THE BRAID–PERMUTATION GROUP

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ABSTRACT: We consider the subgroup of the automorphism group of the free group generated by the braid group and the permutation group. This is proved to be the same as the subgroup of automorphisms of permutation–conjugacy type and is represented by generalised braids (braids in which some crossings are allowed to be "welded"). As a consequence of this representation there is a finite presentation which shows the close connection with both the classical braid and permutation groups. The group is isomorphic to the automorphism group of the free quandle and closely related to the automorphism group of the free rack. These automorphism groups are strongly connected with invariants of classical knots and links in the 3–sphere.

1. Introduction

Let F_n denote the free group of rank n with generators $\{x_1,\ldots,x_n\}$ and let $\mathrm{Aut}F_n$ denote its automorphism group. Let $\sigma_i \in \mathrm{Aut}F_n,\ i=1,2,\ldots,n-1$ be given by

$$\begin{cases} x_i & \mapsto & x_{i+1} \\ x_{i+1} & \mapsto & x_{i+1}^{-1}x_ix_{i+1} \\ x_j & \mapsto & x_j \qquad j\neq i,\ i+1. \end{cases}$$

and let $\tau_i \in \mathrm{Aut}F_n,\ i=1,2,\ldots,n-1$ be given by

$$\begin{cases} x_i & \mapsto & x_{i+1} \\ x_{i+1} & \mapsto & x_i \\ x_j & \mapsto & x_j \qquad j\neq i,\ i+1. \end{cases}$$

The elements $\sigma_i,\ i=1,2,\ldots,n-1$ generate the **braid subgroup** B_n of $\mathrm{Aut}F_n$ which is well known to be isomorphic to the classical braid group of on n strings (for a proof see [B1; 1.10] or [FR; 7.3]), and the elements $\tau_i,\ i=1,2,\ldots,n-1$ generate the **permutation subgroup** P_n of $\mathrm{Aut}F_n$ which is a copy of the symmetric group S_n of degree n. We shall call the subgroup BP_n of $\mathrm{Aut}F_n$ generated by both sets of elements σ_i and $\tau_i,\ i=1,2,\ldots,n-1$ the **braid–permutation group** and this is the subject of this paper.

This group is interesting for a variety of reasons.

Firstly, the elements of BP_n can be pictured in a way that extends the well-known pictures for elements of the braid group. These are braids in which some of the crossings are "welded" and the welded and unwelded crossings interact in an intuitively simple way.

Secondly, there is a simple characterisation of the automorphisms in BP_n. Let $\pi\in S_n$ be a permutation and $w_i,\ i=1,2,\ldots,n$ be words in F_n. Then the assignation

$$x_i \mapsto w_i^{-1}x_{\pi(i)}w_i$$

determines a homomorphism of F_n to itself which is in fact injective (see 2.7). If it is also surjective (and hence an automorphism) then we call it an automorphism of

permutation–conjugacy type. The automorphisms of permutation–conjugacy type define a subgroup PC_n of $\mathrm{Aut}F_n$ which is in fact precisely BP_n.

Thirdly, the pictorial definition of BP_n as equivalence classes of welded braids leads to the following finite presentation which includes the standard presentations for both B_n and S_n. This result was announced in [FRR].

The *generators* are $\sigma_i,\ \tau_i,\ \ i = 1, \ldots, n-1$, as above, and the *relations* are:

$$\begin{cases} \sigma_i\sigma_j &= \sigma_j\sigma_i \quad |i-j|>1 \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \end{cases}$$

The braid relations

$$\begin{cases} \tau_i^2 &= 1 \\ \tau_i\tau_j &= \tau_j\tau_i \quad |i-j|>1 \\ \tau_i\tau_{i+1}\tau_i &= \tau_{i+1}\tau_i\tau_{i+1} \end{cases}$$

The permutation group relations

$$\begin{cases} \sigma_i\tau_j &= \tau_j\sigma_i \quad |i-j|>1 \\ \tau_i\tau_{i+1}\sigma_i &= \sigma_{i+1}\tau_i\tau_{i+1} \\ \sigma_i\sigma_{i+1}\tau_i &= \tau_{i+1}\sigma_i\sigma_{i+1} \end{cases}$$

The mixed relations

Fourthly, BP_n is isomorphic to the automorphism group $\mathrm{Aut}FQ_n$ of the free quandle of rank n, and is closely related to the automorphism group $\mathrm{Aut}FR_n$ of the free rack of rank n (the latter group is the wreath product of BP_n with the integers) and these groups are strongly connected with invariants of classical knots and links in the 3–sphere.

Finally the interpretation of elements of BP_n as welded braids suggests a natural relationship with the **singular braid group** SB_n, of Birman et al., which has applications to the theory of Vassiliev invariants, [Ba], [B2], [Kh]. Indeed these groups have a common subgroup (the braid group) and a common quotient. The connection of BP_n with SB_n and the Vassiliev invariants will be explored in a later paper.

The rest of this paper is organised as follows. In section 2 we prove that the subgroups BP_n and PC_n are identical. This is done by using a variant of the Nielsen theory for free groups. In section 3 we describe welded braid diagrams and the group of welded braids (equivalence classes of diagrams under allowable moves). This is isomorphic to the group presented above and in section 4 we prove that it is isomorphic to BP_n. The proof is diagramatic: using the results of section 2 and 3 we have to show that a welded braid which induces the identity automorphism of F_n can be changed to the trivial braid by allowable moves. Finally in section 5 we interpret the results of previous sections in terms of racks and quandles.

The presentation for BP_n and the proof of this presentation given in this paper are closely related to results for $\mathrm{Aut}FR_n$ given by Krüger [Kr]. However his context is rather different from ours and in particular there is no interpretation in terms of diagrams. The results in section 2 of this paper are similar to results in [H] and [Ko] : our automorphisms of permutation–conjugacy type are called tNt–maps and pseudo-conjugations in [H] and [Ko] respectively.

We would like to thank Gyo Taek Jin for pointing out an inaccuracy in earlier statements of lemmas 4.4 and 4.5.

2. Automorphisms of permuation–conjugacy type

The groups BP_n and PC_n were defined in section 1. The main result of this section is the equality of these two groups:

2.1 Theorem $\qquad BP_n = PC_n$

Since the generators σ_i, τ_i of BP_n are automorphisms of permuation–conjugacy type, BP_n is a subgroup of PC_n; thus we have to show that any automorphism of permuation–conjugacy type is a product of the "elementary" automorphisms σ_i, τ_i. It is convenient to use rather different elementary automorphisms and it is also convenient to use exponential notation for conjugacy.

Notation From now until the end of section 4, the notation x^y will mean the product $y^{-1}xy$.

