Lab 4

CS365

Covariance of X,Y

A measure of how well X and Y vary together.

We are often interested in two or more random variables at the same time.

Consider the following examples:

1. The relationship between height and weight
2. The frequency of exercise and the rate of heart disease
3. Air pollution levels and rate of respiratory illness

Covariance Computation from joint PDF

Let's say we are given the following information

Let X and Y be continuous random variables with joint pdf

$$f_{X,Y}(x,y) = 3x, \qquad 0 \leq y \leq x \leq 1,$$

and zero otherwise.

And we want to compute the covariance of X,Y. Cov(X,Y)

Definition of Covariance

$$Cov_{f_{X,Y}}\left[X,Y\right] \quad = E_{f_{X,Y}}\left[XY\right] - E_{f_X}\left[X\right]E_{f_Y}\left[Y\right]$$

Definition of Covariance

$$Cov_{f_{X,Y}}[X,Y] = E_{f_{X,Y}}[XY] - E_{f_X}[X]\,E_{f_Y}[Y]$$

Expected Value of XY with joint probability density function (pdf), f of X,Y

Definition of Covariance

$$Cov_{f_{X,Y}}[X,Y] \quad = E_{f_{X,Y}}[XY] - E_{f_X}[X] E_{f_Y}[Y]$$

Expected Value of XY with joint probability density function (pdf), f of X,Y

Expected value of X with pdf, f of X

$$f_{X,Y}(x,y) = 3x, \qquad 0 \le y \le x \le 1,$$

Definition of Covariance

$$Cov_{f_{X,Y}}[X,Y] = E_{f_{X,Y}}[XY] - E_{f_X}[X]E_{f_Y}[Y]$$

Expected Value of XY with joint probability density function (pdf), f of X,Y

Expected value of X with pdf, f of X

Expected value of Y with pdf, f of Y

PDF of x and E[x]

$$f_{X,Y}(x,y) = 3x, \qquad 0 \le y \le x \le 1,$$

and zero otherwise.

$$f_X(x) \;= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_0^x 3x dy = 3x^2, \qquad 0 \le x \le 1,$$

$$E_{f_X}[X] \;= \int_{-\infty}^{\infty} x f_X(x)dx = \int_0^1 x \times 3x^2 dx = \left[\frac{3}{4}x^4\right]_0^1 = \frac{3}{4},$$

PDF of Y and E[Y]

$$f_{X,Y}(x,y) = 3x, \qquad 0 \le y \le x \le 1,$$

and zero otherwise.

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_y^1 3x dx = \left[\frac{3}{2}x^2\right]_y^1 = \frac{3}{2}(1-y^2), \quad 0 \le y \le 1,$$

$$E_{f_Y}[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_0^1 y \times \frac{3}{2}(1-y^2)dy = \left[\frac{3}{2}\left(\frac{y^2}{2} - \frac{y^4}{4}\right)\right]_0^1 = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{3}{8},$$

E[XY]

$$f_{X,Y}(x,y) = 3x, \qquad 0 \le y \le x \le 1,$$

and zero otherwise.

$$\begin{aligned}
E_{f_{X,Y}}[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)dydx = \int_0^1\int_0^x xy \times 3xdydx \\
&= \int_0^1 \left\{\int_0^x ydy\right\} 3x^2 dx = \int_0^1 \left[\frac{y^2}{2}\right]_0^x 3xdx = \int_0^1 \frac{x^2}{2} \times 3x^2 dx \\
&= \frac{3}{2}\left[\frac{x^5}{5}\right]_0^1 = \frac{3}{10},
\end{aligned}$$

Cov[XY] final

$$Cov_{f_{X,Y}}[X,Y] = E_{f_{X,Y}}[XY] - E_{f_X}[X]\,E_{f_Y}[Y] = \frac{3}{10} - \frac{3}{4} \times \frac{3}{8} = \frac{3}{160}$$

Remarks

- Events and Random variables
- PDF of continuous random variables
- Chebyshev's inequality

  Let $X_1, X_2, \ldots$ be a sequence of iid RVs with mean μ, and standard deviation σ.

