XVII.—*The Enumeration, Description, and Construction of Knots of Fewer than Ten Crossings.* By Rev. Thomas P. Kirkman, M.A., F.R.S. (Plates XL.–XLIII.)

(Read June 2, 1884.)

1. By a knot of n crossings, I understand a reticulation of any number of meshes of two or more edges, whose summits, all tessaraces (ἀκτή), are each a single crossing, as when you cross your forefingers straight or slightly curved, so as not to link them, and such meshes that every thread is either seen, when the projection of the knot with its n crossings and no more is drawn in double lines, or conceived by the reader of its course when drawn in single line, to pass alternately under and over the threads to which it comes at successive crossings.

The rule for reading such a reticulation of single lines meeting in tessaraces only is this—Coming by the edge or thread pq to the tessarace q, you leave it by the edge of q which makes no angle with pq, nor is part of thread under or over which you pass at q.

2. It is not necessary, after what Professor Tait has written on knots, to prove that every reticulation having only tessarace summits, whether polyedron* or not, if it be one continuous figure and projected to show all its crossings and no more, can be read all through by this alternate under and over, so that all its closed circles, one or more, can be written down in numbered summits, and that the knot can be labelled as unifilar, bifilar, or trifilar, &c.

3. If a thread a of a knot, after passing under or over a thread b, passes over or under b before it meets a third thread c, there is a linkage of two crossings and a *flap* between them. This flap is the small eyelet seen between two links of a slack chain as it lies on the table: it is a 2-gon, a mesh of two edges and of two crossings. Here see art. 9.

4. In our enumeration of knots of n crossings, two, C and C′, are counted as the same one, whenever and only when, in the number, polygonal rank, and order of their meshes C′ is either the exact repetition or the mirrored image of C; and I consider the threads of all the circles of a knot to be tape untwisted.

5. Nothing general seems to have been written upon knots of more than seven crossings. Nor, fortunately for the claims of those knots upon the inte-

* I hope to be pardoned for omitting the *h*. It annoys me to hear the learned say *polyhēdron*. Why not *perihodic* also? or, more learnedly, *perihōdic*?

rested attention of the student, has unanimity been even so far secured. In what LISTING and TAIT have contributed to this theory, there is an affirmation of identity, or of equivalence, denoted by the symbol $=$, between two knots of seven crossings which, by the above definition of sameness, are as unlike as they can be. This, of course, is due to a mere difference in definitions. The right definition in the view of LISTING and TAIT I find not easy to seize, and I cannot work on reticulations between equivalence and identity, nor pause to consider the deformations of a knot A into an equivalent knot B that can be effected by twisting the tape of A. I content myself by exhausting the forms that differ according to my definition; and I leave to a more competent hand the reductions to be made by twisting.

6. The reader will judge for himself whether the number of different unifilar knots of seven crossings is twelve, as I am compelled to believe, or at most eight, as TAIT prefers to say. Whatever be the decision of the reader, I am highly delighted, while attempting to write on a theme so dry and tiresome, that we have, at the outset, such a pretty little quarrel as it stands wherewith to allure his attention.

7. Every polyedron which is an n-acron having only tessarace summits is a *solid* knot of n crossings, on which is neither linkage nor flap. Such a knot can be projected on any of its triangular or m-gonal faces, so as to show all its n crossings, and no more, within that face or at its summits. It has no linear section, *i.e.*, no plane can cut it in space, nor can any closed curve be drawn upon lines of its projection, without meeting it in more than two points, on edge or at crossing.

In a knot which is no polyedron, we call a section that meets it in only two projected summits or crossings, *linear*, as passing through two points only of the figure; although no crossing of a mesh in space can be cut through its opposite angles without making four ends.

8. No projected knot, solid or unsolid, can have a linear section through one edge kl only, and through one crossing q only. For, if it can, let the two threads at q on the left hand of the section be pq and rq, pq passing over rq at q; then pq at the crossing p passed under a thread, and rq at r passed over one. Cut at q, making four ends; reunite into one thread rp the two ends on the left; this shortened thread passes over a thread at r and under one at p, as before, and its further course is unaltered in either direction. The same is true of the shortened thread made by reuniting the two threads on the right of the section.

The figure is now the projection of a knot of $n-1$ crossings having two portions, L on the left and R on the right, which are connected by a single thread kl, the law of under and over being observed in both L and R. The course of the thread kl, pursued along its circle small or great through R from k upon R, must bring it back to l upon L through L. But this is clearly impossible from want of a second connecting thread; which proves the proposition.

9. *Knots Solid, Subsolid, and Unsolid.*—A solid knot is a polyedron, art. 7.

Subsolid knots admit of no linear section but through the two crossings of a flap; through the projections of these a closed curve can be drawn meeting the knot in no third point.

The projections of an unsolid knot admit one or more linear sections either through two crossings not on a flap, or through two edges coming from two crossings *not on the same flap.*

If a projected unsolid V admits a linear section through two edges coming from two crossings on the same flap, V is made up of two portions, K and L, connected like two links of an ordinary chain, so that K (or L) can be set free in its completeness by breaking only one thread of the other. No knot V constructed in these pages is such a compound of K and L. Such a compound is easily drawn; in such a V either K or L can be slipped along the thread of the other, without twisting a tape, so as to occupy, if the other be unifilar, any position upon it. All our unsolids are composite; but no severing of a single thread will ever set free on one of them a portion which is a complete knot.

Of solid knots we are not treating. If the apparent dignity of knots so maintains itself as to make a treatise on these n-acra desirable, it will be no difficult thing to show in a future memoir how to enumerate and construct them to any value of n without omission or repetition. The beginner can amuse himself with the regular 8-edron, which is trifilar, or with the unifilar of eight crossings made by drawing within a square a square askew, and filling up with eight triangles.

10. I consider a knot as given by its projection upon and within any one, 2-gonal or m-gonal, of its meshes drawn large, and as having the symmetry of that projection. Nor do I trouble myself with inquiring how far that symmetry is affected by the law of under and over at the crossings, because, in reading the circle or circles of a projected knot, we can take any crossing q as our first, and can on beginning to read take either of the threads at q as passing under and over the other. A knot in space can be read only as given.

In my description of the symmetry of our reticulations, I shall assume that the reader understands the terms employed. They, with others not wanted here, are necessary and sufficient; they are the only such terms that ever have been proposed; and, for more than twenty years since they were introduced, no more suitable terms have offered in their stead. I am quite ready to use better ones when they are invented. The symmetry, however, of the figures handled in this paper is of itself so evident that the reader will easily satisfy himself, without debate about the terms employed, as to the truth or error of my enumerations.

11. All that we need to add here is on the symmetry of flaps, which are 2-gons, correctly drawn as two curved lines through the same two points.

A flap has the symmetry of its undrawn diagonal d through its two crossings, and may be conceived as standing symmetrically about d, in either of two planes at right angles to each other, which contain d. This d may be asymmetric, or epizonal, or zonal, or zoned polar, or zoneless polar; and the flap is accordingly asymmetric, or epizonal, &c. The two edges of a flap are unlike only when it is asymmetric or epizonal.

In a zonal flap, a single zonal trace passes through the two crossings; in an epizonal flap, a single zone passes between the crossings. In a zoned polar flap, two zonal traces intersect in the centre, the termination of the 2-zoned axis.

In the centre of a zoneless polar flap, an axis of 2-ple repetition terminates.

$_3$A and $_4$A have each one zoned polar flap.
$_5$A has one epizonal flap.
$_6$F has one epizonal and one asymmetric flap.
$_6$G has two different polar, and one zonal flap.
$_6$H has one zonal flap and no other.
$_8$S and $_8$Q have each one zoneless polar and one asymmetric flap; so has $_8$Aj.
$_8$Ak has one asymmetric flap and no more.
$_8$Am has one zoned polar flap and none other—art. 45.
$_8$Bn has one epizonal, two zonal, and one asymmetric flap.
$_8$Bq has one zoned polar and one asymmetric flap.
$_8$Bu has three zonal flaps only.
$_8$Bv and $_8$Bw have each one epizonal and two asymmetric flaps.
$_8$By has one zoned polar and one zonal flap.

12. The construction of polyedra and of other reticulations of n summits is best apprehended by studying their reduction by fixed rules to antecedents or bases of $n-i$ summits. We proceed to the reduction of knots.

Reduction of an Unsolid Knot Q *of* n *Crossings.*

In this are two processes,—the clearing away of concurrences, and the removal of least marginal subsolids.

13. *Concurrences.*—Two or more continuous flaps, each having a crossing common to the next, are a concurrence of two or more, except when two or three flaps are collateral with the same triangular mesh. Three such flaps on a triangle complete the irreducible subsolid $_3$A; and two on a triangle are not counted a concurrence.

When a concurrence of n stands about the $(n+k+2)$-gonal mesh F, their common collateral, F is reduced by deletion of $n-1$ flaps, each conceived to vanish by the union of its two summits, to the $(k+3)$-gon F′, carrying one only of those flaps.

If their collateral F is an $(n+1)$-gon, it is reduced by the vanishing of $n-2$ of the concurrent flaps to a triangle f carrying the remaining two. This f cannot lose a flap without losing an edge and disappearing.

Every concurrence on the unsolid Q of art. 12, whether standing on a marginal or non-marginal component of Q, is to be thus reduced, and Q now becomes an unsolid Q′ without a concurrence, of $n-i$ crossings.

14. *Least Marginal Sections and Least Marginal Subsolids.*—Our unsolid Q′, obtained by deletion of i flaps, has one or more *linear sections*, marginal or not.

By a marginal linear section of Q′ can be cut away one and only one marginal subsolid, on which (art. 9) lies no linear section except through the two crossings of a flap.

By a *least marginal section* of Q′ can be cut away a *least marginal subsolid.*

A marginal subsolid of k crossings all untouched by any possible least section, is a least marginal, when no marginal subsolid of fewer than k crossings untouched by the section can be cut away from Q′ by any kind of section.

15. *The Five Kinds of Linear Section of an Unsolid Knot.*—These are,

ffc,	which is	read,	flap on flap close;
ff,	„	„	flap on flap;
fe,	„	„	flap on edge;
ef,	„	„	edge on flap;
ee,	„	„	edge on edge.

The first letter in these symbols denotes the flap or edge of the least marginal subsolid cut away from, or in construction imposed on, the unsolid or subsolid charged.

16. The linear section *ffc* is the only one that passes through two crossings. After making the section *ffc*, there is a pair of truncated crossings, both on the diminished Q′ and on the subsolid removed. These are completed into tessaraces in the severed portions by replacing each portion on the other by the two pairs of edges of two flaps, at the cost of which two flaps the portions were, in construction at the section *ffc*, united to form Q′.

These pairs are conceived to be so united to the broken threads at the truncated crossings as to complete both the severed portions into two knots. This can always be done, and needs not trouble us, when we are reducing a projected figure of simple lines making tessaraces only.

If the subsolid S removed at this section *ffc* has, when completed by its restored flap, k crossings, the $n-i$ crossings of the undiminished Q′ are made fewer by $k-2$; for Q′ has lost only the crossings of S not on the section.

This S in construction of Q′ was counted as a least marginal charge of $k-2$ crossings. Two crossings are always lost when in construction a subsolid having k crossings is imposed at a section *ffc*.

In reduction at this section *ffc*, this S is a least marginal subsolid of $k-2$ crossings along with others of $k-2$ removed by any section *ff*, *fe*, &c.

17. Our unreduced Q′ of $n-i$ crossings may have in it every kind of least marginal section, art. 15. Such section, not *ffc*, always cuts two edges, making four ends. In every case, *ff*, *fe*, *ef*, or *ee*, those pairs of ends are conceived to be united on either hand into one edge, by which is restored the half-flap or the edge cut away in each portion when united by construction, in order that every summit should be a tessarace in the completed unsolid.

The imposition of a least marginal charge by *ffc* costs four edges of two flaps; by *ff*, *fe*, *ef*, or *ee* it costs only two edges, one on each of the united knots.

No crossing is lost when a charge is imposed by a section *ff*, *fe*, *ef*, or *ee*. All this will be found very clear and easy when we come to constructions, and examples will abound.

18. In this reduction of Q′ all least marginal subsolids, say of k crossings, are to be removed without regard to the number of their meshes, which may differ while each has added k crossings. And care is to be taken that none is cut away which has been loaded with another either on flap or edge, and thus made non-marginal.

