Proof. Ignoring trivial cases, we assume throughout that G has at least two edges. Then G is connected and remains connected when one edge is cut.

The proof is inductive. We construct a strictly increasing sequence, necessarily ending with G, of subgraphs Γ satisfying the following conditions:

(1) Γ has a piecewise linear embedding in $\mathbb{R}^2$, preserving the given unoriented cyclic order at each vertex, which is unique up to homeomorphism of S^2.

(2) The embedded Γ, as a subset of S^2, is the 1-dimensional skeleton of a non-degenerate cell decomposition of S^2.

Here non-degenerate means that the intersection of any two(closed) 2-cells is empty or a 0-cell or a 1-cell, and that each 1-cell is the intersection of two 2-cells. It follows that each 0-cell is on at least three 2-cells and on at least three 1-cells, and must be a vertex of the graph Γ. It is not excluded that two 1-cells may have the same boundary 0-sphere.

To start the induction we first choose a vertex v of G. Since G is connected and has edges, v is one end of some edge. Let u be the other end. Cutting the edge between v and u leaves G connected. Choose a minimal path (least number of edges) from v to u in the cut graph. It is an arc and, with the original edge from v to u, forms a circle (simple closed curve). Cutting the two edges at v on this circle leaves G connected. Join v to the rest of the circle by a minimal path in the cut graph. It is an arc meeting the circle only at v and at its other end w, which is a vertex. The subgraph Γ_0 of G consisting of this arc and the circle is homeomorphic to a graph H consisting of two vertices v, w and three edges, each joining v to w. This graph H, and hence also Γ_0, has a unique piecewise linear embedding in $\mathbb{R}^2$, up to homoemorphism of S^2. It is the 1-skeleton of a cell decomposition of S^2 with three 2-cells, three 1-cells and two 0-cells. Since each vertex of Γ_0 has valency at most 3, the preservation of unoriented cyclic order is trivial. Thus Γ_0 satisfies (1) and (2).

Now take any proper subgraph Γ of G which satisfies (1) and (2).

Case 1. There is a vertex v of Γ which is embedded as a 0-cell of the cell decomposition, and which is an end of an edge of G not in Γ.

Let the other end of this edge be u. If u is not in Γ, there is a minimal path, hence an arc, in the connected set $G\backslash\{v\}$ from u to $\Gamma\backslash\{v\}$, ending, say, at w. If $u \in \Gamma$, take $w = u$. In either case we have an arc from v to w through u, with $w \neq v$ and with only the ends of the arc vw in Γ. Adding the edges and vertices on this arc to Γ, we obtain Γ_1.

G has a piecewise linear embedding in $\mathbb{R}^2$ preserving unoriented cyclic order at its vertices. Hence there is an embedding of Γ which extends to a piecewise linear embedding of Γ_1 preserving unoriented cyclic order at its vertices.

The arc from v to w, apart from its endpoints, must be embedded in some open 2-cell C of the cell decompsition corresponding to Γ. The 0-cell v is on the boundary of at least three 1-cells separating at least three 2-cells. Thus v is on at least three edges of Γ. Hence the unoriented cyclic order of the edges of Γ_1 at v determines the 2-cell C in which vu and vw lie.