Consider the sum $S_n = X_1 + \ldots + X_n$

$$\Pr\left(\left|\frac{S_n}{n} - \mu\right| > \epsilon\right) \leq \frac{\mathbb{V}\mathrm{ar}(S_n/n)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}$$

Bayes Practice Problem

Purpose of this exercise:

1. Practice Bayes with continuous random variables
2. Interpret equations intuitively
3. Intuition behind Bayes theorem

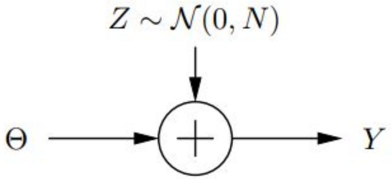
Review:

1. Bayes computes P(A | B) where A is a class, and B is an input
2. Prior - P(A)
3. Likelihood - P(B | A)
4. Posterior = P(A | B)

Bayes Practice Problem

Example: Additive Gaussian Noise Channel

- Consider the following communication channel model



where the signal sent

$$\Theta = \begin{cases} +1, & \text{with probability } p \\ -1, & \text{with probability } 1-p, \end{cases}$$

the signal received (also called observation) $Y = \Theta + Z$, and Θ and Z are independent

Given $Y = y$ is received (observed), find the *a posteriori* pmf of Θ, $p_{\Theta|Y}(\theta|y)$

Describe the situation, what is the diagram showing?

What is the difference between Y and y

Give the equation and describe each of these terms in your own words

Prior: ?
Likelihood: ?
Posterior: ?

Bayes Practice Problem

Example: Additive Gaussian Noise Channel

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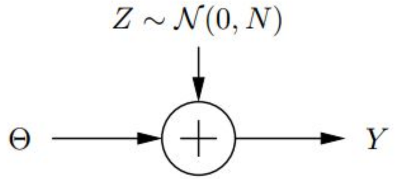

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Given $Y = y$ is received (observed), find the *a posteriori* pmf of Θ, $p_{\Theta|Y}(\theta|y)$

Describe the situation, what is the diagram showing?

Signal (-1 or 1) sent from Theta to Y. Random interfering noise added to the original signal.

For instance, Y might receive 1, but does that mean Theta sent 1, or theta sent -1 and there was a noise value of 2?

Bayes Practice Problem

Example: Additive Gaussian Noise Channel

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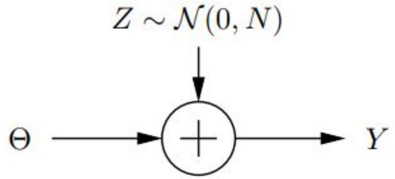

where the signal sent

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the signal received (also called observation) $Y = \Theta + Z$, and Θ and Z are independent

Given $Y = y$ is received (observed), find the *a posteriori* pmf of Θ, $p_{\Theta|Y}(\theta|y)$

Give the equation and describe each of these terms in your own words

Prior:
P(Theta=1) = p

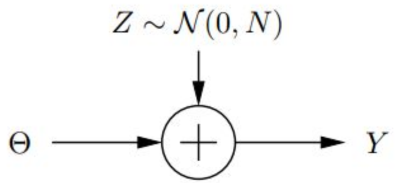
P(Theta = -1) = 1-p

The prior represents the underlying probability for the class' existence

Bayes Practice Problem

Example: Additive Gaussian Noise Channel

- Consider the following communication channel model



where the signal sent

$$\Theta = \begin{cases} +1, & \text{with probability } p \\ -1, & \text{with probability } 1-p, \end{cases}$$

the signal received (also called observation) $Y = \Theta + Z$, and Θ and Z are independent

Given $Y = y$ is received (observed), find the *a posteriori* pmf of Θ, $p_{\Theta|Y}(\theta|y)$

Give the equation and describe each of these terms in your own words

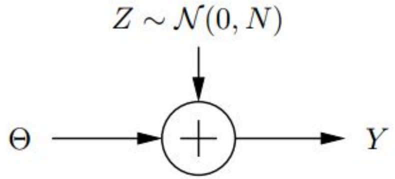
Likelihood:
P(Y | Theta)

For each class, we assume that if that class was the correct class, what are the odds of getting our observation, Y.

Bayes Practice Problem

Example: Additive Gaussian Noise Channel

- Consider the following communication channel model



where the signal sent

$$\Theta = \begin{cases} +1, & \text{with probability } p \\ -1, & \text{with probability } 1-p, \end{cases}$$

the signal received (also called observation) $Y = \Theta + Z$, and Θ and Z are independent

Given $Y = y$ is received (observed), find the *a posteriori* pmf of Θ, $p_{\Theta|Y}(\theta|y)$

Give the equation and describe each of these terms in your own words

Posterior:
P(Theta | Y)

Final probability: given our observation, Y, what is the most likely class it comes from

Solving the Posterior

Prior: P(Theta)

Likelihood: P(Y | Theta)

Posterior: P(Theta | Y)

$$p_{\Theta|Y}(\theta|y) = \frac{f_{Y|\Theta}(y|\theta)p_{\Theta}(\theta)}{\sum_{\theta'} p_{\Theta}(\theta')f_{Y|\Theta}(y|\theta')}$$

