EPFL | MGT-418 : Convex Optimization | Project 4

Questions – Fall 2021

Optimal Power Flow

Description

Consider a power system consisting of N nodes that are connected through a network. At each node, electricity can be demanded and generated. The optimal power flow (OPF) problem aims at satisfying the demand at minimum production cost subject to physical and operational constraints. As seen in class, for linear production costs, the OPF problem can be cast as the *non-convex* quadratic program

$$\begin{array}{lll}
\underset{p^{\mathrm{g}},q^{\mathrm{g}},e,f}{\text{minimize}} & \sum_{n=1}^{N} c_n p_n^{\mathrm{g}} & \\
\text{subject to} & p^{\mathrm{g}}, q^{\mathrm{g}}, e, f \in \mathbb{R}^N & \\
& \sum_{m=1}^{N} G_{nm}(e_n e_m + f_n f_m) - \sum_{m=1}^{N} B_{nm}(e_n f_m - e_m f_n) = p_n^{\mathrm{g}} - p_n^{\mathrm{d}} & \forall n = 1,\ldots,N \\
& -\sum_{m=1}^{N} B_{nm}(e_n e_m + f_n f_m) - \sum_{m=1}^{N} G_{nm}(e_n f_m - e_m f_n) = q_n^{\mathrm{g}} - q_n^{\mathrm{d}} & \forall n = 1,\ldots,N \\
& \underline{p}_n \le p_n^{\mathrm{g}} \le \overline{p}_n & \forall n = 1,\ldots,N \\
& \underline{q}_n \le q_n^{\mathrm{g}} \le \overline{q}_n & \forall n = 1,\ldots,N \\
& \underline{v}_n^2 \le e_n^2 + f_n^2 \le \overline{v}_n^2 & \forall n = 1,\ldots,N
\end{array} \tag{P}$$

with decision variables

$p^{\mathrm{g}}, q^{\mathrm{g}} \in \mathbb{R}^N$:	active power and reactive power generated at each node
$e, f \in \mathbb{R}^N$:	real part and imaginary part of voltage phasors at each node

and given parameters

$c \in \mathbb{R}_+^n$:	production cost of active power at each node
$G, B \in \mathbb{S}^N$:	conductance matrix and susceptance matrix of the network
$p^{\mathrm{d}}, q^{\mathrm{d}} \in \mathbb{R}^N$:	active power and reactive power demanded at each node
$\underline{p}, \overline{p} \in \mathbb{R}^N$:	lower bound and upper bound on the active power production at each node
$\underline{q}, \overline{q} \in \mathbb{R}^N$:	lower bound and upper bound on the reactive power production at each node
$\underline{v}, \overline{v} \in \mathbb{R}^N$:	lower bound and upper bound on the voltage amplitude at each node.

Several heuristic algorithms are available to approximately solve problem (P), however, these methods typically do not provide guarantees on the suboptimality of their solutions. In this project, we will use convex optimization to quantify the suboptimality of a given, heuristic solution to problem (P). More specifically, we will compute three lower bounds L_i, $i = 1, 2, 3$, on the unknown quantity min(P) so as to bound the suboptimality ε of the given solution that attains an objective value $\tilde{P}$ via

$$L_i \le \min(P) \le \tilde{P} \quad \forall i = 1,2,3 \quad \Longrightarrow \quad \varepsilon = \tilde{P} - \min(P) \le \tilde{P} - L_i \quad \forall i = 1,2,3.$$

In terms of power systems, we will first consider the IEEE 9-bus system displayed in Figure 1 and then scale up to the larger IEEE 118-bus system displayed in Figure 2. The parameters of these systems are given in the data structures `IEEE9data.mat` and `IEEE118data.mat`, respectively. A description of the data structures can be found in `datadescription.txt`. The solution times for the larger system can take up to an hour on an average portable computer, so only apply your methods to the larger system once they successfully worked on the smaller system.

<u>Questions</u> (consider the hints on the next page)

