The Kozai-Lidov resonances in stellar dynamics

Background [much from Wikipedia]

In stellar and celestial mechanics, the **Kozai–Lidov mechanism** is a dynamical phenomenon affecting the orbit of a binary system perturbed by a distant third body under certain conditions, causing the binary's orbital argument of pericenter to oscillate about a constant value, which in turn leads to a periodic exchange between its eccentricity ($e = \sqrt{[1-b^2/a^2]}$) and the inclination (ψ) of the orbital plane of the perturber.

The process occurs on timescales *T* much longer than the orbital periods of the binary or the perturber. It can drive an initially near-circular orbit to arbitrarily high eccentricity, and *flip* an initially moderately inclined orbit between a prograde and a retrograde motion. The Kozai-Lidov timescale T_{KL} for these cycles is given by

(1) $$T_{KL} = P_{out}\,\frac{P_{out}}{P}\left(\frac{1+q_{out}}{q_{out}}\right)\left(1-e^2_{out}\right)^{3/2}\,g(e,\psi)$$

where the inner binary of period *P* and eccentricity *e* is perturbed by an outer body of mass m_3 such that

$$q_{out} = m_3 / (\, m_1 + m_2 \,).$$

The period of that outer body is denoted P_{out} . In Eq. (1), the factor $g(e,\psi) \approx 1$ but depends on the precise value of binary eccentricity (in the unperturbed state) and the inclination angle. The geometry of the problem is shown on Fig. 1.

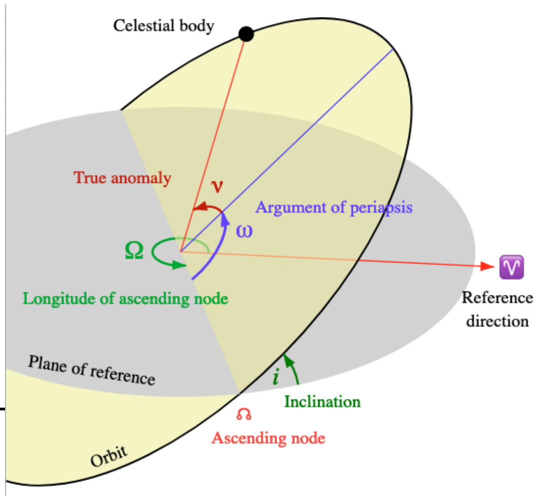


Figure 1. Illustration of the orbital configuration of the system. The inner binary (not shown) orbits in a plane depicted in grey. Its angular frequency Ω is shown in green. The third body (denoted "celestial body") orbits in a plane inclined at an angle ψ = i (inclination) as shown for the yellow circle. The line of nodes joins the two planes, where one seems to pivot with respect to the other. Image credits : Wikipedia / Kozai process.

Details

Let *L* be the angular momentum of the inner binary. Lidov (1961) and Kozai (1962) found that the component of *L* that lies in the orbital plane of the perturber is conserved, while this perturber exerts a torque on the inner binary. This torque can be thought of as slowing down the binary, so that its orbit will become more eccentric ; as a result of action-reaction, the orbital plane of the perturber is also affected, in that it will tend to align itself with the inner binary.

