MA107 Quantitative Methods

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Revision Document

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Abstract

This is a review of mathematics lectures by James Ward in Michaelmas Term of 2018. The notes are mine fully and may not be authentic to the lecturer's as they have been modified.

The format of this material is usually recounted slide by slide but some slides may be merged together as the material fits appropriately with one another.

This material tries to go over key concepts but not all lecture slides. This means topics which were found to be more complex for me or requires formulas that are critical to problem-solving success.

Contents

1   Mathematical Terms and Notation

This covers sets and functions

1.1   Sets and their Members

A set is a collection of objects which is defined in a precise way so that any given object is either in the set or not in it.

- We call the objects in a set its members or elements
- We use curly brackets when indicating a set by listing its members
  Example: $\{1, 2, 3, 4, 5, ...\}$
- In particular, sets may have a finite or infinite number of members

Some sets are important to remember as they have special names:

- $\mathbb{N}$ is the set of **natural numbers** i.e. $\{1, 2, 3, 4, ... \}$
- $\mathbb{Z}$ is the set of **integers** i.e. $\{..., -2, -1, 0, 1, 2, ...\}$
- $\mathbb{R}$ is the set of **real numbers** visualised as the real line
- $\varnothing$ is the **empty set** i.e. $\{ \ \}$

If we want the set of objects in another set, say A, which have a certain property, we use the notation:

$$\{x \in A \mid property \ x \ must \ satisfy\}$$

i.e. this is the set of all members of the set A that have the property.

Lastly, we define $\mathbb{R}_+$ to be the set of non-negative real numbers, i.e.

$$\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$$

1.2   Comparison of Sets

We use the notation $'S \subseteq T'$ to denote that S is a **subset** of T i.e. that all members of the set S are also members of the set T.

- **Example:** $\mathbb{N} \subseteq \mathbb{Z}$, $\mathbb{N} \subseteq R$
- **Example:** $\mathbb{R}_+ \subseteq \mathbb{R}$, $\mathbb{R} \nsubseteq \mathbb{R}_+$

1.3   New sets from old

The **union** of two sets, S and T denoted $S \cup T$ is defined as:

$$S \cup T = \{x \mid x \in S \ or \ x \in T\}$$

The **intersection** of two sets S and T denoted $S \cap T$ is defined as:

$$S \cap T = \{x \mid x \in S \ and \ z \in T\}$$

The **difference** of two sets S and T denoted $S \setminus T$ is defined as:

$$S\backslash T = \{x \mid x \in S \ and \ x \notin T\}$$

The **Cartesian Product** of two sets S and T denoted $S \times T$ is defined as:

$$S \times T = \{(x, y) \mid x \in S \ and \ y \in T\}$$

1.3.1 Cartesian

If we take the Cartesian product of a set with itself i.e. $S \times S$, we usually denote this S^2. This is import as if we take the Cartesian product of $\mathbb{R}$ itself we get:

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$$

This is how we denote the xy-plane in terms of sets e.g. $\mathbb{R}_+^2$ would represent the set of all points in the **non-negative quadrant** of the xy-plane.

2 Functions and Demand and Supply

2.1 Demand

If a product is offered for sale at price p (per unit) what quantity q will the market demand (i.e. purchase)? We use a demand model function to understand this concept:

We denote the demand function as q^D
It is related to price because it is a function of price: $q = q^D(p)$

If the curve is suitably well-behaved we can ask the **inverse question:** if a quantity q is demanded, what price p will the product sell at?

We denote the inverse demand function as p^D
It is related to quantity because it is a function of quantity: $p = p^D(q)$

For example, if we modelled the demand curve by the straight line $5p + q = 40$ we can figure out these two functions be rearranging:

- $q^D(p) = 40 - 5p \ as \ q = 40 - 5p$
- $p^D(q) = \frac{40-q}{5} \ as \ p = \frac{40-q}{5}$

2.2 Supply

If a product can be placed for sale at price p (per unit) what quantity q will the supplier put on the market (i.e. produce)? We use a supply model function to understand this concept:

We denote the demand function as q^S
It is related to price because it is a function of price: $q = q^S(p)$

If the curve is suitably well-behaved we can do the inverse again:

We denote the inverse supply function as p^S
This is a function of quantity: $p = p^S(q)$

The example is below with the supply curve $5p - q = 40$:

- $q^S(p) = 5p - 40 \ as \ q = 5p - 40$
- $p^s(q) = \frac{40+q}{5} \ as \ p = \frac{40+q}{5}$

Hence, from the above two, we can begin to model some functions. Further, well-behaved curves mean that the curve will give us a proper inverse function. The inverses can be verified through using composite functions e.g. $(p^S \circ q^S)(p) = p$

2.3 Equilibrium

The reason we use these models is to inform us on what q and p will actually be achieved in the market. We have two sets:

$$D = \{(q, p) \mid q = q^D(p)\} \text{ and } S = \{(q, p) \mid q = q^S(p)\}$$

This tells us the factors affecting the demand and supply of a product in the market. To find the equilibrium i.e. where everything being supplied is being purchased, we need an intersection of these two sets:

$$E = D \cap S$$

This is called **the equilibrium set** of the market.