When our Q′ is thus reduced, it has become Q_1 an unsolid of $n-j$ crossings $(j>i)$.

This Q_1 will in general have one or more concurrences due to the flaps substituted for subsolids removed. All these are to be cleared away (art. 13), whereby Q_1 becomes Q_1', an unsolid without a concurrence; and Q_1' is to be treated as we treated Q′ in art. 14.

We shall finally arrive by these reductions either at an unsolid of two portions, neither of which is least, which is to be reduced by a final section to two subsolids each of c crossings; or at a ring of flaps which is reducible to the fundamental ${}_3A$; or to a nucleus subsolid or solid knot. We have now to set about the reduction of subsolids.

19. *Reduction of a Subsolid of* n *Crossings by its Leading Flap.*—The rule is—Remove both edges of a leading flap, or of a leading flap when there are co-leaders. By this removal, the two meshes covertical with the deleted flaps lose each a crossing; and if one or both coverticals are triangles, that one or both become flaps. The result obtained is a subsolid of $n-2$ crossings.

20. Every flap can be written AB,CD, where A and B $(A \overline{\gtrless} B)$ are collaterals, and C and D $(C \overline{\gtrless} D)$ are coverticals, of the flap.

We compare first the collaterals of the flaps whose leader is to be found.

If A_1B_1, A_2B_2, &c., are the pairs of collaterals, the leader has the greatest A, no matter what be the coverticals. If several flaps have the greatest A, the leader has the greatest B. If several have both A and B greatest, we compare coverticals. If one has the greatest C, it leads; if more than one, the greatest D gives the lead. If no leader can thus be determined, we have to examine the collaterals of the A's. The leader has more than any other of the greatest of these collaterals; and so on we go over the collaterals of the B's, the C's, and the D's. The leader, if there is only one, is certain to be found.

I have never had occasion to examine the collaterals of AB,CD. If two competitors have these all equal, it is almost a certainty that there is symmetry, and no leader, but a set of co-leaders. Where there is no symmetry, no two edges or flaps on a knot are alike.

It suffices, after writing two or more flaps as equally claiming by their AB, CD to lead, to place a note of interrogation, and to examine the symmetry, which readily betrays itself. The deletion of any one of the co-leaders completes the reduction.

A flap can neither be removed from a knot nor added to it without cutting of threads and reunion of ends. But this does not trouble us here, as by art. 2 we know that every projection making tessaraces only is a true knot, that its circles can be read by the rule of under and over, and that the threads of all the circles can be drawn in double lines as narrow untwisted tapes visibly passing under and over at alternate crossings.

21. In the converse problem of construction, the question is, in how many ways to add, on a knot P' of $n-2$ crossings, a leading flap, so as to construct without risk of repetition a subsolid of n crossings. The note of interrogation written after the comparison of two flaps that can be drawn across two meshes of P' is a presumption of symmetry, which is pretty certain to be verified when we come to draw in turn our new flaps on P', and to examine the constructed P of n crossings as to the leadership of the doubted flap which turns P' into P.

22. Two things are to be noted here, both in reduction and construction. If a flap f on any subsolid is covertical at its crossing a with a triangular mesh abc, which carries a flap f' on the edge bc, since abc cannot lose a crossing by the deletion of f, it thereby becomes itself a flap f'' collateral with the flap f'. Now a pair of collateral flaps is excluded from all our constructions, because it is a circle of two crossings, whose projection represents nothing in space but a movable ring through which one thread once passes. Wherefore the flap f is indelible or a *fixed flap*. It cannot be removed, nor be a competitor for the lead, either in reduction or construction.

When two flaps are collateral with the same triangular mesh, both are *fixed*; for the deletion of either leaves the other hanging by a nugatory crossing which admits a forbidden punctual section. The reader can easily verify this.

By continuing this reduction of a subsolid by removal of the leading flap, we must at last arrive either at a solid knot, or at one of the two irreducibles ${}_3$A and ${}_4$A, of three and of four crossings.

23. *Construction of Knots of* n *Crossings.*—The rules for this are the exact converse of those above given for reduction.

First, to construct the *subsolids* of n crossings, all inferior knots being given with their symmetry, we have in the first place to take in turn every subsolid P′ of $n-2$ crossings, and to determine before we draw them the different leading flaps that can be added on P′. Knowing its symmetry, we can write down and mark on its edges every different pair of points on flap or edge that can be joined as the crossings of a new flap, and also the collaterals and coverticals which this will have. We make a table of the possible leading flaps, with the notes of interrogation that presume symmetry in the P of n crossings to be built on P′. Next we draw the leading flaps, thus constructing and registering the resulting subsolids P.

A caution is required here, for the examination of the claim of a new flap, $ab=3$M, to leadership; a and b being the crossings of the flap, when one of its collaterals is a triangle abc. If c in this triangle is the crossing of a flap cd, cd becomes fixed (art. 22), for it is covertical with a triangle cab, which carries a flap on its edge ab. Care must be taken to exclude this flap cd from claim to leadership over ab.

I was caught in this trap in the art. 41, for I had entered the flap (bd) as led by (56), and thus missed the unifilar ${}_8$G. Professor TAIT found this knot, adding one to my first list of 8-fold knots. He first found also the unifilars ${}_9$Al, ${}_9$Ak, and ${}_9$Bg, omitted by a like error in arts. 51 and 56. He also first found ${}_9$Dm, which I ought to have constructed along with ${}_9$Dl in art. 61.

24. In the second place, we take in turn every unsolid P″ of $n-2$ crossings on which a leading flap or flaps can be drawn so as to abolish all concurrences, and to block linear section. Such leading flaps will be few. Next we draw them all, and thus complete without omission or repetition our list of subsolids P of n crossings. This list is the only difficulty of our work; what follows is for every value of n all easy routine, as we shall see; but it soon becomes too tedious by the enormous number of results to be registered and figured.

25. Next, to construct the *unsolids* of n crossings which have no concurrence, we impose in the first place on the solids and subsolids, and in the second on the unsolids, of $n-i$ crossings, each taken in turn as the subject Q′ to be charged, e charges of least marginal subsolids, all of k crossings, no matter what be the number of their meshes, so as to add to Q′ $ek=i$ crossings, completing an unsolid Q of n crossings without a concurrence.

The e charges may be all or none alike, or all but one alike, &c.; and from our list of subsolids of k crossings must be selected with or without repetition every possible set of e charges. These, as well as their reflected images when required (although such images are neither registered or figured by us), have to be imposed in every different posture, by every kind of possible section, *ffc*, *ff*, &c., on every different set of e flaps or edges that can be selected on the subject knot, and in every different order that symmetry permits without repetition of results (art. 4), so that when the work is done not one of the unsolids Q of n crossings shall have a least marginal subsolid besides the e that we have just imposed, nor have a concurrence upon it.

26. The linear sections by which the charges are imposed may be any of the five of art. 15. But, observe, when we use the section *ffc*, we are to select the charge from our list of subsolids of $k+2$ crossings; because (art. 16) two will be lost. For other sections our selection of the charges will be from those of k crossings.

27. The number of different postures in which a charge can be imposed on Q′ depends on the symmetry of the united portions of the Q completed by the union. Let ϵ denote the edge or flap of the subject and ϵ' that of the charge in all the five sections *ffc*, *ff*, *fe*, *ef*, *ee*. The rules are three—

(1) If one or both of ϵ and ϵ' be zoned polar, only one configuration is possible by the union; no second and different (art. 4) can be formed by turning the charge C through two right angles about ϵ', nor by using C′, the reflected image of C, when C is not C′. Every knot on which is a zone is its own mirrored image.

(2) If neither ϵ nor ϵ' be zoned polar, and if they are not both asymmetric, two and two only different configurations can be made by the above variation of posture of the charge.

(3) If both ϵ and ϵ' are asymmetric, four different configurations can and must be made and registered, due to such variation.

No more results can be obtained by putting for Q′ the reflected image of Q′: nothing is so attainable but repetitions or reflected images of knots already registered.

On almost every subject Q′ and charge C, though having any symmetry, may be found asymmetric, *i.e.*, zoneless and non-polar, flaps and edges, which are to be dealt with by the above rules.

28. The subject Q′ to be charged with a set of least marginal subsolids may have or not have concurrences. All that is required in order that the completed Q shall have no concurrence, is that our number e of charges of k crossings shall be large enough to spoil all concurrences on Q′, as well as to cover at least once every marginal subsolid on Q′ which has fewer than $k+1$ crossings.

In the constructions of this paper, Q and C are one or both symmetric. When asymmetrics come to be handled both as subject and charge, the number of results becomes unmanageably vast long before n the number of crossings is out of its *teens.*

29. The *final operation*, after construction of all knots of n crossings without concurrences, is to take every subsolid and unsolid R in our lists which has $n-c$ crossings and no concurrence, and to add to it in every possible different way c flaps making with one or more on R concurrences of every possible number of two or more flaps, thereby adding c crossings, and completing the number n. This last operation soon becomes impracticable from the number of results.

30. Nothing can be here added that will give so much insight into our subject as the actual construction of knots, to which we now proceed, first to that of subsolids, and next to that of unsolids of the number n in hand of crossings.

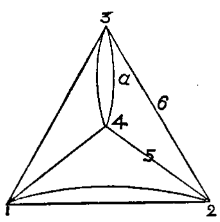
Two Fundamental Subsolid Knots.—The only subsolids that cannot be reduced by deletion of a leading flap (art. 19), are those of three and of four crossings. These, $_3$A and $_4$A(*vide* Plate XL.), are irreducible and fundamental. $_3$A is a 3-zoned monarchaxine, whose principal poles are triangles not plane, which have three common summits and no common edge.

The unsolid $_4$B is formed on $_3$A by art. 29, and has a symmetry of like description. The secondary 2-zoned poles on either are alternately flaps and crossings, being heteroid poles in $_3$A and janal in $_4$B.

31. *Subsolid and Unsolids of Five Crossings.*—The subsolid must be built on $_3$A. The only points that can be here joined by a flap, are either on one flap of $_3$A or on two. We cannot obtain a subsolid by joining the former pair, because the constructed knot would be an unsolid having a concurrence of two (art 13). We join the latter pair, and it matters not whether we draw our flap in the upper or in the lower of the two triangles whose summits are the same three crossings, and whose edges are different halves of the three flaps of $_3$A. Drawing the flap 54, the two flaps of $_3$A connected by it become triangles, and $_5$A is constructed, a 2-zoned monarchaxine heteroid, whose zoned poles are a tessarace and an opposite tetragon. This is the only subsolid of five crossings.

The unsolid $_5$B is by (29) formed on $_4$A, and $_5$C is made on $_3$A.

32. *Knots of Six Crossings.*—The subsolids $_6$A, &c., must be formed on $_4$A and $_4$B. This $_4$A has a janal 2-zoned axis through the centres of the flaps, and two like 2-ple janal zoneless axes through two pairs of opposite mid-edges. It has only one mesh, the monozone triangle 342, and the only pair of points that can be joined are $5a$ and 56.



Drawing $5a$, or rather conceiving it drawn, we write to determine the leading flap,

$$(5a)=43{,}43\,;\quad (12)=43{,}43\,;\quad (5a)>(12)\,?$$

This is read thus—the flap on $5a$ has for collaterals 3 and 4, and for coverticals 3 and 4; so has the flap on 12: which leads ?

Next, conceiving 56 drawn, we write,

$$(56)=43,44;\ (12)=44,43;\ (43)=43,44.$$

Here by art. 19 (12) appears to be leader, until we observe that it is fixed by art. 22, and cannot be a competitor.

We therefore write more correctly,

$$(56)=43,44>(43)=43,44\ ?\ (12) \text{ is fixed};$$

which inquires, Does (56), which is 43,44, lead (43), which is also 43,44? We consider this second as well as the preceding note of interrogation a presumption of symmetry (art. 21).

Drawing the flap ($5a$) we obtain ${}_6$A, and the flap (56) gives us ${}_6$B, on both of which the leading flap so drawn is marked 56. Observe that in our figured subsolids of n crossings, the leading flap is always marked $n(n-1)$. Our presumptions of symmetry are verified in the two results ${}_6$A and ${}_6$B.

The polar edges of the heteroid zoneless axis of ${}_6$A are evident in the figure. The two-zoned axis of ${}_6$B has for faces a flap and an opposite 4-gon. The other two flaps of ${}_6$B are like epizonals.