Elementary automorphisms of permuation–conjugacy type

We shall call the automorphisms $p_{i,k}, s_\pi \in PC_n$ given as follows, **elementary** automorphisms.

$$p_{i,k} : \begin{cases} x_i & \mapsto & x_i^{x_k} \\ x_j & \mapsto & x_j \quad j \neq i \end{cases}$$

$$s_\pi : x_i \mapsto x_{\pi(i)} \quad \pi \in S_n.$$

It is easy to see that $p_{i,k}, s_\pi$ are alternative generators for BP_n indeed

$$\sigma_i = p_{i+1,i}\tau_i \quad \text{and} \quad \tau_i = s_{t_i}$$

where t_i is the transposition $(i, i+1)$ and

$$p_{i,k} = \tau_i\tau_{i+1}\ldots\tau_{k-1}\sigma_{k-1}\tau_{k-2}\ldots\tau_i \quad \text{if} \quad i < k$$

$$p_{i,k} = \tau_{i-1}\tau_{i-2}\ldots\tau_{k+1}\sigma_k\tau_k\ldots\tau_{i-1} \quad \text{if} \quad i > k.$$

(Note that products in these formulae are read from left to right.)

Thus to prove the theorem we have to prove that any automorphism in PC_n is a product of the above elementary automorphisms.

We shall use an adaptation of Nielsen theory following the treatment given in Lyndon and Schupp [LS] pages 4 to 17. Some of the following material appeared in the setting of racks in the appendix to [FR] but for completeness we shall reproduce it here. See also Humphries [H] where a similar theory is developed and applied to the problem of complex monodromy.

Denote the reduced length of the word $w \in F_n$ by $l(w)$. We shall consider sets $\mathbf{u} = \{u_1, \ldots, u_k\}$ of distinct elements of F_n. We use the notation $\mathbf{u}^{-1} = \{u_1^{-1}, \ldots, u_k^{-1}\}$ for the set of inverses and $\mathbf{u}^{\pm 1} = \mathbf{u} \cup \mathbf{u}^{-1}$.

A set $\mathbf{u}$ is called **Nielsen reduced** if the following conditions hold.

N0 If $u \in \mathbf{u}$ then $u \neq 1$.

N1 If $u,v \in \mathbf{u}^{\pm 1}$ and $uv \neq 1$ then $l(uv) \geq \max\{l(u),l(v)\}$.

N2 If $u,v,w \in \mathbf{u}^{\pm 1}$ and $uv \neq 1$ and $vw \neq 1$, then $l(uvw) > l(u) - l(v) + l(w)$.

2.2 Lemma *Let $\mathbf{u} = \{u_1, \ldots, u_k\}$ be Nielsen reduced. If $w = v_1 \cdots v_r$ where $v_i \in \mathbf{u}^{\pm 1}$ and all $v_i v_{i+1} \neq 1$, then $l(w) \geq r$.*

Proof For each $v \in \mathbf{u}^{\pm 1}$ let v_0 be the longest initial segment of v that cancels in any product uv where $u \in \mathbf{u}^{\pm 1}$ and let v_1 be the longest final segment of v that cancels in any product vw where $w \in \mathbf{u}^{\pm 1}$. Note that $v_1 = (v^{-1})_0^{-1}$.

Then we can write $v = v_0 m v_1$ where by N2 $l(m) \geq 1$.

So in the product $w = v_1 \cdots v_r$ there is always at least an irreducible subword $m_1 \cdots m_r$ and the result follows. □

2.3 Corollary *Let $\mathbf{u} = \{u_1, \ldots, u_k\}$ be Nielsen reduced and suppose in addition that $\mathbf{u}$ generates F_n. Then $\mathbf{u}^{\pm 1} = \mathbf{x}^{\pm 1}$ where $\mathbf{x}$ is the basis $\{x_1, \ldots, x_n\}$ of F_n. (Note in particular that $k = n$.)*

Proof Let the basis element x_i be written as a product $x_i = v_1 \cdots v_r$ where $v_i \in \mathbf{u}^{\pm 1}$ and all $v_i v_{i+1} \neq 1$. Then $1 = l(x_i) \geq r$. So r is forced to be unity and $x_i = v_j$ for some j. □

2.4 Definition *Sets of PC-type*

Consider a permutation $\pi \in S_n$ of $\{1, 2, \ldots, n\}$ and n words w_i for $i = 1, 2, \ldots, n$ in the free group F_n. Corresponding to this data is the set of n words $x_{\pi(i)}^{w_i}$ in F_n, where $i = 1, 2, \ldots, n$, obtained by permuting and conjugating the generators. We shall call such a set of words a set of **permutation–conjugacy type**, or **PC–type** for short. We shall also use this name for the set obtained by inverting some of the elements of this set.

Let $\mathbf{u} = \{u_1, u_2, \ldots, u_n\}$ be a set of words of PC–type; we shall consider the following two **double Nielsen transfomations** which preserve PC–type:

T1 replace u_i by $u_j^{-1} u_i u_j$ where $j \neq i$;

T2 replace u_i by $u_j u_i u_j^{-1}$ where $j \neq i$.

2.5 Lemma *Let $\mathbf{u} = \{u_1, u_2, \ldots, u_n\}$ be a set of words of PC–type. Suppose that $l(u_i u_j) < l(u_i)$ then either $l(u_j^{-1} u_i u_j) < l(u_i)$ or $l(u_i u_j u_i^{-1}) < l(u_j)$.*

Proof We first observe that we cannot have $l(u_i) = l(u_j)$ because then $l(u_i u_j) < l(u_i)$ implies that at least half of u_i, u_j cancel in the product and the middle letter of u_i cancels with that of u_j. But this is impossible since the words are of PC–type and their middle letters are different generators.

Now assume that $l(u_j) < l(u_i)$. We shall show $l(u_j^{-1} u_i u_j) < l(u_i)$.

Write $u_i = w^{-1} x w$ where x is one of the generators $x_1, x_2, \ldots x_n$, or an inverse. Then since w has length greater than half of u_j, more than half of u_j cancels with w, i.e.

$$w = ya \quad u_j = a^{-1} z \quad \text{where} \quad l(z) < l(a)$$

Therefore

$$l(u_j^{-1}u_iu_j) = l(z^{-1}aa^{-1}y^{-1}xyaa^{-1}z) = l(z^{-1}y^{-1}xyz) < l(a^{-1}y^{-1}xya) = l(u_i).$$

In the case when $l(u_j) > l(u_i)$ then $l(u_iu_j) < l(u_j)$ and we can show in a similar way that $l(u_iu_ju_i^{-1}) < l(u_j)$. □

2.6 Lemma *Let* $\mathbf{u} = \{u_1, u_2, \ldots, u_n\}$ *be a set of words of PC–type. Then* $\mathbf{u}$ *can be carried by a sequence of moves T1 and T2 above to a set* $\mathbf{v}$ *of words of PC–type which is Nielsen reduced.*

Proof The condition N0 is automatically satisfied so assume that $\mathbf{u}$ does not satisfy N1. Then there is a pair $u,v \in \mathbf{u}^{\pm 1}$ such that $l(uv) < l(u)$ and $uv \neq 1$. Then by the last lemma there is a transformation T1 or T2 which reduces $\sum l(u_i)$. Therefore if we apply T1 and T2 until $\sum l(u_i)$ is minimum the condition N1 will hold.

