

Enumerating the Prime Alternating Knots, Part I

Stuart Rankin, Ortho Flint[†], John Schermann

Department of Mathematics, University of Western Ontario
srankin@uwo.ca, ortho@uwo.ca

Abstract

The enumeration of prime knots has a long and storied history, beginning with the work of T. P. Kirkman [9,10], C. N. Little [14], and P. G. Tait [19] in the late 1800's, and continuing through to the present day, with significant progress and related results provided along the way by J. H. Conway [3], K. A. Perko [17, 18], M. B. Thistlethwaite [6, 8, 15, 16, 20], C. H. Dowker [6], J. Hoste [1, 8], J. Calvo [2], W. Menasco [15, 16], W. B. R. Lickorish [12, 13], J. Weeks [8] and many others. Additionally, there have been many efforts to establish bounds on the number of prime knots and links, as described in the works of O. Dasbach and S. Hougardy [4], D. J. A. Welsh [22], C. Ernst and D. W. Sumners [7], and C. Sundberg and M. Thistlethwaite [21] and others.

In this paper, we provide a solution to part of the enumeration problem, in that we describe an efficient inductive scheme which uses a total of four operators to generate all prime alternating knots of a given minimal crossing size, and we prove that the procedure does in fact produce them all. The process proceeds in two steps, where in the first step, two of the four operators are applied to the prime alternating knots of minimal crossing size n to produce approximately 98% of the prime alternating knots of minimal crossing size $n + 1$, while in the second step, the remaining two operators are applied to these newly constructed knots, thereby producing the remaining prime alternating knots of crossing size $n + 1$. The process begins with the prime alternating knot of four crossings, the figure eight knot.

In the sequel, we provide an actual implementation of our procedure, wherein we spend considerable effort to make the procedure efficient. One very important aspect of the implementation is a new way of encoding a knot. We are able to assign an integer array (called the master array) to a prime alternating knot in such a way that each regular projection, or plane configuration, of the knot can be constructed from the data in the array, and moreover, two knots are equivalent if and only if their master arrays are identical. A fringe benefit of this scheme is a candidate for the so-called ideal configuration of a prime alternating knot.

We have used this generation scheme to enumerate the prime alternating knots up to and including those of 19 crossings. The knots up to and including 17 crossings produced by our generation scheme concurred with those found by M. Thistlethwaite, J. Hoste and J. Weeks (see [8]).

The current implementation of the algorithms involved in the generation scheme allowed us to produce the 1,769,979 prime alternating knots of 17 crossings on a five node beowulf cluster in approximately 2.3 hours, while the time to produce the prime alternating knots up to and including those

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of 16 crossings totalled approximately 45 minutes. The prime alternating knots at 18 and 19 crossings were enumerated using the 48 node Compaq ES-40 beowulf cluster at the University of Western Ontario (we also received generous support from Compaq at the SC 99 conference). The cluster was shared with other users and so an accurate estimate of the running time is not available, but the generation of the 8,400,285 knots at 18 crossings was completed in 17 hours, and the generation of the 40,619,385 prime alternating knots at 19 crossings took approximately 72 hours. With the improvements that are described in the sequel, we anticipate that the knots at 19 crossings will be generated in not more than 10 hours on a current Pentium III personal computer equipped with 256 megabytes of main memory.

1. Introduction

A knot is a smooth embedding of the unit sphere S^1 into \mathbb{R}^3 , and a knot configuration is a projection of a knot into a plane with preimages of size at most two for which a point has preimage of size two only if the point is the image of a crossing. Two knots are said to be equivalent if there exists a homeomorphism of \mathbb{R}^3 onto itself taking one of the knots to the other. It is conventional to indicate on a knot configuration the additional information required to construct a knot equivalent to the one from which the diagram was made. This information takes the form of over/under passes, indicated by a break in the curve in a neighborhood of the point in the preimage which is furthest from the projection plane. This paper provides an effective method for the construction (and hence enumeration) of all prime alternating knots. We rely on the fact that two prime alternating knots are equivalent if and only if they are flype equivalent (the Tait flyping conjecture, proven by W. W. Menasco and M. B. Thistlethwaite [15]).

Up to the present time, the enumeration of knots has primarily relied on a process of representing each configuration of a knot by a sequence of integers. The most efficient of these representation schemes is the Dowker-Thistlethwaite (see [6]) encoding scheme, whereby each configuration of an alternating knot of n crossings is represented by a permutation of the even integers from 2 to $2n$. The standard approach to prime alternating knot enumeration is essentially to produce permutations of the even integers from 2 to $2n$, where n is the crossing size, then eliminate those permutations that are not the Dowker-Thistlethwaite code of any prime alternating knot, and finally, identify amongst the remaining permutations that represented the same alternating knot.

In this work, we introduce an alternative approach which utilizes a new knot configuration encoding, which we call the group code for the knot configuration.

We then introduce an inductive scheme which uses a total of four operators to generate the prime alternating knots of minimal crossing size $n + 1$ from those of minimal crossing size n . Each stage requires two steps. In the first step, we apply two of the four operators to the prime alternating knots of minimal crossing size n to produce a subset of the set of prime alternating knots of minimal crossing size $n + 1$. In our computations so far (up to the knots of 19 crossings), we have found that over 98% of the total number of knots at a given crossing size have been constructed by the first two operators. In the second step, we apply the remaining

two operators to these newly constructed knots to obtain the prime alternating knots of crossing size $n + 1$. The process begins with the prime alternating knot of four crossings, the figure eight knot.

Two of the operators, those that we call D and $ROTS$, are simply specific instances of the general splice operation (see Calvo [2] for an extensive discussion of the splicing operation). A form of D was also used by H. de Fraysseix and P. Ossona de Mendez (see [5]) in their work to characterize Gauss codes. Another one of the operators, OTS , made an appearance in Conway's seminal paper [3]. The fourth operator seems not to have put in an appearance in the literature to date, but it is conceptually related to Conway's notion of tangle insertion in templates.

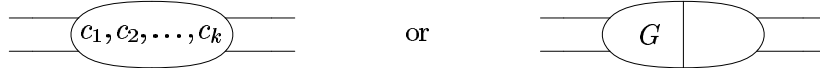
2. Knot encodings

As remarked above, the DT encoding scheme has featured prominently in recent knot enumeration activity. However, our knot operators could not be conveniently represented on the data provided by the DT encoding scheme. Consequently, we introduce a new knot encoding scheme, which we call a group code for the knot configuration.

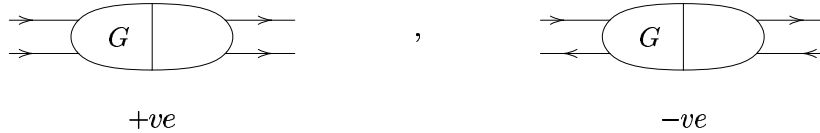
We may regard a knot configuration as a planar graph by considering each crossing as a vertex of the graph and the portion of the curve between two consecutive crossings as an edge between the two vertices. Such a planar graph is always 4-regular.

Definition 1. Given the planar graph for a knot configuration, a k -group, or a group of k , is a maximal length sequence c_1, c_2, \dots, c_k of crossings, subject to the requirement that $k \geq 2$ and for each i from 1 to $k - 1$, c_i and c_{i+1} are joined by two edges. The two edges that are incident to c_1 but not incident to c_2 are called the edges at one end of the group, as are the two edges incident to c_k but not to c_{k-1} . If c_1, c_2, \dots, c_k is a group, any subsequence of the form c_i, c_2, \dots, c_j , where $1 \leq i < j \leq k$ is called a subgroup of the group c_1, c_2, \dots, c_k . More precisely, if $t = j - i + 1$, then c_i, c_2, \dots, c_j is called a t -subgroup of the k -group c_1, c_2, \dots, c_k . The two edges incident to c_i but not to c_{i+1} are called the edges at one end of the subgroup, as are the two edges that are incident to c_j but not to c_{j-1} .

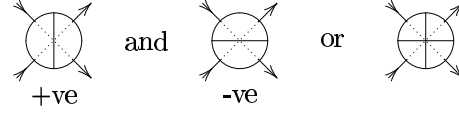
Schematically, we shall illustrate a group by



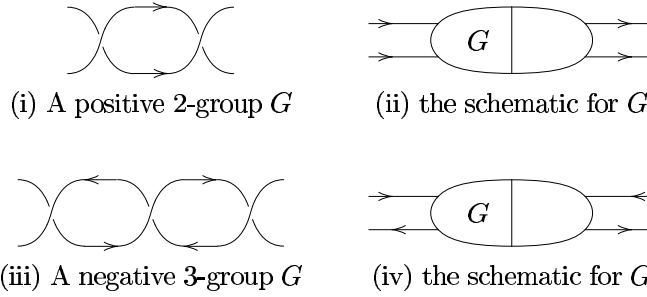
Each group of k is referred to as a positive or negative group according to the following convention: if, upon traversing the knot in either of the two possible directions of travel, it is found that the group is either being entered or exited on both arcs at one end of the group, then the group is said to be positive, otherwise the group is said to be negative.



Any crossing c that is not an element of a group of k for any k shall be called a group of 1, or a loner. The schematic we shall use for a loner is



In certain circumstances, we shall refer to a loner as a positive group, while in other circumstances the same loner may be considered as a negative group. When a loner is being considered as a positive group, then the two adjacent outbound edges (as determined by any traversal of the knot) are the two edges at one end of the group, and the two inbound edges are the two edges at the other end of the group, while if the loner is being considered as a negative group, then an inbound edge and the adjacent outbound edge are considered to be the two edges at one end of the group, with the remaining pair of inbound and outbound edges being the two edges at the other end of the group.



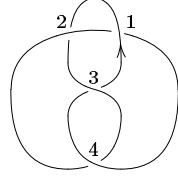
Finally, by a group of the knot configuration, we mean any group of k for $k \geq 1$.

It is clear that each crossing of a knot configuration belongs to exactly one group of the knot configuration, and so the groups provide us with a partition of the set of crossings of the knot configuration.

We are now ready to define the group code for a knot configuration.

Definition 2. Let $C(K)$ denote a configuration of the knot K . For each $k \geq 1$ for which K has at least one group of k , let i_k be the number of groups of k . Label the groups of k in an arbitrary fashion with the labels k_1, k_2, \dots, k_{i_k} . Assign an orientation to K and select arbitrarily a group to start at. Select an outbound edge at one end of this group and, starting with this edge, traverse the knot. Form the sequence of group labels in the order that the groups are encountered in this traversal. Each label will appear exactly twice in this sequence. If a group is negative, then preface both occurrences of the group's label in the sequence with a minus sign. We adopt the convention of labelling any loner as a positive group. The resulting sequence is called a group code for $C(K)$.

One consequence of the fact that the labels k_1, k_2, \dots, k_{i_k} are assigned arbitrarily is that any cyclic permutation of a group code is again a group code for the same knot configuration. This is in keeping with the cyclic structure of a knot, and it is certainly beneficial to view a group code as a cyclic object rather than a linear array.



Gauss code: $\begin{array}{cccccccc} + & - & + & - & + & - & + & - \\ 1, & 2, & 3, & 4, & 2, & 1, & 4, & 3, \end{array}$

$\begin{array}{cccccccc} 1, & 2, & 3, & 4, & 2, & 1, & 4, & 3, \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array}$
 Group code: $-2_1, -2_2, -2_1, -2_2,$

3. The Orbit of a Group

We shall need to have a graph-theoretic description of an m -tangle. A non-trivial m -tangle contains at least one crossing, although it may consist of an $m - 2$ tangle together with an edge that is not incident to any crossing in the tangle. We shall refer to an edge that passes through a tangle without being incident to any vertex of the tangle as a pass-through edge for the tangle. We shall be largely interested in 4-tangles, (in fact, the word tangle shall mean 4-tangle), and a 4-tangle that has an edge that is not incident to any vertex of the tangle is either trivial, or else consists of a 2-tangle and this edge. In a prime knot, there are no 2-tangles, so the smallest m for which there exists an m -tangle is $m = 4$. Thus in a prime knot, a non-trivial 4-tangle must have each edge that enters the 4-tangle be incident to a vertex of the tangle. Moreover, since a prime knot has no 2-tangles, each non-trivial 4-tangle of a prime knot configuration is a connected graph. In the study of prime knots, we shall only have need of connected m -tangles, and a connected m -tangle with at least one crossing can be characterized as a connected induced subgraph of the planar graph, together with all edges incident to exactly one vertex in the vertex set of the subgraph (by induced subgraph, we mean a subgraph formed by choosing a set of vertices and then taking all edges that have both endpoints belonging to the selected set of vertices). An edge that is incident to a vertex of an m -tangle T but that is not an edge of T is said to be incident to T . Thus an m -tangle has m incident edges.

The following two facts about tangles in a prime knot will prove to be quite useful.

Lemma 1. Let $C(K)$ be a configuration of a prime knot K , and let T_1 and T_2 be tangles in $C(K)$ such that $T_1 \cap T_2 \neq \emptyset$, $T_1 - T_2 \neq \emptyset$ and $T_2 - T_1 \neq \emptyset$. Then there are at least two arcs of T_1 each having exactly one endpoint in T_2 , and at least two arcs of T_2 each having exactly one endpoint in T_1 .

Proof. Since T_2 is connected, there must be at least one arc with one endpoint in $T_2 - T_1$ and the other endpoint in $T_2 \cap T_1$. Suppose there is exactly one such arc. Then the other 3 arcs incident to T_1 must have their other endpoints outside of T_2 . We identify four cases.

Case 1: none of these three arcs have endpoints in $T_1 \cap T_2$. Then since $T_1 \cap T_2$ must have at least 4 incident arcs, there must be at least 3 arcs joining vertices in $T_1 \cap T_2$ to vertices in $T_1 - T_2$, whence three of the four arcs incident to T_2 have already been accounted for. Now there must be at least four arcs incident to $T_2 - T_1$, and only one of these can have its other endpoint in $T_1 \cap T_2$. Thus there are at least three arcs incident to $T_2 - T_1$ whose other endpoints are not in T_1 . But then T_2 has at least six incident arcs, which is not possible.

Case 2: exactly one of these three arcs has an endpoint in $T_1 \cap T_2$. Then the remaining two arcs are incident to $T_1 - T_2$, with their other endpoint not in T_2 . Since $T_1 - T_2$ must have at least four incident arcs, there must be at least two arcs joining vertices in $T_1 - T_2$ to vertices in $T_1 \cap T_2$. But then we have accounted for at least three arcs incident to T_2 , and all three are actually incident to $T_1 \cap T_2$. Since $T_2 - T_1$ must have at least four incident arcs, of which exactly one is incident to $T_1 \cap T_2$, there must be at least three arcs joining vertices in $T_2 - T_1$ to vertices neither in T_2 nor in T_1 . But then we have accounted for at least six arcs incident to T_2 , which is not possible.

Case 3: exactly two of these three arcs have an endpoint in $T_1 \cap T_2$. Then the remaining arc is incident to $T_1 - T_2$ with its other endpoint not in T_2 . Since $T_1 - T_2$ must have at least four incident arcs, there must be at least three arcs joining vertices in $T_1 - T_2$ to vertices in $T_1 \cap T_2$. We have now accounted for five arcs incident to T_2 , which is not possible.

