Scuola universitaria professionale
della Svizzera italiana

SUPSI

Università
della
Svizzera
italiana

Istituto Dalle Molle di studi sull'intelligenza artificiale

Applied Operations Research (C-B4051E.1)
Bachelor in Data Science and AI

Lecture 2
Modelling

Matteo Salani
matteo.salani@idsia.ch

Modelling in practice

The main ingredients of a **mathematical model** for a **decision problem** are:

- **Parameters, Data:** these are the invariants of the model and they are defined/indexed on **Domains/Sets** of objects.
- **Variables:** they are elements with a degree of freedom. The decision maker has the power to choose their values.
- **Constraints:** they determine which solution is acceptable and which is not.
- **Objective function:** it discriminates which solution is better than others.

The details of a problem are often implicit and communicated in a non-rigorous way by the domain experts. With practice it is possible to identify common “patterns” in different decision problems.

Modelling in practice - Examples

An investor has 50.000 euros to invest in a number of share of 5 investment funds. The following table reports, for each fund, the type, the cost and the expected annual return.

Fund	Type	Cost	Return
A	Bond Fund	4,5	7%
B	Bond Fund	4	8%
C	Combined	2,5	6%
D	Combined	3	6%
E	Equity Fund	4,5	9%

The investor wants to **maximize the total expected annual return** under the condition that:

- At least 15.000 euros must be invested in bond funds;
- At least 20.000 euros must be invested in combined funds;
- At most 5.000 euros can be invested in equity funds.

Modelling in practice - Examples

A research institute has to decide the number of positions to open for new researchers of first, second and third level. The institute has a maximal annual budget of tre milions of euro for the remuneration of the new researchers. The annual remuneration of a fi
rst, second and third level researcher is equal to 40.000, 30.000 e 25.000 euros, respectively. The total annual expenditure for new researchers of level 2 cannot exceed the 80% of the total annual expenditure for new researchers of first level.Furthermore, the number of researchers of level 2 must be at least the double of the number of new researchers of third level. Finally, the recruitment can be set only if, for each level, at least 6 researchers are hired. Formulate the case as an optimization problem, with the objective of **maximizing the total number** of new researchers hired.

Modelling in practice - Examples

A publishing house has to transport books from 3 deposits (D1, D2, D3) to 4 bookshops (L1, L2, L3, L4). The following table shows the unitary transportation costs (in euros) from each storage to each bookshop, the number of available books in the storages and the demand of each bookshop:

	L1	L2	L3	L4	Availability
D1	0,5	0,8	1	1,5	50
D2	0,7	2	0,8	0,5	100
D3	1	0,5	1,5	0,6	40
Demand	30	70	45	45	

The goal is to minimize the total transportation costs and satisfy the constraints of demand and supply.

Modelling in practice - Generalization

A mathematical programming model has to be general. It is important to make the effort and **abstract** the decision problem from the specific data. In this way it is possible to apply it repeatedly with different data sets. For this purpose it is useful to define sets and parameters and to use them for the formulation of the model:
Back to the bookshop example:

- L: set of the bookshops, index $l = 1, \cdots, |L|$
- D: set of depots, index $d = 1, \cdots, |D|$
- r_l: demand of bookshop $l \in L$ [units]
- s_d: availability at deposit $d \in D$ [units]
- c_{dl}: transportation cost from dept d to bookshop l [euro/unit]
- x_{dl}: amount of books transported from d to l [units]

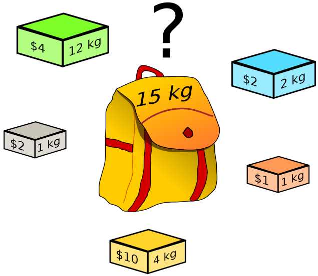
Modelling in practice - Generalization

Specifying the units of measurement helps verifying the correctness of the relations.
Back to the bookshop example:

$$z = \sum_{d \in D} \sum_{l \in L} c_{dl} x_{dl}$$

$$[\text{euros}] = \sum_{d \in D} \sum_{l \in L} [\text{euros/unit}] \cdot [\text{unit}]$$

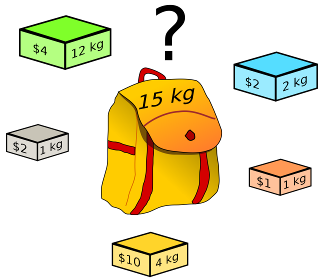
Modelling in practice - Examples - The knapsack problem



Problem definition

A knapsack can bear a given maximum weight. Consider a set of objects with a given weight and a given value. Choose which objects must be put into the backpack in order to maximize the total value without exceeding the maximum weight.