1. **SDP Relaxation**: As discussed in class, problem (P) admits the SDP relaxation

$$\begin{array}{lll} \underset{p^{\mathrm{g}},q^{\mathrm{g}},E,F,H}{\text{minimize}} & \sum_{n=1}^{N} c_n p_n^{\mathrm{g}} & \\ \text{subject to} & p^{\mathrm{g}},\, q^{\mathrm{g}} \in \mathbb{R}^N,\ E, F \in \mathbb{S}^N,\ H \in \mathbb{R}^{N\times N} & \\ & \sum_{m=1}^{N} G_{nm}(E_{nm}+F_{nm}) - \sum_{m=1}^{N} B_{nm}(H_{nm}-H_{mn}) = p_n^{\mathrm{g}} - p_n^{\mathrm{d}} & \forall n = 1,\dots,N \\ & -\sum_{m=1}^{N} B_{nm}(E_{nm}+F_{nm}) - \sum_{m=1}^{N} G_{nm}(H_{nm}-H_{mn}) = q_n^{\mathrm{g}} - q_n^{\mathrm{d}} & \forall n = 1,\dots,N \\ & \underline{p}_n \le p_n^{\mathrm{g}} \le \overline{p}_n & \forall n = 1,\dots,N \quad (\mathrm{R_{SDP}}) \\ & \underline{q}_n \le q_n^{\mathrm{g}} \le \overline{q}_n & \forall n = 1,\dots,N \\ & \underline{v}_n^2 \le E_{nn}+F_{nn} \le \overline{v}_n^2 & \forall n = 1,\dots,N \\ & \begin{pmatrix} E & H \\ H^\top & F \end{pmatrix} \succeq 0, & \end{array}$$

where E_{nm} replaces $e_n e_m$, F_{nm} replaces $f_n f_m$, and H_{nm} replaces $e_n f_m$. Implement this relaxation using YALMIP and solve it with MOSEK first for the IEEE 9-bus system and then for the IEEE 118-bus system. A skeleton of the code is provided in the Matlab file `p4q1.m`. For both systems, report the optimal objective value $\min(\mathrm{R_{SDP}}) =: L_1$, the time $t_{\mathrm{SDP}}^{\mathrm{build}}$ required to build the model (i.e., `yalmiptime`), and the time $t_{\mathrm{SDP}}^{\mathrm{solve}}$ required to solve the model (i.e., `solvertime`).

2. **SOCP Relaxation**: Applying the methods seen in class to further relax the SDP relaxation of a QCQP to an SOCP, we can see that problem (P) also admits the SOCP relaxation

$$\begin{array}{lll} \underset{p^{\mathrm{g}},q^{\mathrm{g}},E,F,H,X}{\text{minimize}} & \sum_{n=1}^{N} c_n p_n^{\mathrm{g}} & \\ \text{subject to} & p^{\mathrm{g}},\, q^{\mathrm{g}} \in \mathbb{R}^N,\ E, F \in \mathbb{S}^N,\ H \in \mathbb{R}^{N\times N},\ X \in \mathbb{S}^{2N} & \\ & \sum_{m=1}^{N} G_{nm}(E_{nm}+F_{nm}) - \sum_{m=1}^{N} B_{nm}(H_{nm}-H_{mn}) = p_n^{\mathrm{g}} - p_n^{\mathrm{d}} & \forall n = 1,\dots,N \\ & -\sum_{m=1}^{N} B_{nm}(E_{nm}+F_{nm}) - \sum_{m=1}^{N} G_{nm}(H_{nm}-H_{mn}) = q_n^{\mathrm{g}} - q_n^{\mathrm{d}} & \forall n = 1,\dots,N \\ & \underline{p}_n \le p_n^{\mathrm{g}} \le \overline{p}_n & \forall n = 1,\dots,N \quad (\mathrm{R_{SOCP}}) \\ & \underline{q}_n \le q_n^{\mathrm{g}} \le \overline{q}_n & \forall n = 1,\dots,N \\ & \underline{v}_n^2 \le E_{nn}+F_{nn} \le \overline{v}_n^2 & \forall n = 1,\dots,N \\ & X = \begin{pmatrix} E & H \\ H^\top & F \end{pmatrix} & \\ & X_{ii} \ge 0 & \forall i = 1,\dots,2N \\ & \left\| \begin{matrix} 2X_{ij} \\ X_{ii} - X_{jj} \end{matrix} \right\|_2 \le X_{ii} + X_{jj} & \forall i,j = 1,\dots,2N,\ i<j. \end{array}$$

Explain why necessarily we have $\min(\mathrm{R_{SDP}}) \ge \min(\mathrm{R_{SOCP}})$. Implement the latter relaxation using YALMIP and solve it with MOSEK first for the IEEE 9-bus system and then for the IEEE 118-bus system. A skeleton of the code is provided in the Matlab file `p4q2.m`. For both systems, report the optimal objective value $\min(\mathrm{R_{SOCP}}) =: L_2$, the time $t_{\mathrm{SOCP}}^{\mathrm{build}}$ needed to build the model (i.e., `yalmiptime`), and the time $t_{\mathrm{SOCP}}^{\mathrm{solve}}$ required to solve the model (i.e., `solvertime`).

