Multivariable Control (ME-422) - Exercise session 2

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1. **Exercise on modal analysis**

   Consider the system $x^+ = Ax$ where

   To find v_i, check for the eigenspace associated to $\lambda = 1$.

   $$\begin{pmatrix} -1.5 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1.5 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \vec{0} \Rightarrow \begin{array}{l} -1.5v_1 = 0 \\ -v_2 = 0 \\ -1.5v_3 + 2\underbrace{v_4}_{\alpha} = 0 \Rightarrow v_3 = \frac{4}{3}\alpha \\ v_4 = \alpha \end{array}$$

   check for v_i s.t. $(A - \lambda I)v_i = 0$

   $$\Rightarrow v = \begin{pmatrix} 0 \\ 0 \\ 4/3 \\ 1 \end{pmatrix} \alpha, \alpha \in \mathbb{R}$$

   $$A = \begin{bmatrix} -0.5 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -0.5 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

   Block diag. matrix $\Rightarrow$ eigenvalues are those of the submatrices:

   $$\left| \begin{bmatrix} -0.5-\lambda & 0 \\ -1 & 1-\lambda \end{bmatrix} \right| = (-0.5-\lambda)(1-\lambda) = 0$$

   $\lambda_1 = -0.5$ $\quad \lambda_3 = -1/2$

   $\lambda_2 = 1$ $\quad \lambda_4 = 1$

   So here, we have two $\lambda_i = 1$. Hence, need to check whether $n_i = \nu_i$ (in which bounded). $n_i = 2 (2\lambda_i = 1)$.

   Tell if $\Rightarrow \nu_i = 2 = n_i \Rightarrow$ all modes associated to $\lambda = 1$ are bounded

   (a) all modes go to zero as $k \to \infty$

   (b) all modes are bounded

   (c) there is at least an unbounded mode.

   Hint: Note that A is block diagonal and its eigenvalues are those of each individual block.

2. **Complex conjugate eigenvalues generate real modes**

   At first sight, it might seem strange that complex conjugate eigenvalues give rise to modes taking values in $\mathbb{R}$ instead of $\mathbb{C}$. This exercise provides some intuition on why this happens.

   Consider the system

   (a) $\hat{x} = Tx \Rightarrow x = T^{-1}\hat{x} \Rightarrow T^{-1}\hat{x}^+ = AT^{-1}\hat{x} \Rightarrow \hat{x}^+ = \overbrace{TAT^{-1}}^{\hat{A}}\hat{x}$

   $x^+ = T^{-1}\hat{x}^+$

   $$\begin{cases} x^+ = Ax \\ y = Cx \end{cases}$$

   $\leftarrow$ no state affected by $\vec{u} \Rightarrow$ FREE states

   $$x(0) = x_0.$$

   (a) Let $A \in \mathbb{R}^{n \times n}$ and $T \in \mathbb{C}^{n \times n}$ be nonsingular matrices.
   Define $\hat{x}(k) = Tx(k) \in \mathbb{C}^n$. Show that $\hat{x}(k)$ follows the dynamics $\hat{x}^+ = \hat{A}\hat{x}$ with $\hat{A} = TAT^{-1}$.

   (b) Consider the case

   $x(1) = Ax_0$, $x(2) = Ax(1) = A^2x_0$, to compute A^k, spectral decomp. of A.

   $|A - \lambda I| = (1-\lambda)(1-\lambda) + 1 = \lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda = 1 \pm j = \sqrt{2}e^{\pm j\pi/4}$

   $$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

   Hence, we get that $\boxed{x(k) = A^k x_0 = T\hat{A}^k T^{-1} x_0}$

   since $A^k = (T\hat{A}T^{-1})(T\hat{A}T^{-1})\ldots(T\hat{A}T^{-1}) = T\hat{A}^k T^{-1}$

   where the eigenvalues of A are $1 \pm j = \sqrt{2}e^{\pm j\frac{\pi}{4}}$ and

   $$A^k = T \begin{bmatrix} (\sqrt{2})^k e^{+kj\pi/4} & 0 \\ 0 & (\sqrt{2})^k e^{-kj\pi/4} \end{bmatrix} T^{-1}$$

   $$\hat{A} = TAT^{-1} = \begin{bmatrix} 1+j & 0 \\ 0 & 1-j \end{bmatrix} \text{ for } T^{-1} = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}.$$

   This shows how the modes are related to the spectral decomposition of A.

   Compute explicitly $x(k)$ and $y(k)$ as a function of x_0.

   For $y(k)$: $y(0) = Cx_0$, $y(1) = Cx(1) = CA\underbrace{x(0)}_{x_0}$

   $\Rightarrow \boxed{y(k) = CA^k x_0 = CT\hat{A}^k T^{-1} x_0} \Rightarrow y(k)$ also depends on the modes.

3. **Satellite simulation**

   Consider the satellite in the figure. The satellite mission requires accurate pointing of a scientific sensor package with respect to a fixed star.

   - θ_2 is the angle deviation of the instrument package from the reference star. It can be measured through devices called "star trackers."

- θ_1 is the angle deviation of the satellite body

By using cold-gas jets, one can produce a torque T_c acting on the satellite body.

The figure also shows an equivalent mechanical model of the satellite. This system is composed by two rotating masses connected by a spring with torque constant κ and viscous-damping constant b.
A discrete-time model of the system with state $x = \begin{bmatrix}\theta_2 & \dot{\theta}_2 & \theta_1 & \dot{\theta}_1\end{bmatrix}$, output $y = \theta_2$, and input $u = T_c$ can be written as

$$\begin{cases} x^+ = \begin{bmatrix} 1 & T & 0 & 0 \\ -T\frac{\kappa}{J_2} & 1 - T\frac{b}{J_2} & T\frac{\kappa}{J_2} & T\frac{b}{J_2} \\ 0 & 0 & 1 & T \\ T\frac{\kappa}{J_1} & T\frac{b}{J_1} & -T\frac{\kappa}{J_1} & 1 - T\frac{b}{J_1} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ T\frac{1}{J_1} \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x, \end{cases}$$

where $T > 0$ is the sampling time and J_1 and J_2 are the inertias of the masses. Set $\kappa = 0.091$, $b = 0.0036$, $J_1 = 1$, $J_2 = 0.1$, and $T = 0.01$.

(a) Check if the system has unbounded modes. Use MATLAB for computations.

(b) Draw the free output between 0 and $200s.$, starting from the initial condition $x = \begin{bmatrix}0 & 0 & 0.1 & 0\end{bmatrix}^T$. Give a physical interpretation to the result.

(c) Draw the impulse response (that is, the output generated by the input $u(0) = 1,\ u(k) = 0,\ k > 0$) and give an interpretation of the results in terms of the satellite motion.

(d) Now consider the output as $y = x_2$. Draw the impulse response and, through the physical interpretation of the results, explain why its asymptotic value is consistent with the graph obtained in the previous point.

Useful MATLAB commands: `eig`, `null`, `ss(A,B,C,D,T)`, `lsim`, `impulse`.