Case 4: all three of the arcs are actually incident to $T_1 \cap T_2$. Since there are at least four arcs incident to $T_1 - T_2$, none of which may have their other endpoints outside of T_1 , there must be at least four arcs joining vertices in $T_1 - T_2$ to vertices of $T_1 \cap T_2$. But then we have accounted for at least seven arcs incident to T_2 , which is not possible.

Since every case leads to a contradiction, we conclude that there must be at least two arcs having one endpoint in $T_2 - T_1$ and the other endpoint in $T_2 \cap T_1$. Since T_1 and T_2 have symmetric roles, it follows as well that there must be at least two arcs having one endpoint in $T_1 - T_2$ and the other endpoint in $T_2 \cap T_1$. \square

Proposition 1. Let $C(K)$ be a configuration of an alternating prime knot, and let T_1 be a non-trivial tangle of K . Cut the four incident arcs to T_1 to allow T_1 to be removed, and replace T_1 by a non-trivial tangle T_2 such that the result is a configuration $C(K_1)$ of an alternating knot K_1 . If T_2 does not contain any 2-tangle, then K_1 is prime.

Proof. Suppose that K_1 is not prime, and let T be a 2-tangle of $C(K_1)$. If $T \cap T_2 = \emptyset$, then T is a 2-tangle in $C(K)$, which is not possible. Thus $T \cap T_2 \neq \emptyset$. If $T \subseteq T_2$, then we are done, so suppose that $T - T_2 \neq \emptyset$. If $T_2 \subseteq T$, then we may undo the surgery to form T' from T by replacing T_2 with T_1 . But then T' is a 2-tangle in $C(K)$, which is not possible. Thus $T_2 - T \neq \emptyset$.

There are at most two arcs incident to $T \cap T_2$ that are actually incident to T . There are three cases to consider.

Case 1: there are exactly two arcs incident to $T \cap T_2$ that are actually incident to T . Since $T - T_2$ is an m -tangle in $C(K)$, $m \geq 4$, there are at least four edges incident to $T - T_2$ each with their other endpoint in $T \cap T_2$. But then they account for all four edges incident to T_2 , in which case $T_2 - T$ has no incident edges. This is not possible.

Case 2: there is exactly one arc incident to $T \cap T_2$ that is incident to T . Then there is one arc incident to T that is actually incident to $T - T_2$. Since $T - T_2$ is an m -tangle in $C(K)$, $m \geq 4$, there are at least four edges incident to $T - T_2$, exactly one of which has an endpoint outside T , whence there are at least three edges incident to $T - T_2$ each with their other endpoint in $T \cap T_2$. But this accounts for at least three edges incident to T_2 . Thus there is at most one edge incident to

T_2 that has an endpoint in $T_2 - T$, which means that $T_2 - T$ is at most a 2-tangle. But then $T_2 - T$ must be a 2-tangle, and it is contained in T_2 , as required.

Case 3: there are no arcs incident to $T \cap T_2$ that are incident to T . Since $T - T_2$ is an m -tangle in $C(K)$, $m \geq 4$, and the two arcs incident to T are actually incident to $T - T_2$, there must be at least two arcs from $T - T_2$ to $T \cap T_2$. But this accounts for at least two of the arcs incident to T_2 , whence there are at most two arcs incident to T_2 that are actually incident to $T_2 - T$. Any additional arcs incident to $T_2 - T$ would have to have their endpoints in $T \cap T_2$, but these would then be arcs incident to $T \cap T_2$ and incident to T , of which there are none. Thus $T_2 - T$ is a 2-tangle contained in T_2 , as required.

This completes the proof that if K_1 is not prime, then T_2 contains a 2-tangle.

□

We remark that in the setting of the preceding proposition, it is possible that certain tangles T_2 might lead to a link or a non-alternating knot. We are only interested in those situations as described in the conditions of the proposition.

Proposition 2. Let $C(K)$ be a configuration of a prime knot K , and let e and f be arcs of $C(K)$. Let a and b be endpoints of e and f , respectively. If T_1 and T_2 are tangles containing a and b with e and f being arcs incident to both T_1 and T_2 , then either $T_1 \subseteq T_2$ or else $T_2 \subseteq T_1$.

Proof. Suppose not. Then $T_1 - T_2$ is an m -tangle with $m \geq 4$ since K is prime. Thus there are at least four arcs incident to $T_1 - T_2$. Since $a, b \in T_2$, these arcs are all different from e and f . But T_1 has four incident arcs, two of which are e and f , so at most two of these m arcs are incident to T_1 , with the having one end point in $T_1 - T_2$ and the other endpoint in $T_1 \cap T_2$. Thus at least two of these m arcs are incident to T_2 . Since T_2 is a 4-tangle, and e and f are incident to T_2 , it must be that there are exactly two arcs that have one endpoint in $T_1 \cap T_2$, and the other endpoint in $T_1 - T_2$. The same reasoning applied to $T_2 - T_1$ establishes that there are exactly two arcs that have an endpoint in $T_1 \cap T_2$ and the other endpoint in $T_2 - T_1$. But then the four arcs incident to T_1 are e , f and the two arcs going from $T_1 \cap T_2$ to $T_2 - T_1$, whence $T_1 - T_2$ has only two incident arcs. This contradicts the fact that $m \geq 4$, so the assumption that neither $T_1 \subseteq T_2$ nor $T_2 \subseteq T_1$ holds is false.

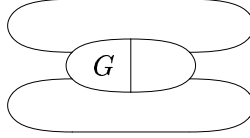
□

Corollary 1. If e and f are edges in a configuration $C(K)$ of a prime knot K , and a and b are endpoints of e and f , respectively, such that there is a tangle T with e and f incident to T and $a, b \in T$, then there is a minimum such tangle.

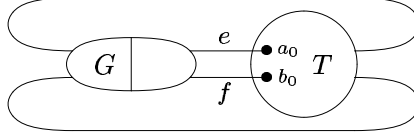
Proof. Of all tangles with this property, let T denote one with fewest vertices. By Proposition 2, if T_1 is any tangle with this property, then either $T \subseteq T_1$ or else $T_1 \subseteq T$. If $T_1 \subseteq T$, then by the minimality of T , we have $T_1 = T$. Otherwise, $T \subseteq T_1$. □

We are now ready to present the definition of the orbit (or flype circuit) of a group in a prime knot.

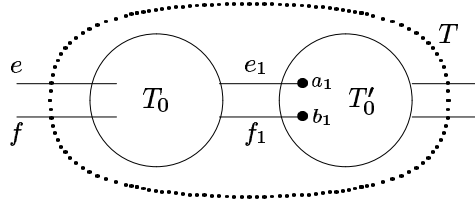
Definition 3. Let $C(K)$ be a configuration of the prime alternating knot K , and let G be a group of $C(K)$. If K is a torus knot, then $C(K)$ is



and we define the orbit of G to be the sequence (G) . Otherwise, K is not a torus knot, and $C(K)$ is

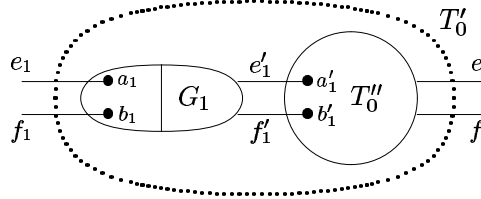


where T is a non-trivial tangle. Let $G_0^* = G_0 = G$. By Corollary 1, there is a tangle T_0 which contains a_0, b_0 and has e, f as incident edges, and which is contained in all other tangles with this property. In particular, $T_0 \subseteq T$ and T is of the form

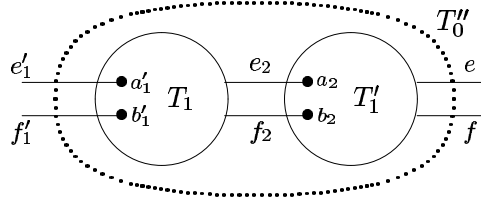


If T'_0 is trivial, then the orbit of G is the sequence (G_0^*, T_0) . Otherwise, T'_0 is non-trivial and we examine the arcs e_1 and f_1 .

Case (1): $a_1 = b_1$. Let G_1 denote the group containing this crossing. Then $G_1 \subseteq T'_0$, and T'_0 has the form

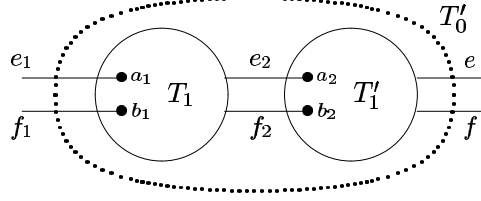


where T''_0 is necessarily a non-trivial tangle. By Corollary 1, there is a tangle T_1 which contains a'_1, b'_1 and has e'_1, f'_1 as incident edges, and which is contained in all other tangles with this property. In particular, $T_1 \subseteq T''_0$ and T''_0 is of the form

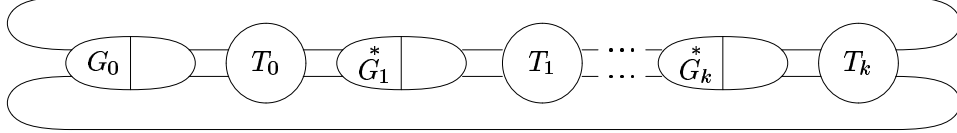


Let $G_1^* = G_1$.

Case (2): $a_1 \neq b_1$. By Corollary 1, there is a tangle T_1 which contains a_1, b_1 and has e_1, f_1 as incident edges, and which is contained in all other tangles with this property. In particular, $T_1 \subseteq T'_0$ and T'_0 is of the form



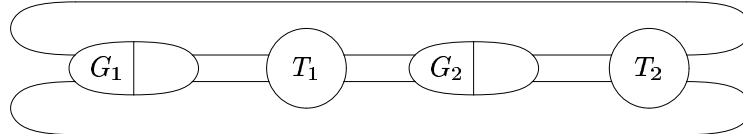
Let $\bar{G}_1 = \{e_1, f_1\}$. In either case, if T'_1 is trivial, we define the orbit of G to be the sequence $(\bar{G}_0, T_0, \bar{G}_1, T_1)$. If T'_1 is non-trivial, we repeat the process. For some k , we end up with T'_k trivial and the process stops. The orbit of G is the sequence $(\bar{G}_0, T_0, \bar{G}_1, T_1, \dots, \bar{G}_k, T_k)$. $C(K)$ has the form



The tangles T_0, T_1, \dots, T_k are called the *min-tangles* of the orbit of G , and for each i , $0 \leq i \leq k$, \bar{G}_i marks *position i* of the orbit. For each min-tangle T_i , the orbit causes the four arcs incident to T_i to be partitioned into two pairs, and each pair shall be referred to as a *position pair* for the min-tangle. For each i , \bar{G}_i either denotes a group or else a pair of arcs. Let $G_F = \{\bar{G}_i \mid \bar{G}_i \text{ is a group}\}$. G_F is called the *full group determined by G* and each \bar{G}_i that is a group is said to be the *component in position i* of the full group. If $G_F = \{G\}$, then G_F is said to be a *full group*. Otherwise, we say that G_F is a *split group*. If $G_F = \{G\}$, we also say that G is a full group, while if G_F is not a full group, then we say that G is a member of the split group.

We remark that if the two arcs on the other side of group G had been chosen to construct the orbit of G , the sequence that would result would be the present sequence presented in the reverse order. Moreover, if for some i , \bar{G}_i^* is a group and we used the two arcs from \bar{G}_i^* to T_i to construct the orbit of the group \bar{G}_i^* , the result would be the sequence $(\bar{G}_i^*, T_i, \bar{G}_{i+1}^*, T_{i+1}, \dots, \bar{G}_k^*, T_k, \bar{G}_0^*, T_0, \dots, \bar{G}_{i-1}^*, T_{i-1})$.

Definition 4. A knot configuration is said to be a *split-group configuration* if there exist groups G_1 and G_2 and non-trivial tangles T_1 and T_2 such that $C(K)$ is of the form



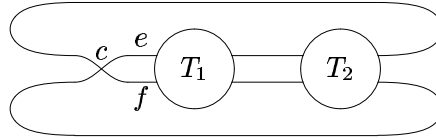
In this case, we say that G_1 and G_2 are part of a split group, with the splitting achieved by tangles T_1 and T_2 . A knot configuration that is not a split-group configuration is said to be a full-group configuration.

Observe that if G_1 and G_2 are part of a split group, with the splitting achieved by tangles T_1 and T_2 , and if G_1 is a group of k and G_2 is a group of l , then a sequence of l flype moves can be applied to flype each crossing of G_2 across T_1 . In the resulting configuration, the groups G_1 and G_2 have become one group G of $k + l$ such that (G, T_1, T_2) is a flype scenario. We say that G_2 has been group flyped (across T_1).

Definition 5. A prime knot configuration is said to be a full-group configuration if every group is a full group. Otherwise, the knot configuration is said to be a split-group configuration.

Our next objective is to establish that if G and G' are groups for which $G_F \neq G'_F$, then the orbit of G and the orbit of G' are in a sense orthogonal to each other. By this we mean that there is a min-tangle T' of the orbit of G' such that $C(K) - T'$ is contained within a min-tangle T of the orbit of G . In order to show this, we shall first of all need to identify the possible flype moves for an individual crossing. In order to do this, we shall introduce a notion of flype scenario for an individual crossing.

Definition 6. Let $C(K)$ be a configuration of a prime knot K , and let c be a crossing of $C(K)$. A crossing flype scenario for c on edges e and f is a sequence (c, e, f, T_1, T_2) , where T_1 and T_2 are tangles, e and f are adjacent arcs at c , incident to T_1 , and $C(K)$ has the form



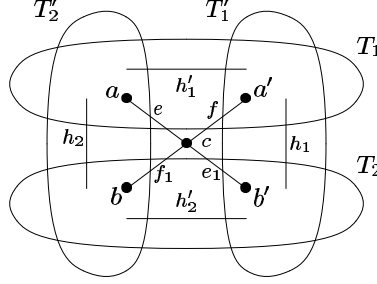
Note that unlike the situation for flype scenario, it is not implied that the endpoints of e and f that belong to T_1 be distinct, nor that the endpoints of the other two arcs incident to c that belong to T_2 be distinct. For example, if K is a 3-crossing torus knot, then we might take c to be the first crossing in the group of three crossings, T_1 to be just the second crossing with its incident arcs and T_2 to be the third crossing with its incident arcs, with e and f being the arcs from c to T_1 .

It turns out that if we label the edges incident to a crossing c in the clockwise direction as e, f, e_1 and f_1 , and there is a crossing flype scenario for c on edges e and f , then there is no crossing flype scenario for c on edges f and e_1 .

Proposition 3. Let $C(K)$ be a configuration of a prime knot K and let c be a crossing of $C(K)$. Let e, f and e_1 be edges incident to c with e and f adjacent, and f and e_1 adjacent. If there is a crossing flype scenario for c on edges e and f , then there is no crossing flype scenario on edges f and e_1 .

