Elements of Signal Theory and Control

Bachelor in Data Science and Artificial Intelligence

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IDSIA - Instituto Dalle Molle di Studi sull'Intelligenza Artificiale

AA 2024-25

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Vector Spaces

Definition

- A vector space V is a set that is closed under finite vector addition and scalar multiplication. If scalars belong to a field $\mathbb{K}$, V is called a vector space over $\mathbb{K}$.
- V is a vector space if addition and scalar multiplication obey the following rules ($\mathbf{x},\mathbf{y},\mathbf{z} \in V$, $a, b \in \mathbb{K}$):
  1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ (commutativity);
  2. $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$ (associativity);
  3. there exists a null vector, $\mathbf{0} \in V$: $\mathbf{0} + \mathbf{x} = \mathbf{x} + \mathbf{0}$;
  4. $\forall$ $\mathbf{x} \in V$, $\exists$ $-\mathbf{x} \in V$: $\mathbf{x} + (-\mathbf{x}) = 0$;
  5. $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$;
  6. $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$;
  7. $a\ (b\ \mathbf{x}) = (a\ b)\ \mathbf{x}$:
  8. $\forall$ $\mathbf{x} \in V$, $1\ \mathbf{x} = \mathbf{x}$.

Examples

1. The set $\mathcal{P}_2 = \{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$ of quadratic polynomials is a vector space under the usual operations of polynomial addition

$$(a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$$

and scalar multiplication.

$$r \cdot (a_0 + a_1x + a_2x^2) = (ra_0) + (ra_1)x + (ra_2)x^2$$

2. The set of 3×3 matrices

$$\mathcal{M}_{3\times 3} = \left\{ \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \middle| \, a_{i,j} \in \mathbb{R} \right\}$$

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Subspace and Basis

- W is a *subspace* of V if $\{a, b \in \mathbb{K} \wedge \mathbf{x}, \mathbf{y} \in W\} \Rightarrow a\mathbf{x} + b\mathbf{y} \ \in W$
- $\{\mathbf{v}_i \ , i = 1 \dots n\} \ \in V$ are *linearly independent* if

$$\sum_{i=1}^{r} c_i \mathbf{v}_i = \mathbf{0} \quad \Longrightarrow \quad c_1 = c_2 = \cdots = c_n = 0 \tag{1}$$

  Otherwise, they are said to be *linearly dependent*.
- The set of vectors $B = \{\mathbf{v}_i \in V, \ i = 1 \dots n\}$ is called a *basis* of V if the vectors are linearly independent and each vector $\mathbf{x} \in V$ can be uniquely expressed as a linear combination of the vectors in B, that is $\mathbf{x} = \sum_{i=1}^{n} c_i \mathbf{v}_i$. The coefficients c_i are called the *coordinates* of the vector $\mathbf{x}$ relative to the basis B.
- The minimal number of vectors generating the vector space V is called the dimension of V.

Subspace

Example: In the vector space $\mathbb{R}^2$, the line $y = 2x$

$$S = \{ \begin{pmatrix} a \\ 2a \end{pmatrix} \mid a \in \mathbb{R} \} = \{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} a \mid a \in \mathbb{R} \}$$

is a subspace. The operations, as required by the definition, are the ones from $\mathbb{R}^2$. We can check all the conditions to show it is a vector space, but the next result gives an easier way.

Example: This subset of $\mathcal{M}_{2\times 2}$ is a subspace.

$$S = \{ \begin{pmatrix} a & b \\ a & b \end{pmatrix} \mid a, b \in \mathbb{R} \} = \{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} a + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} b \mid a, b \in \mathbb{R} \}$$

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Example: This is not a subspace of $\mathbb{R}^3$.

$$T = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + y + z = 1 \}$$

It is a subset of $\mathbb{R}^3$ but it is not a vector space. One condition that it violates is that it is not closed under vector addition: here are two elements of T that sum to a vector that is not an element of T.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

(Another reason that it is not a vector space is that it does not contain the zero vector.)

