$(-1)^{p-t}$, $\phi_p(a_t)=1$ and hence $\phi_p(t)\phi_p(a_t)f(p)=f(t)$. If $t\notin[p, a_p]$ and $a_t\notin[p, a_p]$, then $\phi_p(t)\phi_p(a_t)=\phi_r(t)\phi_r(a_t)$. Thus, for defining $f(t)$ or for checking Rule 2, ϕ_p gives nothing not already given by ϕ_r.

If a is decreasing from $[r, s]$ onto $[a_s, a_r]$, let $p\in[r, s]$. Let $t\notin[p, a_p]$ and $a_t\in[p, a_p]$. Then $\phi_p(t)=\phi_r(t)$, $\phi_p(a_t)=(-1)^{p-r}\phi_r(a_t)$ and $\phi_p(t)\phi_p(a_t)f(p)=\phi_r(t)\phi_r(a_t)f(r)$. If $t\notin[p, a_p]$ and $a_t\notin[p, a_p]$ and t is not on the twist, $\phi_p(t)\phi_p(a_t)=\phi_r(t)\phi_r(a_t)$. If $r\leqslant t<p$, $\phi_p(t)\phi_p(a_t)=1$. Again ϕ_p gives nothing new.

In all cases one needs to check only one ϕ_r for each twist. If 1 is on the twist, ϕ_1 will do. This justifies changing the definition of i_h as follows: Let $A_1=[1, a_1]$, $B_1=\emptyset$, $i_1=1$. Inductively, let

$$A_h=A_{h-1}\cup\{s: i_{h-1}*s\},$$

$$B_h=B_{h-1}\cup\{i_{h-1}, a(i_{h-1})\}$$

$$\cup\{t, a_t: t\notin[i, a_i], a_t\in[i, a_i], a_{t-1}=a_t\pm 1 \ (\mathrm{mod}\ 2n), i=i_{h-1}\},$$

$$i_h=\text{least member of } A_h\backslash B_h \text{ if } A_h\backslash B_h\neq\emptyset.$$

This excludes the possibility, for $i=i_h$, that $a_{i-1}=a_i\pm 1$ (mod $2n$). It ensures that only one ϕ_i is checked for each twist, except that it leaves one unnecessary ϕ_i in certain cases of twists containing 1. We still have $\bigcup_h A_h=\{1, \dots, 2n\}$.

The definition of canonical orientation remains formally the same, but it now depends on the new definition of i_h. But, since the new definition merely excludes certain unnecessary ϕ_i when i is on a twist, the orientation defined is the same as before.

Also Rule 2 remains valid when these unnecessary ϕ_i are excluded. Hence we have the following corollary:

Corollary 3. *Theorem* 3 *remains true with the changed meaning of* i_h.

We have used the canonical orientation only for checking whether S is realizable. It will also be needed for finding a presentation of the knot group, and hence for finding the knot invariants. In fact one can find the group and its invariants for an arbitrary orientation of S, whether S is realizable or not, and for an arbitrary choice of overcrossings [1, p. 86]. This enables one to compare knot groups with other similarly presented groups.

The canonical orientation f for non-realizable S may also be of interest. For example, the sequence S whose standard subsequence is 4 8 2 10 6 is not realizable because (S, f) does not satisfy (i) of Rule 2, where f is the canonical orientation: $f(i)=(-1)^{i-1}$. Taking the odd numbers as overcrossings, (S, f) has the same Alexander polynomial as the reef knot and certain non-composite knots. But the group of this 'knot' is distinguished by its invariants from the group of any knot with up to 12 crossings. This (S, f) can be realized on the torus, and the realization is the projection of an alternating knot in $T^2\times\mathbb{R}^1$.