Proof. Suppose to the contrary that (c, e, f, T_1, T_2) and (c, f, e_1, T'_1, T'_2) are crossing flype scenarios. Let the endpoints of e and f in T_1 be denoted by a and a' , respectively, and let the endpoint of e_1 that is in T'_1 be denoted by b' . Let f_1 denote

the fourth edge incident to c , and denote its other endpoint by b . Then we have a and a' in T_1 , b and b' in T_2 , a' and b' in T'_1 and a and b in T'_2 . Let h_1 and h_2 denote the two edges incident to both T_1 and T_2 , and let h'_1 and h'_2 denote the two edges incident to both T'_1 and T'_2 . Since a and a' are in T_1 , which is connected, there is a path from a to a' using only edges of T_1 . But the graph that is obtained by deleting c and the two edges h'_1 and h'_2 from $C(K)$ has two connected components, namely T'_1 and T'_2 , and a is in T'_2 , while a' is in T'_1 . Thus every path in $C(K)$ from a to a' that does not go through c must use either edge h'_1 or else h'_2 .



Since there is a path from a to a' that uses only edges in T_1 , either h'_1 or h'_2 is an edge in T_1 . A similar argument applied to b and b' shows that either h'_1 or h'_2 must be an edge in T_2 . Thus exactly one of h'_1 and h'_2 belongs to T_1 , the other belongs to T_2 . We may suppose without loss of generality that h'_1 is in T_1 and h'_2 is in T_2 . Similarly, we may suppose without loss of generality that h_1 is in T'_1 and h_2 is in T'_2 . Now a belongs to both T_1 and T'_2 , so $T_1 \cap T'_2$ is an m -tangle for some $m \geq 4$. Now h'_1 is an edge of T_1 and incident to T'_2 , so h'_1 is incident to $T_1 \cap T'_2$. Similarly, h_2 is incident to $T_1 \cap T'_2$. As well, e is incident to $T_1 \cap T'_2$. There must be at least one more edge g incident to $T_1 \cap T'_2$. The edges incident to T_1 are e , f , h_1 and h_2 , and g is not equal to e or h_2 . Since f is incident to T'_1 and not to T'_2 , g is not equal to f . Finally, h_1 is an edge in T'_1 and therefore not incident to T'_2 , whence g is not equal to h_1 . Thus g must be an edge of T_1 . But now a similar argument shows that g must also be an edge of T'_2 , whence g is an edge of $T_1 \cap T'_2$. But this contradicts the fact that g was incident to $T_1 \cap T'_2$, and so it is not possible to have crossing flype scenarios (c, e, f, T_1, T_2) and (c, f, e_1, T'_1, T'_2) . \square

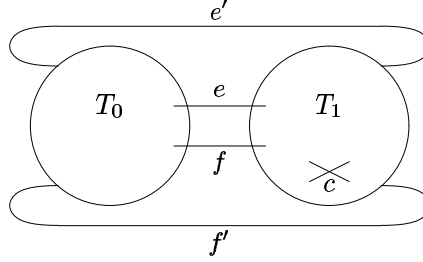
This leads to the next result, which may be thought of as establishing the independence of orbits, or the non-interference of orbits.

Theorem 1. Let $C(K)$ be a configuration of a prime alternating knot K , and let G be a group of K and T_1 be a min-tangle of the orbit of G . Let c be a crossing in T_1 and suppose that T is a tangle over which c may flype. Then either $T \subseteq T_1$ or else $C(K) - T_1 \subseteq T$.

Proof. To begin with, we observe that we may assume without loss of generality that $T \cap T_1 \neq \emptyset$. To see why this is so, we note that c may flype over the tangle $T' = C(K) - T - \{c\}$, and if $T \cap T_1 = \emptyset$, then $T' \cap T_1 \neq \emptyset$. Now, $T \subseteq T_1$ if and only if $C(K) - T_1 \subseteq C(K) - T$, and since $c \in T_1$, we know that $c \notin C(K) - T_1$, whence $C(K) - T_1 \subseteq C(K) - T$ if and only if $C(K) - T_1 \subseteq C(K) - T - \{c\}$. Thus $T \subseteq T_1$ if and only if $C(K) - T_1 \subseteq T'$. Finally, since $T = C(K) - T' - \{c\}$, we obtain by

symmetry that $C(K) - T_1 \subseteq T$ if and only if $T' \subseteq T_1$. Consequently, $T \subseteq T_1$ or $C(K) - T_1 \subseteq T$ if and only if $C(K) - T_1 \subseteq T'$ or $T' \subseteq T_1$.

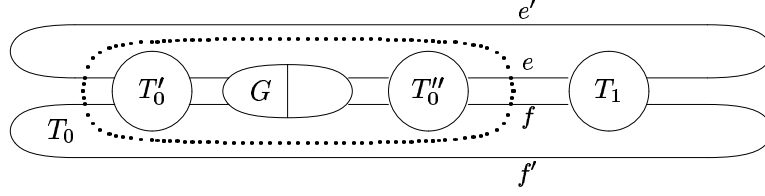
We have established now that we may assume that $T \cap T_1 \neq \emptyset$. If $T \subseteq T_1$, then there is nothing to prove, so we consider the case when T contains at least one crossing that is not in T_1 . We must show that $C(K) - T_1 \subseteq T$. Suppose to the contrary that $C(K) - T_1 \not\subseteq T$, and let $T_0 = C(K) - T_1$. Label one of the position pairs for T_1 as e and f , with the other pair being labelled with e' and f' .



Now, $T_0 \not\subseteq T$ implies that $T_0 - T \neq \emptyset$, whence $T_0 - T$ is an m_1 -tangle and $T_0 \cap T$ is an m_2 -tangle for some even integers $m_1, m_2 \geq 4$. If there was at most one edge joining a vertex of $T_0 - T$ to a vertex of $T_0 \cap T$, then there would be at least 3 edges each with one endpoint in $T_0 - T$ and the other endpoint not in T_0 . Since T_0 has exactly 4 incident edges, this would mean that there is exactly one edge with one endpoint in $T_0 \cap T$ and the other endpoint not in T_0 , in which case $m_2 = 2$, which is not the case. Thus there are at least two edges each with one endpoint in $T_0 - T$ and the other endpoint in $T_0 \cap T$. Such arcs belong to T_0 , so there are at least two arcs of T_0 that are incident to T . Similarly, since $T_1 \cap T \neq \emptyset$ and $T_1 - T \neq \emptyset$, there are at least two arcs of T_1 that are incident to T . But T is a tangle, so there are exactly four arcs incident to T , whence there must be exactly two arcs between $T_0 - T$ and $T_0 \cap T$, and exactly two arcs between $T_1 - T$ and $T_1 \cap T$. Thus of the four arcs incident to T , two are arcs of T_0 and two are arcs of T_1 , whence none of the four arcs e, f, e', f' are incident to T . Since $m_1 \geq 4$, and there are exactly two arcs from $T_0 - T$ to $T_0 \cap T$, at least two of e, f, e', f' are incident to $T_0 \cap T$, hence have both endpoints in T . There are two cases to consider: both arcs of one of the position pairs for T_1 belong to T , or exactly one from each of the two position pairs for T_1 belongs to T .

Case 1: both arcs of one of the position pairs for T_1 belong to T . Without loss of generality, we may assume that e and f belong to T . But then $T_1 \cap T$ is a tangle with e and f incident edges, contradicting the minimality of T_1 . Thus this case can't occur.

Case 2: exactly one from each of the two position pairs for T_1 belongs to T . Suppose that the position pairs have been labelled so that e and e' are the two arcs that belong to T . Since two of the four edges incident to T are edges of T_1 , and the other two are edge of T_0 , we see that f and f' are not in T nor are they incident to T . Thus e and f have different endpoints in T_0 , and e' and f' have different endpoints in T_0 . Thus e and f are not edges incident to the group G , nor are e' and f' incident to G , whence T_0 has a decomposition (obtained from the orbit of G as a tangle T'_0 , the group G and a tangle T''_0 , so $C(K)$ has the structure

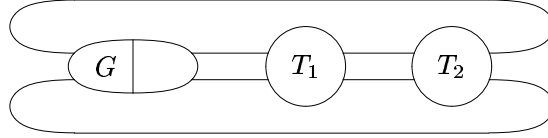


Now $T \cap T_0''$, $T_0'' - T$ and $T - T_0''$ are all nonempty, so by Lemma 1, there are two arcs of T_0'' incident to T . Similarly, there are two arcs of T_0' incident to T . As well, there are two arcs of T_1 incident to T , whence we have accounted for six arcs incident to T . But this is not possible, so the assumption that $C(K) - T_1 \not\subseteq T$ must be false. We conclude therefore that $C(K) - T_1 \subseteq T$. \square

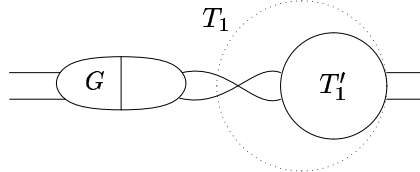
Corollary 2. Let $C(K)$ be a configuration of a prime knot K , and let G be a group of $C(K)$. If T_1 is any min-tangle from the orbit of G and H is any group of $C(K)$ for which $H \cap T_1 \neq \emptyset$, then $H \subseteq T_1$.

Proof. Let H be a group such that $H \cap T_1 \neq \emptyset$, so there is a crossing c of H that is in T_1 . Suppose that H is not contained entirely in T_1 , so there is a crossing d of H that is not in T_1 . Let T denote the connected component of $H - \{c\}$ that contains d . Then T is a tangle over which c may flype. By Theorem 1, we either have $T \subseteq T_1$ or else $C(K) - T_1 \subseteq T$. Since $d \in T - T_1$, it must be that $C(K) - T_1 \subseteq T$. But then $G \subseteq T$, whence d is a crossing in G . By the definition of group, this means that c is in G , which contradicts the fact that T_1 is a min-tangle in the orbit of G . Since this contradiction follows from the assumption that H is not contained entirely in T_1 , we conclude that $H \subseteq T_1$. \square

Definition 7. A *flype scenario* (G, T_1, T_2) in a knot configuration $C(K)$ consists of a group G and two non-trivial tangles T_1 and T_2 such that $C(K)$ is of the form



Observe that T_1 itself cannot be decomposed in the form



In other words, the two arcs that join the group G to T_1 have distinct endpoints in T_1 . Similarly, the two arcs that join the group G to T_2 must have distinct endpoints in T_2 .

It is evident that given a flype scenario (G, T_1, T_2) , each crossing of G can be flyped to the two arcs joining T_1 and T_2 . Conversely, suppose that c is a crossing for which there is a flype move: that is, there is a tangle T across which c can be flyped. Let G denote the group that contains c . There are several possibilities to be investigated.

Case 1: $G \cap T = \emptyset$.

Case 1a) $G \cup T = C(K)$. In this case the flype move does not change the knot configuration.

Case 1b) $G \cup T \neq C(K)$. Then $T_1 = C(K) - (G \cup T)$ is a non-trivial tangle and G , T , and T_1 constitute a flype scenario.

Case 2: $G \cap T \neq \emptyset$ (by which we mean there is at least one crossing that is common to G and to T). There are two subcases to be addressed.

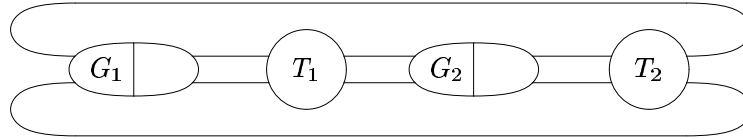
Case 2a) $T \subseteq G$. In this case, the flype move does not change the knot configuration.

Case 2b) $T \not\subseteq G$. Let $T' = T - G$, so that T' is a tangle with $G \cap T' = \emptyset$. Furthermore, at least one end of G is joined to T' . Let c' denote the crossing at an end of G which is joined to T' . There is a finite sequence of flype moves, each of which will leave the knot configuration unchanged, which will flype c to the arcs joining c' to T' . One final flype move will flype c across T' , resulting in the knot configuration that is created by flying c across the original tangle T . We observe that this is the same configuration that would result from flying c' across T' . Accordingly, we replace c by c' and T by T' to obtain a situation where c is to be flyped across T and $G \cap T = \emptyset$. This situation has been handled in Case 1.

The preceding discussion makes it clear that the only flype moves of importance are those identified by a flype scenario.

Theorem 2. Given a knot configuration $C(K)$ of an alternating prime knot K , there is a finite sequence of group flype moves which will transform $C(K)$ into a full-group configuration.

Proof. The proof is by induction on the number of groups in the configuration, and the base case would be a knot configuration with a single group, which is already a full-group configuration. Suppose now that $n \geq 1$ is an integer such that for any prime alternating knot K , any configuration of K with n groups can be transformed into a full-group configuration of K by a finite sequence of group flype moves. Let $C(K)$ be a knot configuration with $n+1$ groups. If $C(K)$ is a full-group configuration, then there is nothing to do. Suppose then that $C(K)$ is a split-group configuration. Let G_1 and G_2 be part of a split group. Then there are non-trivial tangles T_1 and T_2 such that $C(K)$ has the form



and G_2 can be group flyped over T_1 (or T_2) to merge with G_1 . Let G' denote the group that results when G_2 is group flyped over T_1 to merge with G_1 . Then the orbit of G' in the resulting knot configuration $C_1(K)$ contains one fewer groups than the orbit of G_1 had in the original configuration $C(K)$. If H is any group of $C(K)$ that does not belong to the orbit of G_1 , then by Corollary 2, H is contained within some min-tangle of the orbit of G_1 , whence H is contained entirely within either T_1 or else T_2 , in which case H is still a group in $C_1(K)$. Thus the configuration $C_1(K)$ has n groups. By the induction hypothesis, this configuration can be transformed into a full-group configuration by a finite sequence of group flype moves. Consequently, we

obtain a finite sequence of group flype moves that transforms $C(K)$ into a full-group configuration. Now the result follows by induction. \square

Note that in a prime knot, every arc incident to a tangle must terminate at a vertex of the tangle.

We conclude this section with an observation about the orbit of a negative group. To begin with, note that once a knot has been given an orientation, it follows that for any m -tangle, exactly half of the incident arcs are directed into the m -tangle, with the other half of course directed out of the m -tangle. In particular, if G is a group in an oriented prime alternating knot configuration $C(K)$, then for any min-tangle T in the orbit of G , two of the incident arcs are directed into the tangle and two are directed out of the tangle. If G is a positive group, then both arcs of each position pair for T are oriented the same, where as we leave the group to traverse the knot, the first time that we follow an arc incident to T establishes that position pair as being directed into T , while the other position pair will be directed outward from T . On the other hand, if G is a negative group, then the two arcs in both position pairs have opposite orientation.

Proposition 4. Let $C(K)$ be a configuration of a prime alternating knot K , and let G be a negative group of $C(K)$. There is a unique min-tangle T from the orbit of G such that a knot traversal of K begun at G travels both arcs of one position pair of T before it travels an arc from the other position pair of T .

