Stochastic Finance (FIN 519)
Problems and Solutions

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Contents

- **BM** stands for Brownian motion. Assume that B_t is a standard **BM**.
- **RN** and **RV** stand for random number and random variable, respectively.
- $P(\cdot)$ and $E(\cdot)$ are probability and expectation, respectively.
- The PDF and CDF of the standard normal variable are denoted by $n(z)$ and $N(z)$, respectively.
- Assume interest rate and dividend rate are zero in option pricing.
- **SCFA** stands for the exercise problems of Stochastic Calculus and Financial Applications, **HW** homework, **ME** midterm exam, and **FE** final exam.

Probability and Statistics Review

1. **[2018HW 2-1]** In class, we derived the moments of the standard normal distribution:

$$E(Z^{2n}) = (2n-1)(2n-3)\cdots 3\cdot 1 \quad \text{for} \quad Z \sim N(0,1).$$

We can derive the same result using the moment generating function. First, derive the moment generating function,

$$M_X(t) = E(\exp(tX)) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right) \quad \text{for} \quad X \sim N(\mu, \sigma^2).$$

Then, using the Taylor expansion of $M_X(t)$, derive the moment of Z. (After this problem, you can understand SCFA 3.4 better.)

Solution: The PDF of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Therefore,

$$\begin{aligned} M_X(t) = E(\exp(tX)) &= \int_{-\infty}^{\infty} e^{tx} f_X(x)dx \\ &= \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right) \int_{-\infty}^{\infty} f_X(x - \sigma^2 t)dx = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right). \end{aligned}$$

For $Z \sim N(0,1)$, the moment generating function is $M_Z(t) = \exp(t^2/2)$. Expanding $M_Z(t)$,

$$M_Z(t) = 1 + \frac{t^2}{2} + \cdots + \frac{1}{2^n \cdot n!} t^{2n} + \cdots$$

The $2n$-th moment is given as

$$E(Z^{2n}) = \frac{(2n)!}{2^n \cdot n!} = \frac{(2n)!}{(2n)(2n-2)\cdots 2} = (2n-1)(2n-3)\cdots 3 \cdot 1$$

2. **[2019HW 1-1]** Find the moment generating function (MGF) of uniform distribution on $[0,1]$. From the MGF, find the mean and variance of the distribution.

Solution: The MGF is

$$M_U(t) = \int_0^1 e^{tu} du = \frac{1}{t}\left(e^t - 1\right) = 1 + \frac{1}{2}t + \frac{1}{6}t^2 + \cdots.$$

Therefore,

$$E(U) = \frac{1}{2}$$

$$\text{Var}(U) = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$$

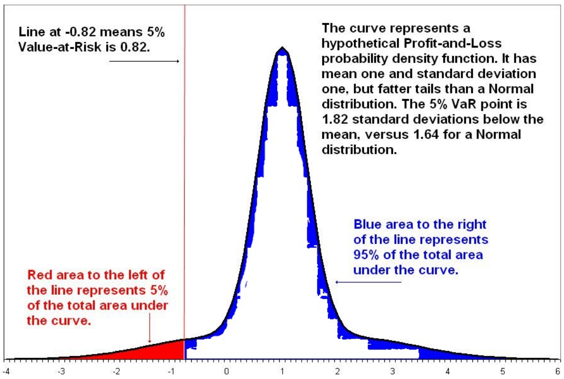
3. **[2019HW 1-2]** Find the moment generating function (MGF) of the exponential distribution with intensity λ. From the MGF, find the mean and variance of the distribution.

> **Solution:** If we let X be the exponential random variable, the MGF is
>
> $$M_X(t) = \int_0^\infty e^{tx}\lambda e^{-\lambda x}dx = \frac{\lambda}{\lambda - t} = 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \cdots \quad (t < \lambda)$$
>
> Therefore,
>
> $$E(X) = \frac{1}{\lambda}$$
> $$\mathrm{Var}(U) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

4. **[2019HW 1-3]** Assume that the random variable X follows the uniform distributions on $[0, 1]$.
   (a) If Y is another random variable given by $Y = X^2$, what is $\rho(X, Y)$, the correlation coefficient between X and Y?
   (b) If Y is another random variable given by $Y = 4X(1 - X)$, what is $\rho(X, Y)$, the correlation coefficient between X and Y?
   (c) Regarding (b), are X and Y independent? We know that the independence between X and Y implies $\rho(X, Y) = 0$. What can you say about the opposite?

> **Solution:**
>
> (a)
>
> $$E(Y) = \int_0^1 x^2 dx = \frac{1}{3}, \quad E(Y^2) = \int_0^1 x^4 dx = \frac{1}{5}, \quad E(XY) = \int_0^1 x^3 dx = \frac{1}{4},$$
> $$\rho = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} = \frac{1/4 - 1/6}{\sqrt{1/12}\sqrt{1/5 - 1/9}} = 96.8\%$$
>
> (b) Because Y is symmetric at $X = 1/2$, the correlation should be zero.
>
> (c) The opposite does not hold. Y is completely dependent on X.

5. **[2018ME, Probability]** Value-at-risk (VaR) (WIKIPEDIA) is a measure of the risk of loss for investments. It estimates how much a set of investments might lose (with a given probability p). VaR is typically used by firms and regulators in the financial industry to gauge the amount of assets needed to cover possible losses. For a given portfolio, time horizon, and probability p, the p-VaR is defined such that the probability of a loss greater than VaR is (at most) p while the probability of a loss less than VaR is (at least) $1 - p$. In other words, p-VaR is the loss <u>at</u> the worst p percentile.



(From Wikipedia. The graph is illustration only. **Ignore** the numbers in the graph.)

Assume that you invest in one share of stock today and that your profit & loss is distributed as $S_T - S_0 = X$ for some random variable X with the CDF, $F_X(x)$, and the PDF, $f_X(x)$.

(a) If $X \sim N(0, 10^2)$ (i.e., $\sigma = 10$), what is your 5%-VaR, $\text{VaR}(p = 0.05)$? You may use $N(-1.64) \approx 0.05$.

(b) Express the put option price with strike price K, $P(K)$, in terms of $f_X(x)$. (You may use integral in the answer.)

(c) Conditional VaR (CVaR or expected shortfall) is another risk measure to improve VaR. It is defined as the expected loss conditional on that the loss is within the worst p percentile. Find the expression for $\text{CVaR}(p)$. You can simplify the expression using $\text{VaR}(p)$ and the put option price, $P(K)$. Between $\text{VaR}(p)$ and $\text{CVaR}(p)$, which one assumes bigger loss?

Solution:

(a) $X \sim 10Z$ where $Z \sim N(0, 1)$.

$$\text{VaR}(0.05) = 10\, N^{-1}(0.05) = 10 \cdot (-1.64) = -16.4$$

(b)

$$\begin{aligned} P(K) &= \int_{x=-\infty}^{K-S_0} (K - S_0 - x) f_X(x) dx \\ \text{or} \quad &= \int_{x=-\infty}^{K} (K - x) f_X(x - S_0) dx \end{aligned}$$

(c) Let $S_T = K$ be the price which give the p-percentile loss. Then, $K - S_0 = \text{VaR}(p)$. We express $\text{CVar}(p)$ using K, and substitute using $K = S_0 + \text{VaR}(p)$ in the end.

$$\begin{aligned} \text{CVar}(p) &= \frac{1}{p} \int_{x=-\infty}^{K-S_0} x f_X(x) dx \\ &= \frac{K - S_0}{p} \int_{x=-\infty}^{K-S_0} f_X(x) dx + \frac{1}{p} \int_{x=-\infty}^{K-S_0} (x - K + S_0) f_X(x) dx \\ &= \frac{\text{VaR}(p)}{p} p - \frac{P(K)}{p} = \text{VaR}(p) - \frac{P(S_0 + \text{VaR}(p))}{p} \end{aligned}$$

CVaR assumes more severe loss.

1 Random Walks and First Step Analysis

1. **[SCFA 1.1]**

Solution: Let $T_{i,j}$ denote the expected time to go from level i to j. We are going to compute the answer as

$$T_{25,18} = T_{25,20} + T_{20,19} + T_{19,18}.$$

First, $T_{25,20} = 15$ from Eq. (1.15):

$$E(\tau \mid S_0 = 0) = \frac{B}{q-p} - \frac{A+B}{q-p}\frac{1-(q/p)^B}{1-(q/p)^{A+B}}$$

with $p = 1/3$, $q = 2/3$, $A = \infty$, $B = 5$. We also know $T_{21,20} = 3$ from $B = 1$. Next, $T_{20,19} = 37$ is calculated from

$$T_{20,19} = \frac{1}{10} \cdot 1 + \frac{9}{10}(1 + T_{21,20} + T_{20,19}).$$

Finally $T_{19,18} = 77$ is obtained from

$$T_{19,18} = \frac{1}{3} \cdot 1 + \frac{2}{3}(1 + T_{20,19} + T_{19,18}).$$

Therefore, $T_{25,18} = 15 + 37 + 77 = 129$.

2. **[SCFA 1.3]**

Solution: Let N_k be the number of visits to the level $k \neq 0$ before returning to 0 for the first time. First, we prove that $P(N_k \geq 1) = 1/(2k)$ for $k \geq 1$. In order for the event, $N_k > 0$, to happen, the first step should be $+1$, ($X_1 = +1$). If the first step is -1, the random walk has to hit 0 before it reaches $k \geq 1$. Given that the first step is $+1$, the probability to hit k ($A = k-1$ more steps up) before hitting 0 ($B = 1$ step down) is given as $1/k$ from Eq. (1.2). Combining the two results together, we have $P(N_k > 0) = 1/(2k)$ for $k \geq 1$.

Next, we prove that

$$P(N_k \geq j+1 \mid N_k \geq j) = \frac{1}{2} + \frac{k-1}{2k}.$$

Imagine that the random walk just hit the level k for the j-th time before hitting 0. If the next step is up, it is guaranteed that it will hit k at least one more time before returning to 0. If the next step is down, we know that the probability to hit k one more time ($A = 1$) before hitting 0 ($B = k-1$) is $(k-1)/k$. Adding the two probabilities together, we obtain the result.

Therefore, we can say that, for $j \geq 0$,

$$\begin{aligned} P(N_k > j) &= P(N_k > 0)P(N_k > 1|N_k > 0) \cdots P(N_k > j|N_k > j-1) \\ &= \frac{1}{2k}\left(\frac{1}{2} + \frac{k-1}{2k}\right)^j = \frac{1}{2k}\left(\frac{2k-1}{2k}\right)^j. \end{aligned}$$

We can prove the final statement:

$$
\begin{aligned}
E(N_k) &= \sum_{j=1}^{\infty} j \cdot P(N_k = j) = \sum_{j=1}^{\infty} P(N_k \geq j) = \sum_{j=0}^{\infty} P(N_k > j) \\
&= \frac{1}{2k} \sum_{j=0}^{\infty} \left(\frac{2k-1}{2k} \right)^j = 1.
\end{aligned}
$$

3. **[2016HW 1, A popular interview quiz, Recurrence relation]**
   Imagine that you keep tossing a fair coin (50% for head and 50% for tail) until you get two heads in a row. On average, how many times do you need to toss the coin?

Solution: Let X be the answer (expected number of tosses) and branch on the following three cases based on the outcomes in the beginning. The head is denoted by **H** and tail by **T**.

1. **T** (Prob = 1/2): You start from the scratch with 1 toss wasted. So the expected number of tosses in the branch is 1+X.
2. **HT** (Prob = 1/4): You start from the scratch with 1 toss wasted. So the expected number of tosses in the branch is 2+X.
3. **HH** (Prob = 1/4): You get two heads in a row in 2 tosses.

Therefore, we obtain the following equation on X

$$
X = \frac{1}{2}(1+X) + \frac{1}{4}(2+X) + \frac{1}{4} \cdot 2
$$

and conclude that $X = 6$.

4. **[2016HW 2-1]** Related to the last statement of Chapter 1 of **SCFA**, prove that the expected time for the gambler to become first positive is infinite: $E(\tau) = \infty$. [Hint: consider the derivative of $\phi(z)$ with respect to z]

Solution:

$$
\begin{aligned}
E(\tau) &= \sum_{k=1}^{\infty} k\, P(\tau = k) = \sum_{k=1}^{\infty} k\, P(\tau = k)\, z^{k-1} \Big|_{z=1} \\
&= \frac{d}{dz}\phi(z)\Big|_{z=1} = -\frac{1-\sqrt{1-z^2}}{z^2} + \frac{1}{\sqrt{1-z^2}}\Bigg|_{z=1} = \infty
\end{aligned}
$$

5. **[2019HW 1-4]** Let

$$S_n = S_0 + X_1 + X_2 + \cdots + X_n, \quad \text{where} \quad X_k = 1\ (p = 0.5) \text{ or } -1\ (q = 1/2)$$

and τ be the first time n when S_n hits either A or $-B$. From lectures, We know that

$$E(\tau|S_0 = k) = (A-k)(B+k)$$

and that it can be understood as the price of the accumulator derivative which pays $\$\,\tau$ when the underlying stock price S_n hits either A or $-B$ for the first time. Imagine that you are the bank who sold the derivative to a client and you want to hedge the derivative by buying or selling some amount of the underlying stock.

(a) On the n-th day with $S_n = k$ ($n < \tau$), how many shares (positive or negative) of the underlying stock do you need to hold to hedge the derivative? (Hint: consider the derivative with respect to k.)

(b) You can verify your answer in (a) under specific scenarios. Assume $S_0 = 0$, $A = 2$, and $B = 1$, and consider the following two scenarios:

1. $S_1 = -1$ ($\tau = 1$). Therefore, you pay \$1 to your client on the day $n = 1$.
2. $S_n = 0,\ 1,\ 0,\ 1,\ 2$ on $n = 0, \cdots, 4$ ($\tau = 4$). Therefore, you pay \$4 to your client on the day $n = 4$.

On each scenario, track your the profit & loss (P&L) from your hedge position and record the accumulated value (of cash and stock) on each day. Do not forget that you received $\$AB$ from your client as the premium, so your initial value is AB. On the day τ, you have to pay $\$\,\tau$ to your client. What is the accumulated cash you hold (after selling the stocks for the hedge)? Is that cash amount same as τ?

Solution:

(a)

$$E(\tau|S_n = k) = n + (A-k)(B+k), \qquad \frac{d}{d\,k}E(\tau|S_n = k) = A - B - 2k$$

Therefore, you have to hold $A - B - 2k$ shares of the underlying stock.

(b) In both scenarios, the accumulated cash is same as the payout, τ.

Scenario 1:

Day (n)	0	1
S_n	0	-1
Hedge (Share)	1	x
P&L from hedge	x	-1
Accumulated value	2	1

Scenario 2:

Day (n)	0	1	2	3	4
S_n	0	1	0	1	2
Hedge (Share)	1	-1	1	-1	x
P&L from hedge	x	1	1	1	-1
Accumulated value	2	3	4	5	4

Note: P&L on day k is the hedge position at $t = k - 1$ times $(S_k - S_{k-1})$. Accumulated value is the sum the cash and the value of the hedge.

