By a suitable bilinear mapping of S^2, regarded as the extended complex plane, we can transform α and β to 0 and ∞ respectively. Let C be a minimal subgraph of H_r, in the sense of least number of edges, which separates 0 from ∞. Then C is a simple closed curve with 0 in the bounded component of its complement. If C is suitably oriented, the winding number of C about 0 is 1. That is, as z goes around C in the positive direction, $\arg z$ increases by 2π.

Let the successive vertices of the directed simple closed curve C be $v_1, v_2, \ldots, v_k$ with edges e_j joining v_j to v_{j+1}, where $v_{k+1} = v_1$. Let $\arg_C z$ be defined unambiguously on C as follows: $0 \leqslant \arg_C v_1 < 2\pi$, and $\arg_C z$ is continuous on C except for a jump of -2π at v_1, with the restriction of $\arg_C z$ to $[v_1, v_2]$ continuous at v_1.

Let $\arg_H z$ be defined on H_r as follows: $0 \leqslant \arg_H z < 2\pi$ at the vertex 1, and $\arg_H z$ varies continuously along the path [1, r]. This defines $\arg_H z$ ambiguously at crossing points of H_r, but unambiguously on the edges.

Let $2\pi d_j$ be the jump of $\arg_H z$ as z passes from e_{j-1} to e_j at v_j. Let $2\pi c_j$ be the jump of $\arg_C z$ at v_j. If we change the function $\arg_H z$ by adding $2\pi a$ to its values on e_j, this will increase d_j by a and decrease d_{j+1} by a. Thus $\sum_1^k d_j$ remains unchanged. Changing $\arg_H z$ on each e_j, so that it agrees with $\arg_C z$ on all the edges of C, leaves $\sum_1^k d_j$ unchanged. Therefore $\sum_1^k d_j = \sum_1^k c_j = -1$. Hence some d_j is odd.

Since $\arg_H z$ is continuous along the path $[1, r]$, and d_j is odd, and hence not 0, v_j is a crossing point of H_r. Let it be the crossing point (i, a_i), where $1 \leqslant i < a_i \leqslant r$. Then $\arg_H z$ is continuous on the closed path $[i, a_i]$ except for a jump of $\pm 2\pi d_j$ at i. Hence the winding number of this closed path about the origin 0 is odd. Hence an arc from 0 to ∞ crosses $[i, a_i]$ an odd number of times.

Returning to the piecewise linear embedding of H_r in $\mathbb{R}^2$, an arc from α to β crosses the loop $[i, a_i]$ an odd number of times. Hence one of α, β is inside the loop, and the other is outside.

Since H_r is embedded with the orientations given by f at its crossing points, and since the path $[1, r]$ includes the loop $[i, a_i]$, the geometric interpretation of ϕ_i deduced above is valid on H_r, and ϕ_i satisfies Rule 2 at the crossing points of H_r.

Now suppose, for example, that α is inside the loop $[i, a_i]$ and β is outside. Then either r is inside the loop, or the path crosses inside the loop at r. In either case, $\phi_i(r) = 1$. Since β is outside, either a_{r+1} is outside or the path $[\beta, r+1]$ is crossing from the outside to the inside at $r+1$. Let $s = r+1$.

First suppose a_s is outside. Then a_s is not in $[i, a_i]$, so $\phi_i(s) = \phi_i(r+1) = \phi_i(r) = 1$. Since $a_s \in H_r$ and a_s is outside, $\phi_i(a_s) = -1$. Thus $\phi_i(s)\phi_i(a_s) = -1$.

Now suppose a_s is on the loop and $[\beta, s]$ is crossing from the outside to the inside. Since a_s is on the loop, $\phi_i(s) = \phi_i(r+1) = -\phi_i(r) = -1$. If $\phi_i(a_s)f(i) = 1$, the inside is on the right at a_s, so $[\beta, s]$ is crossing the path through a_s from left to right, so $f(a_s) = -1$ and $f(s) = 1$. If $\phi_i(a_s)f(i) = -1$, the inside is on the left at a_s, so $[\beta, s]$ is crossing the path through a_s from right to left, so $f(s) = -1$. Thus in each case $\phi_i(s)\phi_i(a_s)f(i) = -f(s)$.

Suppose, if possible, that $r = 2n$. Since $H_r = H_{2n}$ and Rule 2 is satisfied in H_r, if $r+1 = 1 \notin [i, a_i]$, and $a_1 \notin [i, a_i]$, then $\phi_i(1)\phi_i(a_1) = 1$. Also if $1 \notin [i, a_i]$ and $a_1 \in [i, a_i]$,