Proof. Let $(G_0^*, T_0, G_1^*, T_1, \dots, G_k^*, T_k)$ be the orbit of G , so that T_0, T_1, \dots, T_k are the min-tangles of the orbit of G . Observe that we may perform any necessary group flypes to put G_F into full group without changing the min-tangles or the directions assigned to their incident arcs by assigning an orientation to $C(K)$. We

may therefore assume that G is a full group, and that $G_i^* = \{e_i, f_i\}$ for each $i = 1, \dots, k$. As well, let the position pair for T_0 which connect G to T_0 at the beginning of the orbit of G be denoted by $\{e_0, f_0\}$. Give G an orientation and suppose that the arcs have been labelled so that when a knot traversal is performed in the direction determined by the orientation, starting at G , we move initially towards T_0 on e_0 , and that for each $i = 0, 1, \dots, k$, the arc that is used to move from T_i to T_{i+1} (or to G if $i = k$) is e_{i+1} . The orbit of G is thus of the form

Suppose to begin with that as the knot is traversed, the traversal takes us into tangle T_i on e_i and leaves via arc f_{i+1} for every $i = 0, 1, \dots, k$. Then we first return to G via arc e_{k+1} , and since the two arcs e_{k+1} and f_{k+1} meet at a crossing of G , the traversal will not take us out of G via f_{k+1} . It follows that the traversal will take us out of G via e_0 , which means that our knot is a link. Since this is not the case, there must be some index i such that the knot traversal takes us into T_i via arc e_i but then leaves T_i next via arc f_i , whence T_i is a min-tangle with the required properties.

The uniqueness follows similarly from the observation that if there were two such min-tangles, then again we would have to conclude that K is a link, which it isn't. \square

Definition 8. Let $C(K)$ be a configuration of a prime alternating knot K , and let G be a negative group of $C(K)$. The unique min-tangle whose existence was established in Proposition 4 is called the *core* of the orbit of G .

4. The Construction Process

In this section, we introduce the four operators that will be utilized in the construction of the prime alternating knots. Two of the four, D and $ROTS$, take a prime alternating knot configuration with n crossings and return a prime alternating knot configuration with $n + 1$ crossings, while the other two operators do not change the number of crossings.

4.1. The D operator

The D operator can be applied to any crossing of a knot configuration, where the crossing is treated as if it were a negative loner, and the end result is to replace the crossing by a negative group of 2. More precisely, let $C(K)$ be a configuration of a prime alternating knot K , and let x be a crossing of $C(K)$. Label the arcs incident to x as shown in Figure 1 (a), so that under a traversal of the knot, arc a is used to visit x for the first time, and arc b is used to leave x after that visit, while arc c is used for the second visit, with arc d used to leave x after the second visit. Then the result of applying D to x , as shown in Figure 1 (b), is to replace x with a negative 2-group.

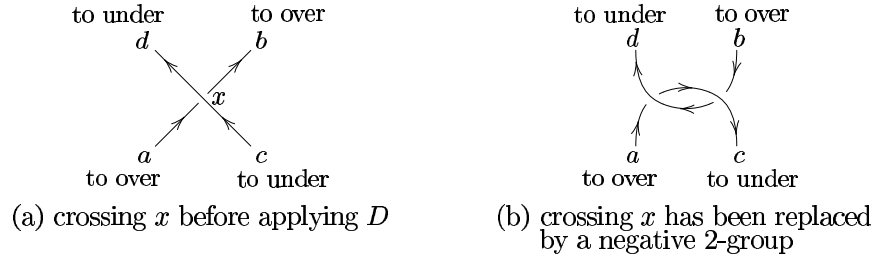


Figure 1: The D operator

For the construction of prime alternating knots, it will suffice to apply D to any one crossing of any negative group or loner, or any positive 2-group, but we prove next that if $C(K)$ is a configuration of a prime alternating knot and D is applied to any crossing of $C(K)$, the result is a configuration $C(K')$ of a prime alternating knot K' of one higher crossing size.

First of all, observe that the result of applying D is a knot K' , rather than a link, since in effect, the curve was cut on both the overpassing strand and the underpassing strand at x , then the loose ends rejoined so that with the orientation as shown in Figure 1, in following a knot traversal we travel from arc a through one arc of the 2-group to arc c , following the portion of the knot which is delineated by arcs b and c in the reverse order to that determined by the orientation of K , then through the other arc of the 2-group and out along arc d , ultimately to return to our starting point via arc a . Moreover, as is shown in Figure 1, the resulting knot is alternating, whence by Proposition 1, K' is prime.

It is also important to observe that this operation is fully reversible, in that if we remove any tangle which consists entirely of two adjacent crossings in a negative

group of a prime alternating knot K' as shown in Figure 1 (b), and replace this tangle with a single crossing, as shown in Figure 1 (a), the result is an alternating knot, hence prime by Proposition 1. We refer to the inverse of D as D^- .

4.2. The *ROTS* operator

This is the second of the two operators that increase the crossing size by one. *ROTS* can be applied to any 2-subgroup (that is, any pair of consecutive crossings) in any group of two or more. In effect, one arc of the 2-subgroup in question is twisted to form a loop, then the other arc is cut, one end is threaded through the loop, and then the two cut ends are rejoined. The twisting and threading is done so that the result is still alternating, as shown in 2. Furthermore, this operation does not change the component count, so when we apply it to a group in a knot, the result is a knot rather than a link.

These observations establish that we can apply Proposition 1 to conclude that we have a configuration of a prime alternating knot. Since we have replaced a tangle that consisted of two crossings with one that consists of three crossings, there is a net increase of one in the crossing size. The tangle shown in Figure 2 (b) is called a *drots tangle*, sometimes abbreviated to *rots tangle*. When the tangle results from an application of *ROTS* to a negative 2-group, we say that it is a positive *drots*-tangle.

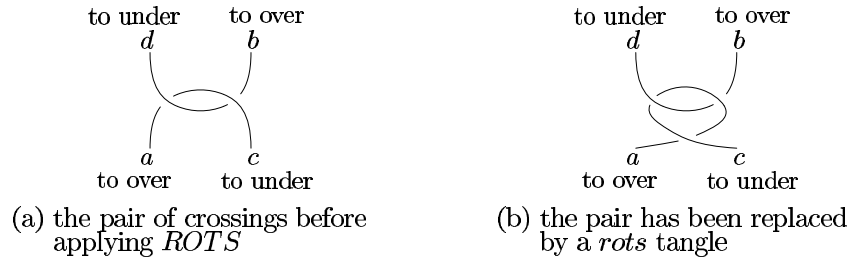


Figure 2: The *ROTS* operator

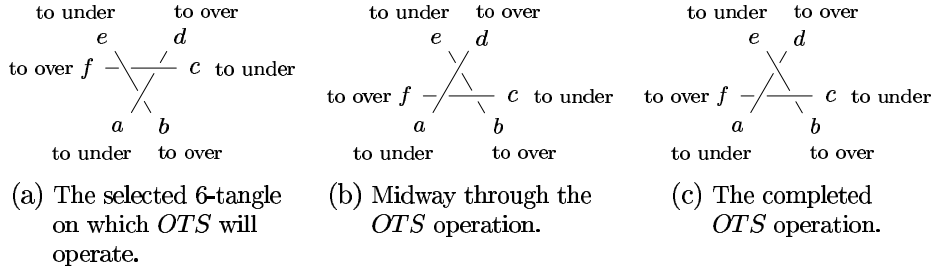
Just as for D , this process is reversible, and the inverse of the *ROTS* operator is denoted by *ROTS* $^-$.

While the *ROTS* operator can be applied to any 2-subgroup, the construction of the prime alternating knots will only use *ROTS* applied to a negative 2-group, or to a 2-subgroup of a negative 3-group. The tangle that results when *ROTS* is applied to a negative 2-group shall be called a positive *rots*-tangle.

4.3. The *OTS* operator

This is the first of the two operators that transform an alternating prime knot into another of the same crossing size. It is also the only operator of the four that does not involve replacing one (4)-tangle by another. A 6-tangle with 3 crossings is called an *OTS* 6-tangle, and the *OTS* operator acts on an *OTS* 6-tangle, replacing it by another *OTS* 6-tangle as illustrated in Figure 3. Since an *OTS* 6-tangle has 3 vertices, all of degree 4 in the knot configuration, the sum of the vertex degrees

in the knot configuration is 12. However, by definition, there are 6 edges incident to the 6-tangle, so the sum of the vertex degrees in the subgraph which is the 6-tangle itself is 6. Thus there are 3 edges in the 6-tangle, and so an *OTS* 6-tangle is a triangle, with each crossing being an endpoint for two of the 6 incident edges. In (a), we have shown an *OTS* 6-tangle, and (b) and (c) illustrate how an *OTS* operation is performed on such a 6-tangle. The operator can be considered to consist of two stages: in the first stage, one of the three strands ad , be , or cf is chosen. The chosen strand is cut, moved to the other side of the crossing formed by the other two strands, and rejoined so as to preserve the over/under pattern on the strand that has been moved. This stage has been completed in Figure 3 (b). At this point, the crossing formed by the other two strands is now an over-pass when it should be an under-pass or vice-versa, so to complete the *OTS* operation, we must apply the unknotting surgery to this crossing. The completed *OTS* operation is shown in Figure 3 (c).

Figure 3: The *OTS* operator

We remark that for a given *OTS* 6-tangle, the *OTS* operation yields the same configuration (not just flype equivalent) independently of which of the three strands is selected. It is also evident that *OTS* is self-inverse.

Since an *OTS* operation can be viewed as the cutting and rejoining one strand, followed by the cutting and rejoining of another strand, it follows that the end result is a configuration of a knot rather than of a link. It remains only to verify that the knot that results is prime. Suppose to the contrary that for some configuration $C(K)$ of a prime alternating knot K , there is an *OTS* 6-tangle O in $C(K)$ such that the knot configuration $C(K')$ that results when the *OTS* operation is performed on O contains a 2-tangle T . Let us denote the three crossings in O by x, y and z . Since $T \cap O = \emptyset$ would mean that T is a 2-tangle in $C(K)$, which is not possible since K is prime, we identify three cases to consider: T contains exactly one, exactly two, and all three crossings of O .

Case 1: T contains exactly one crossing of O , say x . Then the two edges that join a to y and to z , respectively, are the two edges incident to T . If T contains other crossings in addition to x , then the other two edges incident to x must join x to crossings in $T - \{x\}$, whence they are distinct edges and so $T - \{x\}$ is a 2-tangle in $C(K)$ (since the *OTS* operator did not affect any arcs or crossings of $C(K)$ other than those in O). But $C(K)$ has no 2-tangles, so this situation cannot occur. Consider the possibility that T has x as its only crossing. But then the other two arcs incident to x must be one and the same, whence x is a nugatory

crossing, as shown in Figure 4 (a). Upon undoing the *OTS* operation, we find that the 6-tangle O in $C(K)$ must be of the form shown in Figure 4 (b), so that O in fact was a (4-)tangle, rather than a 6-tangle, so this situation does not arise either. This completes the proof that Case 1 cannot occur.

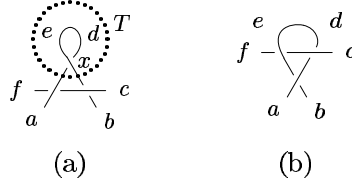


Figure 4: Case 1

Case 2: T contains exactly two crossings of O , say x and y . But then the arc from x to z and the arc from y to z must be the two arcs of $C(K')$ that are incident to T . We may then include z with T to form a larger 2-tangle in $C(K')$ which contains all three of x , y and z . This situation is our third case.

Case 3: T contains all three crossings x , y and z . In this case, undoing the *OTS* operation produces a 2-tangle T' in $C(K)$ which contains the *OTS* 6-tangle O . But since K is prime, $C(K)$ has no 2-tangles, so we obtain a contradiction in this case as well.

It follows now that $C(K')$ can't contain any 2-tangles, and therefore K' is a prime knot.

4.4. The T operator

Given a configuration $C(K)$ of a prime alternating knot K , a turnable-subgroup of $C(K)$ is a subgroup of any size greater than or equal to 2 of a positive group of $C(K)$, or a subgroup of odd size greater than or equal to 3 of any negative group of $C(K)$. The T operator acts on a turnable-subgroup of $C(K)$ via a two stage process. First, label the arcs incident to the turnable-subgroup by the following device: draw a simple closed curve surrounding the subgroup, then follow this curve in the clockwise direction until the first of a pair of edges at one end of the subgroup is encountered. Place the label 1 on this edge, and then continue following this curve, placing the labels 2, 3 and 4 on the remaining three arcs incident to the subgroup as they are encountered. Now at each label c , place a second label c' on the arc situated between the position of the label c and the crossing of the subgroup that the arc meets. Cut each arc between the labels c and c' , then reconnect the cut strands so as to have the arc labelled 1 connect to the arc labelled $2'$, the arc labelled 2 connect to the arc labelled $3'$, the arc labelled 3 connect to the arc labelled $4'$ and the arc labelled 4 connect to the arc labelled $1'$. The effect is to have turned the subgroup one-quarter turn in the counter-clockwise direction. However, arcs 1 and 2 originally met at a crossing at one end of the subgroup, so one was an overpass while the other was an underpass. After the turn, arc 1 will have the overpass/underpass status that was formerly held by arc 2, which means that the resulting configuration is not alternating. Observe that the arcs labelled 1 and 3, respectively, have the same overpass/underpass status at the crossings of the

subgroup that they are incident to, as do arcs labelled 2 and 4. Thus, after the turn, arc 2 has the overpass/underpass status formerly held by arc 3, which is the same as that of the original arc 1 and therefore opposite to that of the original arc 2. The same observation holds for all four arcs. Thus we can make an alternating configuration by cutting one strand at each crossing of the subgroup, moving the strand across the uncut strand and rejoin the cut ends so as to change the cut strand from an underpassing strand to an overpassing strand and vice-versa. If the configuration after the turn was a knot rather than a link, then after completing the second phase of the operation, the result is a configuration of an alternating knot. By Proposition 1, if we apply T to a turnable subgroup of some prime alternating knot, the result is a configuration of a prime alternating knot.

As it turns out, the T operator could be applied to an even subgroup of a negative group, but the result would be a link. We establish now that when T is applied to a turnable-subgroup, the result is a configuration of an alternating knot, hence by Proposition 1, a configuration of a prime alternating knot.

Case 1: the turnable-subgroup is positive. Let k denote the number of crossings in the turnable-subgroup.

Case 1 (a): k is even. Then arcs $1'$ and $3'$ are on the same strand of the subgroup (after the four incident arcs have been cut), and arcs $2'$ and $4'$ are on the other strand. Since the subgroup is positive, when the subgroup has been cut from the configuration, the remainder of the knot decomposes into two strands, and the strand which contains arc 3 is the same strand which contains arc 2, while the strand that contains arc 4 also contains arc 1. When the turnable-subgroup is turned and reattached, a traversal of the configuration beginning at arc 1 in the direction of the reattached subgroup follows arc $2'$ to arc $4'$, which is connected to arc 3, so we continue on to arc 2, which is connected to arc $3'$. Continue the traversal to arc $1'$, which is attached to arc 4, so we are led back to arc 1. Every arc has been traversed, so the result of applying T to an even subgroup of a positive group is a knot configuration.

