Enumerating the Prime Alternating Knots, Part II

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1. Introduction

In Part I, we provided a solution for part of the enumeration problem, in that we described an inductive scheme which used a total of four operators to generate all prime alternating knots of a given minimal crossing size from those of crossing size one less, and we provided a proof there that the procedure does produce them all. Our primary objective in Part II is to present an efficient implementation of this procedure.

One very important aspect of our implementation is the notion of the master array of an alternating knot. The master array of a prime alternating knot is an integer array with the property that each regular projection (plane configuration) at minimal crossing size for the knot can be constructed from the data in the array, and moreover, two knots are flype equivalent if and only if their master arrays are identical. Furthermore, the master array for a prime alternating knot can be calculated from the data describing any minimal crossing configuration for the knot, and we have a procedure for selecting from the master array a preferred configuration, which we suggest could be used as the so-called ideal configuration of the knot. A very efficient way to store the knot would then be to use the Dowker-Thistlethwaite code for this ideal configuration.

An early implementation of this algorithm was used to enumerate the prime alternating knots up to and including those of 19 crossings. It took approximately 2.3 hours of time on a five node beowulf cluster to produce the $1,769,979$ prime alternating knots of 17 crossings. We then went on to produce the prime alternating knots at 18 and 19 crossings using a 48 node beowulf cluster. The cluster was shared with other users and so an accurate estimate of the running time is not available, but the generation of the $8,400,285$ knots at 18 crossings was completed in 17 hours, and the generation of the $40,619,385$ prime alternating knots at 19 crossings took approximately 72 hours. In Part II, we describe improvements to our original implementation that should allow us to produce the prime alternating knots at 19 crossings from those at 18 crossings in about 10 hours on a current Pentium III personal computer equipped with 256 megabytes of main memory.

Since we shall only be working with minimal crossing configurations of alternating knots, it should be understood from now on that when we say configuration, we mean minimal crossing configuration.

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providing the programming expertise (John Schermann having moved on to Open Text Corp.), the procedures described in this paper were implemented in C^{++}. We were able to produce all of the prime alternating knots up to 22 crossings (see Sequence A002864 in [2]) on a 2.8ghz Xeon workstation with 3gB of memory, with the following results.

Crossing Size	Number	Time
18	8,400,285	2m 54s
19	40,619,385	14m 35s
20	199,631,989	1h 29m
21	990,623,857	6h 54m
22	4,976,016,485	35h 44m

The full database is available online at `http://knotilus.math.uwo.ca`, but only for the viewing of individual knots either selected from the database or entered manually. The program suite for the generation of the full database (at present, binaries for several platforms are available) is available for download at `http://www.math.uwo.ca/~srankin/papers/knots/pakg.html`.

2. Master Group Codes and the Master Array

In this section, we introduce the master array for a prime alternating knot. The master array is constructed from any group code for any configuration of the knot. It describes the basic structure of the knot and identifies all possible flype moves for the knot. Consequently, it is possible to construct every configuration of the knot from its master array. Moreover, two knot configurations are flype equivalent if and only if the master arrays that are constructed from the two configurations are identical.

As a first step toward the construction of the master array for a knot, we introduce the notion of a master group code. A master group code is constructed from any configuration for the knot. It is a cyclic code which contains the complete orbit structure of the knot, and therefore any configuration of the knot can be constructed from any master group code for the knot. There are, however, many master group codes for a given knot, and this would lead to extensive computation if one were to try to compare two configurations for flype equivalency by comparing master group codes for the two configurations. The master array is actually a master group code which satisfies certain conditions (which are such that there is exactly one master group code that satisfies the conditions), together with a pointer to the position from which the traversal of the code is to begin, and a direction of traversal.

We begin with the definition of a master group code for an alternating knot. Following that, we present an algorithm for the construction of all master group codes of a given knot. The section concludes with the definition and construction of the master array for a knot.

2.1. *The definition of master group codes*

We have observed that the group orbits of a prime alternating knot can be determined from any configuration of the knot. Furthermore, as established in

Theorem 2 of [1], once the orbits of all groups are known, then it is a simple matter to convert the configuration into a full group configuration. More generally, once the groups and their orbits are identified, then every configuration of the knot can be constructed by flyping the various crossings of the different groups, wherein each crossing can be placed at any position in the orbit of the group of the crossing. It turns out that the decision to deal only with full group configurations provided us with a significant theoretical and computational advantage over the traditional methods in the study of the prime alternating knots.

These observations lead naturally to the notion of what we call a master group code for the knot. In effect, a master group code for a knot describes the orbit structure of each group, the size of each group, and the sign (positive or negative) of each group.

We shall need the following result, which is an immediate consequence of the non-interference of orbits (see Theorem 1 of [1]).

Theorem 1. Let $C(K)$ be a full group configuration of a prime alternating knot K, and let G_1, G_2, G_3 and G_4 be distinct groups in $C(K)$. Further suppose that G_3 and G_4 are connected by an arc e and that G_1 can group flype to e and some other arc, and G_2 can group flype to e and some other arc, so that G_3 and G_4 are in adjacent min-tangles T_1 and T_2, respectively, in the orbit of G_1, and they are in adjacent min-tangles S_1 and S_2, respectively, in the orbit of G_2. Then

(a) G_1 is contained in either S_1 or S_2, and G_2 is contained in either T_1 or T_2;
(b) G_1 is in S_1 if and only if G_2 is in T_2.
(c) G_1 is in S_1 if and only if for any sequence of group flypes applied to $C(K)$ which results in both G_1 and G_2 being group flyped to lie on e, the knot traversal out from the group G_3 in the direction of e encounters G_1 before G_2.

Proof. G_2 belongs to some min-tangle of the orbit of G_1, and by Theorem 1 of [1], G_2 can't flype to any arcs that are not incident to or contained within this min-tangle. Since G_2 can flype onto arc e, which is incident to the two min-tangles T_1 and T_2 of the orbit of G_1, it follows that G_2 is contained either in T_1 or else T_2. Similarly, G_1 belongs either to S_1 or to S_2.

Suppose that G_1 is in S_1. Any group flype move applied to G_2 must leave it still in T_1. Since there is a group flype which will move G_2 onto e, let us apply it. Let e' denote the arc connecting G_3 to G_1 and e'' denote the arc connecting G_1 to G_4 (in effect, e has been replaced by e' and e''). Now e'' is the arc connecting the min-tangle T_1 (or rather, the result of applying the group flype to G_2 in T_1) to the min-tangle T_2, and so we may group flype G_1 to e'' and the other arc joining T_1 to T_2. Now the orbits of G_1 and G_2 are not changed by the group flyping of G_1 and G_2, in the sense that the groups that make up a given min-tangle T of a given group are still the groups that make up the min-tangle after any group flype whatsoever–they may have changed location, but they have not left the min-tangle T. In this last configuration, we have G_3 connected to G_1, which in turn is connected to G_2. Thus G_3 is in one min-tangle of the orbit of G_1 and G_2 is in a different min-tangle of the orbit of G_2. Since G_3 is in T_1, we see that G_2 is not in T_1. Then, since we know that G_2 is in either T_1 or else T_2, we may conclude that G_2 is in T_2, as required. The symmetry in this argument allows us to conclude similarly that G_2 in T_2 implies that G_1 is in S_1.

For the last part, we again suppose that G_1 is in S_1. Apply any sequence of group flypes which results in both G_1 and G_2 being group flyped to the arc e. Then as we traverse the knot from G_3 in the direction of e, we remain in the min-tangle S_1' of the orbit of G_2 which is the image of S_1 under this seqence of group flypes. Since G_1 lies on e, we will encounter G_1 before leaving S_1', hence will encounter G_1 before encountering G_2. Then G_1 and G_4 lie in different min-tangles of the orbit of G_2. Since G_4 is in S_2, this implies that G_1 is in S_1, as required. □

The gist of Theorem 1 is that whenever two or more groups have flype positions with one edge e in common, and the groups are group flyped to the position in their respective orbits at which e appears, then the order in which they appear on that strand in any traversal of the knot is independent of the order in which the group flyping took place.

Definition 1. A *master group code* for a prime alternating knot K is constructed from a full group configuration $C(K)$ of K according to the following procedure. Let C be any group code for the configuration. Recall that in the construction (see Definition 2 of [1]) of the group code, each group G was assigned a label of the form $\pm m_n$, where m denotes the size of the group and n was an index used to distinguish between different groups of the same size. Each pair of consecutive group labels in the group code (recall, this is a cyclic code, so the last group label of the code is followed by the first group label of the code) represents the arc that joins these two groups. For each group $\pm m_n$, list the pairs of arcs which separate consecutive min-tangles in the orbit of the group (excluding the arcs which connect the group itself to the min-tangles on either side of the group). If the orbit is trivial, there will be only one min-tangle in the orbit and so the list will be empty, in which case the two occurrences of the labels m_n are replaced by m_n^0, keeping whatever sign the label had originally. In the case when the orbit is non-trivial, there will be one or more pairs of arcs in the list. In this case, let k denote the number of pairs in the list. Form the labels m_n^i, $i = 0, \ldots, k$. Choose one of these $k+1$ labels and replace both occurrences of m_n in the group code by the selected label m_n^i, keeping the sign of the label being replaced. Then in an arbitrary fashion, assign the remaining k labels to the k pairs of arcs in the list. For each arc of a pair, place the label assigned to the pair between the two consecutive groups in the group code which identify the arc. If the group is a negative group, both arcs should receive a minus sign as part of the assigned label. In the event that more than one label is inserted between two consecutive entries of the group code, the labels must be arranged in the uniquely determined order (see Theorem 1) in which the groups would appear if all were group flyped to their respective orbit positions which involved the arc in question.

The cyclic arrangement of labels that results is a *master group code* for the knot.

In practice, the cyclic arrangement is written as a sequence, with the understanding that the initial entry follows that last entry of the sequence.

Since two full group configurations for the same knot only differ in the position in its orbit that each full group is placed, it follows that all full group configurations for a knot can be constructed from the data stored in any master group code for the knot (and, of course, all split group configurations can be so obtained as well).

2.2. *An Algorithm for the Construction of a Master Group Code*

It should be clear now that any two full group configurations will result in two master group codes which are essentially the same, except for possibly the choice of labelling. There is, however, the possibility that the two group codes that were used to construct the two master group codes for the same knot arose from traversals of the knot in opposite directions. Thus if we were to try to compare two master group codes to determine whether or not they describe the same knot, we not only need to rotate one code to compare to the other, but we must try to decide whether or not we would reverse one before doing the rotations for attempted pattern matching. For much of our work, any master group code would suffice, but it will be necessary for us to compare two configurations for flype equivalence, and that leads to the problems described above. For this reason, it is important to assign to a given knot a representative of the set of master group codes, together with a start point and direction of traversal with the property that two configurations are flype equivalent if and only if the two designated master group codes, complete with start point and direction of traversal, are identical.

Given a master group code, it is a simple matter to indicate a start point and direction of traversal of the code: we simply open the cyclic code at the desired point and write the entries in a sequence to describe the order of traversal. That is, we form a linear array. The problem that remains is how to decide on an appropriate scheme for selecting from all of the arrays that can be formed from all of the different master group codes for the knot the unique one with the desired attributes. We will leave this problem for the last subsection of the current section, and concentrate now on an algorithm for the construction of the master group codes for a knot.

Actually, since it does not take much more work to accomplish this, we shall present an algorithm which takes as its input the group code for a knot configuration (either split or full group) of a prime alternating knot, identifies the orbit of each group, identifies all split groups and recombines the split groups so as to obtain a full group configuration, all the while constructing a master group code for the knot as well.

We start with an array that initially consists of a group code for a configuration of the prime alternating knot. In a sense, the group code serves as a skeleton for the master group code that is to be constructed. The first step is to make a list of all the groups in the knot configuration. Then for each group in the list, we identify the min-tangles of the orbit of the group. In each case, if the group is a split group, then as the various subgroups are encountered during the orbit identification process, they are group flyped to the position of the first subgroup, thereby transforming the split group into a full group. At the end of the orbit identification for the group in question, the size m of its full group will have been determined. If it is the n^{th} group of size m that has been processed so far, and there were $k \geq 1$ positions in the orbit of the group, then in an arbitrary manner, we assign the labels $\pm m_n^0$, $\pm m_n^1, \ldots, \pm m_n^{k-1}$ to the k positions. In the array as constructed so far, this is recorded as follows. If the two arcs that separate the two min-tangles which identify the position to which we wish to assign the label m_n^i

are denoted by e and f, then we place one copy of the label $\pm m_n^i$ between the two groups that are the endpoints of e and one copy of the same label between the two groups that represent the endpoints of f. From this point on, in the identification of the orbits of other groups still to be analyzed, these two labels of the group's position are treated as if the group were actually residing at that position (this will establish the correct order of flyping position on each arc as established in Theorem 1). In actual fact, one or both of the "endpoints" of e and f may be labels that were deposited during the identification of the orbit of a group that came earlier in the list. As we remarked above, once a label is put in place, then for subsequent group orbit analyses, the label is treated as if it were a group (whose name is the full label, so it will not be confused with any other position label for the same group).

The actual procedure to be followed in the identification of the min-tangles of the orbit depends on whether the group is a loner, a positive group or a negative group. If a group is a loner, it is first treated as if it were a negative group and checked to see if it has any flype moves along this alignment. If so, then it does have an orbit along this alignment (and therefore not along its positive alignment), and processing continues to identify the min-tangles of its orbit. On the other hand, if the loner has no flype moves along its negative alignment, it is then treated as a positive group. If it has any flype moves along this alignment, then processing continues to identify the min-tangles of its orbit. If it has no flype moves along either the negative or the positive alignment, then it has a trivial orbit.

The decision to check a loner along its negative alignment first (of course, from a theoretical point of view, the order of checking is not relevant) was empirically based. Our experience has shown the vast majority of loners that do have a nontrivial orbit do so along the negative alignment.

It therefore suffices to describe the methods for identifying positive and negative group orbits.

Case 1: the orbit of a negative group G. In Figure 1, we have illustrated the general situation. The core of G's orbit is denoted by C in the figure. The arcs between two consecutive min-tangles have been labelled so that e_j is the arc going from left to right and f_j is the arc going the opposite direction. For the sake of clarity, we have drawn the diagram as if the arc e_j was above the arc f_j, but of course it could be just the opposite. It also could be that the orbit has only one min-tangle, namely C. Of course, this will come out in the analysis of the orbit.

