CS2040S
Data Structures and Algorithms

Hashing!
(Part 1)

Plan: this week and next

Three (or Four) Days of Hashing

- Applications
- Basic theory
- Handling collisions
- (Hashing in Java)
- Amortized analysis (doubling/shrinking)
- Sets and Bloom filters

Topic of the Week: Hash Tables

Abstract Data Types

Symbol Table

```
public interface  SymbolTable
```

`void`	`insert(Key k, Value v)`	*insert (k,v) into table*
`Value`	`search(Key k)`	*get value paired with k*
`void`	`delete(Key k)`	*remove key k (and value)*
`boolean`	`contains(Key k)`	*is there a value for k?*
`int`	`size()`	*number of (k,v) pairs*

Note: no successor / predecessor queries.

Symbol Table

Examples:

Dictionary:	key = word
	value = definition
Phone Book	key = name
	value = phone number
Internet DNS	key = website URL
	value = IP address
Java compiler	key = variable name
	value = type and value

Implement symbol table with an AVL tree:
(C_I= cost insert, C_S=cost search)

1. $C_I = O(1)$, $C_S = O(1)$
2. $C_I = O(1)$, $C_S = O(\log n)$
3. $C_I = O(1)$, $C_S = O(n)$
4. ✓ $C_I = O(\log n)$, $C_S = O(\log n)$
5. $C_I = O(n)$, $C_S = O(\log n)$
6. $C_I = O(n)$, $C_S = O(n)$

Symbol Table

Implement a symbol table with:

- $C_I = O(1)$
- $C_S = O(1)$

Fast, fast, fast….

Dictionaries vs. Symbol Tables

What can you do with a dictionary but not a symbol table?

Dictionaries vs. Symbol Tables

Sorting with a dictionary:

1) Insert every item into the dictionary.
2) Search for the minimum item.
3) Repeat: find successor

Running time to implement sorting:

With an AVL tree/dictionary?

Dictionaries vs. Symbol Tables

Sorting with a dictionary:

1) Insert every item into the dictionary.
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Running time to implement sorting:

With an AVL tree/dictionary? O(n log n)

With a symbol table?

Dictionaries vs. Symbol Tables

Sorting with a dictionary:

1) Insert every item into the dictionary.

2) Search for the minimum item.

3) Repeat: find successor

Running time to implement sorting:

With an AVL tree/dictionary? $O(n \log n)$

With a symbol table? $O(n^2)$

- No efficient way to find minimum item!
- No ordering of elements.

Sorting (aside)

Isn't O(1) search/insert impossible?

- Sorting takes $\Omega(n \log n)$ comparisons.

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- How do you sort with a symbol table?
- Only search/insert/delete.

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- How do you sort with a symbol table?
- Only search/insert/delete.

(Binary) search takes $\Omega(\log n)$ comparisons.

- Impossible to search in fewer than $\log(n)$ comparisons.
- But a symbol table finds an item in O(1) steps!!
- Conclusion: symbol table is not ***comparison-based***.

Building a Symbol Table

Direct Access Tables

Attempt #1: Use a table, indexed by keys.

0	null
1	null
2	item1
3	null
4	null
5	item3
6	null
7	null
8	item2
9	null

Universe U={0..9} of size $m = 10$.

<u>(key, value)</u>

(2, item1)
(8, item2)
(5, item3)

Assume keys are distinct.

Direct Access Tables

Attempt #1: Use a table, indexed by keys.

0	null
1	null
2	item1
3	null
4	null
5	item3
6	null
7	null
8	item2
9	null

Example: insert(4, Seth)

Direct Access Tables

Attempt #1: Use a table, indexed by keys.

0	null
1	null
2	item1
3	null
4	Seth
5	item3
6	null
7	null
8	item2
9	null

Example: insert(4, Seth)

Time: O(1) / insert, O(1) / search

Direct Access Tables

Problems:

- What if keys are not integers?
  - Where do you put the key/value "**(hippopotamus, bob)**"?
  - Where do you put 3.14159…?

Direct Access Tables

Pythagoras said, “Everything is a number.”

“The School of Athens” by Raphael

Direct Access Tables

Pythagoras said, “Everything is a number.”

- Everything is just a sequence of bits.
- Treat those bits as a number.

- English:
  - 26 letters => 5 bits/letter
  - Longest word = 28 letters (antidisestablishmentarianism?)
  - 28 letters * 5 bits = 140 bits
  - So we can store any English word in a direct-access array of size 2^{140}.

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  - 28 letters * 5 bits = 140 bits
  - So we can store any English word in a direct-access array of size 2^{140}. ≈ number of atoms in observable universe

Direct Access Tables

Problems:

– What if keys are not integers?

- Where do you put the key/value “**(hippopotamus, bob)**”?
- Where do you put 3.14159…?

➔ Can represent anything as a sequence of bits.

– Too much space

- If keys are integers, then table-size > 4 billion

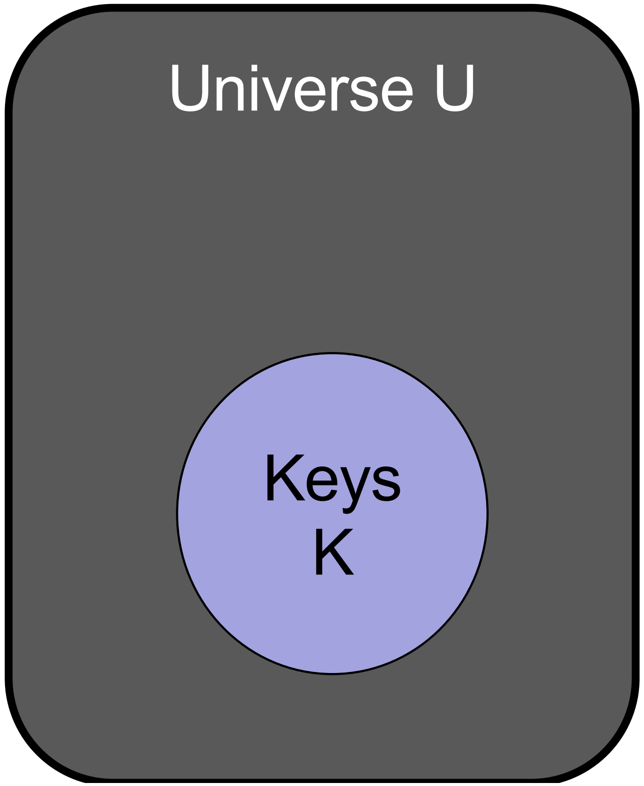
➔ Hashing

Hash Functions

Problem:

e.g., 2^{140}

- Huge universe U of possible keys.
- Smaller number n of actual keys.