Case 1 (b): k is odd. Again, when the turnable-subgroup has been cut from the knot configuration, the resulting four strands have arcs $1'$ and $3'$ on one, arcs $2'$ and $4'$ on another, arcs 1 and 4 on a third, and arcs 2 and 3 on the last strand. When the subgroup is turned and reattached, we begin a traversal at arc 1 and move towards the subgroup. We encounter in the following order arcs $2'$, $4'$, 3, 2, $3'$, $1'$, 4 and finally we arrive back at 1. Again, we have traversed every arc, so the result is a knot configuration. An example of this case is shown in Figure 5.

Case 2: the turnable-subgroup is negative. By definition, the number k of crossings is odd and $k \geq 3$. When the turnable-subgroup is cut out from the knot configuration, arcs $1'$ and $3'$ are on one strand, arcs $2'$ and $4'$ are on a second strand, while arcs 3 and 4 are on another strand and arcs 1 and 2 are on the last strand. When the turnable-subgroup is turned and reattached, a traversal beginning at arc 1 and proceeding in the direction of the subgroup will encounter arcs in the order $2'$, $4'$, 3, 4, $1'$, $3'$, 2 and back to 1, so the entire configuration has been traversed. The result in this case as well is a knot configuration.

Finally, observe that when T is applied to a subgroup of odd size, the turned subgroup has the opposite sign to that of the original (that is, a positive subgroup

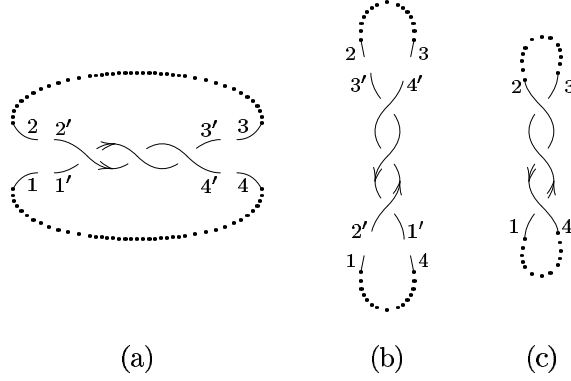


Figure 5: Case 1 (b)

becomes negative, and a negative subgroup becomes positive) while turning a even sized positive subgroup results in a positive subgroup. Consequently, the result of turning a turnable-subgroup is still a turnable-subgroup. If T is then applied to this resulting turnable-subgroup, the original knot configuration is produced, whence T is its own inverse.

5. The Construction Makes Them All

In this section, we shall prove that if all possible applications of both D and $ROTS$ are carried out on the complete set of prime alternating knots of n crossings, and then all possible applications of both OTS and T are carried out on the set of resulting knots of $n + 1$ crossings, then again to the knots that result from this first application of OTS and T , and so on until no further knots are produced, the result is the set of all prime alternating knots of $n + 1$ crossings. In fact, we shall prove that the same outcome results if we only apply D to negative groups (including loners) and positive 2-groups, $ROTS$ to negative 2-groups and negative 3-groups, and T to positive 2-groups. There will be no restrictions on OTS .

It will prove to be convenient to let T_2^+ denote the restriction of T to only positive 2-groups.

To aid us in the proof of this result, we introduce three categories of prime alternating knots, which have the property that each prime alternating knot belongs to exactly one category.

5.1. The K_A configurations

Conceptually, a prime alternating knot configuration is a K_A configuration if it can be obtained from a prime alternating knot configuration of one lower crossing size by a single application of either the D or the $ROTS$ operator (under the restrictions listed at the beginning of this section).

Definition 9. A prime alternating knot configuration $C(K)$ is said to be a K_A configuration if it contains either a negative group of size 2 or greater, or a *rots*-tangle.

Theorem 3. A prime alternating knot configuration is a K_A configuration if it can be obtained from a prime alternating knot configuration of one lower crossing size by a single application of either D to a negative group, a loner, or a positive 2-group, or the $ROTS$ operator to a negative 2-group or 3-group.

Proof. Let $C(K)$ be a K_A configuration. Suppose first of all that $C(K)$ contains a negative group G of size 2 or greater. Then an application of D^- to G results in a configuration $C(K')$ of a prime alternating knot at one lower crossing size containing a negative group G' with the property that if D is applied to G' , the configuration $C(K)$ is reconstructed. Now suppose that $C(K)$ contains a $rots$ -tangle T . Then an application of $ROTS^-$ to T results in a configuration $C(K')$ of a prime alternating knot at one lower crossing size which contains a negative 2-subgroup T' such that if $ROTS$ is applied to T' , $C(K)$ is obtained. If T' is actually a subgroup of a negative group of size at least 4, in which case T is directly attached to a negative 2-group G , and we have already handled the case when $C(K)$ contains a negative group of size 2 or greater. \square

Thus each K_A configuration at a given number of crossings is obtained when D (applied only to negative groups, loners, and positive 2-groups) and $ROTS$ (applied only to negative 2 and 3-groups) are applied to the configurations of the prime alternating knots of one lower crossing size.

5.2. The K_B configurations

To describe the K_B configurations, we require the following concepts.

Definition 10. Let $C(K)$ be a configuration of a prime alternating knot K . An *interleaved 2-sequence* in $C(K)$ is a pair of positive 2-subgroups of $C(K)$ having no crossings in common with the property that, as the knot is traversed, one arc of one of the 2-subgroups is traversed first, then one arc of the other 2-subgroup is traversed next, then the second arc of the first 2-subgroup is traversed and finally the second arc of the second 2-subgroup is traversed.

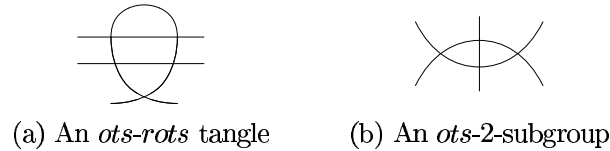


Figure 6: *ots* type tangles

Furthermore, a 6-tangle of the form shown in Figure 6 (a) is called an *ots-rots tangle*, while a 6-tangle of the form shown in Figure 6 (b) is called an *ots-2-subgroup*, with sign that of the underlying 2-subgroup. If the underlying 2-subgroup is actually a group, then the 6-tangle is called an *ots-2-group*.

We observe that any negative group of size 4 or greater provides an instance of an interleaved 2-sequence.

Definition 11. A prime alternating knot configuration $C(K)$ is said to be a K_B configuration if it is not a K_A configuration (that is, contains no negative groups of size 2 or greater, and contains no $rots$ -tangles), but contains one or more of

the following structures: a positive group of size 3, an interleaved 2-sequence, an *ots-rots* tangle, or a negative *ots*-2-group.

We shall prove that for each K_B configuration $C(K)$, there is a K_A configuration $C(K')$ such that there is either a single T_2^+ operation or a single *OTS* operation which transforms $C(K')$ into $C(K)$, and that furthermore, if either T_2^+ or *OTS* is applied to a K_A configuration and the result is not a K_A configuration, then it is a K_B configuration.

By way of illustration, observe that knot 5_1 is a K_B knot of five crossings, since it has both a positive 5-group and an interleaved 2-sequence, but no negative groups nor any *rots*-tangles. Knot 8_{16} (see Figure 7) also qualifies as a K_B knot of eight crossings since it has an *ots-rots* tangle (and also a negative *ots*-2-group, although it does not have any interleaved 2-sequences). Knot 8_{16} does not have any positive groups of size greater than 2, and is not a K_A knot since it has no negative groups nor does it have any *rots*-tangles.

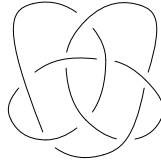
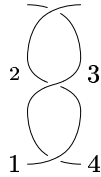


Figure 7: Knot $K_{8_{16}}$

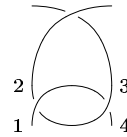
Proposition 5. Let $C(K)$ be a K_B configuration. Suppose that $C(K)$ contains a 2-subgroup G , and that $C(K')$ denotes the configuration that is obtained when T is applied to a 2-subgroup of G . Then the image of G will be a 2-group of the resulting configuration.

Proof. Suppose that the image G' of G upon this application of T is a subgroup of a larger group, as illustrated in Figure 8 (a). Since G' is a subgroup of a larger group in $C(K')$, we see that G formed part of a *rots* tangle, as illustrated in Figure 8 (b). But $C(K)$ is not a K_A configuration, and so $C(K)$ does not have any *rots* tangles. Thus G' could not be a subgroup of a larger group in $C(K')$. \square

Thus, if the T operator is applied to any 2-subgroup of a K_B configuration, the image of the 2-subgroup will be a 2-group of the resulting configuration, whence T can be applied to it to regain G .



(a) G' is part of a larger group



(b) G must be part of a *rots* tangle

Figure 8: Turning a 2-subgroup

Theorem 4. A prime alternating knot configuration $C(K)$ is a K_B configuration if and only if it is not a K_A configuration, but can be obtained from some K_A configuration by a single application of either T_2^+ or OTS .

Proof. Let $C(K)$ be a K_B configuration, whence by definition, $C(K)$ is not a K_A configuration. We must consider four cases.

Case 1: $C(K)$ contains a positive 3-group G . Choose a 2-subgroup of G and apply T to the selected subgroup. Let $C(K')$ denote the configuration that results, and let H denote the positive 2-group which is the image of T applied to the selected 2-subgroup of G . Then H together with the remaining crossing of the original 3-group forms an *ots*-tangle in $C(K')$, whence $C(K')$ is a K_A configuration. If T_2^+ is applied to the positive 2-group H in $C(K')$, the result is the original configuration $C(K)$, and so in this case, $C(K)$ has been obtained by a single application of T_2^+ to some K_A configuration.

Case 2: $C(K)$ contains an interleaved 2-sequence, say G_1 and G_2 . Label the arcs incident to the tangle G_1 with 1, 2, 3 and 4 in preparation for an application of T to G_1 , and begin a traversal of the knot with arc 1 moving towards G_1 . Label the arcs of G_1 and G_2 with e_1, e_2 and f_1, f_2 , respectively, so that the traversal of the knot encounters the arcs in the sequence 1, e_1 , 4, f_1 , 2, e_2 , 3, f_2 , 1, so the fact that G_2 is a positive subgroup is recorded by the data 4, f_1 , 2 and 3, f_2 , 1 in the traversal. Consider what happens when T is applied to G_1 . A traversal of the resulting knot that starts on arc 1 will encounter the arcs of G_1 (turned) and G_2 in the sequence 1, e_2 , 2, f_1 , 4, e_1 , 3, f_2 , 1, so the order of travel through G_2 is 2, f_1 , 4 and 3, f_2 , 1. Thus f_2 is traversed in the same direction as in the original knot configuration, while f_1 is now traversed in the opposite direction to that of the original knot. Since G_2 was originally a positive group, in the configuration $C(K')$ that results from the application of T to C_1 , G_2 has become a negative group, whence $C(K')$ is a K_A configuration. Now, the turned image G'_1 of G_1 is a positive 2-group of $C(K')$, and so we may apply T_2^+ to obtain $C(K)$. Thus in this case as well, $C(K)$ is obtained by applying T_2^+ to some K_A configuration.

Case 3: $C(K)$ contains an *ots-rots* tangle O . Since every *ots-rots* tangle contains an *ots* tangle, we choose to apply OTS to the *ots* tangle that is contained in O . Let $C(K')$ denote the resulting configuration. Now, the application of OTS to O leaves a *rots* tangle in $C(K')$, so $C(K')$ is a K_A configuration. If we apply OTS to the *ots* tangle of $C(K')$ that is the image of O under the initial OTS operation, the result is $C(K)$. Thus every K_B configuration that contains an *ots-rots* tangle is obtained by a single application of OTS to some K_A configuration.

Case 4: $C(K)$ contains a negative *ots*-2-group. Now any *ots* 2-group contains an *ots* tangle, so we apply OTS to the *ots* tangle O that is contained in the negative *ots*-2-group. The result is a configuration $C(K')$ containing a negative 2-group, which is therefore a K_A configuration. When OTS is applied to the image of O in $C(K')$, the original configuration $C(K)$ is obtained. Thus any K_B configuration that contains a negative *ots*-2-group is the image under a single OTS operation of some K_A configuration.

This establishes that for every K_B configuration, there is a K_A configuration such that a single application of either T_2^+ or OTS to this K_A configuration produces the given K_B configuration.

Conversely, suppose that $C(K)$ is not a K_A configuration, but that there is either a K_A knot configuration $C(K_1)$ such that an application of T_2^+ to $C(K_1)$ produces $C(K)$, or else there is a K_A configuration $C(K_1)$ such that an application of $ROTS$ to $C(K_1)$ produces $C(K)$. We show that in either case, $C(K)$ is a K_B configuration.

Case 1: an application of T_2^+ to some positive 2-group G in the K_A configuration $C(K_1)$ produces $C(K)$. Since $C(K)$ is not a K_A configuration, it has no negative groups or *rots* tangles, while $C(K_1)$ is a K_A configuration, and so it must have at least one negative group or *rots* tangle. If $C(K_1)$ contains a *rots* tangle, then T_2^+ must have been applied to the 2-group of the *rots* tangle. Since an application of T_2^+ to the 2-group of a *rots* tangle results in a positive 3-group, it follows that $C(K)$ is a K_B configuration in this case. If $C(K_1)$ does not contain any *rots* tangles, then it must contain a negative group H . But $C(K)$ does not contain any negative groups, so the result of applying T_2^+ to G is to change G into a positive 2-group. But this means that G and H constitute an interleaved-2-sequence, and so in this case as well, $C(K)$ is a K_B configuration.

Case 2: an application of OTS to some *ots* 6-tangle O in the K_A configuration $C(K_1)$ produces $C(K)$. Since $C(K_1)$ is a K_A configuration, it contains either a negative group or a *rots* tangle. If $C(K_1)$ contains a *rots* tangle, then since $C(K)$ does not, we see that the *ots* 6-tangle O must involve crossings of the *rots* tangle. By definition of OTS , O cannot contain both crossings of the positive 2-group of the *rots* tangle. If O contained one crossing of the positive 2-group of the *rots* tangle, then O must contain the third crossing of the *rots* tangle as well, but in such a case, the application of OTS to O would result in a negative 2-group in $C(K)$. Since this is not possible, O cannot contain either crossing of the positive 2-group of the *rots* tangle. But this means that O must contain the third crossing of the *rots* tangle, and so the result of applying OTS to O is an *ots-rots* tangle in $C(K)$, whence $C(K)$ is a K_B configuration.

The last possibility is that $C(K_1)$ contains a negative group G . Since $C(K)$ contains no negative groups, the *ots* 6-tangle O must involve a crossing at one end of G , in which case the application of OTS to O will separate this crossing from the rest of G . If G had been a negative group of size 3 or larger, then $C(K)$ would contain a negative group of size 2 or larger, which is not possible. Thus G must be a negative 2-group and so OTS results in a negative *ots*-2-group. Again, $C(K)$ is a K_B configuration.

Thus in every case, $C(K)$ is a K_B configuration. \square

5.3. The K_C configurations

These configurations make up our last category, and a configuration is said to be a K_C configuration if it is neither a K_A configuration nor a K_B configuration. Thus a configuration is K_C if all groups are of size at most 2, with any 2-groups being positive, and there are no interleaved 2-sequences, nor any *rots* or *ots-rots* tangles nor any negative *ots*-2-groups.