Now consider a triple $u,v,w \in \mathbf{u}^{\pm 1}$ such that $uv \neq 1, vw \neq 1$. Then by N1 we have $l(uv) \geq l(v)$ and $l(vw) \geq l(v)$. It follows that that part of v which cancels in the product uv is no more than half of v. Likewise that part of v which cancels in the product vw is also no more than half of v. So we can write in reduced form $u = ap^{-1}, v = pbq, w = q^{-1}$. Notice that $b \neq 1$ because v is one of a set of PC–words and hence has odd reduced length. So $uvw = abc$ is reduced and $l(uvw) = l(u) - l(v) + l(w) + l(b) > l(u) - l(v) + l(w)$. □

2.7 Corollary *A set of words* $\mathbf{u} = \{u_1, u_2, \ldots, u_n\}$ *of PC–type forms the basis of a free subgroup of* F_n *of rank* n *and hence the endomorphism of* F_n *defined by* $x_i \mapsto u_i$ *is injective.*

Proof By the last lemma we can assume without loss that our set of words is Nielsen reduced and the result now follows from lemma 2.2. □

Proof of theorem 2.1

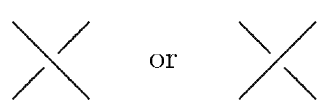
As remarked near the beginning of the section we have to prove that if f is an automorphism of F_n of permutation–conjugacy type then f is a product of elementary automorphisms.

Let $f(x_i) = u_i$ then $\{u_i\}$ is a set of words of PC–type. Now composition of f with the elementary automorphism $p_{i,k}$ for $i \neq k$ realises a double Nielsen transformation on $\{u_i\}$. (If $i = k$ the automorphism has no effect.)

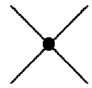
Therefore by lemma 2.6 we may assume that $\{u_i\}$ is Nielsen reduced. But by corollary 2.3 this implies that these words are a permutation of the words x_i, $i = 1, \ldots, n$ and therefore f is the elementary automorphism s_π for suitable π. □

3. Welded braids

A **welded braid diagram** on n strings is a set of n monotone arcs from n points on a horizontal line at the top of the diagram down to a similar set of n points at the bottom of the diagram. The arcs are allowed to cross each other either in a “crossing” thus:



or in a "weld" thus:



An example (on three strings) is illustrated in figure 1 below:

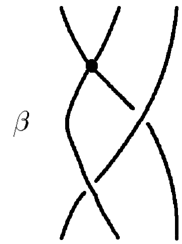


Figure 1 : a welded braid diagram

It is assumed that the crossings and welds all occur on different horizontal levels; thus a welded braid diagram determines a word in the **atomic** diagrams illustrated and labelled in figure 2.

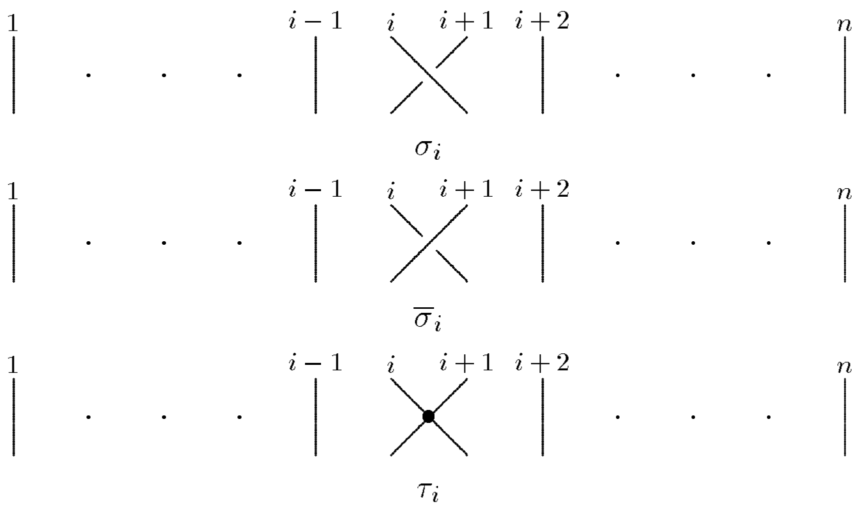


Figure 2 : the three atomic braids

Convention Braids are read *from top to bottom* and words are read *from left to right*.

Thus figure 1 determines the word $\tau_1\overline{\sigma}_2\sigma_1$ which is read: first τ_1 then $\overline{\sigma}_2$ then σ_1.

We identify braid diagrams which determine the same word in the atomic diagrams (in other words diagrams which differ by a planar isotopy through diagrams).

The set of welded braid diagrams (on n strings) forms a semi-group WD_n with composition given by "stacking": if β_1 and β_2 are diagrams then $\beta_1\beta_2$ is the diagram obtained by placing β_1 above β_2 so that the bottoms of the arcs of β_1 coincide with the tops of the arcs of β_2. There is an identity in WD_n, namely the diagram with no crossings.

The notation for atomic diagrams is intended to be confused with the notation for the generators of BP_n because we now consider the homomorphism $\Phi : WD_n \to \mathrm{Aut}F_n$, with image BP_n, defined by mapping σ_i and τ_i to the automorpisms with the same names and $\overline{\sigma}_i$ to σ_i^{-1}.

Thus a welded braid diagram β determines an automorphism $\Phi(\beta)$ of F_n (of permutation–conjugacy type). There is a convenient way to read $\Phi(\beta)$ from the diagram β which we now describe.

Reading the automorphism from the diagram

Label the strings at the bottom of the diagram by the generators $x_1, x_2, \ldots, x_n$ in order and continue to label the subarcs between crossings moving up the diagram and using the following rules (figure 3):

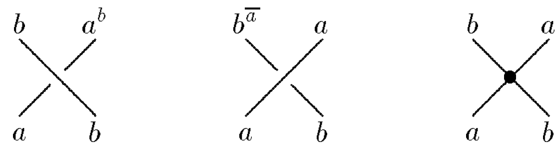


Figure 3 : rules for labelling subarcs

In figure 3 we have used the convention that $\overline{x}$ means x^{-1}. This is useful to avoid double exponents. The top of the strings are thus labelled by words $w_1, w_2, \ldots, w_n$ and the automorphism determined by the diagram is given by $x_i \mapsto w_i$ for $i = 1, \ldots, n$. An example of this process in given in figure 4, where the automorphism corresponding to the diagram in figure 1 is calculated.