Taken together, the Kozai-Lidov effect leads to the relation

$$(1 - e^2)\,\cos^2\psi = C \tag{2}$$

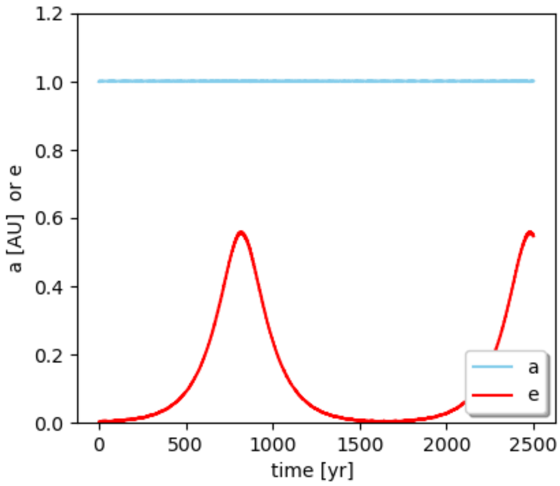


Figure 2. The orbital elements *a* (semi-major axis) and eccentricity *e* of the inner binary plotted against the time in years. Note that the period of the inner binary ~ 9 months < 1 year. In this case the inner had components of mass $m_1 = m_2 = 1$ Solar mass, semi-major axis a = 1 AU and $e = 0$. the outer body (a.k.a. the perturber) had a mass $m_3 = 0.1$ Solar mass and an orbit of semi-major axis = 5 AU and eccentricity $e_{out} = ¼$. The initial inclination angle $\psi = 80^\circ$.

which can be checked empirically to obtain C directly by monitoring the time-evolution of both *e* and inclination angle ψ. An example of this of shown on Fig. 2. Note that these periodic cycles will not develop (or only very mildly) if the inclination angle ψ is less than a critical value given by

$$\psi_{crit} = \arccos\left(\sqrt{\frac{3}{5}}\right). \tag{3}$$

Equations of motion

The equations of motion are those of classical gravity, namely the pair-wise potential, defined in the rest frame of the barycentre of the system.

The set of first-order equations are obtained from the Hamiltonian

$$H = T + W = \sum_i \frac{1}{2} m_i v_i^2 - \sum_{i,j\neq i} G\frac{m_i m_j}{r_{ij}} \tag{4}$$

with *i* the index of stars = 1, 2 or 3, and r_{ij} is the separation between stars *i* and *j.* We can then write in vector form, e.g. for a given reference star ★ :

$$\begin{aligned} \dot{\mathbf{x}}_\star &= [\mathbf{x}_\star, H] = \mathbf{v}_\star \\ \dot{\mathbf{v}}_\star &= [\mathbf{v}_\star, H] = -\sum_{i\neq\star} G\frac{m_i}{r_{i,\star}^2}\,\hat{\mathbf{r}}_{i,\star} \end{aligned} \tag{5}$$

Note that a "hatted" vector has unit length (for space orientation). Key to the problem is that the total mechanical energy and angular momenta are integrals of motion, and so they must be recovered at each time of integration (cf. "Physics of Galaxies" lecture notes and Binney & Tremaine 2008 for details / proofs):

(6) H = E = constant ; **L** = constant (in norm and components).

Numerical integrations

There are many integration techniques that one can use to solve equations (5). We suggest implementing first a low-order but simplectic integrator, before implementing a higher order scheme that allows larger integration steps but is less conservative (it is a semi-implicit scheme).

The two integrators that we will outline here are discussed in details in Binney & Tremaine (2008), §3.4.

1. *The Verlet or leapfrog integrator*

A classic technique consists in shifting the positions and velocities separately, with a half-time step δτ/2 offset between the two procedures : for example, starting with coordinates (x,v) at t_o we may implement the following algorithm:

a. shift the space coordinates to time t' = t_o + δτ/2

$$x' = x + v\,\delta t/2$$

b. With the new set of primed coordinates (for all the bodies, all the components) update the velocity vectors up to time t" = t_o + δτ :

$$v'' = v(t) - \nabla\phi(x', t')\,\delta t$$

The notation is such that primed and double-primed quantities are *not* synchronous, and we have replaced the sum in (5) with the gradient operator for short-hand.

c. Repeat step (a), but with the update velocity field

$$x''(t'') = x'(t') + v''(t'')\,\delta t/2$$

d. Once all coordinates have been updated, rename the position x" —> x, v" —> v, t" —> t_o = t , and repeat from step (a). Note that this way of proceeding leaves the coordinates synchronous at the end of step (c).

The great advantage of this algorithm is that it is easy to implement, and conserves the integrals of motion up to 2nd order in δτ (it is O[$\delta\tau^2$]). But it will require small steps δt for good accuracy.

2. *The Hermite predictor-corrector approach*.

A more accurate technique (allowing for larger step sizes δτ) is the fourth-order Hermite method. In this case, one updates the position and velocity of each body, and then applies a correction to the results for higher accuracy. It is similar in spirit the the ABM scheme that you may have encountered elsewhere: you may just as well implement the ABM algorithm to this problem, if this were more suitable to you.