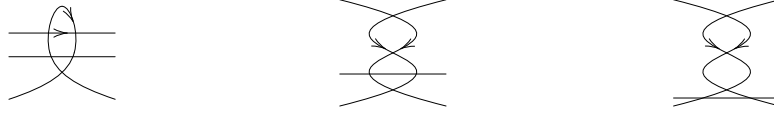
Our objective is to show that for a given K_C configuration, there is a K_B configuration and a finite sequence of T_2^+ and/or OTS operations that will transform the K_B configuration into the given K_C configuration. In fact, the following result

ensures that we can always arrange it so that the first operator in the sequence is OTS .

Proposition 6. Every K_C configuration that can be obtained from a K_B configuration by an application of any finite sequence of T_2^+ operations can also be obtained from a K_B configuration by a single application of OTS .

Proof. It suffices to consider only the case where each application of T_2^+ in the sequence results in a K_C configuration.

Let $C(K)$ denote a K_C configuration which has been obtained from a K_B configuration $C(K_1)$ by a single application of T_2^+ . Thus $C(K)$ has all groups of size at most two, with any 2-groups being positive, and there are no interleaved 2-sequences, nor any *rots* or *ots-rots* tangles nor any negative *ots*-2-groups. Furthermore, $C(K_1)$ does have a positive 2-group G that T_2^+ turns to produce $C(K)$. Since $C(K_1)$ is a K_B configuration, it has no negative groups nor any *rots* tangles, but it must have at least one of the following: a positive group of size at least 3, an *ots-rots* tangle, a negative *ots*-2-group, or an interleaved-2-sequence. Let us examine the possibilities in light of the information that upon a single application of T_2^+ , the resulting configuration is without any of these structures. If $C(K_1)$ had a positive group of size at least 3, then any finite sequence of T_2^+ applications will not affect it, other than to possibly change it to a negative group. Since $C(K)$ has neither, we see that $C(K_1)$ does not have any positive groups of size greater than or equal to 3. Suppose that $C(K_1)$ contains an *ots-rots* tangle. Since $C(K)$ contains no negative groups and no *ots-rots* tangles, it must be that the application of T_2^+ was to the positive 2-group of the *ots-rots* tangle.



(a) the *ots-rots* tangle (b) after turning the 2-group (c) a K_B configuration

Figure 9: Turning the 2-group of an *ots-rots* tangle

This situation is shown, before and after the turn, respectively, in Figure 9 (a) and (b). On the other hand, Figure 9 (c) shows a K_B configuration which will result in the configuration illustrated in (b) upon applying the indicated OTS operation. If one or more additional T_2^+ operations are applied, the end result is either to convert the underlying 3-group to a negative group, or to leave it unchanged as a positive 3-group. In the first case, the resulting knot configuration can also be obtained by a single application of OTS to a K_A configuration, which would mean that the resulting knot is K_B , not K_C . Thus this case can't occur. On the other hand, if the underlying 3-group is positive, then the same argument that we applied for a single application of T_2^+ will also establish that the configuration that results can be obtained by an OTS operation applied to a K_B knot.

We next consider the situation when $K(C_1)$ does not contain any positive groups of size greater than or equal to 3 nor any *ots-rots* tangles. Suppose that $K(C_1)$ contains a negative *ots*-2-group. Since $K(C)$ does not, it must be the case that an application of T_2^+ to some positive 2-group G in $K(C_1)$ will cause the underlying

2-group of the *ots*-2-group to become positive. However, if we were to apply *OTS* to $K(C_1)$ to convert the negative *ots*-2-group to a negative group H (so the resulting configuration is K_A), then apply T_2^+ to the positive 2-group G , the result is a K_B configuration in which the 2-group H is now positive, and thus *OTS* may be applied to this configuration to obtain K_C . Furthermore, if we were to apply subsequent T_2^+ operations, each time resulting in a K_C configuration, we see that none of the subsequent applications can change the sign of the underlying 2-group of the positive *ots*-2-group back to negative, since then the resulting configuration would be K_B . Since all positive 2-groups must be non-interleaved, after the T_2^+ operations have been applied, G remains positive, so the same argument that applied after the first T_2^+ applications is still applicable.

Finally, suppose that $K(C_1)$ does not contain any positive groups of size greater than or equal to 3, any *ots-rots* tangles or any negative *ots*-2-groups. There must then be an interleaved pair of positive 2-groups in $K(C_1)$. However, the only way that an application of T to a positive 2-group can result in an interleaved pair of positive 2-groups is to cause them to change sign. However, $K(C)$ has no negative groups, so this situation cannot occur. Obviously, subsequent T_2^+ operations which are required to produce in a K_C configuration would also not be possible.

This completes the proof that in every situation where an application of any finite sequence of T_2^+ operations to a K_B configuration results in a K_C configuration $C(K)$, there is a corresponding K_B configuration on which *OTS* can be applied, resulting in $C(K)$. \square

Our approach requires that we establish some properties of two specific types of subgraphs that are found in the plane graph that is obtained when a knot configuration is formed.

Definition 12. A plane graph G whose edges are piecewise smooth curves is called a 2-region, respectively a minimal loop, if the following conditions are satisfied:

- (a) G is connected;
- (b) G has a face F whose boundary is a cycle C ;
- (c) there are exactly two vertices on C that have degree 2 in G (called the base vertices of the 2-region), respectively there is exactly one vertex on C that has degree 2 in G (called the base vertex of the minimal loop);
- (d) every non-base vertex on C has degree 3 in G ;
- (e) each vertex that lies in the interior of the region $\mathbb{R}^2 - F$ has degree 4 in G .

The cycle C is called the *boundary* of the 2-region or minimal loop, respectively, while the interior of the region $\mathbb{R}^2 - F$ is called the *interior* of the 2-region, respectively minimal loop (note that in the event that all vertices of G lie on C , then F is not uniquely determined—in such a case, let F be the bounded region). In the case of a 2-region with boundary cycle C and base vertices p and q , the two paths between p to q that C determines are called the boundary paths of G .

More generally, if G is a plane graph with vertex set V , then a subgraph G' of G with vertex set $V' \subseteq V$ is said to be a 2-region (respectively, minimal loop) of G if G' is a 2-region (respectively, minimal loop) and every vertex of G that lies in the interior of G' belongs to V' , and every edge of G that meets the interior of G'

belongs to V' . If G' is a 2-region or minimal loop of G , then G is said to contain the 2-region or minimal loop G' .

A 2-region G is said to be minimal if H is a 2-region of G implies that $H = G$, and a 2-region that has no vertices in its interior is called a 2-group. A minimal loop with a single vertex is said to be trivial.

Note that the interior of a 2-region or minimal loop could be the unbounded region determined by the boundary of the 2-region, respectively minimal loop. Further note that a 2-group has exactly two vertices, which are multiply connected by two edges.

Next, observe that by the handshake lemma, there must be an even number of non-base vertices on the boundary of a 2-region or minimal loop. There is a natural way to pair up the non-base vertices of a 2-region or minimal loop by means of what we shall call arcs in the 2-region or minimal loop.

Definition 13. Let G be a 2-region or minimal loop. An arc in G is a path of maximal length which satisfies the following conditions:

- (a) the initial vertex of the path is a non-base boundary vertex;
- (b) the initial edge of the path is the unique non-boundary edge of G that is incident to the initial vertex;
- (c) each interior (i.e. non-endpoint) vertex of the path is in the interior of the 2-region, respectively minimal loop;
- (d) at each interior vertex u of the path, the two path edges incident to u must be non-adjacent edges at u in G ;
- (e) any non-base boundary vertex on the path is an endpoint of the path.

Proposition 7. Let G be a 2-region or a minimal loop. Then the endpoints of any arc in G are distinct and both are non-base boundary vertices of G . Furthermore, the relation on the set of all non-base boundary vertices of G (if this set is nonempty) given by u is related to v if and only if $u = v$ or u and v are endpoints of an arc of G is an equivalence relation, with each equivalence class of size two. Finally, each vertex of G that lies in the interior of G is on exactly two arcs in G .

Proof. Suppose that G is a 2-region or minimal loop with nonempty set of non-base boundary vertices. Let P be an arc in G with initial vertex v and initial edge e . We first establish that P is not a closed path. For suppose it is. Then there are two edges incident to v that lie in the interior of G . Since v is a non-base boundary vertex, there are two edges of the boundary cycle incident to v , whence v has degree at least 4 in G . Since this is not the case, P is not a closed path. Consequently, the endpoints of P are distinct. Let w denote the other endpoint of P . We must prove now that w is a non-base boundary vertex of G . Suppose not. Since there are only two edges incident to a base vertex, and each such edge has as its other endpoint either a base vertex or a non-base boundary, the only way a path could reach a base vertex would be to pass through a non-base boundary vertex first. But no arc continues on after reaching a non-base boundary vertex, and so w must be a vertex in the interior of G . Let f denote the last edge of P , so that f is incident to w . Now, w has degree 4, and so there is exactly one edge incident to w that is not adjacent to f in G . Denote this edge by f' . If f' were in P , then w would not be an endpoint of P . Thus f' is not in P and so we may extend P by following f' ,

which contradicts the maximality of P . Thus w is a non-base boundary vertex of G .

The preceding discussion actually establishes that for each non-base boundary vertex v , there is a unique arc of G with endpoint v (since there is only one edge available to start the arc, and only one choice of edge to take at each internal vertex of the path). Thus the relation on the set of non-base boundary vertices that is defined by $u = v$ or u and v are endpoints of an arc of G is an equivalence relation, partitioning the set of non-base boundary vertices into cells of size two.

Furthermore, the discussion of the first paragraph of the proof can also be used to establish that each pair of non-adjacent edges incident to a vertex in the interior of G lie on exactly one arc in G , so there are exactly two arcs in G that pass through a given vertex in the interior of G . \square

Corollary 3. Let G be a 2-region for which some arc l_1 in G has both endpoints x and y on the same boundary path L of G . Let l_2 denote the restriction of L to that part which lies between x and y , and let C denote the cycle obtained by following l_1 by l_2 . Let F denote the open region of the plane determined by C which contains the base points of G . Then the graph G' whose vertices and edges are those of G which are in $R = \mathbb{R}^2 - F$ is a 2-region of G .

Proof. Observe that C lies in the closure of the interior of G , but the base vertices of G are not on C . Thus R is contained in the interior of G . The boundary edge of G that is incident to x but which does not belong to l_2 lies in the closure of F and is therefore not in G' , whence x has degree 2 in G' . Similarly, y has degree 2 in G' . As well, each vertex of l_2 other than x and y still has degree 3 in G' . Each vertex of l_1 other than x and y has two edges of G incident to it, one lying in the closure of F , and one lying in R . Thus each vertex of l_1 other than x and y has degree 3 in G' . Each vertex v in the interior of R is in the interior of G , and all four edges of G incident to v are in G' , whence v has degree 4 in G' . Furthermore, v lies on an arc of G , which either has both endpoints on l_2 or else must leave R and thus meet l_1 . In either case, each vertex and edge on such an arc between v and the point of intersection of the arc with either l_1 or l_2 belongs to G' , whence there is a walk in G' from v to x . Thus G' is connected. It follows that G' is a 2-region with interior the interior of R , whence each vertex of G that is in the interior of G' belongs to G' , and so G' is a 2-region of G . \square

Proposition 8. Every non-trivial minimal loop contains a 2-region, and every 2-region contains a minimal 2-region.

Proof. Let G be a non-trivial minimal loop with boundary cycle C and base vertex u . Since G is non-trivial, it must be at least one additional vertex. If G has vertices in its interior, then by Proposition 7, there are arcs in G , and so, again by Proposition 7, there are non-base boundary vertices in G . Let v be a non-base boundary vertex, and let L_1 denote the unique arc in G that has v as initial vertex, and let w denote the other end point of this arc. Then v and w are distinct non-base boundary vertices of G . Let L_2 denote the path obtained by following C from v to w in the direction which does not pass through u . Then L_1 followed by L_2 is a cycle C' in G . Now C' divides the plane into two regions, one of which contains u . Let F denote this region. Let H be the graph whose vertex set consists of all vertices of G that lie either on C' or else in the interior of the region $R = \mathbb{R}^2 - F$,

and whose edges are those edges of G that lie in the region R (which includes the edges of C'). It is straightforward now to verify that v and w have degree 2 in H , any vertex on C' other than v and w has degree 3 in H , every vertex in the interior of R has degree 4 in H , and H is connected, whence H is a 2-region. But then by construction, H is a 2-region of G .

Now suppose that G is a 2-region. Of all 2-regions contained in G (and G is such a 2-region), let H be one with fewest vertices. Suppose that H' is a 2-region of H . Let C_H denote the boundary cycle of H . If $H = C_H$, then $H' = H$ and there is nothing to show. Suppose then that H has non-boundary vertices. Now all vertices of H lie on C_H or in one of the two regions of the plane that C_H determines, and there are vertices of H that are not on C_H . Let R_H denote the region determined by C_H which contains vertices of H . Now there is a cycle $C_{H'}$ in H such that all vertices of H' lie on $C_{H'}$ or in one of the two regions of the plane that $C_{H'}$ determines. If $H' = C_{H'}$, then H' is a 2-group and we let $R_{H'}$ denote the open region determined by $C_{H'}$ that contains no vertices of H . Otherwise, let $R_{H'}$ denote the open region determined by $C_{H'}$ which contains vertices of H' . By definition, every vertex of G that is in the interior of H belongs to H , and every vertex of H that is in the interior of H' belongs to H' . Let x be a vertex of G that lies in the interior of R_H . If x is in H , then x is in H' , as required. Suppose that x is not in H . If $R_{H'} \subseteq R_H$, then x would be in the interior of R_H , whence x belongs to H . Since this is not the case, it must be that $R_{H'}$ is not contained in R_H , whence there is a vertex on the boundary of R_H that is contained in the interior of $R_{H'}$. But such a vertex must then belong to H' and have degree 4 in H' . However, vertices on the boundary of H have degree at most 3 in H , so this is not possible. This contradiction stemmed from the assumption that x was not in H , so we conclude that x must be in H , from which it follows that x is in H' , as required. Thus H' is a 2-region of H such that every vertex of G that is contained in the interior of H' belongs to H' , whence H' is a 2-region of G . By the minimality of H , we must have $H' = H$, whence H is a minimal 2-region. \square

Definition 14. Let $C(K)$ be a configuration of a prime alternating knot K , and let v be a crossing of $C(K)$. Choose any of the four arcs incident to v and construct a closed walk based at v by following a knot traversal in the direction of the chosen arc until v is reached for the first time. Such a closed walk is called a knot circuit based at v .

Proposition 9. Let $C(K)$ be a configuration of a prime alternating knot K . Then

- (a) For every crossing v of $C(K)$, and each knot circuit C based at v , there is a minimal loop of $C(K)$ whose boundary cycle is a subwalk of C ;
- (b) every minimal loop of $C(K)$ contains a 2-region of $C(K)$, and
- (c) every 2-region of $C(K)$ contains a minimal 2-region of $C(K)$.