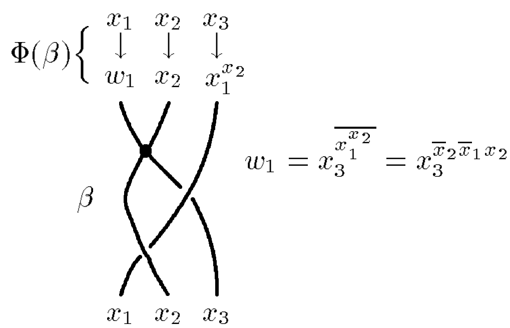


Figure 4

To see that this process gives the correct result it is merely necessary to observe that it is correct for elementary braids (which follows at once from the rules in figure 3) and that the process gives a homomorphism $WD_n \to \mathrm{Aut}F_n$.

To see this consider the effect of stacking the braid β' on top of the braid β. If the i–th point at the bottom of β is labelled x_i and at the top is labelled w_i, then the labels at the top of the combined braid are obtained from those for β' by substituting w_i for x_i. But this is precisely how the composition $\Phi(\beta')\Phi(\beta)$ of the two automorphism of F_n is formed.

Allowable moves on diagrams

We now consider the local changes that can be made to a welded braid diagram β which leave the automorphism $\Phi(\beta)$ unchanged. Four such moves are given in figure 5.

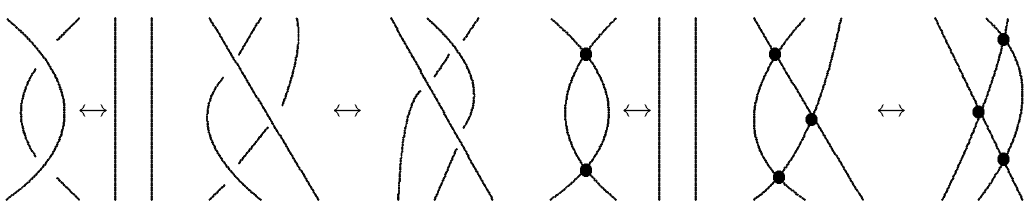


Figure 5

These are moves involving two or three strings which do not mix welded and unwelded crossings. It is easy to check that these leave the induced automorphism unchanged. For the second move in figure 5 this follows from the following easily verified property of conjugation

$$a^{bc} = a^{cb^c}. \tag{3.1}$$

The first two are Reidemeister moves from knot theory. If we identify welded braid diagrams that differ by these Reidemeister moves then we can embed the group of braids into the semigroup of welded braid diagrams. If we also identify those braids that differ by the second two moves then we can give welded braid diagrams the structure of a group.

However there are more local changes that do not change the assigned automorphism. In figure 6 we present two of them.

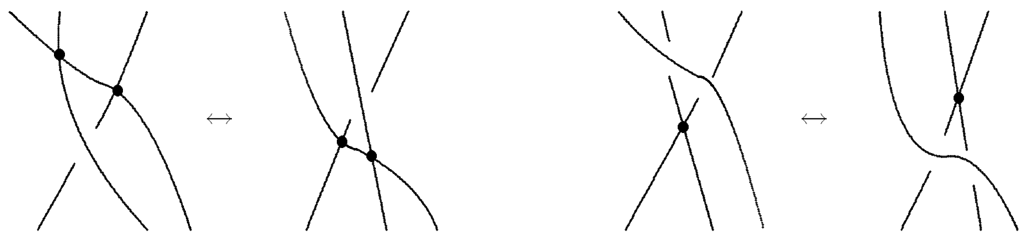


Figure 6a *Figure 6b*

In addition there are moves of a general class involving four strings which allow non-interferring crossings to be reordered. One such move is illustrated in figure 7.

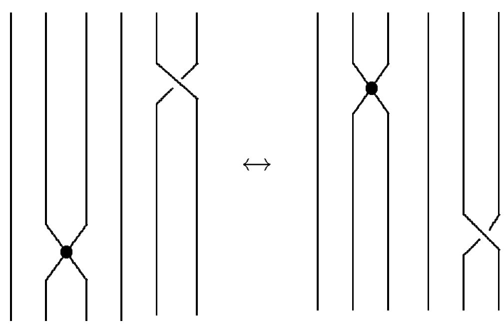


Figure 7

We call the moves illustrated in figure 5, 6 and 7 (and moves similar to figure 7) **allowable moves** and we define a **welded braid** to be an equivalence class of welded braid diagrams under allowable moves. Welded braids on n strings form a group which we shall denote WB_n with composition given (as in WD_n) by stacking. The inverse of a welded braid β is obtained by reflecting β in a horizontal line. That this is an inverse follows from the first and third moves in figure 5.

There are other local moves of the same type as those given above which also do not change the induced automorphism. For instance the moves obtained from the moves in figure 6 by reflection in horizontal or vertical lines. However, as the reader can readily check, these other moves can all be obtained as suitable sequences of the allowable moves given in figures 5 and 6. In figure 8 below, we show as an example how this can be done for a move of the same type as figure 6a.

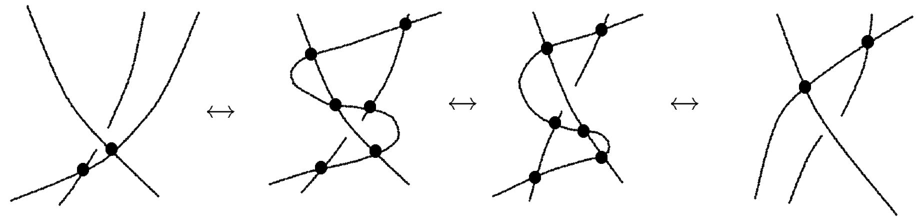


Figure 8

All the allowable moves (and consequent moves such as figure 8) are similar to Reidemeister moves, and therefore one could be forgiven for thinking the following. Welded braids can be defined as the 2–dimensional projection of 3–dimensional welded braids and two welded braid diagrams give the same automorphism of F_n if they are projections of equivalent welded braids — where the equivalence among 3–dimensional welded braids is a natural equivalence, for example isotopy in 3–space.

That this is not so can be seen by observing that the local change pictured in figure 9 **changes** the assigned automorphism.

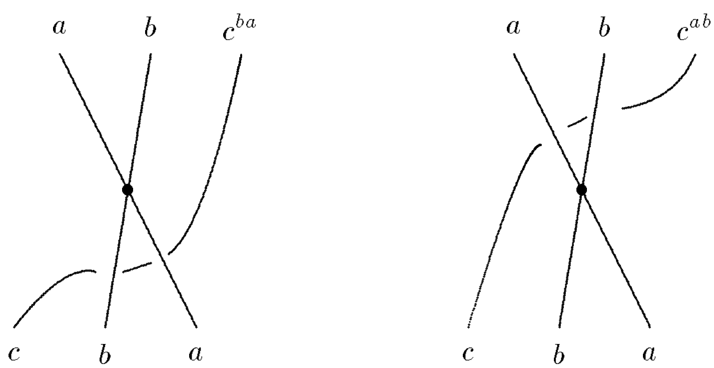


Figure 9 : Two welded braids which do not induce the same element of $Aut F_n$.