Hash Functions

Problem:

- Huge universe U of possible keys.
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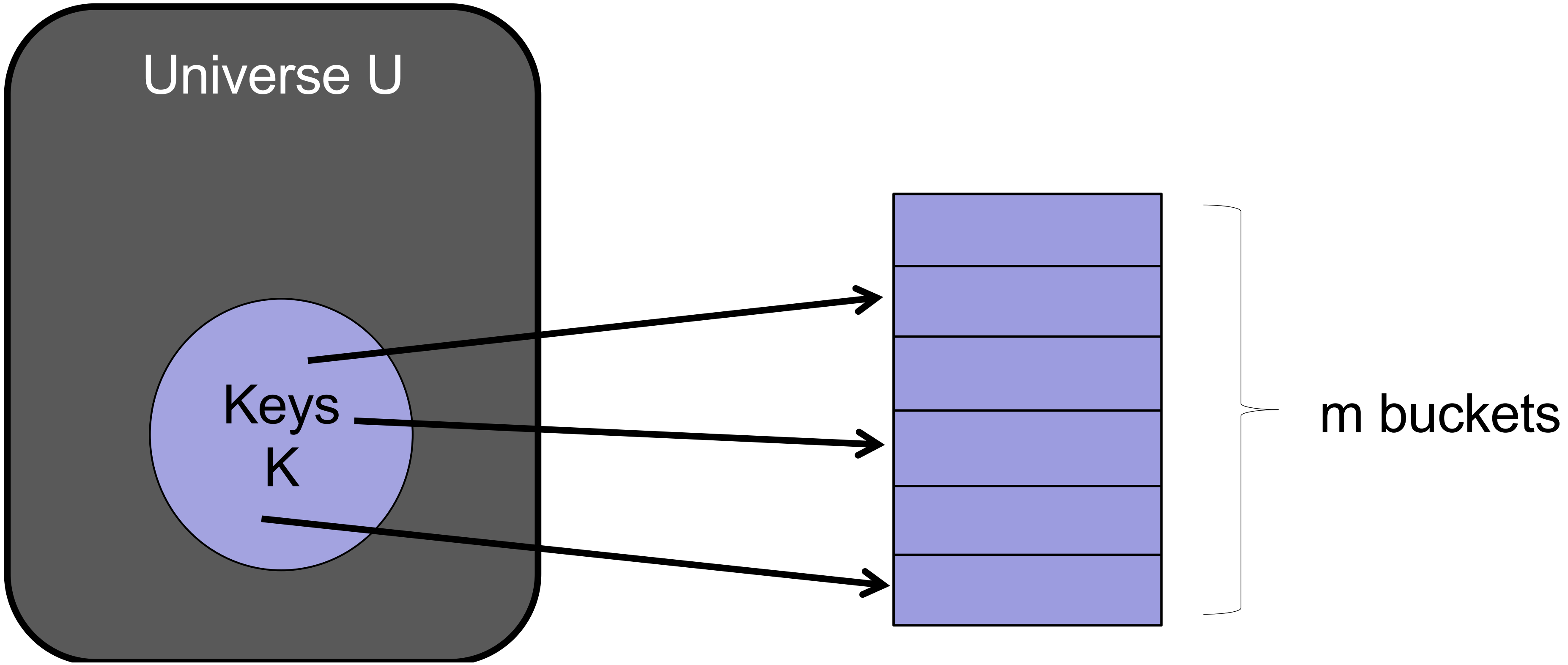

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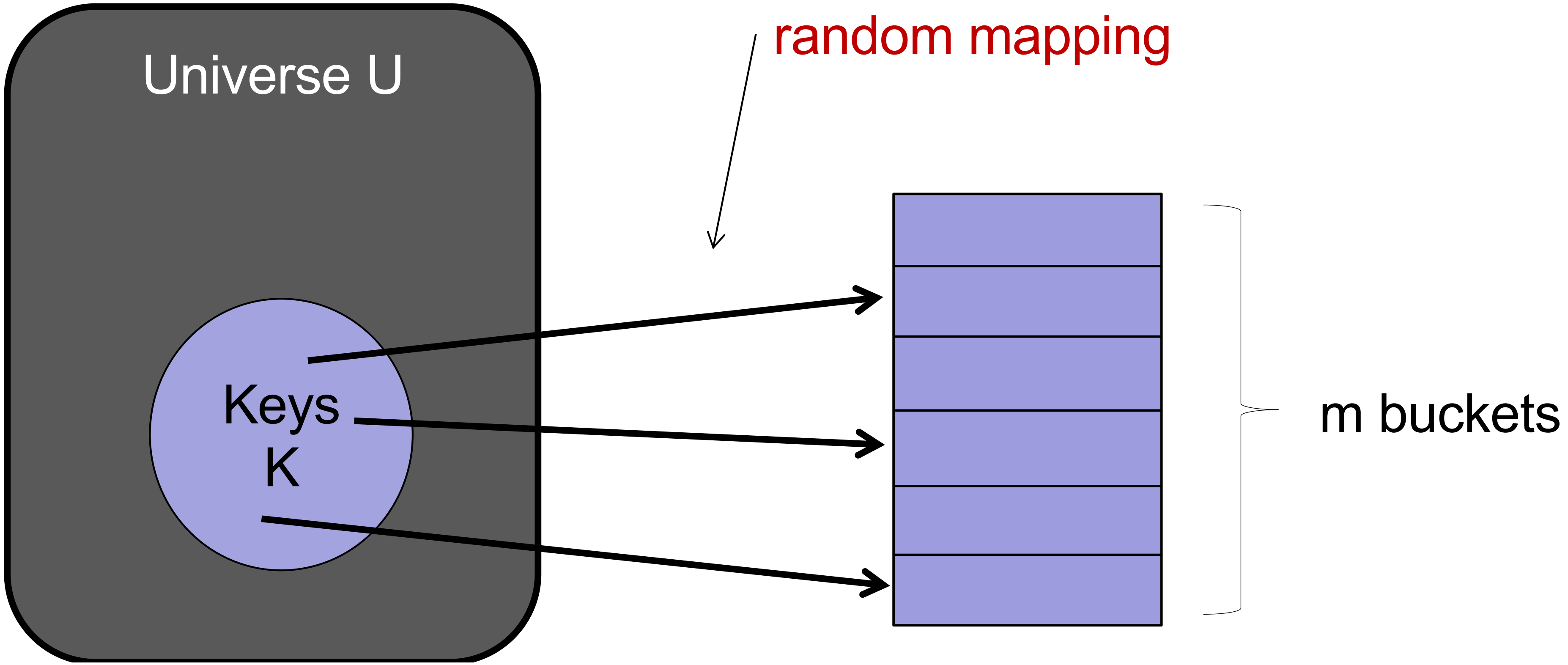
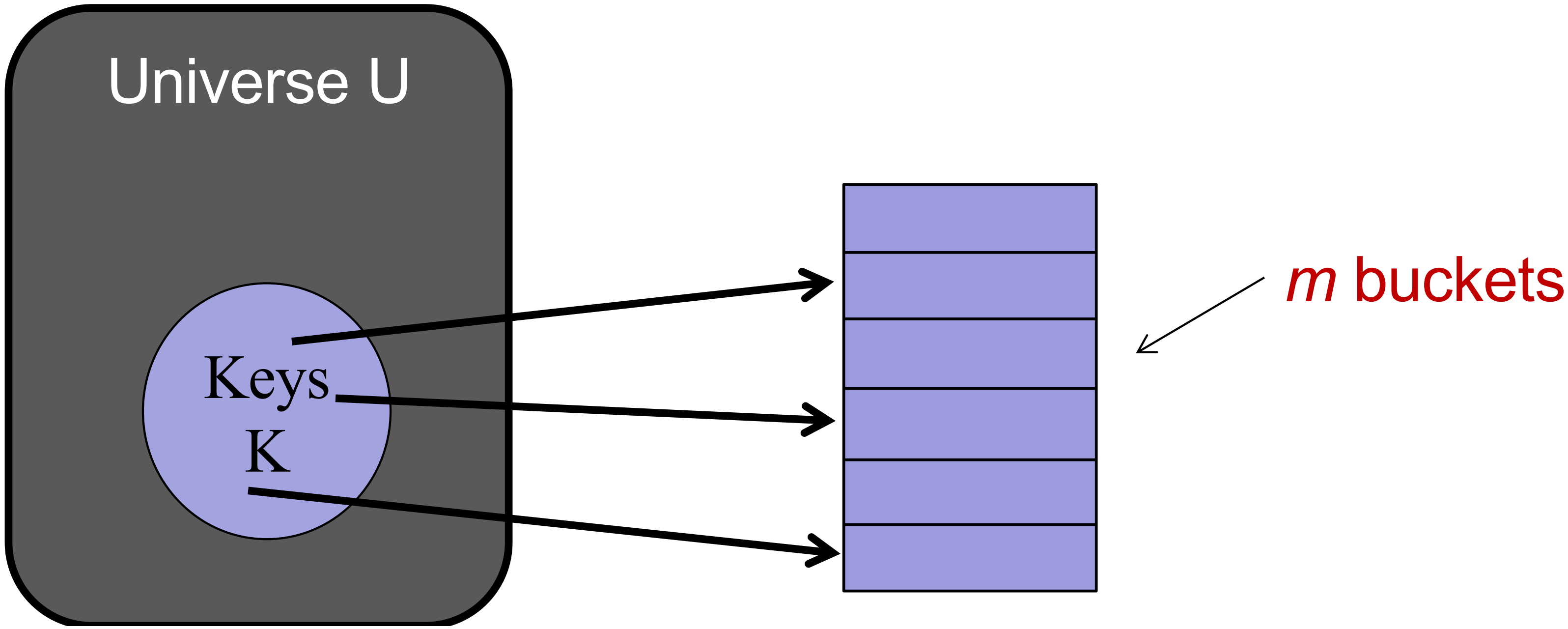