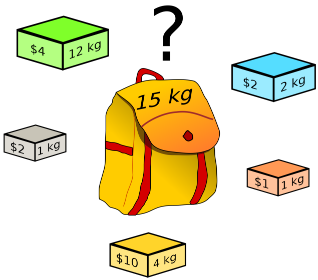
Modelling in practice - Examples - The knapsack problem



Data definition

- N: Set of items
- B: Knapsack capacity
- v_i: Value of item $i = 1..|N|$
- w_i: Weight of item $i = 1..|N|$

Modelling in practice - Examples - The knapsack problem



Decision variables

Is the item i packed?

Yes/no $\rightarrow$ binary variable $x_i \in \{0, 1\}$

$$x_i = \begin{cases} 1 & \text{if item } i \text{ is packed} \\ 0 & \text{otherwise} \end{cases}$$

Modelling in practice - Examples - The knapsack problem

Objective function

Maximize total value of the knapsack

$$\text{Value of item } i = \begin{cases} v_i & \text{if } x_i = 1 \\ 0 & \text{if } x_i = 0 \end{cases}$$

Total value

$$\sum_{i=1}^{|N|} v_i \cdot x_i$$

Objective function

$$z = \max \sum_{i=1}^{|N|} v_i \cdot x_i$$

Modelling in practice - Examples - The knapsack problem

Constraints

Do not exceed available capacity

$$\text{Weight of item } io = \begin{cases} w_i & \text{if } x_i = 1 \\ 0 & \text{if } x_i = 0 \end{cases}$$

Total weight

$$\sum_{i=1}^{|N|} w_i \cdot x_i$$

Capacity constraint

$$\sum_{i=1}^{|N|} w_i \cdot x_i \leq B$$

Modelling in practice - Examples - The knapsack problem

Mathematical model

$$
\begin{array}{lll}
z = \max & \sum_{i=1}^{|N|} v_i \cdot x_i & \\
s.t. & \sum_{i=1}^{|N|} w_i \cdot x_i \leq B & \\
 & x_i \in \{0, 1\} & \forall i \in 1 \cdots |N|
\end{array}
$$

Integer (binary) linear programming model.

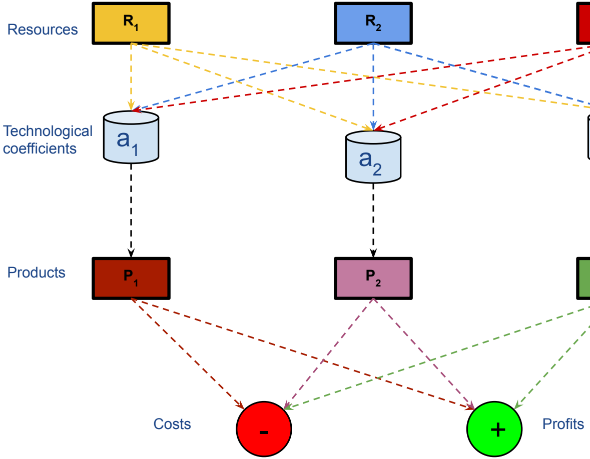
Let's model FantaHockey

With a budget of 120'000 CHF select hockey players in order to form the strongest team, taking care to select

- at least 3 forwards
- at least 2 defenders
- at least 1 goalkeeper
- at most 12 players in total

Product mix (optimal resource allocation)

Determine the product mix which **maximizes** the profit, given the quantities of available raw material and known the sales prices and the production costs of each product.



Product mix

P Set of products
R Set of resources (e.g. raw material)
$A_{|R|\times|P|}$ Technological matrix. [a_{ij} Quantity of resource i used to produce one unit of product j.
p_j Profit for one unit of product j ($j = 1..|P|$)
u_i Availability of resource i ($i = 1..|R|$)
x_j Amount of product j to be produced ($j = 1..|P|$)

$$
\begin{aligned}
z = \max \sum_{j\in P} p_j x_j & \\
s.t. \sum_{j\in P} a_{ij} x_j \leq u_i & \quad \forall i \in R \\
x_j \geq 0 & \quad \forall j \in P
\end{aligned}
$$

Matrix form:

$$
\begin{aligned}
z = \max \ & \mathbf{p}^T \mathbf{x} \\
s.t. \ & A\mathbf{x} \leq \mathbf{u} \\
& \mathbf{x} \geq 0
\end{aligned}
$$

Product Mix: Example

A paint shop produces two types of colourants P_1 and P_2 and uses 3 basic preparations R_1; R_2; R_3. The table reports the quantities (in kg) of the basic preparations required for the production of one litre of each colourant.