6. **[2017ME, Recurrence relation]** A startup company in Shenzhen either fails or succeeds every year with probability of 25% and 75% respectively. If a company is successful, employees spin off a new startup at the end of the year (and it will face with the same fail/success probability afterwards). There is no correlation between companies. Assume that Shenzhen sets off with one startup company when the city was established as a special economic zone in 1980. What is the probability that all startups in Shenzhen eventually fail.

Solution: Let p be the probability and branch on the whether the first company fails or succeeds. If the first company succeeds and there are two companies in Shenzhen next year, the probability that both fail is p^2 due to the independence assumption. Therefore,

$$p = \frac{1}{4} \cdot 1 + \frac{3}{4} \cdot p^2 \quad \Rightarrow \quad 3p^2 - 4p + 1 = (3p-1)(p-1) = 0$$

Among the two solutions, $p = 1/3$ is the right solution.

7. **[2018ME]** In the gambler's ruin problem,

$$S_n = X_1 + \cdots + X_n, \quad X_k = \pm 1 \quad \text{with probability} \quad p : q \ (p+q=1),$$

what is the probability that S_n ever hits a level $A > 0$? How does this probability changes when p changes? (Hint: consider $B \to \infty$ from the results we know from class.)

Solution: The probability of hitting A before hitting $-B$ is given as

$$P(S_\tau = A) = \frac{(q/p)^B - 1}{(q/p)^{A+B} - 1} \quad (p \neq q) \quad \text{or} \quad \frac{B}{A+B} \quad (p = q = 1/2)$$

If we let $B \to \infty$, the probability of S_n ever hitting A is

$$1 \quad \text{if} \quad p \geq 0.5 \quad \text{or} \quad (p/q)^A \quad \text{if} \quad p < 0.5$$

2 First Martingale Steps

1. **[SCFA 2.1]**

Solution: The roots of the equation qualify for the x

$$E(x_1^X) = \frac{0.52}{x} + 0.45x + 0.03x^2 = 1.$$

Form

$$\frac{0.52}{x} + 0.45x + 0.03x^2 - 1 = \frac{x-1}{x}\big(0.03x^2 + 0.48x - 0.52\big)$$

we have the following three values.

$$x = 1,\ \frac{-0.24 \pm \sqrt{0.24^2 + 0.03 \times 0.24}}{0.03} = 1,\ 1.01850,\ -17.0185.$$

We pick $x = 1.01850$ since the change under high powers are reasonable. If we let τ be the first time Gambler's wealth is either 100, 101 or -100, we have the equation from the Martingale property,

$$1 = E(M_\tau) = x^{100}P(S_\tau = 100) + x^{101}P(S_\tau = 101) + x^{-100}P(S_\tau = -100).$$

Letting $p = P(S_\tau = 100) + P(S_\tau = 101)$ and using the fact that $x > 1$,

$$x^{100}p + x^{-100}(1-p) < 1 < x^{101}p + x^{-100}(1-p)$$
$$\frac{1 - x^{-100}}{x^{101} - x^{-100}} < p < \frac{1 - x^{-100}}{x^{100} - x^{-100}} \quad \Rightarrow \quad 0.13531 < x < 0.13788.$$

2. **[SCFA 2.4]**

Solution: From the hint, let us define

$$A_{n+1} = A_n + E[(M_{n+1} - M_n)^2|\mathcal{F}_n]$$

and prove the three required properties:

(i) N_n is a martingale.

$$\begin{aligned} E[N_{n+1} \,|\, \mathcal{F}_n] &= E[M_{n+1}^2 - A_{n+1} \,|\, \mathcal{F}_n] \\ &= E[2M_{n+1}M_n - M_n^2 - A_n \,|\, \mathcal{F}_n] \\ &= 2E[M_{n+1} \,|\, \mathcal{F}_n]\, M_n - M_n^2 - A_n \\ &= M_n^2 - A_n = N_n. \end{aligned}$$

(ii) $A_{n+1} \geq A_n$ is trivial.

(iii) A_n is non-anticipating because it is defined via the expectation under $\mathcal{F}_n$.

3. **[2017HW 1-1]** Using martingale property, re-drive that

$$E(\tau) = AB \quad \text{for} \quad \tau = \min\{n : S_n = A \text{ or } S_n = -B\}.$$

Solution: You can find the answer in **SCFA** Section 2.3.

4. **[2018HW 2-2]** Martingale page (Wikipedia) gives the following example of Martingale. Prove (or disprove) the statement.

> Suppose each amoeba either splits into two amoebas, with probability p, or eventually dies, with probability $q = 1-p$. Let X_n be the number of amoebas surviving in the n-th generation (in particular $X_n = 0$ if the population has become extinct by that time). Let r be the probability of eventual extinction. (Finding r as a function of p is an instructive exercise. Hint: The probability that the descendants of an amoeba eventually die out is equal to the probability that either of its immediate offspring dies out, given that the original amoeba has split.) Then
>
> $$\{\, r^{X_n} : n = 1, 2, 3, \dots \,\}$$
>
> is a martingale with respect to $\{X_n : n = 1, 2, 3, \ldots\}$.

See 2017-ME Recurrence Relation to derive the extinction probability r as a function of p (and q).

Solution: First, we find r in terms of p and q. Since the survival of amoeba's are independent, the probability of n amoebas' extinction is r^n. Branching on the 2nd generation (i.e., multiplication vs death of the first amoeba), we get the following recurrence relation,

$$r = p \cdot r^2 + q,$$

and we can solve $r = q/p$. In fact, the relation above indicate the first step proof of the Martingale:

$$E(r^{X_2}|X_1 = 1) = E(r^{X_1}) = r.$$

For the rest, we do not need $r = q/p$, but just the relation, $1 = pr + q/r$.

At each step, each of X_n amoebas either becomes 2 or 0 with probability of p and q respectively. We can write

$$X_{n+1} = X_n + \sum_{k=1}^{X_n} I_k, \quad r^{X_{n+1}} = r^{X_n} \cdot r^{I_1 + \cdots + I_{X_n}}$$

where I_k takes value of $+1$ or -1 with probability p and q respectively and $\{I_k\}$ are independent events. The probability of j amoebas multiplying by 2 ($X_n - j$ amoebas die) is given by the binomial distribution, $\binom{n}{j} p^j \, q^{X_n - j}$. Therefore,

$$E(r^{I_1+\cdots+I_{X_n}}) = \sum_{j=0}^{X_n} \binom{X_n}{j} p^j \, q^{X_n-j} \cdot \frac{r^j}{r^{X_n-j}} = \sum_{j=0}^{X_n} \binom{X_n}{j} (pr)^j \, \left(\frac{q}{r}\right)^{X_n-j}$$
$$= \left(pr + \frac{q}{r}\right)^{X_n} = 1.$$

Therefore,

$$E(r^{X_{n+1}}|X_n) = r^{X_n} \cdot E(r^{I_1+\cdots+I_{X_n}}) = r^{X_n}.$$

5. **[2016ME(ASP), Martingale, Polya's urn]**
A box has 1 red ball and 9 blue balls. Pick up one ball randomly. If it is red, put it back and add one more red ball into the box. If it is blue, put it back and add one more blue ball into the box. If Y_n is the proportion of the red balls in the box after the process is repeated n times ($Y_0 = 0.1$), show that $\{Y_n\}$ is a martingale. So what is the expected number of the red balls after you repeat the process 100 times?

Solution:

$$Y_{n+1} = \begin{cases} \frac{(n+10)Y_n+1}{n+11} & \text{if a red ball is picked with probability } Y_n \\ \frac{(n+10)Y_n}{n+11} & \text{if a blue ball is picked with probability } 1 - Y_n \end{cases}$$

Therefore,

$$E(Y_{n+1}|Y_n) = \frac{(n+10)Y_n + 1}{n+11} Y_n + \frac{(n+10)Y_n}{n+11}(1 - Y_n) = Y_n,$$

so Y_n is a martigale. The expected number of the red balls at $n = 100$ is

$$E\big((10 + 100)Y_{100}\big) = 110\, Y_0 = 110 \times 0.1 = 11.$$

6. **[2016ME(ASP), Wald's equation]**
When $\{X_k\}$ are independent identically distributed random variable and N is a random variable taking positive integer values, Wald's equation says

$$E(X_1 + X_2 + \cdots X_N) = E(N)\, E(X_1)$$

if either (i) N is independent from $\{X_k\}$ or (ii) N is a stopping time with respect to $\{X_k\}$. Consider an example where $X_k = 0$ or 1 with 50% and 50% probability and N is given as

$$N = X_1 + X_2 + 1.$$

Obviously, $E(X_k) = 1/2$ and $E(N) = 2/2+1 = 2$. Find $E(X_1+X_2+\cdots+X_N)$ and explain why Wald's equation does not hold in this example.

Solution: We can branch on the first scenarios (with probability 1/4) depending on the outcome of X_1 and X_2: 0–0, 0–1, 1–0 and 1–1.

$$\begin{aligned}E(X_1 + X_2 + \cdots + X_N) &= \frac{1}{4}E(0) + \frac{1}{4}E(0+1) + \frac{1}{4}E(1+0) + \frac{1}{4}E(1+1+X_3)\\ &= \frac{1}{4}(0+1+1+2+\frac{1}{2}) = \frac{9}{8} \neq E(X_k)E(N) = 1.\end{aligned}$$

Wald's equation does not hold because $N = X_1 + X_2 + 1$ is neither
(i) independent from $\{X_k\}$: N is defined via X_1 and X_2
nor (ii) a stopping time w.r.t. $\{X_k\}$: it looks into the future, i.e., X_2 at $k = 1$.

7. **[2016ME, Wald's equation]**
When $\{X_k\}$ are independent identically distributed random variable and N is a random variable taking positive integer values, Wald's equation says

$$E(X_1 + X_2 + \cdots X_N) = E(N)\, E(X_1)$$

if either (i) N is independent from $\{X_k\}$ or (ii) N is a stopping time with respect to $\{X_k\}$.
Consider an example where $X_k = 0$ or 1 with 50% and 50% probability and N is given as

$$N = X_2 + 1.$$

Obviously, $E(X_k) = 1/2$ and $E(N) = 1/2 + 1 = 3/2$. Find $E(X_1 + X_2 + \cdots + X_N)$ and explain why Wald's equation does not hold in this example. If N is given instead as

$$N = X_1 + 1,$$

does Wald's equation hold? Is N a stopping time?

Solution: Branching on the value of X_2:

$$N = \begin{cases} 1 & (X_2 = 0, \quad \text{Prob} = 1/2) \\ 2 & (X_2 = 1, \quad \text{Prob} = 1/2), \end{cases}.$$

we compute

$$\begin{aligned}E(X_1 + X_2 + \cdots + X_N) &= \frac{1}{2}E(X_1) + \frac{1}{2}E(X_1 + 1)\\ &= \frac{1}{2}(\frac{1}{2} + \frac{1}{2} + 1) = 1 \neq E(X_k)E(N) = \frac{3}{4}.\end{aligned}$$

Wald's equation does not hold because $N = X_1 + X_2 + 1$ is
(i) not independent from $\{X_k\}$ as N is defined via X_2 and (ii) not a stopping time w.r.t. $\{X_k\}$ because we need future information X_2 in order to determine N (X_1 is not enough to tell $N = 1$ or not.)

If $N = X_1 + 1$, N is a stopping time. Based on X_1 we can tell whether $N = 1$ or not: $N = 1$ if $X_1 = 0$ and $N \neq 1$ (in fact, $N = 2$) if $X_1 = 1$. Therefore Wald's inequality should hold. We can directly verify that

$$E(X_1 + X_2 + \cdots + X_N) = \frac{1}{2}E(X_1) + \frac{1}{2}E(X_1 + X_2) = \frac{1}{2}(\frac{1}{2} + \frac{1}{2} + \frac{1}{2}) = \frac{3}{4}.$$

3 Brownian Motion

1. **[SCFA 3.1, Brownian Bridge]**

Solution:

(a) This problem is based on a series representation of BM (WIKIPEDIA). See p.286 of **SCFA** also.

$$B_t = tZ_0 + \sum_{k=0} + \sqrt{2}\sum_{k=1}^{\infty} Z_k \frac{\sin \pi k t}{\pi k}$$

for independent standard normal random variables, $\{Z_{k\geq 0}\}$. (But I think the author somehow dropped this in the current version of **SCFA**.) So, $\Delta_0(t) = t$ and $\lambda_0 = 1$. Because $B_1 = Z_0$ (from $\Delta_{k\geq 1}(1) = 0$), the first term can be expressed as $\lambda_0 Z_0 \Delta_0(t) = t\, B_1$. Therefore,

$$U_t = B_t - tB_1$$

(b)

$$\begin{aligned}\mathrm{Cov}(U_s, U_t) &= E\Big((B_s - sB_1)(B_t - tB_1)\Big) = E\Big(B_sB_t - sB_1B_t - tB_sB_1 + stB_1^2\Big)\\ &= \min(s,t) - s\min(1,t) - t\min(s,t) + st = s(1-t)\end{aligned}$$

(c) We need to find any set of functions, $g(\cdot)$ and $h(\cdot)$, such that

$$\mathrm{Cov}(X_s, X_t) = g(s)g(t)\min(h(s), h(t)) = s(1-t) \quad \text{for} \quad s \leq t.$$

If we narrow down the search by assuming $h(\cdot)$ is monotonically increasing,

$$\mathrm{Cov}(X_s, X_t) = g(s)g(t)h(s) = s(1-t),$$

so we get

$$g(t) = 1-t, \quad h(s) = \frac{s}{1-s},$$

where $h(s)$ is indeed an increasing function. Therefore we obtained a representation of Brownian bridge,

$$X_t = (1-t)B_{\frac{t}{1-t}}$$

Since U_{1-t} is also a Brownian bridge due to the symmetry,

$$X_{1-t} = tB_{\frac{1-t}{t}}, \quad \Big(g(t) = t,\ h(t) = \frac{1-t}{t}\Big)$$

is also a valid solution.

(d) We use the inequality $s/(1+s) \leq t/(1+t)$ if $0 \leq s \leq t$.

$$\begin{aligned}\mathrm{Cov}(Y_s, Y_t) &= \mathrm{Cov}\Big((1+s)U_{\frac{s}{1+s}}(1+t)U_{\frac{t}{1+t}}\Big) = (1+s)(1+t)\mathrm{Cov}(U_{\frac{s}{1+s}}, U_{\frac{t}{1+t}})\\ &= (1+s)(1+t)\,\frac{s}{1+s}\Big(1 - \frac{t}{1+t}\Big) = s = \min(s,t).\end{aligned}$$

2. **[SCFA 3.2, Cautionary Tale]**

Solution: Suppose X is a standard normal, consider an independent U such that $P(U = 1) = 1/2 = P(U = -1)$, and set $Y = UX$. Then, Y is also a standard normal as X and $-X$ are also standard normal.

In order to show X and Y are not independent, we need to show

$$P(I_X \,\&\, J_Y) \neq P(I_X)P(J_Y)$$

for some event I_X and I_Y regarding X and Y respectively.

