the other vertices. This embedding is unique up to orientation preserving homeomorphism of S^2.

Note that if two directed arcs cross each other in the oriented plane, the first crosses the second from right to left if and only if the second crosses the first from left to right. Let $\rho:[0, 2n]\to\mathbb{R}^2$ be a realization of S in the oriented plane. We identify 0 with $2n$ and use numbers mod $2n$. Let $f(i)=1$ if the arc $\rho([a_i-1, a_i+1])$ crosses the arc $\rho([i-1, i+1])$ from right to left, $f(i)=-1$ otherwise. Then $f(i)=-f(a_i)$. We can choose the orientation of $\mathbb{R}^2$, or the realization ρ, so that $f(1)=1$.

For any S, realizable or not, we define an *orientation* to be a function $f:\{1, 2, \dots, 2n\}\to\{-1, 1\}$ such that $(i) f(1)=1$; $(ii) f(i)=-f(a_i)$ for $i=1, 2, \dots, 2n$.

If f is an orientation of S, a *realization* of (S, f) is a realization ρ of S in the oriented plane $\mathbb{R}^2$ such that $\rho([a_i-1, a_i+1])$ crosses $\rho([i-1, i+1])$ from right to left if and only if $f(i)=1$. (S, f) is called *realizable* if there exists a realization of (S, f).

The following is a restatement of Corollary 1.1.

Corollary 1.2. *If S is realizable, there is a unique orientation f such that (S, f) is realizable, and the realization of (S, f) is unique up to orientation preserving homeomorphism of S^2.*

For any S, we define a sequence of functions $\phi_i:\{1, 2, \dots, 2n\}\to\{-1, 1\}$ as follows:

(i) $\phi_i(i)=1$;

(ii) $\phi_i(r)=\begin{cases}-\phi_i(r-1) & \text{if } a_r\in\{i, \dots, a_i\},\\ \phi_i(r-1) & \text{otherwise.}\end{cases}$

This is consistent because $\phi_i(r)$ changes sign an even number of times before returning to $\phi_i(i)=1$. We have $\phi_i(a_i)=-1$ and $\phi_i(i-1)=-1$.

The *loops* of the realization ρ are the closed paths $\rho([i, a_i])$. We use the following convention. The component of the complement of the loop $\rho([i, a_i])$ which contains the edges $\rho([i-1, i])$ and $\rho([a_i, a_i+1])$, except their end points, is said to be *outside* the loop, and the components reached from this one by crossing the loop an even number of times are also *outside*. The other components are *inside*.

By abuse of language we write $[i, a_i]$ for the loop $\rho([i, a_i])$. Then $\phi_i(a_i)=-1$ where the path $\rho([0, 2n])$ passes outside the loop $[i, a_i]$. When this path crosses from the outside to the inside, or vice versa, $\phi_i(r)$ changes sign. Thus if $r\in\{a_i, a_i+1, \dots, i-1\}$, $\phi_i(r)=-1$ if the path at r or just beyond is outside, $\phi_i(r)=1$ if it is inside.

Note that if $\rho(r)$ is not on the loop then, since $\rho(r)$ and $\rho(a_r)$ are the same point, they are both inside or both outside. Hence $\phi_i(r)\phi_i(a_r)=1$.

If $f(i)=1$, the arc through a_i crosses the arc through i from right to left, and the component to the right of $\rho([i, i+1])$ is inside. Also, since $\phi_i(i)=1$, $\phi_i(i)f(i)=1$. If $\phi_i(i)f(i)=-1$, the component to the left of $\rho([i, i+1])$ is inside. Where the loop crosses itself, $\phi_i(r)f(i)$ changes sign and the inside changes from left to right, or vice versa. Thus if $r\in\{i, \dots, a_i-1\}$, $\phi_i(r)f(i)=1$ if the inside is to the right at r, or just beyond; $\phi_i(r)f(i)=-1$ if the inside is to the left.