3 Taxation

Two questions on supply and demand sets:

1. If market conditions change, what happens to the equilibrium?
2. If markets start away from the equilibrium, can it move towards it?

3.1 Excise Tax

This is an example of how market conditions can change. The theory goes as follows: A market is governed by a supply function $q^S(p)$ and the demand function $q^D(p)$. If a tax of T is imposed per unit on the product sold, the supplier will pay T per unit sold. If a price of p is charged, the supplier gets $p - T$ per unit from the sale. The price p is the price per unit seen by consumers and $p - T$ is the effective price per unit seen by suppliers.

The market is now disturbed and we get new but related supply and demand functions:

- **New Demand Function:** $q_T^D(p) = q^D(p)$
- **New Supply Function:** $q_T^S(p) = q^S(p - T)$

Only the supply function changes if an excise tax is introduced. We might want to find the new equilibrium which can be solved by:

$$q^D(p_T^*) = q^S(p_T^* - T) \text{ or } q_T^D(p_T^*) = q_T^S(p_T^*)$$

An example is given below:

$$D = 5p + q = 40 \text{ and } S = 15p - 2q = 20$$
$$q^D(p) = 40 - 5p \text{ and } q^S(p) = \tfrac{15}{2}p - 10$$
Equating the two we can get p^* and q^*
$$p^* = 4 \text{ and } q^* = 20$$
Now suppose that we introduce an excise tax into the market to discourage consumption or gain government revenue
The new demand function: $q_T^D(p) = q^D(p) = 40 - 5p$
The new supply function: $q_T^S(p) = q^S(p - T) = \tfrac{15}{2}(p - T) - 10$
To solve for the equilibrium we equate: $40 - 5p_T^* = \tfrac{15}{2}(p_T^* - T) - 10$
Solving we can get $p_T^* = 4 + \tfrac{3}{5}T$ and using $q_T^D = 40 - 5(4 + \tfrac{3}{5}T) = 20 - 3T$
Hence $E = (20 - 3T, 4 + \tfrac{3}{5}T)$ in the form $E = (q, p)$

From this example, we can see that quantity in the market is reduced by $3T$ and the price increases by $\frac{3}{5}T$ which means that **not all the tax is passed onto consumers.**

Moreover, the tax cannot be arbitrarily high if we want our market to function as the equilibrium quantity must be positive. Hence:

$$q_T^* > 0 \rightarrow 20 - 3T > 0 \rightarrow T < \tfrac{20}{3}$$

Thus the maximum tax that can be charged in this market is $\frac{20}{3}$ units of tax.

Tax revenue can also be maximised through optimisation:

$$R_T = q_T^* \times T$$
$$R_T = (20 - 3T)T = 20T - 3T^2$$
$$\text{Differentiating: } \tfrac{\delta T}{\delta R} = 20 - 6T$$
$$\text{Finding maximum: } 6T = 20, \ T = \tfrac{10}{3}$$
$$\text{Revenue: } 20(\tfrac{10}{3}) - 3(\tfrac{10}{3})^2 = \tfrac{100}{3}$$

Alternatively you can solve this by completing the square to get $-3(T - \tfrac{10}{3})^2 + \tfrac{100}{3}$

Sometimes you will also see Tax as a percentage of the price. This would lead to a slightly different equation than $p - T$:

$$\text{Suppliers see it as: } p - rp = p(1 - r)$$
$$\text{So the supply function would be } q_r^S(p) = q^S(p(1 - r))$$

4 Recurrence

We want to solve recurrence of the form $y_t = ay_{t-1} + b$. This is for $t \geq 1$.

