EPFL | MGT-418 : Convex Optimization | Project 3

Questions – Fall 2021

Image Denoising and Reconstruction

Part 1: Image Denoising

Description

A gray scale image is an $m \times n$ matrix of intensities (ranging from 0 to 255). Given a noisy gray scale image X^{noisy}, our goal is to construct an image X that resembles the unknown original image X^{true}. To this end, in analogy to the total variation reconstruction in the lecture, we will minimize the sum of the Frobenius norm of $X - X^{\text{noisy}}$, *i.e.*,

$$\left\|X - X^{\text{noisy}}\right\|_{\text{F}} = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}\left(X_{i,j} - X_{i,j}^{\text{noisy}}\right)^2\right)^{\frac{1}{2}},$$

and a total variation regularization term of the image X, *i.e.*,

$$\|X\|_{\text{TV}} = \sum_{j=1}^{n}\sum_{i=1}^{m-1}|X_{i+1,j} - X_{i,j}| + \sum_{i=1}^{m}\sum_{j=1}^{n-1}|X_{i,j+1} - X_{i,j}|.$$

Thus, we aim to solve the convex optimization problem

$$\underset{X \in \mathbb{R}^{m \times n}}{\text{minimize}} \quad \left\|X - X^{\text{noisy}}\right\|_{\text{F}} + \rho \|X\|_{\text{TV}}, \tag{1}$$

where $\rho > 0$ is the regularization weight.

Questions

1.1. **SOCP Reformulation**: Verify that problem (1) is equivalent to a second-order cone program.

1.2. **Denoising**: Load the image of a dog (`dog.png`), see Figure 1a, and create a noisy image by $X^{\texttt{noisy}} = X^{\texttt{true}} + 20 * \texttt{randn}(\texttt{size}(X^{\texttt{noisy}}))$, see Figure 1b. Solve problem (1) with $\rho = 25 * 10^{-4}$ to denoise the noisy image. A skeleton of the code you will have to implement is provided in the Matlab file `p3q12.m`. Display the true image, the noisy image and the image obtained by solving problem (1).

1.3. **Regularization Weight**: Solve problem (1) with $\rho \in [10^{-4}, 10^{-1}]$ (discretize the range to 40 discrete points by `logspace(-4,-1,40)`) to denoise the noisy image of the dog obtained in 1.2. Compute and plot the values

$$f(\rho) = \frac{\left\|X(\rho) - X^{\text{true}}\right\|_{\text{F}}}{\left\|X^{\text{noisy}} - X^{\text{true}}\right\|_{\text{F}}}$$

as a function of the regularization weight ρ. Display the image obtained by solving problem (1) for $\rho = 10^{-4}$, $\rho = 10^{-1}$ and for the ρ that minimizes $f(\rho)$. Briefly comment on the results.

Part 2: Image Reconstruction

Description

A color image is an $m \times n \times 3$ matrix of RGB values (ranging from 0 to 255). Given a subset of pixels $X_{ij}^{\text{true}} \in \mathbb{R}^3$, $(i,j) \in \mathcal{S}$, where $\mathcal{S} \subset \{1,\dots,m\} \times \{1,\dots,n\}$, our goal is to construct an image X that resembles the unknown original image X^{true} by filling in the missing pixels $(i,j) \in \{1,\dots,m\} \times \{1,\dots,n\} \setminus \mathcal{S}$. To this end, we will minimize the total variation of the image X while ensuring that X captures the information about the known pixels. Thus, we aim to solve the convex optimization problem

$$\underset{X \in \mathbb{R}^{m\times n\times 3}}{\text{minimize}} \quad \|X\|_{\text{TV}} \quad \text{subject to} \quad X_{ij} = X_{ij}^{\text{true}} \quad \forall (i,j) \in \mathcal{S}, \tag{2}$$

where

$$\|X\|_{\text{TV}} = \sum_{k=1}^{3} \left(\sum_{j=1}^{n} \sum_{i=1}^{m-1} |X_{i+1,j,k} - X_{i,j,k}| + \sum_{i=1}^{m} \sum_{j=1}^{n-1} |X_{i,j+1,k} - X_{i,j,k}| \right).$$

Questions

2.1. **LP Reformulation**: Verify that problem (2) is equivalent to a linear program.

2.2. **Reconstructing Mona Lisa**: Load the image of Mona Lisa (`monalisa.png`), see Figure 2a. Construct the partial image through

$$X_{ij}^{\text{par}} = \begin{cases} X_{ij}^{\text{true}} & \text{with a chance of } 40\% \\ (255,255,255) & \text{with a chance of } 60\%, \end{cases}$$

where $(255,255,255)$ represents the color white. Solve problem (2) to reconstruct the image using the known pixels. A skeleton of the code you will have to implement is provided in the Matlab file `p3q22.m`. Display the original image, the partial image and the reconstructed image.

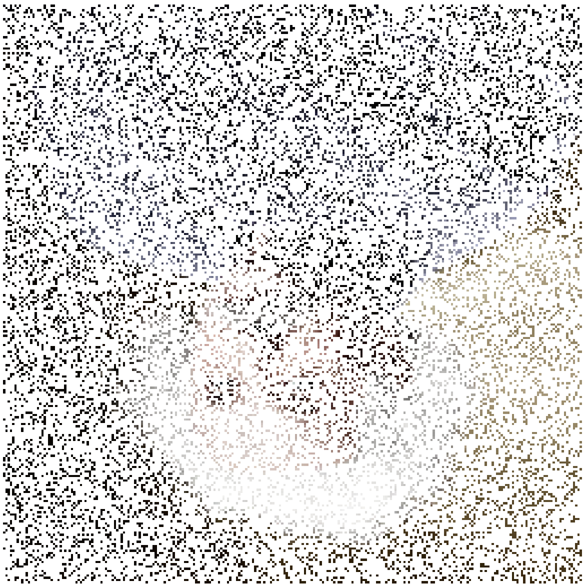
2.3. **Reconstructing the Unknown**: Load the partial image (`unknown.png`), see Figure 2b, that contains approximately 20% of the original image's pixels. Solve problem (2) to reconstruct the image. A skeleton of the code you will have to implement is provided in the Matlab file `p3q23.m`. Display the partial image and the reconstructed image.

(a) True image.

(b) Noisy image.

Figure 1: Image Denoising.

(a) Mona Lisa.



(b) Unknown image.

Figure 2: Image Reconstruction.