It was possible to connect by a flap the two edges of the flap 43 in ${}_4$A. But this could have completed an unsolid having a linear section through 3 and 4; and completed it wrong, because no unsolid is ever made by so adding a flap.

33. We next take the unsolid ${}_4$B, considering whether or no a flap can be drawn on it to make it a subsolid of six crossings. Readily we perceive that by joining two opposite flaps we can both spoil the concurrence and block the linear section. This gives us ${}_6$C, which has all the symmetry of the wedge which it becomes when an edge is removed from every flap. The three, ${}_6$A ${}_6$B and ${}_6$C, are all the subsolids of six crossings.

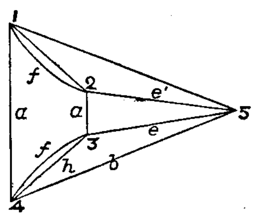
34. We seek now the unsolids of six crossings. To obtain them by least marginal charge or charges (art. 25), we have to lay 2 upon 4 and 3 upon 3.

There is but one charge that can add two crossings only, ${}_4$A*ffc*, which means ${}_4$A imposed (art. 15), by the section *ffc*. Imposing this on ${}_4$A we get ${}_6$D, a zoned triaxine, whose three janal 2-zoned axes have for poles, one two tessaraces, a second two flaps, and the third two 4-gons symmetrical but not plane, which have two common summits and no common edge.

Laying next on ${}_3$A the charge ${}_3$A*ff* (art. 15), we obtain ${}_6$E, another zoned triaxine, whose janal poles are two edges, two tessaraces, and two hexagons alike and non-planar. There is in truth no least marginal subsolid in either ${}_6$D or ${}_6$E, the two halves of the knot being identical in each. But it is instructive, and involves no error, here to consider them as cases of the linear sections *ffc* and *ff*.

35. We have constructed all the knots of six crossings that are without a concurrence, viz., ${}_6$A, ${}_6$B, ${}_6$C, ${}_6$D, ${}_6$E. Those having concurrences are obtained, ${}_6$F on ${}_5$A, ${}_6$G and ${}_6$H on ${}_4$A, and ${}_6$I on ${}_3$A. These nine, ${}_6$A . . . ${}_6$I, along with the solid knot ${}_6$J, are the ten possible knots of six crossings. Four of them, as TAIT has found and drawn them, are unifilar, viz., ${}_6$A, ${}_6$E, ${}_6$F, ${}_6$G; and this is read on the figures. The number 12 on each shows that there are 12 steps in the circle of the knot, which passes twice through every crossing, once over and once under. ${}_6$B, ${}_6$D, and ${}_6$H are bifilars; ${}_6$C and ${}_6$J are trifilars.

36. *Knots of Seven Crossings.*—The subsolids ${}_7$A, &c., must be built on ${}_5$A, &c. The only lines that can be drawn on ${}_5$A here given are *ff* and *aa*, each 44; and *af*, *ae'*, *ee'* *be*, *bh*, each 34.



By *ff*, which has no rival, we get ${}_7$A; whose 2-zoned poles are the flap and the tessarace 3333,

$$(aa)=44>(12), \text{ or } (34)=43;$$
$$(af)=53, 43>(12)=53, 43\ ?$$
$$(bh)=43, 43>(12)=43, 43\ ?$$

For the rest, *ae'*, *be*, and *ee'*,

$$(43)=53>(ae')=43;$$
$$(43)=44>(be)=43 \text{ and } >(ee')=43.$$

We have to draw, besides *ff*, the flaps (*aa*) (*af*) and (*bh*), expecting symmetry with the two last, which we soon find.

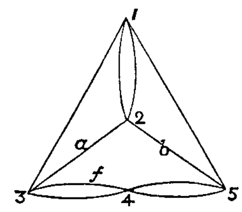
By (*aa*) we get ${}_7$B,

whose 2-zoned poles are this flap and a tessarace.

By (*af*) comes ${}_7$C, monozone;
By (*bh*) " ${}_7$D,

whose zoneless 2-ple poles are a 4-gon and a 4-ace. Thus there are four subsolids, ${}_7$A, ${}_7$B, ${}_7$C, ${}_7$D, reducible by the leading flap to ${}_5$A.

37. On ${}_5$B, annexed, as we cannot allow a concurrence, we can draw only (*af*) and (*bf*),



$$(af)=53>(12)=43, \text{ and } (45) \text{ is fixed (art. 22)}.$$
$$(bf)=44, 43>(45)=44, 43\ ?\ \big((12)=34\big),$$

for (45) is not fixed when (*bf*) is drawn.

We have to draw (*af*) and (*bf*) looking for symmetry in the latter.

(af) gives us ${}_7$E, asymmetric;
(bf) „ ${}_7$F, monozone.

Thus there are six subsolids, ${}_7$A ${}_7$B . . . ${}_7$F, of which we read on their figures that ${}_7$C, ${}_7$D, and ${}_7$E are unifilar.

38. To obtain the unsolids ${}_7$G, &c., without concurrences, we have to lay 2 upon 5, 3 upon 4, and 2 . 2 upon 3,

${}_4$A*ffc* on ${}_5$A gives ${}_7$G, monozone.

This is 2 upon 5. We cannot lay the same charge on ${}_5$B destroying the concurrence, without completing an unsolid having two marginal charges of which we have just imposed only one; which is forbidden (art. 25).

For 3 upon 4,

${}_3$A*ff* on ${}_4$A gives ${}_7$H:
${}_3$A*fe* on ${}_4$A gives ${}_7$I; *vide* the figures.

Observe that we can impose ${}_3$A only by the sections *ff* and *fe*; for it has no edge to lose at a section *ef* or *ee*; and if we attempt to lay it on a flap *h* by *ffc*, we merely turn *h* into a concurrence of two, which is not permitted as a result of any charging operation.

In ${}_7$H and ${}_7$I the 2-zoned and the zoneless 2-ple axes of ${}_4$A, after being loaded symmetrically by ${}_3$A, retain their repeating polarity, but from being janal have become heteroid, not janal.

For 2 . 2 upon 3, we lay on ${}_3$A the two charges ${}_4$A^2*ffc*, which stands for twice ${}_4$A*ffc*. The result is ${}_7$J, in which one flap is zoned polar, and two are epizonal. Thus there are four unsolids, ${}_7$G, ${}_7$H, ${}_7$I, ${}_7$J, without concurrence.

39. For unsolids, ${}_7$K, &c., having concurrences, we obtain on ${}_6$A, ${}_7$K; on ${}_6$B, ${}_7$L and ${}_7$M; on ${}_6$C, ${}_7$N; on ${}_6$D, ${}_7$P; on ${}_6$E, ${}_7$Q; on ${}_5$A ${}_7$R and ${}_7$S; on ${}_4$A, ${}_7$T and ${}_7$U; on ${}_3$A, ${}_7$V; eleven of them.

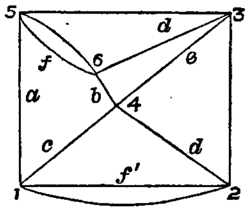
Thus we have 21 knots of seven crossings, of which 6 are subsolids and 15 are unsolids. Their symmetry and circles are to be read on their figures.

Twelve of them, ${}_7$C, ${}_7$D, ${}_7$E, ${}_7$G, ${}_7$H, ${}_7$I, ${}_7$L, ${}_7$P, ${}_7$S, ${}_7$T, ${}_7$U, ${}_7$V, are unifilar, of which all but ${}_7$I have been found and figured by TAIT. See Plate XV., *Trans. R.S.E.*, 1876–77, for his eleven figured unifilars, and his reduction of them to eight.

40. The meaning of the symbols *ffc*, *ff*, and *fe* is clear from the figures ${}_7$G, ${}_7$H, ${}_7$I. In reduction of ${}_7$G, after making the linear section, two flaps have to be restored. Also after section of the two parallels in ${}_7$H, the cut portions have to be united to make two flaps on the severed knots; and after section in ${}_7$I they have to be united to restore the flap on the charge ${}_3$A and the edge on ${}_4$A.

We shall see an example of the section *ef* in ${}_9$D*g*, and of *ee* in ${}_8$A*k* and ${}_9$D*i*; *vide* the figures.

41. *Construction of Knots of Eight Crossings.*—For the subsolids ${}_8A$, &c., we have to draw leading flaps on ${}_6A$, &c. In ${}_6A$, which is figured here, the 2-ple zoneless heteroid axis passes through a and e. The only faces are 5641, 634, and 563, and there is one flap which is asymmetric, having different edges marked f and f'. The only different lines that can be drawn are



fa, *fb*, *ca*, *cb*, each 35;
fc and *ab*, each 44; and
f'd, *f'c*, *be*, *bd*, *de*, *dc*, each 34.

The two *d*'s are the same point repeated in the repeated triangles 563 and 124.

$(fa) = 53, 53 > (12) = 53, 53$?
$(fb) = \quad 53 > (12) = 43$;
$(ca) = 53, 54 > (56) = 53, 53$; (12) is fixed (art. 22);
$(cb) = 53, 44 > (56) = 53, 44$?
$(fc) = 44, 43 > (12) = 44, 43$?
$(f'd) = 43, 43 > (12) = 43, 43$?
$(bd) = 43, 54 > (12) = 43, 53$; (56) is fixed (art. 23).

For the rest,

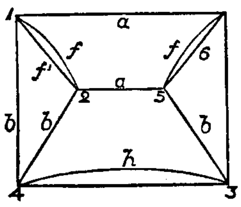
(12) leads (*ab*); and (56) leads (*f'c*) (*be*), (*de*) and (*dc*).

We have to draw seven flaps, expecting symmetry in four cases—

(*fa*) gives us ${}_8A$;		(*ca*) gives us ${}_8E$;	
(*fc*) " ${}_8B$;		(*cb*) " ${}_8F$;	
(*fb*) " ${}_8C$;		(*bd*) " ${}_8G$;	
(*f'd*) " ${}_8D$;			

whose symmetry and circles are read on their figures, where the zoneless poles on ${}_8A$, ${}_8B$, and ${}_8D$ are 55 and 33, 44 and 33, 44 and 44, and the leading flaps are marked 78.

42. We take next ${}_6B$, whose only faces are the 2-zoned polar 1256, and the monozone faces 4253 and 124. The only lines drawable are,



[The 6 should be at the base corner.]

fa, *ba*, *hb*, each 35,
ff, *aa*, *bb*, *ha*, each$=$44,
and *f'b*, *bb*, each$=$43.

Here (43) is fixed for every flap that we can draw, except (*ff*) and (*bb*), each$=$44.

$(ba) = 53, 54 > (56) = 53, 54$? (12) and (43) are fixed (arts. 22, 23).
$(hb) = \quad 53 > (12) = 44$ and $> (56) = 43$.
$(bb) = 44, 44 > (12)$ or (56) or (43) ? each$=$44, 44.
$(ff) = 44, 33 > (43) = 44, 33$?
$(aa) = 44 \quad > (12) = 43$ and $> (56) = 43$.

For the rest,

(56) leads (fa) and ($f'b$) from the flap (12),

and (12) leads (bb)=43.

We have to draw five flaps, presuming symmetry with (ba), (bb), and (ff).

(ff) gives us ${}_8$H,

whose three 2-zoned janal axes terminate in the centres of the zoned polar flaps and of two pairs of edges 44, 44; and 33, 33.

(hb) gives us ${}_8$I, asymmetric,

(bb) gives us ${}_8$J,

whose two janal zoned poles are 4-gons, the four like janal zoneless 2-ple poles being edges 44. We often mean by pole the polar face summit or edge in which an axis ends.

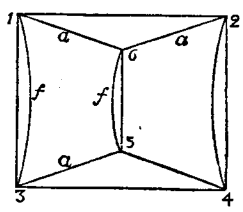
(aa) gives us ${}_8$K,

having two different zoned polar flaps:

(ba) gives us ${}_8$L,

whose zoneless poles are 44 and 55.

43. Our next base is ${}_6$C, which has one flap, one triangle, and one 4-gon. The only different lines that can be drawn are ff and aa, each 44, and fa and aa, each 43. The two f's are alike.



(ff) =44, 33 > (23)=43, 33 ?

(aa)=44 > (32) or (56) or (14) ? each =44;

not (fa)=53 > (14)=54;

nor (aa)=43 > (14)=54.

We have two flaps only to draw.

(ff) gives ${}_8$M,

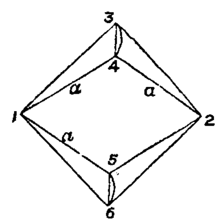
whose zoned poles are the flaps, the four like zoneless 2-ple poles being tessaraces.