Need to find: Prior, Likelihood

Solving the Posterior - prior

$$p_{\Theta|Y}(\theta|y) = \frac{f_{Y|\Theta}(y|\theta)p_\Theta(\theta)}{\sum_{\theta'} p_\Theta(\theta') f_{Y|\Theta}(y|\theta')}$$

Prior: P(Theta)

Likelihood: P(Y | Theta)

Posterior: P(Theta | Y)

Prior:

We know the prior: P(Theta=1) = p, P(Theta= -1) = 1-p

$$\Theta = \begin{cases} +1, & \text{with probability } p \\ -1, & \text{with probability } 1-p, \end{cases}$$

Solving the Posterior - Likelihood

Prior: P(Theta)

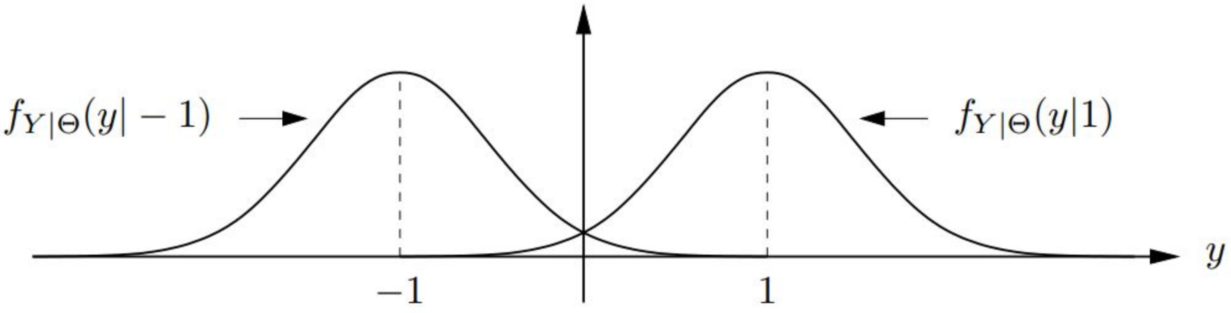
Likelihood: P(Y | Theta)

Posterior: P(Theta | Y)

$$\Theta = \begin{cases} +1, & \text{with probability } p \\ -1, & \text{with probability } 1-p, \end{cases}$$

$$Z \sim \mathcal{N}(0, N)$$

Likelihood: P(Y | Theta)



Therefore, $Y|\{\Theta = +1\} \sim \mathcal{N}(+1, N)$. Also, $Y|\{\Theta = -1\} \sim \mathcal{N}(-1, N)$

Solving the posterior - final

$$G(x,y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

$$p_{\Theta|Y}(\theta|y) = \frac{f_{Y|\Theta}(y|\theta)p_\Theta(\theta)}{\sum_{\theta'} p_\Theta(\theta') f_{Y|\Theta}(y|\theta')}$$

prior

$$p_{\Theta|Y}(1|y) = \frac{\frac{p}{\sqrt{2\pi N}} e^{-\frac{(y-1)^2}{2N}}}{\frac{p}{\sqrt{2\pi N}} e^{\frac{-(y-1)^2}{2N}} + \frac{(1-p)}{\sqrt{2\pi N}} e^{\frac{-(y+1)^2}{2N}}}$$

$$= \frac{pe^y}{pe^y + (1-p)e^{-y}} \text{ for } -\infty < y < \infty$$

Bayes' Formula Problem

- Approximately 1% of some population have a disease.
- There exists a laboratory test for the disease
- A person with the disease has a 90% chance of getting a positive result from the test
- A person without the disease has a 10% chance of getting a positive result from the test
- What is the probability that someone has the disease given they just had a positive test?

Bayes’ Formula Problem

- Approximately 1% of some population have a disease.
- There exists a laboratory test for the disease
- A person with the disease has a 90% chance of getting a positive result from the test
- A person without the disease has a 10% chance of getting a positive result from the test
- What is the probability that someone has the disease given they just had a positive test?
- Spend some time to compute this information on your own
- $P(A \mid B) = P(A)*P(B \mid A) / P(B)$

Bayes Solution

- $P(A \mid B) = P(A)*P(B \mid A) / P(B)$
- $P(A)$ = Chance they have the disease
- $P(B)$ = Chance of positive test
- $P(B \mid A)$ = Chance of positive test given they have disease

Bayes Solution

- P(A | B) = P(A)*P(B | A) / P(B)
- P(A) = Chance they have the disease
- P(B | A) = Chance of positive test given they have disease
- P(B) = Chance of positive test

- P(A) = 0.01
- P(B | A) = 0.9
- P(B) = P(A)*P(B | A) + P(not A)*P(B | not A) = (0.009) + (0.099) = .108

- P(A | B) = (.01*.09)/(.01*.09 + .99*.1) = .009/.108 = 8.3%