and they have the same SDE from (a) and (b). Therefore, the distributions of X_t and Y_t are the same.

3. (4 points) **[Option theta under the BSM model]** Remind that the present value of the call option under the BSM model is

$$C = S_0 N(d_1) - e^{-rT}KN(d_2) \quad \text{where} \quad d_{1,2} = \frac{\log(S_0\, e^{rT}/K)}{\sigma\sqrt{T}} \pm \frac{1}{2}\sigma\sqrt{T}.$$

Derive the theta, that is the derivative of C with respect to the (decreasing) time-to-maturity:

$$\Theta = -\frac{\partial C}{\partial T}.$$

Hint: Since d_1 and d_2 are the functions of T, you should also differentiate d_1 and d_2 with respect to T.

Solution: Since

$$\frac{\partial d_{1,2}}{\partial T} = -\frac{\log(S_0/K)}{2\sigma T\sqrt{T}} + \frac{r}{2\sigma\sqrt{T}} \pm \frac{\sigma}{4\sqrt{T}},$$

let

$$A = -\frac{\log(S_0/K)}{2\sigma T\sqrt{T}} + \frac{r}{2\sigma\sqrt{T}}, \quad B = \frac{\sigma}{4\sqrt{T}}.$$

From [**2016FE, Option delta under the BSM model**] and [**2016FE, Option vega under the BSM model**], we also know that

$$S_0\, e^{rT} n(d_1) = K n(d_2)$$

Now we can calculate Θ as

$$\begin{aligned}
\Theta = -\frac{\partial C}{\partial T} &= -S_0 n(d_1)\frac{\partial d_1}{\partial T} + e^{-rT} K n(d_2)\frac{\partial d_2}{\partial T} - rKe^{-rT}N(d_2) \\
&= -S_0 n(d_1)(A+B) + e^{-rT}Kn(d_2)(A-B) - rKe^{-rT}N(d_2) \\
&= -S_0 n(d_1)(2B) - rKe^{-rT}N(d_2) \\
&= -\frac{S_0\, n(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2).
\end{aligned}$$

Alternatively, Θ is expressed as

$$\Theta = -\frac{Ke^{-rT}n(d_2)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2) = -Ke^{-rT}\left(\frac{n(d_2)\sigma}{2\sqrt{T}} + rN(d_2)\right).$$

4. (6=2x3 points) [**Implied volatility bound**] Suppose that a call option with strike K and time-to-maturity T is traded at the price C in the market. The implied volatility is σ such that the Black–Scholes formula with σ yields the observed price C. Because there is no analytic solution for $\sigma = \sigma(C, K, S_0, T)$, we solve σ using numerical methods such as bisection or Newton's method. For such numerical root-finding, knowing the bounds of σ can reduce computation. While we can use a simple lower bound $\sigma_L = 0$, we are going to find an upper bound σ_U in this question so that $0 \le \sigma \le \sigma_U$. Assume that $r = q = 0$.

   (a) Before finding σ_U, let us consider a special case first. If C is the market price of the at-the-money option ($K = S_0$), it is possible to obtain the implied volatility σ directly. Find $\sigma = \sigma(C, S_0, T)$. You can use the inverse normal CDF, $N^{-1}(\cdot)$, in your answer. **Hint:** $N(x) + N(-x) = 1$.

   (b) When you hold a call option with strike K, you can maximize the option value by exercising it when $S_T \ge K$. But, imagine that you **incorrectly** exercise the option when $S_T \ge S_0$. In this way, your payout out can be even negative. Derive the option value (i.e., the expected payout) under the incorrect exercise policy.

   (c) Using the result from (b), find an upper bound σ_U. **Hint:** (i) The price from (b) is always lower than (or equal to) the usual BSM price with the same volatility because of the incorrect exercise policy. (ii) The option value is an increasing function of volatility.