For any $h > 0$,

$$\begin{aligned} P(X > h \,\&\, Y > h) &= \frac{1}{2}P(X > h \,\&\, X > h \mid U = 1) \\ &\quad + \frac{1}{2}P(X > h \,\& -X > h \mid U = -1) \\ &= \frac{1}{2}(1 - N(h)) + 0. \end{aligned}$$

However,

$$P(X > h)P(Y > h) = (1 - N(h))(1 - N(h))$$

is not same as the previous value. Therefore

3. **[SCFA 3.3, Multivariate Gaussians]**

Solution:

(a) Let us work on each components of the vectors and matrices; $V = (v_i)$, $\mu = (\mu_i)$, $A = (a_{ij})$ and $\Sigma = (\sigma_{ij})$.

$$E\Big((AV)_i\Big) = E\Big(\sum_j a_{ij}V_j\Big) = \sum_j a_{ij}E\Big(V_j\Big) = \sum_j a_{ij}\mu_j = (A\mu)_i$$
$$E\big(AV\big) = A\mu$$

$$\begin{aligned} \mathrm{Cov}\Big((AV)_i, (AV)_j\Big) &= \mathrm{Cov}\Big(\sum_l a_{il}V_l, \sum_m a_{jm}V_m\Big) = \sum_{l,m} a_{il}\mathrm{Cov}(V_l, V_m)a_{jm} \\ &= \sum_{l,m} a_{il}\sigma_{lm}a_{jm} = (A\Sigma A^T)_{ij} \end{aligned}$$

Therefore,

$$\mathrm{Cov}\Big(AV, AV\Big) = A\Sigma A^T.$$

(b)

$$E(X \pm Y) = E(X) \pm E(Y) = 0 \pm 0 = 0$$
$$\mathrm{Var}(X \pm Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) \pm 2\,\mathrm{Cov}(X, Y) = 1 + 1 + 0 = 2$$
$$\mathrm{Cov}(X + Y, X - Y) = \mathrm{Var}(X) - \mathrm{Var}(Y) = 1 - 1 = 0$$

(c) When $\mathrm{Cov}(X,Y)=0$, the covariance matrix Σ is given as

$$\Sigma = \begin{pmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{pmatrix}, \quad \Sigma^{-1} = \begin{pmatrix} 1/\sigma_{11} & 0 \\ 0 & 1/\sigma_{22} \end{pmatrix}, \quad \det \Sigma = \sigma_{11}\,\sigma_{22}$$

The joint density function can be factored to the product of the single variable density function,

$$\begin{aligned} f(x,y) &= \frac{1}{2\pi\sqrt{\sigma_{11}\,\sigma_{22}}} \exp\Big(-\frac{(x-\mu_X)^2}{2\sigma_{11}} - \frac{(y-\mu_Y)^2}{2\sigma_{22}} \Big) \\ &= \frac{1}{2\pi\sqrt{\sigma_{11}}} \exp\Big(-\frac{(x-\mu_X)^2}{2\sigma_{11}} \Big) \frac{1}{2\pi\sqrt{\sigma_{22}}} \exp\Big(-\frac{(y-\mu_Y)^2}{2\sigma_{22}} \Big) = f(x)f(y). \end{aligned}$$

Therefore X and Y are independent.

(d) We first find the matrix A such that, for the independent standard normal variables W and Z,

$$\begin{pmatrix} X-\mu_X \\ Y-\mu_Y \end{pmatrix} = A \begin{pmatrix} W \\ Z \end{pmatrix}$$

has the given covariance matrix

$$\begin{pmatrix} \sigma_{XX} & \sigma_{XY} \\ \sigma_{XY} & \sigma_{YY} \end{pmatrix} = A\,I\,A^T = A\,A^T.$$

One of the solution from Cholesky decomposition is

$$A = \begin{pmatrix} \sqrt{\sigma_{XX}} & 0 \\ \sigma_{XY}/\sqrt{\sigma_{XX}} & \sqrt{\sigma_{YY} - \sigma_{XY}^2/\sigma_{XX}} \end{pmatrix}.$$

Conditional on that $X=x$,

$$Y = \mu_Y + \frac{\sigma_{XY}}{\sqrt{\sigma_{XX}}}\frac{x-\mu_X}{\sqrt{\sigma_{XX}}} + Z\sqrt{\sigma_{YY} - \sigma_{XY}^2/\sigma_{XX}}.$$

Therefore

$$\begin{aligned} E(Y|X=x) &= \frac{\sigma_{XY}}{\sigma_{XX}}(x-\mu_X) = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)}(x-\mu_X) \\ \mathrm{Var}(Y|X=x) &= \sigma_{YY} - \frac{\sigma_{XY}^2}{\sigma_{XX}} = \mathrm{Var}(Y) - \frac{\mathrm{Cov}^2(X,Y)}{\mathrm{Var}(X)} \end{aligned}$$

4. **[SCFA 3.4, Auxiliary Functions and Moments]**

Solution:

$$E(e^{tz}) = \int_{-\infty}^{\infty} e^{tz}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz = \int_{-\infty}^{\infty} e^{t^2/2}\frac{1}{\sqrt{2\pi}}e^{-(z-t)^2/2}dz = e^{t^2/2}$$

If M_n is the $n-$th moment,

$$E(e^{tz}) = \sum_{k=0}^{\infty} M_k \frac{t^k}{k!} = 1 + \frac{t^2}{2} + \frac{1}{2!}\left(\frac{t^2}{2}\right)^2 + \frac{1}{3!}\left(\frac{t^2}{2}\right)^3 + \cdots$$

By matching the coefficients, we get

$$\begin{aligned} &M_0 = 1 \\ &M_1 = M_3\ (= M_{2k-1}) = 0 \\ &M_2 = 1 \\ &M_4 = 4!/(2!\ 2^2) = 3 \\ &M_6 = 6!/(3!\ 2^3) = 15. \end{aligned}$$

For $t > 0$,

$$E(e^{tz^4}) = \int_{-\infty}^{\infty} e^{tz^4} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \to \infty,$$

so the MGF of Z^4 does not exist.

5. **[2019HW 2-1]** For a standard BM, B_t, find the probability that

$$B_1 + B_2 + B_3 \geq 3.$$

Solution: Let $x = B_1$, $y = B_2 - B_1$, and $z = B_3 - B_2$, then x, y, and z follow $\mathcal{N}(0,1)$ and they are independent. Since,

$$B_1 + B_2 + B_3 = x + (x + y) + (x + y + z) = 3x + 2y + z \sim \mathcal{N}(0, 14),$$

the probability that $B_1 + B_2 + B_3 \geq 3$ is $1 - N(3/\sqrt{14}) = 0.2113$.

6. **[2016ME, Standard BM]**
If B_t is a standard BM, determine whether each of the followings is a standard BM or not. Provide a brief reason for your answer.
   (a) $4B_{t/2}$
   (b) $tB_{1/t}$ with $B_0 = 1$.
   (c) $2(B_{1+t/4} - B_1)$
   (d) $\sqrt{t}Z$ for a standard normal RV Z

Solution:
(a) No. $\mathrm{Var}(4B_{t/2}) = 16 \times t/2 = 8t \neq t$.

(b) No. B_0 should be 0.

(c) Yes. $B_{1+t/4} - B_1$ is equivalent to $B_{t/4}$ and $2B_{t/4}$ is equivalent to B_t.

(d) No. For any value of Z, $\sqrt{t}Z$ is not a stochastic process. For example, $\sqrt{s}Z$ and $(\sqrt{t} - \sqrt{s})Z$ for $s < t$ are correlated.

7. **[2017ME, Standard BM]** If B_t is a standard BM, determine whether each of the followings is a standard BM or not. Provide a brief reason for your answer.

(a) $\dfrac{1}{2}B_{4t}$

(b) $\dfrac{1}{2}(B_{1+2t} - B_1)$

(c) $\dfrac{3}{5}B_t + \dfrac{4}{5}W_t$, where W_t is another BM independent from B_t.

Solution:

(a) **Yes**. $\text{Var}(\frac{1}{2}B_{4t}) = \frac{1}{4} \cdot 4t = t$.

(b) **No**. It is equivalent to $\frac{1}{2}B_{2t}$, however $\frac{1}{2}B_{2t}$ is not a standard BM.

(c) **Yes**. $aB_t + bW_t$ is a standard BM when $a^2 + b^2 = 1$.

8. **[2018ME, Standard BM]** If B_t is a standard BM, determine whether each of the followings is a standard BM or not. Explain briefly why it is a standard BM or not.

(a) $(1/\sqrt{2})B_{2t}$

(b) $\begin{cases} B_t & \text{if} \quad t \le \tau_a \\ 2a - B_t & \text{if} \quad t > \tau_a \end{cases}$, where τ_a is the first time B_t hitting the level a.

(c) $\frac{1}{13}(5B_t + 12W_t)$ where W_t is another standard BM independent from B_t.

(d) $B_{2t} - B_t$

Solution:

(a) Yes. The scaling property.

(b) Yes. The reflection principle.

(c) Yes.

$$\frac{5^2 + 12^2}{13^2} = 1$$

(d) No. Let $X_t = B_{2t} - B_t$.

$$\begin{aligned} \text{Cov}(X_s, X_t) &= E((B_{2s} - B_s)(B_{2t} - B_t) = E(B_{2s}B_{2t} - B_{2s}B_t - B_sB_{2t} + B_sB_t) \\ &= 3\min(s,t) - \min(2s,t) - \min(s,2t) \end{aligned}$$

If X_t is a standard BM, $\text{Cov}(X_1, X_2) = \min(1,2) = 1$. However, $\text{Cov}(X_1, X_2) = 3 - 2 - 1 = 0$.

9. **[2016ME, Average of a BM path]**
If B_t for $0 \le t \le 1$ is a standard BM, what is the distribution of the average of the BM values observed at three different times, $T = 1/3, 2/3$ and 1,

$$A = \frac{1}{3}\Big(B_{\frac{1}{3}} + B_{\frac{2}{3}} + B_1\Big)?$$

Please make sure to provide the mean and the standard deviation of the distribution.

Solution:

$$\begin{aligned}\mathrm{Var}\Big(\frac{1}{3}(B_{1/3} + B_{2/3} + B_1)\Big) &= \frac{1}{9}E\Big((B_{1/3} + B_{2/3} + B_1)^2\Big) \\ &= \frac{1}{9}E\Big(B_{1/3}^2 + B_{2/3}^2 + B_1^2 + 2B_{1/3}(B_{2/3} + B_1) + 2B_{2/3}B_1\Big) \\ &= \frac{1}{9}\Big(\frac{1}{3} + \frac{2}{3} + 1 + 2 \cdot \frac{1}{3} \cdot 2 + 2 \cdot \frac{2}{3}\Big) = \frac{1}{9}\frac{14}{3} = \frac{14}{27}.\end{aligned}$$

4 Martingales: The next steps

1. **[SCFA 4.6]**

Solution: The stopped process is a martingale. By the symmetry, $\mathrm{P}(B_\tau = A) = \mathrm{P}(B_\tau = -A) = 0.5$.

$$1 = E(X_\tau) = \frac{1}{2}e^{\alpha A}E(e^{-\alpha^2\tau/2}) + \frac{1}{2}e^{-\alpha A}E(e^{-\alpha^2\tau/2}).$$

Therefore, we have

$$E\left(e^{-\alpha^2\tau/2}\right) = \frac{1}{\cosh(\alpha A)}$$

or

$$\phi(\lambda) = E(e^{-\lambda\tau}) = \frac{1}{\cosh(A\sqrt{2\lambda})}.$$

In order to calculate $E(\tau^2)$, we need to obtain the x^4 term in the expansion of $1/\cosh(x)$ given that $\sqrt{\lambda}$ appears in the expression. From the expansion, $\cosh x \sim 1 + x^2/2! + x^4/4! + \cdots$,

$$\begin{aligned}\frac{1}{\cosh x} &\sim \frac{1}{1 + (x^2/2! + x^4/4! + \cdots)} = 1 - \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \left(\frac{x^2}{2!} + \cdots\right)^2 \\ &= 1 - \frac{x^2}{2} + \frac{5}{24}x^4 + \cdots.\end{aligned}$$

Finally we get

$$E(\tau^2) = 2\frac{5}{24}(A\sqrt{2\lambda})^4|_{\lambda=1} = \frac{5}{3}A^4$$

For the non-symmetric case ($A \neq B$), we can not use $\mathrm{P}(B_\tau = A) = \mathrm{P}(B_\tau = -B) = 0.5$ anymore.

2. **[2016ME, Martingale related to BM]**
If B_t is a standard BM, find the value of the coefficient λ in order for each of the following expressions to be a martingale.
   (a) $B_{at}^2 - \lambda t$
   (b) $\exp(-B_{at} + \lambda t)$

Solution:

(a) $\lambda t = E(B_{at}^2) = at$. Therefore, $\lambda = a$.

(b) $\sqrt{a}B_t$ is a BM equivalent to $-B_{at}$. Therefore, $\lambda = -a/2$.

Bachelier (Normal) Model

1. **[2016HW 3-1, Digital option]** Derive the (forward) price of the digital(binary) call/put option struck at K at maturity T. The digital(binary) call/put option pays \$1 if S_T is above/below the strike K, i.e. $1_{S_T \ge K}/1_{S_T \le K}$.

Solution: Similarly following the derivation of the call price, the digital call option price is

$$C_D(K) = E(1_{S_T \ge K}) = P(S_T \ge K) = \int_{-d_\mathrm{N}}^{\infty} n(z)dz = 1 - N(-d_\mathrm{N}) = N(d_\mathrm{N}).$$

and the digital put option price is

$$P_D(K) = E(1_{S_T \le K}) = P(S_T \le K) = \int_{-\infty}^{-d} n(z)dz = N(-d_\mathrm{N}) = 1 - N(d_\mathrm{N}),$$

where d_N is given as

$$d_\mathrm{N} = \frac{S_0 - K}{\sigma\sqrt{T}}.$$

Notice that $N(d_\mathrm{N})$ has another meaning as the probability of the stock price ends up in-the-money in addition the delta of the call option.

2. **[2016HW 3-2, Asset-or-nothing option]** The payoff of the call option, $\max(S_T - K, 0)$ can be decomposed into two parts,

$$S_T \cdot 1_{S_T \ge K} \quad - \quad K \cdot 1_{S_T \ge K}.$$

The first payout is the payout of the **asset-or-nothing** call option and the second payout if the binary call option multiplied with $-K$. Under normal model, what is the price of the asset-or-nothing call option?

Solution: From the binary call option price above and the (regular) call option price from the class,

$$C(K) = (F - K)N(d_\mathrm{N}) + \sigma\sqrt{T}\, n(d_\mathrm{N}),$$

we conclude that

$$C_{\text{A-or-N}} = F\, N(d_\mathrm{N}) + \sigma\sqrt{T}\, n(d_\mathrm{N}).$$

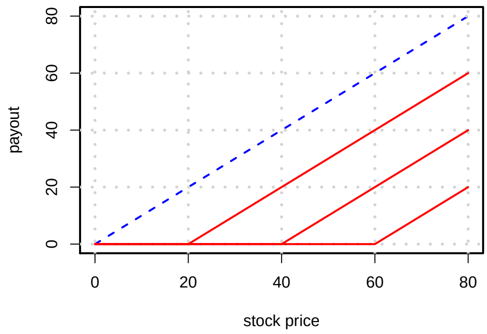
3. **[2016HW 4-4, Maximum call option value]**

   (a) Pleased with the outstanding performance of the Stochastic Finance class, Professor is going to give each student a gift of **EITHER** one share of Tencent stock **OR** one unit of the call option on Tencent stock struck just at 1 yuan (so the call option is deep in-the-money) with maturity at the end of this module. Which gift has more financial value? (Assume that Tencent pays no dividend. No calculation required. Use your common sense.)