Proof. Let v be a crossing of $C(K)$ and let C denote a knot circuit based at v . For each crossing x on C (including $x = v$), start at x and follow C until either x or v is reached, whichever comes first. If the result is a circuit based at x (which will be the case if $x \neq v$ and x is encountered first), then let c_x denote the number of crossings that were encountered (excluding x itself) during this traversal, otherwise let $c_x = \infty$. Of all crossings of C , choose one, say w , for which c_w is smallest. Let

C' denote the closed walk in $C(K)$ that is obtained by beginning at w and following C until w is reached again. Then by the minimality of c_w , C' must be a cycle. Let F denote the region determined by C' that contains the two edges incident to w that are not on C' , and let V' be the set consisting of all crossings of $C(K)$ that lie on C' or in the interior of the region $R = \mathbb{R}^2 - F$. Let H be the plane graph whose vertex set is V' and whose edge set consists of all edges of $C(K)$ whose endpoints are in V' and which lie in R . We claim that H is a minimal loop in $C(K)$ with boundary cycle C' .

First of all, observe that H is a plane graph with a face F whose boundary is the cycle C' , and that w has degree 2 in H . Next, consider a vertex u on C' different from w . Since C' is a cycle, there are two edges of C' that are incident to u , and since these two edges were successively travelled in the knot circuit C , they are not adjacent edges at u . Consequently, one of these two edges, e_1 say, lies in F and the other, e_2 say, lies in R . Then e_2 belongs to H while e_1 does not. It follows that there are exactly three edges incident to u in H ; namely the two on C' and e_2 , the one which lies in R . Thus u has degree 3 in H . Finally, consider any vertex y of H which lies in the interior of R . Now y has degree 4 in $C(K)$, and all four edges that are incident to y have endpoints that are either in the interior of R or else lie on C' , whence all four edges of $C(K)$ that are incident to y belong to H . Thus y has degree 4 in H . Furthermore, any path in H of maximal length with initial vertex y must terminate at a vertex of odd degree in H . Since the only vertices of odd degree in H lie on C' , it follows that there is a walk in H from y to w , whence H is connected. This completes the proof that H is a minimal loop with boundary cycle C' .

That H is a minimal loop of $C(K)$ follows now from the fact that the interior of H is the interior of R , and every vertex of $C(K)$ that is in R was put in V' , the set of vertices of H .

Thus H is a minimal loop of $C(K)$ with boundary cycle C' , which is a closed subwalk of C .

Next, suppose that G is a minimal loop of $C(K)$, so there exists a cycle C_G in $C(K)$ such that all vertices of G lie in one of the two regions of the plane that C_G determines. Let R_G denote this region. By Proposition 8, G contains a 2-region H , whence there is a cycle C_H in G such that all vertices of H lie in one of the two regions of the plane that C_H determines. Let R_H denote this region. By definition, every vertex of $C(K)$ that is in the interior of G belongs to G , and every vertex of G that is in the interior of H belongs to H . Let x be a vertex of $C(K)$ that lies in the interior of H ; that is, in the interior of R_H . If x is in G , then x is in H , as required. Suppose that x is not in G . If $R_H \subseteq R_G$, then x would be in the interior of R_G , whence x belongs to G . Since this is not the case, it must be that R_H is not contained in R_G , whence there is a vertex on the boundary of R_G that is contained in the interior of R_H . But such a vertex must then belong to H and have degree 4 in H . However, vertices on the boundary of G have degree at most 3 in G , so this is not possible. This contradiction stemmed from the assumption that x was not in G , so we conclude that x must be in G , from which it follows that x is in H , as required. Thus H is a 2-region contained in G such that every vertex of $C(K)$ that is contained in the interior of H belongs to H , whence H is a 2-region contained in

$C(K)$.

An argument similar to the preceding one will establish that if G is a 2-region of $C(K)$, and H is a minimal 2-region contained in G , then every vertex of $C(K)$ that is contained in the interior of H must belong to G and thus to H , whence H is a minimal 2-region of $C(K)$. \square

Thus every configuration of a prime alternating knot has minimal loops, 2-regions and therefore minimal 2-regions. We remark that an empty 2-region of a prime alternating knot configuration is a 2-subgroup.

We next observe that the *OTS* operation on a knot configuration has an analog for 2-regions and minimal loops.

Definition 15. Let G be a 2-region or a minimal loop. An *ots*-triangle in G is a face of degree 3 none of whose boundary edges belong to the boundary of a 2-group of G , while a face of degree 3 which does have a boundary edge in common with a face of degree 2 is called a *rots*-triangle.

Let O be an *ots*-triangle in G , say with boundary edges B_1 , B_2 and B_3 . There are three possible situations: none, exactly one, or exactly two of the edges B_1 , B_2 , B_3 is a boundary edge of G . We define the *ots* operation in each of these three cases. Let V denote the vertex set of G and E denote the edge set of G .

Case 1: none of B_1 , B_2 , B_3 is a boundary edge of C . Now each pair of these edges have exactly one endpoint in common. Since a boundary vertex has at most one edge lying in the interior of G incident to it, we see that none of these three common endpoints is a boundary vertex of G , and so the compact set $B_1 \cup B_2 \cup B_3$ is contained in the interior of G . Furthermore, each of the three common endpoints has degree 4, so at each there are two additional incident edges. Since the edge curves are smooth, there is an open neighborhood U of O whose intersection with each of these six additional edges is connected, and which does not meet any other edge curve of G . Arbitrarily choose one pair of boundary edges of O , and suppose that the edges were labelled so that B_1 and B_2 are the chosen edges. Let a denote the common endpoint of B_1 and B_2 , and let e and f denote the two edges that are incident to a in addition to B_1 and B_2 . Further suppose that all labelling has been done so that in a clockwise scan at a , the edges are encountered in the order B_1 , e , f and B_2 . Choose a point $x \neq a$ in $e \cap U$, and choose a point $y \neq a$ in $f \cap U$. Let B'_3 be a smooth curve from x to y within U that does not meet any curve of G other than e at x and f at y . Let b denote the common endpoint of B_2 and B_3 , and let c denote the common endpoint of B_3 and B_1 . Further, let g and h denote the two edges incident to b other than B_2 and B_3 , labelled in order in the clockwise direction, and let i and j denote the two edges incident to c other than B_3 and B_1 , labelled in order in the clockwise direction. Let j' denote a smooth curve with endpoint x that agrees with j outside of U and within U meets no edge curve of G except for e at x , and let g' denote a smooth curve with endpoint y that agrees with g outside of U and within U meets no edge curve of G except for f at y . Let the portion of e from a to x be denoted by B'_1 and denote the remaining portion of e by e' . Similarly, let the portion of f from a to y be denoted by B'_2 and denote the remaining portion of f by f' . Finally, let $i' = i \cup B_1$ and $h' = h \cup B_2$. Let G' denote the plane graph with piecewise smooth edge curves whose vertex set is $(V - \{b, c\}) \cup \{x, y\}$ and edge set $(E - \{B_1, B_2, B_3, e, f, g, h, i, j\}) \cup \{B'_1, B'_2, B'_3, e', f', g', h', i', j'\}$. Since G'

agrees with G outside of U , it follows that G' is a 2-region, respectively minimal loop, said to be obtained from G by an *ots* operation (on O).

Case 2: exactly one of B_1, B_2, B_3 is a boundary edge of G . Suppose that the curves were labelled so that B_3 is the boundary curve of G . Label the vertices and edges as in Case 1, with the only differences stemming from the fact that the endpoints of B_3 are boundary vertices of G (non-base, since neither B_1 nor B_2 is a boundary edge of G), so there is no edge h or i . Carry out the construction as in Case 1 (so there is no corresponding h' or i'), and let G' denote the plane graph with piecewise smooth edge curves whose vertex set is $(V - \{a, b, c\}) \cup \{x, y\}$ and edge set $(E - \{B_1, B_2, B_3, e, f, g, i\}) \cup \{B'_3, e', f', g', i'\}$. Since G' agrees with G outside of U , it follows that G' is a 2-region, respectively minimal loop, said to be obtained from G by an *ots* operation (on O). Note that the number of vertices of G' is one less than the number of vertices of G . In this case, we say that vertex a has been *ots*-ed out of G .

Case 3: two of the edges B_1, B_2, B_3 are boundary edges of G . Suppose that the curves have been labelled so that B_3 is not a boundary edge of G . Let the endpoints of B_3 be b and c . Then b and c are non-base boundary vertices, while the common endpoint of B_1 and B_2 is a base boundary vertex. Each of b and c have one additional boundary edge incident to them. Let h be the additional boundary edge incident to b and let i be the additional boundary edge incident to c , and set $h' = h \cup B_2$ and $i' = i \cup B_1$. Let G' denote the plane graph with piecewise smooth edge curves whose vertex set is $V - \{b, c\}$ and edge set $(E - \{B_1, B_2, B_3, h, i\}) \cup \{h', i'\}$. Since G' agrees with G outside of U , it follows that G' is a 2-region, respectively minimal loop, said to be obtained from G by an *ots* operation (on O).

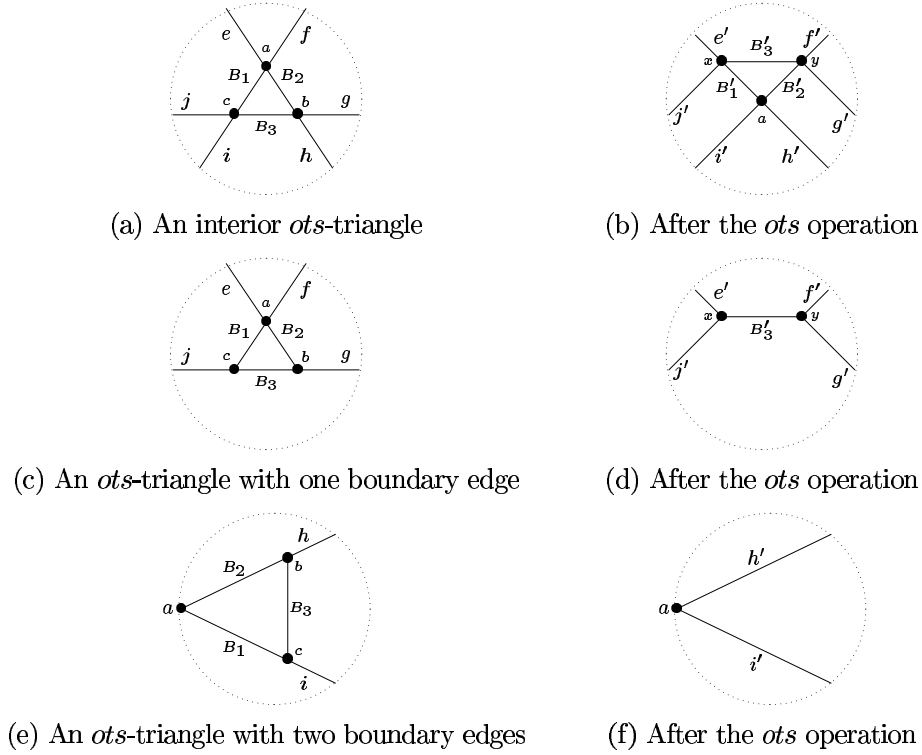
In Cases 2 and 3, we say that the *ots*-triangle is on the boundary of G .

For example, in Figure 10 (a) and (b), an *ots*-triangle in the interior of a 2-region or minimal loop and the result of applying an *ots* operation to the *ots*-triangle are shown, while in (c) and (d), an *ots*-triangle with exactly one edge on the boundary of the 2-region or minimal loop, together with the outcome of an *ots* operation to this *ots*-triangle are shown. Finally, in (e) and (f), an *ots*-triangle with two edges on the boundary of the 2-region or minimal loop and the effect of applying an *ots* operation to the *ots*-triangle are shown.

We remark that the plane graphs that result from each of the three possible *ots* operations that may be performed on an *ots*-triangle that is contained in the interior of a 2-region or minimal loop are isomorphic via an isomorphism that is the restriction of a homeomorphism of the plane onto itself that fixes all points in an open neighborhood of the compact set that consists of the *ots*-triangle together with the six additional edges incident to the vertices of the *ots*-triangle.

The introduction of the notion of *ots* operation on a 2-region or a minimal loop is in recognition of the local nature of the *OTS* operation on a prime knot configuration. In order to establish that each K_C configuration can be obtained by a finite sequence of *OTS* and/or T_2^+ operations applied to a K_B configuration, we shall require the following result about minimal 2-regions.

Theorem 5. Given a minimal 2-region, there is a finite sequence of *ots* operations which will transform the minimal 2-region into an empty 2-region.

Figure 10: *ots* operations in a 2-region or a minimal loop

Proof. The proof will be by induction on the number of vertices in the interior of the minimal 2-region. Suppose now that we have a minimal 2-region G which contains no vertices in its interior. If G consists only of the two boundary edges, then G is empty and we are done. Suppose that G has edges other than boundary edges. For such an edge, both endpoints must be boundary vertices since G has no vertices in its interior. Let p and q denote the base vertices of G , and let L_1 and L_2 denote the two boundary paths from p to q . By Corollary 3, a non-boundary edge of G must have one endpoint on L_1 and one endpoint on L_2 . Of all such edges of G , choose the one, e say, whose endpoint on L_1 , v say, is closest to p . Then there are no vertices on L_1 between p and v . If the endpoint of e that lies on L_2 is denoted by w , then there can be no vertices on L_2 between p and w , since such a vertex would be the endpoint of an arc of G which must, by Corollary 3, meet L_1 at a vertex between p and x , which is not possible. Thus e and the two boundary edges incident to p form an *ots*-triangle. Let G' denote the 2-region that is formed by performing an *ots* operation on this *ots*-triangle. Then G' is a minimal 2-region with no vertices in its interior, having one less non-boundary edge than G . Induction on the number of non-boundary edges establishes the base case of the main induction argument.

In order to prove the inductive step, we must first establish that any minimal 2-region with at least one non-boundary edge contains an *ots*-triangle on a boundary.

This problem can be modelled as follows. Let l be a line in the plane through which pass finitely many given lines in the plane, such that no three of the lines (including l) are coincident. If some pair of lines other than l intersect at a point X , then there is a triangle on l with third vertex lying on the same side of l as does X and which is such that none of the lines pass through the interior of this triangle. Such a triangle can be found by examining the finite set of all triangles based on l that are formed by the given lines and choosing one of least height.

We now return to the proof of the theorem. Suppose that $k \geq 0$ is an integer such that for every minimal 2-region with k or fewer interior vertices, there is a finite sequence of *ots* operations that will transform the minimal 2-region into an empty 2-region. Consider any minimal 2-region G with $k + 1$ interior vertices. If there are *ots*-triangles for which two of the three edges of the *ots*-triangle are boundary edges of G , then we may perform an *ots* operation on such an *ots*-triangle to remove l from G , leaving a minimal 2-region still with $k + 1$ interior vertices. Thus we may assume that no *ots*-triangle of G is such that two of the edges are boundary edges of G . Since $k + 1 \geq 1$, G has at least one non-boundary edge, whence there is an *ots*-triangle on a boundary. Choose one *ots*-triangle on a boundary and apply the *ots* operation to this *ots*-triangle. The result is a minimal 2-region G' with k interior vertices, so there is a finite sequence of *ots*-operations which will transform G' into an empty 2-region. Thus there is a finite sequence of *ots*-operations that will transform G into an empty 2-region. This completes the proof of the inductive step. The theorem follows now by induction. \square

Our objective is to prove that for any K_C configuration $C(K)$, there is a K_B configuration $C(K')$ and a finite sequence of *OTS* and/or T_2^+ operations which will transform $C(K')$ into $C(K)$. For this, we require one additional result, which involves the notion of a *t* operation on a 2-region or minimal loop.