Solution:

(a) Because $d_{1,2} = \pm\sigma\sqrt{T}/2$ when $K = S_0$, the at-the-money call option price under the BSM model is

$$C = S_0(N(d_1) - N(-d_1)) = S_0(2N(d_1) - 1)$$

Therefore, the implied volatility is given by

$$\sigma = \frac{2}{\sqrt{T}} N^{-1}\left(\frac{C + S_0}{2S_0}\right).$$

(b) If you (incorrectly) exercise the call option when $S_T \geq K^*$ in general, the option value is

$$C^* = S_0 N(d_1^*) - KN(d_2^*) \quad \text{for} \quad d_{1,2}^* = \frac{\log(S_0/K^*)}{\sigma\sqrt{T}} \pm \frac{\sigma\sqrt{T}}{2}.$$

Note that the only modifications from the BSM formula are d_1 and d_2 because they are related to the exercise boundary. If $K^* = S_0$, the price is given by

$$C^* = S_0 N(d_1^*) - KN(-d_1^*) \quad \text{for} \quad d_1^* = \frac{\sigma\sqrt{T}}{2}.$$

(c) The implied volatility of the incorrect price C^* from (b) is given by

$$C^* = S_0 N(d_1^*) - KN(-d_1^*) = (S_0 + K)N(d_1^*) - K$$
$$\sigma* = \frac{2}{\sqrt{T}} N^{-1}\left(\frac{C^* + K}{S_0 + K}\right).$$

Because of the exercise decision is suboptimal, this price C^* is lower than the BSM price (with correct exercise). In other words, if $C^* = C$, the implied volatility σ^* under the incorrect exercise should be higher than the implied volatility σ under the correct exercise. Therefore, σ^* for C is an upper bound of σ:

$$\sigma \leq \sigma_U = \frac{2}{\sqrt{T}} N^{-1}\left(\frac{C + K}{S_0 + K}\right).$$

Here we note that (i) when C approaches the maximum call option value S_0, $\sigma_U \to \infty$ as expected, and (ii) when $K = S_0$, the upper bound is the exact implied volatility ($\sigma = \sigma_U$) in (a).

5. (4=2x2 points) **[Equivalent Martingale Measure]** Let β_t ($\beta_0 = 1$) be the value of the saving account and $P(t, T)$ be the price of the zero-coupon bond maturing at T, observed at time t. Give **True or False**. If false, briefly explain why.

(a) For a stock price S_t and the risk-neutral measure Q,

$$(S_0 - K)^+ = E^Q\left(\frac{(S_T - K)^+}{\beta_T}\right)$$

(b) For the stochastic process of a security price X_t,

$$X_0 = P(0, T)E^Q(X_T)$$

Solution:

(a) False. If $S_0 < K$ for instance, the left-hand side is zero, but the right-hand side is positive. The statement is true for the option price X_t that pays $(S_T - K)^+$ at T.

$$X_0 = E^Q\left(\frac{X_T}{\beta_T}\right) = E^Q\left(\frac{(S_T - K)^+}{\beta_T}\right)$$

(b) False. The expectation should be under the T-forward measure:

$$X_0 = P(0,T)E^T(X_T)$$

6. (6 points) **[Interest rate caplet pricing]** We are going to price *caplet*, that is an option on the deposit rate L between T and $T+\Delta$. As in the class, let $L(t,T)$ be the forward rate of making a deposit from T to $T+\Delta$ observed at t. If you hold a caplet with strike price K with maturity T, you have a right (not an obligation) against a bank to make a deposit of 1 at the deposit rate K for the period Δ. Of course, you will exercise this option only when K is higher than the current deposit rate $L(T,T)$ at $t = T$ offered by the bank. So your payout of the caplet is

$$\text{Payout} = \Delta(K - L(T,T))^+.$$

However, caplet is different from equity options is that, although the caplet expiry is at $t = T$, the payout is paid at $t = T + \Delta$ when the deposit matures.

Suppose that the SDE for r_t is given by the Ho–Lee model:

$$r_t = r_0 + \alpha t + \beta B_t^Q \quad (dr_t = \alpha dt + \beta\, dB_t^Q),$$

where B_t^Q is a standard BM under the risk-neutral measure. Follow the next steps to price the caplet under this model. You may use the results of **[2017FE, Interest rate and bond price SDE]** and **[2018FE, T-forward measure]**.

(a) From no-arbitrage replication, we know that

$$L(t,T) = \frac{P(t,T) - P(t,T+\Delta)}{\Delta \cdot P(t,T+\Delta)},$$

where $P(t,T)$ is the time t price of the zero-coupon bond maturing at T. Derive the SDE for $L(t,T)$ under the risk-neutral measure (i.e., using B_t^Q).