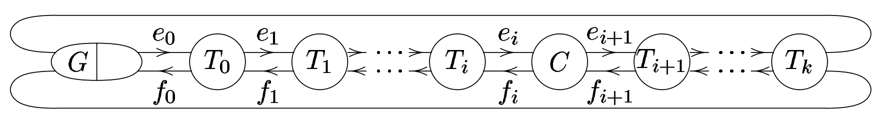


Figure 1: The orbit of a negative group

If the group code is followed from the first arc of G to the second arc of G, then we will traverse both arcs of every group belonging to any min-tangle of the orbit of G that lies between G and the core of the orbit of G; namely the tangles T_0, $T_1, \ldots, T_i$ as shown in Figure 1. Any group for which exactly one of its group arcs is encountered during the traversal from the first arc of G through the core and back to the second arc of G must belong to the core itself. If we were to continue the traversal from the second arc of G through the core and back to the first arc of G (thereby completing the knot traversal), all the while making note of the groups for which exactly one of the group arcs was encountered in this part of the traversal, exactly the same collection of groups would be formed.

For simplicity, we shall cycle the current array to bring one of the arcs of G to the front, and we shall refer to this arc as the first arc of G. Further, let us refer to the portion of the array that lies between the first and the second arcs of G as the first section, and the portion which lies between the second arc of G and the end of the array (the group arc which precedes the first arc of G) as the second section.

With this terminology, the first step in the identification of the orbit of G is to determine which groups of the configuration have one arc in each section. Having done this, we split the analysis into two parts.

Part I. This part describes the processing of the second section. It begins with the identification of the two arcs which separate the core from the first min-tangle which follows it (e_{i+1} and f_{i+1} in Figure 1). Observe that this is not the min-tangle that is encountered after the core has been entered for the first time. The knot traversal that the group code provides starts at the first arc of G and proceeds through possibly several min-tangles (T_0, $T_1, \ldots, T_i$) before entering the core C. After exiting C for the first time, we return to the second arc of G through these same min-tangles (in the reverse order), then through possibly more min-tangles (T_k, $T_{k-1}, \ldots, T_{i+1}$) until the core is re-entered by means of arc f_{i+1}, traversed and exited by means of arc e_{i+1}, then back through T_{i+1}, $T_{i+2}, \ldots, T_k$ and finally to the first arc of G to complete the traversal.

Once e_{i+1} and f_{i+1} have been identified, the next step will be to identify the entry and exit pair which connect T_{i+1} to T_{i+2}, then the pair which connect T_{i+2} to T_{i+3} and so on, until we have worked our way completely through the second section. The last pair of arcs that this process identifies is the pair that connects the last min-tangle of this portion of the orbit to the group G.

So we must begin by finding the two arcs e_{i+1}, f_{i+1} which separate the core from the min-tangle that follows the core in the second section. We have already identified the groups for which one arc lies in the first section and one lies in the second section–these groups we know to be in the core, and we shall refer to them as the *starter core group arcs.* Imagine the strand of the knot which starts at f_{i+1}, runs through the core, and finishes at e_{i+1}. This strand will at intervals intertwine with the strand which starts at e_i, runs through the core and stops at f_i, thereby forming the starter core groups. As well, it may intertwine with itself to form additional groups. Thus we see that we are searching for the shortest strand which contains the starter group arcs and which is closed in the sense that any non-starter group which has an arc on this strand has both arcs on the strand. To begin with, we extract from the current array the shortest subsequence S_1 that contains all of the starter group arcs. There are two possibilities: either S_1 is closed, or else there exists at least one group which has exactly one of its arcs in S_1. In the latter case, we extend S_1 to S_2, the shortest subsequence of the current array that contains the starter arcs and both arcs of each group in S_1 other than the starter groups. If S_2 is not closed, repeat this process. Eventually we arrive at a closed subsequence S of the current array. Then f_{i+1} is the arc which leads into the first group arc of S, and e_{i+1} is the arc which leads out of the last group of S.

Next, we must find the pair of arcs e_{i+2} and f_{i+2} that separate T_{i+1} from T_{i+2} (if indeed there is another min-tangle in this segment of the orbit). Form a new set S_1 by adjoining to the set S either the group arc that is adjacent to e_{i+1} but not in the core (that is, not in S) or the group arc that is adjacent to f_{i+1} but not in the core. Again, the selected group arc might be a position marker rather than one of the group arcs from the original group code. Close S_1 as before to obtain a subsequence of the current array. The arcs at each end are the sought-after pair e_{i+2}, f_{i+2}, unless the tangle that lies between the pair e_{i+1} and f_{i+1} and the pair just determined is actually another subgroup of G (it is readily verified that such a tangle is actually a subgroup of the negative group if and only if the tangle itself is just a negative group). In the event that the tangle is a subgroup of G, we delete this group from the current array and from the control list, and the number of crossings it contributes to the full group is recorded.

Continue this process until the subsequence that is formed is the entire second section, at which point this part is complete.

Part II. This part describes the processing of the first section, whereby the min-tangles which preceed the core are identified. This is accomplished by applying the procedure of Part I to the group code (or more precisely, the group code as modified by the construction procedure so far) that is obtained by writing the group code in the reverse order. That is, we construct a temporary working copy of the current state of the master group code by starting at the initial group code, and write the array in reverse order, so that what was called the first section is now the second section. Now apply Part I to this array, but place all labels in the appropriate position in the current array, not the temporary working copy.

We illustrate this process with an example. Consider the knot configuration shown in Figure 2. This is a prime alternating 11 crossing knot in full group. We have shown a Gauss labelling on the diagram, and if we start at the crossing labelled 1 and proceed in the direction of the crossing labelled 2 on the strand taking us to crossing labelled 3, we obtain the following group code for the configuration.

$$-2_1, -2_2, 2_3, -2_2, 2_4, -2_1, 3_1, 2_3, 2_4, 3_1$$

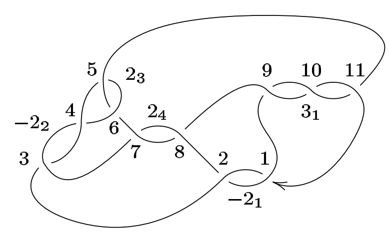


Figure 2: A configuration of a prime 11 crossing knot

The control list of groups is $-2_1, -2_2, 2_3, 2_4, 3_1$, and so we would begin by finding the orbit of the negative group $G = -2_1$. No cycling of the array is necessary, since one of the group arcs of -2_1 is the first entry in the array. The first and second sections, respectively, are

$$-2_2, 2_3, -2_2, 2_4 \quad \text{and} \quad 3_1, 2_3, 2_4, 3_1$$

and so the starter group arcs are 2_3 and 2_4. Since the sequence $S_1 = 2_3, 2_4$ is closed, $S = S_1$ and we have found the entry and exit arcs for this side of the core of the orbit of G to be the arc from 11 to 5 and the arc from 8 to 9, respectively. In this particular example, the two adjacent arcs to the subsequence S belong to the same group, 3_1. We form $S_1 = 3_1, 2_3, 2_4$. This is not closed, since the second arc of group 3_1 is missing. Upon closing it up, we obtain $S = 3_1, 2_3, 2_4, 3_1$. The tangle which has just been determined has incident arcs $(5, 11)$ and $(8, 9)$ as determined earlier, and the arcs $(1, 9)$ and $(1, 11)$ which have just been determined. This tangle consists of just the group 3_1, which must therefore be checked to see if it is a subgroup of the full group determined by -2_1. Since it is a positive group, it is not a subgroup of the negative group to which -2_1 belongs, so it is a min-tangle in the orbit of -2_1. Finally, since S is equal to the second section, part I has been completed.

For part II, we reverse the array to obtain

$$-2_1, 3_1, 2_4, 2_3, 3_1, -2_1, 2_4, -2_2, 2_3, -2_2$$

with first and second sections

$$3_1, 2_4, 2_3, 3_1 \quad \text{and} \quad 2_4, -2_2, 2_3, -2_2,$$

respectively. We know that the starter arcs are 2_3 and 2_4, so $S_1 = 2_4, -2_2, 2_3$. This is not closed, since the other arc for group -2_2 is missing. Upon closing S_1, we obtain $S = 2_4, -2_2, 2_3, -2_2$, which is the second section. Thus the entry and exit arcs for this side of the core of $G = -2_1$ are $(2, 3)$ and $(8, 2)$, respectively. Since we have exhausted the second section (the first section of the actual array, since we are doing part II), there are no additional min-tangles. Thus the orbit of $G = -2_1$ is of the form

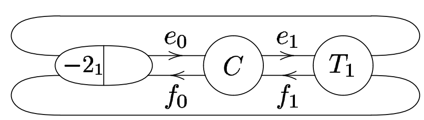


Figure 3: The orbit of -2_1

and the updated array, showing the additional position of group -2_1 is

$$-2_1^0, -2_2, 2_3, -2_2, 2_4, -2_1^0, 3_1, -2_1^1, 2_3, 2_4, -2_1^1, 3_1.$$

Now that the orbit of -2_1 has been identified, we can remove it from the control list. The revised control list is $-2_2, 2_3, 2_4, 3_1$, and the next group in the list is also a negative group. We cycle the current array so as to start with one of the group arcs for $G = -2_2$, say

$$-2_2, 2_3, -2_2, 2_4, -2_1^0, 3_1, -2_1^1, 2_3, 2_4, -2_1^1, 3_1, -2_1^0.$$

The first and second sections are 2_3 and $2_4, -2_1^0, 3_1, -2_1^1, 2_3, 2_4, -2_1^1, 3_1, -2_1^0$, respectively, and there is only one starter group arc; namely 2_3. We have $S_1 = 2_3 = S$, and the entry and exit arcs from this side of the core are $(11, 5)$ and $(6, 7)$, respectively. The adjacent group arcs are -2_1^1 and 2_4, so we choose one, say -2_1^1, and form $S_1 = -2_1^1, 2_3$. Since the other arc of group -2_1^1 is missing, S_1 is not closed. We find the other arc of group -2_1^1 and form $S_2 = -2_1^1, 2_3, 2_4, -2_1^1$. Since the other arc of group 2_4 is missing, we find it and form $S_3 = 2_4, -2_1^0, 3_1, -2_1^1, 2_3, 2_4, -2_1^1$. Since S_3 is missing the other arc from each of groups -2_1^0 and 3_1, we form $S_4 = 2_4, -2_1^0, 3_1, -2_1^1, 2_3, 2_4, -2_1^1, 3_1, -2_1^0$, which is closed. Thus $S = S_4$ is the second section, so this part is done. For part II, we reverse the sequence to obtain

$$-2_2, -2_1^0, 3_1, -2_1^1, 2_4, 2_3, -2_1^1, 3_1, -2_1^0, 2_4, -2_2, 2_3,$$

whose second section is simply 2_3. Thus there is nothing to do in part II. The core of the orbit of group -2_2 is just the group 2_3. The updated array, showing the additional position of group -2_2 is

$$-2_2^0, 2_3, -2_2^0, 2_4, -2_1^0, 3_1, -2_1^1, -2_2^1, 2_3, -2_2^1, 2_4, -2_1^1, 3_1, -2_1^0.$$

We delete the group -2_2 from the control list, thereby obtaining the revised control list $2_3, 2_4, 3_1$. Since the remaining groups are positive, we must wait until after the method for identifying the orbit of a positive group has been discussed before completing the construction of the master group code for this example.

Case 2: the orbit of a positive group G. The situation for a positive group is considerably simpler than that of a negative group. As before, we cycle the current array so as to begin with one of the arcs of G, which we shall refer to as the first arc of G, and we refer to the portion of the array that lies between the first and the second arcs of G as the first section, and the portion which lies between the second arc of G and the end of the array as the second section. The arcs labelled e_i join group arcs that are in the first section, while the arcs labelled f_i join group arcs that are in the second section (again, we have taken some artistic liberties, and listed e_i as if it was always above f_i in the configuration, but of course this is not necessarily the case).

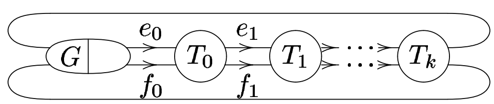


Figure 4: The orbit of a positive group

Suppose that we have identified the arcs e_i and f_i that identify the i^{th} position of the group G (that is, the two arcs that separate min-tangle T_i from min-tangle T_{i+1}, if $i > 0$, or the pair of arcs e_0 and f_0 that connect G to T_0). Let S_i^1 denote the sequence of all group arcs that belong to the min-tangles between G and the position determined by e_i and f_i in the order that they are encountered on the strand which leaves G along e_0, and let S_i^2 denote the sequence of all group arcs that belong to the same min-tangles but are encountered on the strand which leaves G along f_0. Thus S_i^1 is an initial segment of the first section, and S_i^2 is an initial segment of the second section, and these two segments constitute a closed pair, by which we mean that any group that has at least one of its arcs appear in either S_i^1 or S_i^2 also has the other arc appearing in either S_i^1 or S_i^2. Note that S_0^1 and S_0^2 are empty. If either of S_i^1 or S_i^2 is equal to the section to which it belongs, the orbit has been completely identified. Let S_{i+1}^1 and S_{i+1}^2 be the shortest initial segments of the first and second sections, respectively, for which S_i^1 is a proper initial segment of S_{i+1}^1 and S_i^2 is a proper initial segment of S_{i+1}^2 which are a closed pair in the sense described above. The groups whose arcs appear in S_{i+1}^1 or S_{i+1}^2 but not in S_i^1 or S_i^2 form a tangle which is either a min-tangle in the orbit of G or a subgroup of the full group determined by G. It is easy to verify that the tangle is a subgroup of G if and only if the tangle consists of a single positive group, in which case its two group arcs are deleted from the current array and from the first and second sections, it is removed from the control list, the number of crossings it contributes to the size of the full group to which G belongs is recorded, and we reset $S_{i+1}^1 = S_i^1$ and $S_{i+1}^2 = S_i^2$, and repeat the process (in effect, we have group flyped the subgroup to amalgamate it with the original group G). On the other hand, if the tangle is a min-tangle in the orbit of G, then we record the pair of arcs e_i and f_i as a position pair for the orbit of G, and set e_{i+1} to be the arc joining the last group arc of S_{i+1}^1 to the next group arc to be encountered on that strand (either the next group arc in the first section, or, if the first section has been exhausted, to the arc of G that follows the first section), and set f_{i+1} to be the arc joining the last group arc of S_{i+1}^2 to the next group arc to be encountered on that strand (either the next group arc in the second section, or, if the second section has been exhausted, to the arc of G that follows the second section), and repeat the process, replacing i by $i+1$.

When the identification of the orbit is complete, the size of the full group, say m, is known, and the full group is assigned the index of the next group of size m to be labelled, say n. Suppose that there are k positions in the orbit of G. The current array is then modified as follows: the two arcs of G are replaced by the label m_n^0, and if $k > 1$, then for each arc in each pair e_i, f_i, $i = 1, \ldots, k-1$, the label m_n^i is inserted between the two group arcs that are joined by the arc in question. This completes the processing of the orbit of G.

