- Data $\underline{x} \in \mathbb{R}^{m \times S}$

$$\underline{x} = \underbrace{(\underline{x}_0, \underline{x}_1, \underline{x}_2, \dots, \underline{x}_{S-1})}_{S} \updownarrow m$$

TIMES: $t_0, t_1, \dots, t_{S-1}$

- q-fold concatenation (q odd):

$\underline{x} \to \underline{X} \in \mathbb{R}^{mq \times s}$ $\qquad s = S - q + 1$

$$\underline{X} = \begin{pmatrix} \underline{x}_{q-1} & \underline{x}_q & \underline{x}_{q+1} & & \underline{x}_{S-1} \\ \underline{x}_{q-2} & \underline{x}_{q-1} & \underline{x}_q & & \underline{x}_{S-2} \\ \underline{x}_{q-3} & \underline{x}_{q-2} & \underline{x}_{q-1} & & \underline{x}_{S-3} \\ \dots & \dots & \dots & & \\ \underline{x}_{\frac{(q-1)}{2}} & & \dots & \dots & \underline{x}_{S-\frac{q+1}{2}} \\ \dots & & & & \\ \underline{x}_2 & \underline{x}_3 & \underline{x}_4 & & \\ \underline{x}_1 & \underline{x}_2 & \underline{x}_3 & & \\ \underline{x}_0 & \underline{x}_1 & \underline{x}_2 & & \underline{x}_{s-1} \end{pmatrix} \updownarrow mq$$

$\longleftrightarrow$ $s := S - q + 1$

$S-1-(q-1)$
$= S - q$
$= s - 1$

Column 0: $\bar{t}_0$ is the average of $(t_0, t_1, \dots t_{q-1})$

Column 1: $\bar{t}_1$ is the average of $(t_1, t_2, \dots t_q)$

- SUBSPACE PROJECTION (LOW-PASS FILTERING)

$$\underline{\Phi} = (\underline{\varphi}_0, \underline{\varphi}_1, \underline{\varphi}_2, \cdots, \underline{\varphi}_{2j_{max}}) \updownarrow n$$

$$\xleftrightarrow{\quad 2j_{max}+1 \quad}$$

$$\underline{A} = \underline{X}\,\underline{\phi}$$

$\underline{\phi}$: $n \times (2j_{max}+1)$

$\underline{X}$: $mq \times n$

$\underline{A}$: $mq \times (2j_{max}+1)$

$$\underline{A} = (\underline{a}_0, \cdots, \underline{a}_{2j_{max}})$$

$$\underline{a}_i = \underline{X}\,\underline{\varphi}_i$$

- SVD OF $\underline{A}$

$$\underline{A} = \underline{U}\;\underline{S}\;\underline{V}^T \qquad \underline{S} = \begin{pmatrix} \sigma_0 & & 0 \\ & \sigma_1 & \\ 0 & \cdots & \sigma_{2j_{max}} \end{pmatrix}$$

$\underline{A}$: $mq \times (2j_{max}+1)$

$\underline{U}$: $mq \times (2j_{max}+1)$

$\underline{S}$: $(2j_{max}+1) \times (2j_{max}+1)$

$\underline{V}$: $(2j_{max}+1) \times (2j_{max}+1)$

$$\underline{U} = (\underline{u}_0, \underline{u}_1, \cdots, \underline{u}_{2j_{max}})$$

$\underline{u}_i$: TOPOGRAMS

$$(\underline{A}^T\underline{A}) = (USV^T)^T(USV^T) = VS^TU^TUSV^T = VS^2V^T$$

$\underline{A}^T\underline{A}$: $(2j_{max}+1) \times (2j_{max}+1)$

$$(A^TA)_{ij} = (A^TA)_{ji} = \underline{a}_i^T\underline{a}_j$$

Eigendecomposition of small, symmetric matrix $A^TA \rightarrow \underline{V}, \underline{S}^2$

$\underline{U} = \underline{A}\,\underline{V}\,\underline{\underline{S}}^{-1}$

Chronograms: $\underbrace{(V^T \Phi^T)}_{(2j_{max}+1) \times s}$

The chronograms are the rows of $V^T\Phi^T$ (s-element long)

- RECONSTRUCTION

$$\underline{\tilde{X}} = \underline{X}\,\underline{\Phi}\,\underline{\Phi}^T = \underline{A}\,\underline{\Phi}^T = \underline{U}\,\underline{\underline{S}}\,\underline{V}^T\underline{\Phi}^T$$

- LOW PASS FILTERING

$\underline{\Phi} \in \mathbb{R}^{s \times k}$ $\quad k = 2j_{max}+1$

If $k = s$ then $\underline{\Phi}\,\underline{\Phi}^T = \text{Identity}$ → NO LOW-PASS FILTERING

→ normally $k \ll s$

- SPECTRAL FILTERING:

In the reconstruction: $\underline{\tilde{X}} = \underline{U}\,\underline{\underline{S}}\,\underline{V}^T\underline{\Phi}^T$

only the first $\tilde{r} \ll 2j_{max}+1$ modes are kept

$$\underbrace{\underline{\tilde{X}}}_{mq \times s} = \underbrace{\underline{U}}_{mq \times \tilde{r}}\ \underbrace{\underline{\underline{S}}}_{\tilde{r} \times \tilde{r}}\ \underbrace{(\underline{V}^T\underline{\Phi}^T)}_{\tilde{r} \times s}$$

$\underline{\tilde{X}} \in \mathbb{R}^{mq \times s}$ has the same 'structure' as $\underline{X}$ (p. 1).

$\rho = 0$ → $\underline{x}$ corresponds to the 'central block' of $\underline{\tilde{X}}$.

$$
\underline{\tilde{X}} = \begin{pmatrix}
\underline{\tilde{x}}_{q-1} & \cdots & \underline{\tilde{x}}_{S-1} \\
\underline{\tilde{x}}_{q-2} & & \underline{\tilde{x}}_{S-2} \\
\underline{\tilde{x}}_{q-3} & & \\
\cdots & & \cdots \\
\boxed{\underline{\tilde{x}}_{\frac{q-1}{2}} \qquad \qquad \underline{\tilde{x}}_{S-\frac{q+1}{2}}} & & \\
\cdots & & \cdots \\
\underline{\tilde{x}}_{1} & & \\
\underline{\tilde{x}}_{0} & & \underline{\tilde{x}}_{s-1}
\end{pmatrix}
$$

$$\xleftrightarrow{\quad s \quad}$$

Reconstruction with $p=0$
corresponds to extracting
the 'central block'
of $\underline{\tilde{X}}$
$\rightarrow \underline{\tilde{x}} \in \mathbb{R}^{m \times s}$