MR0171275 (30 #1506) 55.20
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On a certain numerical invariant of link types.
Trans. Amer. Math. Soc. **117** 1965 387–422

This paper is concerned with the signature $\sigma(l)$ of the quadratic form of a link l with μ components. Trotter showed [Ann. of Math. (2) **76** (1962), 464–498; MR0143201 (26 #761)] that for $\mu = 1$, σ is an invariant, and this is shown here (Theorem 3.1) to be true also for $\mu > 1$.

By the "reduced Alexander polynomial" $\Delta(t)$ of l is meant a generator of the principal ideal $\mathcal{E}_1(t, \cdots, t)$; thus for a knot it is the usual Alexander polynomial, and for $\mu > 1$ it is $(t - 1)\Delta(t, \cdots, t)$. It is normalized in this paper by the following conditions: $\Delta(t)$ has no terms of negative degree, $\Delta(0) \neq 0$ (unless $\Delta(t) = 0$, of course), and $\Delta(-1) \geq 0$. If l is a non-splittable alternating link, then $\Delta(0)\Delta(-1) > 0$ (Lemma 5.1). If l is a non-splittable special alternating link, then the degree of $\Delta(t)$ is $|\sigma(l)|$ (Lemma 5.2). If l is a non-splittable amphicheiral link, then $\sigma(l) = 0$ (Lemma 5.8). Consequently, of the non-splittable special alternating links only the trivial ones are amphicheiral (Theorem 5.5). If k is any knot, $\sigma(k) \equiv 0$ or $2 \pmod 4$ according as $\Delta(-1) \equiv 1$ or $3 \pmod 4$ [cf. the reviewer, Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Inst., 1961), pp. 168–176, problem 12; Prentice-Hall, Englewood Cliffs, N.J., 1962; MR0140100 (25 #3523)].

The reviewer has suggested three generalizations of the concept of slice link (Properties (1) = (2), (3) and (4) [ibid., p. 173, following problem 25]), and the author names them as follows: (1) = (2) slice link in the strong sense, (3) slice link, (4) slice link in the weak sense. If l is a slice link in the weak sense, then $|\sigma(l)| \leq \mu - 1$ (Theorem 8.5); if l is a slice link, then $\sigma(l) = 0$ (Theorem 8.8); if l is a slice link in the strong sense, then not only is $\sigma(l) = 0$ but $\mathcal{E}_{\mu-1}(t, \cdots, t) = 0$ and $\mathcal{E}_{\mu}(t, \cdots, t)$ is a principal ideal generated by an element of the form $f(t)f(t^{-1})$, where $f(t)$ is an integral polynomial for which $f(1) = 1$ (Theorem 8.4) [ibid., problem 26]. Consequently, if k is a slice knot, then $\sigma(k) = 0$ (Theorem 8.3). In particular, the granny knot is not a slice knot (Theorem 8.9) [this solves problem 23, ibid.]. Whether l is a slice link or not, $\mathcal{E}_{\mu}(t, \cdots, t) \neq 0$ (Lemma 6.1). Also $\sigma(l_1 \# l_2) = \sigma(l_1) + \sigma(l_2)$ (Corollary 7.4).

Now let l be oriented and denote its genus by $h(l)$ and the minimum of the genera of the locally flat surfaces in half 4-space that are bounded by l by $h^*(l)$ [ibid., p. 172, following problem 23]. We have (as a consequence of a somewhat stronger inequality) $|\sigma(l)| \leq 2h^*(l) + \mu - 1$ (Theorem 9.1). If k is a special alternating knot, then $h^*(k) = h(k)$ (Theorem 9.3) [ibid., problems 12 and 24].

Finally, consider the unknotting number $u(k)$ of a knot [H. Wendt, Math. Z. **42** (1937), 680–696]. We have $|\sigma(k)| \leq 2u(k)$ (Theorem 10.1) and $h^*(k) \leq u(k)$ (Theorem 10.2). Consequently if k is the torus knot of type $(2n + 1, 2)$, its unknotting number is $|n|$ (Corollary 10.3).

Reviewed by *R. H. Fox*