We illustrate this procedure by completing the construction of the master group code for the knot shown in Figure 2. We had left off with the control list $2_3, 2_4, 3_1$, so the positive group 2_3 is the next to be processed. We cycle the current array so as to begin with an arc of group 2_3. The result is:

$$2_3, -2_2^0, 2_4, -2_1^0, 3_1, -2_1^1, -2_2^1, 2_3, -2_2^1, 2_4, -2_1^1, 3_1, -2_1^0, -2_2^0$$

and so $-2_2^0, 2_4, -2_1^0, 3_1, -2_1^1, -2_2^1$ and $-2_2^1, 2_4, -2_1^1, 3_1, -2_1^0, -2_2^0$ are the first and second sections, respectively. We start with S_0^1 and S_0^2 empty, and look for the shortest initial subsequences of the first and second sections that contain S_0^1 and S_0^2, respectively, and which form a closed pair. We find that S_1^1 is the first section and S_1^2 is the second section, so the orbit of 2_3 has a single min-tangle. We replace both occurrences of 2_3 by 2_3^0, and remove group 2_3 from the control list to complete the processing of the orbit of group 2_3. The control list is now $2_4, 3_1$, so we cycle the current array so as to begin with an arc of group 2_4, say

$$2_4, -2_1^0, 3_1, -2_1^1, -2_2^1, 2_3^0, -2_2^1, 2_4, -2_1^1, 3_1, -2_1^0, -2_2^0, 2_3^0, -2_2^0,$$

and we find that $-2_1^0, 3_1, -2_1^1, -2_2^1, 2_3^0, -2_2^1$ and $-2_1^1, 3_1, -2_1^0, -2_2^0, 2_3^0, -2_2^0$ are the first and second sections. We start with S_0^1 and S_0^2 empty and look for the shortest initial subsequences S_1^1 and S_1^2 of the first and second sections which properly contain S_0^1 and S_0^2, respectively, and which form a closed pair. We find that $S_1^1 = -2_1^0, 3_1, -2_1^1$ and $S_1^2 = -2_1^1, 3_1, -2_1^0$, so e_1 is the arc that joins -2_1^1 to -2_2^1 in the first section, and f_1 is the arc that joins 2_1^0 to -2_2^0 in the second section. At the next step, we find that S_2^1 is the first section, so S_2^2 must be the second section, and the orbit has been completely identified. There are two min-tangles in the orbit of 2_4. We replace both occurrences of 2_4 by 2_4^0, and place the label 2_4^1 between the first occurrences of -2_1^1 and -2_2^1, and between the second occurrences of -2_1^0 and -2_2^0. The result is

$$2_4^0, -2_1^0, 3_1, -2_1^1, 2_4^1, -2_2^1, 2_3^0, -2_2^1, 2_4^0, -2_1^1, 3_1, -2_1^0, 2_4^1, -2_2^0, 2_3^0, -2_2^0.$$

We may now remove group 2_4 from the control list to complete the processing of the orbit of group 2_4. The control list is now 3_1, so we cycle the current array so as to begin with an arc of group 3_1, say

$$3_1, -2_1^1, 2_4^1, -2_2^1, 2_3^0, -2_2^1, 2_4^0, -2_1^1, 3_1, -2_1^0, 2_4^1, -2_2^0, 2_3^0, -2_2^0, 2_4^0, -2_1^0$$

with $-2_1^1, 2_4^1, -2_2^1, 2_3^0, -2_2^1, 2_4^0, -2_1^1$ and $-2_1^0, 2_4^1, -2_2^0, 2_3^0, -2_2^0, 2_4^0, -2_1^0$ the first and second sections, respectively. We start with S_0^1 and S_0^2 empty and look for the shortest initial subsequences S_1^1 and S_1^2 of the first and second sections which properly contain S_0^1 and S_0^2, respectively, and which form a closed pair. Since the first arc of the second section is -2_1^0, and the second occurrence of that group is at the end of the second section, we see that S_1^1 and S_1^2 must be the first and second sections, respectively, and the processing of the orbit of group 3_1 is complete–its orbit has a single min-tangle. We replace both occurrences of 3_1 by 3_1^0, and remove group 3_1 from the control list to complete the processing of the orbit of group 3_1. Since the control list is now empty, the construction of the master group code is complete. The master group code that we have constructed from the group code

$$-2_1, -2_2, 2_3, -2_2, 2_4, -2_1, 3_1, 2_3, 2_4, 3_1$$

is

$$3_1^0, -2_1^1, 2_4^1, -2_2^1, 2_3^0, -2_2^1, 2_4^0, -2_1^1, 3_1^0, -2_1^0, 2_4^1, -2_2^0, 2_3^0, -2_2^0, 2_4^0, -2_1^0.$$

Once we have a master group code for a full group knot configuration, then by selecting one position for each group and then writing out the selected group arcs in the order that they appear in the master group code, we will produce a group code for a full group configuration of the knot, and every full group configuration of the knot can be obtained in this way.

For example, if we select the zero position for groups 3_1, -2_2 and 2_3, and position one for each of -2_1 and 2_4, we obtain the group code

$$3_1, -2_1, 2_4, 2_3, -2_1, 3_1, 2_4, -2_2, 2_3, -2_2.$$

We have shown this knot in Figure 5.

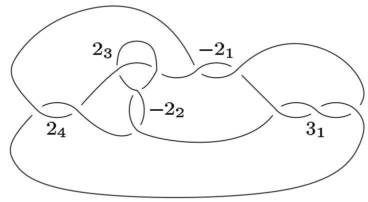


Figure 5: A new configuration for the prime 11 crossing knot

2.3. *The Master Array for a Knot*

Each master group code for a knot provides the orbit structure for the knot, detailing for each group its size, sign (positive or negative) and the min-tangles of its orbit, and any two master group codes for a knot have the same structure, in the sense that any one can be obtained from any other by an appropriate relabelling and possibly a reversal of direction of traversal.

However, comparing two master group codes to determine whether or not they represent the same knot is a computationally demanding task. As an alternative, we introduce a procedure to produce an array from a master group code, where this array has the property that two master group codes represent configurations of the same knot if and only if the constructed arrays for the two master group codes are identical. This array shall be called the master array for the knot.

As a first step, we intoduce a linear ordering on the labels in a master group code. Recall that each label is an ordered triple (m, n, i), where m is a signed integer whose magnitude is the size of the group that the label represents, n is a positive integer which is the index of this particular group of size $|m|$, and i is a non-negative integer which represents a position in the orbit of the group. We shall use the usual ordering of integers to lexicographically order these triples.

Now, let M be a master group code for a knot K. Form the set of all sequences that can be obtained by selecting an entry in M and a direction to traverse M (recall that M is a cyclic code), then recording the entries of M as they are encountered when the cycle is followed in the selected direction from the selected starting point. Working with this set corresponds to rotating (and possibly reversing) a master group code in order to compare it to some other master group code. The problem of the arbitrariness of the labelling still remains. To deal with this, we relabel each sequence in our set. For each sequence, we work from the beginning, changing each label $\pm m_n^i$ as it is encountered in the sequence according to the following scheme. We leave m unchanged (since this is the size of the group), but n and i will be changed to p and j, respectively, where this label represents the $j+1$ orbit position of the p^{th} group of size m that has been encountered by the time this occurrence of the label $\pm m_n^i$ has been reached. Thus after relabelling, all groups of a given size m will now appear in the sequence in the order m_1, m_2, and so on, and the positions of the group labelled m_j will now be labelled in order of appearance as m_j^0, m_j^1, and so on.

Now, since each sequence in this set consists of labels $\pm m_n^i$, which we may consider as ordered triples $(\pm m, n, i)$ of integers, we could lexicographically order the set of sequences that we have obtained from the master group code (where we consider the triples to be linearly ordered by the lexicographical ordering on $\mathbb{Z}^3$ that is obtained from the usual ordering on $\mathbb{Z}$). Then (of many possible choices), we might choose the unique minimum sequence as our designated master array. While this would be reasonable in principle, the set of sequences that we must analyze is large enough to present computational headaches. Consequently, we introduce a scheme for selecting a special subset of the set of all sequences that we obtain from a master group code, and then apply the lexicographical ordering to select the least element of this special subset. This least sequence is the one that we call the master array for the knot. Since different master group codes for the same knot will all produce the same set of sequences, it follows that this scheme will result in equal master arrays when applied to two master group codes for the same knot. Morever, since the master array is a sequence which, when converted to a cycle by having the first element of the sequence follow the last element of the sequence, does yield a master group code for the knot, it follows that different master arrays correspond to different knots.

Definition 2. Let K be a prime alternating knot and let M be a master group code for K. A *ladder* of M is an interval $m_i, m_{i+1}, \dots, m_{i+k}$ of consecutive entries of M (where the indices are to be calculated modulo the length of M to account for the fact that M is a cyclic code) for which no two entries in the interval are of the form $(\pm m, n, i)$ and $(\pm m, n, j)$ for any values of i and j. The *length* of a ladder is the sum of the sizes of the groups that appear in the ladder. Finally, the *maximum ladder length* of K, denoted by $\ell_{\max}(K)$, or simply $\ell_{\max}$ when K is understood, is defined to be the maximum of the lengths of the ladders of M.

As is implied in the definition of $\ell_{\max}$, it is independent of the particular master group code from which it is computed.

Now, of all of the relabelled sequences that we have formed from the master group code that we are working with, we select only those that begin with an $\ell_{\max}$ ladder; that is, a ladder of length $\ell_{\max}$. This set of sequences shall be denoted by $L_{\max}$.

Definition 3. The *master array* for a given knot K is the least element in the lexicographically ordered set $L_{\max}$.

We shall finish up this section by illustrating the construction of the master array from the master group code

$$3_1^0, -2_1^1, 2_4^1, -2_2^1, 2_3^0, -2_2^1, 2_4^0, -2_1^1, 3_1^0, -2_1^0, 2_4^1, -2_2^0, 2_3^0, -2_2^0, 2_4^0, -2_1^0.$$

for the knot shown in Figure 2. The first job is to determine the ladders of greatest length. Conceptually, for each entry in the master group code, we determine the longest ladder that begins at the selected entry (following a clockwise direction–the direction proceeding from left to right in the linear representation of the cycle as shown above). From these, we determine those of maximum length, and then for each ladder of maximum length, we form two sequences, one obtained by starting at the beginning of the ladder and proceeding in the clockwise direction around the cycle, the other obtained by starting at the end of the ladder and proceeding in the counterclock-wise direction around the cycle. The resulting set of sequences, after relabelling, is $L_{\max}$. We then seek the least element of this set.

Starting with the element 3_1^0, we find the ladder $3_1^0, -2_1^1, 2_4^1, -2_2^1, 2_3^0$ which can't be extended further, since the next entry is -2_2^1 and the group -2_2 has already made an appearance. The length of this ladder is $3+2+2+2+2=11$. Since we are searching for ladders of greatest length, we can skip every ladder which begins at any point between 3_1^0 and the first appearance of the label -2_2^1, since they will necessarily be shorter. Thus the next starting point that we check will be the entry immediately after the first appearance of -2_2^1, namely 2_3^0. We find the ladder $2_3^0, -2_2^1, 2_4^0, -2_1^1, 3_1^0$ which can't be extended, since the next entry is -2_1^0, and -2_1^1 already appears in the ladder. This ladder also has length 11. We move ahead to the entry that follows -2_1^1, and find the ladder $3_1^0, -2_1^0, 2_4^1, -2_2^0, 2_3^0$, which has length 11. We may again skip ahead to the start point 2_3^0, and there we find the ladder $2_3^0, -2_2^0, 2_4^0, -2_1^0, 3_1^0$, a ladder of length 11 as well. Since the group that stops this ladder is -2_1, we skip ahead to the first entry that follows -2_1^0. Since this is the entry 3_1^0 which is the first entry that we considered, we have determined that $\ell_{\max} = 11$ for this knot. Morever, we have found 4 maximum length ladders in the master group code. Since we are using lexicographical ordering, a group of 3 will lose out to a group of 2, so we don't need to expend the effort to relabel the sequences that begin with a group of 3. Each of the four ladders of length 11 produces one sequence that starts with a group of 2 and sequence that starts with a group of 3. After discarding the four sequences that begin with a group of 3, we are left with 4 sequences that begin with a group of 2. The one that we obtain from the first $\ell_{\max}$ ladder that was found above is (before relabelling)

$$2_3^0, -2_2^1, 2_4^1, -2_1^1, 3_1^0, -2_1^0, 2_4^0, -2_2^0, 2_3^0, -2_2^0, 2_4^1, -2_1^0, 3_1^0, -2_1^1, 2_4^0, -2_2^1,$$

which, after relabelling, becomes

$$2_1^0, -2_2^0, 2_3^0, -2_4^0, 3_1^0, -2_4^1, 2_3^1, -2_2^1, 2_1^0, -2_2^1, 2_3^0, -2_4^1, 3_1^0, -2_4^0, 2_3^1, -2_2^0.$$

The next $\ell_{\max}$ ladder provides the sequence (before relabelling)

$$2_3^0, -2_2^1, 2_4^0, -2_1^1, 3_1^0, -2_1^0, 2_4^1, -2_2^0, 2_3^0, -2_2^0, 2_4^0, -2_1^0, 3_1^0, -2_1^1, 2_4^1, -2_2^1,$$

which, after relabelling, results in the same sequence as we obtained above. The remaining two $\ell_{\max}$ ladders also produce this sequence after relabelling, so the master array for the 11 crossing knot shown in Figure 2 (and of course, for the knot shown in Figure 5) is

$$2_1^0, -2_2^0, 2_3^0, -2_4^0, 3_1^0, -2_4^1, 2_3^1, -2_2^1, 2_1^0, -2_2^1, 2_3^0, -2_4^1, 3_1^0, -2_4^0, 2_3^1, -2_2^0.$$

3. An Implementation of the Construction, Part I: Full Group Configurations Suffice

In many of the various attempts to enumerate alternating prime knots over the years, a major stumbling block was the large amount of redundant work that had to be carried out. A primary feature of our knot encoding scheme allows us to work with a given knot in a full group configuration only. By not having to work with split group configurations, a tremendous amount of unnecessary work is avoided.

In this section, we prove that it suffices to apply the operators introduced above to knots in full group configuration only, and in the case of D, to a single crossing of each group that is eligible for D to act on. Furthermore, we shall examine each operator in order to identify situations that we can be assured will result in redundant constructions. This will enable us to instruct each operator to bypass certain situations, thereby obtaining a more efficient construction process.

For conceptual purposes, we quickly review the basic construction process here. The construction begins with the application of D and $ROTS$ to the knot configurations of n crossings. That is, for each negative group, each loner and each positive 2-group of any knot configuration of n crossings, D is applied to any one crossing of the selected group.

After all possible applications of D have been made, the next step is to apply the $ROTS$ operator to each negative group size 2 or 3 of any knot configuration of n crossings.

The combined result of applying D and $ROTS$ as described above is a very large collection of prime alternating knots of $n+1$ crossings (for example, in our computation of the $40,619,385$ prime alternating knots of 19 crossings, $39,722,121$ were constructed by D and $ROTS$).