Definition 16. Let G be a 2-region or a minimal loop with edge set E , and let G' be a 2-group contained in G whose base vertices x and y are interior vertices of G . Let e and f denote the two boundary edges of G' , labelled in that order clockwise at x . Further let the remaining two edges of G that are incident to x be denoted by e_1 and e_2 in clockwise order, and the remaining two edges of G that are incident to y be denoted by f_1 and f_2 in clockwise order. Let U be an open neighborhood of the closure of the interior of G' which meets no other edges of G than e_1, e_2, f_1 and f_2 (in addition to containing e and f). Delete that part of e_2 and of f_2 which lies in U , except for the endpoints x and y , and extend $e_2 - U$ smoothly into U to terminate at y without meeting e_1, e or f_1 , and extend $f_2 - U$ smoothly into U to terminate at x without meeting f_1, f or e_1 . Denote the two extensions of e_2 and f_2 by e'_2 and f'_2 , respectively. Then the graph H whose vertex set is that of G but whose edge set is $(E - \{e_2, f_2\}) \cup \{e'_2, f'_2\}$ is a 2-region or minimal loop respectively, said to be obtained from G by turning the 2-group G' , or by applying the *t* operation to G' .

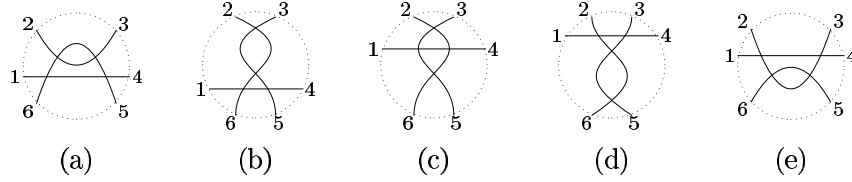
Theorem 6. Let G be a nontrivial minimal loop. Then there is a finite sequence of *ots* and/or *t* operations that will transform G into a minimal loop with either a 2-group one of whose edges is a boundary edge of the minimal loop, or with two 2-groups having a vertex in common, or a *rots*-triangle.

Proof. The proof will be by induction on the number of vertices in the interior of G .

For the base case, consider any nontrivial minimal loop G with no interior vertices. By Proposition 8, G contains a 2-region H , which must then have no vertices in its interior and so must be a 2-group in G . Since G itself is not a 2-region, not both boundary edges of H are boundary edges of G . Thus one boundary edge of the 2-group is a boundary edge of G , and the other is not, whence H is a 2-group one of whose edges is a boundary edge of G .

Now suppose that $k \geq 0$ is an integer such that for any minimal loop with at most k vertices in its interior, there is a finite sequence of *ots* and/or *t* operations that will transform the minimal loop into one which contains either a 2-group having one edge a boundary edge of the minimal loop, or two 2-groups with a vertex in common, or a *rots*-triangle. Consider a minimal loop G with $k + 1$ vertices in its interior. By Proposition 8, G contains a minimal 2-region. Of all minimal 2-regions of G , let H denote one with fewest vertices. By Theorem 5, there is a finite sequence of *ots* operations that will transform H into a 2-group. Suppose that H has an *ots*-triangle O . Then H is not a 2-group, and by the minimality of H in G , G contains no 2-groups. Thus O is an *ots*-triangle of G and so any *ots* operation on O in H can be obtained as the restriction of an *ots* operation on O in G . Furthermore, an *ots* operation on an *ots*-triangle in H will not create a 2-group outside of H . Thus there is a finite sequence of *ots* operations on G that will result in a minimal loop G' containing exactly one 2-group. If one of the edges of the 2-group is a boundary edge of G' , we are done. Suppose that this is not the case. Let u and v denote the vertices of the 2-group. Then both u and v are interior vertices of G' . Let the four edges of G' that are not edges of the 2-group but are incident to either u or v be labelled, in clockwise order, as e_1, f_1, e_2 , and f_2 . Let U be an open neighborhood containing the two edges of the 2-group which meets no other edges of G except for e_1, f_1, e_2 and f_2 . Delete all points on edges of G that lie in U , and choose a point x in U . Extend $e_1 - U, f_1 - U, e_2 - U$ and $f_2 - U$ smoothly into U without meeting, terminating at x . Let e'_1, f'_1, e'_2 and f'_2 denote these extensions of e_1, f_1, e_2 and f_2 , respectively. Let G'' be the plane graph whose vertices are x together with the vertices of G' except for u and v , and whose edges are e'_1, f'_1, e'_2 and f'_2 , together with the edges of G' except for the six edges that are incident to either u or v . Then G'' is a minimal loop with at most k interior vertices, whence by the induction hypothesis, there is a finite sequence of *ots* and/or *t* operations on G'' that will transform it into a minimal loop with either a 2-group having one edge on the boundary, or two 2-groups with a common vertex, or a *rots*-triangle. We shall try to lift each of these *ots* and/or *t* operations on G'' to corresponding operations on G' . It is clear that if an *ots* or *t* operation is being applied to an *ots*-triangle or 2-group, respectively, to which the vertex x does not belong, the *ots* or *t* operation can be lifted directly to G' . On the other hand, suppose that there is an *ots*-triangle O in G'' which has x as one of its vertices, and an *ots* operation is to be performed on O .

We shall always choose to perform the *ots* operation that moves the side opposite the 2-group that is represented by x . This might require a *t* operation if the 2-group is not aligned the right way, followed by two *ots* operations to move the edge across both vertices of the 2-group. Note that since x is an interior vertex of G'' , after the *ots* operation on G'' , x is either still an interior vertex or it has been *ots*-ed out of

Figure 11: Lifting an *ots* operation involving the 2-group vertex

G'' . When we lift the *ots* operation to G' , the end result is that either both vertices of the 2-group are interior vertices or else both have been *ots*-ed out of the minimal loop. Thus every *ots* operation can be successfully lifted, resulting in possibly a *t* operation, but always in two *ots* operations on G' .

Now consider the possibility that a 2-group of G'' has x as a vertex, and the *t* operator is to be performed on this 2-group. At this stage in the sequence of *ots* and/or *t* operations applied to G' , we have obtained either two 2-groups with a common vertex (x represents one of the 2-groups), or else we have a *rots*-triangle, depending on the alignment of the 2-group represented by x .

Thus if the sequence of *ots* and/or *t* operations that are to be performed on G'' never needs to apply *t* to a 2-group that has x as one of its vertices, then the sequence of *ots* and/or *t* operations lifts to give a finite sequence of *ots* and/or *t* operations on G' that results in a minimal loop containing a 2-group with one edge on the boundary of the minimal loop. On the other hand, if the sequence of *ots* and/or *t* operations that are to be performed on G'' does at some point apply *t* to a 2-group that has x as one of its vertices, then the sequence of *ots* and/or *t* operations lifts to give a finite sequence of *ots* and/or *t* operations on G' that results in a minimal loop containing either two 2-groups with a common vertex, or a *rots*-triangle. This completes the proof of the inductive step. \square

Definition 17. Let $C(K)$ be a K_C configuration of a prime knot K . If G is a positive 2-group of $C(K)$ with crossings u and v and edges e and f , then a knot circuit based at either u or v with initial edge either e or f is called a 2-circuit in $C(K)$. The number of crossings on a 2-circuit is called the weight of the 2-circuit.

Theorem 7. Let $C(K)$ be a K_C configuration of a prime alternating knot K . Then there is a K_B configuration $C(K')$ and a finite sequence of *OTS* and/or T_2^+ operations which transforms $C(K')$ into $C(K)$.

Proof. Let $C(K)$ be a K_C configuration of a prime alternating knot K . By Proposition 9, $C(K)$ contains a minimal 2-region G , and then by Theorem 5, there is a finite sequence of *ots* operations which will transform G into an empty 2-region. Now, an *ots*-triangle which is not on the boundary of a minimal 2-region of $C(K)$ is an *OTS* 6-tangle and the *ots* operation on such an *ots*-triangle is exactly the result of an *OTS* operation applied to the *OTS* 6-tangle. On the other hand, an *ots*-triangle which is on the boundary might fail to be an *OTS* 6-tangle, but only if it is actually a *rots*-triangle. Since $C(K)$ is a K_C configuration, it has no *rots*-tangles, so if a *rots*-tangle appears, it was as a result of an *OTS* operation. Prior to such an *OTS* operation, the *rots*-tangle would have been an *ots-rots* tangle, so there would be a finite sequence of *OTS* operations that would create an *ots-rots* tangle

from crossings of G . If no such *ots-rots* tangle is created as the *OTS* operations are applied to G to empty it out, then we arrive at an empty 2-region, which is then a 2-subgroup. If it is negative, then prior to the last *OTS* operation, we would have obtained a negative *ots*-2-subgroup. Otherwise, the finite sequence of *ots* operations corresponds to a finite sequence of *OTS* operations which transform the minimal 2-region into a positive 2-subgroup. This positive 2-subgroup is either a positive 2-group, or else it is a 2-subgroup of a positive 3-group (since $C(K)$ had no positive groups of size greater than 2). Consequently, from G we determine that either there is a finite sequence of *OTS* operations that transforms $C(K)$ into a K_B configuration by virtue of creating an *ots-rots* tangle, a negative *ots*-2-subgroup, or a positive 3-group, in which case $C(K)$ is obtained by the inverse sequence of *OTS* operations applied to the K_B configuration, or else there is a finite sequence of *OTS* operations that when applied to $C(K)$ result in a K_C configuration $C(K')$ containing a positive 2-group. The inverse sequence of *OTS* operations will transform $C(K')$ into $C(K)$.

It therefore suffices to prove that every K_C configuration that contains a positive 2-group is obtained by applying some finite sequence of *OTS* and/or T_2^+ operations to some K_B configuration. We shall prove by induction on k that for every integer $k \geq 3$, if $C(K)$ is a K_C configuration of a prime alternating knot K with a 2-circuit of weight k , then there is a K_B configuration $C(K')$ and a finite sequence of *OTS* and/or T_2^+ operations that transforms $C(K')$ into $C(K)$.

The statement is vacuously true for $k = 3$, since the existence of a 2-circuit of weight 3 would imply the existence of a 2-tangle, and a prime knot has no 2-tangles (we are working with knots of 4 crossings or more). Suppose now that $k \geq 3$ is an integer such that for every K_C configuration $C(K)$ having a 2-circuit of weight at most k , there is a K_B configuration $C(K')$ and a finite sequence of *OTS* and/or T_2^+ operations that will transform $C(K')$ into $C(K)$. Let $C(K)$ be a K_C configuration with a 2-circuit C of weight $k + 1$. By Proposition 9, there is a minimal loop G of $C(K)$ whose boundary cycle C' is a subwalk of C . By Theorem 6, there is a finite sequence of *ots* and/or t operations that will transform G into a minimal loop with either a 2-group one of whose edges is a boundary edge of the minimal loop, or with two 2-groups having a vertex in common, or a *rots*-triangle. It is evident that any t operation on a minimal loop or 2-region of a knot configuration which contains no negative groups is the restriction of a T_2^+ operation on the knot configuration, but it is possible that an *ots*-triangle of a minimal loop or 2-region of a knot configuration is not an *OTS* 6-tangle of the knot configuration. However, in such a case, the *ots*-triangle would be such that two of its vertices were actually vertices of a 2-group in the knot configuration, whence the knot configuration would be a *rots* tangle. Let us start with $C(K)$ and follow the sequence of *ots* and/or t operations which will transform G into a minimal loop with either a 2-group one of whose edges is a boundary edge of the minimal loop, or with two 2-groups having a vertex in common, or a *rots*-triangle. Let $C(K_0) = C(K)$, $C(K_1), \dots, C(K_n)$ denote the configurations formed by the sequence of operations, either carried out to completion or until an *ots* operation is reached where the corresponding *ots*-triangle is actually part of a *rots* tangle in the knot configuration produced by the preceding operation. Suppose that $C(K_n)$ is not a K_C configuration. Since $C(K_0)$ is a K_C configuration, there is a least index i such that $C(K_{i-1})$ is a K_C

configuration but $C(K_i)$ is not a K_C configuration. Since no single OTS or T_2^+ operation will convert a K_C configuration into a K_A configuration, it must be that $C(K_i)$ is a K_B configuration. Thus the sequence of operations is reversible (it suffices to observe that no T_2^+ operation resulted in a group of size greater than 2) and so we have a finite sequence of OTS and/or T_2^+ operations which will convert the K_B configuration $C(K_i)$ into $C(K)$.

Otherwise, $C(K_n)$ is a K_C configuration, whence it contains no *rots* tangles. It follows that the sequence of *ots* and/or *t* operations has been successfully carried out as OTS and/or T_2^+ operations on $C(K)$. Moreover, the inverse of the sequence of OTS and/or T_2^+ operations is a sequence of OTS and/or T_2^+ operations that will transform $C(K_n)$ into $C(K)$. It therefore will suffice to establish that there is a K_B configuration and a finite sequence of OTS and/or T_2^+ operations that will transform the K_B configuration into $C(K_n)$. Now, since $C(K_n)$ is a K_C configuration, it does not contain any 3-groups nor *rots* 6-tangles, nor any negative groups, so we must have obtained a minimal loop containing a positive 2-group H with one of its edges on the boundary of the minimal loop. Moreover, if the edge of H which is not on the boundary of the minimal loop is not on the 2-circuit C , then H and the 2-group that gave rise to C would constitute an interleaved 2-sequence, whence $C(K_n)$ would be a K_B configuration. Since $C(K_n)$ is a K_C configuration, it follows that the edge of H which is not on the boundary of the minimal loop is on the 2-circuit C . Let v denote the first vertex of H to be encountered as C is traversed. Let C' denote the walk obtained by starting at v and following C along the first edge incident to v that is traversed by C until v is reached again (as it must be, since both edges of H will be traversed by C). Then C' is a 2-circuit of $C(K_n)$. Moreover, since every crossing of C' is a crossing of C , and the initial vertex of the 2-circuit C does not belong to C' , it follows that the weight of C' is less than $k+1$, the weight of C . But then $C(K_n)$ is a K_C configuration with a 2-circuit of weight at most k , whence by the induction hypothesis, there is a K_B configuration $C(K')$ and a finite sequence of OTS and/or T_2^+ operations which will transform $C(K')$ into $C(K_n)$, as required.

This completes the proof of the induction step, and the result follows now by induction. \square

We remark that for any K_C configuration, it follows from Proposition 6, that there is a K_B configuration and a sequence of OTS/T operations, the first of which is OTS , which when applied to the K_B configuration results in the given K_C configuration.

6. Summary

This completes the proof that the algorithm does indeed produce all prime alternating knots of minimal crossing size $n+1$ from those of minimal crossing size n . In the sequel, an implementation of the algorithm is presented.

7. References

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