There is however a way to regard welded braids as 3–dimensional objects if desired. Think of the braids as comprising strings embedded in half of 3–space, namely the half *above* the plane of the diagram, and think of the strings as free to move above the plane except at the welds, which are to be regarded as small spots of glue holding the strings down onto the plane. Thus strings are not allowed to move behind welds, as happens in figure 9.

We can now state the main result of the paper:

3.2 Main Theorem *Two welded braid diagrams determine the same automorphism of F_n if and only if they can be obtained from each other by a finite sequence of allowable moves.*

The proof of the theorem comprises the next section. We finish this section by remarking that the theorem implies the presentation for BP_n given in section 1. It is not hard to see that the group WB_n of welded braids has this presentation. The elementary braids σ_i and τ_i generate WB_n and the allowable moves each correspond to one of the relations listed in section 1. For example the moves in figure 6 correspond to the second and third "mixed relations" respectively.

Indeed welded braids can be regarded as a convenient way of illustrating this presentation.

Now we have the homomorphism $\Phi : WD_n \to \mathrm{Aut}F_n$ with image BP_n. Since allowable moves leave the induced automorphism unaltered this factors via a homomorphism $WB_n \to \mathrm{Aut}F_n$ with the same image. The theorem implies that this homomorphism is injective and hence WB_n and BP_n are isomorphic. Moreover the generators of WB_n and BP_n (with the same names) correspond, hence BP_n has the presentation given.

4. Proof of the main theorem

In order to prove the main theorem (Theorem 3.2) it is sufficient to show that any welded braid which induces the identity automorphism of the free group F_n can be reduced to the identity braid by a finite sequence of allowable moves.

Let $C = \{x_i^w \,|\, w \in F_n\}$ denote the set of conjugates of generators of F_n. If $a = x_i^w$ is in C let $L(a) = l(w)$ where w is chosen to have minimal length $l(w)$. Note that for example $x_i^{x_i w} = x_i^w$.

If a, b belong to C then we write $a \prec b$ if $L(b^a) < L(b)$ and we write $\overline{a} \prec b$ if $L(b^{\overline{a}}) < L(b)$. The following technical lemmas will be useful.

Let a, b, c be arbitary elements of C.

4.1 Lemma *If $a \prec b$ $(\overline{a} \prec b)$ and $a = x_i^w$ for some w then $b = x_j^{u\overline{x}_i w}$ $(b = x_j^{u x_i w})$ for some u where $u\overline{x}_i w$ $(u x_i w)$ is reduced.*

Proof If $b = x_j^v$ then $b^a = x_j^{v\overline{w}x_i w}$. If $l(v\overline{w}x_i w) < l(v)$ it follows that v ends in $\overline{x}_i w$. The case when $\overline{a} \prec b$ is similarly proved. □

4.2 Lemma *If $a \prec b$ and $b \prec c$ then $a \prec c$ and $a \prec c^b$.*

Proof If $a = x_i^w$ then by the above $b = x_j^{u\overline{x}_i w}$ and $c = x_k^{v\overline{x}_j u\overline{x}_i w}$. So $c^a = x_k^{v\overline{x}_j u w}$, $c^{ba} = x_k^{vuw}$ and $c^b = x_k^{vu\overline{x}_i w}$. Using the fact that $l(\overline{x}_i w) > l(w)$ the result follows.

4.3 Lemma *If $a \prec b$ and $\overline{b} \prec c$ then $a \prec c$ and $a \prec c^{\overline{b}}$.*

Proof The proof is similar to 4.2. □

4.4 Lemma *If $a \prec b$, $c \prec b$ and $a \neq c$ then $L(a) \neq L(c)$. If $L(a) < L(c)$ then $a \prec c$ and $L(b^{ca}) < L(b)$.*

Proof Let $a = x_i^w$, $c = x_j^u$ in reduced form. From the hypotheses it follows that the exponent of b ends in $\overline{x}_i w$ and also $\overline{x}_j u$. Because $a \neq c$ $L(a) \neq L(c)$. If $L(a) < L(c)$ it follows that $u = s\overline{x}_i w$ and $b = x_k^{v\overline{x}_j s\overline{x}_i w}$, $c = x_j^{s\overline{x}_i w}$. Then $c^a = x_j^{sw}$ so $L(c^a) < L(c)$. We have $b^{ca} = x_k^{vsw}$ which gives the second conclusion. □

4.5 Lemma *If $\overline{a} \prec b$, $c \prec b$ and $L(a) < L(c)$ then $\overline{a} \prec c$ and $L(b^{c\overline{a}}) < L(b)$.*

Proof The proof is similar to 4.4. □

4.6 Lemma *The following three pairs of conditions are contradictory.*

(1) $a \prec b$ and $b \prec a$.

(2) $a \prec b$ and $\overline{b} \prec a$.

(3) $a \prec b$ and $\overline{a} \prec b$.

Proof For the pairs (1) and (2) lemmas 4.2 and 4.3 respecively imply that $a \prec a$ which is impossible. For (3) lemma 4.1 implies that $b = x_j^{u_1 \overline{x}_i w} = x_j^{u_2 x_i w}$. A simple argument using the minimal length representation of the exponent shows that $b = x_i^w = a$ which is again a contradiction. □

We can assign a non negative integer $L(\kappa)$, the **length** to any endomorphism κ of F_n of PC–type by the formula: $L(\kappa) = \sum L(\kappa(x_i))$. Clearly $L(\kappa) = 0$ if and only if κ is a permutation of the generators.

Now let β be a welded braid diagram which represents the identity automorphism. We assume that all (welded and unwelded) crossings have distinct y–coordinates. Let us suppose that the unwelded crossings occur with y–coordinates: $y_1, \ldots, y_{k-1}$. We choose the notation so that

$$0 = y_0 < y_1 < y_2 < \ldots < y_{k-1} < y_k = 1$$

where y_0 is the top level of the braid and y_k is the bottom level of the braid. *Remember the y–coordinate increases as we go down the page.* Any horizontal line whose height is none of these critical values and also does not meet a weld will divide the welded braid β into an upper welded braid β' and a lower welded braid β'' so that $\beta = \beta'\beta''$. If t is the height of the horizontal line define $L(t) = L(\Phi(\beta''))$. Notice that the function $L(t)$ changes only at the critical values. The values of L for $0 < t < y_1$ and for $y_{k-1} < t < 1$ will be 0. If $k > 1$ let us take one of the maximal L–valued intervals, say $[y_s, y_{s+1}]$. The value of s cannot be 0 or $k-1$. We will show that we can change the welded braid by allowable moves so that the value of L is reduced within this interval and is unaltered elsewhere.

