S is realizable. Hence, by Theorem 2, S is realizable if and only if (S, f) satisfies Rule 2.

Here we use a slightly different form of Rule 2, justified by the symmetries of $\phi_i(s)\phi_i(a_s)$. When $s<i<a_s<a_i$, let $t=a_i$. Then $\phi_i(s)\phi_i(a_s)=\phi_t(s)\phi_t(a_s)=-\phi_s(t)\phi_s(a_t)$ and $f(i)=-f(t)$, so

$$\phi_i(s)\phi_i(a_s)f(i)f(s)=\phi_s(t)\phi_s(a_t)f(s)f(t).$$

When $s<i<a_i<a_s$, let $t=a_s$. Then

$$\phi_i(s)\phi_i(a_s)=\phi_i(t)\phi_i(a_t)=\phi_t(i)\phi_t(a_i).$$

When $s<a_s<i<a_i$, let $t=a_i$. Then $\phi_i(s)\phi_i(a_s)=\phi_s(i)\phi_s(a_i)=\phi_s(t)\phi_s(a_t)$. In all cases, $i=i_h$ for some h. This completes the proof.

Note that the definition of the canonical orientation and the checking of Rule 2 involve the same function $\phi_i(s)\phi_i(a_s)$. They can conveniently be combined in a single operation.

This completes the theoretical justification of an adequate method of tabulating knot projections. The following remarks leading to Corollary 3 are merely to make the computing somewhat more efficient.

The sequence S may contain a *twist*, that is, a maximal pair of intervals $[r, s]$ and $[a_r, a_s]$ or $[a_s, a_r]$, where $r \neq s$, which are mapped montonically onto each other by the involution a.

Consider first a twist not containing 1 and a_1. There is a first i_h such that $[i, a_i]$, where $i=i_h$, contains part but not all of the twist. Then $[i, a_i]$ contains, say, $[r, s]$, and $[a_i, i]$ contains $[a_r, a_s]$ or $[a_s, a_r]$. Then ϕ_i determines $f(a_q)=\phi_i(a_q)\phi_i(q)f(i)$ for $r \leqslant q \leqslant s$ and, since $\phi_i(q)=\phi_i(r)$ and $\phi_i(a_q)=(-1)^{q-r}\phi_i(a_r)$, therefore $f(a_q)=(-1)^{q-r}f(a_r)$ and $f(q)=(-1)^{q-r}f(r)$.

If 1 is on the twist with, say, $1 \in [r, s]$, and if the mapping a is increasing from $[r, s]$ onto $[a_r, a_s]$, then ϕ_1 determines $f(t)=(-1)^{t-1}$ for $r \leqslant t \leqslant s$. In case a is increasing from $[1, a_1]$ onto $[a_1, s]$, then the whole knot is a twist with $a_1=n+1$, and hence with n odd, and $f(t)=(-1)^{t-1}$ for all t.

If $1 \in [r, s]$ and a is decreasing from $[r, s]$ onto $[a_s, a_r]$ then the orientation at i_2 is determined by ϕ_1 and the orientation on the twist, except at 1 and a_1, is determined by ϕ_i when $i=i_2$. Indeed $i=i_2 \in [1, a_1]$, $a_i \notin [1, a_1]$ and $f(a_i)=\phi_1(a_i)\phi_1(i)f(1)=\phi_1(i)\phi_1(a_i)$. Hence

$$f(i)=-\phi_1(i)\phi_1(a_i)=\phi_i(1)\phi_i(a_1) \quad \text{and} \quad \phi_i(1)\phi_i(a_1)f(i)=1=f(1).$$

Thus the orientation determined by ϕ_i on the rest of the twist agrees at 1 and a_1 with that already determined. Hence $f(t)=(-1)^{t-1}$ for $t \in [r, s]$. Thus in all the cases, f is alternating on the twist.

Now suppose a is increasing from $[r, s]$ onto $[a_r, a_s]$. Let $p \in [r, s]$. Let $t \notin [p, a_p]$ and $a_t \in [p, a_p]$, with $t \notin [r, p-1]$. Then $\phi_p(t)=(-1)^{p-r}\phi_r(t)$, $\phi_p(a_t)=\phi_r(a_t)$, $f(p)=(-1)^{p-r}f(r)$, and hence $\phi_p(t)\phi_p(a_t)f(p)=\phi_r(t)\phi_r(a_t)f(r)$. If $t \in [r, p-1]$, then $\phi_p(t)=$