Example: The vector space of quadratic polynomials $\mathcal{P}_2 = \{ a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R} \}$ has a subspace comprised of the linear polynomials $L = \{ b_0 + b_1 x \mid b_0, b_1 \in \mathbb{R} \}$. To verify that, take scalars $r, s \in \mathbb{R}$ and consider a linear combination.

$$r(b_0 + b_1 x) + s(c_0 + c_1 x) = (rb_0 + sc_0) + (rb_1 + sc_1)x$$

The right side is a linear polynomial with real coefficients, and so is a member of L. Thus L is closed under linear combinations.

Example: Another subspace of $\mathcal{P}_2$ is the set of quadratic polynomials with all three coefficients equal.

$$M = \{ a + ax + ax^2 \mid a \in \mathbb{R} \} = \{ (1 + x + x^2)a \mid a \in \mathbb{R} \}$$

Verify that it is a subspace by taking two scalars $r, s \in \mathbb{R}$ and considering a linear combination of polynomials with all three coefficients the same.

$$r(a + ax + ax^2) + s(b + bx + bx^2) = (ra + sb) + (ra + sb)x + (ra + sb)x^2$$

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Vector decomposition in a given basis

Let $\{\mathbf{v}_i \ , i = 1 \ldots n\}$ be a basis of V. Then $\mathbf{x} \in V$ is given by

$$\mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$$

To obtain the coordinates α_i of $\mathbf{x}$, one has to solve the corresponding linear system

Example: Writing the vector $\mathbf{v} = (4, -3, 2)$ using the basis $v_1 = (1, 0, 0)$, $v_2 = (1, 1, 0)$ e $v_3 = (1, 1, 1)$, one obtains the following linear system:

$$\begin{cases} x + y + z & = 4 \\ x + y & = -3 \\ x & = 2 \end{cases} \quad \Rightarrow \quad \begin{aligned} x &= 2 \\ y &= -5 \\ z &= 7 \end{aligned}$$

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Exercises

1. Write the vector $\mathbf{v} = (1, -2, 5)$ as linear combination of the vectors $e_1 = (1, 1, 1)$, $e_2 = (1, 2, 3)$ and $e_3 = (2, -1, -1)$.
   Is $\{e_1, e_2, e_3\}$ a basis for $\mathbb{R}^3$?
2. For which value of k belongs the vector $\mathbf{v} = (1, -2, k)$ to the subspace of R^3 generated by the vectors $\mathbf{x} = (3, 0, -2)$ and $\mathbf{y} = (2, -1, -5)$?
3. Show that the vectors $e_1 = (1, -1, 0)$, $e_2 = (1, 3, -1)$ and $e_3 = (5, 3, -2)$ are not a basis of $\mathbb{R}^3$.
4. Let W be the space generated by the following polynoms:

$$p_1 = t^3 - 2t^2 + 4t + 1 \quad , \quad p_2 = 2t^3 - 3t^2 + 9t - 1$$

$$p_3 = t^3 + 6\ t - 5 \quad , \quad p_4 = 2t^3 - 5t^2 + 7\ t + 5$$

   Determine the dimension of W and find a basis for this space.

Norms and scalar products

Norm and distance

- V is called a normed space, if to each vector there corresponds a certain non-negative real number called the norm of the vector, such that the following conditions are satisfied:
  1. $\|\mathbf{x}\| = 0 \iff \mathbf{x} = \mathbf{0}$ (identity axiom);
  2. $\|\lambda \mathbf{x}\| = |\lambda| \, \|\mathbf{x}\|$ (homogeneity axiom);
  3. $\|\mathbf{x} + \mathbf{y}\| \leqslant \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangle inequality).
- The distance ρ between two vectors is defined by $\rho(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$
- One defines in $\mathbf{C}^n$ the *p-norm* of a vector $\mathbf{x} = [\xi_1, \ldots, \xi_n]^T$ by the formulas

$$\|\mathbf{x}\|_p = (| \xi_1 |^p + \cdots + | \xi_n |^p)^{\frac{1}{p}} \, , \quad \|\mathbf{x}\|_\infty = \max_i | \xi_i |$$

The same definitions hold for the distance between two vectors.

