Let τ and τ' be two piecewise linear embeddings of Γ_1 in $\mathbb{R}^2$ preserving the unoriented cyclic order of edges at each vertex. Since Γ satisfies (1) there is a homeomorphism $h:S^2 \to S^2$ such that $h \circ \tau$ agrees with τ' on Γ. If C is the 2-cell which must contain $\tau(vw)$, then $h(C)$ is the 2-cell which must contain $\tau'(vw)$. There is a homeomorphism $h_1:S^2 \to S^2$ which agrees with h on the complement of the interior of C and agrees with $\tau' \circ \tau^{-1}$ on $\tau(vw)$ and hence on all of $\tau(\Gamma_1)$. Then $h_1 \circ \tau = \tau' : \Gamma_1 \to S^2$, so Γ_1 satisfies (1).

If w is a 0-cell of the embedded Γ, we change the cell decomposition by introducing a new 1-cell vw, which divides the 2-cell C into two 2-cells. If w is not a 0-cell, it lies inside some 1-cell. We divide the 1-cell, at a new 0-cell w, into two 1-cells. Then the new 1-cell vw divides C into two 2-cells. In either case we have a new non-degenerate cell decomposition whose 1-skeleton is Γ_1. Thus Γ_1 satisfies (2).

Case 2. The assumption of Case 1 does not hold but there is a 1-cell containing more than one edge of Γ.

We cut the two end edges of this 1-cell. Since G with these cuts is still connected, we take a minimal arc in the cut graph G from the interior of the 1-cell to the rest of Γ, say from a vertex v inside the 1-cell to a vertex w of Γ not inside the 1-cell. This arc has only its end points in Γ. Since Γ is not in Case 1, w is not a 0-cell but, like v, it lies inside a 1-cell. Hence w is not even an end point of the 1-cell through v. Adding the arc vw to Γ, we obtain Γ_1.

The embedding of G gives us an embedding of Γ which extends to a piecewise linear embedding of Γ_1 preserving unoriented cyclic order at its vertices.

The arc vw, apart from its end points, must be embedded in some open 2-cell C. The 1-cell containing v is the intersection of two 2-cells, one of which must be C. Since w is in C but not in this 1-cell, w is not in the other 2-cell. The arc vw can be embedded only in the cell C. As in Case 1, it follows that Γ_1 satisfies (1).

The 1-cell containing v is divided into two 1-cells by a new 0-cell at v. The 1-cell containing w is divided into two 1-cells by a new 0-cell at w. Then C is divided into two 2-cells by the new 1-cell vw. We have a new non-degenerate cell decomposition of S^2 whose 1-skeleton is Γ_1. Thus Γ_1 satisfies (2).

There are no other cases because, since Γ is a proper subgraph of G and G is connected, there is an edge of G which is not in Γ but has an end point v in Γ. If v is a 0-cell, we are in Case 1. Otherwise, v is inside a 1-cell which must have more than one edge of Γ, so we are in Case 1 or in Case 2.

Thus there exists a subgraph Γ_0 of G satisfying (1) and (2). And for each proper subgraph Γ of G satisfying these conditions, there is a larger subgraph Γ_1 satisfying the conditions. Hence G satisfies (1) and (2). In particular, G satisfies (1), which proves the lemma, and hence also the theorem.

Corollary 1.1. *If S is realizable and G is the 4-valent graph corresponding to S, there is a piecewise linear embedding of G in the oriented plane $\mathbb{R}^2$ which preserves the cyclic order of the edges at vertex* 1, *and preserves the unoriented cyclic order at*