(aa)=44 gives ${}_8$N,

whose eight janal secondary 2-zoned poles are alternately flaps and 4-gons.

44. We have no more subsolids of six crossings. Of the unsolids, we find only four, ${}_6$D, ${}_6$E, ${}_6$F, and ${}_6$G, on which a flap can be drawn to block linear section.

We have on ${}_6$D, in 1aa, and 14aa,



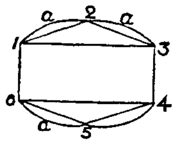
(aa)=53 > (34) and > (56), each=43.

(aa)=44 > (34) or (56) each=43.

By (aa)=53 we get ${}_8$P.

By (aa)=44 we get ${}_8$Q, *vide* the figures.

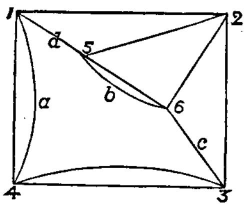
We have on $_6$E,



$(aa)=64>(23)$ or $(45)=63$.
$(aa)=55>(12)$ or $(45)=53$.

By $(aa)=64$ we get $_8$R, and
by $(aa)=55$ we get $_8$S.

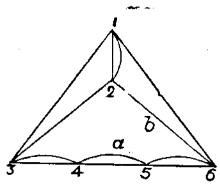
Upon $_6$F we see that (ab), (ad), (ac), are the only flaps that can block linear section and exclude concurrence,



$(ab)=54, 33>(43)=54, 33$?
$(ac)=54 \quad >(43)=44$,
not $(ad)=63 \quad >(43)=64$.

Drawing the flaps (ab) and (ac) we get $_8$T and $_8$U.

Upon $_6$G only (ab) can exclude concurrence. Here (34) is fixed (art. 22).



$(ab)=54>(56)=54$? $(12)=34$.

Drawing (ab) we obtain $_8$V symmetric as we expected, having three different flaps, one zonal, one epizonal, and one asymmetric.

One more subsolid, $_8$W, is built on $_6$J, which has only one edge and one angle.

We have constructed twenty-two subsolids of eight crossings, $_8$A . . . $_8$W, whose symmetry and circles are seen on their figures. Seven of them are unifilar.

In naming these knots I use an alphabet of 25 letters, omitting the letter O. Thus,

A, B, . . . *z*; A*a*, A*b*, . . . A*z*; B*a*, B*b*, . . . B*z*;

are each a set of 25.

45. For the unsolids $_8$X, &c., of eight crossings which have no concurrence, we have to lay 2 upon 6, 3 upon 5, 4 and 2·2 upon 4.

For 2 upon 6 we can impose $_4$A*ffc* on $_6$A once, on $_6$B twice, on $_6$C once, on $_6$E once, and on $_6$F once, but on no other of six crossings, without violating the rules in arts. 25, 28.

$_4$A*ffc* on $_6$A gives $_8$X;
" $_6$B " $_8$Y and $_8$Z;
" $_6$C " $_8$A*a*;
" $_6$E " $_8$A*b*;
" $_6$F " $_8$A*c*;

whose descriptions are read on their figures. The monozone $_8$A*b* has four different flaps, one epizonal upon the marginal charge, and three zonals, in the zonal plane.

For 3 upon 5,

${}_{3}Aff$ on ${}_{5}A$ gives ${}_{8}Ad$;
${}_{3}Afe$ on ${}_{5}A$ gives ${}_{8}Af$ and ${}_{8}Ae$;
${}_{3}Afe$ on ${}_{5}B$ attempted gives ${}_{8}Z$

erroneously, or leaves a concurrence. ${}_{5}Affc$ on ${}_{5}A$ gives ${}_{8}Ag$, by arts. 16, 17, and ${}_{8}Ah$, art. 27.

For 4 upon 4,

${}_{4}Aff$ on ${}_{4}A$ gives ${}_{8}Ai$;
${}_{4}Afe$ on ${}_{4}A$ „ ${}_{8}Aj$;
${}_{4}Aee$ on ${}_{4}A$ „ ${}_{8}Ak$ and ${}_{8}Al$.

The janal zoned poles on ${}_{8}Ai$ are two flaps, two edges of the 4-gon, and two opposite not plane 6-gons. The two poles of ${}_{8}Aj$ are a flap and an edge 33: on ${}_{8}Ak$ the janal zoneless poles are edges 33: on ${}_{8}Al$ are two pairs (66) and (33) of janal zoneless 2-ple poles; and a third pair are two 6-gons not plane.

There are two constructions by the charge ${}_{4}Aee$, because neither ϵ nor ϵ' (art. 27) is zoned polar.

For 2·2 on 4 (art. 38),

${}_{4}A^{2}ffc$ on ${}_{4}A$ gives ${}_{8}Am$,

which has all the symmetry of ${}_{4}A$; the four 2-ple zoneless janal poles are where they were, and the zoned janal poles are the two flaps of the imposed charges.

Finally,

${}_{4}A^{2}ffc$ on ${}_{4}B$ gives ${}_{8}An$,

having a janal zoned pole in each flap, and another pair in opposite non-plane 6-gons.

${}_{8}Ap$ is the only solid knot of eight crossings, a 4-zoned monarchaxine homozone, whose eight identical janal 2-ple zoneless poles bisect eight polar edges 33. Thus we have constructed seventeen unsolids without concurrence, ${}_{8}X$ ${}_{8}Y$ . . . ${}_{8}An$, of which nine are unifilar.

46. We complete our list of unsolids by art. 29—

${}_{7}A$ gives ${}_{8}Aq$;	${}_{7}J$ gives ${}_{8}Bg$, ${}_{8}Bh$;
${}_{7}B$ „ ${}_{8}Ar$, ${}_{8}As$;	${}_{6}A$ „ ${}_{8}Bi$, ${}_{8}Bj$;
${}_{7}C$, ${}_{8}At$;	${}_{6}B$ „ ${}_{8}Bk$, ${}_{8}Bl$, ${}_{8}Bm$, ${}_{8}Bn$;
${}_{7}D$ „ ${}_{8}Au$;	${}_{6}C$ „ ${}_{8}Bp$, ${}_{8}Bq$;
${}_{7}E$ „ ${}_{8}Av$, ${}_{8}Aw$, ${}_{8}Ax$;	${}_{6}D$ „ ${}_{8}Br$, ${}_{8}Bs$;
${}_{7}F$ „ ${}_{8}Ay$, ${}_{8}Az$;	${}_{6}E$ „ ${}_{8}Bt$, ${}_{8}Bu$;
${}_{7}G$ „ ${}_{8}Ba$, ${}_{8}Bb$;	${}_{5}A$ „ ${}_{8}Bv$, ${}_{8}Bw$;
${}_{7}H$ „ ${}_{8}Bc$, ${}_{8}Bd$;	${}_{4}A$ „ ${}_{8}Bx$, ${}_{8}By$, ${}_{8}Bz$;
${}_{7}I$ „ ${}_{8}Be$, ${}_{8}Bf$;	${}_{3}A$ „ ${}_{8}Ca$.

Of these 36 we have figured only half, the 18 of them which are unifilar;

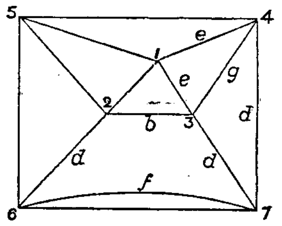
as in the unifilars appears to lie mainly the interest of these knots. The 18 plurifils (if ever they deserve a name) can easily be drawn by the student with the aid of the above list; and they must be figured by him who constructs the 10-fold knots, for on many of them flaps can be drawn to make subsolids of ten crossings.

We have found of eight crossings,

1 solid knot, unifilar,
22 subsolids, ${}_8A \ldots {}_8W$,
17 unsolids without concurrences
36 unsolids with concurrences;

in all 76 8-fold knots, of which 35 are unifilar.

47. *Construction of Knots of Nine Crossings.*—The subsolids ${}_9A$, &c., are to be formed by drawing leading flaps on ${}_7A$, &c.



The only lines that differ on ${}_7A$ here figured are,

fb and *dd*, each=44;
df, *dd*, *db*, *dg*, *ge*, *ee*, *eb*, each=34:

2376, 347, 123, and 134 are the only faces. The *d*'s are alike, all on the same asymmetric edge 34, which has two different sides.

(*fb*)=44, (*fd*)=43, and (*dd*)=43, have no competitors; for this (*dd*) fixes (67) (art. 23),

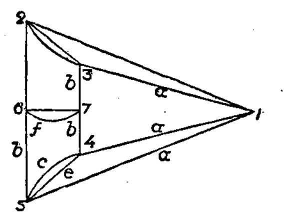
$$(dd)=44, 44>(67)=44, 44\ ?$$

All the lines =34, except *fd* and *dd*, are led by (67). We have four flaps to draw, expecting symmetry with (*dd*)=44.

By (*dd*)=44 we get ${}_9A$; by (*dd*)=43 we get ${}_9D$;
" (*fb*)=44 " ${}_9B$; " (*fd*) , ${}_9C$.

The symmetry and circles of these four knots are read on their figures.

48. We consider next ${}_7B$, here drawn. This has the faces 1473, 6745, and 541, with one zoned polar and one epizonal flap. The different lines that can be drawn are,



ab, *aa*, *fb*, *bb*, and *cb*, each=53,
ab, *fc*, and *bb*, each=44,
aa and *ae*, each=43.

In *b7b* (*bb*) =53, 55>(23) or (54) each 53 54; (67) is fixed (art. 23)
(*aa*)=53 >(23, 67 or 54) each=44;
(*fc*) =44 >(23)=43;
In 67*bb* (*bb*) =44, 55>(67)=44, 55 ? (45)=(23)=43.

For the rest,

(54) leads (fb), $(aa)=44$ and $(ab)=44$,
(67) leads $(ab)=35$, (ea) and (cb).

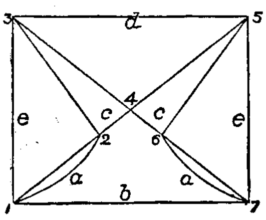
We have to draw $(bb)=53$, (aa), (fc) and $(bb)=44$, expecting symmetry with the last,

$(bb)=53$ gives ${}_9$E; $(bb)=44$ gives ${}_9$G;
(fc) „ ${}_9$F; (aa) „ ${}_9$H;

whose symmetry and circles are on their figures.

49. ${}_7$C annexed has monozone faces 1357, 12467, 354, and asymmetric faces 456, 567, and the flap.

The different lines on it are,



ab, ac, cc, each 36;
aa, ac, bc, each 45;
bd, ee, each 44;
ed, eb, each 35;

besides lines 34 that can be drawn in triangles.

	(aa) has no rival:
in $1ab$,	$(ab)=63, 53>(67)=63, 53$?
in $2ac$,	$(ac)=63, 43>(67)=63, 43$?
in $24ca$	$(ac)= \quad 54>(67)=53$;
in $12eb$	$(bc)= \quad 54>(67)=53$ or $(12)=43$;
	$(cc)=63, 44>(12)$ or (67) each $=63, 44$?

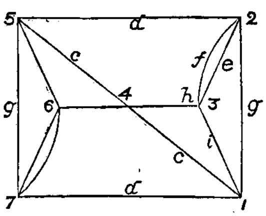
For the rest, (12) or (67) leads them all, as well as all flaps on lines 34.

We have to draw six flaps, looking for symmetry with three,

	(aa) gives us ${}_9$I;	(ab) gives us ${}_9$J;
In $2ac$	(ac) „ ${}_9$K;	(ae) „ ${}_9$L in $24ca$.
in $21bc$	(bc) „ ${}_9$M;	(ce) „ ${}_9$N.

The symmetry and circles of these are on the figures.

50. Next comes ${}_7$D here drawn. The polar 4-gon is 5217, and the asymmetric faces are 5234, 123, 143. The only different lines to be drawn are,



fd, fh, ch, cd, dg_1 in $1dg$, and dg_7 in $7dg$, all 53;
fc, dd, dh, gg, all 44;
eg, ei, gi, hc, hi, ci, all 43; 16 lines.