Hash Functions

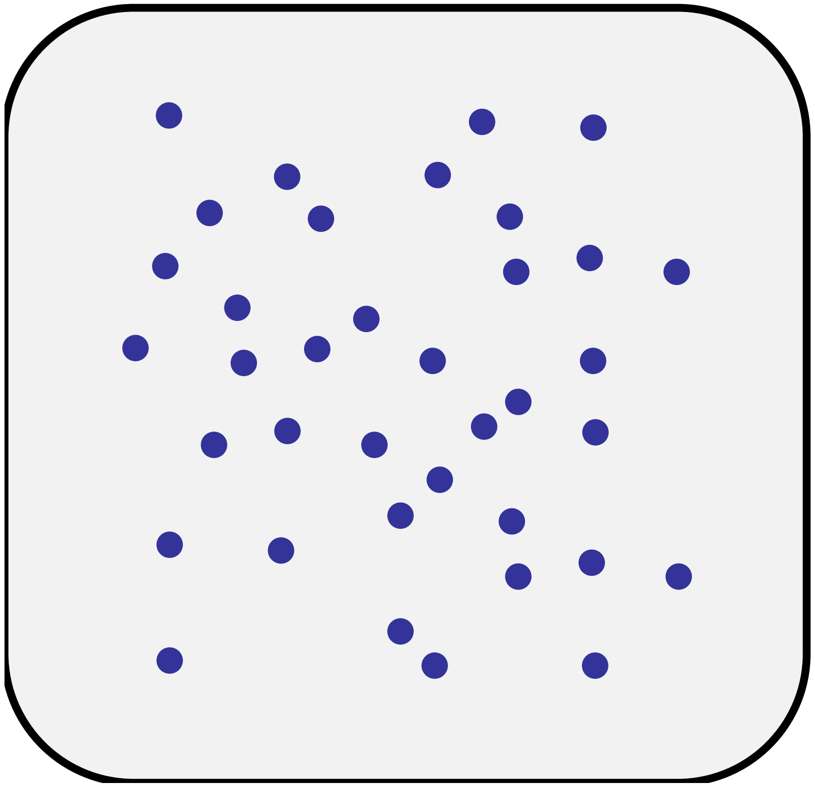
Define hash function $h : U \rightarrow \{1..m\}$

- Store key k in bucket $h(k)$.
- Time complexity:
  - Time to compute h + Time to access bucket
- Usually: assume hash function has cost 1 to compute.

unless otherwise specified, e.g., long strings.



Hash Functions



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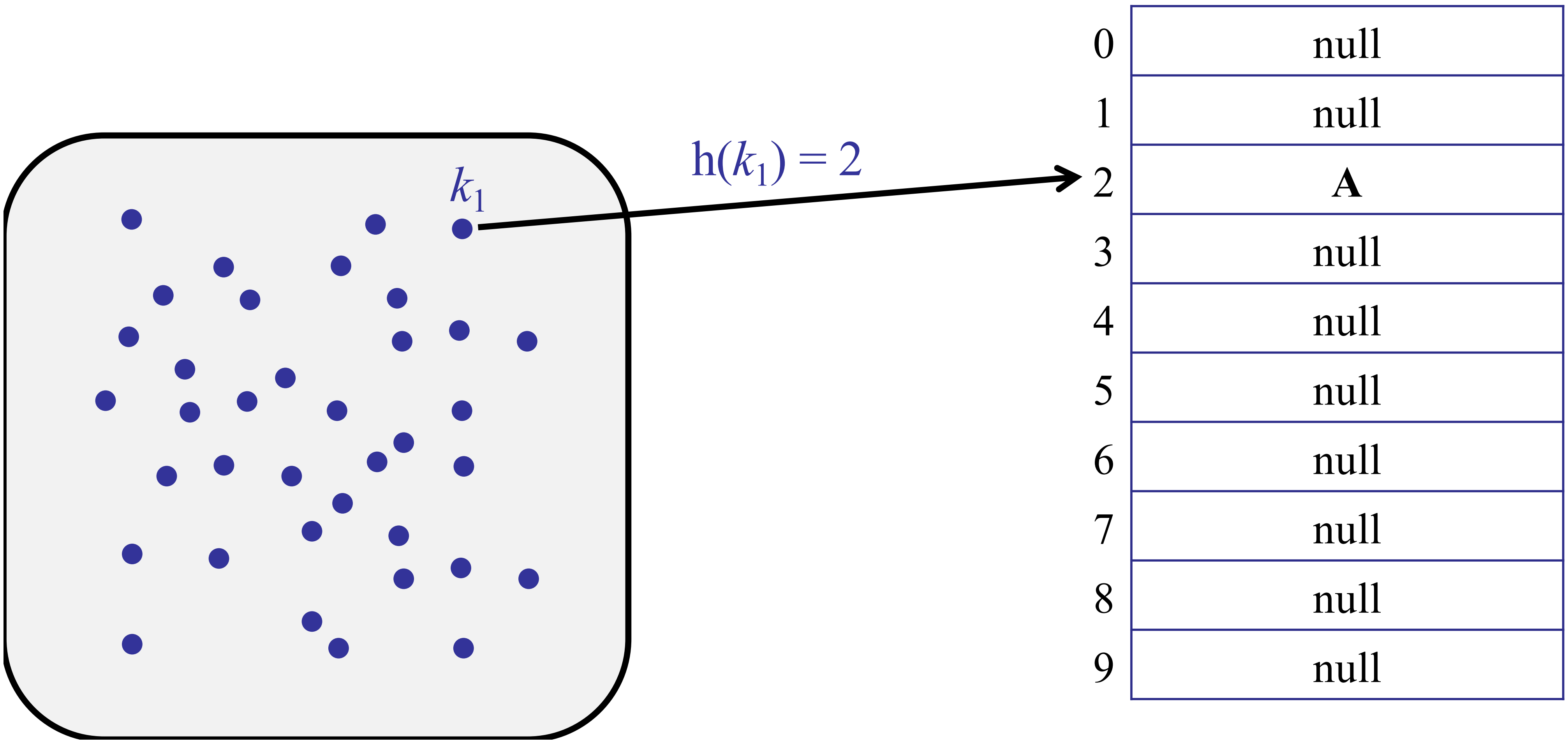
Hash Functions

