Fourier Series

Trigonometric polynomials

- The vector system $\{e^{ikx}, k \in \mathbb{Z}\}$ is orthogonal in $L^2([-\pi,\pi])$:

$$\langle e^{inx}, e^{imx} \rangle = \int_{-\pi}^{\pi} e^{inx} e^{-imx} dx = 2\pi\delta_{nm}$$

- The $2n+1$ functions $\{e^{ikx}, k = 0, \pm 1, \ldots, \pm n\}$ generate the subspace $\mathcal{F}_n$ in L^2. The elements of $\mathcal{F}_n$ are called trigonometric polynomials of order $\leqslant n$, because the orthogonal basis of real-valued functions is constituted by the set

$$\{1,\ \cos x,\ \sin x, \ldots,\ \cos nx,\ \sin nx\} \quad .$$

$$\begin{aligned} \int_{-\pi}^{\pi} \cos kx\ \sin lx\ dx &= 0 & \int_{-\pi}^{\pi} \cos kx\ \cos lx\ dx &= \pi\delta_{kl} \\ \int_{-\pi}^{\pi} \cos kx\ dx = \int_{-\pi}^{\pi} \sin kx\ dx &= 0 & \int_{-\pi}^{\pi} \sin kx\ \sin lx\ dx &= \pi\delta_{kl}\ . \end{aligned}$$

- The degree n- Fourier polynomial of a function $f \in L^2$ is given by its orthogonal projection $s_n^{[f]}(x)$ on the subspace $\mathcal{F}_n$; we will also call it $s_n(x)$. It means that:

$$s_n(x) = \sum_{k=-n}^{n} \frac{\langle f(x), e^{ikx} \rangle}{\|e^{ikx}\|^2} e^{ikx}$$

Therefore:

$$s_n(x) := \sum_{k=-n}^{n} c_k \; e^{ikx} \quad \text{with} \quad c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt \; , \quad k \in \mathbb{Z}$$

With the real basis, the series is written as:

$$s_n(x) := \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

where

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \; dt \; , \quad k = 0, \ldots, n$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt \; dt \; , \quad k = 1, \ldots, n$$

Definition

- $\lim_{n\to\infty} \; s_n^{[f]}(x)$ is called the Fourier series of the function f.
- Example: $V = L^2([-\pi,\pi])$. Find the Fourier series of $f(x) :=\mid x \mid$.
- Solution: it is an even function, therefore we use the real basis because

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \mid t \mid \sin kt \; dt = 0 \; , \quad \forall k \; .$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \mid t \mid \cos kt \; dt = \frac{2}{\pi}\int_{0}^{\pi} t \cos kt \; dt = 2\frac{(-1)^k - 1}{\pi k^2}$$

$$\begin{aligned} \mid x \mid &= \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos(2k+1)x \\ &= \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \dots\right) \end{aligned}$$

Graphical analysis

Let us look at the graphical representation of the first terms of the Fourier series of $y = |x|$:

$$s_1(x) = \frac{1}{2}\pi - 4\,\frac{\cos(x)}{\pi} \qquad s_3(x) = \frac{1}{2}\pi - 4\,\frac{\cos(x)}{\pi} - \frac{4}{9}\,\frac{\cos(3\,x)}{\pi}$$