- If $a = 1$ then this is an **arithmetic progression**
- If $b = 0$ then this is a **geometric progression**

4.1 Step 1: Time Independent Solution

Let's see if there is a time independent solution y^* i.e. we have $y_t = y^*$ for all $t \geq 0$. This means that y_t is a constant independent of t. If this is the case we have:

$$y_t = y^* \text{ and } y_{t-1} = y^*$$

That means our equation becomes $y^* = ay^* + b \rightarrow (1 - a)y^* = b \rightarrow y^* = \frac{b}{1-a}$. This works as long as $a \neq 1$

4.2 The Discrepancy

This is the difference from the solution we seek and the time independent solution we have such that $z_t = y_t - y^*$. This means we have:

$$z_t = y_t - y^* \rightarrow z_0 = y_0 - y^*$$
$$\text{This means we have found: } z_t = a^t z_0 \rightarrow y_t - y^* = a^t(y_0 - y^*)$$
$$\text{More neatly: } y_t = y^* + (y_0 - y*)a^t$$
$$\textbf{This is the time dependent solution to the recurrence equation:}$$
$$y_t = ay_{t-1} + b$$
$$\text{This can help us solve any recurrence equation of the same form.}$$

What happens in the long term depends on the geometric variable:

- $a > 1$, $a^t \to \infty$ as $t \to \infty$
- $0 < a < 1$, $a^t \to 0$ as $t \to \infty$
- $-1 < a < 0$, $a^t \to 0$ as $t \to \infty$ but oscillates around 0
- $-1 < a$, $a^t \to 0$ as $t \to \infty$ but oscillates around 0

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5   Applications

5.1   Annuity

Investment decisions and annuities can be calculated with a recurrence equation:

- $y_t = (1 + r)y_{t-1} - I$ where $(1 + r)$ is the interest rate and I is the amount taken out of the annuity
- This is of the form $y_t = ay_{t-1} + b$ which we can solve
- $a = (1 + r)$ and $b = (-)I$
- $y^* = \frac{b}{1-a} = -I1 - (1 + r) = \frac{I}{r}$
- With our solution of form $y_t = y^* + (y_0 - y^*)a^t$
- We have: $y_t = \frac{I}{r} + (P - \frac{I}{r})(1 + r)^t$

Through this we can figure out our Principal required for a set I to be taken out or the I we can take out for a given Principal:

- $P = \frac{I}{r}[1 - \frac{1}{(1+r)^n}]$ which is form 1
- $I = \frac{Pr(1+r)^n}{(1+r)^n - 1}$ which is form 2

Both of these can be figured out from the recurrence equation above.

5.2   Compounding

Compounding intervals can be different and provide different outcomes:

- **Annually:** $P(1 + r)^1$ where 1 is the number of years
- **Semi-annually:** $P(1 + \frac{r}{2})^2$ where 2 is the number of compounds in a year $\to$ after n years we have $P(1 + \frac{r}{2})^{2n}$
- **In General 1:** $P(1 + \frac{r}{m})^m$ where m is the number of compounding intervals
- **In General 2:** $P(1 + \frac{r}{m})^{mn}$ where n is years
- **In General 3:** Where $m \to \infty$ we get: Pe^{rn} where e^r is the rate of growth for a given interest rate compounded continuously

5.3 Present Values

- Capital Growth would be a function of the Principal such that: $C(P) = Pe^{rn}$

- Present Value would be a function of Capital Growth such that: $P(C) = Ce^{-rn}$

Simple, present value can be found from a given amount at the end of a period such that: $\frac{x}{(1+r)^n}$. However, this does not take into account present values of annuities. This is more complex and can be seen below:

$$\text{Payment 1 of an annuity has the value of } \frac{I}{(1+r)}$$
$$\text{Payment 2 of an annuity has the value of } \frac{I}{(1+r)^2}$$
$$\text{The sum of all payments: } \frac{I}{(1+r)} + \frac{I}{(1+r)^2} + \cdots + \frac{I}{(1+r)^n}$$
$$\text{This is a geometric series with first term } \frac{I}{1+r} \text{ and common ratio } \frac{1}{1+r}$$
$$\text{Hence we can find it with the formula: } \frac{I}{1+r}\left(\frac{1-(\frac{1}{1+r})^n}{1-\frac{1}{1+r}}\right)$$
$$\text{More simply: } \frac{I}{r}\left(1 - \frac{1}{(1+r)^n}\right)$$

5.4 Cobweb Model

In general, the Cobweb Model explains how markets converge to the equilibrium or how it may oscillate away or around it.

$$q_t = q^S(p_{t-1}) \text{ and } p_t = p^D(q_t) \text{ if } p_0 \neq p^* \text{ for } t \geq 1$$
$$\text{Supply Function: } q^S(p) = mp - n$$
$$\text{Demand Function: } q^D(p) = k - lp$$
$$\text{Equilibrium price: } q^S(\bar{p}) = q^D(\bar{p}) \rightarrow m\bar{p} - n = k - l\bar{p} \rightarrow \bar{p} = \frac{k+n}{m+l} \text{ Equilibrium}$$
$$\text{quantity: } \bar{q} = k - l\frac{k+n}{m+l} = \frac{km-ln}{m+l}$$
$$q^S(p) = mp - n \rightarrow q_t = mp_{t-1} - n$$
$$q^D(p) = k - lp \rightarrow p^D(q) = \frac{k-q}{l} \rightarrow p_t = \frac{k-q_t}{l}$$
$$p_t = \frac{k-(mp_{t-1}-n)}{l} = -\frac{m}{l}p_{t-1} - + \frac{k+n}{l}$$
$$\text{This gives us a form in a recurrence equation which we can then solve with a time}$$
$$\text{independent solution: } p^* = \frac{k+n}{m+l} \text{ which is unsurprisingly the equilibrium.}$$