The next stage in our construction consists of repeatedly performing the following action: for each positive 2-group of each configuration in the collection of all knots of $n+1$ crossings that have been constructed so far, apply T and, if the result is a new knot (which is determined by computing its master array and comparing it to those already in the collection of all knot produced so far), add it to the collection. This process terminates when the collection reaches a stage where subsequent applications of T do not produce any new knots.

Now we are ready to apply OTS for the first time. We examine each knot configuration of each knot in the current collection to identify the OTS scenarios and to each one, we apply OTS. Each new knot configuration is added to the collection, and the enlarged collection becomes the current collection. When no new knot can be created by an application of OTS to any configuration of any knot in the collection, this stage is complete.

The construction is completed by iterating the T and OTS operators as described above, each operator only being applied to the knots produced by the other operator in the preceding run. Eventually, a stage is reached at which no new knots can be produced by applying either T or OTS, and at this point, the construction of all of the prime alternating knots of the given crossing is complete.

We now begin the process of examining the operators in the order that they will be applied in order to identify some of the redundant constructions and establish some means whereby we can prevent most of the unnecessary work from being done. In particular, for each operator, we shall establish that it suffices to apply the operator only to full group configurations, and that, with the possible exception of the OTS operator, the configuration that results from an application of an operator to a full group configuration is again a full group configuration. Note that of the four operators, OTS is the only one that does not operate on a group, and we shall see that this makes it somewhat more complicated to deal with.

3.1. *The D operator*

Observe that if D is applied to two different crossings in the same subgroup of a negative group, the resulting configurations are identical, since in both cases, the size of the subgroup simply increased by 1.

On the other hand, if D is applied to a crossing of a positive group of 2 or more, the result is to replace the crossing by a group of 2 which forms a min-tangle in the orbit of the group to which the crossing had belonged (now a group of size one smaller). It follows that the result of applying D to two different crossings of a positive group is to produce two flype equivalent configurations.

As for the issue of full group versus split group configurations, it suffices to observe that if D is applied to a crossing of a subgroup at some position of the group's orbit, and subsequently the other subgroups of the group are flyped to this position, the result is the same as first flyping the other subgroups to the selected position, then applying D to the originally selected crossing.

Furthermore, we observe that if D is applied to a full group configuration, the result is still a full group configuration.

Since we require a single configuration of a prime alternating knot from which to compute its master array, it is not necessary to apply D to more than one crossing from a given group, nor is it necessary to apply D to any split-group configurations. Thus we need only apply D to full group configurations.

Even though the decision to work only with full group configurations does reduce the number of configurations of a given knot that must be examined for potential applications of D, there will in general be many full group configurations of the knot. However, when we consider any group G to which D can be applied, and any flype move which does not move any crossings of G along its orbit, then the effect of the flype move is restricted to some min-tangle of the orbit. It follows that the D operation and the flype move commute with each other in such a case. Thus it suffices to extract the group code for a single configuration from the master array of the given knot, examine the selected configuration for a group G to which D is to be applied, then form all configurations that can be obtained by group flyping G to its various orbit positions. Apply D to the group G in each of these configurations, and if the resulting knot is a new one, add it to the knot database.

3.2. *The $ROTS$ operator*

Just as for D, $ROTS$ operates on a group, so any flype moves that do not involve the group will commute with $ROTS$. This observation implies that we need only investigate full group configurations, with the possible exception of a negative group of size at least four. If we have a negative group of 4 or more crossings, we could flype all but three of its crossings to a new position, thereby creating a situation where $ROTS$ can be applied. However, upon performing $ROTS$ on the negative 3-group, we flype the crossings back to their original position, which results in a negative 2-group attached to the *rots* tangle. This same knot configuration will be obtained from the K_A knot configuration that results from applying D to the loner or negative group that is obtained when D^- is applied to the negative group that is attached to the *rots* tangle. Consequently, we see that it suffices to apply $ROTS$ only to full group configurations. Furthermore, the configuration that results from an application of $ROTS$ to a full group configuration is itself a full group configuration.

Given a master array for a knot, it suffices to select any one full group configuration and search for a negative group of 2 or 3. If such a negative group is found, we shall make all possible full group configurations from the initial one by group flyping the group to its various positions. Then $ROTS$ will be applied to the group in each of these configurations. Any new knots that result will be added to the knot database.

3.3. *The T operator*

Just as for D and $ROTS$, T operates on a group and so T will commute with any flype operation that does not involve any crossings of the group that T is operating on. Consider any full group configuration that has a positive group G of size at least 3. Flype all but two of the crossings of G to some common position in the orbit of G, and let G' denote the positive group that is formed by these crossings in their new position. What remains is a positive 2-group for T to operate on. In other words, by splitting this group, we have created a chance to apply T that did not exist in a full group configuration. We shall show that the result of applying T to the positive 2-group that results when we split a positive group in this fashion will already have been obtained by the application of either $ROTS$ (if G is a 3-group) or D. In the configuration that results when T has been applied to the positive 2-group, the positive 2-group has been replaced by its turned image (still a positive 2-group), which now becomes another min-tangle in the orbit of G', while G' is now a negative group. Flype G' back to its original position, and denote this knot configuration by $C_1(K)$. In $C_1(K)$, the tangle which consists of the crossing of G' that is adjacent to the turned 2-group together with the turned 2-group is a *rots*-tangle. If G' is a loner, this same configuration would have been obtained by $ROTS$, while if G' is not a loner, then it is a negative group, and so the configuration would have been produced by an application of D to the configuration from $C_1(K)$ obtained by reducing the size of the negative group G' (that is, by applying D^- to G').

It therefore suffices to apply T to positive 2-groups in full group configurations only. The result of applying T to a positive 2-group G in some full group configuration is a split group configuration if and only if G is a core min-tangle in the orbit of some negative group or loner. Thus if we take care to select the full group configuration to which T is to be applied on G so that if G is the core tangle of some negative group or loner, the negative group or loner is placed in a position on one side of G or the other. We shall see shortly that rather than apply T to a group code, we in fact will apply T to a master group code, and as a result, we will be in a position to instantly detect those situations when the result of turning a particular positive 2-group will result in the creation of a larger positive group.

With this precaution, it follows that the result of applying T to a full group configuration will again be a full group configuration.

We have also established that from each master array, we may select just one full group configuration, and for each positive 2-group in that configuration, we form all full group configurations from the selected one by group flyping the positive 2-group to each of the various positions of its orbit. We apply T to the positive 2-group in each of the these configurations, and any new knots that are so obtained are added to the knot database.

3.4. *The* OTS *operator*

While OTS does not operate on groups, it is nevertheless evident that OTS will commute with any flype that does not involve the three crossings in the OTS 6-tangle on which OTS will operate. Furthermore, since no two crossings in an OTS 6-tangle can belong to the same group, it turns out that if c_1, c_2 and c_3 are the three crossings which make up an OTS 6-tangle, then performing the OTS operation on this 6-tangle will remove each of c_1, c_2 and c_3 from their respective groups, but does not interfere with the orbits of the respective groups. Thus the OTS operation on this OTS 6-tangle will commute with group flypes which put the groups of the crossings c_1, c_2 and c_3 into full group, with the respective groups positioned at the locations of c_1, c_2 and c_3. In other words, it suffices to only apply OTS to full group configurations. However, unlike the case for the other three operators, from each master array we must construct each and every full group configuration and search each one for OTS 6-tangles on which to apply OTS (later on, we shall show that many of these configurations do not have to be constructed, and even in those that do, many if not most of the OTS 6-tangles they contain will not have to be considered).

There is an interesting observation to be made at this point. The configuration that results from an application of OTS need not be a full group configuration, which presents us with a complication, since we wish to be working only with full group configurations. However, we shall see shortly that one of the major benefits of master group code is that we may apply the operator in question to a master group code, rather than to a group code. Consequently, we shall have a simple mechanism available to us to detect those instances when a group will become larger as a result of an OTS operation, and we shall then see that it is extremely easy to handle this situation.

Once we have a master group code for the resulting knot, we construct its master array and submit it to the knot database.

4. An Implementation of the Construction, Part II: Major Reductions in the Work Done by the Operators

As we have described the process so far, all of the operators will be performing a tremendous amount of redundant work. Some indication of the redundancy is available from the run times for our calculations of the prime alternating knots up to and including those at 19 crossings. To facilitate the discussion, we shall refer to the two operators D and $ROTS$ combined as $DROTS$. In the production of the 19 crossing knots, we found that the run time for T was about one half that of $DROTS$ to complete, while the application of OTS to the combined output from $DROTS$ and T took about the same amount of time as $DROTS$ and T combined. As a point of interest, we note that in every case so far (which is up to and including the knots at 19 crossings), all knots have been produced by the end of the first application of

T and OTS. We speculate that this will always be the case, but this is yet to be proven.

We return to the assessment of the rate at which T and OTS are producing knots. T and OTS account for less than 2% of the total knot production, yet they take over 60% of the production time! Our objective is to bring the amount of work done by T and OTS more in line with their rate of production. In fact, it turns out that a careful analysis of each operator permits us to dramatically reduce the amount of work that each must do. In the production of the prime alternating knots of 19 crossings prior to the implementation of the refinements to be described below, the number of knots that were built was approximately 8 times the total number of knots at the crossing size. Preliminary calculations indicate that the refinements will bring the ratio down to less than 2.

As we progress through these discussions, the value of the master array will become increasingly more evident. Of course, it has tremendous value by virtue of the fact that two prime alternating knots are flype equivalent if and only if their master arrays are identical, but now we shall see the returns deriving from the fact that the master array simultaneously describes all possible (minimal crossing) configurations of the knot. The general theme will be to establish that the four operators can be made to operate on the master array, rather than a configuration, and this observation alone accounts for a monumental reduction in the amount of work to be done in the generation of the prime alternating knots of $n+1$ crossings from (the master arrays of) the prime alternating knots of n crossings.

The following terminology will be useful for the presentation of the reduction rules.

Definition 4. Any tangle of the form shown in Figure 6 is called a *drots* tangle, positive if the 2-group is positive and negative if the 2-group is negative.

Moreover, in the master array of a knot, any 2-group which serves as a min-tangle for some group shall be called a *drots* 2-group, with the same sign as the 2-group. The group in whose orbit a *drots* 2-group appears as a min-tangle is called the *orbiter* of the *drots* 2-group.

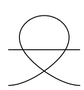
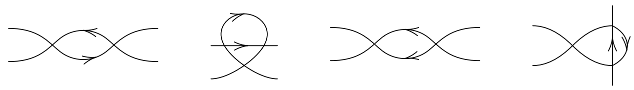


Figure 6:

If a 2-group is a *drots* 2-group in a master array, then in any configuration of the knot that has at least one of the crossings of the orbiter group in a position adjacent to the *drots* 2-group, the *drots* 2-group together with such a crossing will form a *drots* tangle. We note that a *drots* 2-group itself has trivial orbit. Furthermore, any orbiter group must be either a loner or a negative group.

Note that a positive *drots* tangle is simply a $ROTS$ tangle, and a $ROTS$ tangle results when the $ROTS$ operator is applied to a negative 2-group. On the other hand, a negative *drots* tangle results from applying D to a positive 2-group. In Figure 7 (i) and (ii) we show a negative 2-group and the result of applying $ROTS$ to it, while in (iii) and (iv), we show a positive 2-group and the result of applying D to it. It follows that any knot whose master array contains a *drots* 2-group can be obtained by an application of one of D or $ROTS$.

4.1. *Reduction in the work to be done by D on negative groups and loners.*



(i) (ii) (iii) (iv)

Figure 7: How *drots* tangles arise

Observe that as things now stand, a knot of $n+1$ crossings with two or more negative groups will be constructed at least once per negative group during the application of D to the various knots of n crossings, so there is considerable obvious redundancy in D. We shall set out some rules which will allow D to decide to pass over an opportunity to be applied to a negative group in some knot, with the guarantee that the knot that would result from the skipped application of D will be produced by some other application of D.

Suppose that we have a master array for a prime alternating knot K of n crossings. Let G denote a negative group or a loner. We set out the following conditions under which D will be applied to G. We shall let K_1 denote the knot in full group configuration that would be obtained by an application of D to G in some full group configuration of K, and let G_1 denote the negative group that would be obtained from G by this application of D.

We remark that our approach will be to work on master arrays, thereby obtaining a master group code for the knot that results from the application of D. The application of D to a master array is extremely efficient, and while we could examine the master array of the original knot to see it meets certain conditions which determine whether or not we should apply D, it is probably more efficient to actually perform D and examine the resulting master group code that results to determine whether to subject it to further processing and submit it to the database of all knots constructed so far, or to discard it. We shall take the latter approach.

(a) if the size of the negative group G_1 is greatest among all negative groups in K_1, and G_1 is the only negative group of this size in K_1, apply the following test.

    (i) if the original group G has a trivial flype orbit, then construct the master array for K_1 and submit it to the database.
    (ii) if the original group G has a nontrivial flype orbit, then submit K and G to the symmetry test outlined below to determine in which positions of the orbit of G in K we must place G and apply D.

(b) if the size m of the negative group G_1 is greatest among all negative groups in K_1, but K_1 has more than one negative group of this size, let $\mathcal{H}$ denote the collection of all of the negative groups in $C(K_1)$ of size m that have a nontrivial flype orbit, and $\mathcal{J}$ denote the collection of all of the negative groups in $C(K_1)$ of size m that have a trivial flype orbit (except for those negative 2-groups which belong to a *drots* tangle in the case $m=2$), and apply the following test.

    (i) if $G_1 \in \mathcal{J}$ and $\mathcal{H} \neq \emptyset$, then do not apply D to G in any position of its orbit (the orbit of G will be trivial as well if G is a group of 2 or more, but if G is a loner, it is possible that G could have had an orbit in the positive direction).

(ii) if $G_1 \in \mathcal{J}$ and $\mathcal{H} = \emptyset$ (so $\mathcal{J} - \{ G_1 \} \neq \emptyset$), then perform a group LNR competition (described below) between G_1 and each group belonging to $\mathcal{J} - \{ G_1 \}$ (note that there is only one position of each such group). If G_1 is the winner of the competition, then compute the master array for K_1 and submit it to the database.

(iii) if $G_1 \in \mathcal{H}$ and $\mathcal{H} - \{ G_1 \} \neq \emptyset$, then compare the size of the orbit of G_1 to that of the other groups in $\mathcal{H}$. If the orbit of G_1 is not of maximal size, then do not apply D to G in any position. On the other hand, if the orbit of G_1 is of maximal size k, then submit G_1 to a core LNR competition (described below) against each of the groups in $\mathcal{H} - \{ G_1 \}$ whose orbit size is k . If G_1 is the competition winner, then send G off to the symmetry test described below to determine the positions in the orbit of G to which D is to be applied.

In all other cases, submit K and G to the symmetry test described below, and perform D on G in every position indicated by the symmetry test, calculate the master array for the resulting knot, and submit it to the database.

We shall first describe the LNR competition scheme, and then present the symmetry test.

