Probability and Statistics Review
Stochastic Finance (FIN 519)

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- Random Variable (RV): U, X, Y, Z
- Probability density function (PDF): $f_X(x)$
- Cumulative distribution function (CDF): $F_X(x) = \int f_X(x)dx$
- Standard deviation, variance:

$$\text{Var}(X) = E((X-\bar{X})^2) = E(X^2) - E(X)^2, \quad \sigma_X = \sqrt{Var(X)}$$

- (Centralized) Moments: $M_k(X) = E((X-\bar{X})^k) = \int (x-\bar{X})^k f_X(x)dx$
- Moment generating function (MGF): $M_X(t) = E(e^{tX})$

$$M_X(t) = 1 + tM_1 + \frac{t^2}{2!}M_2 + \cdots + \frac{t^k}{k!}M_k$$

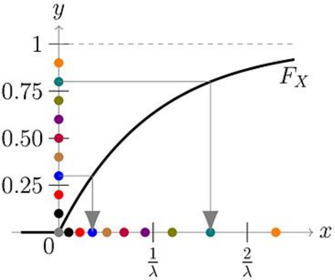
- Characteristic function (CF): $\phi_X(t) = E(e^{itX}) + \cdots$
- Covariance: $\text{Cov}(X,Y) = E((X-\bar{X})(Y-\bar{Y})) = E(XY) - E(X)E(Y)$
- Correlation: $\rho(X,Y) = \text{Cov}(X,Y)/\sqrt{\text{Var}(X)\text{Var}(Y)} = \text{Cov}(X,Y)/(\sigma_X\sigma_Y)$

Properties

- Support: $[0, 1]$
- PDF: $f(x) = 1$
- CDF: $F(x) = x$
- Mean: $E(U) = 1/2$
- Var: $\text{Var}(U) = 1/12$

Uniform distribution is a fundamental RV which can be generated by computer. Once U is generated, any RV X is generated by inverse transform sampling

$$X = F_X^{-1}(U)$$



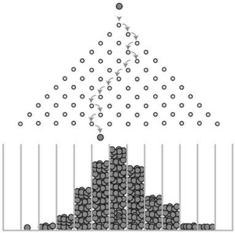
Prob. Distributions: True/False, Up/Down, Win/Lose, etc

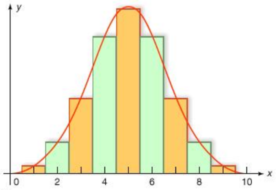
Bernoulli distribution

- $P(X = 1) = p, \quad P(X = 0) = q = (1 - p)$
- $E(X) = p, \quad \mathrm{Var}(X) = pq$

Binomial distribution

- $Y = \sum_1^n X_k \sim N(n, p)$ for i.i.d. Bernoulli $\{X_k\}$ with p.
- $P(Y = k) = \binom{n}{k} p^k q^{(n-k)}$
- $E(Y) = np, \quad \mathrm{Var}(Y) = \sum_1^n \mathrm{Var}(X_k) = npq.$
- Approximated as normal dist. for large n: $B(n, p) \approx N(np, npq)$





Prob. Distribution: Event (default, arrival) at a constant rate λ

Exponential distribution

- Distribution for the survival time or the interval between the events, T
- PDF: $f(t) = \lambda e^{-\lambda t}$, CDF: $F(t) = 1 - e^{-\lambda t}$
- $E(T) = 1/\lambda$, $\mathrm{Var}(T) = 1/\lambda^2$.
- Memoryless: past events have no impact on the future!

Poisson distribution (discrete)

- The number of occurrences X of a Poisson-type event in a unit time interval $T = 1$
- PDF: $P(X = k) = \lambda^k e^{-\lambda}/k!$
- $E(X) = \mathrm{Var}(X) = \lambda$

Gamma distribution

- The distribution of time X before the next k Poisson-type events occur
- $X \sim \Gamma(\alpha, \beta)$ where $\alpha = k$, $\beta = \lambda$.
- PDF: $f(x) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}$ for $x \geq 0$ and $\alpha, \beta > 0$.
- $E(X) = \alpha/\beta, \mathrm{Var}(X) = \alpha/\beta^2$.

- $X \sim N(\mu, \sigma^2), \quad Z \sim N(0,1)$
- PDF: $f_X(x) = \frac{1}{\sqrt{2\sigma^2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) = \frac{1}{\sigma} n(\frac{x-\mu}{\sigma})$
- CDF: $F_X(x) = N(\frac{x-\mu}{\sigma})$
- MGF: $M_X(x) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right), \quad M_k = \sigma^k (k-1)!!$ for even k.
- Skewness: $s = M_3/\sigma^3 = 0$, Kurtosis $\kappa = M_4/\sigma^4 = 3$ (Ex-kurtosis: 0).

Variations

- Multivariate normal distribution: $(X_1, \cdots, X_n)$
- Log-normal distribution: $Y \sim e^{\mu + \sigma Z}$ for standard normal Z.

Conditional Probability and Independence

Conditional Probability

A probability of an event A given that an event B has occurred.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Independence

The two events A and B are (statistically) independent if $\boxed{P(A \cap B) = P(A)P(B)}$.
Equivalently,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A) \quad \text{if} \quad P(B) \neq 0$$
$$\text{and} \quad P(B|A) = P(B) \quad \text{if} \quad P(A) \neq 0$$

- Joint CDF: $F_{X,Y}(x,y) = F_X(x)F_Y(y)$ $\big(\mathrm{P}(X \le x, Y < y) = \mathrm{P}(X \le x)P(Y \le y)\big)$
- Joint PDF: $f_{X,Y}(x,y) = f_X(x)f_Y(y)$
- $E(XY) = E(X)E(Y)$, $\mathrm{Cov}(X,Y) = \rho(X,Y) = 0$. However, $\rho(X,Y) = 0$ does not imply independence.