EPFL | MGT-418 : Convex Optimization | Project 4

Answers – Fall 2020

Optimal Power Flow

Answer to Question 1

An implementation of ($\mathrm{R_{SDP}}$) can be found in the Matlab script `p4a1.m`. On a 3.40 GHz i7 computer with 16GB RAM, a single run of the script results in the following quantities:

IEEE 9-bus:	$L_1 := \min(\mathrm{R_{SDP}}) = 373.83\,[\$]$,	$t^{\text{build}}_{\text{SDP}} = 0.58\,[s]$,	$t^{\text{solve}}_{\text{SDP}} = 0.03\,[s]$
IEEE 118-bus:	$L_1 := \min(\mathrm{R_{SDP}}) = 86\text{'}298\,[\$]$,	$t^{\text{build}}_{\text{SDP}} = 0.32\,[s]$,	$t^{\text{solve}}_{\text{SDP}} = 1\text{'}394\,[s]$.

Answer to Question 2

Observe that the constraints of ($\mathrm{R_{SDP}}$) imply the constraints of ($\mathrm{R_{SOCP}}$). Hence, the feasible region of ($\mathrm{R_{SOCP}}$) is (weakly) larger than the feasible region of ($\mathrm{R_{SDP}}$). As the objective functions coincide, we thus must have $\min(\mathrm{R_{SDP}}) \geq \min(\mathrm{R_{SOCP}})$. An implementation of ($\mathrm{R_{SOCP}}$) is provided in the Matlab script `p4a2.m`. On a 3.40 GHz i7 computer with 16GB RAM, a single run of the script results in the following quantities:

IEEE 9-bus:	$L_2 := \min(\mathrm{R_{SOCP}}) = 362.22\,[\$]$,	$t^{\text{build}}_{\text{SDP}} = 0.92\,[s]$,	$t^{\text{solve}}_{\text{SDP}} = 0.01\,[s]$
IEEE 118-bus:	$L_2 := \min(\mathrm{R_{SOCP}}) = 84\text{'}868\,[\$]$,	$t^{\text{build}}_{\text{SDP}} = 496\,[s]$,	$t^{\text{solve}}_{\text{SDP}} = 1.7\,[s]$.

Answer to Question 3.1

We first state problem (P) in standard form and associate Lagrange multipliers with the constraints,

$$
\begin{array}{lll}
\underset{p^{\mathrm{g}},q^{\mathrm{g}},e,f}{\text{minimize}} & \sum_{n=1}^N c_n p^{\mathrm{g}}_n & \\
\text{subject to} & p^{\mathrm{g}}, q^{\mathrm{g}}, e, f \in \mathbb{R}^N & \\
& \sum_{m=1}^N G_{nm}(e_n e_m + f_n f_m) - \sum_{m=1}^N B_{nm}(e_n f_m - e_m f_n) - p^{\mathrm{g}}_n + p^{\mathrm{d}}_n = 0 & \forall n = 1,\dots,N \quad (\alpha_n \text{ free}) \\
& -\sum_{m=1}^N B_{nm}(e_n e_m + f_n f_m) - \sum_{m=1}^N G_{nm}(e_n f_m - e_m f_n) - q^{\mathrm{g}}_n + q^{\mathrm{d}}_n = 0 & \forall n = 1,\dots,N \quad (\beta_n \text{ free}) \\
& \underline{p}_n - p^{\mathrm{g}}_n \leq 0 & \forall n = 1,\dots,N \quad (\gamma^-_n \geq 0) \\
& p^{\mathrm{g}}_n - \overline{p}_n \leq 0 & \forall n = 1,\dots,N \quad (\gamma^+_n \geq 0) \\
& \underline{q}_n - q^{\mathrm{g}}_n \leq 0 & \forall n = 1,\dots,N \quad (\delta^-_n \geq 0) \\
& q^{\mathrm{g}}_n - \overline{q}_n \leq 0 & \forall n = 1,\dots,N \quad (\delta^+_n \geq 0) \\
& \underline{v}^2_n - e^2_n - f^2_n \leq 0 & \forall n = 1,\dots,N \quad (\phi^-_n \geq 0) \\
& e^2_n + f^2_n - \overline{v}^2_n \leq 0 & \forall n = 1,\dots,N \quad (\phi^+_n \geq 0).
\end{array}
$$