From this:

- If $m < l$ then p_t will oscillate decreasingly to $\bar{p}$ so the market is stable

- If $m = l$ then p_t will oscillate constantly around $\bar{p}$ so the market is cyclic

- If $m > l$ then p_t will oscillate increasingly around $\bar{p}$ so the market is unstable

6 Differentiation and Optimisation

6.1 Differentiation Basics

There are four ways in which a suitably well defined function can have a point $x = c$ which makes $f'(c) = 0$

- Point of inflection from positive to positive gradient
- Point of inflection from negative to negative gradient
- Local minimum i.e. negative gradient to positive
- Local maximum i.e. positive gradient to negative

We can also use the second derivative test to see whether or not the points are **mins** or **maxs.**

- $f''(c) > 0$ then we have a local minimum
- $f''(c) < 0$ then we have a local maximum
- $f''(c) = 0$ this tells us **nothing**

So our procedure for optimising a function is:

1. Find the stationary points using the **first order condition**
2. Find out what they are using out **tests**
3. Evaluate the function at the stationary points
4. Evaluate the function at the end-points of the interval (if any)
5. Determine the global maximum or global minimum (if any)

6.2 Optimisation

6.2.1 Profit Maximisation

Suppose a firm incurs as cost $C(q)$ when it supplies an amount q of its product. It is also a monopoly so the price will be determined by the inverse demand function $p^D(q)$. This means that revenue would be $qp^D(q)$. Such that the profit function is:

$$\Pi(q) = R(q) - C(q) = qp^D(q) - C(q)$$

How do we maximise this function? Given our cost curve is $C(q) = q^3 - 10q^2 + 25q + 10$ and our inverse demand function is $p^D(q) = 10 - q$

$$\Pi(q) = q(10 - q) - (q^3 - 10q^2 + 25q + 10)$$
$$\Pi(q) = 10q - q^2 - q^3 + 10q^2 - 25q - 10$$
$$\Pi(q) = -q^3 + 9q^2 - 15q - 10$$
$$\Pi'(q) = -3q^2 + 18q - 15$$

Factorised: $(-3q + 3)(q - 5)$ and knowing our constraint $0 \leq q \leq 10$

Testing for Mins and Maxs with $\Pi''(q) = -6q + 18$

$3q = 3 \therefore q = 1$ and $q = 5$

For $q = 1$, $-6 + 18 > 0 \therefore local\ minimum$ so $-(1)^3 + 9(1)^2 - 15(1) - 10 = -15$

For $q = 5$, $-30 + 18 < 0 \therefore local\ maximum$ so $-(5)^3 + 9(5)^2 - 15(5) - 10 = 15$

Test end points: $0 + 0 + 0 - 10 = -10$ and $-(10)^3 + 9(10)^2 - 15(10) - 10 = -260$

Hence Global Max: 15 and Global Min: -260

6.2.2   Discounting

Suppose we have the option to buy an antique. Market information about this antique suggests that its value in dollars will be given by: $V(t) = 2000 + 500t$ if we were to buy it now and sell it after t years. We also have a bank account which pays continuously compounded interests at a rate of 5% per annum. Should we buy the antique? And, if we do, what is the best time to sell it?

With interest at 5% we can see our present value of investing the 2000 principal in a bank account:
$$C(P) = Pe^{0.05t} \text{ and Present Value: } P(C) = e^{-0.05t}$$
So if I know that, I can get $V(t)$ in t years time from selling our antique:
$$P(t) = V(t)e^{-0.005} \text{ which is the function we want to maximise}$$

From the above we get a local maximum of $t = 16$ from $P(t) = (2000 + 500t)(e^{-0.05t})$ where we can apply the product rule to get $-100e^{-0.05t} - 25te^{-0.05t} + 500e^{-0.05t} = (400 - 25t)(e^{-0.05t})$. This gives us a value of $t = 16$. What maximises our value?

When $t = 16$ we get $[2000 + 500(16)][e^{-0.05(16)}] = 4,493.29 > P(0) = 2000$ when $t = 0$
Hence we should buy the antique and sell it after 16 years.