$(fd)=53>(67)=43$;
$(fh)=53>(67)=43$;
$(cd)=53, 54>(32)=53, 53$; $(67)=43$;
$(dg_1)=53, 54>(67)=53, 53$, or $(32)=44$;
$(dg_7)=53 \quad >(32)=43$; and (67) is fixed (art. 23);

in $c4h$, $(ch) = 53, 44 > (32) = 53, 44$? (67) is 43 ;
$(fc) = 44 \quad > (76) = 43$;
$(dh) = 44 \quad > (76 \text{ or } 32) = 43$;
$(gg) = 44, 44 > (76 \text{ or } 32) = 44, 43$
$(eg) = 43, 53 > (76) = 43, 53$?
$(ei) = 43, 43 > (76) = 43, 43$?
$(hi) = 43, 54 > 43, 43$; (23) is fixed (art. 23).

For the rest,

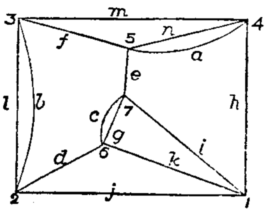
(32) leads dd, gi, ci, and $hc = 43$.

We have twelve flaps to draw, expecting with three at least symmetry :

(fd) gives ${}_9P$;	(fc) gives ${}_9V$;
(fh) „ ${}_9Q$;	(dh) „ ${}_9W$;
(cd) „ ${}_9R$;	(gg) „ ${}_9X$;
(gd_1) „ ${}_9S$;	(eg) „ ${}_9Y$;
(ch) „ ${}_9T$;	(ei) „ ${}_9Z$;
(dg_7) „ ${}_9U$;	(hi) „ ${}_9Aa$.

The poles of ${}_9X$ are a tessarace and the leading flap. The symmetry and circles of all are read on figures.

51. Our next subsolid base is E_7 appended, on which no two edges are alike. Thirty-one different flap-lines can be drawn on it, namely,



bd, bf, cd, cc, fe, all 63 ;
bc, be, cf, fd, de, all 54 ;
ae, ah, lm, lj, hm, hj, hi, ie, all 53 ;
ai, hl, eh, jm, all 44 ;
fm, nf, nm, gk, gi, ik, jd, jk, kd, all 43 ;

the lines lm, &c., in the base being supposed dotted.

$(bf) = 63, 43 > (67) = 63, 43$? $54 = 44$;
$(cd) = 63 \quad > (54) = 43$;
$(ce) = 63 \quad > (54) = 53$;
$(bc) = 54 \quad > (54) = 43$;
$(be) = 54 \quad > (54 \text{ or } 67) = 53$;
$(cf) = 54, 43 > (23) = 54, 43$? $(54) = 44$;
$(fd) = 54 \quad > (23 \text{ or } 54) = 44$ and $> (67) = 53$;
$(de) = 54 \quad > (54) = 53$, or $(67) = 43$; (23) is fixed ;
$(lm) = 53, 43 > (67) = 53, 43$? $(67) = 44$;
$(dk) = 43, 64 > (54) = 43, 64$? (67) and (23) are fixed
$(ie) = 53, 64 > (54) = 53, 64$? „ „

For the rest,

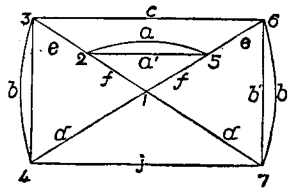
(23) leads nf, ah, ae, ai, mh, ef, mj, mn ;
(67) „ hl, he, hi, bd, lj, jd, mf ;
(54) „ gi, gk, ik, jk, jh.

We have eleven flaps to draw, with five queries about symmetry, which speedily reveals itself,

(*bf*) gives ${}_9\mathrm{A}b$;	(*fd*) gives ${}_9\mathrm{A}h$;
(*cd*) „ ${}_9\mathrm{A}c$;	(*de*) „ ${}_9\mathrm{A}i$;
(*ce*) „ ${}_9\mathrm{A}d$;	(*lm*) „ ${}_9\mathrm{A}j$;
(*bc*) „ ${}_9\mathrm{A}e$;	(*dk*) „ ${}_9\mathrm{A}k$;
(*be*) „ ${}_9\mathrm{A}f$;	(*ie*) „ ${}_9\mathrm{A}l$.
(*cf*) „ ${}_9\mathrm{A}g$;	

The symmetry and circles of all are written on their figures. The poles of ${}_9\mathrm{A}j$ are the tessarace common to the two 5-gons, and the flap which connects them: those of ${}_9\mathrm{A}k$ are a 6-gon and a 4-ace.

52. The sixth and last subsolid is ${}_7\mathrm{F}$, which has only one asymmetric face (1567), and one asymmetric flap (67). The flap (25) is epizonal. Eighteen different flap-lines can be drawn:



ae, b'd, b'e, bj, bc, ef, ec, fd, all 53 (*bj* and *bc* dotted below);
ac, bb, b'f, ee, ed, cj, all 44;
ff, a'f, dd, dj, all 43.

$(bb)=44>(25)=43$;
$(b'd)$ in $b'd7=53>(25)=43$ and $>(34)=44$.

For the rest,

(25) leads *b'e, b'f, cj, ed*;
(34) leads *ae, ee, ec, ac, a'f, dd,* and *dj* in $7dj$;
(67) leads *fd, bc* in $3bc$, *ff, fe,* and *bj* in $4bj$.

We have two flaps to draw,

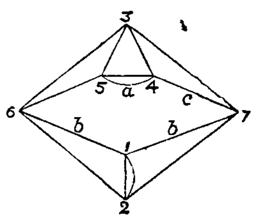
(*bb*) giving ${}_9\mathrm{A}m$ and (*b'd*), giving ${}_9\mathrm{A}n$,

whose circles and symmetry are read on the figures.

53. We betake ourselves next to the unsolids of seven crossings, which, by a leading flap, can become subsolids. These are

${}_7\mathrm{G}$, ${}_7\mathrm{H}$, ${}_7\mathrm{I}$, ${}_7\mathrm{J}$, ${}_7\mathrm{K}$, ${}_7\mathrm{L}$, ${}_7\mathrm{M}$, ${}_7\mathrm{N}$, ${}_7\mathrm{Q}$, ${}_7\mathrm{R}$, ${}_7\mathrm{S}$

On ${}_7\mathrm{G}$ annexed can be drawn to block the linear section only three lines, which are



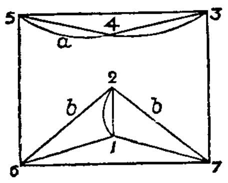
$(ab)=54>(12)=43$;
in $cb7$, $(cb)=63{,}44>(54)=63{,}43$ and $>(12)=43$;
in $cb17$, $(cb)=54>(54)=53$ and $>(12)=43$.

We draw three flaps,

(*ab*) giving ${}_9\mathrm{A}p$, (*cb*) giving ${}_9\mathrm{A}q$, and (*cb*) giving ${}_9\mathrm{A}r$;

whose description is seen on their figures.

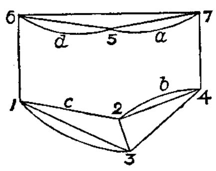
54. On $_7$H, here seen, we can draw two flaps only,



$(ab) = 64$ (in $56ba$) $> (43) = 63$ or $(21) = 43$.
$(ab) = 55$ (in $562ba$) $> (43) = 53$ or $(21) = 43$.

One (ab) gives $_9$As, the other gives $_9$At.

Four flaps can be drawn on $_7$I annexed,

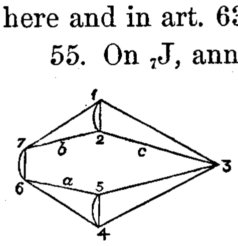


$(ab) = 64 > (56) = 63$ or $(13) = 53$;
$(db) = 55 > (57) = 53$ or $(13) = 53$;
$(ac) = 55 > (24$ or $56) = 53$ and $> (13) = 54$;
$(dc) = 64 > (57$ or $24) = 63$ and $> (13) = 54$.

Here (ab) gives $_9$Au; (ac) gives $_9$Aw;
(db) „ $_9$Av; (dc) „ $_9$Ax.

Professor TAIT does not allow $_7$H and $_7$I to be different knots, giving a reason at p. 158, *Trans. R.S.E.* 1876–7, which appears to me sufficient wherever it can be verified without twisting the tape, or breaking the law of alternate over and under. It is true that on the knot in space whose projection is $_7$H, the three crossings 543 which are found on the thread 67 between 6 and 7, can by slipping of the thread be made to appear on the thread 71; so that the order of the crossings shall be changed from 165435437.., the thread passing over at 15347, to 167543543.., the thread passing over at 17534, *i.e.*, making two consecutive overs at 7 and 5. The resultant figure in space, although it would have $_7$I for its projection, would, if I am in the right, be no knot. If I had not drawn both $_7$H and $_7$I, I should have missed some unifilar 9-fold knots, both here and in art. 63.

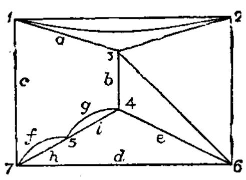
55. On $_7$J, annexed, can be drawn only two lines to make a subsolid,



$(ab) = 54, 44 (67) = 54, 44$? $(54 = 12) = 43$.
$(ac) = 54, 44 > (12$ or $54) = 43$; (67) is fixed.

(ab) gives $_9$Ay, and (ac) gives $_9$Az.

On $_7$K, here seen, can be drawn flaps only from f, g, h, or i, so as to spoil both concurrence and linear section,



$(fa) = 54, 43 > (54) = 54, 33$; $(12) = 44$;
$(fb) = 54 > (54) = 44$; $(12) = 43$;
$(ga) = 54, 43 > (57) > 54, 43$? $(12) = 44$;
$(gc) = 54 > (57) = 44$, or $(12) = 53$.

A flap (fg) can be drawn, but the linear section 74 would remain. For the rest,

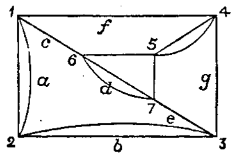
(45) leads fc, he, hd; (75) leads bg, ic, id.

Here

(fa) gives $_9\mathrm{B}a$; (ga) gives $_9\mathrm{B}c$;
(fb) „ $_9\mathrm{B}b$; (gc) „ $_9\mathrm{B}d$.

The circles and symmetry are on the figures.

56. On $_7$L, here given, no line can be drawn to spoil the concurrence and the linear section but from a or b,

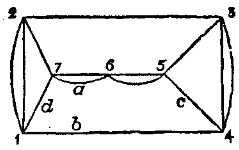


$(ae) = 54 \quad > (23) = 44$ and $> (67) = 53$; (54) is fixed (art. 22).
not $(ac) = 63 \quad > (23) = 64$;
not $(ad) = 54, 33 > (23) = 54, 43$;
not $(bg) = 53, 53 > (67) = 53, 54$;
not $(bf) = 44 \quad > (67) = 53$.

Here

(ae) gives us $_9\mathrm{B}e$.

On $_7$M, here seen, the leading flap must be drawn from a.

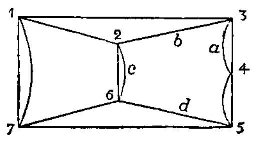


$(ad) = 63 \quad > (12) = 44$, or $(34) = 43$;
$(ab) = 54 \quad > (12$ or $34) = 53$; (56) is fixed (art. 22).
$(ac) = 54, 43 > (56) = 54, 43$? $(34) = 44$.

Here

(ad) gives $_9\mathrm{B}f$, (ab) $_9\mathrm{B}g$, and (ac) $_9\mathrm{B}h$, the latter symmetrical.

57. On $_7$N, annexed,



$(ad) = 54 > (42, 26, 17) = 54$?
(26) leads (ab), and (45) leads (ac).

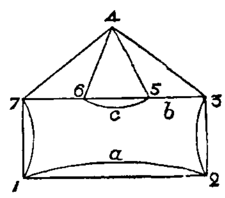
The only leader

(ad) gives us $_9\mathrm{B}i$, symmetric.

On $_7$P no flap can spoil both concurrence and linear section.

On $_7$Q there can be drawn only one leading flap—

$(ab) = 56$, giving $_9\mathrm{B}j$.

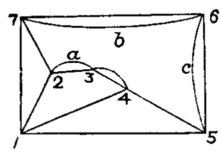


On $_7$R, here given,

$(ac) = 55 > (32$ or $17) = 55$?
(ab) is led by (17).

Here (ac) gives us $_9\mathrm{B}k$, symmetric.

On $_7$S, annexed,



$(ac) = 55 > (76$ or $34) = 54$.
$(ab) = 64 > (56$ or $34) = 64$?

