1. Find the new coordinates of the vector $a = 5\ e_1 + e_2 + 14\ e_3$ in relation to the following basis:

$$v_1 = 3\ e1 + e_2 + 5\ e_3, \quad v_2 = 2\ e_1 - e_2 + 4\ e_3, \quad v_3 = e_1$$

Sol: $(2,\ 1,\ -3)$

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2. Let us consider the following three polynoms in $\mathcal{P}_2$ defined by

$$p_1(t) = t^2 - 3t + 2\ , \quad p_2(t) = -2t^2 + 3t - 1, \quad p_3(t) = t^2 - 2t + 1 \quad .$$

The set $\{p_1, p_2, p_3\}$ is a basis of $\mathcal{P}_2$? Sol: no, it is not a basis

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3. Let W be the subspace of R^4 generated by the following vectors:

$$v_1 = (1, -2, 5, -3)\ , \quad v_2 = (2, 3, 1, -4)\ , \quad v_3 = (3, 8, -3, -5)$$

a) Find the dimension of W; Sol: dim=2

b) Write down an orthogonal basis of W; Sol: $(1, -2, 5, -3)$ and $(5, 11, -2, -9)$

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4. Consider the following two vectors in R^3:

$$v_1 = (1, 2, 1)\ , \quad v_2 = (1, 1, 1)$$

a) Compute the length of v_1 and v_2.

b) Determine the orthogonal projection of v_1 on v_2

c) Find an orthonormal basis of the subspace of R^3 generated by v_1 and v_2

Sol: (a) $\sqrt{6}, \sqrt{3}$ (b) $(4/3,\ 4/3,\ 4/3)$ (c) $\{1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\}, \{-1/\sqrt{6}, 2/\sqrt{6}, -1/\sqrt{6}\}$

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5. In the vectorspace $V = L^2[-2, 2\ ]$, the standard integral scalar product is defined as

$$< f, g >:= \int_a^b f(x)\overline{g(x)}dx \quad \text{for} \quad V = L^2[a, b]\ .$$

Let us consider the subspace $\mathcal{P}_2$ spanned by the vectors

$$p_0(t) = 1,\ p_1(t) = t,\ p_2(t) = t^2\ ,$$

which corresponds to the so-called standard basis.

a) Calculate an orthonormal basis $s_0(t), s_1(t)$ and $s_2(t)$ of $\mathcal{P}_2$ where $s_0(t)$ is a constant function

b) Find the orthogonal projection of the function $\cos(\pi t/4)$ into this subspace $\mathcal{P}_2$.

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6. Let us consider the vector space V of the linear functions and a vector in V given by the function

$$f_1(x) = 1 + x$$

The norm in V is given by the standard one in $L^2\ [-1, 1]$.

a) Find a fucntion $f_2(x)$ which is orthogonal to $f_1(x)$.

b) Find the orthogonal projection of the function $g(x) = \sin(\pi x/2)$ into the subspace V using as basis f_1 e f_2.

Sol: (a) $1 - 3x$ (b) $12x/\pi^2$

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7. Let us consider the vector subspace V generated by the following two functions:

$$f_1(x) = x \quad , \qquad f_2(x) = \sin(\pi x)$$

a) Find the norm of the two function using the standard norm in $L^2\ [0, 1]$.

b) Find the orthogonal projection of the function $h(x) = \frac{1}{3}$ in the subspace V.

Sol: (a) $1/\sqrt{3}$; $1/\sqrt{2}$ (b) $(3(\pi^2 - 8)x + 2\pi \sin \pi x)/6(\pi^2 - 6)$

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8. Verify by direct computation that:

$$\{1, \ \cos x, \ \sin x, \ldots, \ \cos nx, \ \sin nx\} \ .$$

is a set of orthogonal vectors in relation to the standard scalar product in $L^2([-\pi, \pi])$.

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9. Find the Fourier series of the periodic function $f(x) = x^2$ defined in $[0, 2\pi)$.

Using that series, show that:

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

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10. Find the Fourier series of the periodic functions

$$a) \ \ f(t) = \cos^2 t$$

$$b) \ \ f(t) = \cos(t/2)$$

defined in the interval $-\pi \leq x \leq \pi$.

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11. Compute the Fourier series of the periodic function defined in $[-1, 1)$ by

$$f(x) = 1 - x$$

Sol: $f(x) \sim 1 + 2\sum_{k=1}^{\infty}(-1)^k(\frac{\sin \pi k x}{\pi k})$

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12. The function $g(t)$ is defined by:

$$g(t) = \begin{cases} -1 & \text{if} \quad -1 \leq \mathrm{t} < 0 \\ 0 & \text{if} \quad \mathrm{t} = 0 \\ 2 & \text{if} \quad 0 \leq \mathrm{t} < 1 \end{cases}$$

a) Compute the coefficients of the Fourier series of $g(t)$

b) Write down explicitely the first three terms ($k = 0, 1, 2$) of the series

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13. Consider the periodic function with a period of length π defined as

$$f(t) = (t + \frac{\pi}{4})(\frac{3\pi}{4} - t) \ , \quad -\frac{\pi}{4} \leq t \leq \frac{3\pi}{4}$$

a) Sketch the function for at least 3 periods.

b) Write the Fourier series of the function