Using this notation, Lagrangian function takes the following form

$$
\begin{aligned}
& L(p^{\mathrm{g}}, q^{\mathrm{g}}, e, f, \alpha, \beta, \gamma^-, \gamma^+, \delta^-, \delta^+, \phi^-, \phi^+) = \\
& + \textstyle\sum_{n=1}^N c_n p^{\mathrm{g}}_n \\
& + \textstyle\sum_{n=1}^N \alpha_n \big(\sum_{m=1}^N G_{nm}(e_n e_m + f_n f_m) - \sum_{m=1}^N B_{nm}(e_n f_m - e_m f_n) - p^{\mathrm{g}}_n + p^{\mathrm{d}}_n \big) \\
& + \textstyle\sum_{n=1}^N \beta_n \big(-\sum_{m=1}^N B_{nm}(e_n e_m + f_n f_m) - \sum_{m=1}^N G_{nm}(e_n f_m - e_m f_n) - q^{\mathrm{g}}_n + q^{\mathrm{d}}_n \big) \\
& + \textstyle\sum_{n=1}^N \gamma^-_n (\underline{p}_n - p^{\mathrm{g}}_n) + \sum_{n=1}^N \gamma^+_n (p^{\mathrm{g}}_n - \overline{p}_n) \\
& + \textstyle\sum_{n=1}^N \delta^-_n (\underline{q}_n - q^{\mathrm{g}}_n) + \sum_{n=1}^N \delta^+_n (q^{\mathrm{g}}_n - \overline{q}_n) \\
& + \textstyle\sum_{n=1}^N \phi^-_n (\underline{v}^2_n - e^2_n - f^2_n) + \sum_{n=1}^N \phi^+_n (e^2_n + f^2_n - \overline{v}^2_n).
\end{aligned}
$$

Next, we rearrange terms according to the primal variables they contain. Note that, while the primal variables p^{g} and q^{g} are individually separable, the primal variables e and f are coupled, and thus they cannot be separated from each other. The rearranged Lagrangian function then reads

$$\begin{aligned}
&L(p^{\mathrm{g}}, q^{\mathrm{g}}, e, f, \alpha, \beta, \gamma^-, \gamma^+, \delta^-, \delta^+, \phi^-, \phi^+) = \\
&+ (c - \alpha - \gamma^- + \gamma^+)^\top p^{\mathrm{g}} + (-\beta - \delta^- + \delta^+)^\top q^{\mathrm{g}} \\
&- e^\top \mathrm{diag}(\phi^-) e - f^\top \mathrm{diag}(\phi^-) f + e^\top \mathrm{diag}(\phi^+) e + f^\top \mathrm{diag}(\phi^+) f && \text{(T1)} \\
&+ \textstyle\sum_{n=1}^N \alpha_n \big(e_n \sum_{m=1}^N G_{nm} e_m + f_n \sum_{m=1}^N G_{nm} f_m - e_n \sum_{m=1}^N B_{nm} f_m + f_n \sum_{m=1}^N B_{nm} e_m \big) && \text{(T2)} \\
&+ \textstyle\sum_{n=1}^N \beta_n \big(- e_n \sum_{m=1}^N B_{nm} e_m - f_n \sum_{m=1}^N B_{nm} f_m - e_n \sum_{m=1}^N G_{nm} f_m + f_n \sum_{m=1}^N G_{nm} e_m \big) && \text{(T3)} \\
&+ \alpha^\top p^{\mathrm{d}} + \beta^\top q^{\mathrm{d}} + (\gamma^-)^\top \underline{p} - (\gamma^+)^\top \overline{p} + (\delta^-)^\top \underline{q} - (\delta^+)^\top \overline{q} + \underline{v}^\top \mathrm{diag}(\phi^-) \underline{v} - \overline{v}^\top \mathrm{diag}(\phi^+) \overline{v}.
\end{aligned}$$

The terms gathered in (T1) can be expressed as the following quadratic form

$$\text{(T1)} = e^\top \mathrm{diag}(\phi^+ - \phi^-) e + f^\top \mathrm{diag}(\phi^+ - \phi^-) f = \begin{pmatrix} e \\ f \end{pmatrix}^\top \begin{bmatrix} \mathrm{diag}(\phi^+ - \phi^-) & 0 \\ 0 & \mathrm{diag}(\phi^+ - \phi^-) \end{bmatrix} \begin{pmatrix} e \\ f \end{pmatrix}.$$