	P1	P2
R1	1	1
R2	1	2
R3	0	1

The quantities of available basic preparations (in kg) are known for each month:.

R1	R2	R3
750	1000	400

The sales prices of colourants P1 and P2 are 12CHF/kg and 14CHF/kg, respectively. Determine the optimal production strategy.

Product mix: Solution of the example

Some elements of the problem:

- Domains of the problem: Colorants and preparations.
- Data of the problem: Quantity of available resources (for each month), resource consumption in the production (technical coeffcients), pro
  t for each unit of product.
- Objective of the problem: **Maximize** the total monthly revenue.
- Variables of the problem: Quantities produced per month.

$$
\begin{aligned}
z = \max 12 \cdot x_1 + 14 \cdot x_2 & \\
s.t.\ 1 \cdot x_1 + 1 \cdot x_2 &\leq 750 \\
1 \cdot x_1 + 2 \cdot x_2 &\leq 1000 \\
0 \cdot x_1 + 1 \cdot x_2 &\leq 400 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Product mix: Solution of the example

$$
\begin{aligned}
z = \max 12 \cdot x_1 + 14 \cdot x_2 \\
s.t.\ 1 \cdot x_1 + 1 \cdot x_2 &\leq 750 \\
1 \cdot x_1 + 2 \cdot x_2 &\leq 1000 \\
0 \cdot x_1 + 1 \cdot x_2 &\leq 400 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Optimal solution: $z = 9.500$ CHF, $x_1 = 500$, $x_2 = 250$.

- R1 and R2 are used entirely, 250 kg of R3 are used.
- The constraints related to R1 and R2 are **active** (scarce resources). The constraint related to R3 is **not active** (the resource is not scarce).

Another example: textile company

Let's consider a textile company:

- Product set: The company has to produce two types of fabric, 1 and 2
- Resource consumption:
  - 28 Kg of wool and 7 Kg of cotton are needed to produce 100 Kg of fabric 1
  - 7 Kg of wool and 14 Kg of cotton are needed to produce 100 Kg of fabric 2
  - 3 working hours are needed to produce 100 Kg of fabric 1 or 2
- Weekly available resources: 168 Kg of wool, 84 Kg of cotton, 42 working hours
- Profit: 20 CHF/Kg for fabric 1, 10 CHF/Kg for fabric 2
- Objective of the company: define the weekly production plan that maximizes the total revenue.

Another example: textile company

Mathematical model

$$
\begin{aligned}
z = \max\ & 2000 \cdot x_1 + 1000 \cdot x_2 \\
s.t.\ & 28 \cdot x_1 + 7 \cdot x_2 \leq 168 \\
& 7 \cdot x_1 + 14 \cdot x_2 \leq 84 \\
& 3 \cdot x_1 + 3 \cdot x_2 \leq 42 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Optimal solution: $z = 13714.28$ CHF, $x_1 = 5.14$, $x_2 = 3.42$.

Product mix - example with fixed costs

A carpenter has 2000 kg of wood and 210 kg of glue available for the production of chairs and tables. For the production of a chair and a table the required quantity of wood is 10 kg and 25 kg, and the quantity of glue is 1.2 kg and 2.2 kg, respectively. The expected revenue is 95 CHF for a chair and 150 CHF for a table. The woodworker's objective is to maximize the total revenue. In order to start the production, machines must be prepared with a fixed cost of 75 CHF for chairs and of 120 CHF for tables.

Product mix - example with fixed costs

Frequently, decision problems about the production of goods involve fixed costs. Fixed costs don't depend on the produced quantity. For example, they take into account the preparation costs of the machines.

These costs are usually de
fined as **setup costs**.

Let f_j be the fixed cost to activate the production of item j ($f_j \geq 0$). Then:

$$z = \max \sum_{j \in P} \begin{cases} p_j \cdot x_j - f_j & \text{if } x_j > 0 \\ 0 & \text{if } x_j = 0 \end{cases}$$

The objective function is no longer linear!

Product Mix - Fixed Costs

By means of **auxiliary binary variables**, the problem is transformed into a mixed integer linear problem..