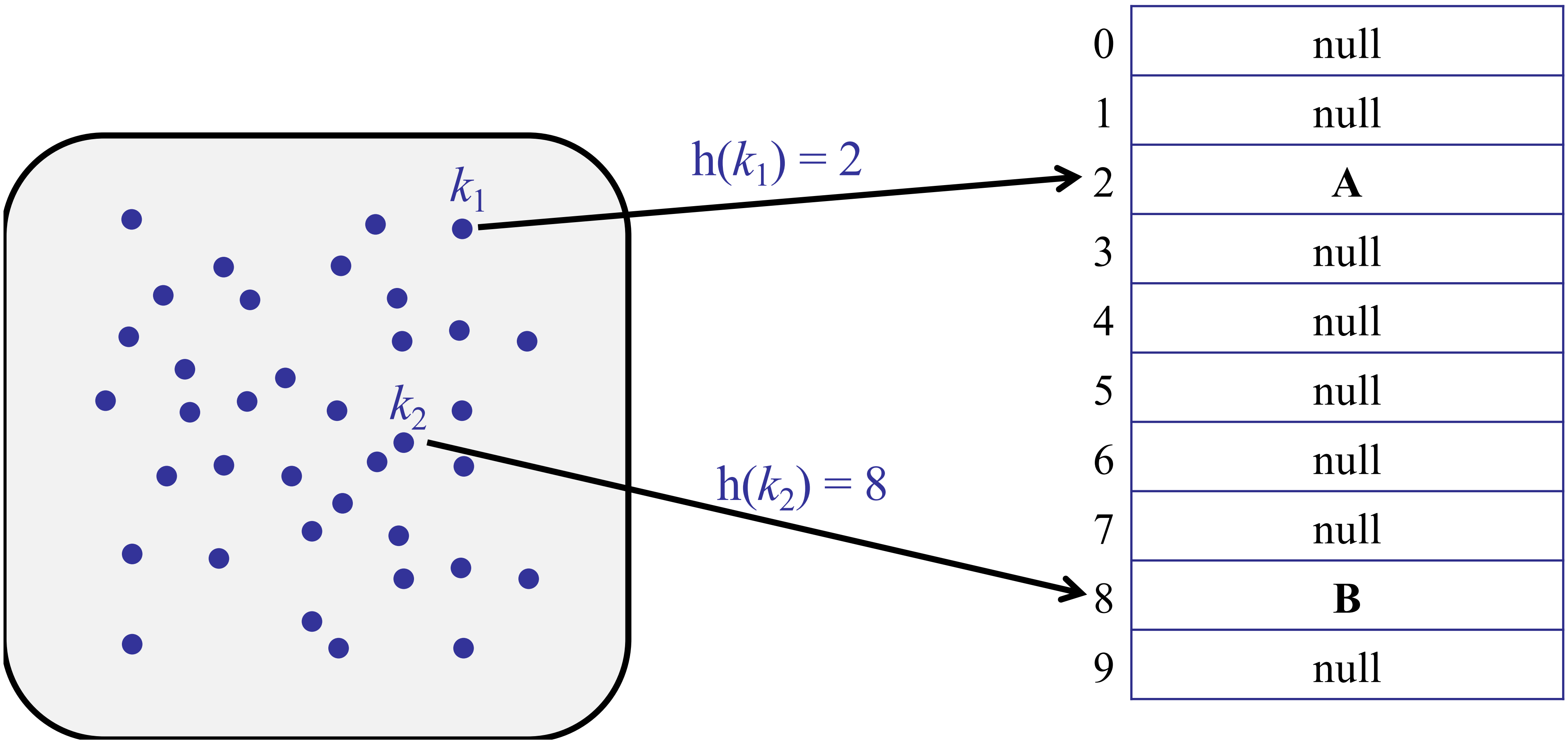
insert(k_1, A)



Hash Functions

insert(k_1, A)

insert(k_2, B)