(b) What is the upper limit of the call option value under the normal model? In normal model, under which circumstance the option is more valuable than the underling stock itself? How does it affect your choice of the gift in the previous question?

Solution:

(a) Given that a stock price can not go below zero, the payoff of a stock is always greater than that of a call option with any strike K. Therefore, the underlying stock is more valuable. See the plot below for the payoff of a stock (dashed blue) versus that of the options with $K = 20, 40$ and 60.



(b) The price of a call option (and put option as well) is unbounded (can go to infinity) under the normal model. This happens when volatility is very high. As $\sigma \to \infty$, $d \to 0$, $N(d) \to 0$ and $n(d) \to 1/\sqrt{2\pi}$. Therefore $C \approx 0.4\sigma\sqrt{T} \to \infty$. Intuitively, it is because a call option gives you a protection against the negative underlying price (underlying asset becoming liability), which is possible under the normal model. On the other hand, under Black-Scholes-Merton model the call option price is always bounded by the price of the underlying asset as mentioned in the class. Since a stock value can not be negative, you still better of by choosing a stock rather than a call option.

4. **[2019HW 2-2]** Under the Bachelier (normal) model, the undiscounted price of call option is given by

$$C_\mathrm{N}(K) = (S_0 - K)N(d_\mathrm{N}) + \sigma\sqrt{T}\, n(d_\mathrm{N}) \quad \text{for} \quad d_\mathrm{N} = \frac{S_0 - K}{\sigma_\mathrm{N}\sqrt{T}}$$

(a) Show that the delta, the sensitivity on S_0, is

$$\frac{\partial C_\mathrm{N}}{\partial S_0} = N(d_\mathrm{N})$$

(b) Show that the vega, the sensitivity on σ, is

$$\frac{\partial C_\mathrm{N}}{\partial \sigma_\mathrm{N}} = \sqrt{T}\, n(d_\mathrm{N})$$

In the questions above, make sure to apply the derivative to d_N since d_N depends on S_0 and σ_N.

Solution:

(a)

$$
\begin{aligned}
\frac{\partial C_{\rm N}}{\partial S_0} &= N(d_{\rm N}) + (S_0-K)n(d_{\rm N})\frac{\partial d_{\rm N}}{\partial S_0} + \sigma\sqrt{T}(-d_{\rm N})n(d_{\rm N})\frac{\partial d_{\rm N}}{\partial S_0}\\
&= N(d) + \Big((S_0-K)n(d_{\rm N}) - \sigma\sqrt{T}\, d_{\rm N}\, n(d_{\rm N})\Big)\frac{\partial d_{\rm N}}{\partial S_0}\\
&= N(d_{\rm N}) + 0\cdot\frac{\partial d_{\rm N}}{\partial S_0} = N(d_{\rm N})
\end{aligned}
$$

(b)

$$
\begin{aligned}
\frac{\partial C_{\rm N}}{\partial \sigma_{\rm N}} &= (S_0-K)n(d_{\rm N})\frac{\partial d_{\rm N}}{\partial \sigma_{\rm N}} + \sqrt{T}\, n(d_{\rm N}) + \sigma_{\rm N}\sqrt{T}\,(-d_{\rm N})n(d_{\rm N})\frac{\partial d_{\rm N}}{\partial \sigma_{\rm N}}\\
&= \sqrt{T}\, n(d_{\rm N}) + \Big((S_0-K) - \sigma\sqrt{T}\, d_{\rm N}\Big)\, n(d)\frac{\partial d_{\rm N}}{\partial \sigma_{\rm N}}\\
&= \sqrt{T}\, n(d_{\rm N}) + 0\cdot n(d_{\rm N})\frac{\partial d_{\rm N}}{\partial \sigma_{\rm N}} = \sqrt{T}\, n(d_{\rm N})
\end{aligned}
$$

5. **[2019HW 2-3, A simple credit default swap]** Assume that the stock price of a firm follows an arithmetic BM,

$$S_t = S_0 + \sigma_{\rm N} B_t \quad \text{where } B_t \text{ is a standard BM}$$

and that the firm is considered default if $S_t = 0$. You want to buy a credit default swap (CDS) from an investment bank. You continuously pay at the rate of λ to the bank as premium (i.e., pay λdt during the time period between t and $t+dt$). The bank will pay you \$$M$ if the firm defaults. We want to determine the fair value of λ. Assume that the continuously compounded interest rate is $r>0$.

(a) What is the expected value of the payout (\$$M$) at the default event.

(b) What is the expected value of the premium payment.

(c) Determine the fair rate λ by equating the two values above.

(Hint: use the result from **Hitting time of a level** in SCFA Section 4.5)

Solution: The Laplace transform of the first hitting time τ is given by

$$E(e^{-r\tau}) = e^{-(S_0/\sigma_{\rm N})\sqrt{2r}}.$$

(a) The expectation of the CDS payout is

$$M\cdot E(e^{-r\tau}) = Me^{-(S_0/\sigma_{\rm N})\sqrt{2r}}.$$

(b) The present value of the premium paid until τ is

$$\int_0^\tau e^{-rt}\lambda dt = \frac{\lambda}{r}(1-e^{-r\tau}).$$

Taking the expectation,

$$E\left(\frac{\lambda}{r}(1-e^{-r\tau})\right) = \frac{\lambda}{r}\left(1 - E\left(e^{-r\tau}\right)\right) = \frac{\lambda}{r}\left(1 - e^{-(S_0/\sigma_{\rm N})\sqrt{2r}}\right)$$

(c) By equating the results of (a) and (b),

$$\frac{\lambda}{r}\left(1 - e^{-(S_0/\sigma_{\rm N})\sqrt{2r}}\right) = Me^{-(S_0/\sigma_{\rm N})\sqrt{2r}},$$

we obtain λ as

$$\lambda = \frac{rMe^{-(S_0/\sigma_{\rm N})\sqrt{2r}}}{1 - e^{-(S_0/\sigma_{\rm N})\sqrt{2r}}} = \frac{rM}{e^{(S_0/\sigma_{\rm N})\sqrt{2r}} - 1}.$$

6. **[2016ME(ASP), Asian option]** Asian option is an option where the payoff at maturity T is derived from the average of the underlying prices at a given set of times before and at the maturity,

$$\left(\frac{1}{N}\sum_{k=1}^{N} S(t_k) - K\right)^+ \quad \text{for} \quad 0 < t_1 < \cdots < t_N = T.$$

When $N = 4$ and $t_k = k/4$ $(T = 1)$ (a quarterly averaged Asian option), find the price of the Asian option. Assume the underlying stock price follows BM process, $dS(t) = \sigma\, dB(t)$, and the option price is given as $C = 0.4\,\sigma\sqrt{T}$ (at-the-money strike, zero interest rate and zero divided rate). How much the price of this Asian option is cheaper (or more expensive) to that of the European option with the same maturity and the same volatility?

Solution:

$$\begin{aligned}
&\mathrm{Var}\Big(\frac{\sigma}{4}(B_{1/4} + B_{2/4} + B_{3/4} + B_{4/4})\Big) = \frac{\sigma^2}{16}\mathrm{Var}(B_{1/4} + B_{2/4} + B_{3/4} + B_{4/4}) \\
&= \frac{\sigma^2}{16}E\Big((B_{1/4} + B_{2/4} + B_{3/4} + B_{4/4})^2\Big) \\
&= \frac{\sigma^2}{16}E\Big(B_{1/4}^2 + B_{2/4}^2 + B_{3/4}^2 + B_{4/4}^2 + 2B_{1/4}(B_{2/4} + B_{3/4} + B_{4/4}) \\
&\qquad\qquad + 2B_{2/4}(B_{3/4} + B_{4/4}) + 2B_{3/4}B_{4/4}\Big) \\
&= \frac{\sigma^2}{16}\Big(\frac{1}{4} + \frac{2}{4} + \frac{3}{4} + \frac{4}{4} + 2\frac{1}{4}\cdot 3 + 2\frac{2}{4}\cdot 2 + 2\frac{3}{4}\cdot 1\Big) = \frac{\sigma^2}{16}\frac{30}{4} = \frac{15\sigma^2}{32}.
\end{aligned}$$

The option price is given as

$$\text{Price of Asian option} = 0.4\sqrt{\frac{15}{32}}\,\sigma.$$

Asian option is about 31.5% $(= 1 - \sqrt{15/32})$ cheaper than European option with the same expiry, 0.4σ.

7. **[2017ME(ASP), Asian option]**
Asian option is an option where the payoff at maturity T is derived from the average of the underlying prices at a given set of times before and at the maturity,

$$\left(\frac{1}{N}\sum_{k=1}^{N} S(t_k) - K\right)^+ \quad \text{for} \quad 0 < t_1 < \cdots < t_N = T.$$

When $N = 3$ and $t_k = (k+3)/3$ ($T = 2$), find the price of the at-the-money Asian option. Assume the underlying stock price follows an arithmetic BM process (normal model), $dS_t = \sigma\, dW_t$, and the option price is given as $C = 0.4\,\sigma\sqrt{T}$ (at-the-money strike, zero interest rate and zero divided rate). How much the price of this Asian option is cheaper (or more expensive) than that of the European option with the same maturity ($T = 2$) and the same volatility?

Solution:

$$\begin{aligned}\mathrm{Var}\Big(\frac{1}{3}(B_{4/3} + B_{5/3} + B_2)\Big) &= \frac{1}{9}E\Big((B_{4/3} + B_{5/3} + B_2)^2\Big)\\ &= \frac{1}{9}E\Big(B_{4/3}^2 + B_{5/3}^2 + B_2^2 + 2B_{4/3}(B_{5/3} + B_2) + 2B_{5/3}B_2\Big)\\ &= \frac{1}{9}\Big(\frac{4}{3} + \frac{5}{3} + 2 + 2\cdot\frac{4}{3}\cdot 2 + 2\cdot\frac{5}{3}\Big) = \frac{1}{9}\cdot\frac{41}{3} = \frac{41}{27}.\end{aligned}$$

The option price is given as

$$\text{Price of Asian option} = 0.4\sqrt{\frac{41}{27}}\,\sigma.$$

Asian option is about 12.8% ($= 1 - \sqrt{41/54}$) cheaper than European option with the same expiry, $0.4\sqrt{2}\,\sigma$.

8. **[2017ME, Basket/Spread option under normal model]** Let W_t and Z_t are standard **BM**s with correlation ρ, i.e., $E(W_t\, Z_t) = \rho\, t$. For (b) and (c), assume that two stock prices follow

$$S_{1t} = 100 + 20W_t, \quad S_{2t} = 80 + 10Z_t.$$

and you may use the simple ATM option price formula, $C = P = 0.4\sigma\sqrt{T}$.

(a) Calculate $\mathrm{Var}(a\, W_T + b\, Z_T)$ for some constants a and b.

(b) Consider a spread option, whose payout is on *the difference of the prices*, $\max\big(S_{1T} - S_{2T} - K, 0\big)$. What is the price of the ATM option ($K = 20$) as a function of ρ? What is the min and max price and when do you get the those values?

(c) Consider a basket option, whose payout is on *the weighted average of the prices*, $\max\big((S_{1T} + S_{2T})/2 - K, 0\big)$. What is the price of the ATM option ($K = 90$) as a function of ρ? What is the min and max price and when do you get the those values?

Solution:

(a)

$$\mathrm{Var}(a\,W_T + b\,Z_T) = (a^2 + 2\rho ab + b^2)T$$

(b) $\mathrm{Var}(S_{1T} - S_{2T}) = \mathrm{Var}(a\,W_T + b\,Z_T)$ with $a = 20$ and $b = -10$. Therefore,

$$C = 0.4\sqrt{(a^2 + 2\rho ab + b^2)T} = 4\sqrt{(5 - 4\rho)T},$$

and

$$4\sqrt{T}\ \ (\rho = 1,\text{ i.e., correlated}) \le C \le 12\sqrt{T}\ \ (\rho = -1,\text{ i.e., anti-correlated}).$$

(c) $\mathrm{Var}((S_{1T} + S_{2T})/2) = \mathrm{Var}(a\,W_T + b\,Z_T)$ with $a = 10$ and $b = 5$. Therefore,

$$C = 0.4\sqrt{(a^2 + 2\rho ab + b^2)T} = 2\sqrt{(5 + 4\rho)T},$$

and

$$2\sqrt{T}\ \ (\rho = -1,\text{ i.e., anti-correlated}) \le C \le 6\sqrt{T}\ \ (\rho = 1,\text{ i.e., correlated}).$$

9. **[2018ME, Merton's model]** From an accounting standpoint, a firm's equity (stock) value, S, is equal to $A - D$, where A is total asset value and D is total debt. When A goes below D, the firm defaults with equity value $S = 0$. In 1974, Merton proposed a model to price the current equity value S_0 as the expected asset value A_T above the constant debt value D, i.e., the call option on the asset A_T stuck at D, at some time T (expiry):

$$S_0 = E(\max(A_T - D, 0)) \quad \text{where} \quad A_0 > D.$$

If A_t follows a geometric BM, the resulting equity price is same as the call option price formula by Black and Scholes (1973). Thus, the formula is called Black-Scholes-Merton formula (and Merton was awarded Nobel prize with Scholes in 1997).

Instead, in this problem, assume that A_t follows an arithmetic BM with volatility σ:

$$A_t = A_0 + \sigma B_t \quad (\text{assume} \quad r = 0).$$

(a) What should be the current equity value S_0? You may use the result from class.

(b) Under this framework, we can also derive the corporate bond (issued by the firm) maturing at T. At the maturity $t = T$, the firm pays \$1 to the bond holder. If the firm defaults before T, however, the bond value becomes zero. Assume that the risk-free interest rate is zero. (Hint: use the result on the probability of the first hitting time, $P(\tau_a > T)$)

Solution:

(a) It is same as the call option value under normal model:

$$S_0 = (A_0 - D)N(d_{\rm N}) + \sigma\sqrt{T}\,n(d_{\rm N}) \quad \text{where} \quad d_{\rm N} = \frac{A_0 - D}{\sigma\sqrt{T}}$$

(b) For a standard BM, the probability for the first time hitting the level a is given as

$$P(\tau_a > T) = 2N\left(\frac{|a|}{\sqrt{T}}\right) - 1$$

The bond price is same as the probability of A_t hitting D happening later than T. Therefore,

$$\text{Corporate bond price} = 2N\left(\frac{A_0 - D}{\sigma\sqrt{T}}\right) - 1$$

10. **[2019ME, Equity Linked Note]** An equity-linked note (ELN) is a debt instrument, usually a bond, that differs from a standard fixed-income security in that the final payout is based on the return of the underlying equity, which can be a single stock, basket of stocks, or an equity index. Equity-linked notes are a type of structured products (WIKIPEDIA). We (as a security firm) want to design and sell an ELN based on a stock following a BM:

$$S_t = S_0 + \sigma\, B_t.$$

This note has coupon N periods, $t = k\Delta t$ for $k = 1, 2, \cdots, N$. At $t = 0$, investors buy this note at the price of P for the notional value of \$1. At the end of the k-th period, $t = k\Delta t$, it pays coupon μ if the stock price did not fall more than δ (i.e., if $S_{k\Delta t} - S_{(k-1)\Delta t} \geq -\delta$) and continues to the next period. At the maturity $t = N\Delta t$, it pays $1 + \mu$ (1 is the redemption of the notional value). If the price falls more than δ at $t = k\Delta t$, (i.e., $S_{k\Delta t} - S_{(k-1)\Delta t} < -\delta$), the note terminates immediately by redeeming $(1 - L)$ (at the loss of L).