(b) What is the SDE for $L(t,T)$ under the $(T+\Delta)$-forward measure (i.e., with $B_t^{T+\Delta}$)?

(c) Using the $(T+\Delta)$-forward measure, derive present value of the at-the-money caplet? The at-the-money strike is $K = L(0,T)$.

Solution:

(a) Let

$$H_t = 1 + \Delta L(t,T) = \frac{P(t,T)}{P(t,T+\Delta)}$$

and we first derive the SDE of H_t. From **[2018FE, T-forward measure]**,

$$d\log P(t,T) = (r_t - \sigma^2/2)dt + \sigma\, dB_t^Q \quad (\sigma = -\beta(T-t)),$$
$$d\log P(t,T+\Delta) = (r_t - \sigma'^2/2)dt + \sigma'\, dB_t^Q \quad (\sigma' = -\beta(T+\Delta-t)),$$

we get

$$\begin{aligned} d\log H_t &= -\frac{1}{2}(\sigma^2 - \sigma'^2)dt + (\sigma - \sigma')dB_t^Q, \\ \frac{dH_t}{H_t} &= -\frac{1}{2}(\sigma^2 - \sigma'^2)dt + \frac{1}{2}(\sigma - \sigma')^2 dt + (\sigma - \sigma')dB_t^Q \\ &= -\sigma'(\sigma - \sigma')dt + (\sigma - \sigma')dB_t^Q \\ &= \Delta\beta^2(T + \Delta - t)dt + \Delta\beta dB_t^Q \end{aligned}$$

Therefore,

$$\begin{aligned} dL(t,T) &= \frac{1 + \Delta L(t,T)}{\Delta}(\Delta\beta^2(T + \Delta - t)dt + \Delta\beta dB_t^Q) \\ &= (1 + \Delta L(t,T))(\beta^2(T + \Delta - t)dt + \beta dB_t^Q) \end{aligned}$$

(b) The numeraire $P(t, T + \Delta)$ has the volatility $\sigma' = -\beta(T + \Delta - t)$. The BM under the $(T + \Delta)$-forward measure and the BM under the risk-neutral measure are related by

$$dB_t^Q = dB_t^{T+\Delta} - \beta(T + \Delta - t)dt.$$

Therefore, the SDE for $L(t,T)$ under the $(T + \Delta)$-forward measure is

$$dL(t,T) = \beta(1 + \Delta L(t,T))\, dB_t^{T+\Delta}.$$

Note that $L(t,T)$ is martingale under the $(T+\Delta)$-forward measure and this is already expected because of $P(t, T + \Delta)$ in the denominator of the equation for $L(t,T)$.

(c) We use H_t for the caplet pricing because H_t follows a geometric BM with volatility $\Delta\beta$:

$$\frac{dH_t}{H_t} = \Delta\beta\, dB_t^{T+\Delta}.$$

$$\begin{aligned} C &= P(0, T + \Delta)E^{T+\Delta}\left(\Delta \cdot (L(0,T) - L(T,T))^+\right) \\ &= P(0, T + \Delta)E^{T+\Delta}\left((1 + \Delta L(0,T) - H_T)^+\right) \end{aligned}$$

The bond price $P(0, T + \Delta)$ is given from **[2017FE, Interest rate and bond price SDE] (c)**:

$$P(0, T + \Delta) = \exp\left(-(T + \Delta)r_0 - \frac{1}{2}\alpha(T + \Delta)^2 + \frac{1}{6}\beta^2(T + \Delta)^3\right).$$

The expectation part is obtained by the BS formula for the put option with strike and spot $(1 + \Delta L(0,T))$:

$$\begin{aligned} E^{T+\Delta}\left((1 + \Delta L(0,T) - H_T)^+\right) &= (1 + \Delta L(0,T))N(d_1) - (1 + \Delta L(0,T))N(-d_1) \\ &= (1 + \Delta L(0,T))(2N(d_1) - 1) \end{aligned}$$

where

$$d_1 = \frac{\Delta\beta\sqrt{T}}{2}.$$

Finally, the caplet price is given by

$$C = \exp\left(-(T + \Delta)r_0 - \frac{1}{2}\alpha(T + \Delta)^2 + \frac{1}{6}\beta^2(T + \Delta)^3\right)(1 + \Delta L(0,T))(2N(d_1) - 1)$$