then $\phi_i(1)\phi_i(a_1)f(i)=f(1)$. If $i=1$, then $[2n, 1]$ comes from outside to 1 and $\phi_1(r)=\phi_1(2n)=-1$, so both α and β would have to be outside. Hence $r<2n$ and $s\leqslant 2n$. Thus $1\leqslant i<a_i<s\leqslant 2n$ and $1\leqslant a_s<s\leqslant 2n$. This completes the proof.

Lemma 3.1. *Let B be a proper subset of $\{1, 2, \ldots, 2n\}$ which is mapped onto itself by the involution a, and let C be the complementary set. Then some pair i, a_i in B separates some pair s, a_s in C in the cyclic order* mod $2n$.

Proof. B is divided by C into intervals $B_1, B_2, \ldots, B_k$ which in this order are separated by successive intervals $C_1, C_2, \ldots, C_k$ (mod $2n$);

$$\{1, \ldots, 2n\}=B_1\cup C_1\cup B_2\cup C_2\cup\cdots\cup B_k\cup C_k.$$

By Rule 1, there exists $i\in B_1$ with $a_i\notin B_1$. Choose r minimal positive so that some i in some B_h or C_h has a_i in B_{h+r} or C_{h+r}. We may assume $i\in B_h$, $a_i\in B_{h+r}$. By Rule 1, some $s\in C_h$ has $a_s\notin C_h$. By the minimality of r, $a_s\notin C_{h+1}\cup\cdots\cup C_{h+r-1}$. Hence $a_s\notin [i, a_i]$, so i, a_i separate s, a_s.

Let us write $i*s$ to mean that i, a_i separate s, a_s. We define inductively subsets A_h, B_h of $\{1, \ldots, 2n\}$ and elements $i_h\in A_h$ as follows, starting from $A_1=\{1, a_1\}$, $B_1=\emptyset$, $i_1=1$. For $h>1$, let $A_h=A_{h-1}\cup\{s: i_{h-1}*s\}$, $B_h=B_{h-1}\cup\{i_{h-1}, a(i_{h-1})\}$, and let i_h be the least member of $A_h\backslash B_h$ if this set is not empty.

Lemma 3.2. *The sequence $i_1, i_2, \ldots$ ends with i_n.*

Proof. Since, from the definition, all i_h and $a(i_h)$ are distinct, there cannot be more than n values of h. Clearly i_1 exists. By Lemma 3.1, since for $1<h\leqslant 2n$, B_h is a proper subset of $\{1, \ldots, 2n\}$, there is some $i\in B_h$ and $s\notin B_h$ such that i, a_i separate s, a_s. If $i<a_i$, $i\in\{i_1, \ldots, i_{h-1}\}$. Hence s and a_s are in A_h and hence in $A_h\backslash B_h$. So i_h exists for $h\leqslant n$.

For each S, whether realizable or not, we define a canonical orientation f inductively as follows. Let $f(1)=1$, $f(a_1)=-1$. For s and a_s in $A_h\backslash A_{h-1}$, we may assume that $s\notin[i, a_i]$, $a_s\in[i, a_i]$, where $i=i_{h-1}$. Then let $f(s)=\phi_i(s)\phi_i(a_s)f(i)$ and $f(a_s)=-f(s)$. This defines f on $\bigcup_1^n A_h\backslash A_{h-1}=A_n=\{1, \ldots, 2n\}$.

Theorem 3. *S is realizable if and only if (S, f) is realizable, where f is the canonical orientation. Hence S is not realizable if and only if there exist h and s such that, for $i=i_h$, $s<i$ and either $a_s\in[i, a_i]$ and $\phi_i(s)\phi_i(a_s)f(i)=-f(s)$* or *$s<a_s\notin[i, a_i]$ and $\phi_i(s)\phi_i(a_s)=-1$.*

Proof. If S is realiable then, by Corollary 1.2, there is a unique orientation g such that (S, g) is realizable. We have $f(1)=1=g(1)$ and $f(a_1)=-1=g(a_1)$. Assume that $f(j)=g(j)$ for all $j\in A_{h-1}$. For $s\in A_h\backslash A_{h-1}$ with $s\notin[i, a_i]$, where $i=i_{h-1}$, $f(s)=\phi_i(s)\phi_i(a_s)f(i)=\phi_i(s)\phi_i(a_s)g(i)=g(s)$ and $f(a_s)=-f(s)=-g(s)=g(a_s)$. Thus, by induction, $f=g$. Hence (S, f) is realizable. Trivially, if (S, f) is realizable,