The terms gathered in (T2) can be brought into the following quadratic form, where the last equality invokes the symmetric counterpart mentioned in the third hint,

$$\begin{aligned}
\text{(T2)} &= \textstyle\sum_{n=1}^N e_n (\alpha_n \sum_{m=1}^N G_{nm} e_m) + \sum_{n=1}^N f_n (\alpha_n \sum_{m=1}^N G_{nm} f_m) \\
&\quad - \textstyle\sum_{n=1}^N e_n (\alpha_n \sum_{m=1}^N B_{nm} f_m) + \sum_{n=1}^N f_n (\alpha_n \sum_{m=1}^N B_{nm} e_m) \\
&= e^\top \mathrm{diag}(\alpha) G e + f^\top \mathrm{diag}(\alpha) G f - e^\top \mathrm{diag}(\alpha) B f + f^\top \mathrm{diag}(\alpha) B e \\
&= \begin{pmatrix} e \\ f \end{pmatrix}^\top \begin{bmatrix} \mathrm{diag}(\alpha) G & -\mathrm{diag}(\alpha) B \\ \mathrm{diag}(\alpha) B & \mathrm{diag}(\alpha) G \end{bmatrix} \begin{pmatrix} e \\ f \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} e \\ f \end{pmatrix}^\top \begin{bmatrix} \mathrm{diag}(\alpha) G + G \mathrm{diag}(\alpha) & -\mathrm{diag}(\alpha) B + B \mathrm{diag}(\alpha) \\ \mathrm{diag}(\alpha) B - B \mathrm{diag}(\alpha) & \mathrm{diag}(\alpha) G + G \mathrm{diag}(\alpha) \end{bmatrix} \begin{pmatrix} e \\ f \end{pmatrix}.
\end{aligned}$$

Similarly, the terms gathered in (T3) can be brought into the quadratic form below, where the last equality invokes the symmetric counterpart mentioned in the third hint,

$$\begin{aligned}
\text{(T3)} &= -\textstyle\sum_{n=1}^N e_n (\beta_n \sum_{m=1}^N B_{nm} e_m) - \sum_{n=1}^N f_n (\beta_n \sum_{m=1}^N B_{nm} f_m) \\
&\quad - \textstyle\sum_{n=1}^N e_n (\beta_n \sum_{m=1}^N G_{nm} f_m) + \sum_{n=1}^N f_n (\beta_n \sum_{m=1}^N G_{nm} e_m) \\
&= -e^\top \mathrm{diag}(\beta) B e - f^\top \mathrm{diag}(\beta) B f - e^\top \mathrm{diag}(\beta) G f + f^\top \mathrm{diag}(\beta) G e \\
&= \begin{pmatrix} e \\ f \end{pmatrix}^\top \begin{bmatrix} -\mathrm{diag}(\beta) B & -\mathrm{diag}(\beta) G \\ \mathrm{diag}(\beta) G & -\mathrm{diag}(\beta) B \end{bmatrix} \begin{pmatrix} e \\ f \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} e \\ f \end{pmatrix}^\top \begin{bmatrix} -\mathrm{diag}(\beta) B - B \mathrm{diag}(\beta) & -\mathrm{diag}(\beta) G + G \mathrm{diag}(\beta) \\ \mathrm{diag}(\beta) G - G \mathrm{diag}(\beta) & -\mathrm{diag}(\beta) B + B \mathrm{diag}(\beta) \end{bmatrix} \begin{pmatrix} e \\ f \end{pmatrix}.
\end{aligned}$$

For ease of notation, define the symmetric matrix

$$\begin{aligned}
Q := \begin{bmatrix} \mathrm{diag}(\phi^+ - \phi^-) & 0 \\ 0 & \mathrm{diag}(\phi^+ - \phi^-) \end{bmatrix} &+ \frac{1}{2} \begin{bmatrix} \mathrm{diag}(\alpha) G + G \mathrm{diag}(\alpha) & -\mathrm{diag}(\alpha) B + B \mathrm{diag}(\alpha) \\ \mathrm{diag}(\alpha) B - B \mathrm{diag}(\alpha) & \mathrm{diag}(\alpha) G + G \mathrm{diag}(\alpha) \end{bmatrix} \\
&+ \frac{1}{2} \begin{bmatrix} -\mathrm{diag}(\beta) B - B \mathrm{diag}(\beta) & -\mathrm{diag}(\beta) G + G \mathrm{diag}(\beta) \\ \mathrm{diag}(\beta) G - G \mathrm{diag}(\beta) & -\mathrm{diag}(\beta) B + B \mathrm{diag}(\beta) \end{bmatrix}.
\end{aligned}$$

With this notation, the Lagrangian function can be expressed as

$$\begin{aligned}
&L(p^{\mathrm{g}}, q^{\mathrm{g}}, e, f, \alpha, \beta, \gamma^-, \gamma^+, \delta^-, \delta^+, \phi^-, \phi^+) = \\
&+ (c - \alpha - \gamma^- + \gamma^+)^\top p^{\mathrm{g}} + (-\beta - \delta^- + \delta^+)^\top q^{\mathrm{g}} + \begin{pmatrix} e \\ f \end{pmatrix}^\top Q \begin{pmatrix} e \\ f \end{pmatrix} \\
&+ \alpha^\top p^{\mathrm{d}} + \beta^\top q^{\mathrm{d}} + (\gamma^-)^\top \underline{p} - (\gamma^+)^\top \overline{p} + (\delta^-)^\top \underline{q} - (\delta^+)^\top \overline{q} + \underline{v}^\top \mathrm{diag}(\phi^-) \underline{v} - \overline{v}^\top \mathrm{diag}(\phi^+) \overline{v}.
\end{aligned}$$