Here, the notation will follow that of the textbook by Aarseth (2003). The accelerations are denoted **a** and an overlayed "dot" means "derivative wrt time".

If we start at time *t,* coordinates (**x**, **v**), we may *predict* (subscript p) the new coordinates at t + δτ with

(7)
$$\mathbf{x}_p = \mathbf{x} + \delta t\,\mathbf{v} + \frac{\delta t^2}{2}\,\mathbf{a} + \frac{\delta t^3}{6}\,\dot{\mathbf{a}}$$
$$\mathbf{v}_p = \mathbf{v} + \delta t\,\mathbf{a} + \frac{\delta t^2}{2}\,\dot{\mathbf{a}}$$

To do that, we compute the acceleration $\mathbf{a}_\bigstar$ from the sums in (5); the time-derivatives are

(8)
$$\dot{\mathbf{a}}_\star = -\sum_{i\neq\star} \frac{Gm_i}{r^3_{i,\star}}\left[\mathbf{v}_{i,\star} - 3\frac{\mathbf{v}_{i,\star}\cdot r_{i,\star}}{r_{i,\star}}\,\hat{\mathbf{r}}_{i,\star}\right]$$

for a particular star '★'. If we recompute the accelerations and their derivatives at the *predicted coordinates* ($\mathbf{x}_p$, $\mathbf{v}_p$), at time t' = t + δτ, and denote them with "primes", then the corrected coordinates can be computed with the following (Hermite) terms

(9)
$$\mathbf{a}^{(2)} = -\frac{6(\mathbf{a}-\mathbf{a}') + \delta t\,(4\dot{\mathbf{a}} + 2\dot{\mathbf{a}}')}{\delta t^2}$$
$$\mathbf{a}^{(3)} = \frac{12(\mathbf{a}-\mathbf{a}') + 6\delta t\,(\dot{\mathbf{a}} + \dot{\mathbf{a}}')}{\delta t^3}$$

The final, corrected coordinates are now:

(10)
$$\mathbf{x}'(t') = \mathbf{x}_p + \frac{\delta t^4}{24}\mathbf{a}^{(2)} + \frac{\delta t^5}{120}\mathbf{a}^{(3)}$$
$$\mathbf{v}'(t') = \mathbf{v}_p + \frac{\delta t^3}{6}\mathbf{a}^{(2)} + \frac{\delta t^4}{24}\mathbf{a}^{(3)}\ .$$

and note that the corrections on the velocities proceed by direct derivation of the corrections on the coordinates, **x**. The same holds for the prediction, Eq. (7).

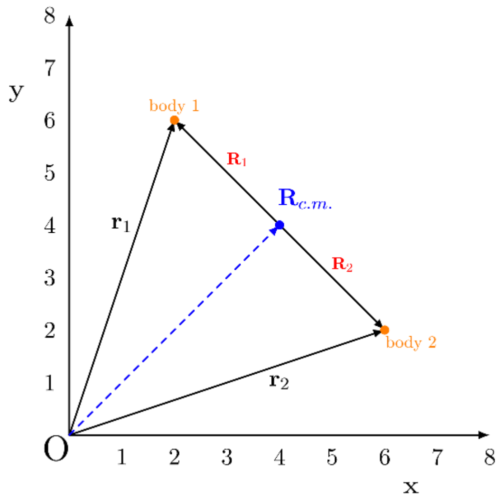
Strategy

A good way to proceed is probably to exploit the hierarchy of complexity that we have, both in the nature of the problem (binary + perturber) and the algorithmic integration schemes proposed.

(I) **Simplify the problem**: inner binary alone.

- Now that we know the problem, the first step consists in validating your integrators. The case of a binary star on a circular orbit is ideal since the solution **r** = constant holds for the two stars (but the norm $r_1 \neq r_2$ if $m_1 \neq m_2$). Set your system at rest, and make sur to include the three dimensions (even if the motion is two-dimensional).
- **Centre of mass frame.** It is useful to reset the coordinates of the bodies such that the barycentre lies at (0,0) and is at rest. Given any set of coordinates (**r**,**v**) in an inertial frame *O* at rest, we locate the centre of mass at $\mathbf{R}_{cm} = (m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2) / (m_1 + m_2)$ and the relative coordinates follow from $\mathbf{R} = \mathbf{r} - \mathbf{R}_{cm}$ (for each star; see sketch Fig. 3).