Scalar product

- A function $< \, , \, >: V \times V \to \mathbf{C}$ is called a *scalar product* on V, if the following properties are satisfied:
  1. $\mathbf{x} \neq \mathbf{0} \Rightarrow \langle \mathbf{x}, \mathbf{x} \rangle > 0$ (positive definite);
  2. $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$ (hermitian);
  3. $\langle \alpha \mathbf{x} + \beta \mathbf{y}, \mathbf{z} \rangle = \alpha \langle \mathbf{x}, \mathbf{z} \rangle + \beta \langle \mathbf{y}, \mathbf{z} \rangle$ (sesquilinear form);
- A scalar product induces a norm on V given by $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.
  These norms are called *hilbertian norms*.
- $| \langle \mathbf{x}, \mathbf{y} \rangle | \leqslant \|\mathbf{x}\| \|\mathbf{y}\|$ (Cauchy-Schwarz inequality)
- Two vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be *orthogonal* when their scalar product is zero.

Norm and scalar product: exercises

1. Show that the following is a scalar product in $\mathbb{R}^2$:

$$< u, v >= x_1 y_1 - x_1 y_2 - x_2 y_1 + 3 x_2 y_2,$$

   with $u = (x_1, x_2)$ and $v = (y_1, y_2)$.

2. Let V be the vector space of the polynoms, where the scalar product is given by $< f, g >= \int_0^1 f(t)\ g(t)\ dt$.
   - Show that $<>$ defines a scalar product
   - $f(t) = t + 2$ and $g(t) = t^2 - 2t - 3$, find (i)$< f, g >$ and (ii)$\|f\|$

Orthonormal basis decomposition

If the basis $\{\mathbf{e}_i \ , i = 1 \dots n\}$ is *orthonormal*, that is:

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}$$

then, the vector $\mathbf{v} \in V$ can be written as:

$$\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{e}_i$$

and the coordinates of $\mathbf{v}$ in the orthonormal basis $\{\mathbf{e}_i\}$ are given by:

$$\langle \mathbf{v}, \mathbf{e}_j \rangle = \sum_{i=1}^{n} \alpha_i \langle \mathbf{e}_i, \mathbf{e}_j \rangle = \sum_{i=1}^{n} \alpha_i \delta_{ij} = \alpha_j$$

Exercises

- Show that

$$e_1 = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$$
$$e_2 = (-2/\sqrt{6}, 1/\sqrt{6}, 1/\sqrt{6})$$
$$e_3 = (0, -1/\sqrt{2}, 1/\sqrt{2})$$

is an orthonormal basis of $\mathbb{R}^3$.

- Find the coordinates of the vector $\mathbf{v} = (1, 0, 1)$ in the previous basis.

$$\alpha_1 = \langle \mathbf{v}, \mathbf{e}_1 \rangle = 2/\sqrt{3}$$

$$\alpha_2 = \langle \mathbf{v}, \mathbf{e}_2 \rangle = -1/\sqrt{6}$$
$$\alpha_3 = \langle \mathbf{v}, \mathbf{e}_3 \rangle = 1/\sqrt{2}$$

Orthogonal projection

- The *orthogonal complement* of the set $W \subset V$ is the set $W^{\perp}$ defined by

$$W^{\perp} := \{\mathbf{x} \in V : \langle \mathbf{x}, \mathbf{y} \rangle = 0, \forall \mathbf{y} \in W\}$$

  $W^{\perp}$ is a subspace of V (even if W is not)

