Knots as processes: a new kind of invariant

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Abstract. We exhibit an encoding of knots into processes in the π-calculus such that knots are ambient isotopic if and only their encodings are weakly bisimilar.

1 Introduction and motivation

Compositionality.

1.1 Overview and summary of contributions

2 The π-calculus in a nutshell

2.1 π-calculus

π-calculus

$$
\begin{array}{rcl}
P, Q & ::= & 0 \\
& | & x[y] \\
& | & x(y)\,.\,P \\
& | & (\nu\, x)P \\
& | & P \mid Q \\
& | & !P
\end{array}
$$

Note well: names are quoted *ρ-calculus* processes.

2.2 Structural congruence

Definition 1. *The* structural congruence*, $\equiv$, between processes is the least congruence closed with respect to alpha-renaming, satisfying the abelian monoid laws for parallel (associativity, commutativity and* 0 *as identity), and the following axioms:*

1. *the scope laws:*

$$\begin{aligned} (\nu\ x)0 &\equiv 0, \\ (\nu\ x)(\nu\ x)P &\equiv (\nu\ x)P, \\ (\nu\ x)(\nu\ y)P &\equiv (\nu\ y)(\nu\ x)P, \\ P \mid (\nu\ x)Q &\equiv (\nu\ x)P \mid Q,\ \textit{if}\ x \notin \mathcal{FN}(P) \end{aligned}$$

2. *the recursion law:*

$$!P \equiv P \mid !P$$

2.3 Operational semantics

The operational semantics is standard.

$$\frac{}{x[z] \mid x(y)\,.\,P \to P\{z/y\}} \qquad \text{(COMM)}$$

In addition, we have the following context rules:

$$\frac{P \to P'}{P \mid Q \to P' \mid Q} \qquad \text{(PAR)}$$

$$\frac{P \to P'}{(\nu\ x)P \to (\nu\ x)P'} \qquad \text{(NEW)}$$

$$\frac{P \equiv P' \qquad P' \to Q' \qquad Q' \equiv Q}{P \to Q} \qquad \text{(EQUIV)}$$

Again, we write $\Rightarrow$ for $\to^*$, and rely on context to distinguish when $\to$ means reduction in the π-calculus and when it means reduction in the ρ-calculus. The set of π-calculus processes will be denoted by $Proc_\pi$.

3 Stuff about knots

A knot is usually represented by a knot projection, which satisfies some basic general position properties.

3.1 Reidemeister moves

There are three Reidemeister moves that may be performed upon a knot projection, which we denote by $\Omega_1, \Omega_2, \Omega_3$. These are illustrated in the figure (Figure 1).

Fig. 1. The three Reidemesiter moves

3.2 The Dowker-Thistlethwaite code

The Dowker-Thistlethwaite code ("DT") of a knot K is obtained as follows:

Choose a point on K and begin traversing K , counting each crossing you pass through. If K has n crossings, then (since every crossing is visited twice) the count ends at $2n$. Label each crossing with the value of the counter when it is visited (each crossing is labeled twice). Finally, when labeling a crossing with an even number, prepend with the label with a minus sign if traversing "under" the crossing. All crossings end up being labeled by a pair of integers whose absolute values run, *in toto*, from 1 to $2n$. It is easy to see that each crossing is labeled with one odd integer and one even integer. For each odd integer j between 1 and $2n-1$ inclusive, let $\mathrm{pairedWith}(j)$ be the even integer with which it is paired. The DT code is the sequence $1, \mathrm{pairedWith}(1), 3, \mathrm{pairedWith}(3), \ldots, 2n-1, \mathrm{pairedWith}(2n-1)$.

4 The encoding

5 Ambient isotopy as weak bisimilarity

It is well-known that two knot presentations are ambient isotopic if and only if there is a sequence of Reidemeister moves transforming one presentation to the other. There are three types of Reidemeister moves and within each type we have a move and its inverse, as shown in the figure.

We first establish a lemma to our purposes, that shows we can clean up our knot diagram in a particularly suitable way.

Given a knot K and a diagram $D(K)$ of K, we call the application of a Reidemeister move ρ a *neatening* move if the resulting diagram $D'(K)$ has a crossing number less than or equal to the crossing number of K i.e. any neatening of move of the type Ω_1 or Ω_2 can occur in only one direction. Let $\hat{\rho} = \rho_1 \cdots \rho_n$ be a sequence of successive Reidemeister moves on the diagram $D(K)$ (with $n = 0$ being the identity move). We call $\hat{\rho}$ a *neatening isotopy* if either $n = 0$ each ρ_i is a neatening move. If, in addition, at least one of the ρ_i is a move of type Ω_1 or Ω_2, we call $\hat{\rho}$ a *cleaning isotopy* of $D(K)$.

For a given diagram $D(K)$, the collection of all its neatening isotopies can be given a partial ordering: given neatening isotopies $\hat{\rho} = \rho_1 \cdots \rho_n$ and $\hat{\sigma} = \sigma_1 \cdots \sigma_m$, we put $\hat{\rho} \leq \hat{\sigma}$ when $n \leq m$ and $\rho_i = \sigma_i$ for all $i \leq n$.

Lemma 1. *For any knot K and knot diagram $D(K)$, we may reduce $D(K)$ via a neatening isotopy to a diagram $D'(K)$ having the property that no neatening isotopy of $D'(K)$ is a cleaning isotopy of $D'(K)$.*

Proof. If the collection of cleaning isotopies of $D(K)$ is an empty one, then $D(K)$ is the desired diagram. So assume there is at least one cleaning isotopy of $D(K)$.

The collection of cleaning isotopies of $D(K)$ forms a partially ordered set. Since each cleaning isotopy reduces the number of crossings in the diagram by at least 1, each chain in the partially order set has length bounded by the number of crossings in $D(K)$. Select any maximal element $\hat{\rho}$ of the partially ordered set. Let $D'(K)$ be the diagram derived from $D(K)$ by applying $\hat{\rho}$.

Observation: because we know that the Reidemeister moves result in bisimilar processes, $[|D(K)|] \sim [|D'(K)|]$.

We abuse notation, and write $[|K|]$ to mean $[|D'(K)|]$ with $D'(K)$ reduced as per 1.

Suppose K_0, K_1 such that $[|D(K_0)|] \sim [|D(K_1)|]$. Using Lemma 1, we have $[|K_0|] \sim [|K_1|]$.

For any crossing $C_i(x_0, x_1, y_0, y_1)$ of $[|K_{i \bmod 2}|]$ there is exactly one transition possible. Because $[|K_0|] \sim [|K_1|]$ this transition must be mimicked by some crossing $C_j(u_0, u_1, v_0, v_1)$ of $[|K_{i+1 \bmod 2}|]$. Following this transition enables the unique transition of the 'next' crossing. **what about the 'previous'? or is 'next' going to be made precise in a previous section? or am i just being thick headed?** This must be mimicked by some crossing in $[|K_1|]$. Continuing in this way we visit every crossing exactly twice before we arrive at the state $[|K_0|]$ again. Thus, the bisimulation establishes a bijection between the crossings of $D'(K_0)$ and $D'(K_1)$ that preserves the polarity (i.e. over/under relationships) and the connections between the crossings. Thus, $D'(K_0)$ is in the same ambient isotopy class as $D'(K_1)$.

5.1 Encoding the Reidemeister moves

6 Conclusions and future work

Compositionality.

Generalizing crossings.

Braids and tangles.

Geometry and concurrency has been investigated in one direction. This work suggests that the other might also be fruitful: algebras with dynamics as invariants for spatial systems.

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