3. **Dualization**:

   3.1 Show that the Lagrangian dual of problem (P) can be stated as shown below.

$$\begin{array}{lll} \underset{\substack{\alpha,\beta,\gamma^-,\gamma^+,\\ \delta^-,\delta^+,\phi^-,\phi^+}}{\text{maximize}} & (p^{\mathrm{d}})^\top \alpha + (q^{\mathrm{d}})^\top \beta + \underline{p}^\top \gamma^- - \overline{p}^\top \gamma^+ + \underline{q}^\top \delta^- - \overline{q}^\top \delta^+ + \underline{v}^\top \mathrm{diag}(\phi^-)\underline{v} - \overline{v}^\top \mathrm{diag}(\phi^+)\overline{v} & \\ \text{subject to} & \alpha, \beta \in \mathbb{R}^N,\ \gamma^-, \gamma^+, \delta^-, \delta^+, \phi^-, \phi^+ \in \mathbb{R}_+^N & \\ & c - \alpha - \gamma^- + \gamma^+ = 0 & \\ & \beta + \delta^- - \delta^+ = 0 & (\mathrm{D}) \\ & \begin{bmatrix} \mathrm{diag}(\alpha)G + G\mathrm{diag}(\alpha) & -\mathrm{diag}(\alpha)B + B\mathrm{diag}(\alpha) \\ \mathrm{diag}(\alpha)B - B\mathrm{diag}(\alpha) & \mathrm{diag}(\alpha)G + G\mathrm{diag}(\alpha) \end{bmatrix} & \\ & + \begin{bmatrix} -\mathrm{diag}(\beta)B - B\mathrm{diag}(\beta) & -\mathrm{diag}(\beta)G + G\mathrm{diag}(\beta) \\ \mathrm{diag}(\beta)G - G\mathrm{diag}(\beta) & -\mathrm{diag}(\beta)B - B\mathrm{diag}(\beta) \end{bmatrix} & \\ & + 2\begin{bmatrix} \mathrm{diag}(\phi^+ - \phi^-) & 0 \\ 0 & \mathrm{diag}(\phi^+ - \phi^-) \end{bmatrix} \succeq 0 & \end{array}$$

3.2 Implement the Lagrangian dual (D) using YALMIP and solve it with MOSEK first for the IEEE 9-bus system and then for the IEEE 118-bus system. A skeleton of the code is provided in the Matlab file `p4q3.m`. For both systems, report the optimal objective value $\max(\mathrm{D}) =: L_3$, the time $t_{\mathrm{D}}^{\mathrm{build}}$ required to build the model (i.e., `yalmiptime`), and the time $t_{\mathrm{D}}^{\mathrm{solve}}$ required to solve the model (i.e., `solvertime`).

4. **Evaluation and Comparison**: Complete the table below with the results that you have obtained. Briefly comment on how the different methods compare to each other. Which method yields the tightest lower bound (and thus the best suboptimality bound)? Which method is most convenient in terms of runtime? Which method strikes the best trade-off in your opinion? Also, feel free to comment on any further, particular insights that you had.

IEEE	Heuristic	SDP Relaxation			SOCP Relaxation			Dualization		
System	$\tilde{P}$ [\$]	L_1 [\$]	$t_{\mathrm{SDP}}^{\mathrm{build}}$ [s]	$t_{\mathrm{SDP}}^{\mathrm{solve}}$ [s]	L_2 [\$]	$t_{\mathrm{SOCP}}^{\mathrm{build}}$ [s]	$t_{\mathrm{SOCP}}^{\mathrm{solve}}$ [s]	L_3 [\$]	$t_{\mathrm{D}}^{\mathrm{build}}$ [s]	$t_{\mathrm{D}}^{\mathrm{solve}}$ [s]
9-bus	373.83									
118-bus	86'300									

Hints

1. If you run the solver using the command `diagnosis = optimize(con, obj, opt_set)` you can retrieve `yalmiptime` and `solvertime` via `diagnosis.yalmiptime` and `diagnosis.solvertime`.

2. Deriving the Lagrangian dual of problem (P) can be challenging. The closest reference that we discussed so far is the two-way partitioning problem of Exercise 2 in Tutorial 5. You may want to have a look at the corresponding solution for inspiration. The key idea when taking the infimum over e and f is to rewrite the bilinear terms of the Lagrangian as a quadratic form in the joint vector $\binom{e}{f} \in \mathbb{R}^{2N}$, *i.e.*, to find a (not necessarily symmetric) matrix $Q \in \mathbb{R}^{2N \times 2N}$ such that the bilinear terms of the Lagrangian amount to $\binom{e}{f}^\top Q \binom{e}{f}$.