Assume that the discounting rate for one period, Δt, is r. So, the present value of \$1 paid after time Δt is $1/(1 + r)$. To simplify notation, you can use $D = 1/(1 + r)$.

The basic design of this ELN is that investors receive coupon μ higher than the risk-free rate r if stock market does not crash. However, they take a risk of heavy loss L if market crashes.

(a) Obtain the price P by calculating the expected value of the payout of this ELN.

(b) Assume the following specific parameters:

$$N = 8,\ \Delta t = 0.25,\ S_0 = 100,\ \sigma = 10,\ \delta = 10,\ D = 0.97\ (r \approx 3\%),\ L = 0.5.$$

That is, this ELN observes the price every 3 months and the maturity is 2 years. Given the parameters, determine the return μ to make the price of this ELN par (i.e., $P = 1$). You may use spreadsheet. How does it compares to the risk-free rate $r \approx 3\%$?

(c) Right after clients buy this ELN at the price $P = 1$, the volatility of the underlying stock suddenly increased to $\sigma = 20$ due to the spread of the Corvid-19 virus. (Assume that $S_0 = 100$ is unchanged.) What is client's loss?

Solution:

(a) From the independent increment of BM, the probability p to pay the coupon μ is same at every period. The probability, p, and $q = 1 - p$ is given by

$$
\begin{aligned}
p &= P(S_{k\Delta t} - S_{(k-1)\Delta t} \ge -\delta) = P(\sigma B_{\Delta t} \ge -\delta) \\
&= 1 - N\left(-\frac{\delta}{\sigma\sqrt{\Delta t}}\right) = N\left(\frac{\delta}{\sigma\sqrt{\Delta t}}\right).
\end{aligned}
$$

The present value of the ELN is decomposed into three components:

- The coupon μ paid at the k-th period ($k = 1, \cdots, n$):

  $$\text{Present Value} = \mu D^k, \quad \text{Probability} = p^k$$

  The expectation is given by

  $$\sum_{k=1}^{n} \mu D^k \cdot p^k = \mu(pD)\frac{1-(pD)^n}{1-pD}$$

- The early terminated redemption with loss, $(1 - L)$, at the k-th period ($k = 1, \cdots, n$):

  $$\text{Present Value} = (1-L)D^k, \quad \text{Probability} = p^{k-1}q.$$

  The expectation is give by

  $$\sum_{k=1}^{n}(1-L)D^k \cdot p^{k-1}q = (1-L)\frac{q}{p}\sum_{k=1}^{n}(pD)^k = (1-L)(qD)\frac{1-(pD)^n}{1-pD}$$

- The redemption of the notional \$1 at the n-th period (maturity):

  $$\text{Present Value} = D^n, \quad \text{Probability} = p^n.$$

  The expectation is $(pD)^n$.

The price P is the sum of the three expected values:

$$
\begin{aligned}
P &= \mu(pD)\frac{1-(pD)^n}{1-pD} + (1-L)(qD)\frac{1-(pD)^n}{1-pD} + (pD)^n \\
&= \Big(\mu p + (1-L)q\Big)\, D\, \frac{1-(pD)^n}{1-pD} + (pD)^n.
\end{aligned}
$$

(b) The coupon μ to satisfy $P = 1$ is obtained as

$$
\begin{aligned}
\Big(\mu p + (1-L)q\Big)\, D\, \frac{1-(pD)^n}{1-pD} + (pD)^n &= 1 \\
\mu p + (1-L)q &= \frac{1-pD}{D} \\
\mu &= \frac{1}{p}\left(\frac{1-pD}{D} - (1-L)q\right)
\end{aligned}
$$

Using the parameter values, the coupon should be $\mu = 4.234\%$. The intermediate values are

$$p = N(2) = 97.725\% \quad \text{and} \quad q = 1 - p = 2.275\%.$$

(c) When σ is suddenly jump to $\sigma = 20$, the price drops down to $P = 0.6867$. The loss is about 31%. The intermediate values are

$$p = N(1) = 84.134\% \quad \text{and} \quad q = 1 - p = 15.866\%.$$

5 Richness of Paths

1. **[An extra problem]** Derive the probability results on the running maximum and the first hitting time to the BM with the volatility σ. Using the scaling, $\sigma B_t = B_{\sigma^2 t}$, you are going to replace t with $\sigma^2 t$.

Solution: For the CDF (on both time and space) and the PDF on space, the simple replacement works.

$$P(\sigma B_t^* < x) = P(\tau_x > t) = 2N(x/\sigma\sqrt{t}) - 1.$$

$$f_{\sigma B_t^*}(x) = \frac{2}{\sigma\sqrt{t}} n\left(\frac{x}{\sigma\sqrt{t}}\right)$$

For the PDF on time, however, we need to consider the normalization because the time is scaled. The original PDF $f_{\tau_x}(t)$ satisfies $\int_0^\infty f_{\tau_x}(t)dt = 1$. After the scaling,

$$\int_0^\infty f_{\tau_x}(\sigma^2 t)dt = \frac{1}{\sigma^2},$$

so the new PDF should be

$$\sigma^2 f_{\tau_x}(\sigma^2 t) = \frac{x}{\sigma t^{3/2}}\, n\left(\frac{x}{\sigma\sqrt{t}}\right).$$

2. **[2016ME(ASP). Maximum of a BM with drift]**

Proposition The maximum of a BM with drift $\mu < 0$, $B^* = \max_{0\le t\le\infty}(B_t + \mu t)$, has exponential distribution

$$P(B^* > x) = e^{2\mu x} \quad (x \ge 0)$$

(a) Using the above proposition, prove that

$$P(B_t \le at + b \text{ for all } t > 0) = 1 - e^{-2ab} \quad \text{for} \quad a, b > 0$$

(b) Using a proper change of variable, scaling of BM and etc, extend the result of (a) for BM with volatility, σB_t.

Solution:

(a)

$$\begin{aligned} P(B_t \le at + b \text{ for all } t \ge 0) &= P(B_t - at \le b \text{ for all } t \ge 0) \\ &= 1 - P(B_t - at > b \text{ for some } t \ge 0) \\ &= 1 - P(B^* > b) = 1 - e^{-2ab} \end{aligned}$$

(b) Method 1:

$$P(\sigma B_t \le at + b) = P\left(B_t \le \frac{a}{\sigma}t + \frac{b}{\sigma}\right) = 1 - e^{-2ab/\sigma^2}$$

Method 2: We use $\sigma B_t \sim B_{\sigma^2 t}$ ($t' = \sigma^2 t$):

$$P(\sigma B_t \le at + b) = P(B_{\sigma^2 t} \le at + b) = P\left(B_{t'} \le \frac{a}{\sigma^2} t' + b\right) = 1 - e^{-2ab/\sigma^2}$$

3. **[2018ME, Call option on maximum]** Assume that a stock price follows a BM, $S_t = S_0 + \sigma B_t$. As in the text book, assume that $S_t^* = \max_{0 \le s \le t} S_s$. Calculate the call option price whose payout at expiry $t = T$ is given by the maximum value on the path

$$\max(S_T^* - K, 0) \quad \text{where} \quad K > S_0$$

Intuitively, this option should be more expensive that the regular call option whose payout is given by the final price S_T. By how much more is it more expensive? (Hint: In class and textbook, we derived the PDF of B_T^*. Properly adjust σ.)

Solution: The PDF for the maximum of BM, B_t^*, is given as

$$f(x) = \frac{2}{\sqrt{t}} n\left(\frac{x}{\sqrt{t}}\right) \quad \text{for} \quad x \ge 0$$

This is equivalent to the normal distribution, $N(0, t)$, defined on the positive side only (that is why factor 2 is multiplied). Therefore, $S_T^* = S_0 + \sigma\sqrt{T}\, z$ where the PDF of z is $2n(z)$ and $z \ge 0$. The option price is twice as expensive as that of the regular call option:

$$C(K) = 2(S_0 - K)N(d_{\text{N}}) + 2\sigma\sqrt{T}\, n(d_{\text{N}}) \quad \text{where} \quad d_{\text{N}} = \frac{S_0 - K}{\sigma\sqrt{T}}$$

4. **[2019ME, Put option on minimum]** Assume that the stock price follows a BM, $S_t = S_0 + \sigma B_t$. Assume that $S_t^m = \min\{S_s : 0 \le s \le t\}$. Calculate the put option price whose payout at expiry $t = T$ is given by the minimum value along the path

$$\max(K - S_T^m, 0) \quad \text{where} \quad K < S_0$$

Intuitively, this option should be more expensive that the regular put option whose payout is given by the final price S_T. By how much is it more expensive?

Solution: Let $W_t = -B_t$, then W_t is also a standard BM. The minimum of B_t is the negative of the maximum of W_t:

$$B_t^m = \min\{B_s : 0 \le s \le t\} = -\max\{W_s : 0 \le s \le t\} = -W_t^M$$

Since the PDF for W_t^M, is given by

$$f(x) = \frac{2}{\sqrt{t}} n\left(\frac{x}{\sqrt{t}}\right) \quad \text{for} \quad x \ge 0,$$

the PDF for B_t^m is same as

$$f(x) = \frac{2}{\sqrt{t}}\, n\left(\frac{x}{\sqrt{t}}\right) \quad \text{for} \quad x \le 0.$$

because $n(x) = n(-x)$. This is two times the PDF for B_t defined on the negative side only. The minimum put option price is twice as expensive as that of the regular put option:

$$P(K) = 2(K - S_0)N(-d_{\text{N}}) + 2\sigma\sqrt{T}\, n(d_{\text{N}}) \quad \text{where} \quad d_{\text{N}} = \frac{S_0 - K}{\sigma\sqrt{T}}$$

5. **[2016ME(ASP), Down-and-out digital option, Joint distribution of B_t^m and B_t]** We are going to derive the price of the binary call option with knock-out (down-and-out) feature under the Bachelier model. Assume that the underlying stock follows the process $S_t = S_0 + \sigma B_t$. The option will pay you \$1 at the expiry T if $S_T > K$ for a strike price K **and** the stock price S_t has never been below L for $L < \min(S_0, K)$ anytime before the expiry, $0 \le t \le T$. In other words, this option knocks out (expires worthless) if S_t falls below L any time before the expiry T. Let the running maximum and minimum of B_t

$$B_T^M = \max_{0 \le t \le T} B_t \quad \text{and} \quad B_T^m = \min_{0 \le t \le T} B_t.$$

(a) In the class (and in the textbook), we derived the joint CDF for B_T and B_T^M,

$$P(B_T^M < v,\ B_T < u) = N\Big(\frac{u}{\sqrt{T}}\Big) - N\Big(\frac{u - 2v}{\sqrt{T}}\Big) \quad \text{for} \quad v \ge \max(0, u)$$

Using this result, derive the joint CDF for BM with volatility, σB_t,

$$P(\sigma B_T^M < v,\ \sigma B_T < u).$$

(b) Using the symmetry that $-\sigma B_t$ has the same distribution as σB_t, drive the probability

$$P(\sigma B_T^m > v, \sigma B_T > u) \quad \text{for} \quad v \le \min(0, u).$$

(c) Finally find the price of the binary call option with knock-out feature? Assume that interest rate and divided rate is zero. How much is this derivative cheaper (or more expensive) than the regular binary call option **without** knock-out feature?

Solution:

(a)

$$P(\sigma B_T^M < v,\ \sigma B_T < u) = P(B_T^M < v/\sigma,\ B_T < u/\sigma) = N\Big(\frac{u}{\sigma\sqrt{T}}\Big) - N\Big(\frac{u - 2v}{\sigma\sqrt{T}}\Big)$$

(b) Under the reflection, $B_t \to -B_t$, we get

$$(-B)_T^m = \min_{0 \le t \le T}(-B_t) = -\max_{0 \le t \le T} B_t = -B_T^M.$$

Applying the reflection to the probability,

$$
\begin{aligned}
P(\sigma B_T^m > v,\ \sigma B_T > u) &= P(\sigma(-B)_T^m > v,\ \sigma(-B_T) > u) \\
&= P(-B_T^M > v/\sigma,\ -B_T > u/\sigma) \\
&= P(B_T^M < -v/\sigma,\ B_T < -u/\sigma) \\
&= N\Big(\frac{-u}{\sigma\sqrt{t}}\Big) - N\Big(\frac{2v-u}{\sigma\sqrt{t}}\Big).
\end{aligned}
$$

(c) Plug in $K - S_0 \to u$ and $L - S_0 \to v$ to get

$$
\text{Price with Knockout} = N\Big(\frac{S_0 - K}{\sigma\sqrt{t}}\Big) - N\Big(\frac{2L - K - S_0}{\sigma\sqrt{t}}\Big)
$$

The price without knockout is the first term, so the knockout feature is cheapening the price by $N\Big(\frac{2L-K-S_0}{\sigma\sqrt{t}}\Big)$.

6. **[2017HW 2-3, Bachelier knock-out (down-and-out) option]** Derive the price of down-and-out call option with knock-out strike L and option strike K. (Obviously, $L < S_0$ and $L < K$) See the derivation for up-and-out call option (2016HW 3-3) and down-and-out digital option (2016ME(ASP)).