The abbreviation LNR stands for a *longest non-repeating sequence*, by which we mean a longest possible subsequence of consecutive entries from a specified starting point in a master group code with a specified direction of traversal, subject to the condition that no two entries of the sequence belong to one group (where for the purpose of this concept, all positions of a group are considered as belonging to the group). A group LNR competition consists of the following steps. At each end arc of each group in the competition, traverse the arc in the direction away from the group, and examine the first group to be encountered. Stop the competition for any strand for which the group encountered is not the smallest (in the lexicographical ordering) of those encountered by all of the strands for all of the groups in the competition. If after this first stage, there are two or more strands still active among all of the groups in the competition, and at least one of these belongs to G_1, carry out the next stage, whereby each strand that is still in the competition is traversed to the next group to be encountered, and the assessment is repeated (but now, a strand will also be stopped if the group encountered is a repetition of one encountered earlier on the strand, since that would cause the sequence of groups encountered along the strand to fail to be a LNR). The process stops when G_1 no longer has any strands in the competition, but there are still active strands, in which case G_1 is declared to have lost the competition, or until all strands have been stopped, and G_1 had at least one strand active up to the end, in which case G_1 has won (or tied, but we declare a tie to be a win) the competition.

A core LNR competition is a minor variant of the group LNR competition described above. The competition is applicable only to negative groups with a non-trivial flype orbit, and instead of starting at the group ends and following the strands out from the group, we go to the core tangle for the group, and follow the outbound strands from the core tangle through the copy of the group that sits on either side of the core tangle, taking the resulting location as our starting point for each strand in the competition. The direction of travel on the strand will be away from the core tangle.

We now describe the symmetry test. The underlying observation is that a group G may have two or more positions, but it may happen that as a result of symmetry within one or more of the min-tangles in the orbit of G, if D is applied to G in two different positions in its orbit, the resulting knots are flype equivalent. Accordingly, we now proceed to identify certain such situations.

Definition 5. Let $C(K)$ be a full group configuration of a prime alternating knot K, G be a group in $C(K)$, and let T be a min-tangle in the orbit of G. The following process shall be referred to as *rotating* T. Cut the four arcs incident to T and label each cut arc i with two labels, one on either side of the cut so that the side that is no longer incident to T is labelled by i' and the side that is still incident to T is labelled by i''. Then rotate T as if it were involved in a flype of G from the pair of arcs on one side of T to the pair of arcs on the other side of T, and connect the cut arcs in such a way that if the arcs on a side were labelled i and j, then i' is connected to j'' and j' is connected to i''.

Since one of the two arcs on one side of T represents an overpass of G, while the other arc on the same side represents an underpass of G, it follows that rotating a min-tangle of a group in a configuration of a prime alternating knot results in a configuration of an alternating knot, necessarily prime.

Definition 6. Let $C(K)$ be a full group configuration of a prime alternating knot K, G be a group in $C(K)$, and T be a min-tangle in the orbit of G. Let K_1 denote the prime alternating knot whose configuration is obtained from $C(K)$ by rotating T. T is said to be *flype-symmetric* if and only if $K_1 = K$.

The significance of a flype-symmetric min-tangle T in the orbit of G, where G is either a negative group or else a loner with an orbit in the negative direction, is that applying D to G whether G is adjacent to T on one side or the other results in the same knot.

Definition 7. Let K be a prime alternating knot with master array M, and let G be a group in $C(K)$. Any choice of two positions in the orbit of G provides a splitting of the orbit of G into two tangles, each referred to as the complement of the other (if T denotes one of the two tangles, we shall denote the other one, the complement of T, by T^c). Suppose that two positions in the orbit of G have been chosen. Let T denote one of the two tangles that results. M determines an orientation of K, and according to this orientation, two of the four arcs incident to T are entering T, while the other two are exiting T. Let e_{in} denote an arc entering T and let e_{out} denote the arc by which T is first exited after T has been entered by e_{in}. The sequence s in M that is delimited by e_{in} and e_{out} is called a *master array sequence determined by* T. Let $[s]$ denote the sequence that is obtained from s by traversing s, relabelling the groups so that the i^{th} group of size k that is encountered in the traversal of s receives label k_i (with the correct sign), and the positions of group k_i are labelled in the order that they are encountered in the traversal of s.

Furthermore, let $[s]^r$ denote the sequence that is obtained by applying the same procedure to the same sequence of the master array, but traversed in the opposite direction.

Proposition 1. Let $C(K)$ be a configuration of a prime alternating knot K, G a group in $C(K)$, and T a min-tangle in the orbit of G. Further let s and t denote the two master array sequences that are determined by T. Then T is flype-symmetric if:

(a) G is negative, T is the core of G's orbit, $[s] = [s]^r$, and $[t] = [t]^r$, or
(b) G is negative, T is not the core of G's orbit, and $[s] = [t]^r$, or
(c) G is positive, and $[s] = [t]$.

Proof. Suppose G is negative, T is the core tangle of G's orbit, and $[s] = [s]^r$, $[t] = [t]^r$. Then upon rotating T, the segments s and t in the master array M get replaced by s reversed and t reversed, respectively to produce a master group code of the resulting knot. Since $[s] = [s]^r$ and $[t] = [t]^r$, the master array that results from this master group code is again M, whence T is flype-symmetric.

Now suppose that G is negative, T is not the core tangle of G's orbit, and $[s] = [t]^r$. Upon rotating T, the segments s and t in M are replaced by t reversed and s reversed, respectively, providing a master group code for the resulting knot. Since $[s] = [t]^r$, we also have $[t] = [s]^r$, and so the master array that is obtained from this master group is M, whence T is flype-symmetric.

Finally, suppose that G is positive, and that $[s] = [t]$. Upon rotating T, the segments s and t in M get replaced by t and s, respectively, providing a master group code for the resulting knot. Since $[s] = [t]$, the master array that is obtained from this master group code is again M, so T is flype-symmetric. □

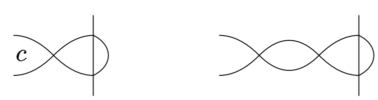
The way in which the above information is utilized in the application of D on a negative group G or a loner with an orbit in the negative direction is as follows. After having decided that we shall apply D to the negative group G in the knot K, we locate G in its zero position in the master array of K, and mark this position as one in which we shall perform the D operation. We then traverse the knot from this position in the orbit of G in the direction determined by the master array and examine the first min-tangle T that is encountered. If T is flype-symmetric, then we do not need to place G on the arcs on the other side of T and perform the D operation on G in this position. We check the conditions outlined in 1 and if met, we know that T is flype-symmetric and we move on to repeat the process with the next min-tangle in the orbit of G. If the conditions are not met, we mark this location as one where we shall perform the D operation on G. We then continue on to investigate the next min-tangle. This process is repeated until the orbit has been completely examined. For each position in the orbit of G that has been marked by the procedure outlined above, D is applied to G in that position, and the master array of the resulting knot is constructed and the knot is submitted to the database.

4.2. *Reduction in the work to be done by D on positive 2-groups*

If a prime alternating knot K contains a positive 2-group G such that if D was to be applied to G, the resulting knot K_1 would contain either a negative 2-group that is not in a negative *drots* tangle, or a negative group of size at least three, then the resulting knot would already have been made by D on negative groups, and so we shall not apply D to such a positive 2-group. We remark that it suffices to check G in any position of its orbit in order to determine whether or not any of these conditions are met. On the other hand, suppose that K_1 contains no negative groups other than negative 2-groups in negative *drots* tangles. We may still opt to defer to one of these by setting up a group LNR competition between G_1 and the other negative 2-groups. As before, if G_1 wins the competition, then we construct the master array for K_1 and submit it to the database, otherwise we choose not to process the application of D to G.

4.3. *Reductions in the work to be done by* $ROTS$

First Reduction for $ROTS$. If a master array contains two or more negative groups, then the master array will not be submitted to $ROTS$. For the result of applying $ROTS$ to a negative 2-group or a negative 3-group would still leave a negative group elsewhere in the knot, which means that the resulting knot would already have been produced by D.
Second Reduction for $ROTS$. If a master array contains just one negative group, and that one negative group is the negative 2-group of a negative *drots*-tangle, shown in Figure 8 (i), then this master array will not be submitted to $ROTS$. For if we were to apply $ROTS$ to this negative 2-group, the result would be as shown in Figure 8 (ii). But this last tangle is exactly what would result from an application of D to c. It follows therefore that if we were to apply $ROTS$ to such a negative 2-group, the knot that would result would already have been constructed by D, and so this would be redundant work. Thus we shall never apply $ROTS$ in this situation.



(i) (ii)

Figure 8: A second redundant $ROTS$ scenario

4.4. *Reductions in the work to be done by* T *and* OTS

Our first step is to remove from the $DROTS$ output those master arrays that we can guarantee will not produce any new knots by an application of either T or OTS.

Our first reduction stems from the observation that if a master array that was produced by $DROTS$ contains a negative *drots* 2-group, then since the orbiter group of the *drots* 2-group is either negative or else a loner, it follows that any application of T will not operate on the *drots* 2-group nor on its orbiter group, nor will the orbit of the orbiter group be affected. Thus the master array of any knot that is produced by an application of T to any configuration that is obtained from a master array that contains a negative *drots* 2-group will contain a *drots* 2-group (which might be positive since the application of T could change the sign of one or more groups in the configuration). Thus all knots that T can produce from such a master array will already have been produced by $DROTS$.

Next, consider the result of applying OTS to a configuration obtained from a master array which contains a negative *drots* 2-group. If OTS is applied to an OTS 6-tangle which does not involve either of the crossings of the *drots* 2-group, then the resulting configuration has a negative 2-group. In such a case, the knot that the resulting configuration represents will already have been produced by D. Suppose now that there is an OTS 6-tangle which involves some of the crossings of the *drots* 2-group. Since no OTS 6-tangle can contain two crossings from the same group, this OTS 6-tangle must contain exactly one crossing from the negative *drots* 2-group. The following proposition will allow us to deal with this situation.

Proposition 2. Let $C(K)$ be a full group configuration of a prime alternating knot K and suppose that G_D is a *drots* 2-group (positive or negative) such that there is an OTS 6-tangle that contains a crossing of G_D. Then the orbiter group G of G_D must be a loner adjacent to G_D.

Proof. Let c, x and y be crossings which form an OTS 6-tangle, and suppose that c is in G_D (so neither x nor y can belong to G_D). Suppose that neither x nor y is a crossing of G. Since G_D is a min-tangle in the orbit of G, x and y belong to min-tangles T_1 and T_2, respectively, in the orbit of G. Now T_1 and T_2 are adjacent to G_D, one on each side, so the group G lies between T_1 and T_2. Thus there can be no edge directly joining x to y, so this situation can't arise. Accordingly, at least one, and therefore exactly one of x and y belong to G, whence G is adjacent to G_D. Suppose that x is in G. Then y is in a min-tangle in the orbit of G, on the other side of G_D from G. Thus for x and y to be adjacent, it must be that G consists only of x; that is, G is a loner adjacent to G_D, as required. □

Thus in our current situation, the orbiter group must be a loner in a position adjacent to the *drots* 2-group, and the OTS 6-tangle is as shown in Figure 9 (i). The result after OTS has been performed is shown in (ii). Note that there is a *drots* tangle in (ii), so the resulting knot would already have been built by $DROTS$.

(i) (ii)

Figure 9: Negative *drots* tangles and OTS

Thus any master array which contains a negative *drots* tangle will not produce any new knots from applications of either T or OTS, and so we shall remove such master arrays from the collection that will be given to T and subsequently to OTS.

So now let us examine the situation when we apply the T operator to the master arrays that have been produced by $DROTS$ and which are without negative *drots* 2-groups.

We examine each such master array in turn.

First Reduction for T. Consider the case of a master array which contains at least two positive *drots* 2-groups. The master array of any knot that results from an application of T to any full group configuration of such a knot will necessarily contain a *drots* 2-group (possibly negative, since applying T can cause changes in the sign of a group) and so it would already have been produced by $DROTS$. Thus we may skip any such master array.

Second Reduction for T. It is a consequence of Proposition 6 of [1] that it is not necessary to iterate T in the application of T immediately after $DROTS$.

Configuration dependent reductions for T. Any master array that does not get set aside as a result of the reduction described above is now examined for configuration dependent reductions. If the master array contains exactly one positive *drots* 2-group, then even though it may contain other positive 2-groups, the only positive 2-group which might lead to a new knot upon application of T is the positive *drots* 2-group (which necessarily has trivial orbit). Even this application may not be necessary, and we shall submit such a knot to the following test before applying T–the point is that there is only one group to examine in this particular situation.

If the master array contains at most one positive *drots* 2-group, then for each positive 2-group G, unless there is a positive *drots* 2-group, in which case we restrict G to just the positive *drots* 2-group, we check to see if at least one of the following conditions holds.

(a) there is a negative group both of whose arcs are in the same section of G.
(b) there is a positive group with an arc in each section of G.
(c) turning G results in a positive group G' having the turned G as a subgroup, but there is a positive group H of size larger than that of G' in the resulting knot.

If any of these conditions hold for the group G under consideration, then we do not apply T to G. For if either of the first two conditions holds, then the configuration that would result from turning G would have a negative group, so it would already have been produced by D. On the other hand, suppose that the last condition holds. Then the configuration $C(K)$ that results from turning G would also be produced by an application of T to the knot $C(K_1)$ obtained from $C(K)$ by turning an end 2-subgroup of the positive group H (which must be of size at least 3) to create a positive 2-group H' in a $ROTS$ tangle. Thus $C(K_1)$ would be produced by $DROTS$, and $C(K)$ would be produced from $C(K_1)$ by turning H', producing a unique positive group of largest size. Our convention requires that we not apply T to G, in deference to this other application of T.

For each positive 2-group G which is to have T applied to it, we apply T to the group in each of its positions, and for each of the resulting knots, we construct the master array and submit it to the database.

When the first round of T applications is complete, we take the master arrays that were produced by $DROTS$ but which have no negative *drots* 2-groups, together with any new knots that were produced by T as the set of knots that OTS is to work on. We remark that any knots that T produces will not have any negative (or positive) *drots* 2-groups, so it is not necessary to examine them for the purpose of eliminating any that might have a negative *drots* 2-group.

However, before we submit any knots to OTS, we perform a further round of reductions that are unique to OTS.

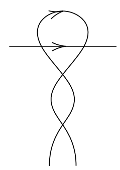


Figure 10:

First OTS reduction: any master array that contains a positive *drots* 2-group whose orbiter group is of size at least 2 can be skipped by OTS. To see why, let G_D denote a positive 2-group whose orbiter group G has size at least 2 (see Figure 10).

Observe first of all that any OTS 6-tangle that does not involve a crossing from G_D can involve at most one crossing from G, and the result of applying OTS to such an OTS 6-tangle would be a configuration which (is flype equivalent to a configuration which) has a $ROTS$ tangle and would therefore have been produced by $DROTS$. On the other hand, if there is an OTS 6-tangle which involves a crossing of G_D, then by Proposition 2, the orbiter would be a loner. Since this is not the case, there can be no OTS 6-tangle which involves a crossing of G_D.