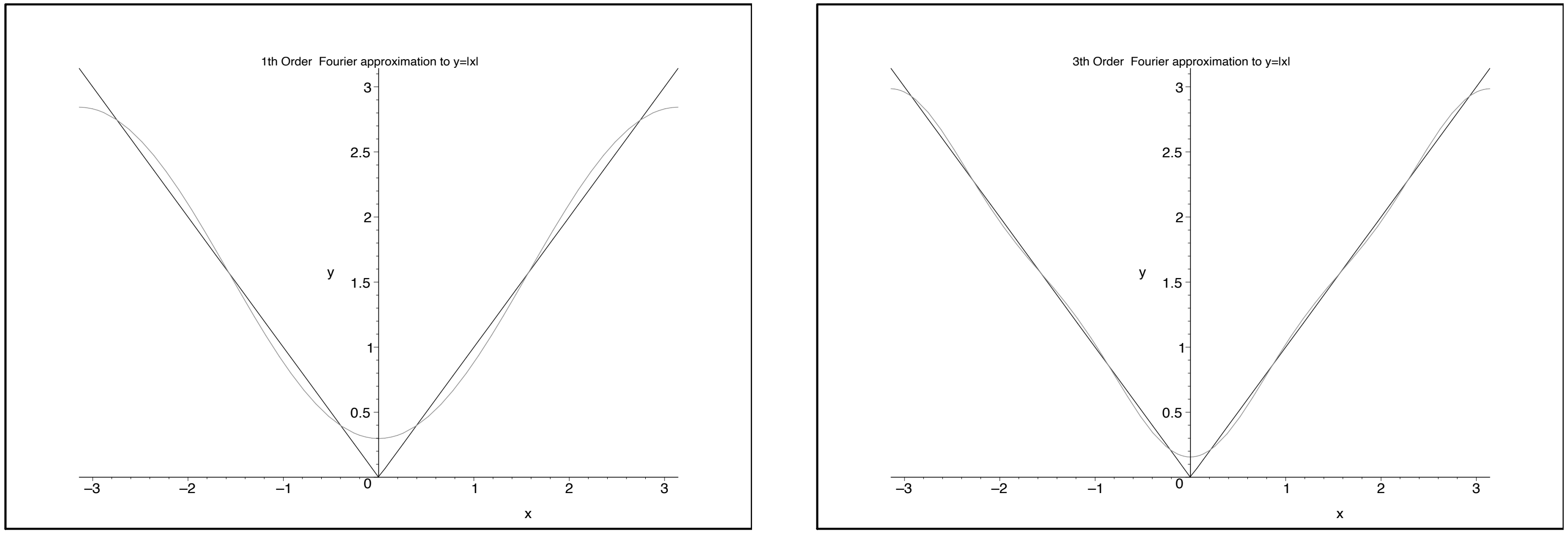


Figura: The 1th and 3th order approximation to $y = |x|$

Arbitrary period

- The previous results can be easily generalized to periodic functions with arbitrary period $T > 0$. The space $L^2([0, T])$ or $L^2([-T/2, T/2])$ admits as orthogonal basis the following set of complex-valued functions:

$$\{e^{in\omega t}, n \in \mathbb{Z}\} \quad \text{where} \quad \omega := 2\pi/T \quad \text{and} \quad \|e^{in\omega t}\|^2 = T$$

- This implies the following Fourier series:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{ik\omega x} \quad \text{with} \quad c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-ik\omega t} dt$$

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\omega x + b_k \sin k\omega x) \quad \text{with}$$

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t) \cos k\omega t \; dt\,, \quad b_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t) \sin k\omega t \; dt$$

- $a_k = c_k + c_{-k}$, $b_k = i(c_k - c_{-k})$, $c_k = \frac{1}{2}(a_k - ib_k)$, $c_{-k} = \frac{1}{2}(a_k + ib_k)$. The above formulas are valid for every $k \in \mathbb{Z}$ putting $b_0 = 0$.

Example: f(x)=x

- Example: $V = L^2([-1,1])$. Find the Fourier series of $f(x) := x$.
- Solution: it is an odd function, therefore again we use the real basis with $T = 2$. With $\omega = \pi$, the set of real-valued $\{1,\ \cos \pi x,\ \sin \pi x, \dots,\ \cos n\pi x,\ \sin n\pi x\}$ becomes even orthonormal. It follows:

$$
\begin{aligned}
a_k &= \int_{-1}^{1} x \cos k\pi x \; dx = 0 \qquad \forall k \ , \\
b_k &= \int_{-1}^{1} x \sin k\pi x \; dx = -2\,\frac{(-1)^k}{k\,\pi}
\end{aligned}
$$

This leads to the following Fourier series:

$$
\begin{aligned}
x &= \sum_{k=1}^{n} -2\,\frac{(-1)^k \sin(k\pi x)}{k\pi} \\
&= 2\,\frac{\sin(\pi x)}{\pi} - \frac{\sin(2\pi x)}{\pi} + \frac{2}{3}\frac{\sin(3\pi x)}{\pi} - \frac{1}{2}\frac{\sin(4\pi x)}{\pi} + \dots
\end{aligned}
$$

Example: f(x)=x

- Example: $V = L^2([-1,1])$. Find the Fourier series of $f(x) := x$.
- Solution: it is an odd function, therefore again we use the real basis with $T = 2$. With $\omega = \pi$, the set of real-valued $\{1, \cos \pi x, \sin \pi x, \ldots, \cos n\pi x, \sin n\pi x\}$ becomes even orthonormal. It follows:

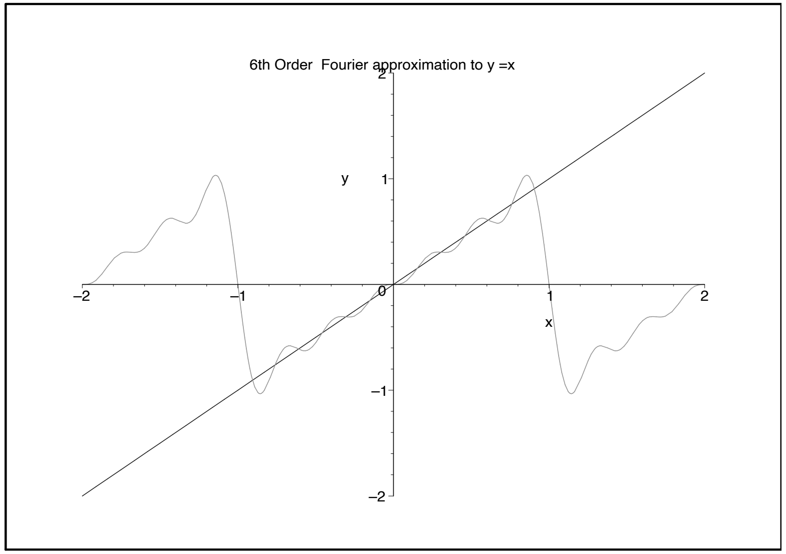
$$
\begin{aligned}
a_k &= \int_{-1}^{1} x \cos k\pi x \; dx = 0 \qquad \forall k \;, \\
b_k &= \int_{-1}^{1} x \sin k\pi x \; dx = -2 \, \frac{(-1)^k}{k\,\pi}
\end{aligned}
$$

This leads to the following Fourier series:

$$
\begin{aligned}
x &= \sum_{k=1}^{n} -2 \, \frac{(-1)^k \sin(k\pi x)}{k\pi} \\
&= 2 \, \frac{\sin(\pi x)}{\pi} - \frac{\sin(2\pi x)}{\pi} + \frac{2}{3} \frac{\sin(3\pi x)}{\pi} - \frac{1}{2} \frac{\sin(4\pi x)}{\pi} + \ldots
\end{aligned}
$$

Example: graphical representation

let us look at the 6th order approximation of the function and the corresponding error function.



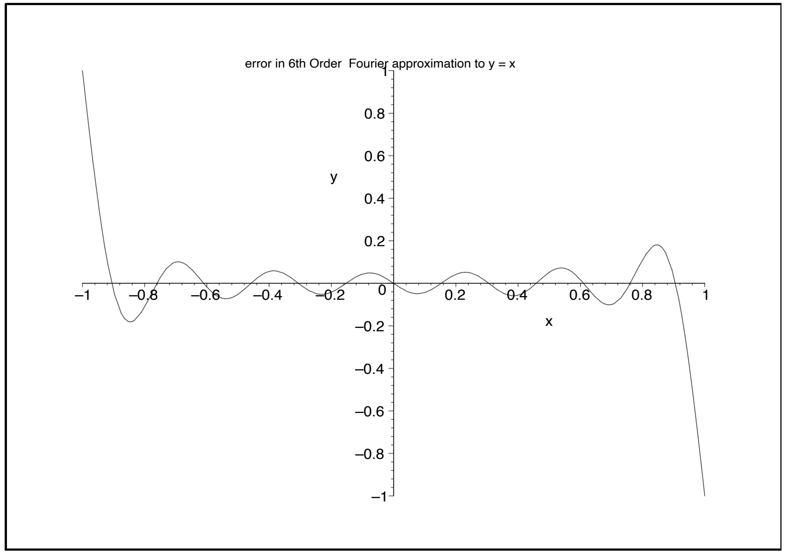


Figura: The 6th order approximation to $y = x$ with the corresponding error

Exercise

- Find the Fourier series of the piecewise function sign(x) defined by

$$f(x) = \begin{cases} -1 & \text{if} \quad -1 \leqslant x < 0 \\ 0 & \text{if} \quad x = 0 \\ 1 & \text{if} \quad 0 < x \leqslant 1 \end{cases} \tag{2}$$

- Solving the previous problem leads to the series:

$$\text{sign}(x) = 4\left(\frac{\sin(\pi x)}{\pi} + \frac{\sin(3\pi x)}{3\pi} + \frac{\sin(5\pi x)}{5\pi} + \frac{\sin(7\pi x)}{7\pi} + \dots\right) \tag{3}$$