A binary variable y_j is defined for each product j.

$$y_j = \begin{cases} 1 & \text{if production of } j \text{ is activated} \\ 0 & \text{otherwise} \end{cases}$$

The objective function is linear again.

$$z = \max \sum_{j \in P} (p_j \cdot x_j - f_j \cdot y_j)$$

Variables x and y have to be connected by suitable constraints

$$x_j \leq M_j \cdot y_j \qquad \forall j = 1, \cdots, P$$

Fixed Costs - the "big M" constraint

$$x_j \leq M_j \cdot y_j \qquad \forall j = 1, \cdots, P$$

- If $x_j > 0$ (that is we want to produce j) then it must be $y_j = 1$. Therefore, the fixed cost f_j is counted in the objective function.
- If $x_j = 0$ (that is we **don't** want to produce j) the optimal assignment for the variable y_j is $y_j = 0$. Also the assignment $y_j = 1$ is feasible, but it is not optimal (the costs are minimized)!

The value of M_j (big M) has to be large enough, that is M_j must be bigger than any possible value of the variable x_j. In a product mix, for example, it is possible to define:

$$M_j = \min_{i=1,\cdots,R} \left(\frac{u_i}{a_{ij}} \right)$$

Blending problem: definition

Data:

- Given a set of available resources (input) and a set of desired components (output)
- Given the unit purchase cost of each available resource and the concentration of each component in the resources
- Given the demand for each desired output component

Objective:

- Determine the optimal purchase quantities of the resources, in such a way that the requirements are satisfied and the total costs **minimized**.

Blending problem: mathematical model

I Set of resources (input)
U Set of desired components (output)
$F_{|U| \times |I|}$ Resource matrix. [f_{ui} ratio of component u ucontained in resource i].
c_i Unit purchase cost of resource i ($i = 1..|I|$)
r_u Required quantity of component u ($u = 1..|U|$)
x_i Quantity of resource i to be mixed ($i = 1..I$)

$$
\begin{aligned}
z = \min \ & \sum_{i \in I} c_i x_i \\
s.t. \ & \sum_{i \in I} f_{ui} x_i \geq r_u \quad && \forall u \in U \\
& x_i \geq 0 && \forall i \in |I|
\end{aligned}
$$

Matrix formulation:

$$
\begin{aligned}
z = \min \ & \mathbf{c}^T \mathbf{x} \\
s.t. \ & F\mathbf{x} \geq \mathbf{r} \\
& \mathbf{x} \geq 0
\end{aligned}
$$

Blending problem: Example of the diet problem

A diet prescribes that minimum daily amounts of nutrients must be ingested. Five basic foods with their nutritional properties, costs per portion and maximum number of portions tolerated are given:

	Bread	Milk	Eggs	Meat	Cake
calories	110	160	180	260	420
proteins	4	8	13	14	4
calcium	2	285	54	80	22
cost	2	3	4	19	20
portions	4	8	3	2	2

The requirements are 2000 calories, 50 g protein, 700 mg calcium. **Minimize** the cost of meeting the diet.

Blending problem: Solution of the diet problem

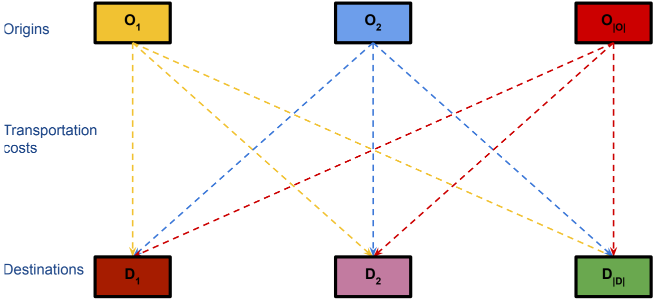
- Domains Food type (Aliments, A), Nutrients (N).
- Data Food cost c_a ($\forall a \in A$), Daily limit u_a ($\forall a \in A$), Required nutrients r_n ($\forall n \in N$), Quantity of nutrient n in food f a, $f_{n,a}$ ($\forall n \in N, a \in A$)
- Variables Quantity of food x_a ($\forall a \in A$).