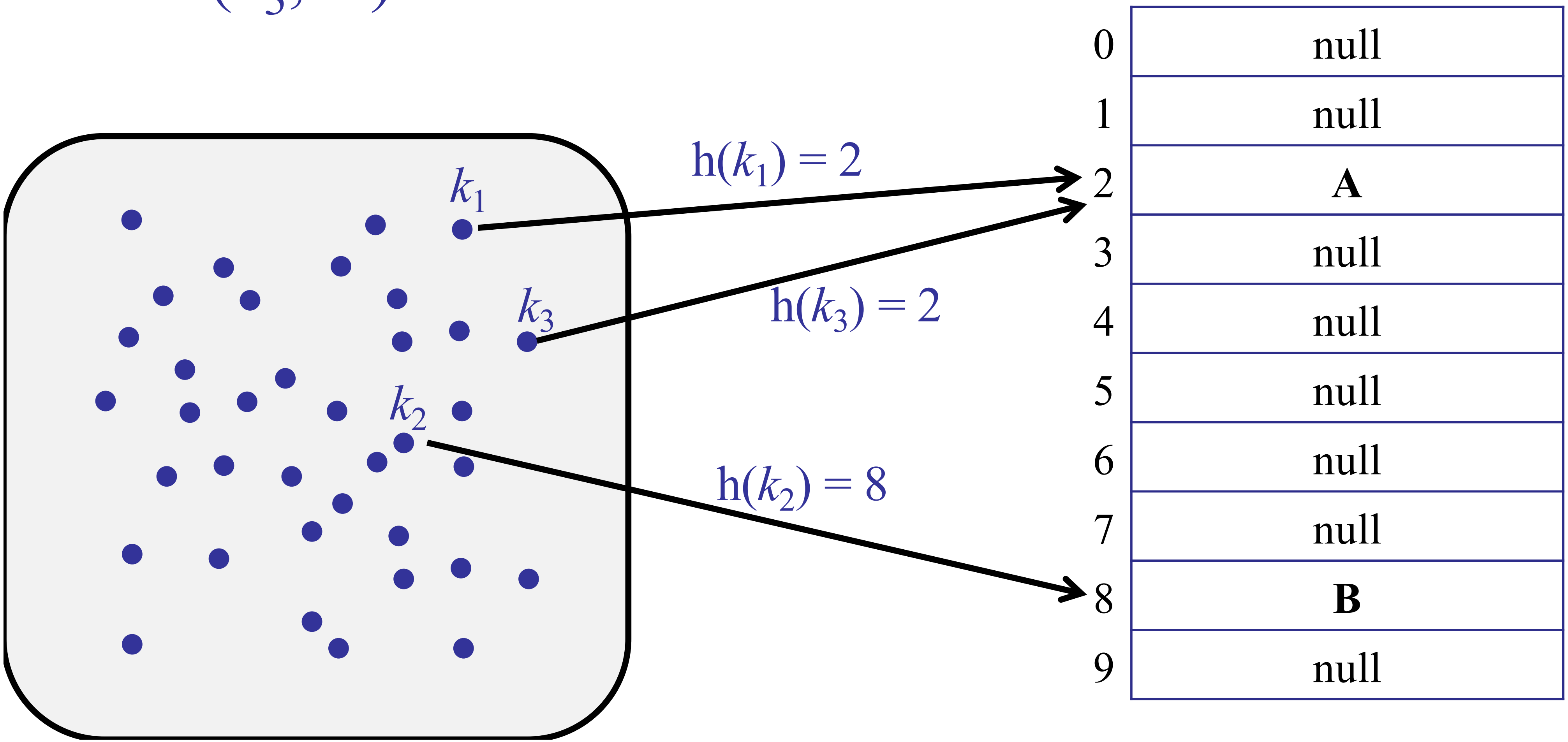
Hash Functions

insert(k_1, A)

insert(k_2, B)

insert(k_3, C)

Collision!



Hash Functions

Collisions:

- We say that two distinct keys k_1 and k_2 collide if:

$$h(k_1) = h(k_2)$$

Can we choose a hash function with no collisions?

1. Yes
2. Sometimes, if we choose carefully
3. No, impossible

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Hash Functions

Collisions:

- We say that two distinct keys k_1 and k_2 collide if:

$$h(k_1) = h(k_2)$$

- Unavoidable!
  - The table size is smaller than the universe size.
  - The pigeonhole principle says:
    - There must exist two keys that map to the same bucket.
    - Some keys must collide!

Coping with Collision

Idea: choose a new, better hash functions

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- Hard to find.
- Requires re-copying the table.
- Eventually, there will be another collision.

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Idea: chaining (today)

- Put both items in the same bucket!

Idea: open addressing (next week)

- Find another bucket for the new item.

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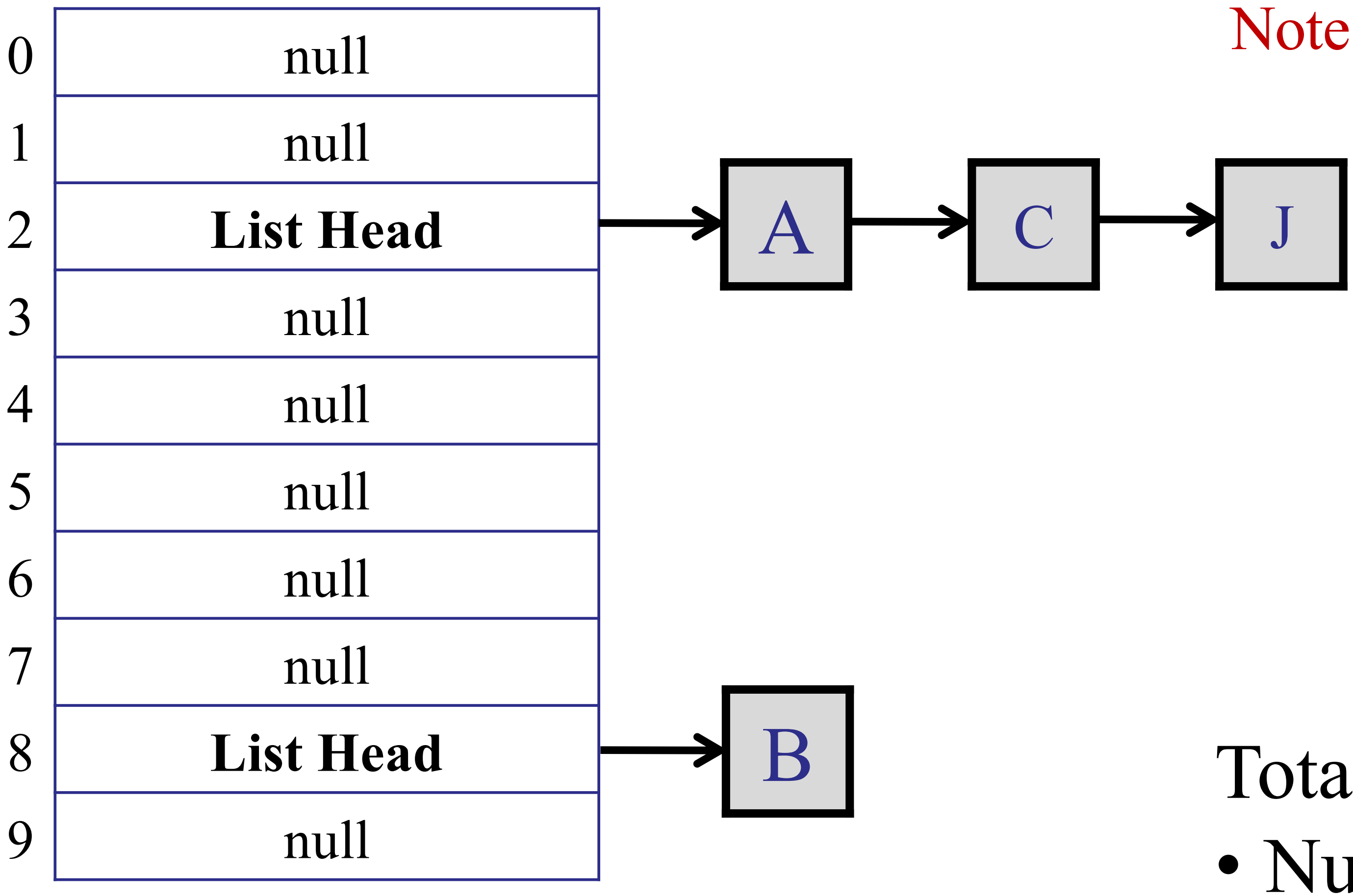
Chaining

Each bucket contains a linked list of items.