Here (ac) gives $_9\mathrm{B}l$, and (ab) $_9\mathrm{B}m$, the latter symmetric.

We have constructed by their leading flaps sixty-three subsolids of nine crossings, of which thirty are unifilars, bearing on their figures the number 18.

58. We demand next the number of the unsolid 9-fold knots, and first, of those which have no concurrence.

To construct these we are to lay 2 upon 7, 3 upon 6, 4 and 2·2 upon 5, and 3·2 upon 3.

For 2 upon 7 (art. 34) imposing ${}_4Aff c$, we get

On ${}_7A$, ${}_9Bn$;	On ${}_7I$, ${}_9Cb$, ${}_9Cc$;
„ ${}_7B$, ${}_9Bp$ and ${}_9Bq$;	„ ${}_7K$, ${}_9Cd$, ${}_9Ce$;
„ ${}_7C$, ${}_9Br$;	„ ${}_7L$, ${}_9Cf$;
„ ${}_7D$, ${}_9Bs$;	„ ${}_7M$, ${}_9Cg$;
„ ${}_7E$, ${}_9Bt$, ${}_9Bu$, ${}_9Bv$;	„ ${}_7N$, ${}_9Ch$;
„ ${}_7F$, ${}_9Bw$, ${}_9Bx$;	„ ${}_7Q$, ${}_9Ci$;
„ ${}_7G$, ${}_9By$;	„ ${}_7R$, ${}_9Cj$.
„ ${}_7H$, ${}_9Bz$, ${}_9Ca$;	

On ${}_7J$ we do nothing, because we cannot cover both its least marginal charges; and nothing on ${}_7P$, because we cannot both spoil the concurrence and cover the least marginal charge.

59. For 3 upon 6, the charge must be ${}_5Affc$ (art. 26) or ${}_3Aff$, or ${}_3Afe$,

${}_5Affc$ on ${}_6A$ gives ${}_9Ck$ and ${}_9Cl$;
„ ${}_6B$ „ ${}_9Cm$, ${}_9Cn$, ${}_9Cp$;
„ ${}_6C$ „ ${}_9Cq$.

On ${}_6D$ we cannot cover both marginals ${}_4Affc$.

On ${}_6E$ we cannot cover both.

On ${}_6F$, ${}_5Affc$ imposed to spoil the concurrence would be ${}_5A^2ffc$ on ${}_3A$ wrongly constructed by one charge ${}_5Affc$ only.

On ${}_6G$ we cannot both spoil the concurrence and cover the least marginal.

On ${}_6H$ it requires two charges to spoil the two concurrences.

Next, for 3 upon 6 again,

${}_3Aff$ on ${}_6A$ gives ${}_9Cr$;
${}_3Afe$ on ${}_6A$ „ ${}_9Cs$, ${}_9Ct$, ${}_9Cu$, ${}_9Cv$, ${}_9Cw$;

for the e charged on ${}_6A$ (art. 41) is in turn every different edge, a,b,c,d,e.

${}_3Aff$ on ${}_6B$ gives ${}_9Cx$, ${}_9Cy$;
${}_3Afe$ on ${}_6B$ „ ${}_9Cz$, ${}_9Da$;
${}_3Aff$ on ${}_6C$ „ ${}_9Db$;
${}_3Afe$ on ${}_6C$ „ ${}_9Dc$;
${}_3Aff$ on ${}_6F$ „ ${}_9Dd$;
${}_3Afe$ on ${}_6J$ „ ${}_9De$.

60. We have next to lay 4 and 2·2 upon 5,

${}_4\mathrm{A}ff$ on ${}_5\mathrm{A}$ gives ${}_9\mathrm{D}f$;
${}_4\mathrm{A}ef$ on ${}_5\mathrm{A}$ „ ${}_9\mathrm{D}g$;
${}_4\mathrm{A}fe$ on ${}_5\mathrm{A}$ „ ${}_9\mathrm{D}h$;
${}_4\mathrm{A}ee$ on ${}_5\mathrm{A}$ „ ${}_9\mathrm{D}i$.
${}_4\mathrm{A}^2ffc$ on ${}_5\mathrm{A}$ gives ${}_9\mathrm{D}j$ (art. 38).
${}_4\mathrm{A}^2ffc$ on ${}_5\mathrm{B}$ „ ${}_9\mathrm{D}k$.

${}_5\mathrm{B}$ was made (art. 31) by adding a concurrence to ${}_4\mathrm{A}$; but it is also ${}_4\mathrm{A}ffc$ on ${}_3\mathrm{A}$, though improperly made (art. 38); and as we have two charges to impose, we can both spoil the concurrence and cover the least marginal of ${}_5\mathrm{B}$, thus making ${}_9\mathrm{D}k$.

61. Finally, we have to lay 2·3 and 3·2 upon 3,

${}_5\mathrm{A}^2ffc$ on ${}_3\mathrm{A}$ gives ${}_9\mathrm{D}l$ and ${}_9\mathrm{D}m$.
${}_3\mathrm{A}^2ff$ on ${}_3\mathrm{A}$ gives ${}_9\mathrm{D}n$. These lay 2·3 on 3.
${}_4\mathrm{A}^3ffc$ on ${}_3\mathrm{A}$ gives ${}_9\mathrm{D}p$. This lays 3·2 on 3.

In ${}_9\mathrm{D}l$, ${}_9\mathrm{D}m$, and ${}_9\mathrm{D}n$ the symmetry is maintained about one of the 2-zoned axes of ${}_3\mathrm{A}$, though not a 2-zoned symmetry.

We have constructed 115 9-fold knots without concurrences, of which 63 are subsolids, and 52 are unsolids without concurrences. Among the 63 are 30 unifilars, and among the 52 are 25, making 55 unifilars without concurrences.

62. There remain only the 9-fold unsolids which have concurrences. The number of ways in which a knot K′ of $n-c$ crossings can be made by adding c concurrences of flaps into K of n crossings is easily seen when the symmetry of K′ is given, K′ having no concurrence—

${}_8\mathrm{A}$ gives ${}_9\mathrm{D}q$;
${}_8\mathrm{B}$ „ ${}_9\mathrm{D}r$;
${}_8\mathrm{C}$ „ ${}_9\mathrm{D}s$, ${}_9\mathrm{D}t$;
${}_8\mathrm{D}$ „ ${}_9\mathrm{D}u$;
${}_8\mathrm{E}$ „ ${}_9\mathrm{D}v$, ${}_9\mathrm{D}w$, ${}_9\mathrm{D}x$;
${}_8\mathrm{F}$ „ ${}_9\mathrm{D}y$, ${}_9\mathrm{D}z$;
${}_8\mathrm{G}$ „ ${}_9\mathrm{E}a$, ${}_9\mathrm{E}b$, ${}_9\mathrm{E}c$;
${}_8\mathrm{H}$ „ ${}_9\mathrm{E}d$;
${}_8\mathrm{I}$ „ ${}_9\mathrm{E}e$, ${}_9\mathrm{E}f$, ${}_9\mathrm{E}g$;
${}_8\mathrm{J}$ „ ${}_9\mathrm{E}h$;
${}_8\mathrm{K}$ „ ${}_9\mathrm{E}i$, ${}_9\mathrm{E}j$, ${}_9\mathrm{E}k$;
${}_8\mathrm{L}$ „ ${}_9\mathrm{E}l$, ${}_9\mathrm{E}m$;
${}_8\mathrm{M}$ „ ${}_9\mathrm{E}n$;
${}_8\mathrm{N}$ „ ${}_9\mathrm{E}p$;
${}_8\mathrm{P}$ „ ${}_9\mathrm{E}q$, ${}_9\mathrm{E}r$;
${}_8\mathrm{Q}$ „ ${}_9\mathrm{E}s$, ${}_9\mathrm{E}t$;
${}_8\mathrm{R}$ „ ${}_9\mathrm{E}u$, ${}_9\mathrm{E}v$;
${}_8\mathrm{S}$ „ ${}_9\mathrm{E}w$, ${}_9\mathrm{E}x$;
${}_8\mathrm{T}$ „ ${}_9\mathrm{E}y$;
${}_8\mathrm{U}$ „ ${}_9\mathrm{E}z$, ${}_9\mathrm{F}a$, ${}_9\mathrm{F}b$;
${}_8\mathrm{V}$ gives ${}_9\mathrm{F}c$, ${}_9\mathrm{F}d$, ${}_9\mathrm{F}e$;
${}_8\mathrm{W}$ „ ${}_9\mathrm{F}f$;
${}_8\mathrm{X}$ „ ${}_9\mathrm{F}g$, ${}_9\mathrm{F}h$;
${}_8\mathrm{Y}$ „ ${}_9\mathrm{F}i$, ${}_9\mathrm{F}j$, ${}_9\mathrm{F}k$;
${}_8\mathrm{Z}$ „ ${}_9\mathrm{F}l$, ${}_9\mathrm{F}m$;
${}_8\mathrm{A}a$ „ ${}_9\mathrm{F}n$, ${}_9\mathrm{F}p$;
${}_8\mathrm{A}b$ „ ${}_9\mathrm{F}q$, ${}_9\mathrm{F}r$, ${}_9\mathrm{F}s$;
${}_8\mathrm{A}c$ „ ${}_9\mathrm{F}t$, ${}_9\mathrm{F}u$, ${}_9\mathrm{F}v$;
${}_8\mathrm{A}d$ „ ${}_9\mathrm{F}w$, ${}_9\mathrm{F}x$;
${}_8\mathrm{A}e$ „ ${}_9\mathrm{F}y$, ${}_9\mathrm{F}z$;
${}_8\mathrm{A}f$ „ ${}_9\mathrm{G}a$, ${}_9\mathrm{G}b$, ${}_9\mathrm{G}c$;
${}_8\mathrm{A}g$ „ ${}_9\mathrm{G}d$;
${}_8\mathrm{A}h$ „ ${}_9\mathrm{G}e$;
${}_8\mathrm{A}i$ „ ${}_9\mathrm{G}f$;
${}_8\mathrm{A}j$ „ ${}_9\mathrm{G}g$, ${}_9\mathrm{G}h$;
${}_8\mathrm{A}k$ „ ${}_9\mathrm{G}i$;
${}_8\mathrm{A}l$ „ ${}_9\mathrm{G}j$;
${}_8\mathrm{A}m$ „ ${}_9\mathrm{G}k$;
${}_8\mathrm{A}n$ „ ${}_9\mathrm{G}l$, ${}_9\mathrm{G}m$.

63. We have next to add two concurrences of flaps to all of ${}_7A$, &c., on which is no concurrence—

${}_7A$ gives ${}_9Gn$;
${}_7B$ „ ${}_9Gp$, ${}_9Gq$, ${}_9Gr$, ${}_9Gs$;
${}_7C$ „ ${}_9Gt$, ${}_9Gu$;
${}_7D$ „ ${}_9Gv$, ${}_9Gw$;
${}_7E$ „ ${}_9Gx$, ${}_9Gy$, ${}_9Gz$, ${}_9Ha$, ${}_9Hb$, ${}_9Hc$;
${}_7F$ „ ${}_9Hd$, ${}_9He$, ${}_9Hf$, ${}_9Hg$;
${}_7G$ „ ${}_9Hh$, ${}_9Hi$, ${}_9Hj$;
${}_7H$ „ ${}_9Hk$, ${}_9Hl$, ${}_9Hm$;
${}_7I$ „ ${}_9Hn$, ${}_9Hp$, ${}_9Hq$, ${}_9Hr$;
${}_7J$ „ ${}_9Hs$, ${}_9Ht$, ${}_9Hu$, ${}_9Hv$.

The number of results in any of the above cases of this article is that of the different flaps which can be made a concurrence of three plus the number of different pairs of flaps that can be made each a concurrence of two.

64. We have next to place three concurrences of flaps on ${}_6A$, &c., four on ${}_5A$, five on ${}_4A$, and six on ${}_3A$—

${}_6A$ gives ${}_9Hw$, ${}_9Hx$;
${}_6B$ „ ${}_9Hy$, ${}_9Hz$, ${}_9Ia$, ${}_9Ib$, ${}_9Ic$, ${}_9Id$;
${}_6C$ „ ${}_9Ie$, ${}_9If$, ${}_9Ig$;
${}_6D$ „ ${}_9Ih$, ${}_9Ii$;
${}_6E$ „ ${}_9Ij$, ${}_9Ik$.

