0.1 Reflective rewrite systems

Ground terms and reflections

Definition 1 (ground terms). *Given a signature* $\Sigma = (\Sigma_i)_{i \in \mathcal{N}}$ *with* $\Sigma_i \cap \Sigma_j = \emptyset$ *when* $i \neq j$*, we define* $\mathcal{L}(\Sigma)$*, the* ground terms *of* Σ*, by the following rules*

$$\textsc{constants}\quad \frac{s \in \Sigma_0}{s \in \mathcal{L}(\Sigma)}$$

$$\textsc{ground terms}\quad \frac{s_j \in \mathcal{L}(\Sigma) \qquad 0 \leq j < i \qquad s \in \Sigma_i}{s(s_0, ..., s_{i-1}) \in \mathcal{L}(\Sigma)}$$

If $s \in \Sigma_i$ *we use* $|s| \triangleq i$ *to denote the* arity *of* s*.*

Remark 1. For purposes of this note we ignore issues of polymorphism and associativity and commutativity, etc. The $\mathcal{L}(\Sigma)$ is describing the set of terms before any such relations are imposed. That is, $\mathcal{L}(\Sigma)$ is the free construction over some signature, Σ.

Remark 2. As usual, we abuse notation, appearing to conflate Σ with $\bigcup_i \Sigma_i$ and write expressions such as $s \in \Sigma$.

Next, we define $\mathcal{L}_Q(\Sigma)$, the *reflection* of Σ, by

Definition 2 (terms and codes).

$$\textsc{constants}\quad \frac{s \in \Sigma_0}{s \in \mathcal{L}_Q(\Sigma)}$$

$$\textsc{terms}\quad \frac{((s_j \in \mathcal{L}_Q(\Sigma)) or (s_j = \ulcorner t \urcorner, t \in \mathcal{L}_Q(\Sigma))) \qquad 0 \leq j < i \qquad s \in \Sigma_i}{s(s_0, ..., s_{i-1}) \in \mathcal{L}_Q(\Sigma)}$$

$$\textsc{dequoted terms}\quad \frac{s = \ulcorner t \urcorner, t \in \mathcal{L}_Q(\Sigma)}{\urcorner s \ulcorner \in \mathcal{L}_Q(\Sigma)}$$

and impose the relation $\ulcorner\urcorner s \ulcorner\urcorner \equiv s$.

We call $\ulcorner s \urcorner$ *the* code *of* s *and set* $\ulcorner \mathcal{L}_Q(\Sigma) \urcorner \triangleq \{\ulcorner s \urcorner \mid s \in \mathcal{L}_Q(\Sigma)\}$, *to denote the set of* codes*. Similarly, we call* $\urcorner\ulcorner t \urcorner\ulcorner$ *a decoding of* $\ulcorner t \urcorner$.

Remark 3. We do *not* require $\urcorner\ulcorner s \urcorner\ulcorner \equiv s$.

Remark 4. Intuitively, the set of codes corresponds to a set of names or variables. The set of terms does not intersect the set of codes, i.e. $\mathcal{L}_Q(\Sigma) \cap \ulcorner \mathcal{L}_Q(\Sigma) \urcorner = \emptyset$. Further, if we let $\mathcal{T}(s)$ denote the evident syntax tree of a term, then nodes of $\mathcal{T}(s)$ labeled with $\ulcorner - \urcorner$ cannot have children labeled with $\ulcorner - \urcorner$; rather, there must be an intervening term constructor.

Said, if we define a labeling of the tree with O for nodes labled with term constructors and P we see that a node labeled P must always have a child labeled O. Moreover, terms correspond to trees with roots labeled O and codes correspond to trees with roots labeled P. This alternation discipline is why we cannot achieve the construction merely by enriching Σ_1 with another operator.

Following this intuition of codes as names or variables we define the set of *free codes* of a term.

Definition 3 (free codes).

$$\frac{|s| = 0}{\mathcal{FC}(s) \triangleq \emptyset}\ \textsc{constants}$$

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$$\mathcal{FC}(s(t_0, ..., t_{i-1})) \triangleq \{t_j \mid t_j = \ulcorner u \urcorner\} \cup \bigcup_{\{t_k \mid t_k \neq \ulcorner u \urcorner\}} \mathcal{FC}(t_k)$$

$$\frac{s = \urcorner\ulcorner t \urcorner\ulcorner}{\mathcal{FC}(s) \triangleq \{\ulcorner t \urcorner\}}\ \textsc{decodes}$$

Theorem 1. $\ulcorner s \urcorner \notin \mathcal{FC}(s)$

Proof. By structural induction.

Definition 4 (open term constructors). *Let* C *be a set of codes. Suppose* $\exists k \in \Sigma$ *such that* $0 < |k| < |C|$*. We can form the set of terms* $\mathcal{O}(C) = \{k(t_0, ..., t_{|k|-1}) \mid t_i = k(c_0, ..., c_{|k|-1}), \bigcup_i \mathcal{FC}(t_i) = C\}$.