So we are considering only master arrays in which any positive *drots* 2-group has a loner as its orbiter group.

Second OTS reduction: if a master array contains a negative group of size at least 3, a positive group of size at least 4, or more than two positive *drots* 2-groups, or if the sum of the number of negative 2-groups, positive 3-groups and positive *drots* 2-groups exceeds 3, then no OTS operation can destroy all of the scenarios which would allow the resulting knot to be obtained from $DROTS$. For if an OTS 6-tangle does not involve the crossing of the loner orbiter of a positive *drots* 2-group (in whatever configuration), or does not involve a crossing of a negative group of size at least 2, or a crossing of a positive group of size at least 3, then after OTS operates on this OTS 6-tangle, there will still be a positive *drots* tangle (possibly after flyping), a negative group of size at least 2, or a positive group of size at least 3, respectively, in the configuration that results. Such a configuration could be obtained by $ROTS$ in the case of a positive *drots* tangle, D in the case of a negative group, or T after $ROTS$ on a negative 2-group, in the case of a positive group of size at least 3. In any event, such a knot would already have been made before OTS begins, so any such master array can be skipped by OTS.

At this point, we have eliminated all master arrays which contain more than two positive *drots* 2-groups.

Third OTS reduction: Consider a master array which contains a positive *drots* 2-group in the setting illustrated in Figure 11 (i). This is intended to show that the loner c is the orbiter group of the positive *drots* 2-group, and the crossing x (which is either a loner or a crossing in a negative group of 2) has a position in its orbit which has it adjacent to the positive *drots* 2-group as shown in (i).

Then c has two positions in its orbit, the one illustrated in (i) and its position after flyping across the *drots* 2-group. Both positions of c lead to the same tangle as shown in (i). Any OTS 6-tangle must involve c, in whatever position c is in, since otherwise the OTS operation will result in a configuration that either has a *drots* tangle or is flype equivalent to a configuration that has a *drots* tangle, and the knot

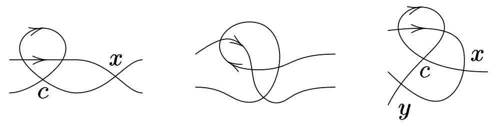


Figure 11: An OTS scenario involving a positive *drots* 2-group

represented by such a configuration has already been produced by $DROTS$. If x is not adjacent to the *drots* 2-tangle, as shown in (ii) (or on the other side of the *drots* 2-group), then c is not in any OTS 6-tangle and so we do not need to perform any OTS operation on such a configuration. If x is in the position shown in (i), there is the obvious OTS 6-tangle that involves c, x and one of the crossings of the *drots* 2-group. The result of performing the OTS operation on this OTS 6-tangle is shown in (ii). The resulting configuration contains a negative 2-group, and thus the knot it represents has already been produced by $DROTS$. There is one other possible OTS 6-tangle that could be formed with c. If the positive *drots* 2-group is actually in the setting shown in (iii), then x has only two positions in its orbit, the position shown in (iii) and its position after it is flyped across the *drots* tangle (every configuration of this knot will actually have c adjacent to the positive *drots* 2-group to form a positive *drots* tangle). If the OTS operation is performed on the OTS 6-tangle c, x and y as shown in (iii), the resulting configuration has a positive 3-group, hence is also obtainable by T following $ROTS$ on a negative 2-group. On the other hand, if we have a configuration with y still adjacent to x, but flyped across the *drots* 2-group, then y is no longer adjacent to c. Since we need only perform an OTS operation on an OTS 6-tangle which contains c, there is nothing to do for this configuration. If in addition, we flype c across the positive *drots* 2-group, then again we find two OTS 6-tangles involving c, one of which results in a negative 2-group after the OTS operation, the other resulting in a positive 3-group after the OTS operation. Thus in every case, the resulting knot will already have been produced by $DROTS$ and possibly T. It follows that it is not necessary to apply OTS to any configuration obtained from a master array which contains a positive *drots* 2-group in the setting shown in (i). Since the positive *drots* 2-groups of this type are so important for their role in the reduction of the work done by OTS, we shall give them a name as well.

Definition 8. A positive *drots* 2-group whose orbiter group is a loner and for which the *drots* 2-group together with its orbiter form a min-tangle in the orbit of some group G is called a *tight drots 2-group.* A tangle which consists of the tight *drots* 2-group, its loner orbiter and one crossing from G (as shown in Figure 11 (i)) is called a *tight drots tangle.*

Up to this point, we have determined that we need not submit any master array from $DROTS$ to OTS if it contains any of the following:

(i) a negative *drots* 2-group,
(ii) a positive *drots* 2-group whose orbiter group is not a loner,
(iii) a negative group of size at least 3,
(iv) a positive group of size at least 4,
(v) 3 or more positive *drots* 2-groups,
(vi) any combination of four or more of negative 2-groups, positive 3-groups, and positive *drots* 2-groups,
(vii) a tight *drots* 2-group.

Fourth OTS reduction: At this point, the master arrays that we are considering have at most two positive *drots* 2-groups. We examine those which have exactly two positive *drots* 2-groups (each necessarily with orbiter a loner). The only OTS 6-tangles that we need consider must involve the two orbiter loners (each in either position of its orbit). Thus if no configuration of the knot has the two orbiter loners adjacent to each other, there will be no OTS 6-tangle to consider, and so we may forego the application of OTS to any configuration obtained from such a master array.

Suppose now that we have a master array with exactly two positive *drots* 2-groups such that the two orbiter loners are adjacent in some configuration. We consider the various possible scenarios.

(i) (ii)

Figure 12: The two possible OTS scenarios involving two positive *drots* 2-groups with adjacent orbiters

First of all, suppose that the master array contains exactly two positive *drots* 2-groups such that some configuration of the knot has the tangle shown in Figure 12 (i). By virtue of the earlier reduction rules, we know that the knot does not contain any tight *drots* 2-groups, so we can be assured that the tangle is not as shown in either Figure 13 (i) or (ii). Thus the two orbiter loners do not participate in an OTS 6-tangle in any configuration where they are adjacent to each other and each is adjacent to its *drots* 2-group. We must consider the possibility that in some other configuration, the two orbiter loners might participate in an OTS 6-tangle; that is, it might be possible to start with the two positive *drots* 2-groups with their loners in the position as shown in Figure 13 (i) and flype one or both of the orbiters to create an OTS 6-tangle which contains the two orbiters.
At this stage, it is known that the knot does not have any tight *drots* 2-groups, and so this could not occur if only one of the orbiters was flyped. We therefore consider the possibilities that arise when both are flyped.

In Figure 14, one min-tangle, T, of the orbit of orbiter loner c is shown (by means of a dotted outline). The orbiter loner d is in the min-tangle T of c's orbit, so d can only have flype positions within T. Thus if c and d are able to flype so as

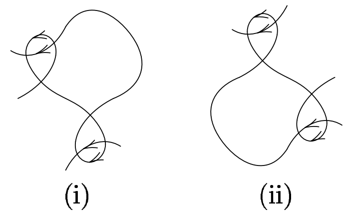


Figure 13:

to participate in an OTS 6-tangle, the only possible flype position for c would be the pair of arcs e_5 and e_6 on the side of T opposite to that of c, and then d must have the opportunity to flype to a position on arcs e_2 and e_3 where involves exactly one of arcs e_2 and e_3 coincides with one of the arcs e_5 and e_6 (it is not possible for d to have a flype position on the pair of arcs e_5 and e_6, since if it did, then c and d would belong to the same group). In Figure 14, that part of the orbit of d which consists of all min-tangles between d and the position on arcs e_2 and e_3 is indicated by the tangle (also outlined by a dotted curve) which lies immediately above d. There are four cases in total to consider, e_2 coincides with either e_5 or e_6, or else e_3 coincides with either e_5 or e_6.

Consider the possibility that e_2 coincides with e_5. Then e_3 can't coincide with e_6, and e_1 can't coincide with (that is, be connected directly to) e_6, since if it were, the e_1-e_6 arc would form a crossing with the e_2-e_5 arc, which is not possible since T is a min-tangle, and the two arcs on one side of a min-tangle do not meet in a crossing inside the min-tangle. Thus e_1 must cross the e_2-e_5 arc to meet up with arc e_3 and therefore arc e_4 must coincide with arc e_6. But this implies the existence of a 2-tangle inside T, which is not possible.

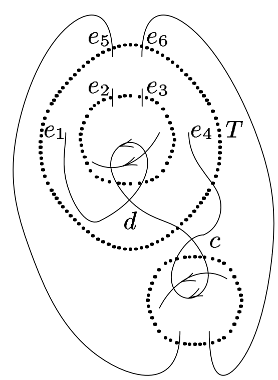


Figure 14:

Next, consider the possibility that e_2 coincides with e_6. Then e_1 does not coincide with e_5, nor can e_4 coincide with e_5, otherwise arcs e_5 and e_6 form a crossing inside T and this is not possible. Thus e_5 must coincide with e_1 and so e_3 must coincide with e_4 (note that in this case, e_3 cannot be the arc crossing the loop to form the positive *drots* 2-group for which d is the loner orbiter, since that would imply that the knot had a tight *drots* 2-group). But then arc e_1-e_5 will not meet arc e_3-e_4 after d is flyped to lie on arcs e_2 and e_3, and thus when c is flyped to lie on arcs e_5 and e_6, c and d will be joined by arc e_2-e_6, with arcs e_1-e_5 and e_3-e_4 being the other two arcs that would have to belong to an OTS 6-tangle involving c and d in their new positions. Thus this case also does not lead to any OTS 6-tangles involving c and d. The remaining two cases are similar, with none resulting in an OTS 6-tangle involving c and d.

Thus any of the master arrays that we are still considering that contain exactly two positive *drots* 2-groups with a configuration as shown in Figure 12 (i) will not need to be submitted to OTS.

We now consider a knot with exactly two positive *drots* 2-groups for which there is a configuration as shown in Figure 12 (ii). We need only consider those OTS 6-tangles which contain each of the two orbiter loners in some position. In particular, if the two loners are in the position as shown in Figure 12 (ii), the only possible OTS 6-tangle they could participate in would require a crossing as shown in Figure 15, where the crossings that make up the OTS 6-tangle are labelled b, c and d, with c and d being the two orbiter loners. Our immediate task is to determine what other, if any, OTS 6-tangles might involve c or d, in whatever positions they occupy of their respective orbits. Now, both c and d belong to a min-tangle T in the orbit of b. Let e and f denote the incident arcs on the side of T that is opposite to b. Since neither c nor d can flype to arcs outside of T, other than possibly the arcs that are incident to T, it suffices to examine T.

Suppose that it is possible to flype both c and d in such a way that in their new positions, they have an arc in common (so that, in their new positions, c and d are adjacent). Then there must exist tangles T_1 and T_2, each consisting of one or more min-tangles from the orbits of c and d, respectively, with c adjacent to T_1 and d adjacent to T_2, and one of the incident arcs on the other side of T_1 is coincident with one of the incident arcs on the other side of T_2, as shown in Figure 16.

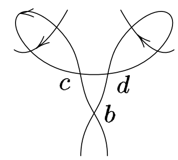


Figure 15:

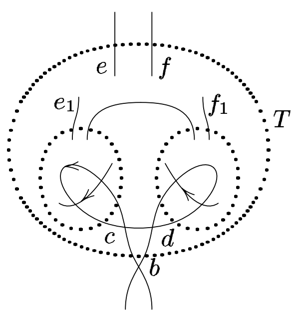


Figure 16:

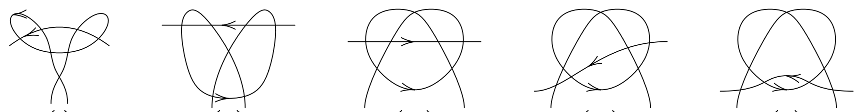
Since b could then be flyped to arcs e_1 and f_1, it follows that e_1 must coincide with e and f_1 must coincide with f, whence c and d in their new positions would form an OTS 6-tangle with b flyped to arcs e and f. There is no other possible OTS 6-tangle that c and d (in any of their positions) can participate in. For such a master array, it suffices to consider only OTS 6-tangles which involve c, d and the crossing b, and there are at most two such OTS 6-tangles.

There is a special case of this last scenario. If the arc that cuts the loop to form the positive *drots* 2-group of d proceeds directly to cut the loop to form the positive *drots* 2-group of c, as shown in Figure 17 (i), then it is not necessary to send the master array to OTS.

For the result of performing OTS on the OTS 6-tangle which involves the two orbiter loners is shown in Figure 17 (ii). However, the tangle shown in Figure 17 (v) contains a negative 2-group, so it will already have been produced by D. The results of applying three OTS operations in succession (and the reductions that we have implemented above for OTS will not prevent these OTS operations from being performed) to the tangle shown in (v) are displayed in (iv), (iii) and (ii), respectively, so the OTS operation on the OTS 6-tangle shown in (i) does not need to be performed.

In summary, if a master array contains exactly two positive *drots* 2-groups, then there is only one scenario that will cause us to submit the master array to OTS, and that is the one illustrated in Figure 15, but not as shown in Figure 17 (i).

Configuration dependent reductions for OTS. The observation to be made



(i) (ii) (iii) (iv) (v)

Figure 17:

here for a given master array M is that if M contains a negative 2-group, then we need only consider OTS 6-tangles that contain a crossing from any negative 2-group of M, and if M contains a positive 3-group, then we need only consider OTS 6-tangles that contain a crossing from any positive 3-groups of M, and if M contains a positive *drots* 2-group, then we need only consider OTS 6-tangles that contain the orbiter loner of any positive *drots* 2-group of M. Moreover, if the master array contains exactly two positive *drots* 2-groups, then we shall only apply OTS to an OTS 6-tangle that is comprised of the loner orbiters of the two positive *drots* 2-groups, together with the third crossing as shown in Figure 16, and there can be at most two such 6-tangles, one which is as illustrated in Figure 15, with one additional OTS 6-tangle possible if crossing b can flype to arcs e and f, and at the same time, the loner orbiters of the two positive *drots* 2-groups can be flyped onto arcs e and f, respectively.

Of course, if the master array is without any one of the above-mentioned objects, the corresponding requirement is not relevant. In particular, in a master array all of whose groups are loners or positive 2-groups, then every OTS 6-tangle must have OTS applied to it.

5. An Implementation of the Construction, Part III: the Operators at Work

In this, the final section, we describe an efficient implementation of the algorithms described in the preceding sections. As the construction of the knots at a given crossing size proceeds, it is important to keep in memory a database of the knots of that crossing size that have been constructed in order that as each successive knot is produced, it can be compared against the database to determine whether it is a new knot, in which case it is added to the database. However, the size of such a database becomes an issue even at 18 crossings. Our first reduction in memory requirements stems from the observation that the master array is completely determined by any configuration of the knot, and thus to compare master arrays, it suffices to compare the special configuration obtained from each master array by placing each group in its zero position. In fact, this might even be considered to be the so-called *ideal configuration* of a prime alternating knot. For our internal comparisons, we shall store the zero position configuration, and in fact, we store this in its Dowker-Thistlethwaite code.