Solution: Assume $B_T^M = \max_{0\le t\le T} B_t$ and $B_T^m = \min_{0\le t\le T} B_t$. From textbook and class, we know

$$
P(\sigma B_T^M < v,\ \sigma B_T < u) = P(B_T^M < v/\sigma,\ B_T < u/\sigma) = N\Big(\frac{u}{\sigma\sqrt{T}}\Big) - N\Big(\frac{u-2v}{\sigma\sqrt{T}}\Big),
$$

where $N(\cdot)$ is the normal distribution CDF. Using the reflection, $B_t \to -B_t$, we get

$$
(-B)_T^m = \min_{0\le t\le T}(-B_t) = -\max_{0\le t\le T} B_t = -B_T^M
$$

and

$$
P(\sigma B_T^m > v,\ \sigma B_T > u) = N\Big(\frac{-u}{\sigma\sqrt{T}}\Big) - N\Big(\frac{2v-u}{\sigma\sqrt{T}}\Big).
$$

As the stock price is given as $S_T = S_0 + \sigma B_T$,

$$
P(S_T^m > v,\ S_T > u) = N\Big(\frac{S_0-u}{\sigma\sqrt{T}}\Big) - N\Big(\frac{2v-u-S_0}{\sigma\sqrt{T}}\Big).
$$

The probability density function on u with the joint condition, $\sigma B_T^m > v$ is obtained from the partial derivative w.r.t. u (with negative sign),

$$
f(u) = \frac{1}{\sigma\sqrt{T}}\left(n\Big(\frac{S_0-u}{\sigma\sqrt{T}}\Big) - n\Big(\frac{2v-u-S_0}{\sigma\sqrt{T}}\Big)\right) \quad \text{for} \quad -\infty < v \le u.
$$

Let $z = (u - S_0)/\sigma\sqrt{T}$, $d = (S_0 - K)/\sigma\sqrt{T}$ and $d^* = (S_0 - L)/\sigma\sqrt{T}$. Then, the down-and-out call option price is given as

$$\begin{aligned} C(K, L) = & \int_{u=K}^{\infty} (u - K)\, f(u) du = \int_{z=-d}^{\infty} (S_0 - K + \sigma\sqrt{T}\, z)(n(z) - n(z + 2d^*)) dz \\ =& (S_0 - K) N(d) + \sigma\sqrt{T}\, n(d) - (S_0 - K - 2d^*\sigma\sqrt{T}) N(d - 2d^*) \\ & - \sigma\sqrt{T}\, n(d - 2d^*). \end{aligned}$$

The first two terms are exactly the regular call option price, $C(K) = (S_0 - K)N(d) + \sigma\sqrt{T}\, n(d)$. Therefore, the down-and-out option is cheaper than the regular option by $(S_0 - K - 2d^*\sigma\sqrt{T})N(d - 2d^*) + \sigma\sqrt{T}\, n(d - 2d^*)$.

We can verify two cases:

1. If $L \to -\infty$ ($d^* \to \infty$), $C(K, L) = C(K)$ because the probably of being knocked out is zero. It is indeed the case because $N(d - 2d^*) = n(d - 2d^*) = 0$.
2. If $L \to S_0$ from below ($d^* \to 0$) on the other hand, the knock-out probability approaches to 100%, so the price should be zero. This is also the case from the formula.

7. **[2016HW 3-3, Bachelier knock-out (up-and-out) option]**
Using the joint distribution of B_t and $B_t^* = \max_{0 \le s \le t} B_s$, derive the price of the call option struck at K and knock-out at H ($> K$). First, generalize the joint CDF function $P(u < B_t, v < B_t^*)$ to σB_t. Next, derive the pdf on u by taking derivative on u. Then, integrate the payoff $(S_T - K)^+$ from K to H. (Assume $r = q = 0$. Otherwise the problem is too complicated.)

Solution:

$$\begin{aligned} P(S_T^* < v, S_T < u) &= P(\sigma B_T^* < v - S_0,\ \sigma B_T < u - S_0) \\ &= P(B_T^* < (v - S_0)/\sigma,\ B_T < (u - S_0)/\sigma) \\ &= N\Big(\frac{u - S_0}{\sigma\sqrt{T}}\Big) - N\Big(\frac{u - 2v + S_0}{\sigma\sqrt{T}}\Big) \end{aligned}$$

The probability density function on u conditional on $S_T^* < H$ is obtained from the partial derivative w.r.t. u,

$$f(u) = \frac{1}{\sigma\sqrt{T}} \left(n\Big(\frac{u - S_0}{\sigma\sqrt{T}}\Big) - n\Big(\frac{u - 2H + S_0}{\sigma\sqrt{T}}\Big) \right) \quad \text{for} \quad -\infty < u \le v$$

With variables, $z = (u - S_0)/\sigma\sqrt{T}$. $d = (S_0 - K)/\sigma\sqrt{T}$ $d^* = (S_0 - H)/\sigma\sqrt{T}$, the knock-out call option price is given as

$$
\begin{aligned}
C(K,H) = \int_K^H (u-K)\, f(u)du = &\int_{-d}^{-d^*} (S_0 - K + \sigma\sqrt{T}\, z)\big(n(z) - n(z+2d^*)\big)dz \\
=&(S_0-K)\int_{-d}^{-d^*} \big(n(z) - n(z+2d^*)\big)dz \\
&+\sigma\sqrt{T}\int_{-d}^{-d^*} \big(z\, n(z) - (z+2d^*)\, n(z+2d^*) + 2d^*\, n(z+2d^*)\big)dz \\
=&(S_0-K)\big(N(-d^*) - N(-d) - N(d^*) + N(-d+2d^*)\big) + \sigma\sqrt{T}\Big(-n(-d^*) \\
&\quad + n(-d) + n(d^*) - n(-d+2d^*) + 2d^* N(d^*) - 2d^* N(-d+2d^*)\Big) \\
=&(S_0-K)\big(N(d) - 2N(d^*) + N(2d^*-d)\big) \\
&+\sigma\sqrt{T}\big(n(d) - n(2d^*-d) + 2d^* N(d^*) - 2d^* N(2d^*-d)\big)
\end{aligned}
$$

We can verify two trivial cases:

1. If $K = H$ ($d = d^*$), the option price should be zero

$$C(K, H=K) = 0.$$

2. If $H = \infty$ ($d^* = -\infty$), the price is same as the price of the regular call option

$$C(K, H=\infty) = (S_0-K)N(d) + \sigma\sqrt{T}\, n(d).$$

We also note that the difference in the prices of the knock-out option and the regular option is given as

$$C(K) - C(K,H) = (S_0-K)\big(2N(d^*) - N(2d^*-d)\big) + \sigma\sqrt{T}\big(n(2d^*-d) - 2d^* N(d^*) + 2d^* N(2d^*-d)\big).$$

8. **[2019ME, First hitting time of a BM with drift]** From SCFA Chapter 5, we derived that the probability density of the first-hitting time τ to the level $\delta > 0$ is given by

$$f_\tau(t) = \frac{\delta}{\sqrt{2\pi t^3}} e^{-\delta^2/2t} = \frac{\delta}{\sqrt{t^3}}\, n\left(\frac{\delta}{\sqrt{t}}\right). \tag{1}$$

We also know from **Hitting time of a level** in Chapter 4.5 that the Laplace transform of τ is given by

$$E(e^{-r\tau}) = e^{-\delta\sqrt{2r}}$$

and that the result is interpreted as the present value (discounted with the interest rate r) of a derivative paying \$1 at the event. We are going to generalize the result to the first hitting time of a drifted BM, $B_t + \gamma t$ ($\gamma > 0$). The probability density for τ,

$$\tau = \min\{t : B_t + \gamma t = \delta\} \quad (\delta > 0)$$

is given by

$$f_\tau(t) = \frac{\delta}{\sqrt{2\pi t^3}} \exp\left(-\frac{(\gamma t-\delta)^2}{2t}\right) = \frac{\delta}{\sqrt{t^3}}\, n\left(\frac{\delta-\gamma t}{\sqrt{t}}\right), \tag{2}$$

and you can use this result without proof.

(a) What is the price of the derivative, $E(e^{-r\tau})$?
Hint: Consider the Laplace transform of Eq. (1).

(b) (3 points) Obtain the mean and variance of τ, $E(\tau)$ and $\mathrm{Var}(\tau)$.
Hint: if $L(r) = E(e^{-r\tau})$ is the Laplace transform of τ, $L(-r)$ is the moment generating function because $L(-r) = E(e^{r\tau})$.

(c) We further generalize the results to derivative pricing. Assume that a stock follows the process, $S_t = S_0 + \mu t + \sigma B_t$ and the derivative pays \$1 when S_t hits K ($> S_0$) for the first time, i.e., $S_\tau = K$. What is the price of the derivative at $t = 0$? (Modify the result from (a).) Assume that you sold the derivative to clients and that you need to hedge the position using the underlying stock. How many shares of the underlying stock do you need to long or short?

(d) From class we know that the CDF for Eq. (1) is

$$P(\tau \le t) = 2 - 2N(\delta/\sqrt{t}),$$

where $N(\cdot)$ is the normal cumulative distribution function. Can you derive the CDF for the generalized density function, Eq. (2)? (This question might be challenging. Try it only when you have extra time.)

Solution: The distribution of τ, Eq. (2), is known as the inverse Gaussian distribution (WIKIPEDIA). The inverse Gaussian distribution is typically parameterized by μ and λ:

$$f_\tau(t) = \sqrt{\frac{\lambda}{2\pi t^3}} \exp\left(-\frac{\lambda(t-\mu)^2}{2\mu^2\, t}\right).$$

The parameters (μ, λ) are related to our parameters (γ, δ) by

$$\mu = \delta/\gamma, \quad \lambda = \delta^2.$$

Any established results for the inverse Gaussian distribution can be expressed in terms of (γ, δ) using the above formula. The distribution is called *inverse* Gaussian because it describes the (first-hitting) time of BM at a fixed location, whereas the Gaussian distribution describes the location at a fixed time. The answers to this problem is well-known properties of the inverse Gaussian distribution. But we can derive the solution from the knowledge obtained in class except (d).

(a) The density function, Eq. (1) and its Laplace transform are expressed as

$$E(e^{-r\tau}) = \int_{t=0}^{\infty} e^{-rt} f_\tau(t)dt = \int_{t=0}^{\infty} \frac{\delta}{\sqrt{2\pi t^3}} \exp\left(-\frac{\delta^2}{2t} - rt\right) dt = e^{-\delta\sqrt{2r}}$$

Based on this, we can derive the Laplace transform of Eq. (2):

$$
\begin{aligned}
E(e^{-r\tau}) &= \int_{t=0}^{\infty} \frac{\delta}{\sqrt{2\pi t^3}} \exp\left(-\frac{(\gamma t-\delta)^2}{2t} - rt\right) dt \\
&= \int_{t=0}^{\infty} \frac{\delta e^{\gamma\delta}}{\sqrt{2\pi t^3}} \exp\left(-\frac{\delta^2}{2t} - (r+\frac{\gamma^2}{2})t\right) dt \\
&= \int_{t=0}^{\infty} \frac{\delta e^{\gamma\delta}}{\sqrt{2\pi t^3}} \exp\left(-\frac{\delta^2}{2t} - r't\right) dt \quad \left(r' = r + \frac{\gamma^2}{2}\right) \\
&= \exp\left(\gamma\delta - \delta\sqrt{2r'}\right) \\
&= \exp\left(\gamma\delta\left(1-\sqrt{1+2r/\gamma^2}\right)\right).
\end{aligned}
$$

(b) From Taylor's expansion,

$$
1-\sqrt{1+\frac{2r}{\gamma^2}} = 1-\left(1+\frac{r}{\gamma^2}-\frac{r^2}{2\gamma^4}+\cdots\right) = -\frac{r}{\gamma^2}+\frac{r^2}{2\gamma^4}+\cdots,
$$

the Laplace transform is expanded into

$$
E(e^{-r\tau}) = 1+\gamma\delta\left(-\frac{r}{\gamma^2}+\frac{r^2}{2\gamma^4}+\cdots\right)+\frac{1}{2}\left(\frac{\delta^2}{\gamma^2}r^2+\cdots\right)+\cdots.
$$

Therefore, the first two moments and the variance are given by

$$
M_1 = \frac{\delta}{\gamma}, \quad M_2 = \frac{\delta}{\gamma^3}+\frac{\delta^2}{\gamma^2}, \quad \mathrm{Var} = \frac{\delta}{\gamma^3}.
$$

(c) From

$$
S_\tau = S_0 + \mu\tau + \sigma B_\tau = K \quad \Longrightarrow \quad (\mu/\sigma)\tau + B_\tau = (K-S_0)/\sigma,
$$

we can use

$$
\delta = \frac{K-S_0}{\sigma}, \quad \gamma = \frac{\mu}{\sigma}.
$$

From the result of (a),

$$
P = E(e^{-r\tau}) = \exp\left(\frac{\mu(K-S_0)}{\sigma^2}\left(1-\sqrt{1+\frac{2r\sigma^2}{\mu^2}}\right)\right)
$$

The amount of the underlying stock to hold for hedging the position is given by

$$
\frac{\partial P}{\partial S_0} = -\frac{\mu}{\sigma^2}\left(1-\sqrt{1+\frac{2r\sigma^2}{\mu^2}}\right)\exp\left(\frac{\mu(K-S_0)}{\sigma^2}\left(1-\sqrt{1+\frac{2r\sigma^2}{\mu^2}}\right)\right)
$$

(d) The CDF of the inverse Gaussian distribution is give by

$$
F(t) = N\left(\gamma\sqrt{t}-\frac{\delta}{\sqrt{t}}\right) + e^{2\gamma\delta}N\left(-\gamma\sqrt{t}-\frac{\delta}{\sqrt{t}}\right).
$$

This CDF was originally found by

Shuster, J. (1968). On the inverse Gaussian distribution function. *Journal of the American Statistical Association*, 63(324), 1514–1516. https://doi.org/10.1080/01621459.1968.10480942

It can be also derived using Girsanov theorem in Chapter 13.

When $\gamma = 0$, the CDF $F(t)$ is reduced to the case we learned in class:

$$F(t) = P(\tau \le t) = N\left(-\frac{\delta}{\sqrt{t}}\right) + N\left(-\frac{\delta}{\sqrt{t}}\right) = 2 - 2N\left(\frac{\delta}{\sqrt{t}}\right)$$

6 Itô Integration

1. **[SCFA 6.1]**

Solution: The mean of the two expressions are zero.

$$\operatorname{Var}\left(\int_0^t |B_s|^{\frac{1}{2}} dB_s\right) = E\left(\int_0^t |B_s| ds\right) = \int_0^t E(|B_s|)ds = \int_0^t \sqrt{\frac{2s}{\pi}}ds = \frac{2}{3}\sqrt{\frac{2}{\pi}}\,t^{\frac{3}{2}}$$

$$\begin{aligned}\operatorname{Var}\left(\int_0^t (B_s+s)^2 dB_s\right) &= E\left(\int_0^t (B_s+s)^4 ds\right)\\ &= \int_0^t E(B_s^4 + 4sB_s^3 + 6s^2B_s^2 + 4s^3B_s + s^4)ds\\ &= \int_0^t (3s^2 + 0 + 6s^2\cdot s + 0 + s^4)ds = \frac{1}{5}t^5 + \frac{3}{2}t^4 + t^3\end{aligned}$$

2. **[SCFA 6.2]**

Solution: For I_1,

$$E(I_1) = \int_0^t E(B_s)ds = \int_0^t 0\,ds = 0.$$

Using Itô's lemma applied to sB_s, $d(sB_s) = sdB_s + B_s ds$, we can express I_1 as

$$I_1 = tB_t - \int_0^t sdB_s = t\int_0^t dB_s - \int_0^t sdB_s = \int_0^t (t-s)dB_s,$$

where we used a trick of $B_t = \int_0^t dB_s$ in order to make the expression suitable for Ito's isometry. We get

$$\operatorname{Var}(I_1) = \int_0^t (t-s)^2 ds = \frac{1}{3}t^3$$

For I_2,

$$E(I_2) = \int_0^t E(B_s^2)ds = \int_0^t s\,ds = \frac{t^2}{2}.$$

Using Itô's lemma applied to sB_s^2, $d(sB_s^2) = B_s^2 ds + 2sB_s dB_s + sds$, we can express I_2 as

$$I_2 = tB_t^2 - 2\int_0^t sB_s dB_s - \frac{t^2}{2}.$$

We apply a similar trick, $d(B_s^2) = 2B_s dB_s + ds$, to replace B_t^2 with a more suitable expression for Itô's isometry,

$$I_2 = t\left(2\int_0^t B_s dB_s + t\right) - 2\int_0^t sB_s dB_s - \frac{t^2}{2} = 2\int_0^t (t-s)B_s dB_s + \frac{t^2}{2},$$

where we can reconfirm that $E(I_2) = t^2/2$. Finally,

$$\mathrm{Var}(I_2) = E\left(\left(I_2 - \frac{t^2}{2}\right)^2\right) = 4\int_0^t E\Big((t-s)^2 B_s^2\Big) ds = 4\int_0^t (t-s)^2 s\, ds = 4\cdot\frac{t^4}{12} = \frac{t^4}{3}$$