${}_5A$ gives ${}_9Il$, ${}_9Im$, ${}_9In$;
${}_4A$ „ ${}_9Ip$, ${}_9Iq$, ${}_9Ir$;
${}_3A$ „ ${}_9Is$.

Finally, there is one solid knot, ${}_9It$.

The number of 9-fold knots that have concurrences is 128, of which we have figured only the 70 of them which are unifilars. The rest will have to be drawn if the census of unifilars is carried to higher values. This can easily be done.

We have found 244 knots of nine crossings, viz. :—

1 solid knot,
63 subsolids,
52 unsolids without concurrences,
128 unsolids with concurrences.

Of these 244—

$30+25+70+1=126$ are unifilar.

I think that no difficulty will present itself in the construction of higher n-fold knots, which has not been met in the preceding pages.

Here follow the abbreviations used in the descriptions of symmetry :—

Monch. for Monarchaxine.	*Mox.* for Monaxine.
Triax. or *Tri.* for Triaxine.	*Moz.* for Monozone.
Triarch. for Triarchaxine.	*Hom.* for Homozone.
Zo. for Zoned.	*Het.* for Heteroid, not janal.
Az. for Azonal, or Zoneless.	*2p.* for 2-ple, repetition about an axis.

POSTSCRIPT, September 1, 1884.

65. As it is a brief matter, it may be worth the while to show how all solid knots can be constructed without omission or repetition.

Solid Knots, Prime and Non-prime.—A solid knot Q of n crossings is prime or non-prime according as it has or has not a crossing or summit A3B3, A and B being any meshes.

Lowest Triangle of a Solid Knot Q.—It is easily proved that no solid knot has fewer than eight triangles. The triangle L of Q is ABC, DEF, where ABC are the three covertical faces and DEF the collaterals of L, the lesser being written before the greater in both triplets.

If L′ be another triangle of Q, the lower of LL′ has the least A, whatever be the other five faces. If $A=A'$, the lower has the least B. If also $B=B'$, the lower has the least C. If ABC and A′B′C′ are alike, the lower has the least D, and so on.

If the six faces are alike in both, it is wisest, and almost sure to be right in construction, to presume that L and L′ are identical, or one the reflected image of the other, by the symmetry of Q, which is soon decided. If they are not thus proved alike, an examination of the collaterals of ABCDEF cannot fail to determine the lower triangle. The one whose A, or, if required, whose B, &c., has the lowest collaterals is the lower.

66. *Reduction of a Solid Knot* Q.—The simple rule is, efface the edges of the lowest triangle L of Q, or of a lowest when Q has a symmetry.

Such reduction of a prime solid knot Q of n crossings gives us a subsolid or an unsolid knot P of n–3 crossings, which has one, two, or three flaps, according as the effaced L had one, two, or three covertical triangles: and L must have one, or, by our first definition (65), it cannot be lowest on Q.

Such reduction of a non-prime Q gives either a non-prime P′ or a prime solid knot P of n–3 crossings; but I am not certain that this can ever be P′.

67. *Construction of Solid Knots* Q *of* n *Crossings.*—The rule is the converse of the preceding. Add to the subject P of n–3 crossings, whether P be solid, subsolid, or unsolid, a lowest triangle of the result Q, occupying three mid-edges of a mesh of P. I am not certain that when P is non-prime such addition can ever be made.

In order that Q may be solid, P must have fewer than four flaps, which, if more than one, must be collaterals of one mesh. If P has only one flap, it is collateral with two meshes, alike or different. Such a P must not be unsolid.

It may be that several different lowest triangles of Q may be drawn upon P, giving as many different Q, or that no lowest of Q can be drawn on P. In this case P is no base, and in construction is useless. No knot Q is reducible to it by deletion of a lowest triangle of Q. Examples are given below.

68. We proceed to construct on our figured knots P every possible solid knot Q.

Our only knots which have fewer than four flaps, all of which stand about one mesh, are ${}_3$A; ${}_5$A; ${}_6$F, ${}_6$J; ${}_7$A, ${}_7$C; ${}_8$R, ${}_8$T, ${}_8$W, ${}_8$A*d*, ${}_8$A*g*, ${}_8$A*p*, ${}_8$A*q*, ${}_8$A*t*; ${}_9$A, ${}_9$B, ${}_9$C, ${}_9$I, ${}_9$J, ${}_9$K, ${}_9$L, ${}_9$N, ${}_9$B*m*, ${}_9$B*n*, ${}_9$D*e*, ${}_9$D*l*, ${}_9$E*y*, ${}_9$F*f*, ${}_9$G*d*, ${}_9$I*t*; thirty of them, of which ${}_8$A*q*, ${}_9$E*y*, ${}_9$F*f* are not figured, but can easily be drawn (arts. 46, 62) on ${}_7$A, ${}_8$T, and ${}_8$W—

${}_3$A gives the solid ${}_6$J;
${}_5$A „ ${}_8$A*p*;
${}_6$F „ ${}_9$I*t*; see the three figures;
${}_7$A „ ${}_{10}$A, zo. tri. $4^4 3^6$, (446);
${}_7$C „ ${}_{10}$B, 5 zo. monch. hom. $5^2 3^{10}$, (20);
and also ${}_{10}$C, az. tri. $4^4 3^8$, (6, 14);
${}_8$R gives the solid ${}_{11}$A, 2 zo. mox. het. $4^5 3^8$, (6, 6, 10);
${}_8$A*d* „ ${}_{11}$B, 2 zo. mox. het. $5^2 4 3^{10}$, (4, 4, 14);
${}_8$A*t* „ ${}_{11}$C, moz. $5 4^3 3^9$, (22);
${}_9$A „ ${}_{12}$A, moz. $5 4^4 3^9$, (4, 20);
${}_9$B „ ${}_{12}$B, az. tri. $4^6 3^8$, (24);
${}_9$J „ ${}_{12}$C, asym. $5^2 4^2 3^{10}$, (24);
„ „ ${}_{12}$D, 6 zo. hom. $6^2 3^{12}$, (888);
${}_9$K „ ${}_{12}$E, moz. $5 4^4 3^9$, (10, 14);
${}_9$L „ ${}_{12}$F, 2p. mox. het. $4^6 3^8$, (6, 18);
${}_9$N „ ${}_{12}$G, 3 zo. monch. $4^6 3^8$, (6666);
${}_9$B*m* „ ${}_{12}$H, zo. triarch. $4^6 3^8$, (6, 6, 6, 6);
${}_9$D*e* „ ${}_{12}$I, 2p. mox. het. $5^2 4^2 3^{10}$, (4, 4, 16);
${}_9$E*y* „ ${}_{12}$J, 2 zo. mox. het. $5^2 4^2 3^{10}$, (10, 14);
${}_9$F*f* „ ${}_{12}$K, 2p. mox. moz. $4^6 3^8$, (24).

We have thus twenty prime solid knots, of fewer than thirteen crossings, made on eighteen of our thirty inferior knots. The remaining twelve, viz.—

$${}_6J\,,\ {}_8T\,,\ {}_8W\,,\ {}_8Ag\,,\ {}_8Ap\,,\ {}_8Aq\,,\ C_9\,,\ {}_9I\,,\ {}_9Bn\,,\ {}_9Dl\,,\ {}_9Gd\,,\ {}_9It\,,$$

are found to be no bases.

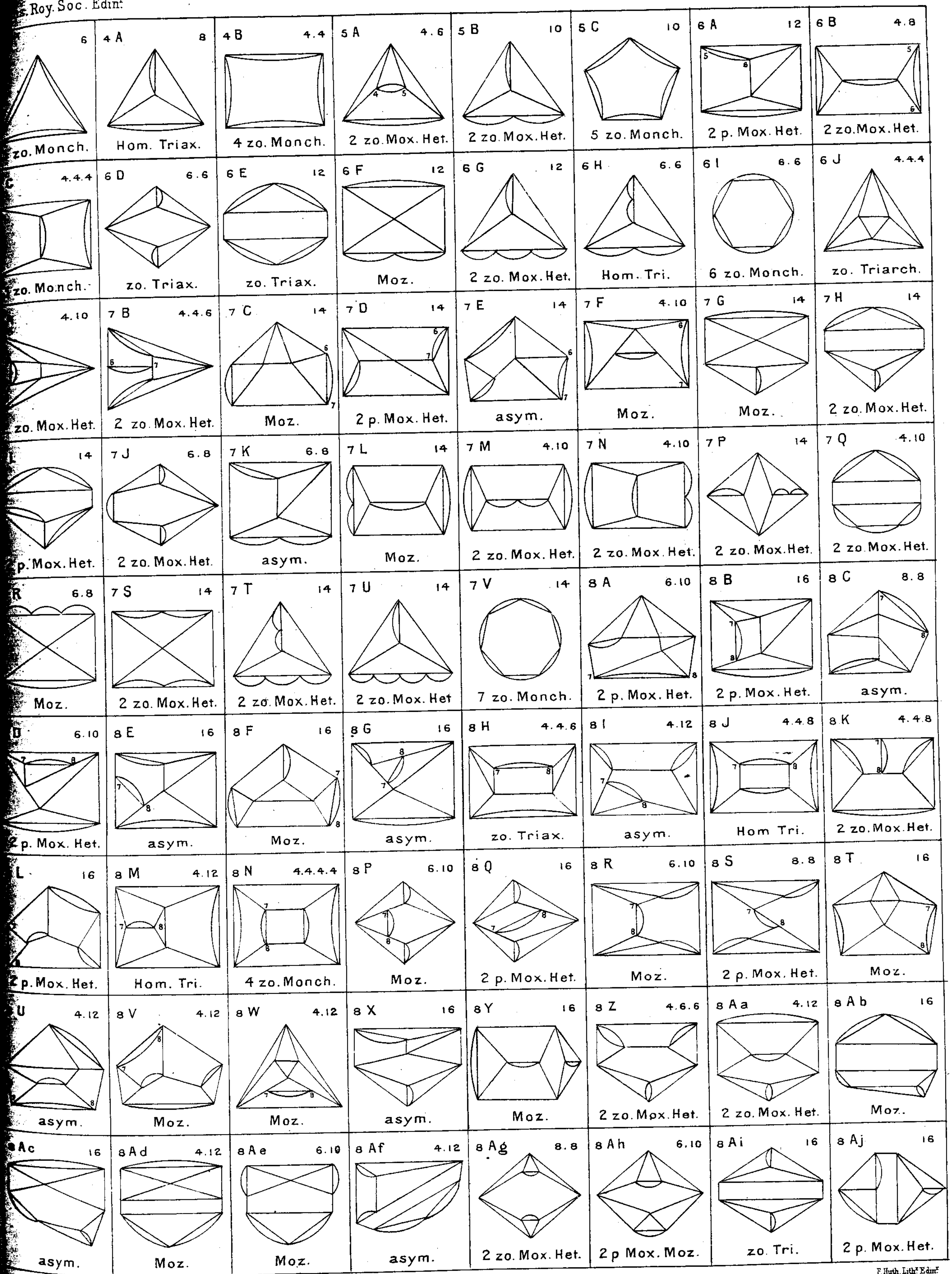
In ${}_{10}A$ above, 4^4 means 4444, and the circles are in parentheses.

No non-prime solid knot has fewer than sixteen crossings. The simplest is $4^{10}3^8$, 4 zo. monch., in which two opposite 4-gons are each covertical with four triangles, the triangles being four pairs of collaterals.

In order that a prime solid knot P may be a base, it must have not more than three summits A3B3, which must be so placed that, by drawing a lowest triangle of the non-prime Q to be formed, every pair of covertical triangles shall disappear.