Exercise

- Find the Fourier series of the piecewise function sign(x) defined by

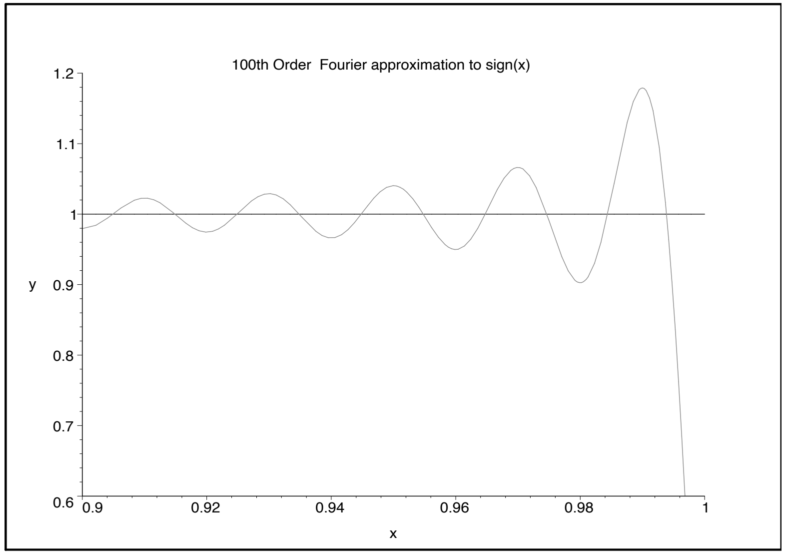
$$f(x) = \begin{cases} -1 & \text{if} \quad -1 \leqslant x < 0 \\ 0 & \text{if} \quad x = 0 \\ 1 & \text{if} \quad 0 < x \leqslant 1 \end{cases} \tag{2}$$

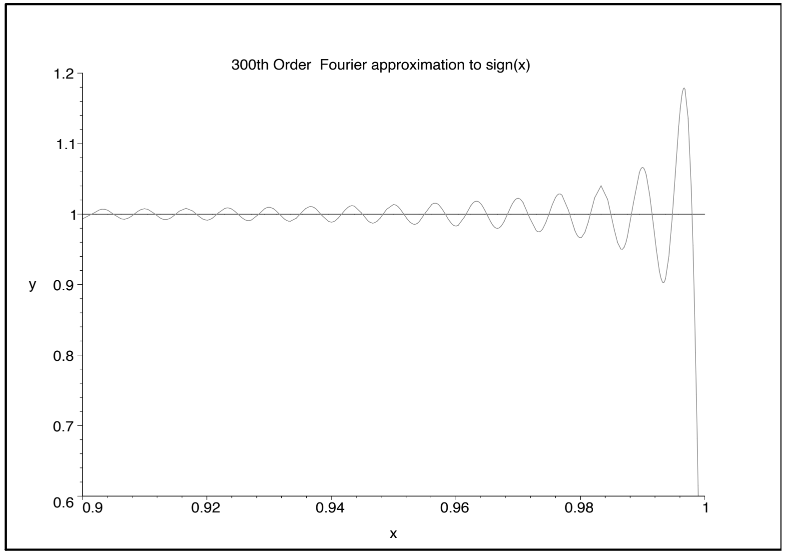
- Solving the previous problem leads to the series:

$$\text{sign}(x) = 4\left(\frac{\sin(\pi x)}{\pi} + \frac{\sin(3\pi x)}{3\pi} + \frac{\sin(5\pi x)}{5\pi} + \frac{\sin(7\pi x)}{7\pi} + \dots\right) \tag{3}$$

Gibbs phenomenon

The following two pictures show the 100th and 300th order approximation near the discontinuity:





Gibbs phenomenon

The Fourier series approximation displays an overshoot in discontinuities. While the convergence of the Fourier series to the function improves anywhere else, the height of the overshoot does not decrease with augmenting the number of terms. This behavior is called *Gibbs Phenomenon*. It shows that the Fourier series does not converge uniformly to a discontinous function in an arbitrary small interval of the discontinuity point.

Formally, the following theorem can be proven:

Let f be a piecewise smooth function 2π periodic. A each discontinuity point x_0, its Fourier series either overshoots or undershoots $f(x_0^+)$ by about 9% of the magnitude of the discontinuity jump

Interestingly, the overshoot factor depends only on the type of discontinuity and not on the values of the function.