3. **[SCFA 6.3]**

Solution: At any time s, X_s and B_s has the same distribution, normal distribution with mean 0 and variance s, so $E(f(B_s)) = E(f(X_s))$ and

$$E(U_t) = \int_0^t E(f(B_s))ds = \int_0^t E(f(X_s))ds = E(V_t)$$

For variance, simply let $f(x) = x$. Using that $V_t = \int_0^t \sqrt{s}\, Z ds = \frac{2}{3} t^{\frac{3}{2}}\, Z$,

$$\mathrm{Var}(V_t) = \frac{4}{9} t^3.$$

According to **Exercise 6.2**, however,

$$\mathrm{Var}(U_t) = \frac{1}{3} t^3 \neq \mathrm{Var}(V_t).$$

4. **[2016HW 4-1, Itô's isometry]** Find the mean and variance of the following stochastic integral

$$Y_t = \int_0^t e^{B_s} dB_s$$

Solution: The mean is zero because an Itô's integral is a martingale. Alternatively, it is because $dY_t = e^{B_t} dB_t$ has no drift term (i.e., dt). The variance can be calculated using Itô's isometry:

$$\mathrm{Var}(Y_t) = E\Big[\Big(\int_0^t e^{B_s} dB_s\Big)^2\Big] = E\Big[\int_0^t e^{2B_s} ds\Big] = \int_0^t E(e^{2B_s})ds = \int_0^t e^{2s} ds = \frac{1}{2}(e^{2t} - 1)$$

5. **[2016ME(ASP), Volatility on holidays]**
In the class, we covered the relation between the daily volatility σ_d and the annual volatility σ_y as

$$\sigma_y = \sqrt{256}\, \sigma_d = 16\, \sigma_d,$$

where we assume the stock price is moving only on the 256 trading days in one year.

In this problem, we want to make it slightly more complicated. The stock price is usually more volatile on Mondays because new information, e.g., news related to the stock, economy, etc, is accumulated over the weekend and make the price move on Monday when the stock market is open. What would be the relation between the daily volatility σ_d and the annual volatility σ_y if we assume Monday's price is 50% more volatile than the other trading days, i.e., $1.5\,\sigma_d$ on Mondays? Assume there are 52 Mondays in one year (so the rest of the trading days are $256 - 52 = 204$).

Solution:

$$\sigma_y^2 = \left(204 \times 1 + 52 \times 1.5^2\right)\sigma_d^2 = 321\sigma_d^2$$

Therefore,

$$\sigma_y = \sqrt{321}\sigma_d \approx 17.92\,\sigma_d$$

6. **[2017ME, Exponentially decaying volatility]** Assume that a stock follows BM with time-varying volatility:

$$dS_t = \sigma(t)dB_t \quad \text{for} \quad \sigma(t) = a + be^{-\lambda t} \quad (a, b > 0)$$

What is the ATM call option price at expiry T? What is the equivalent normal model volatility? In other words, what value of σ_N gives the same option price when the stock price follows $dS_t = \sigma_\text{N}\,dW_t$ at $t = T$.

Solution: The variance of S_t computed as

$$\begin{aligned}\text{Var}(S_T) &= \int_0^T \sigma^2(t)dt = \int_0^T (a^2 + 2ab\,e^{-\lambda t} + b^2\,e^{-2\lambda t})dt \\ &= a^2T + \frac{2ab}{\lambda}(1 - e^{-\lambda T}) + \frac{b^2}{2\lambda}(1 - e^{-2\lambda T}).\end{aligned}$$

The option price is given as

$$C = 0.4\sqrt{a^2T + \frac{2ab}{\lambda}(1 - e^{-\lambda T}) + \frac{b^2}{2\lambda}(1 - e^{-2\lambda T})}.$$

From $\sigma_\text{N}^2 T = \text{Var}(S_T)$,

$$\sigma_\text{N} = \sqrt{a^2 + \frac{2ab}{\lambda T}(1 - e^{-\lambda T}) + \frac{b^2}{2\lambda T}(1 - e^{-2\lambda T})}$$

7. **[2017ME, Stochastic integral]** Assume that you follow a trading strategy based on *momentum*, where you long more stock if the stock price is up and short more if the stock price is down. If S_t is the process for the stock price, the amount you long/short is $S_t - S_0$.
   (a) If X_T is the profit and loss from this strategy at time $t = T$, express X_T using stochastic integral.
   (b) When the stock follows a BM, $S_t = S_0 + \sigma B_t$, what is X_T? You may directly use the result from the class.

(c) Imagine a scenario where the stock price goes up a lot but loses later recovering the original price, $S_T = S_0$. How much do you profit or lose at $t = T$? Intuitively explain why you profit or lose? (The opposite is called *mean-reversion* strategy, where you long/short by $S_0 - S_t$.)

(d) Calculate the variance of X_T. You can either use the result from (b) or use Itô's isometry.

Solution:

(a)

$$X_T = \int_0^T (S_t - S_0) dS_t$$

(b)

$$X_T = \sigma^2 \int_0^T B_t \, dB_t = \frac{\sigma^2}{2}(B_T^2 - T)$$

(c) Since $B_T = B_0 = 0$, the momentum strategy loses by $\sigma^2 T/2$ (where as the mean-reversion strategy profits the same amount). The strategy profits when the stock price goes up but loses when it goes down. Intuitively, loss is bigger than profit because you start from zero long position when the price is up where as you already have some short position when the price is down.

(d) From Itô's isometry,

$$\text{Var}(X_T) = \sigma^4 \int_0^T E(B_t^2) dt = \sigma^4 \int_0^T t \, dt = \frac{\sigma^4}{2} T^2$$

From (b),

$$\begin{aligned} \text{Var}(X_T) &= \frac{\sigma^4}{4} \text{Var}(B_T^2 - T) = \frac{\sigma^4}{4} E(B_T^4 - 2T\, B_T^2 + T^2) \\ &= \frac{\sigma^4}{4} E(3T^2 - 2T \cdot T + T^2) = \frac{\sigma^4}{2} T^2 \end{aligned}$$

8. **[2018ME, Time-dependent volatility]** The at-the-money options on Meituan Dianping (IPO in September 2018 on Hong Kong stock exchange) with three-month ($T = 1/4$) and one-year ($T = 1$) maturities are currently trading at the prices of 4.0 and 6.4 Hong Kong dollars, respectively. Assume that the stock price follows $dS_t = f(t)\, dB_t$ and that the option price can be approximated with 0.4 stdev(S_T). Find the piecewise constant instantaneous volatility $f(t)$ that satisfies the observed option prices.

Solution: We need to find

$$f(t) = \begin{cases} a & \text{if} \quad 0 \le t \le 0.25 \\ b & \text{if} \quad 0.25 \le t \end{cases}$$

For the two options,

$$
\begin{aligned}
4 &= 0.4\sqrt{0.25\,a^2} \\
6.4 &= 0.4\sqrt{0.25\,a^2 + 0.75\,b^2}
\end{aligned}
$$

We get $a = 20$ and $b = \sqrt{208} \approx 14.42$

7 Localization and Itô's integral

1. **[SCFA 7.1]**

Solution: The function τ_t is given by

$$\tau_t = \mathrm{Var}(B_{\tau_t}) = \mathrm{Var}(Y_t) = \mathrm{Var}(X_t) = \int_0^t e^{2s} ds = \frac{1}{2}(e^{2t} - 1).$$

$E(X_t^2) = E(Y_t^2)$ because

$$E(X_t^2) = \int_0^t e^{2s} ds = \frac{1}{2}(e^{2t} - 1), \quad E(Y_t^2) = \tau_t = \frac{1}{2}(e^{2t} - 1).$$

$E(X_t^4)$ and $P(X_t \geq 1)$ are given by

$$E(X_t^4) = E(Y_t^4 = B_{\tau_t}^4) = 3\tau_t^2 = \frac{3}{4}(e^{2t} - 1)^2$$
$$P(X_t \geq 1) = P(Y_t = B_{\tau_t} \geq 1) = 1 - N(1/\sqrt{\tau_t}) = 1 - N(\sqrt{2/(e^{2t} - 1)}).$$

Note the difference between this problem and **SCFA Corollary 7.1** (time-changed BM). Given B_t is a standard BM,

$$X_t = \int_0^t f(s) dB_s, \quad \text{and} \quad \tau(t) = v = \mathrm{Var}(X_t) = \int_0^t f^2(s) ds,$$

this exercise problem is effectively stating that X_t and $B_{\tau(t)}$ are same processes. Whereas, the Corollary 7.1 states that $X_{\tau^{-1}(v)}$ and B_v are same processes where $\tau^{-1}(\cdot)$ is the inverse function of $\tau(\cdot)$, i.e., $t = \tau^{-1}(v)$. Although they look different in forms, the intuitions behind them are same in that the variance of X_t can be used as a new *time scale* of a standard BM.

8 Itô's Formula

1. **[SCFA 8.2]**

Solution: (The notation $h \in C^1(\mathbb{R}^+)$ means that the function $h(s)$ is differentiable for $s > 0$.) From the SDE of $h(t)B_t$,

$$d(h(t)B_t) = h(t)dB_t + h'(t)B_t dt$$

we get

$$\int_0^t h(s)dB_s = h(t)B_t - \int_0^t h'(s)B_s\,ds.$$

Using that $B_T = \int_0^T dB_t$, we also get another useful result:

$$\int_0^T h'(s)B_s\,ds = h(T)B_T - \int_0^T h(t)dB_t = \int_0^T (h(T) - h(t))dB_t$$

2. **[SCFA 8.4]** (The sub-problem (b) is understood better after 2017HW 3-2 is solved.)

Solution:

(a) If $f(t,x) = \phi(t)\psi(x)$, the condition $f_t = -\frac{1}{2}f_{xx}$ yields to

$$-2\frac{\phi_t(t)}{\phi(t)} = \frac{\psi_{xx}(x)}{\psi(x)} = \lambda,$$

where λ is a constant. If $\lambda > 0$ ($\lambda = \alpha^2$ for some α), we get the GBM solution,

$$\phi(t) = \phi(0)e^{-\alpha^2 t/2} \quad \text{and} \quad \psi(x) = Ce^{\alpha x} + De^{-\alpha x} \quad \text{for some constants} \quad C, D$$

$$M_t = (Ce^{\alpha B_t} + De^{-\alpha B_t})e^{-\alpha^2 t/2}$$

If $\lambda < 0$ ($\lambda = -\alpha^2$ for some α), we get

$$M_t = (C\cos(\alpha B_t) + D\sin(\alpha B_t))e^{\alpha^2 t/2}$$

If $\lambda = 0$, we get $\phi(t) = \phi(0)$ and $\psi(x) = Cx + D$, therefore we have

$$M_t = CB_t + D.$$

(b)

$$\begin{aligned} M_t =& 1 + (\alpha B_t - \alpha^2 t/2) + \frac{1}{2}(\alpha B_t - \alpha^2 t/2)^2 + \frac{1}{6}(\alpha B_t - \alpha^2 t/2)^3 \\ &+ \frac{1}{24}(\alpha B_t - \alpha^2 t/2)^4 + \cdots \\ =& 1 + (B_t)\alpha + \frac{1}{2}(B_t^2 - t)\alpha^2 + \frac{1}{6}(B_t^3 - 3tB_t)\alpha^3 + \frac{1}{24}(B_t^4 - 6tB_t^2 + 3t^2)\alpha^4 + \cdots \end{aligned}$$

We get the first five martingales as below:

$$\begin{aligned}
H_0(t, B_t) &= 1 \\
H_1(t, B_t) &= B_t \\
H_2(t, B_t) &= \frac{1}{2}(B_t^2 - t) \\
H_3(t, B_t) &= \frac{1}{6}(B_t^3 - 3tB_t) \\
H_4(t, B_t) &= \frac{1}{24}(B_t^4 - 6tB_t^2 + 3t^2).
\end{aligned}$$

3. **[2016HW 4-3. SDE]** For the following functions $f(t, x)$, find the stochastic differential equation (SDE) of the stochastic process $Y_t = f(t, B_t)$ where B_t is a standard BM. If $f(x) = x^2$, for example, the SDE is

$$dY_t = d\, f(B_t) = d\, B_t^2 = 2B_t dB_t + dt.$$

(a) $f(t, x) = x^3 - 3tx$
(b) $f(t, x) = e^{t/2} \sin(x)$

Solution: The function $f(t, x)$ was constructed in such a way that $dY_t = df(t, B_t)$ becomes a martingale.

(a)

$$df(t, x = B_t) = (3x^2 - 3t)dB_t + \frac{1}{2}(6x)(dB_t)^2 - 3xdt = 3(B_t^2 - t)dB_t$$

(b)

$$\begin{aligned}
df(t, x = B_t) &= e^{t/2} \cos(x)dB_t - \frac{1}{2}e^{t/2} \sin(x)(dB_t)^2 + \frac{1}{2}e^{t/2} \cos(x)dt \\
&= e^{t/2} \cos(B_t)dB_t
\end{aligned}$$

4. **[2017HW 3-2]** For a standard BM B_t, let

$$N_t = B_t^3 - 3t\, B_t.$$

(i) Prove that N_t is a martingale. (Hint: use **SCFA Proposition 8.1**) (ii) By applying Itô's lemma, express N_t as a stochastic integration. (iii) Calculate the variance of N_t.

Solution: We set $N_t = f(t, B_t)$ where $f(t, x) = x^3 - 3tx$. Applying Itô's lemma,

$$dN_t = 3(B_t^2 - t)dB_t + 3B_t(dB_t)^2 - 3B_t dt = 3(B_t^2 - t)dB_t$$

As there is no drift term, N_t is a martingale and is represented as a stochastic integral:

$$N_t = \int_0^t 3(B_s^2 - s)dB_s.$$

The variance is calculated as

$$\begin{aligned}\mathrm{Var}(N_t) &= \int_0^t E[3^2(B_s^2 - s)^2]ds = 9\int_0^t (E(B_s^4) - 2sE(B_s^2) + s^2)ds \\ &= 9\int_0^t (3s^2 - 2s^2 + s^2)ds = 6t^2\end{aligned}$$

5. **[2016ME(ASP), Itô's calculus]** Which of the following quantity is same as $(dB_t)^2$?
A. $\sqrt{dt}$ B. dt C. $dt/2$ D. $(dt)^2/2$

Solution: B.