Theorem 2. *Let s be a term.* $\forall t \in \mathcal{O}(\mathcal{FC}(s)).\ulcorner t \urcorner \notin \mathcal{FC}(s)$.

Proof. By structural induction.

Remark 5. As the free codes correspond to the notion of free names or free variables of a term, these two theorems give us a supply of 'fresh' codes analogous to fresh names or variables.

Substitution

Definition 5 (renaming – the noun). *A* renaming, σ, *is a partial map,* $\sigma : \ulcorner \mathcal{L}_Q(\Sigma) \urcorner \rightarrow \ulcorner \mathcal{L}_Q(\Sigma) \urcorner_\perp$, *from codes to codes.*

Definition 6 (substitution – the verb). *We define,* $s \cdot \sigma$, *the application of a renaming to a term, via recursive traversal of the term structure.*

$$\text{CONSTANTS}\quad \frac{|s| = 0}{s \cdot \sigma \triangleq s}$$

$$\text{TERMS}\quad s(t_0, ..., t_{i-1}) \cdot \sigma \triangleq s(t_0 \cdot \sigma, ..., t_{i-1} \cdot \sigma)$$

$$\text{CODES}\quad \frac{\sigma(\ulcorner t \urcorner) = \perp}{\ulcorner t \urcorner \cdot \sigma \triangleq \ulcorner t \urcorner} \qquad \text{CODES}\quad \frac{\sigma(\ulcorner t \urcorner) = s}{\ulcorner t \urcorner \cdot \sigma \triangleq s} \qquad \text{DECODES}\quad \frac{\sigma(\ulcorner t \urcorner) = \perp}{\urcorner\ulcorner t \urcorner\ulcorner \cdot \sigma \triangleq \urcorner\ulcorner t \urcorner\ulcorner} \qquad \text{DECODES}\quad \frac{\sigma(\ulcorner t \urcorner) = \ulcorner u \urcorner}{\urcorner\ulcorner t \urcorner\ulcorner \cdot \sigma \triangleq u}$$

Remark 6. The application of a renaming does more than shuffle names: when it encounters a decoding it substitutes in a term.

0.2 Basic rewriting

Definition 7. *Let* $\rightarrow \subset \mathcal{L}_Q(\Sigma) \times \mathcal{L}_Q(\Sigma)$. *We say a term s rewrites to a term t ($s \rightarrow t$) when* $\exists (l, r) \in \rightarrow, \sigma. l \cdot \sigma = s, r \cdot \sigma = t$. *We also use* $s \stackrel{(l,\sigma,r)}{\rightarrow} t$ *to label the rewrite with the witness.*

0.3 Binders

Definition 8 (binders). *Given a signature, Σ, we define $\mathcal{L}_Q^\lambda(\Sigma)$ by the following rules*

$$\frac{s \in \Sigma_0}{s \in \mathcal{L}_Q^\lambda(\Sigma)}\ \textsc{constants}$$

$$\frac{((s_j \in \mathcal{L}_Q^\lambda(\Sigma)) or (s_j = \ulcorner t \urcorner, t \in \mathcal{L}_Q^\lambda(\Sigma))) \qquad 0 \leq j < i \qquad s \in \Sigma_i}{s(s_0, ..., s_{i-1}) \in \mathcal{L}_Q^\lambda(\Sigma)}\ \textsc{terms}$$

$$\frac{s = \ulcorner t \urcorner, t \in \mathcal{L}_Q^\lambda(\Sigma)}{\urcorner s \ulcorner \in \mathcal{L}_Q^\lambda(\Sigma)}\ \textsc{dequoted terms}$$

$$\frac{s = \ulcorner s' \urcorner, s' \in \mathcal{L}_Q^\lambda(\Sigma) \qquad t \in \mathcal{L}_Q^\lambda(\Sigma)}{\lambda s.t \in \mathcal{L}_Q^\lambda(\Sigma)}\ \textsc{abstractions}$$

Definition 9 (free codes). *We extend the previous definition with the following rule.*

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$$\mathcal{FC}(\lambda s.t) \triangleq \mathcal{FC}(t) \setminus \{s\}$$

Definition 10 (beta-reduction). $(\lambda s.t)u \to_\beta t\{\ulcorner u \urcorner / s\}$

Definition 11 (closed terms). *A term, $t \in \mathcal{L}_Q^\lambda(\Sigma)$, is closed $\Leftrightarrow$ $\mathcal{FC}(t) = \emptyset$. We denote the set of closed terms with $\overline{\mathcal{L}_Q^\lambda(\Sigma)}$.*

0.4 Duality

Definition 12 (dual system). $\overline{\mathcal{L}_Q^\lambda(\Sigma)}^\perp \subset [\overline{\mathcal{L}_Q^\lambda(\Sigma)} \to \ulcorner \overline{\mathcal{L}_Q^\lambda(\Sigma)} \urcorner]$ *is defined by* $t^\perp(u) \triangleq \ulcorner (tu) \urcorner$

References