The other observation that leads to reduced database memory requirements derives from the fact (to be established in this section) that we can have the operators act on master arrays to produce master group codes for the knots that result from the operator application, and there is a strong connection between the master array of the knot being operated on and the resulting master array. We shall exploit this connection to obtain a method for structuring the database so that only part of it needs to be available at any one stage of the construction process. This partitioning of the database is done according to the number of groups in a group code for the knot.

Definition 9. Let K be a prime alternating knot. Then the number of groups in any full group configuration of K is called the *group number* of K, denoted by $\mathrm{gn}(K)$.

We shall begin the process of generating the prime alternating knots of $n+1$ crossings with the application of $DROTS$, which takes as its input the collection of all prime alternating knots of n crossings, partitioned by group number. The knots that are produced by $DROTS$ are the K_A knots of $n+1$ crossings, and each new K_A knot will be assessed according to the reduction rules for T and OTS to see whether or not it should be put into either or both of the input queues for T and for OTS, as well as being written to the $n+1$ crossing K_A directory, in files according to group number.

Our approach will be to set k equal to the value of the smallest group number for the knots at n crossings, and for each n crossing knot K of group number k, its master array M is sent simultaneously to the four $DROTS$ operator situations; namely D on negative groups and loners, D on positive 2-groups, $ROTS$ on negative 2-groups and $ROTS$ on negative 3-groups. Each of these operators will examine M and decide, in view of their respective restrictions, whether or not the operator should be applied to one or more of the knot configurations of K. Any knot that does result from an application of one of these four operators is then examined to see whether or not it has already been produced. If not, then it is added to the database in memory, examined to see whether or not it should be subsequently submitted to one or both of the operators T and OTS, and ultimately saved to disk, classified by its group number within the collection of K_A knots. When the last n crossing knot of group number k has been processed, we increment k and repeat the process.

The effect of each of the various operators in the $DROTS$ family has on the group number of a knot is easily described. To begin with, D acting on negative groups and loners does not change the group number of the knot on which it operates, while D acting on positive 2-groups and $ROTS$ acting on negative 2-groups increases the group number by one (the 2-group being operated on is converted into a loner and a 2-group, the 2-group becoming a new min-tangle in the orbit of the loner). The last operator is $ROTS$ acting on negative 3-groups, and it results in a negative 3-group being replaced by a loner in the flype orbit of the original 3-group, and a new min-tangle added to this orbit, the min-tangle being a positive *drots*-tangle. Thus $ROTS$ acting on negative 3-groups causes an increase of two in the group number of the knot.

The significance of the preceding observations about the change in group number when a knot is acted on by a $DROTS$ operator is revealed when we consider the K_A knot database that is being kept in memory. When the last n crossing knot of group number k has been processed, we may purge the K_A knots with group number k from our database. When $DROTS$ has finished processing the last n crossing knot, the database can be completely purged, for our reduction scheme assures us that no subsequent operation will ever produce a K_A knot. For the range of crossing sizes that we have computed, this results in a reduction of approximately 80% in main memory requirements.

We have one last general remark to make. As we have structured our operators, once a master array has been made, it is submitted to the database routine. It is the responsibility of the database routine to determine first whether the knot is new and therefore to be added to the database or that it already exists and the submitted master array is to be discarded. If the knot is new, the database routine will compute the Dowker-Thistlethwaite code for the zero position configuration and add it to the database. As well, it will determine the group number of the submitted knot and write it to the appropriate disk file, according to type (K_A, K_B, or K_C) and group number. Furthermore, during the construction of the K_A knots, the database routine will be responsible for building the input files for the applicaton of T and OTS.

At the completion of the construction, we will have all K_A knots, all K_B knots, and all K_C knots, each collection classified according to group number.

5.1. *D on negative groups and loners*

D receives the master array M of a knot of n crossings, and searches for negative groups and loners. For each negative group or loner with a flype orbit in the negative direction, a master group code for the knot that results from the application of D to the group or loner in question in position zero is constructed from M as follows. Let G denote the group in question. The first occurrence of a group arc of G in position zero is located, and then the second of G's arcs in position zero is located. The segment of the master array that lies between these two positions referred to as the first segment for G in the master array. Reverse the order of the entries in the first segment, and then replace the label G in each of its positions by the next available label for a group of size one greater than the size of G, recording it as a negative group. Finally, a sign change must be recorded for each of the groups in the core tangle whose arcs were what we called the starter core group arcs (those groups which have one arc in the first section and one arc in the second section of G's orbit).

On the other hand, if G is a loner with either an orbit in the positive direction or with a trivial flype orbit, then a master group code for the knot that results from an application of D to G is obtained from M as follows: first, delete all positions of G except for the one in zero position, then reverse the first segment for G in the master array and replace the label G by the next available label for a group of size two, recording it as a negative group, and finally, change the sign of each group that had one arc in the first section of the loner and one arc in the second section of the loner.

In either case, the result is a master group code for the knot that results from the application of D to G. The master group code can now be checked according to the reduction criteria described in the preceding section to see if we should proceed further with this group. If so, and G is a negative group or a loner with an orbit in the negative direction, then we examine the orbit of G for flype-symmetric tangles in order to identify the actual positions in the orbit of G at which we shall apply D, while if G is a loner with an orbit in the positive direction, we shall apply D to G in each of its positions, in the manner described above for G in position zero. When the positions have been identified, we then construct from M a master group code for each knot that is produced by applying D to G in each of the identified positions. The resulting master group codes are converted to master arrays and submitted to the database.

5.2. D on positive 2-groups

Suppose that G is a positive 2-group in the knot K of n crossings. We may form a master group code for the knot that results from an application of D to G in position j of its orbit as follows. Let m denote the number of loners in K, and suppose that G was the i^{th} 2-group to have been labelled. Further, suppose that the orbit of G had k positions. In M, locate both arcs of G in position j. These two positions partition the orbit of G into two segments, and we choose one and reverse it. Then replace the first label 2_i^j by $-1_{m+1}^j, -2_i^0, -1_{m+1}^j$ and the second label 2_i^j by $-1_{m+1}^k, -2_i^0, -1_{m+1}^k$. Next, for every $0 < t \neq j$, replace each occurrence of 2_i^t by -1_{m+1}^t. Finally, each group that had one arc in the first section for G in the master array and one arc in the second section must have a change of sign recorded.

To begin with, we apply the above process to G in position zero, and then apply the reduction criteria for D on positive 2-groups. If the reduction criteria do not stop D from being applied to G, then we apply D to G in each position and submit the resulting negative 2-group to a group LNR competition against the other negative 2-groups (in this setting, all the negative 2-groups in the resulting knot will be 2-groups in a negative *drots* tangle, and so will have trivial flype orbit). If the group that results from the application of D to G in a given position wins its group LNR competition, then the master array of the knot that resulted from the application of D to G in the given position is calculated and submitted to the knot database.

5.3. $ROTS$ on negative 2 and 3-groups

Given the master array M of an n crossing knot K, we check first to see if any reduction rules apply. If M contains two or more negative groups, or just one negative group, but that negative group either has size greater than 3 or else is the negative 2-group of a negative *drots* tangle, then M is not submitted to $ROTS$.

Once it has been determined that M is to be submitted to $ROTS$, the location of the unique negative group G in M is known, and G is either a 2-group or a 3-group. If G is a 2-group, then we apply $ROTS$ to the group once for each position of the group in its orbit. A master group code for the knot that results from the application of $ROTS$ to G in position i is obtained from M as follows. First, delete all copies of G in positions other than i. The result of applying $ROTS$ to G is to replace G by a positive 2-group and a loner. Let m denote the number of loners in M, and let $G = -2_k$. The first occurrence of -2_k^i in M is replaced by $-1_{m+1}^0, 2_k^0, -1_{m+1}^0$, and the second occurrence of -2_k^i in M is replaced by $-1_{m+1}^1, 2_k^0, -1_{m+1}^1$.

If the 2-group -2_k has a non-trivial orbit, then the result is a master group code for the knot that results from the application of $ROTS$ to the negative group in any one position of its orbit. However, if -2_k had a trivial orbit, it is possible that it could be a min-tangle in what we might call a virtual orbit, by which we mean an orbit with no group. In this case, the virtual orbit will become part of the orbit of the new loner that results from the application of $ROTS$, and in addition, the newly created positive group (2_k) will be a min-tangle in the orbit of the new loner, so there will one additional position. Thus in the event that -2_k has a trivial flype orbit, we must complete the formation of a master group code for the knot that results from the application of $ROTS$ to -2_k by determining the rest of the orbit of the newly created loner. Upon doing that, we will have a master group code for the resulting knot, and we then construct the master array from this master group code and submit it to the database.

It remains to deal with the case when G is a 3-group. The application of $ROTS$ to a negative 3-group replaces the 3-group by a positive 2-group with trivial flype orbit, a loner with a two position flype orbit (one min-tangle in the orbit of this loner is the positive 2-group that was created), and a loner that inherits the orbit of G. We apply $ROTS$ to G in each position of its orbit. A master group code for the knot that results from the application of $ROTS$ to G in position i is obtained from M as follows. Let m and t denote the number of loners, respectively 2-groups, in M, and suppose that $G = -3_k$. In M, replace the first occurrence of -3_k^i by $-1_{m+2}^0, -1_{m+1}^0, 2_{t+1}^0, -1_{m+1}^0, -1_{m+2}^1$, and replace the second occurrence of -3_k^i by $-1_{m+2}^1, -1_{m+1}^1, 2_{t+1}^0, -1_{m+1}^1, -1_{m+2}^0$. Replace every occurence of -3_k^j for $j \neq i$ by 1_{m+2}^j.

In this case, the result is a master group code for the knot that results from the application of $ROTS$ to the negative 3-group in one position of its orbit. The master array is then constructed from the master group code, and the knot is submitted to the database.

Once D and $ROTS$ are finished, all K_A knots have been constructed. The database can now be purged completely in readiness for the construction of the K_B and K_C knots.

Recall that as it processed the K_A knots, the database routine was preparing the input files for T and OTS. In the case of T, each K_A master array M is checked to see if it contains a negative $DROTS$ 2-group, or no positive 2-groups, or two or more positive $DROTS$ 2-groups. If any of these conditions are met, then M is not submitted to T. Thus the input file for T will consist of all master arrays from the $DROTS$ output that contain no negative $DROTS$ 2-groups, at least one positive 2-group, and at most one positive $DROTS$ 2-group.

5.4. T on positive 2-groups

Consider the master array M of an $n+1$ crossing K_A knot that has been submitted to T.

Case 1: M contains a single positive $DROTS$ 2-group G. Then that is the only 2-group that T need be applied to. The positive $DROTS$ 2-group G necessarily has trivial orbit, and turning G will increase the size of the orbiter group H by 2 crossings. Furthermore, this enlarged group will retain the orbit of H. The group number of the new knot will be one smaller than the group number of K. We may readily construct from M a master group code for the knot that results from the application of T to G. To begin with, G is denoted by 2_i for some index i, and in M, we shall find 2_i surrounded by the arcs of some position r of its orbiter group H, which must be a negative group, say $H = -m_j$. Thus M will have the form

$$\ldots, -m_j^r, 2_i^0, -m_j^r, \underbrace{\ldots\ \ldots}_{\text{Section 1}}, -m_j^s, 2_i^0, -m_j^s, \ldots$$

Reverse the subsequence labelled section 1 above. Then, if there are already q groups of size $m+2$ in K, replace the subsequence $-m_j^r, 2_i^0, -m_j^r$ by $(m+2)_{q+1}^r$, replace the subsequence $-m_j^s, 2_i^0, -m_j^s$ by $(m+2)_{q+1}^s$, and replace every other position $-m_j^t$ by $(m+2)_{q+1}^t$. Finally, for each of the remaining groups, any that had one arc in section 1 and one arc in the other section must have their sign changed. This completes the construction of a master group code for the knot that has been produced from K by turning G.

This master group code is then submitted to the procedure that will apply the reduction rules to determine whether to construct its master array and submit it to the database procedure or to discard it.

Case 2: M does not contain any positive $DROTS$ 2-groups. For each position of each positive 2-group G, we perform the following procedure. Suppose that G is the 2-group 2_i and we are considering position r. In M, we locate all positions 2_i^s for $s \neq r$ and remove them. Then locate the subsequence contained between the first and second occurrences of the arcs 2_i^r and reverse it. Next, examine the code to locate any groups that exactly one group arc in the subsequence that was reversed and change the sign of the group. Finally, replace the two arcs 2_i^r by 2_i^0. The result is almost a master group code for the knot that results from turning G in position r. All of the orbit information for every group except the turned group is already in place; that is, the array that has been constructed is that array that would have been built if we had started with a group code for the knot and constructed a master group code from it, leaving the group 2_i for last. In effect, we are picking up the process at that point. We complete the construction of a master group code for the knot that results from this application of T by identifying the orbit of 2_i. It is worthwhile to note here that if the group 2_i had a non-trivial orbit, then when it is turned in any position of its orbit, the resulting 2-group will have trivial orbit, and so no further work is required; that is, we will already be in possession of a master group code for the resulting knot.

This master group code is then submitted to the procedure that will apply the reduction rules to determine whether to construct its master array and submit it to the database procedure or to discard it. We point out that in Case 2 above, if the group 2_i has a non-trivial flype orbit, it is sufficient to apply the configuration dependent reduction criteria to 2_i in any one position of its orbit. If it is rejected in that position, it will be rejected in every position, and we can bypass the application of T to group 2_i in any position. Furthermore, although we have not described them here, it is evident that we could set up group LNR competitions for T on positive 2-groups similar in principle to those we have described for D.

The procedure that applies the reduction rules will examine the master group codes that are submitted to it from T. If the master group code contains any negative group, the master group code is discarded. If the master group code contains a positive group of size greater than the group that results from turning G (which is a 2-group if G was not the 2-group of a positive $DROTS$ tangle, but it could have size much larger than 2 if G was the 2-group of a positive $DROTS$ tangle), the master group code is discarded. Otherwise, the master array is constructed from the master group code, and the knot is submitted to the database. If the knot is not already in the database, the database routine will insert it, and also write it to disk as a K_B knot, classified according to its group number. As well, it will be submitted to the routine that is preparing the input files for the initial OTS application.

5.5. *OTS*

Unlike the other operators, OTS does not work directly on groups, but rather on 6-tangles. Consequently, OTS has the potential to be more expensive to perform on group code. We shall therefore expend considerable effort towards reducing the work that OTS must do. Recall that each knot that was produced by $DROTS$ was examined to see if it contained a negative $DROTS$ tangle, and only if it did not was it forwarded to the routine that is preparing the input file for the application of OTS. As well, the output from the application of T is sent to this routine as well (no knot that is produced by the application of T as described above will contain a negative $DROTS$ tangle, so it is not necessary to examine these knots for negative $DROTS$ tangles).