9 Stochastic Differential Equations

1. **[SCFA 9.1]**

Solution: This is slightly modified from the OU process with the extra βdt term. We use the same initial guess, $e^{\alpha t}X_t$, for the OU process.

$$\begin{aligned} d\big(e^{\alpha t}X_t\big) &= \alpha e^{\alpha t}X_t dt + e^{\alpha t}dX_t + \frac{1}{2}0 \cdot (dX_t)^2 \\ &= e^{\alpha t}\big(\alpha X_t dt - \alpha X_t dt + \beta dt\big) + \sigma e^{\alpha t} dB_t \\ &= \beta e^{\alpha t}dt + \sigma e^{\alpha t}dB_t. \end{aligned}$$

Therefore, we get

$$\begin{aligned} e^{\alpha t}X_t - x_0 &= \frac{\beta}{\alpha}\big(e^{\alpha t} - 1\big) + \sigma \int_0^t e^{\alpha s} dB_s \\ X_t &= e^{-\alpha t}x_0 + \frac{\beta}{\alpha}\big(1 - e^{-\alpha t}\big) + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} dB_s. \end{aligned}$$

2. **[SCFA 9.2]**

Solution: We first guess from the traditional calculus. We let $x = X_t$ and solve

$$e^{-t^2/2}dx = txe^{-t^2/2} \quad \Rightarrow \quad \frac{dx}{x} = -t\,dt$$

Luckily we can solve this to have $x = Ce^{t^2/2}$ or $e^{-t^2/2}x = C$ for some constant C, so we stochastically differentiate $e^{-t^2/2}X_t$ to get

$$d\big(e^{-t^2/2}X_t\big) = -te^{-t^2/2}X_t dt + e^{-t^2/2}\big(tX_t\,dt + e^{t^2/2}dB_t\big) = dB_t.$$

We can solve the SDE as

$$e^{-t^2/2}X_t - X_0 = B_t \quad \Rightarrow \quad X_t = e^{t^2/2}(B_t + X_0).$$

3. **[SCFA 9.3]**

Solution: The guess from the traditional calculus is

$$\frac{dx}{x} = \frac{-2}{1-t}\,dt \quad \Rightarrow \quad x = C(1-t)^2.$$

Therefore we start by differentiating $(1-t)^{-2}X_t$:

$$d\left(\frac{X_t}{(1-t)^2}\right) = 2\frac{X_t}{(1-t)^3}dt + \frac{1}{(1-t)^2}\Big(-2\frac{X_t}{1-t}dt + \sqrt{2t(1-t)}\,dB_t\Big) = \frac{\sqrt{2t}}{(1-t)^{3/2}}dB_t$$

and finally solve the SDE as

$$X_t = (1-t)^2 \int_0^t \frac{\sqrt{2u}}{(1-u)^{3/2}} dB_u.$$

Since the integrand $\sqrt{2u}(1-u)^{-3/2}$ depends only on the time variable u, X_t is a Gaussian process with the variance

$$\begin{aligned}
\mathrm{Var}(X_t) &= (1-t)^4 \int_0^t \frac{2u}{(1-u)^3} du = (1-t)^4 \int_{1-t}^1 \frac{2(1-u')}{u'^3} du \quad (u' = 1-u) \\
&= (1-t)^4 \left(1 - \frac{2}{1-t} + \frac{1}{(1-t)^2}\right) = (1-t)^4 \frac{t^2}{(1-t)^2} = t^2(1-t)^2
\end{aligned}$$

The covariance can be obtained similarly. Assuming that $s < t$,

$$\begin{aligned}
\mathrm{Cov}(X_s, X_t) = E(X_s X_t) &= (1-s)^2(1-t)^2\, E\left[\int_0^s \frac{\sqrt{2u}}{(1-u)^{3/2}} dB_u \int_0^t \frac{\sqrt{2v}}{(1-v)^{3/2}} dB_v\right] \\
&= (1-s)^2(1-t)^2\, E\left[\left(\int_0^s \frac{\sqrt{2u}}{(1-u)^{3/2}} dB_u\right)^2\right] \\
&= (1-s)^2(1-t)^2 \cdot \frac{s^2}{(1-s)^2} = s^2(1-t)^2.
\end{aligned}$$

The covariance is square of that of the Brownian bridge $\mathrm{Cov}(X_s, X_t) = s(1-t)$.

4. **[SCFA 9.6]**

Solution: We first derive the mean and the variance of lognormal distribution, $Y \sim \exp(\mu + \sigma Z)$, where Z is a standard normal (see the same result at (WIKIPEDIA)):

$$E(Y) = e^{\mu + \sigma^2/2}$$

and

$$\begin{aligned}
\mathrm{Var}(Y) &= e^{2\mu} E((e^{\sigma Z} - e^{\sigma^2/2})^2) = e^{2\mu} E(e^{2\sigma Z} - 2e^{\sigma Z + \sigma^2/2} + e^{\sigma^2}) \\
&= e^{2\mu}(e^{2\sigma^2} - 2e^{\sigma^2} + e^{\sigma^2}) = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1).
\end{aligned}$$

Back to the problem, for $t = kh$ and $s = (k-1)h$,

$$R_k(h) + 1 = \frac{X_t}{X_s} = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) h + \sigma(B_t - B_s)\right) = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) h + \sigma\sqrt{h} Z\right),$$

where Z is standard normal distribution. Since $R_k(h)+1$ is a lognormal distribution with $\mu := (\mu - \sigma^2/2)h$ and $\sigma := \sigma\sqrt{h}$, we obtain the mean and the variance of $R_k(h)+1$

as

$$E(R_k(h)+1) = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)h + \frac{\sigma^2}{2}h\right) = e^{\mu h}$$
$$\mathrm{Var}(R_k(h)+1) = e^{2\mu h}(e^{\sigma^2 h} - 1).$$

Therefore,

$$E(R_k(h)) = e^{\mu h} - 1 \quad \text{and} \quad \mathrm{Var}(R_k(h)) = e^{2\mu h}(e^{\sigma^2 h} - 1)$$

and it follows that

$$\mathrm{Var}(R_k(h)) = (1 + E(R_k(h))^2(e^{\sigma^2 h} - 1)$$
$$\sigma^2 = \frac{1}{h}\log\left(1 + \frac{\mathrm{Var}(R_k(h))}{(1+E(R_k(h))^2}\right).$$

From sample data, we can estimate the mean and the variance as

$$E(R_k(h)) = \frac{1}{n}\sum_{k=1}^{n} R_k(h), \quad \mathrm{Var}(R_k(h)) = \frac{1}{n-1}\sum_{k=1}^{n}(R_k(h) - E(R_k(h))).$$

Below are the s and σ values for various values of $E(R_k(h))$ and $\mathrm{Stdev}(R_k(h))$:

(All numbers are in the unit of %. $h = 1/12$.)

$\mathrm{Stdev}(R_k(h)) \rightarrow$		5	10	15	20	25
$E(R_k(h))$	s	17.3	34.6	52.0	69.3	86.6
-2		17.7	35.3	52.7	70.0	87.0
0		17.3	34.6	51.7	68.6	85.3
2	σ	17.0	33.9	50.7	67.3	83.7
4		16.6	33.2	49.7	66.0	82.1
6		16.3	32.6	48.8	64.8	80.6

The values of s and σ are not significantly different unless the average return $E(R_k(h))$ is high.

5. **[2018HW 3-4, Exponential Ornstein-Uhlenbeck process]** Solve the following SDE:

$$\frac{dP_t}{P_t} = \alpha(\mu - \log P_t)dt + \sigma dB_t.$$

What are $E(P_t)$ and $\mathrm{Var}(P_t)$ as $t \to \infty$? (Hint: use $X_t = \log P_t$.)

Solution: The SDE for X_t satisfies

$$dX_t = \frac{dP_t}{P_t} - \frac{(dP_t)^2}{2P_t^2} = \alpha\left(\mu - \frac{\sigma^2}{2\alpha} - X_t\right)dt + \sigma dB_t.$$

This is the Ornstein-Uhlenbeck with $X_\infty = \mu - \frac{\sigma^2}{2\alpha}$. Therefore,

$$X_t = X_\infty + e^{-\alpha t}(X_0 - X_\infty) + \frac{\sigma e^{-\alpha t}}{\sqrt{2\alpha}} B_{e^{2\alpha t}-1}$$
$$\log P_t = \mu - \frac{\sigma^2}{2\alpha} + e^{-\alpha t}\left(\log P_0 - \mu + \frac{\sigma^2}{2\alpha}\right) + \frac{\sigma e^{-\alpha t}}{\sqrt{2\alpha}} B_{e^{2\alpha t}-1}.$$

The mean and variance as $t \to \infty$ are given as

$$E(X_t) = X_\infty = \mu - \frac{\sigma^2}{2\alpha}, \quad \mathrm{Var}(X_t) = \frac{\sigma^2}{2\alpha}.$$

For P_t, we apply the properties of the lognoraml distributions:

$$E(P_t) = \exp\left(X_\infty + \frac{1}{2}\mathrm{Var}(X_t)\right) = \exp\left(\mu - \frac{\sigma^2}{4\alpha}\right).$$

$$\mathrm{Var}(P_t) = \left(\exp\left(\frac{\sigma^2}{2\alpha}\right) - 1\right)\exp\left(2\mu - \frac{\sigma^2}{2\alpha}\right) = \left(1 - \exp\left(-\frac{\sigma^2}{2\alpha}\right)\right) e^{2\mu}$$

6. **[2016ME(ASP), SDE modified from SCFA 9.2]** Solve the SDE for the starting value X_0

$$dX_t = -t\, X_t\, dt + \sigma e^{-t^2/2}\, dB_t.$$

What is the mean and the variance of the process when $t \to \infty$?

Solution: From $e^{t^2/2}(dx + tdx) = d(e^{t^2/2}x)$, we apply Itô's lemma to our initial guess of $e^{t^2/2}X_t$;

$$d\Big(e^{t^2/2}X_t\Big) = te^{t^2/2}\, X_t\, dt + e^{t^2/2}dX_t = e^{t^2/2}\Big(tX_t dt + dX_t\Big) = \sigma dB_t$$
$$e^{t^2/2}X_t - X_0 = \sigma B_t$$
$$X_t = e^{-t^2/2}(X_0 + \sigma B_t),$$
$$E(X_t) = e^{-t^2/2}X_0 \to 0 \text{ as } t \to \infty$$
$$\mathrm{Var}(X_t) = e^{-t^2}\sigma^2\, t \to 0 \text{ as } t \to \infty$$

7. **[2016ME(ASP), SDE between normal and lognormal]** Solve the SDE for the starting value X_0 and $0 < \beta < 1$,

$$\frac{dX_t}{X_t^\beta} = \sigma\, dB_t + \frac{\sigma^2 \beta}{2} X_t^{\beta-1}\, dt$$

Solution: We apply Itô's lemma to $X_t^{1-\beta}$,

$$
\begin{aligned}
d(X_t^{1-\beta}) &= (1-\beta)X_t^{-\beta}dX_t - \frac{1}{2}(1-\beta)X_t^{-1-\beta}\beta(dX_t)^2 \\
&= (1-\beta)X_t^{-\beta}dX_t - \frac{1}{2}(1-\beta)\beta X_t^{-1-\beta}\cdot X_t^{2\beta}\sigma^2 dt \\
&= (1-\beta)\Big(\sigma\, dB_t + \frac{\sigma^2\beta}{2}X_t^{\beta-1}\, dt\Big) - \frac{1}{2}(1-\beta)\beta X_t^{-1-\beta}\big(X_t^{\beta}\sigma\big)^2 dt \\
&= (1-\beta)\sigma dB_t.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
X_t^{1-\beta} &= X_0^{1-\beta} + (1-\beta)\sigma B_t \\
X_t &= \Big(X_0^{1-\beta} + (1-\beta)\sigma B_t\Big)^{1/(1-\beta)}.
\end{aligned}
$$

10 Arbitrage and SDEs

11 The Diffusion Equation

1. **[2018ME(ASP), Option vega under the BSM model]** Derive that the vega of a call option (i.e., sensitivity with respect to the volatility σ) is

$$V = \frac{\partial C}{\partial \sigma} = S_0\, n(d_1)\sqrt{T} = Ke^{-rT} n(d_2)\sqrt{T}.$$

Remind that the call option price under the BSM model is

$$C = S_0 N(d_1) - e^{-rT} K N(d_2) \quad \text{where} \quad d_{1,2} = \frac{\log(S_0\, e^{rT}/K)}{\sigma\sqrt{T}} \pm \frac{1}{2}\sigma\sqrt{T}$$

Since the terms d_1 and d_2 are implicit functions of σ, you should also differentiate d_1 and d_2.

Solution: Using the properties

$$\frac{\partial d_{1,2}}{\partial \sigma} = -\frac{\log(S_0\, e^{rT}/K)}{\sigma^2\sqrt{T}} \pm \frac{1}{2}\sqrt{T} = -\frac{d_{2,1}}{\sigma}$$

and

$$d_1^2 - d_2^2 = (A+B)^2 - (A-B)^2 = 4AB = 2\log(S_0\, e^{rT}/K) \quad \Rightarrow \quad \frac{n(d_2)}{n(d_1)} = \frac{S_0\, e^{rT}}{K}$$

we compute the vega as

$$\begin{aligned} V &= \frac{\partial}{\partial \sigma}\big(S_0 N(d_1) - e^{-rT} K N(d_2)\big) = S_0 n(d_1)\frac{-d_2}{\sigma} - e^{-rT} K n(d_2)\frac{-d_1}{\sigma} \\ &= S_0 n(d_1)\left(-\frac{d_2}{\sigma} + \frac{Kn(d_2)}{S_0 e^{rT}\, n(d_1)}\frac{d_1}{\sigma}\right) = S_0 n(d_1)\left(-\frac{d_2}{\sigma} + \frac{d_1}{\sigma}\right) \\ &= S_0 n(d_1)\sqrt{T} = Ke^{-rT} n(d_2)\sqrt{T}. \end{aligned}$$

12 Representation Theorems

1. **[2017HW 3-5, Martingale representation theory]**
For a standard BM B_t $(0 \le t \le T)$, find the martingale representation of $X_t = E(B_T^3|\mathcal{F}_t)$. (In class, we did the same for $X_t = E(B_T^2|\mathcal{F}_t)$)

Solution: Using the short notation $E_t(\,\cdot\,) = E(\,\cdot\,|\mathcal{F}_t)$,

$$\begin{aligned} X_t &= E_t((B_t + B_T - B_t)^3) = B_t^3 + 3B_t^2 E_t(B_T - B_t) + 3B_t\, E_t((B_T - B_t)^2) + E_t((B_T - B_t)^3) \\ &= B_t^3 + 0 + 3(T-t)B_t + 0 = B_t^3 + 3(T-t)B_t. \end{aligned}$$

In particular, $X_0 = 0$. From the SDE,

$$dX_t = 3B_t^2 dB_t + 3B_t(dB_t)^2 + 3(T-t)dB_t - 3B_t dt = 3(B_t^2 + T - t)dB_t,$$

the martingale representation is

$$X_T = \int_0^T 3(B_t^2 + T - t)\, dB_t$$

13 Girsanov Theory