Let ς_i and ς_j be the unwelded crossings at level y_s and y_{s+1} respectively. Then the braid between levels y_s and y_{s+1} has only welded crossings in other words it is a permutation τ say. We will endevour to reduce the number of welds in τ to a minimum.

Let x be the number of strings which are involved with ς_i and ς_j. The integer x can take the values $2, 3$ or 4. By allowable moves that do not essentially alter the function L we can assume that the remaining $n - x$ strings do not have any welded crossings in the interval $[y_s, y_{s+1}]$. For example figures 6 and 8 show how to move a welded string past an unwelded crossing.

Now we consider the three cases $x = 2, 3, 4$. We will assume that we have used allowable the moves to minimize the number of welds in the interval $[y_s, y_{s+1}]$.

- **Case** $x = 2$

The maximum number of welds occuring now is 1. The possible cases up to a simple symmetry are displayed in figure 10. The remaining $n - 2$ strings are not shown.

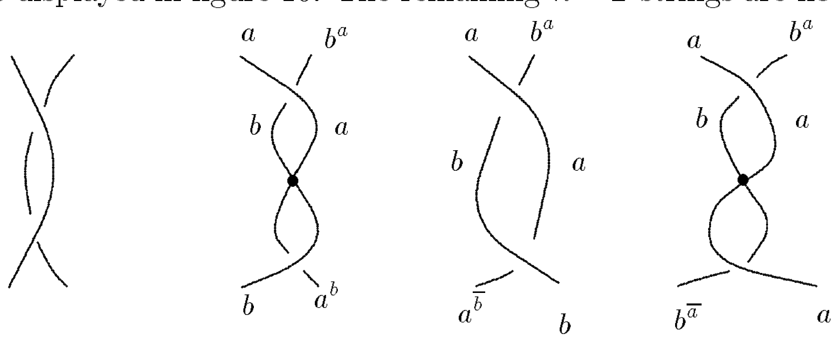


Figure 10

The first case can easily be simplified. It is not difficult to show with the help of the lemmas above that the other three cases can not occur if the interval defines a maximum of the function L. For example the second picture above gives the conditions $a \prec b$ and $b \prec a$ which contradicts lemma 4.6 (1) (see the notation of the picture).

- **Case** $x = 3$

The number of possibilities to be considered can be reduced up to a simple symmetry to the following 9 illustrated in figure 11 below.

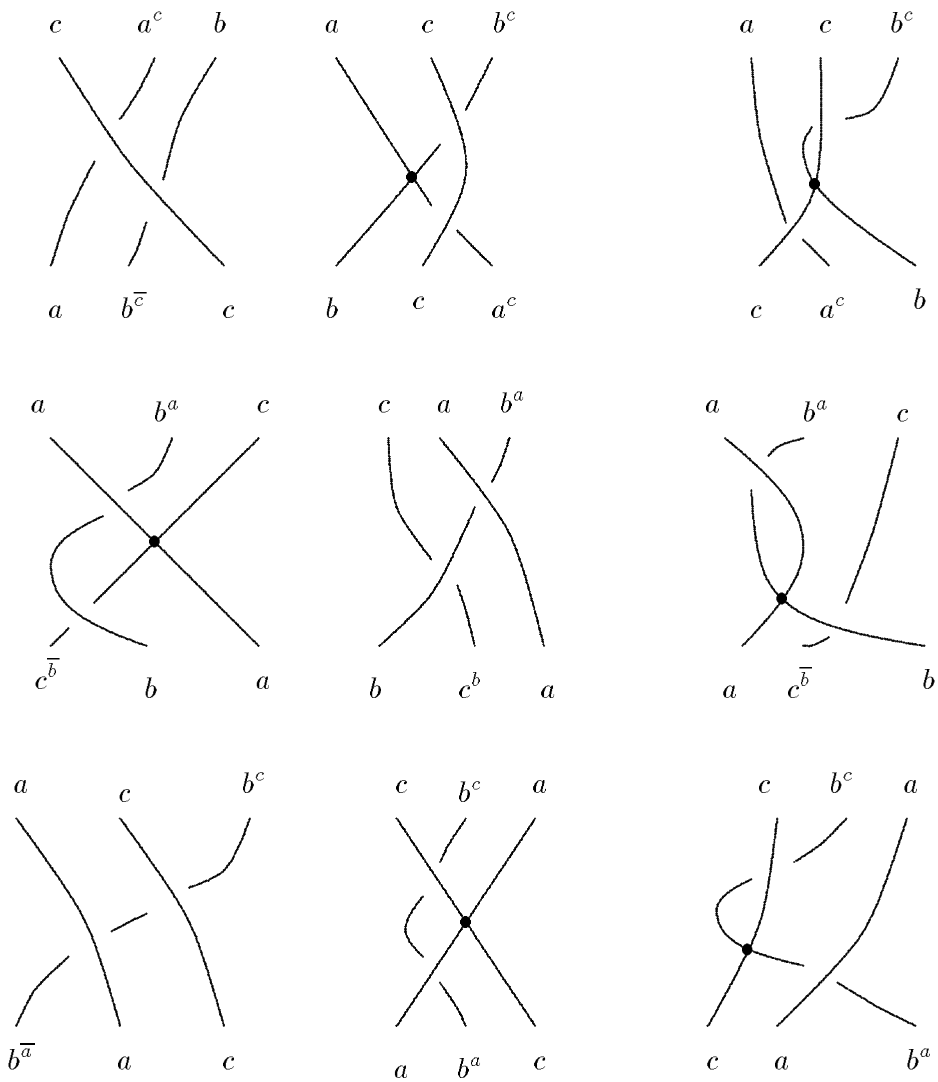


figure 11

These can be grouped into three sets of 3 by the following description. The first 3 occur where the overcrossing string at the bottom is an overcrossing string at the top. For the next 3 the overcrossing string at the bottom is an undercrossing string at the top and for the last 3 the undercrossing string at the bottom is the undercrossing string at the top.

The fourth possibility is that the undercrossing string at the bottom is an overcrossing string at the top. However this becomes the middle case by reflection in a horizontal line.

We now change the first six cases by allowable moves so that they look like the following illustrated in figure 12 below.

