EPFL | MGT-418 : Convex Optimization | Project 3

Answers – Fall 2021

Image Denoising and Reconstruction

Part 1: Image Denoising

Answer to Question 1.1

For a matrix $Q \in \mathbb{R}^{m \times n}$, define its vectorized representation as $\text{vec}(Q) := [Q_{:,1}; \ldots; Q_{:,n}] \in \mathbb{R}^{mn}$, that is, $\text{vec}(Q)$ is a column vector containing the vertically stacked columns of Q. Observe then that

$$\left\|X - X^{\text{noisy}}\right\|_{\text{F}} = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{i,j} - X_{i,j}^{\text{noisy}})^2\right)^{\frac{1}{2}} = \|\text{vec}(X) - \text{vec}(X^{\text{noisy}})\|_2.$$

We can now introduce an epigraphical variable t to deal with the Forbenius-norm term and constrain it via the second-order cone constraint $\|\text{vec}(X) - \text{vec}(X^{\text{noisy}})\|_2 \leq t$. Recalling Exercise 4 in Tutorial 4, we can also introduce auxiliary variables $Y \in \mathbb{R}^{m-1 \times n}$, $Z \in \mathbb{R}^{m \times n-1}$ and apply the usual reformulation technique to linearize the $\|\cdot\|_1$-type terms. This yields the second-order cone program

$$\begin{array}{lll}
\underset{\substack{X \in \mathbb{R}^{m\times n}, t\in\mathbb{R}, \\ Y \in \mathbb{R}^{m-1\times n}, \\ Z\in\mathbb{R}^{m\times n-1}}}{\text{minimize}} & t + \rho\left(\displaystyle\sum_{i=1}^{m-1}\sum_{j=1}^{n} Y_{i,j} + \sum_{i=1}^{m}\sum_{j=1}^{n-1} Z_{i,j}\right) & \\
\text{subject to} & \|\text{vec}(X) - \text{vec}(X^{\text{noisy}})\|_2 \leq t & \\
& -Y_{i,j} \leq X_{i+1,j} - X_{i,j} \leq Y_{i,j} & \forall i = 1, \ldots m-1,\ j = 1, \ldots, n \\
& -Z_{i,j} \leq X_{i,j+1} - X_{i,j} \leq Z_{i,j} & \forall i = 1, \ldots m,\ j = 1, \ldots, n-1.
\end{array}$$

Answer to Question 1.2

The required steps are implemented in the Matlab script `p3a12.m`. Figure 1 below illustrates the true image, the noisy image, and the denoised image obtained by solving problem (1).

(a) True image. (b) Noisy image. (c) Denoised image.

Figure 1: Image Denoising.

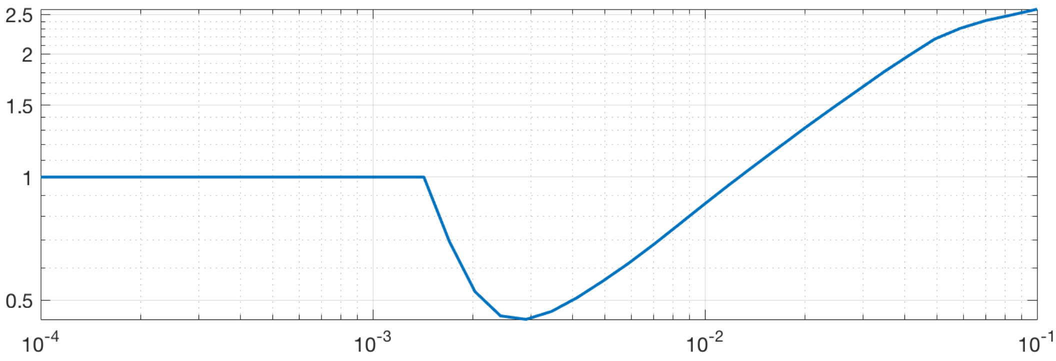


Figure 2: Logarithmic swipe over various regularization weights $\rho \in [10^{-4}, 10^{-1}]$.

Answer to Question 1.3

The required steps are implemented in the Matlab script `p3a13.m`. Figure 2 above displays the value of $f(\rho)$ for $\rho \in [10^{-4}, 10^{-1}]$. Figure 3 below shows the images obtained solving problem (1) for $\rho = 10^{-4}$, $\rho = 10^{-1}$, and $\rho = 0.0029$, respectively. If the regularization weight ρ is chosen to be small, changes in brightness are penalized less severely. The optimization then chooses X very close to X^{noisy}, which is why the noise is (almost) not removed. If the regularization weight ρ is chosen to be big, changes in brightness are severely penalized. The optimization then choses X to have few, localized brightness changes and essentially ignores the information contained in X^{noisy}. For this specific image, roughly, the best trade-off is achieved by $\rho = 0.0029$. The regularization then provides enough incentive for the optimization to eliminate the sudden brightness changes induced by the noise. At the same time, the regularization is not too strong, so the optimization still picks up the useful signals in X^{noisy}.