Therefore, the infimum of the Lagrangian function over the primal variables is finite provided that

$$c - \alpha - \gamma^- + \gamma^+ = 0, \qquad -\beta - \delta^- + \delta^+ = 0, \qquad Q \succeq 0,$$

and the dual objective function evaluates to

$$\begin{aligned} g(\alpha,\beta,\gamma^-,\gamma^+,\delta^-,\delta^+,\phi^-,\phi^+) &= \inf_{p^{\mathrm{g}},q^{\mathrm{g}},e,f} L(p^{\mathrm{g}},q^{\mathrm{g}},e,f,\alpha,\beta,\gamma^-,\gamma^+,\delta^-,\delta^+,\phi^-,\phi^+) \\ &= \begin{cases} \alpha^\top p^{\mathrm{d}} + \beta^\top q^{\mathrm{d}} + (\gamma^-)^\top \underline{p} - (\gamma^+)^\top \overline{p} + (\delta^-)^\top \underline{q} - (\delta^+)^\top \overline{q} + \underline{v}^\top \mathrm{diag}(\phi^-)\underline{v} - \overline{v}^\top \mathrm{diag}(\phi^+)\overline{v} \\ \qquad \text{if } c - \alpha - \gamma^- + \gamma^+ = 0,\ -\beta - \delta^- + \delta^+ = 0,\ Q \succeq 0, \\ -\infty \quad \text{otherwise.} \end{cases} \end{aligned}$$

Recalling the nonnegativity conditions on γ^-, γ^+, δ^-, δ^+, ϕ^- and ϕ^+, the suggested Lagrangian dual now follows from the above dual objective function.

Answer to Question 3.2

An implementation of (D) is provided in the Matlab script `p4a3.m`. On a 3.40 GHz i7 computer with 16GB RAM, a single run of the script results in the following quantities:

$$\begin{aligned} &\text{IEEE 9-bus:} && L_3 := \max(\mathrm{D}) = 373.83\,[\$], && t^{\mathrm{build}}_{\mathrm{SDP}} = 0.11\,[s], && t^{\mathrm{solve}}_{\mathrm{SDP}} = 0.01\,[s] \\ &\text{IEEE 118-bus:} && L_3 := \max(\mathrm{D}) = 86\text{'}297\,[\$], && t^{\mathrm{build}}_{\mathrm{SDP}} = 0.62\,[s], && t^{\mathrm{solve}}_{\mathrm{SDP}} = 0.55\,[s]. \end{aligned}$$

Answer to Question 4

A summary of the obtained results can be found in Table 1 below. The tightest lower bound is achieved by the SDP relaxation ($\mathrm{R_{SDP}}$), and also, roughly, by the dual (D). The best performance in terms of runtime is achieved by the dual (D), which clearly also strikes the best trade-off.

It is interesting to observe that the SDP relaxation ($\mathrm{R_{SDP}}$) is built quickly, but takes quite some time to solve. In contrast, the SOCP relaxation ($\mathrm{R_{SOCP}}$) takes quite some time to be built, but solves quickly. The best of both worlds is eventually achieved by the dual (D) which both builds and solves quickly. Interestingly, one can show that the Lagrange dual problem of the dual (D) is, in fact, the SDP relaxation ($\mathrm{R_{SDP}}$). The contrast in the respective solution times is thus all the more striking.

Table 1: Summary of the obtained results.

IEEE	Heuristic	SDP Relaxation			SOCP Relaxation			Dualization		
System	$\tilde{P}$ [$]	L_1 [$]	$t^{\mathrm{build}}_{\mathrm{SDP}}$ [s]	$t^{\mathrm{solve}}_{\mathrm{SDP}}$ [s]	L_2 [$]	$t^{\mathrm{build}}_{\mathrm{SOCP}}$ [s]	$t^{\mathrm{solve}}_{\mathrm{SOCP}}$ [s]	L_3 [$]	$t^{\mathrm{build}}_{\mathrm{D}}$ [s]	$t^{\mathrm{solve}}_{\mathrm{D}}$ [s]
9-bus	373.83	373.83	0.58	0.03	362.22	0.92	0.01	373.83	0.11	0.01
118-bus	86'300	86'298	0.32	1'394	84'868	496	1.7	86'297	0.62	0.55