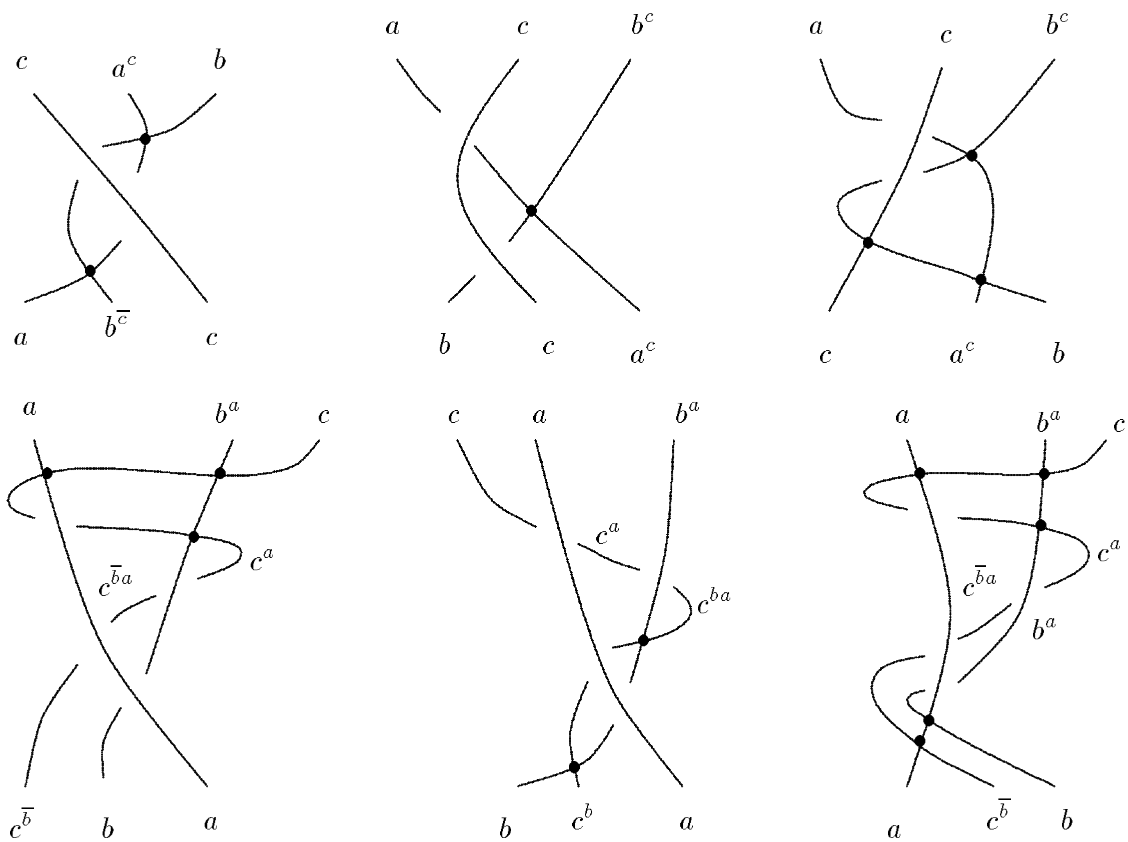


Figure 12

The relevant change to the last three depends on the length of a and c. If the lengths $L(a)$ and $L(c)$ are equal then Lemma 4.4 gives a contradiction. We give the appropriate changes if $L(a) < L(c)$ in figure 13. The reader can easily construct the changes in the opposite case.

After these changes have been completed it only remains to check that the value of L has been decreased in the interval. This can be done using the above lemmas and may safely be left to the reader. For convenience we give the notation in the pictures so that they fit the notations of lemmas 4.1–4.6.

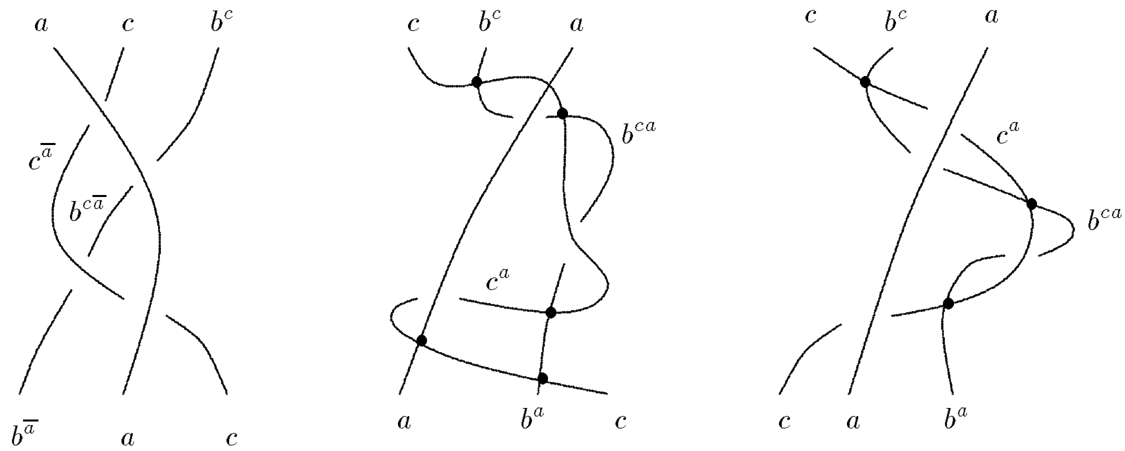


Figure 13

- **Case** $x = 4$

By minimality there is only one possibility which is illustrated in figure 14 below.

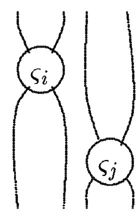


Figure 14

We change the situation by interchanging the heights of ς_i and ς_j. This has the curious effect of decreasing the value of L in the interval as the reader may verify.

We now repeat this argument until the value of L is constant (which must be zero). This implies that all crossings are welded and therefore the induced automorphism is a permutation. Since this is the identity permutation we can reduce the welded braid to the identity by a sequence of the two right hand moves in figure 5.

This completes the proof of the main theorem. $\square$

5. Racks and Quandles

In this section we turn to the theory which originally prompted our interest in the subgroups of $\operatorname{Aut} F_n$ studied in the previous sections. We have throughout written a^b for the conjugate $b^{-1}ab$ in a group. This leads naturally to the binary operation of a rack which can be thought of as generalised conjugation. A rack or quandle is an algebraic gadget intimately associated with a knot or link. Racks and quandles have been defined by many authors. We shall recap the theory briefly and refer to [FR] for full details.

Note that *from now on exponential notation is no longer used as a shorthand for conjugation in groups.*

Basic definitions

A **rack** R is a set with a binary operation. This operation — which we will write exponentially — is subject to the following axioms:

(i) for all b and c in R there exists a unique $a \in R$ such that

$$a^b = c$$

(the element a will be denoted $a = c^{\overline{b}}$)

(ii) for all $a, b, c \in R$

$$a^{bc} = a^{cb^c}$$

where we have adopted the usual conventions for an operation written exponentially: $x^{yz} := (x^y)^z$ and $x^{y^z} := x^{(y^z)}$.

If Q is a rack and satisfies the additional axiom

(iii) $a^a = a$ for all $a \in Q$

then it is called a **quandle**.