- $V_n \subset V$ generated by the set $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$,

$$\forall \mathbf{x} \in V \; \exists \mathbf{y} \in V_n : \mathbf{x} = \mathbf{y} + \mathbf{z}, \text{ with } \mathbf{z} \in V_n^{\perp}$$

  $\mathbf{y}$ is the *orthogonal projection* of $\mathbf{x}$ on V_n and it holds

$$\|\mathbf{x} - \mathbf{y}\| \leqslant \|\mathbf{x} - \mathbf{y}'\|, \; \forall \mathbf{y}' \in V_n \quad .$$

  If $\{\mathbf{x}_i \;, i = 1 \ldots n\}$ is an orthogonal family, then $\mathbf{y}$ is given by

$$\mathbf{y} = \sum_{k=1}^{n} \frac{\langle \mathbf{x}, \mathbf{x}_k \rangle}{\|\mathbf{x}_k\|^2} \mathbf{x}_k$$

Orthogonal projection

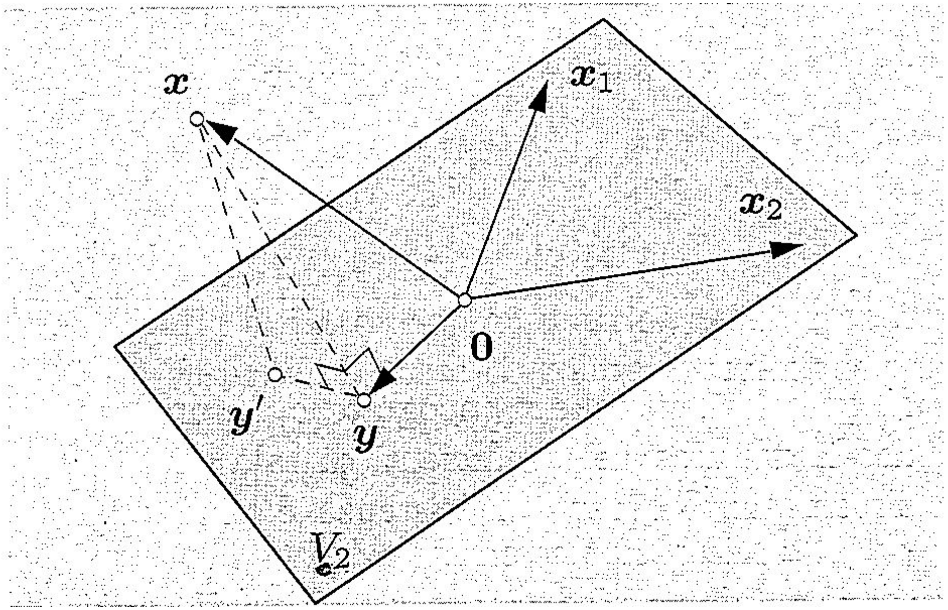

Figura: The orthogonal projection y of x on V_2 is also the point of V_2 with the minimal distance to x

Orthogonal projection: exercises

1. Find the orthogonal projection of the vector $\mathbf{v} = (3, -2, 5)$ in the subspace generated by the vectors $\mathbf{v}_1 = (1, 0, -1)$ and $\mathbf{v}_2 = (-2, -1, 4)$.
2. An important class of vector spaces with complex scalar product is that of Hilbert Spaces, which an example is given by $V = L^2[a, b]$, defined as follows:

$$V = L^2[a, b] = \{f : [a, b] \to \mathbf{C}, \int_a^b \mid f(x) \mid^2 \; dx < \infty\}$$

$$\Rightarrow \; \langle f, g\rangle := \int_a^b f(x)\overline{g(x)}dx$$

Find in $L^2[0, 1]$ the orthogonal projection of the function $f(t) = \sin(\pi t/2)$ in the subspace of dimension 2 generated by the function $x_1(t) = 1$ and $x_2(t) = t$.