$$
\begin{aligned}
z = \min 2 \cdot x_P + 3 \cdot x_L + 4 \cdot x_U + 19 \cdot x_C + 20 \cdot x_D \\
s.t.\ 110 \cdot x_P + 160 \cdot x_L + 180 \cdot x_U + 260 \cdot x_C + 420 \cdot x_D \geq 2000 \\
4 \cdot x_P + 8 \cdot x_L + 13 \cdot x_U + 14 \cdot x_C + 4 \cdot x_D \geq 50 \\
2 \cdot x_P + 285 \cdot x_L + 54 \cdot x_U + 80 \cdot x_C + 22 \cdot x_D \geq 700 \\
0 \leq x_P \leq 4, 0 \leq x_L \leq 8, 0 \leq x_U \leq 3, 0 \leq x_C \leq 2, 0 \leq x_D \leq 2
\end{aligned}
$$

$$
\begin{aligned}
z = \min \sum_{a \in A} c_a \cdot x_a & \\
s.t.\ \sum_{a \in A} f_{n,a} \cdot x_a \geq r_n & \qquad \forall n \in N \\
0 \leq x_a \leq u_a & \qquad \forall a \in A
\end{aligned}
$$

Transportation problem: definition

We call transportation problem the problem of transporting goods from a given number of origins to a given number of destinations. The available quantities in the origins and the required quantities in the destinations are known. Knowing the unit transportation cost for each origin/destination pair, the requirement is to plan the shipments with minimal transportation costs.



Transportation problem: mathematical model

- O: Set of origins
- D: Set of destinazions
- $C_{|O|\times|D|}$: Unit transportation cost. $[c_{ij}]$ unit transportation cost from i to j, $\forall i \in O, j \in D$.
- **d**: d_i availability of goods at the origin i
- **b**: b_j demand of goods at the destination j

$$
\begin{aligned}
\min z = & \sum_{i\in O}\sum_{j\in D} c_{ij}x_{ij} & \\
s.t. & \sum_{j\in D} x_{ij} = d_i & \forall i \in O \\
& \sum_{i\in O} x_{ij} = b_j & \forall j \in D \\
& x_{ij} \geq 0 & \forall i \in O\forall, j \in D
\end{aligned}
$$

TPransportation problem: mathematical model when $\sum_{i \in O} d_i \geq \sum_{j \in D} b_j$

$$
\begin{aligned}
\min z = & \sum_{i \in O} \sum_{j \in D} c_{ij} x_{ij} & \\
s.t. & \sum_{j \in D} x_{ij} \leq d_i & \forall i \in O \\
& \sum_{i \in O} x_{ij} \geq b_j & \forall j \in D \\
& x_{ij} \geq 0 & \forall i \in O \forall, j \in D
\end{aligned}
$$

Transportation problem: the bookshop example of lecture 1

Inventory management (lot sizing)

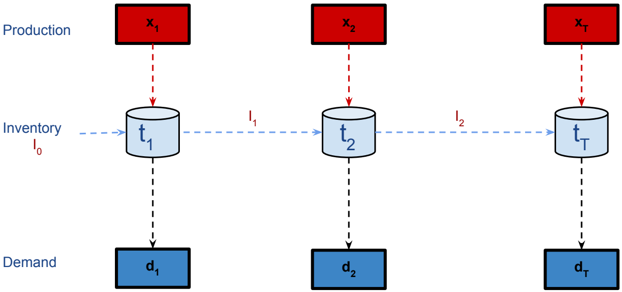
Inventory management is about planning the production of a single product for a certain time horizon::

- The time horizon is divided in periods (e.g. weeks, months).
- Let T be the set of periods, $T = \{1, \cdots, |T|\}$
- Expected demand: d_t is the expected demand for the product in period t
- Storage costs are known: h_t is the storage cost per product unit in the period t
- Production costs are known: c_t is the production cost per unit in period t

Inventory management (lot sizing)

Variables:

- **Decision variables**: x_t quantity produced in period t
- **Auxiliary variables**: I_t quantity of the product available in the inventory at the end of period t.



Inventory management (lot sizing)

Constraints:

- A stockout condition must be avoided (no negative inventory).
- Flow conservation constraints: the continuity equation, $I_t = I_{t-1} + x_t - d_t$, must hold for all periods: the stock after period t is equal to the stock after period $t-1$ minus the quantity of product sold, plus the quantity produced.
- There are no limits on the production capacity in the periods (uncapacitated lot sizing).

Inventory management (lot sizing)

Basic model

$$
\begin{aligned}
z = \min \sum_{t \in T} & (c_t x_t + h_t I_t) & \\
s.t.\ I_t &= I_{t-1} + x_t - d_t & \forall t = 1, \cdots |T| \\
I_t &= I_0 = 0 & \\
x_t &\geq 0 & \forall t \in T
\end{aligned}
$$

Note: It is not necessary to impose a constraint on the value of $I_{|T|}$. It will be 0 anyway. Why?