Notice that rule (ii) (the **rack law**) has exactly the same form as property 3.1 (of conjugation) and indeed the conjugation operation in a group defines a quandle. The hyperplane reflections associated with a realisation of a Coxeter group gives an example of a rack which is not a quandle. For more examples and theory see [FR].

As usual in universal algebra the concept of free rack and free quandle can be defined. Free racks and quandles $FQ(S)$ and $FR(S)$ on a set S are constructed as follows. $FR(S)$ is the set $S \times F(S) = \{(a, w) = a^w | a \in S, w \in F(S)\}$ with the operation defined by:

$$(a^w)^{b^u} := a^{w\overline{u}bu}$$

$FQ(S)$ is $FR(S)$ modulo the equivalence generated by $a^a = a$ for all $a \in S$. It follows that the elements of $FQ(S)$ are equivalence classes of the elements $(a, w) = a^w$ under the equivalence given by $a^w \sim a^{a^n w}$.
If S is the finite set $\{x_1, \ldots, x_n\}$ then we will use the notation

$$FQ_n = FQ(\{x_1, \ldots, x_n\})$$

for the free quandle on $\{x_1, \ldots, x_n\}$ and

$$FR_n = FR(\{x_1, \ldots, x_n\})$$

for the free rack on $\{x_1, \ldots, x_n\}$.

Let $\partial : FQ_n \to F_n$ from the free quandle to the free group be defined by $\partial(a^w) = w^{-1}aw$. Then $\partial : FQ_n \to F_n$ is injective and any automorphism $\phi : FQ_n \to FQ_n$ induces an automorphism $\phi_\sharp : F_n \to F_n$ such that the diagram below commutes.

$$\begin{tikzcd} FQ_n \arrow[r, "\phi"] \arrow[d, "\partial"'] & FQ_n \arrow[d, "\partial"] \\ F_n \arrow[r, "\phi_\sharp"] & F_n \end{tikzcd}$$

The correspondence $\phi \to \phi_\sharp$ embeds $\mathrm{Aut}\, FQ_n$ as a subgroup of $\mathrm{Aut}\, F_n$.

It is known that $\mathrm{Aut}\, FQ_n$ is isomorphic to BP_n, see [FR]. We can see this using the methods of section 2 as follows. Let $\phi : FQ_n \to FQ_n$ be an automorphism. Suppose $\phi(x_i) = x_{\pi(i)}^{w_i}$. Then the set $\{x_{\pi(i)}^{w_i}\}$ is of PC–type and generates F_n.

However such sets $\{x_{\pi(i)}^{w_i}\}$ determine PC_n and conversely. So $\mathrm{Aut}\, FQ_n$ is isomorphic to PC_n but we already know that PC_n is BP_n.

Now consider the group $\mathrm{Aut}\, FR_n$. We will see shortly that this is a wreath product of $\mathrm{Aut}\, FQ_n$ with the integers.

Let σ_i and τ_i where $i = 1, 2, \ldots, n-1$ be the elements of $\mathrm{Aut}\, FQ_n$ and $\mathrm{Aut}\, FR_n$ which are defined in an exactly similar way to their namesakes in $\mathrm{Aut}\, F_n$. The **elementary automorphisms** $p_{i,k}$ and s_π of FQ_n and FR_n are also defined in exactly the same way as for F_n, see section 2. The methods of section 2 show that these elementary automorphism generate the two automorphism groups (see also the appendix to [FR]).

Now let ρ_i, where $i = 1, 2, \ldots, n$, be the elements of $\mathrm{Aut}\, FR_n$ defined by

$$\begin{cases} x_i & \mapsto & x_i^{x_i} \\ x_j & \mapsto & x_j \quad j \neq i. \end{cases}$$

In other words ρ_i is $p_{i,i}$. It is easily seen that

$$\rho_i \sigma_i = \sigma_i \rho_{i+1},\ \rho_i \tau_i = \tau_i \rho_{i+1} \text{ and } \rho_i \rho_j = \rho_j \rho_i.$$

Let $\eta \colon \mathrm{Aut} FR_n \to \mathrm{Aut} FQ_n$ be the natural map. Then the kernel of η is R the subgroup of $\mathrm{Aut}\, FR_n$ generated by the ρ_i. Clearly R is isomorphic to the lattice group $\mathbb{Z}^n$. Thus $\mathrm{Aut}\, FR_n$ is a semi-direct product of $\mathrm{Aut}\, FQ_n$ with R; moreover the action on R permutes the factors, in other words $\mathrm{Aut}\, FR_n$ is the permutation wreath product of $\mathrm{Aut}\, FQ_n$ with $\mathbb{Z}$.

In terms of presentations we have the following.

5.1 Theorem *The group $AutFQ_n$ has a finite presentation of the following form. The generators are σ_i and τ_i where $i = 1, 2, \ldots, n-1$.*

The relations are

$$\begin{cases} \sigma_i \sigma_j & = \sigma_j \sigma_i \quad |i-j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i & = \sigma_{i+1} \sigma_i \sigma_{i+1} \end{cases}$$

The braid relations

$$\begin{cases} \tau_i^2 & = 1 \\ \tau_i \tau_j & = \tau_j \tau_i \quad |i-j| > 1 \\ \tau_i \tau_{i+1} \tau_i & = \tau_{i+1} \tau_i \tau_{i+1} \end{cases}$$

The permutation group relations

$$\begin{cases} \sigma_i \tau_j & = \tau_j \sigma_i \quad |i-j| > 1 \\ \tau_i \tau_{i+1} \sigma_i & = \sigma_{i+1} \tau_i \tau_{i+1} \\ \sigma_i \sigma_{i+1} \tau_i & = \tau_{i+1} \sigma_i \sigma_{i+1} \end{cases}$$

The mixed relations for $AutFQ_n$

The group $AutFR_n$ *has the same generators and relations but in addition the generators* ρ_i, *where* $i = 1, 2, \ldots, n$ *and relations*

$$\rho_i\rho_j = \rho_j\rho_i.$$

The ρ_i *commute*

$$\begin{cases} \rho_i\sigma_i & = \sigma_i\rho_{i+1} \\ \rho_i\tau_i & = \tau_i\rho_{i+1} \end{cases}$$

The mixed relations for $AutFR_n$

□

Final remarks The theory of welded braids developed in this paper has a corresponding theory of welded knots and links. For example gluing the top of a welded braid to the bottom yields a welded link in half 3–space. This can be generalised to yield a theory of welded links in any 3–manifold with boundary. This theory is important because any such link has a fundamental rack: for a glued braid this can be calculated by quotienting the free rack by the automorphism which the welded braid determines. (This automorphism can be computed from the welded braid by the same method as we gave in section 3 for computing the induced automorphism of the free group.) The theory of racks can then be applied to yield invariants for these generalised links. We shall investigate this further in a subsequent paper.

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