Figure 3. Coordinates of body 1 and 2 of a binary star, defined in a frame O ; only the x-y axes are shown. The coordinates of the centre of mass (c.m.) are shown in blue, the bodies are represented with filled orange circles. The relative coordinates are shown with capital letters, so for instance $\mathbf{R}_1$ is the relative position of body 1 of mass m_1 with respect to $\mathbf{R}_{cm}$. The same construction applies to the velocities, with $\mathbf{V}_{cm} = d\,\mathbf{R}_{cm} / dt$ etc.



- **Angular momentum.** In the frame at rest with the barycentre, the binary star's angular momentum reads

(11)
$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 = \sum_{i=\{1,2\}} \mathbf{R}_i \times \mathbf{p}_i$$

where we wrote $\mathbf{p}_i = m_i \mathbf{R}_i$, the momentum of star *i* measured in frame of the centre of mass.

:: Each component of **L** is a conserved quantity, and hence its magnitude is, too. ::

- **Orthogonal initial configuration.** Determine the initial conditions and recover a closed orbit for the binary stars. A simple way to setup the initial conditions is to fix the semi-major axis *a* parallel to one of the Cartesian axes of the coordinate system, and set the initial velocity vector **v** orthogonal to it for each star: this effectively fixes the phase of the orbit at the apocenter.

    - Determine the angular momentum and total energy E < 0 of the system ;
    - Verify that E is independent of L or *e* :

$$E = -\frac{Gm_1m_2}{2a} \,.$$

- Integrate the binary star's orbit for a circular orbit (*e* = 0) and find the number of steps needed to maintain a relative error on the total energy δE / E of 10^{-5} or less ;
- Reduce the total angular momentum **L** of the binary so that *e > 0*. Are you able to recover a closed orbit with the same number of integration steps as before ? Since the minimum distance r —> r (1 - e), it may be useful to think in terms of the time interval needed to trace a half-circle of radius r (1 - e) and use that information to inform the number of steps required to trace a full orbit.
- Once you have success with the Verlet integrator, try using the Hermite predictive step alone, and the the predictor/correcteur method, Eqs. (7) to (10). The goal is to determine if the Hermite scheme allows larger time steps for the same precision δE / E as with the Verlet integrator.

(II) **Add a third, mass-less body**. Once you are satisfied that your integration proceeds with reasonable accuracy, add a third body with $m_3 = 0$ on some outer orbit ($a_{out} > a$), starting in a plane at an inclination angle ψ of your choice. Does the body m_3 remain in the same orbital plane, ψ = constant ?

(III) **Try to recover** the time series for the eccentricity of the inner binary shown on Fig. 2 (the configuration is indicated in the caption). How does ψ vary over time for that case ?

The problem of the Kozai-Lidov orbit coupling is a special case of the more general problem of orbital resonances : the response of an orbit to an external perturbation repeating at a fixed frequency, commensurate with the orbital period. The short bibliography below outlines some applications of this mechanism.

Bibliography

General references: books, reviews

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Aarseth, S.J. 2003, *Gravitational N-body Simulations,* Cambridge: The University press

The original paper by Kozai :

Kozai, Y., 1962. *Secular perturbations of asteroids with high inclination and eccentricity. Astron.* J. 67, 591-598.

An application to chaotic motion (technically advanced, concepts interesting) :

Li, G., Naox, S., Holman, M. & Loeb. A. 2014, *Chaos in the test particle eccentric Kozai-Lidov mechanism*, The Astrophysical Journal, 791:86 (10pp), 2014

An early application to planetary system dynamics (emphasis on stability) :

Innanen, K.A. et al. 1997, *The Kozai mechanism and the stability of planetary orbits in binary star systems*, The Astronomical Journal, Vol 113, no 5, p. 1915

Introduction to the ideas : see e.g. the Wikipedia pages. Some background material was lifted from there.