Now let $r \notin [i, a_i]$, $a_r \in [i, a_i]$. If the arc through r crosses the loop from outside to inside and the inside is to the right at r, or if it crosses from inside to outside and the inside is to the left at r, then $\phi_i(r)$ and $\phi_i(a_r)f(i)$ have the same sign and hence $\phi_i(r)\phi_i(a_r)f(i)=1$. Also since the arc through r crosses from left to right, $f(a_r)=-1$ and $f(r)=1$. Similarly, if $\phi_i(r)\phi_i(a_r)f(i)=-1$, then $f(r)=-1$. Thus $\phi_i(r)\phi_i(a_r)f(i)=f(r)$.

Thus if (S,f) is realizable, (S,f) must satisfy the following condition:

Rule 2. *For all i and s in $\{1,\ldots,2n\}$ such that $i<a_i<s$ and $a_s<s$,*
(i) $\phi_i(s)\phi_i(a_s)=1$ *if* $a_s \notin [i, a_i]$;
(ii) $\phi_i(s)\phi_i(a_s)f(i)=f(s)$ *if* $a_s \in [i, a_i]$.

The conditions $i<a_i<s$ and $a_s<s$ make Rule 2 easier to check. They are justified by Theorem 2 below. They can also be justified by the symmetries of the function $\phi_i(s)\phi_i(a_s)$. Obviously $\phi_i(s)\phi_i(a_s)=\phi_i(a_s)\phi_i(s)$. It can be verified that if i, a_i separate s, a_s in the cyclic order mod $2n$,

$$\phi_i(s)\phi_i(a_s)=\phi_{a(i)}(s)\phi_{a(i)}(a_s) \quad\text{and}\quad \phi_i(s)\phi_i(a_s)=-\phi_s(i)\phi_s(a_i).$$

If the cyclic order is i, a_i, s, a_s, then

$$\phi_i(s)\phi_i(a_s)=-\phi_{a(i)}(s)\phi_{a(i)}(a_s) \quad\text{and}\quad \phi_i(s)\phi_i(a_s)=\phi_s(i)\phi_s(a_i).$$

For example, if the cyclic order is i, a_s, a_i, s, $\phi_i(s)\phi_i(a_s)=(-1)^u$, where u is the number of elements t of $[s+1, a_s]$ for which $a_t \in [i, a_i]$. And $\phi_s(i)\phi_s(a_i)=(-1)^v$, where v is the number of elements t of $[s, a_s]$ for which $a_t \in [i+1, a_i]$. Since $a_s \in [i, a_i]$ but $a_i \notin [s, a_s]$, $v=u+1$ and hence $\phi_i(s)\phi_i(a_s)=-\phi_s(i)\phi_s(a_i)$.

Theorem 2. *Rule 2 is a necessary and sufficient condition for the realizability of (S,f).*

Proof. Necessity has been shown. It remains to be proved that if (S,f) is not realizable, there exist i and s such that $i<a_i<s$ and $a_s<s$ and either
(i) $a_s \notin [i, a_i]$ and $\phi_i(s)\phi_i(a_s)=-1$, or
(ii) $a_s \in [i, a_i]$ and $\phi_i(s)\phi_i(a_s)f(i)=-f(s)$.

Let G be the 4-valent graph obtained from $[0, 2n]$ by identifying 0 with $2n$ and identifying each i with a_i. Let H_r be the image of the closed interval $[1, r]$ in G. There is some r such that H_r has a piecewise linear embedding in $\mathbb{R}^2$ with the orientations where it crosses itself consistent with f, but H_{r+1} does not have such an embedding. Hence the embedding of H_r cannot be extended by a piecewise linear embedding of $[r, r+1]$ so that the resulting embedding of H_{r+1} is consistent with f. Clearly it would be possible to extend to $[r, r+1]$ if $a_{r+1} \notin H_r$; hence $a_{r+1} \leq r$.

Let α, β be points of the edge $[r, r+1]$ with $r<\alpha<\beta<r+1$. We can extend the embedding of H_r to include a linear embedding of $[r, \alpha]$. We can also embed $[\beta, r+1]$, with $r+1$ at a_{r+1}, but otherwise not meeting H_r, so that the orientation at $r+1$ is consistent with $f(r+1)$. Since $[r, r+1]$ cannot be embedded, the points α and β of $\mathbb{R}^2$ cannot be joined by an arc in $\mathbb{R}^2 \backslash H_r$.