The job of the routine that is preparing the input file for the OTS application is to apply the reduction criteria described in the preceding section to each submitted knot, discarding those that do not meet the criteria for submission to OTS. In the course of the examination of a knot for the purpose of applying these criteria, a considerable amount of information about the knot is gathered. This information is pertinent to the identification of the OTS 6-tangles to which the OTS operation is to be applied for those knots that do get submitted to OTS. Thus the input file for the first application of OTS will contain not only the master arrays of the knots to which OTS is to be applied, but as well, each master array will be accompanied by additional data that will help to identify the OTS 6-tangles to which OTS is to be applied.

As each knot that is constructed by the first application of OTS is submitted to the database routine, it will be examined to see if it is new and if so, it will be inserted in the database and written to disk as a K_B knot classified by group number. Furthermore, it will be added to the input file for the next application of OTS, which is to say that OTS is to be applied iteratively, until no new knots are constructed by OTS. Only the very first application of OTS will produce K_B knots, as all subsequent applications of OTS to knots that have been produced by earlier OTS applications will only produce K_C knots (actually, K_B knots may be constructed, but they will be discarded by the database routine, since we have retained the K_B knots in the database. After OTS has finished processing its input file, it will restart with the new input file that was constructed during the just-completed OTS run. This will continue until OTS encounters an empty input file.

We emphasize again that after the first, non-iterative application of OTS, all K_B knots will have been constructed, and there are no reduction criteria applicable for any subsequent OTS applications.

The combined output from all of the OTS rounds is also written to a file to serve as the input for the subsequent application of T, which shall be referred to as iterative T. By this we mean that T is applied to the combined output of the preceding OTS run, then T is applied to the new knots that were produced by this application of T, and so on until we reach a stage where the application of T to the new knots that were produced by the preceding application of T does not result in any new knots. The combined output of iterative T becomes the input for the next application of iterative OTS. The construction will continue in this way, alternating between iterative T and iterative OTS, with the output from each operator serving as the input for the next round of the other operator. The process stops when no new knots are produced during the application of one of the operators.

Let us examine how the routine that will prepare the input file for the first, non-iterative application of OTS to the knots that have been produced by $DROTS$ and T. Each master array that is submitted to the routine will be examined group by group. In doing so, counts are maintained of the number of negative 2-groups that are not negative *drots* 2-groups, the number of positive 3-groups, and the number of positive $DROTS$ 2-groups with loner orbiter group that have been encountered, and for each such object, a record of that object and its index in the master array is constructed. If at any point in the examination, we encounter a negative group of size at least three, a positive group of size at least 4, a positive $DROTS$ 2-group whose orbiter group has size at least 2, or a tight $DROTS$ 2-group, or the number of positive $DROTS$ 2-groups with loner orbiter exceeds 2, or the sum of the number of negative 2-groups that are not *drots* 2-groups, the number of positive 3-groups and the number of positive $DROTS$ 2-groups with loner orbiter exceeds 4, then the master array is discarded.

If we reach the end of the examination of the master array without discarding it, then we know that the master array array contains at most 2 positive $DROTS$ 2-groups. For those that do contain 2 positive $DROTS$ 2-groups, we perform an additional check to see if the 2 positive $DROTS$ 2-groups are in the configuration shown in Figure 15, but not as shown in Figure 17 (i). If not, then the master array is discarded.

For a master array M that survives the application of the reduction rules, the next step is to identify the actual OTS 6-tangles in the master array to which OTS is to be applied. The following observations are pertinent. For any negative 2-group or positive 3-group G of M, it is necessary only to consider OTS 6-tangles that contain a crossing of G, and for any positive $DROTS$ 2-group H of M, it is necessary only to consider OTS 6-tangles that contain the loner orbiter of H. Moreover, if M contains two positive $DROTS$ 2-groups, then we shall apply OTS to only two OTS 6-tangles; namely the one that consists of the loner orbiters of the two positive $DROTS$ 2-groups together with the third crossing as shown in Figure 15, and the one obtained by putting the two orbiter loners in their other position, and flyping the third crossing to the position in its orbit that has it make an OTS

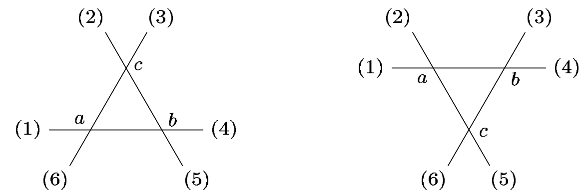
6-tangle with the two loner orbiters again. If the groups in M are either loners or positive 2-groups, and none of the positive 2-groups are positive $DROTS$ 2-groups, then we must apply OTS to every OTS 6-tangle.

There remains the issue of identifying the OTS 6-tangles in the master array. Of course, in the first application of OTS, we may have many restrictions on the OTS 6-tangles to which OTS is to be applied, while in subsequent applications, every OTS 6-tangle must have OTS applied to it. The master array is a master group code, and an OTS 6-tangle involves 3 different groups. It is important to be able to work with the master array rather than an individual group code, since any flype operation that does not involve any of the crossings of an OTS 6-tangle will commute with OTS applied to that OTS 6-tangle. Accordingly, if we apply OTS to the same OTS 6-tangle in two configurations that differ only by flypes that do not involve the crossings of the OTS 6-tangle, the resulting knots will be flype-equivalent. In such a case, it is not necessary to perform OTS on both of the initial configurations. Moreover, the orbit structure of the knot that results from an application of OTS will be very similar to the orbit structure of the original knot. Since we do not want to rediscover information that is already in our possession, we shall identify the various OTS 6-tangles in the master array to which we are to apply OTS, and for each such 6-tangle, we construct a master group code for the knot that is obtained upon the application of OTS to the selected OTS 6-tangle. Just as was the case for T, we shall find ourselves with most of the orbit structure already known to us, with the orbits of at most three groups to be determined before we have a master group code for the knot that results from the application of OTS.

We now describe the process whereby we are able to identify OTS 6-tangles in the master array, and to describe the effect of applying OTS to a selected OTS 6-tangle in the master array. As we mentioned above, the result is a group code that is most of the way through the process of being converted to a master group code for the newly constructed knot. We then describe how to identify the groups that still need to have their orbits determined, which then allows us to finish the job of constructing a master group code for the newly constructed knot.

To begin with, we shall describe the procedure for identifying an OTS 6-tangle when given a Gauss code for a knot configuration $C(K)$, following which, we describe the changes that must be made to this Gauss code in order to construct a Gauss code for the knot configuration that results from the application of OTS to a selected OTS 6-tangle in $C(K)$. To find an OTS 6-tangle when given a Gauss code for a knot configuration, we work our way through the code, checking each arc to see if the two crossings that determine the arc participate in an OTS 6-tangle. Suppose that we are considering the arc formed by crossings a and b, with b following a in the Gauss code. We are looking to see if there is a crossing c as shown in Figure 18 (i). If so, the three crossings form an OTS 6-tangle, and to perform OTS on it, we have chosen to move the arc determined by a and b (recall that the outcomes of moving any one of the three arcs determined by the three crossings in the OTS 6-tangle are all identical). The result of moving the arc ab is shown in Figure 18 (ii).

We shall consider the (cyclic) Gauss code to begin with the appearance of the



(a) An OTS 6-tangle (b) After applying OTS

Figure 18: OTS in action

crossing a that is immediately followed by crossing b. We scan the rest of the code, searching for the second occurrences of crossings a and b. Once they have been found, we examine the crossings on either side of the second occurrence of crossing a and on either side of the second occurrence of crossing b to see if there is a crossing c which has one of its occurrences adjacent to the second occurrence of a and its other occurrence adjacent to the second occurrence of b. In the knot traversal, strand (4) could connect next to any of strands (2), (3), (5) or (6), and we examine each of these possibilities. For example, if strand (4) connects next to strand (5), then there are two subcases to consider, depending on whether strand (2) connects next to strand (3) or to strand (6). In the event that strand (2) connects to strand (3), we find that the code, which begins with a, b, continues with a segment of code that we shall refer to as S_1, then we find the pair b, c, followed by a segment of code that we shall refer to as S_2, then the pair c, a, followed by the final segment of code, which we shall refer to as S_3. Upon consulting Figure 18, we see that after performing the OTS operation, the code becomes ab, S_1, ca, S_2, bc, S_3. We present the corresponding information in tabular form for all four cases, each with two subcases.

(a)	(4) → (5)	(2) → (3)	Before OTS	$ab, S_1, bc, S_2, ca, S_3$
			After OTS	$ab, S_1, ca, S_2, bc, S_3$
(b)		(2) → (6)	Before OTS	$ab, S_1, bc, S_2, ac, S_3$
			After OTS	$ab, S_1, ca, S_2, cb, S_3$
(c)	(4) → (6)	(3) → (2)	Before OTS	$ab, S_1, ac, S_2, cb, S_3$
			After OTS	$ab, S_1, cb, S_2, ac, S_3$
(d)		(3) → (5)	Before OTS	$ab, S_1, ac, S_2, bc, S_3$
			After OTS	$ab, S_1, cb, S_2, ca, S_3$
(e)	(4) → (2)	(5) → (3)	Before OTS	$ab, S_1, cb, S_2, ca, S_3$
			After OTS	$ab, S_1, ac, S_2, bc, S_3$
(f)		(5) → (6)	Before OTS	$ab, S_1, cb, S_2, ac, S_3$
			After OTS	$ab, S_1, ac, S_2, cb, S_3$
(g)	(4) → (3)	(6) → (2)	Before OTS	$ab, S_1, ca, S_2, cb, S_3$
			After OTS	$ab, S_1, bc, S_2, ac, S_3$
(h)		(6) → (5)	Before OTS	$ab, S_1, ca, S_2, bc, S_3$
			After OTS	$ab, S_1, bc, S_2, ca, S_3$

Figure 19: Performing OTS operations on a Gauss code

From the data in Figure 19, we extract the following rule, which describes in every case how to perform the OTS operation. The pair of crossings between segments S_1 and S_2 and the pair of crossings between segments S_2 and S_3 change places, with the crossing in common, c being placed in such a way that it does not occupy the same relative position (first or second crossing) between S_1 and S_2 as it did in the original pair between S_1 and S_2, nor does it occupy the same relative position (first or second crossing) between S_2 and S_3 as it did in the original pair between S_2 and S_3.

Let us now examine a group code for a knot, or more generally, the master array for the knot, as it provides a copy of each group in every position that the group can be placed. Our first objective is to identify the OTS 6-tangles from the master array, and then for each OTS 6-tangle, we explain how to transform the master array into a master group code for the knot that results from the application of OTS to the selected OTS 6-tangle. In the search for an OTS 6-tangle, we shall examine adjacent groups G_1 and G_2 in the master array, say with crossing c_1 at the end of G_1 adjacent to crossing c_2 at the end of G_2, and we look to see if there is a third group G_3 such that of the two arcs at one end of G_3, leaving crossing c_3 of the third group, say, one is incident to c_1 and the other is incident to c_2 (if at least two of the groups are loners, it is possible that another position of one of the loners might mask an OTS situation, and so one must check for this). If so, then c_1 and c_2 form an OTS 6-tangle with c_3, otherwise c_1 and c_2 do not participate in an OTS 6-tangle. To perform the OTS operation once we have found such crossings c_1, c_2 and c_3, we begin by creating three loners in the master group code: one at the end of G_1 next to c_1 (this loner is actually c_1), (this loner is actually c_2), and one at the end of G_3 adjacent to c_3 (this loner is actually c_3). Note that each crossing will actually generate two entries in the master group code, one for each arc of the group in question. Next, we decrement the crossing size of each position of G_1, G_2 and G_3 (using the next available label for groups of the respective sizes), and if this results in a group with no crossings (that is, we had started with a loner), then we simply remove all positions of that group. We then treat the crossings c_1, c_2 and c_3 just as if they were crossings in a Gauss code, and perform the OTS operation. For any two of the three crossings of the OTS 6-tangle, if the two adjacent arcs of the OTS 6-tangle that are incident to the two selected crossings are coincident at their other end, being arcs at the end of a group G_4, then after the OTS operation, the third crossing of the OTS 6-tangle becomes another crossing of the group G_4. Accordingly, we must relabel group G_4 to indicate that its size is now one greater than it was prior to the OTS operation. The orbit of this enlarged copy of G_4 is unchanged, so G_4 must have its size increased by 1 in every position, using the next available label for a group of that size.

On the other hand, if the two adjacent arcs of the OTS 6-tangle that are incident to the two selected crossings are not coincident at their other ends, then the third crossing becomes a loner after the OTS operation. Note that if any of G_1, G_2 or G_3 is not a loner, then the crossings from the other two must become loners after the OTS has been performed. Accordingly, if two or more of G_1, G_2 and G_3 are not loners, then all three crossings c_1, c_2 and c_3 become loners after the application of OTS.

Once we apply OTS, taking care of the possible enlarged groups that might arise, the result is almost a master group code for the knot that has been obtained by the application of OTS. All that remains is to determine the orbits of any (at most three) loners that were created by the application of OTS. Thus after applying OTS to the master array, a very small amount of additional work will produce a master group code for the resulting knot.

When we are examining the master array for OTS 6-tangles, we shall take advantage of the information that we have compiled for the master array during the preparation of the input file for the initial round of OTS. It will often be the case that, as a result of the OTS reductions, an OTS 6-tangle will not actually have OTS performed on it. Recall that for a given master array M, if M contains at least one negative 2-group, then we need only consider OTS 6-tangles that contain a crossing from each negative 2-group of M, and if M contains at least one positive 3-group, then we need only consider OTS 6-tangles that contain a crossing from each positive 3-group of M, and if M contains a positive $DROTS$ 2-group, then we need only consider OTS 6-tangles that contain the orbiter loner of each positive $DROTS$ 2-group of M. Moreover, if the master array contains exactly two positive $DROTS$ 2-groups, we will only apply OTS to an OTS 6-tangle that involves the two loners that orbit the two positive $DROTS$ 2-groups. As shown earlier, there are at most two such OTS 6-tangles.

Of course, any knot that is produced from the initial round of OTS will consist entirely of loners or positive 2-groups, with no positive *drots* tangles, so every OTS 6-tangle will have to have OTS applied to it.

Once the master group code has been contructed during an application of OTS, it is standardized to produce the master array, which is then submitted to the database routine. The database routine will compare the master array to the knot database, and if the knot is a new one, it will be inserted into the database in memory, written to the directory of K_B or K_C knots, as appropriate, organized according to group number, and written to the input file for the subsequent round of OTS.

We continue applying the cycle of alternating iterative OTS, followed by iterative T, with each operator being applied only to the new knots that were produced by the preceding application of the other operator, until no new knots are produced. At this point, the construction is complete.

6. References

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