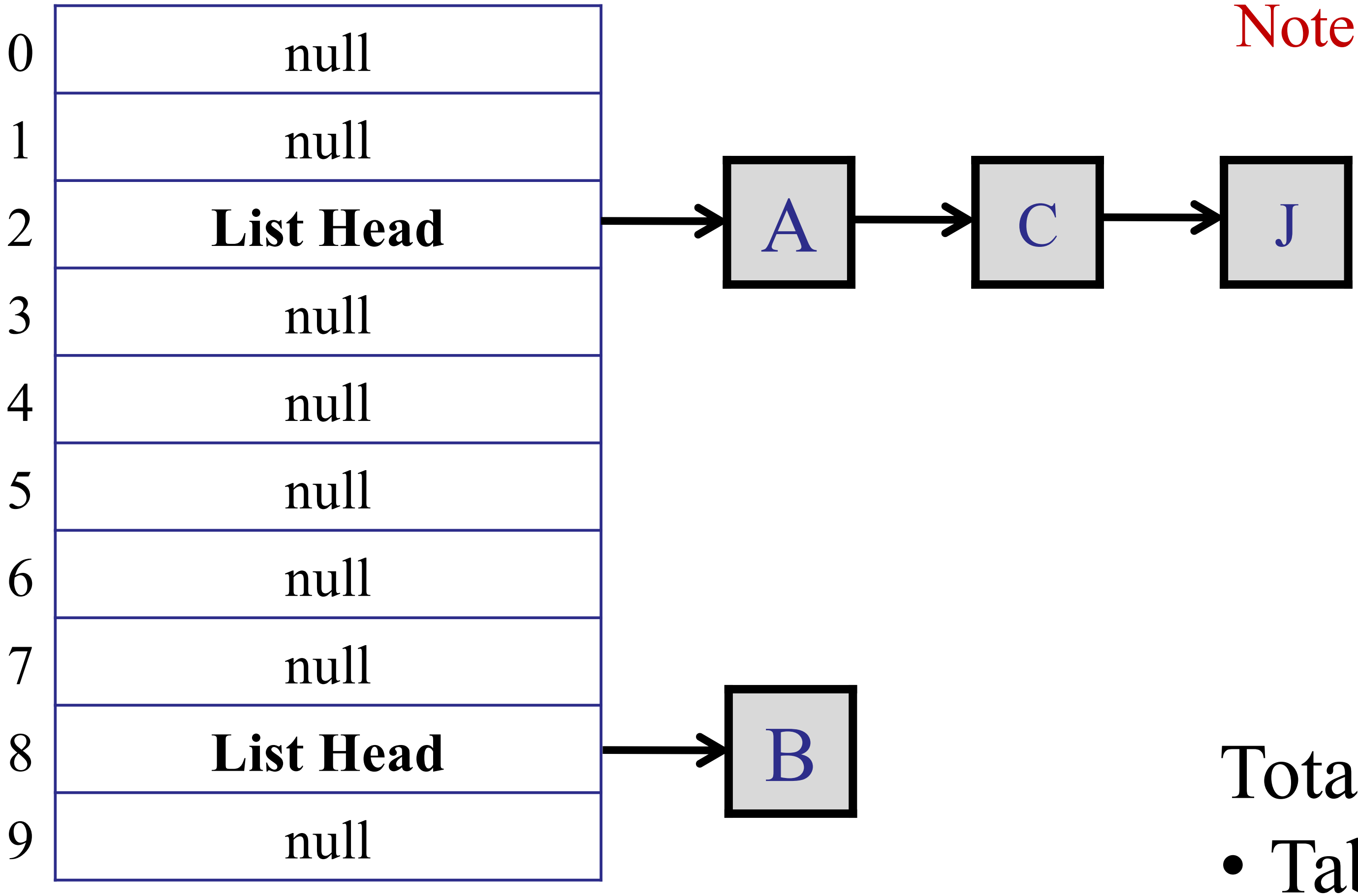
Note: h(A) == h(C) == h(J)

Total space:

- Number buckets: m
- Number entries: n

Chaining

Each bucket contains a linked list of items.



Note: h(A) == h(C) == h(J)

Total space: $O(m + n)$

- Table size: m
- Linked list size: n

Hashing with Chaining

Operations:

- insert(key, value)
  - Calculate h(key)
  - Lookup h(key) and add (key,value) to the linked list.

- search(key)
  - Calculate h(key)
  - Search for (key,value) in the linked list.

What is the worst-case cost of inserting a (key, value)? Assume cost(h) is cost of computing the hash function.

1. $O(1 + \text{cost}(h))$
2. $O(\log n + \text{cost}(h))$
3. $O(n + \text{cost}(h))$
4. $O(n\ \text{cost}(h))$
5. $O(n^2)$.

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Do we care about duplicates?

➔ If so, the cost of insert is higher because we need to search for duplicates.

Hashing with Chaining

Operations:

- insert(key, value)
  - Calculate h(key)
  - Lookup h(key) and add (key,value) to the linked list.

  (Note: this allows duplicate keys. Need to specify more precisely the behavior or insert!)

- search(key)
  - Calculate h(key)
  - Search for (key,value) in the linked list.

Always O(1).

What is the worst-case cost of searching a (key, value)?

1. $O(1 + \text{cost}(h))$
2. $O(\log n + \text{cost}(h))$
3. $O(n + \text{cost}(h))$
4. $O(n * \text{cost}(h))$
5. We cannot determine it without knowing h.

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Hashing with Chaining

Operations:

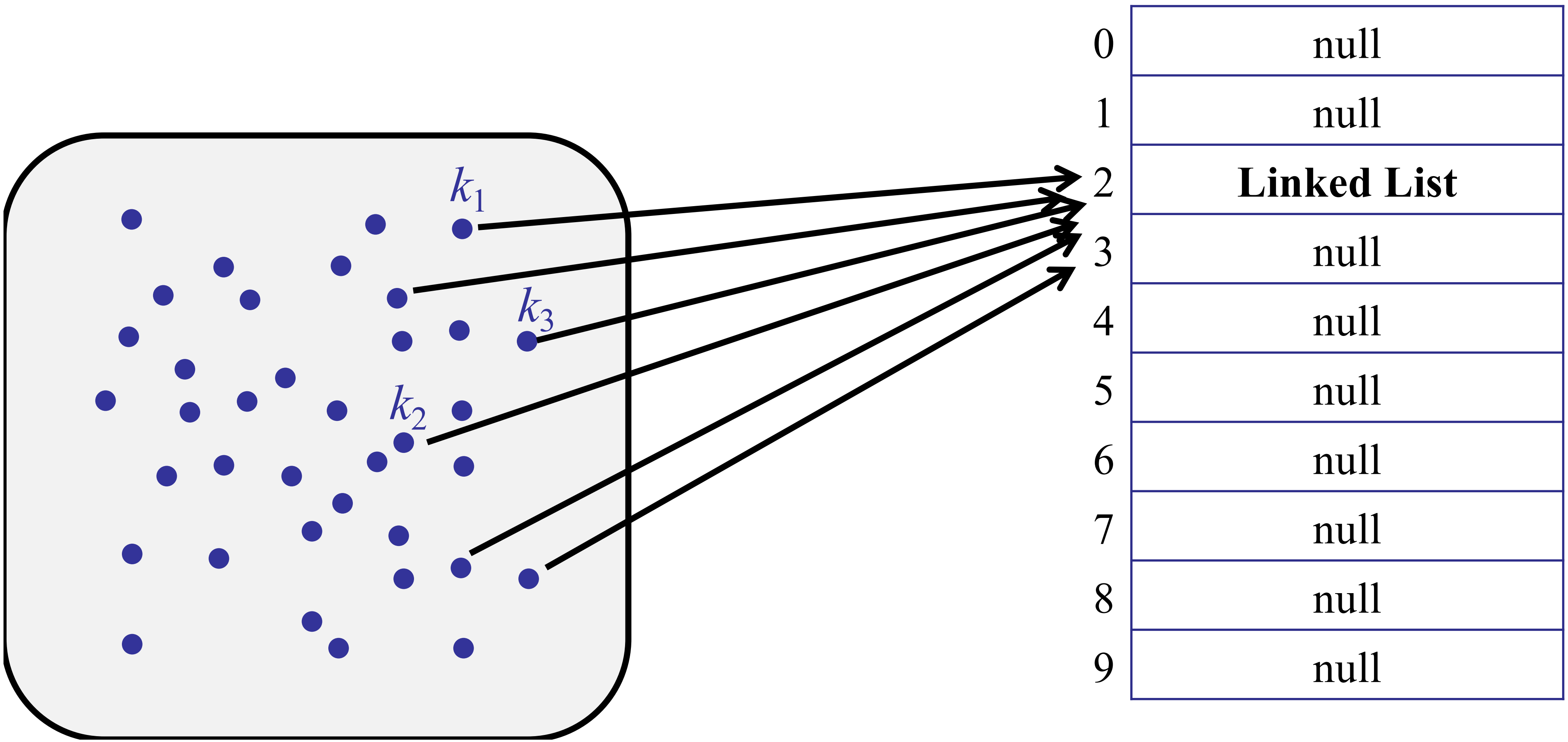
- insert(key, value)
  - Calculate h(key)
  - Lookup h(key) and add (key,value) to the linked list.