(a) Denoised with $\rho = 10^{-4}$. (b) Denoised with $\rho = 10^{-1}$. (c) Denoised with $\rho = 0.0029$.

Figure 3: Visual effect of different regularization strengths.

Part 2: Image Reconstruction

Answer to Question 2.1

As in Question 1.1, we introduce auxiliary variables $Y \in \mathbb{R}^{m-1 \times n \times 3}$, $Z \in \mathbb{R}^{m \times n-1 \times 3}$ to obtain

$$\begin{array}{lll} \underset{\substack{X \in \mathbb{R}^{m \times n \times 3} \\ Y \in \mathbb{R}^{m-1 \times n \times 3}, \\ Z \in \mathbb{R}^{m \times n-1 \times 3}}}{\text{minimize}} & \displaystyle\sum_{k=1}^{3} \left(\sum_{i=1}^{m-1} \sum_{j=1}^{n} Y_{i,j,k} + \sum_{i=1}^{m} \sum_{j=1}^{n-1} Z_{i,j,k} \right) & \\ \text{subject to} & X_{ij} = X_{ij}^{\text{true}} & \forall (i,j) \in \mathcal{S} \\ & -Y_{i,j,k} \le X_{i+1,j,k} - X_{i,j,k} \le Y_{i,j,k} & \forall i = 1, \dots m-1,\ j = 1, \dots, n,\ k = 1,2,3 \\ & -Z_{i,j,k} \le X_{i,j+1,k} - X_{i,j,k} \le Z_{i,j,k} & \forall i = 1, \dots m,\ j = 1, \dots, n-1,\ k = 1,2,3. \end{array}$$

Answer to Question 2.2

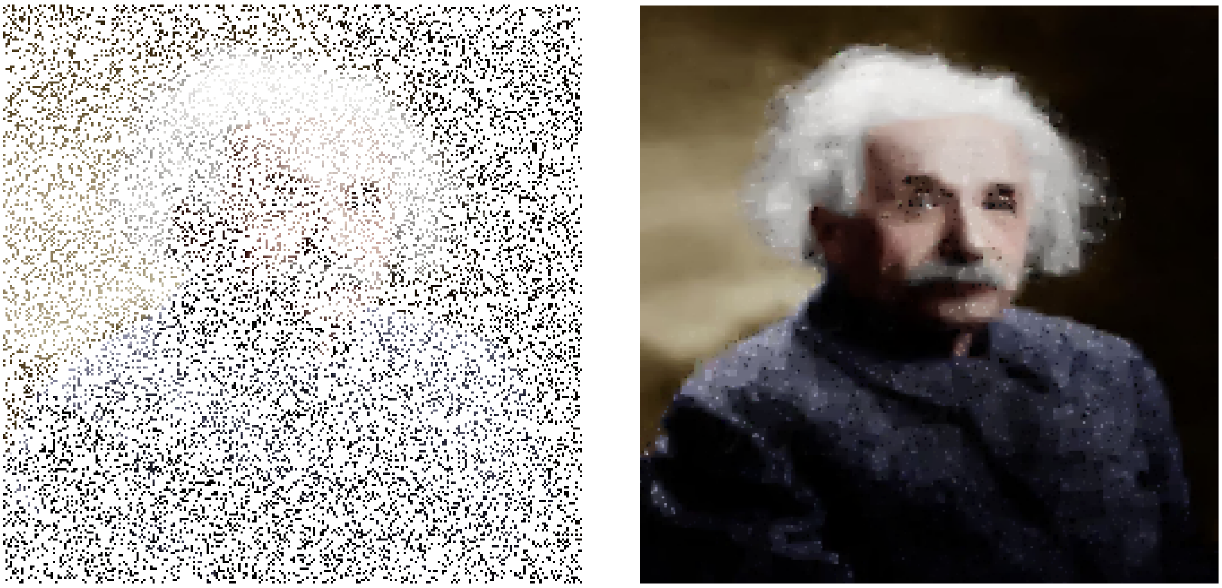
The required steps are implemented in the Matlab script `p3a22.m`. Figure 4 below illustrates the true image, the partial image, and the reconstructed image obtained by solving problem (2).

(a) True image. (b) Partial image. (c) Reconstructed image.

Figure 4: Reconstructing Mona Lisa.

Answer to Question 2.3

The required steps are implemented in the Matlab script `p3a23.m`. Figure 5 below displays the partial image and the reconstructed image obtained by solving problem (2). Can you recognize a well-known physicist?



(a) Partial image. (b) Reconstructed image.

Figure 5: Reconstructing the Unknown.