Inventory management with fixed costs (lot sizing)

What if producing in a given period t implies fixed costs q_t

$$
\begin{aligned}
z = \min \sum_{t \in T} (c_t x_t + q_t y_t + h_t I_t) & \\
s.t.\ I_t = I_{t-1} + x_t - d_t & \qquad \forall t = 1, \cdots |T| \\
I_t = I_0 = 0 & \\
x_t \leq M \cdot y_t & \qquad \forall t \in T \\
y_t \in \{0, 1\} & \qquad \forall t \in T \\
x_t \geq 0 & \qquad \forall t \in T
\end{aligned}
$$

Lot Sizing and Product mix: the basic models can be combined!

The company **LightWeight** produces and markets 4 types of precision scales. The production of each type is organised in batches. The production of each batch requires calibration instruments which have to be prepared and require staff for the setup. Once prepared, they are available for all product types. The company has to plan the annual production, knowing the monthly demand of each product. Each product unit uses 3 components which have a limited availability.

Lot Sizing and Product mix: the basic models can be combined!

Domains:

- Production periods: T : 1..12
- Products:P : 1..4
- Components: C : 1..3

Costs:

- Fixed production costs: The preparation of the instruments causes a fixed cost s independent from the period and the product e.g., $s = 5000$CHF.
- Variable production costs: each product unit has a production cost$c_t, \ \forall t \in T$ which depends on the production period but not on the product type.
- Inventory costs: the firm has a storage with a maximal capacity of q units. Each product unit stored has an inventory cost h independent from the production period and from the type of product.
- Extra inventory costs:The company can decide to rent extra storage space $\hat{q}$ at a monthly cost $\hat{g}$.

Lot Sizing and Product mix: the basic models can be combined!

Problem data:

- Estimated demand in units: $d_{p,t}\ \forall p \in P,\ \forall t \in T$

	1	2	3	4	5	6	7	8	9	10	11	12
1	4	1	2	1	3	3	5	2	3	3	5	2
2	3	1	3	2	1	2	3	1	3	2	1	2
3	1	1	4	2	1	2	2	1	1	1	2	2
4	1	2	4	1	5	2	2	1	1	3	1	2

- Components per product unit: $a_{c,p}\ \forall c \in C,\ \forall p \in P$

		P			
		1	2	3	4
	1	1	1	1	1
C	2	2	2	3	3
	3	7	7	6	6

- Monthly availability of the components: $r_c\ \forall c \in C$

C	1	2	3
r_c	5	6	12

Lot Sizing and Product mix: the basic models can be combined!

Variables:

- $x_{p,t} \geq 0, \ \forall p \in P, \ \forall t \in T$: Quantity of product p produced in the period t
- $I_{p,t} \geq 0, \ \forall p \in P, \ \forall t \in \{0\} \cup T$: Quantity of product p stored in the inventory in the period t (the additional period $\{0\}$ is needed to model the status of the inventory at the beginning of the planning period).
- $y_t \in \{0,1\}, \ \forall t \in T$: Activation of the production in period t.
- $w_t \in \{0,1\}, \ \forall t \in T$: Rental of extra storage space in the period t
- $M >> \ \sum_{p \in P} \sum_{t \in T} d_p, t$: "big-M" constant.

Lot Sizing and Product mix: the basic models can be combined!

$$
\begin{aligned}
\min z = \sum_{p \in P} \sum_{t \in T} (c_t \cdot x_{p,t} + h \cdot I_{p,t}) \quad + \quad & \sum_{t \in T} (s \cdot y_t + \hat{g} \cdot w_t) \\
s.t.\ I_{p,t} = I_{p,t-1} + x_{p,t} - d_{p,t} \quad & \forall p \in P, \forall t \in T \\
I_{p,0} = 0 \quad & \forall p \in P \\
x_{p,t} \leq M \cdot y_t \quad & \forall p \in P, \forall t \in T \\
\sum_{p \in P} I_{p,t} \leq q + \hat{q} \cdot w_t \quad & \forall t \in T \\
\sum_{p \in P} a_{c,p} \cdot x_{p,t} \leq r_c \quad & \forall c \in C, \forall t \in T \\
y_t, w_t \in \{0, 1\} \quad & \forall t \in T \\
x_{p,t} \geq 0 \quad & \forall p \in P, \forall t \in T \\
I_{p,t} \geq 0 \quad & \forall p \in P, \forall t \in \{0\} \cup T
\end{aligned}
$$