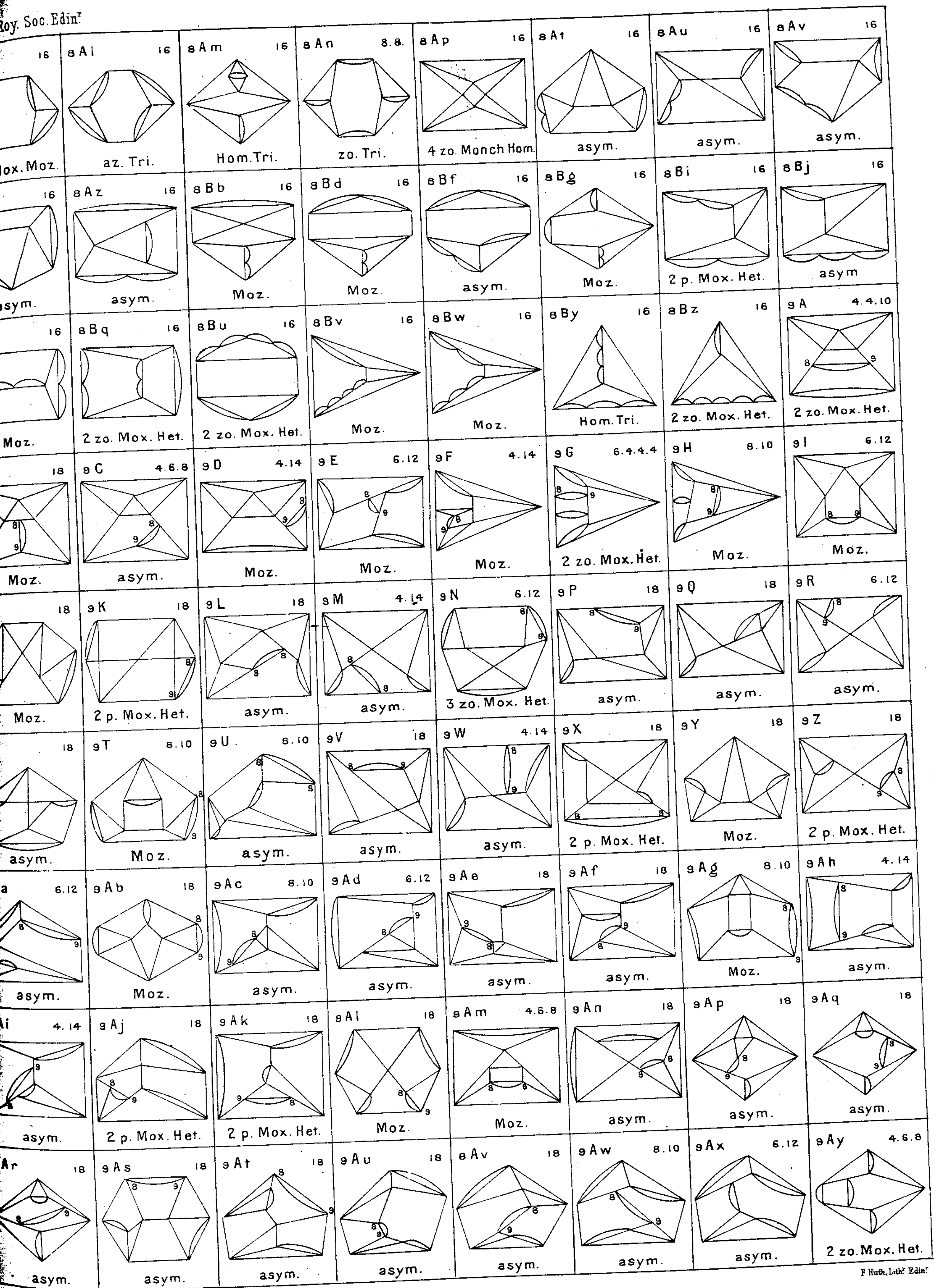
All non-primes can be easily constructed by our simple rule without omission or repetition when the primes of more than twelve crossings are before us.

This may suffice on solid knots until their value in electricity and magnetism is so enhanced as to call for a formal treatise on the whole subject.



Kirkman del.

F. Huth, Lith.[r] Edin.[r]



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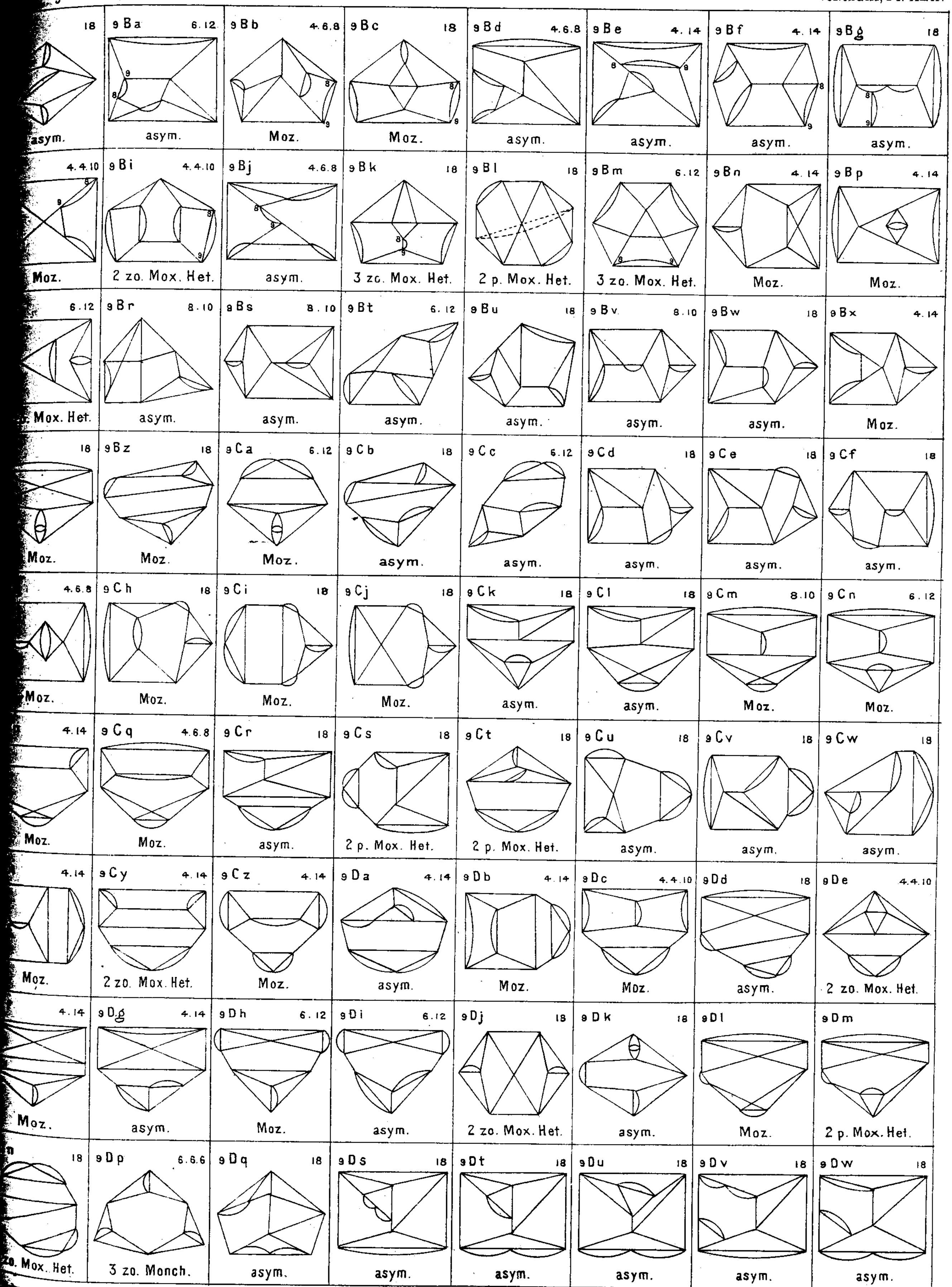

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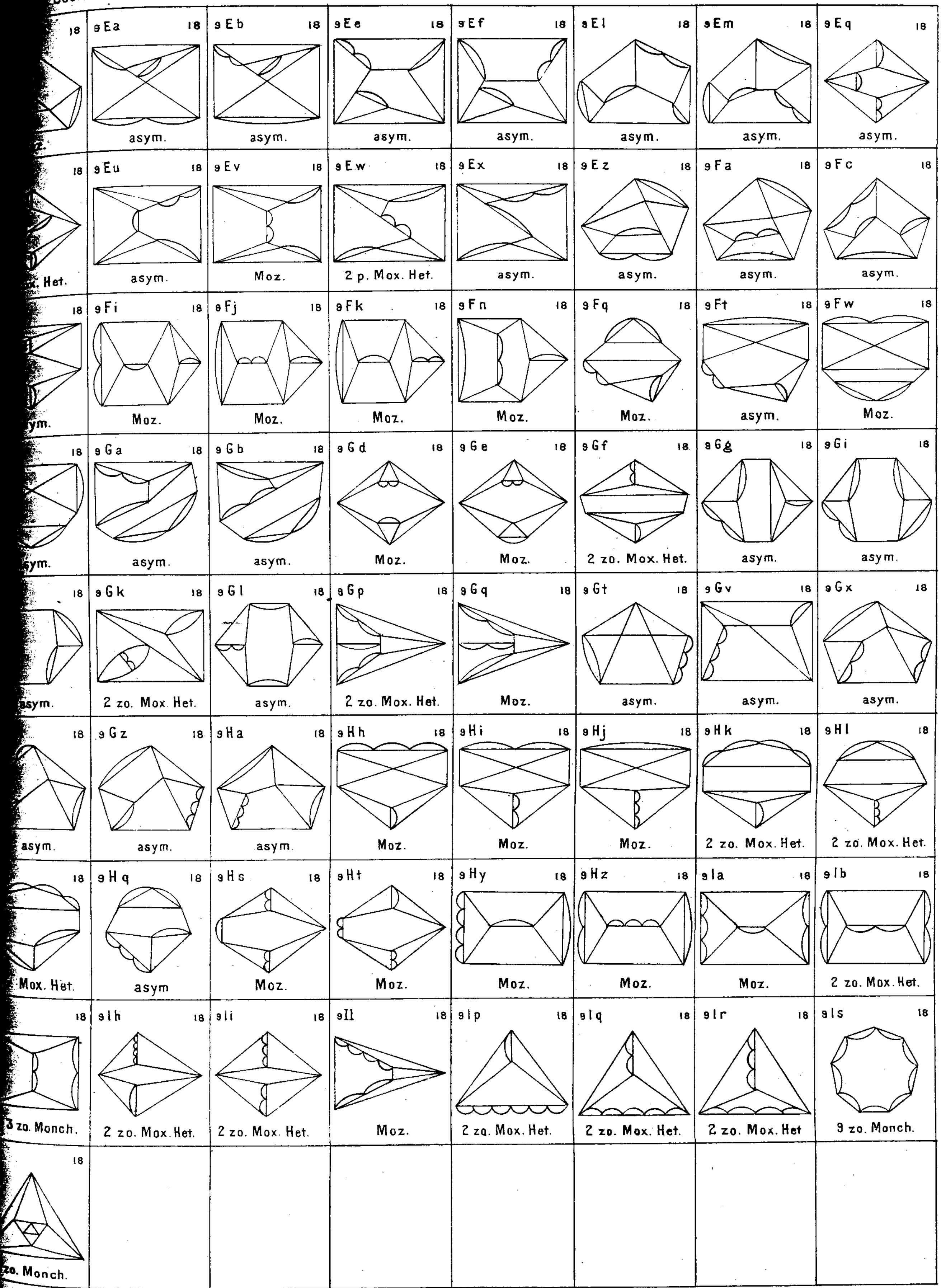
18
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9 Ea 18
asym.
9 Eb 18
asym.
9 Ee 18
asym.
9 Ef 18
asym.
9 El 18
asym.
9 Em 18
asym.
9 Eq 18
asym.
18
Het.
9 Eu 18
asym.
9 Ev 18
Moz.
9 Ew 18
2 p. Mox. Het.
9 Ex 18
asym.
9 Ez 18
asym.
9 Fa 18
asym.
9 Fc 18
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18
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9 Fi 18
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9 Fj 18
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9 Fk 18
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9 Fn 18
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9 Fq 18
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9 Ft 18
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9 Gb 18
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9 Ge 18
Moz.
9 Gf 18
2 zo. Mox. Het.
9 Gg 18
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9 Gi 18
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9 Gk 18
2 zo. Mox. Het.
9 Gl 18
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9 Gp 18
2 zo. Mox. Het.
9 Gq 18
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9 Gt 18
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9 Gv 18
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9 Gx 18
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9 Gz 18
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9 Ha 18
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9 Hh 18
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9 Hi 18
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9 Hy 18
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9 Hz 18
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9 Ib 18
2 zo. Mox. Het.
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3 zo. Monch.
9 Ih 18
2 zo. Mox. Het.
9 Ii 18
2 zo. Mox. Het.
9 Il 18
Moz.
9 Ip 18
2 zq. Mox. Het.
9 Iq 18
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9 Ir 18
2 zo. Mox. Het
9 Is 18
9 zo. Monch.
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zo. Monch.