- search(key) → time depends on length of linked list
  - Calculate h(key)
  - Search for (key,value) in the linked list.

Hashing with Chaining

Assume all keys hash to the same bucket!

- Search costs O(n)
- Oh no!



Let's be optimistic today.

The Simple Uniform Hashing Assumption

- Every key is equally likely to map to every bucket.
- Keys are mapped independently.

Intuition:

- Each key is put in a random bucket.
- Then, as long as there are enough buckets, we won't get too many keys in any one bucket.

Why don’t we just insert each key into a random bucket (instead of using a hash function h)?

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Why don't we just insert each key into a random bucket (instead of using hash function h)?

1. It would be slow to insert.
2. Computers don't have a real source of randomness.
3. By choosing the keys carefully, a user could force the random choices to create many collisions.
4. Searching would be very slow.
5. None of the above.

Let's be optimistic today.

The Simple Uniform Hashing Assumption

- Assume:
  - n items
  - m buckets
- Define: load(hash table) = n/m

  = average # items / bucket.

- Expected search time = 1 + *expected # items per bucket*

hash function + array access

linked list traversal

Probability Theory

Set of outcomes for $X = (e_1, e_2, e_3, \ldots, e_k)$:

- $\Pr(e_1) = p_1$
- $\Pr(e_2) = p_2$
- …
- $\Pr(e_k) = p_k$

Expected outcome:

$$E[X] = e_1p_1 + e_2p_2 + \ldots + e_kp_k$$

Probability Theory

Linearity of Expectation:

- $E[A + B] = E[A] + E[B]$

Example:

- A = # heads in 2 coin flips
- B = # heads in 2 coin flips
- $A + B$ = # heads in 4 coin flips

$E[A+B] = E[A] + E[B] = 1 + 1 = 2$

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  - m buckets
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- Expected search time = 1 + *expected # items per bucket*

hash function + array access

linked list traversal

A little more probability

Indicator random variables

$$X(i, j) = 1 \quad \text{if item } i \text{ is put in bucket } j$$
$$= 0 \quad \text{otherwise}$$

Pr(X(i, j) == 1) = ?

1. 1/m
2. 1/n
3. 1/(m+n)
4. m/n
5. n/m
6. log(n)

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A little probability

Indicator random variables

$$X(i, j) = 1 \quad \text{if item } i \text{ is put in bucket } j$$
$$= 0 \quad \text{otherwise}$$

$$\mathbf{Pr}(X(i, j) == 1) = 1/m$$

A little probability

Indicator random variables

$X(i, j) = 1$ if item i is put in bucket j

$\quad\quad\quad = 0$ otherwise

Pr$(X(i, j) == 1) = 1/m$

E$(X(i, j)) =$ **??**

A little probability

Indicator random variables

$X(i, j) = 1$ if item i is put in bucket j

$= 0$ otherwise

$\mathbf{Pr}(X(i, j)==1) = 1/m$

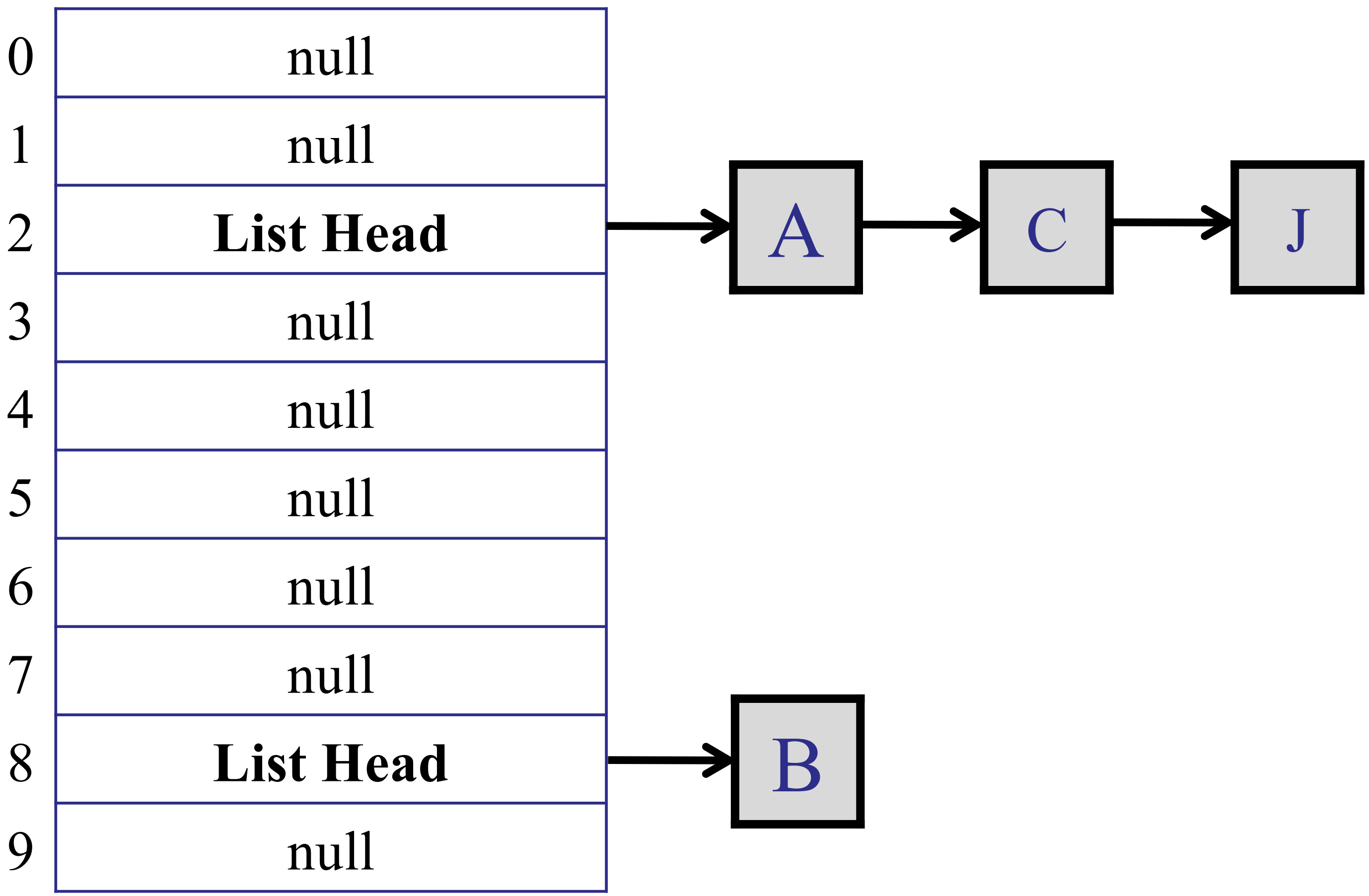
$\mathbf{E}(X(i, j)) = \mathbf{Pr}(X(i, j)==1)*1 + \mathbf{Pr}(X(i, j)==0)*0$

$= \mathbf{Pr}(X(i, j)==1)$

$= 1/m$

A little probability

What is the expected number of items in a bucket?