3. Once such a matrix Q has been found, without loss of generality one can and should replace it with its symmetric counterpart $Q_{\mathrm{sym}} = \frac{1}{2}(Q + Q^\top)$. The equivalence holds because

$$v^\top Q_{\mathrm{sym}} v = v^\top (\tfrac{1}{2}(Q + Q^\top))v = \tfrac{1}{2}(v^\top Q v + v^\top Q^\top v) = \tfrac{1}{2}(v^\top Q v + v^\top Q v) = v^\top Q v \quad \forall v \in \mathbb{R}^{2N}.$$

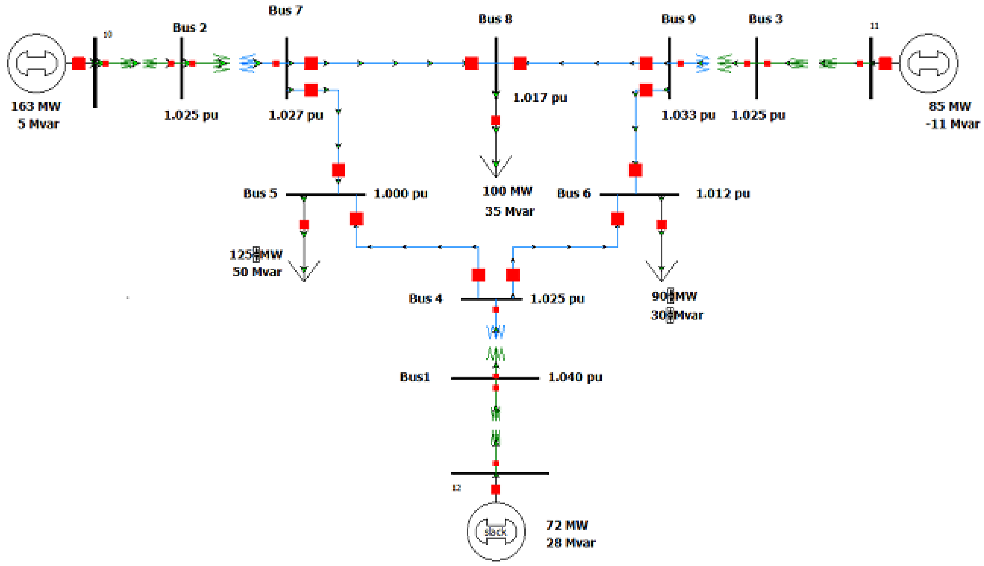

Figure 1: Schematic representation of the IEEE 9-bus system.

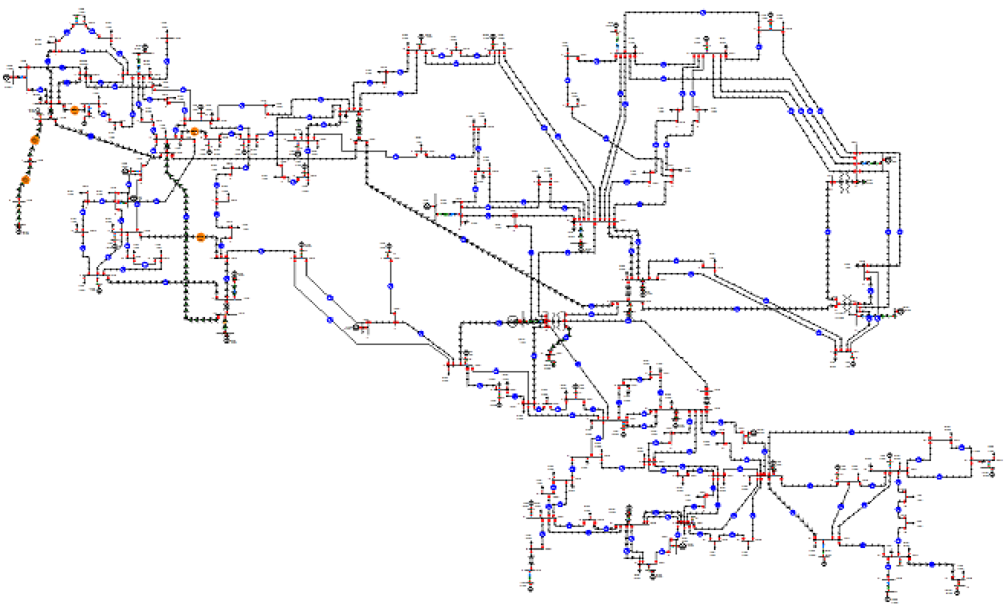

Figure 2: Schematic representation of the IEEE 118-bus system.