A little probability

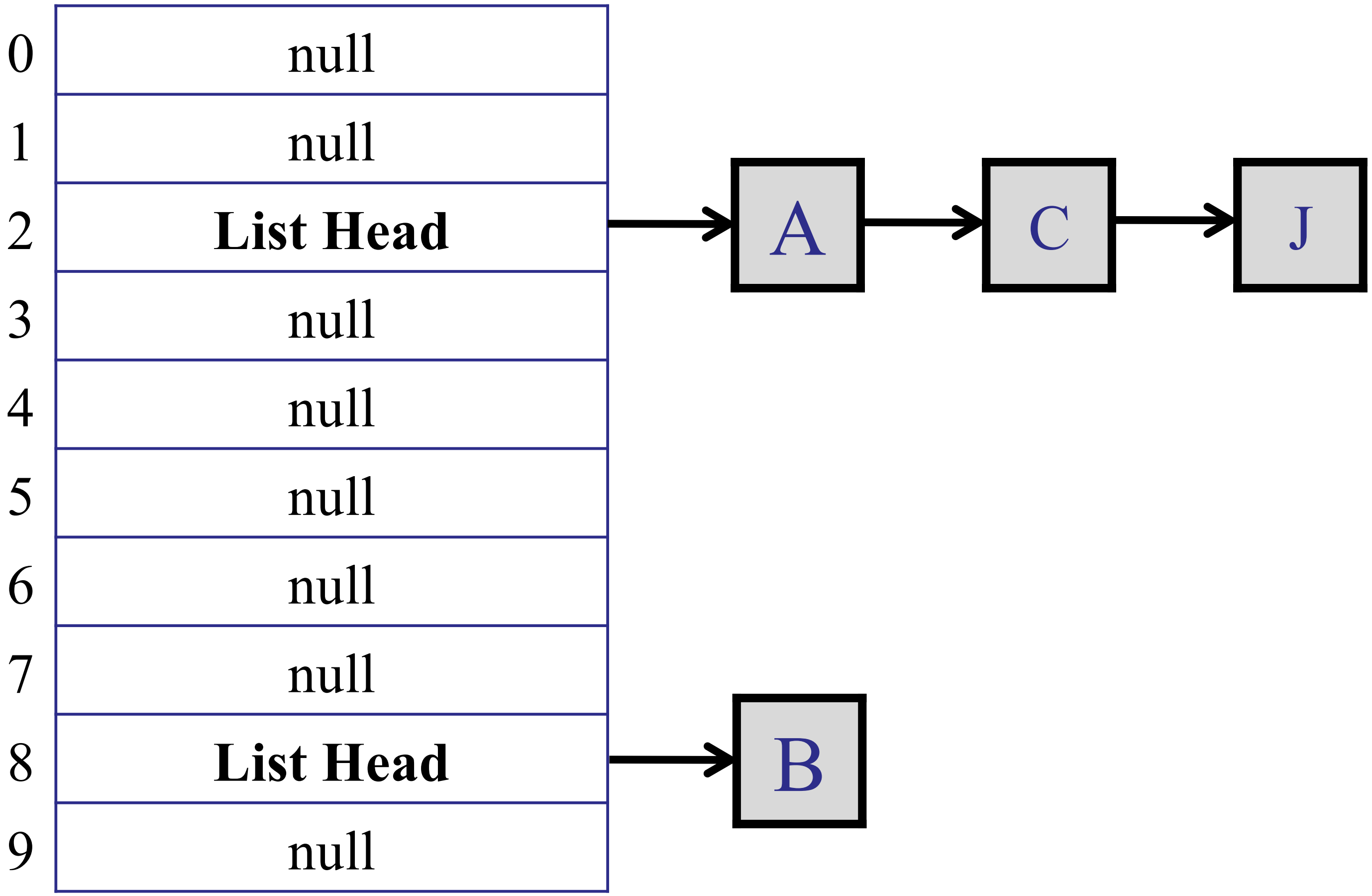
Indicator random variables

$$
\begin{aligned}
X(i, j) &= 1 \quad \text{if item } i \text{ is put in bucket } j \\
&= 0 \quad \text{otherwise}
\end{aligned}
$$

$\Sigma_i X(i, b) =$ number of items in bucket b

A little probability

Each item contributes `1’ to the bucket it is in..



A little probability

Indicator random variables

$X(i, j) = 1$ if item i is put in bucket j

$= 0$ otherwise

$\Sigma_i X(i, b) =$ number of items in bucket b

A little probability

Calculate expected number of items per bucket:

Expected (Σ_i X(i, b)) =

A little probability

Calculate expected number of items per bucket:

$$\mathbf{E}(\Sigma_i X(i, b)) = \Sigma_i \mathbf{E}(X(i, b))$$

Linearity of expectation: $E(A + B) = E(A) + E(B)$

A little probability

Calculate expected number of items per bucket:

$$\mathbf{E}(\Sigma_i X(i, b)) = \Sigma_i \mathbf{E}(X(i, b))$$

$$= \Sigma_i 1/m$$

$$= n/m$$

Let's be optimistic today.

The Simple Uniform Hashing Assumption

- Assume:
  - n items
  - m buckets
- Define: load(hash table) = n/m

  = average # items / buckets.

- Expected search time = $1 + n/m$

linked list traversal

hash function + array access

Let's be optimistic today.

The Simple Uniform Hashing Assumption

- Assume:
  - n items
  - $m = \Omega(n)$ buckets, e.g., $m = 2n$

- Expected search time $= 1 + n/m$

$$= O(1)$$

Hashing with Chaining

Searching:

- Expected search time = $1 + n/m$ = O(1)
- Worst-case search time = O(n)

Inserting:

- Worst-case insertion time = O(1)

Hashing with Chaining

What if you insert n elements in your hash table?

What is the expected *maximum* cost?

Hashing with Chaining

What if you insert n elements in your hash table?

What is the expected *maximum* cost?

- Analogy:
  - Throw n balls in m = n bins.
  - What is the maximum number of balls in a bin?

Cost: O(log n)

Hashing with Chaining

What if you insert n elements in your hash table?

What is the expected *maximum* cost?

- Analogy:
  - Throw n balls in m = n bins.
  - What is the maximum number of balls in a bin?

Cost: $\Theta(\log n / \log\log n)$

Hashing: Recap

Problem: coping with large universe of keys

- Number of possible keys is very, very large.
- Direct Access Table takes too much space

Hash functions

- Use hash function to map keys to buckets.
- Sometimes, keys collide (inevitably!)
- Use linked list to store multiple keys in one bucket.

Analyze performance with simple uniform hashing.

- Expected number of keys / bucket is $O(n/m) = O(1)$.