Oplossingen Oefeningen Grondslagen 1: Lambda-calculus

Oefening 69

a) Neen, een λ-expressie kan nooit op haakjes eindigen

c) Neen, een λ-expressie kan nooit op haakjes eindigen

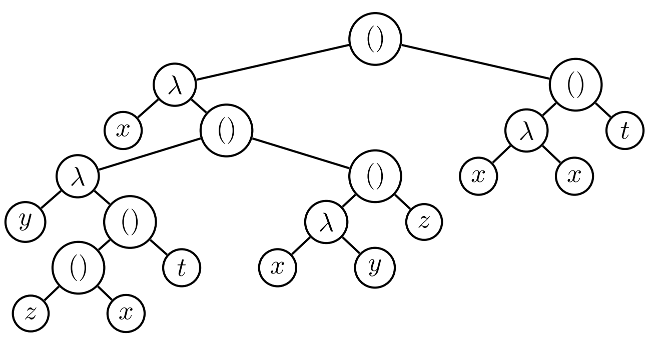
b) OK

d) OK

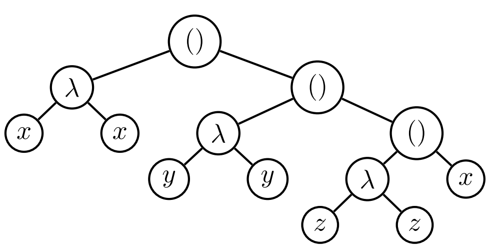
e) OK

Oefening 70

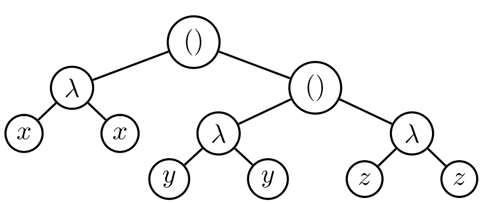
a)



b)

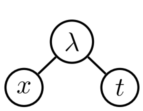


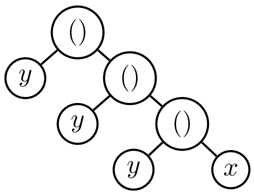
c)

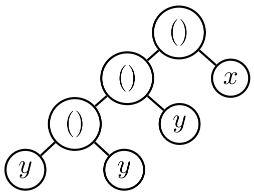


d)

e)



f) 

g) 

Oefening 72

a) $\lambda x.\lambda y.\lambda z.((=)((+)(^2)x)(^2)y)(^2)z$

b) $\lambda x.\lambda y.\lambda z.((-)x)((||||)y)z$

Oefening 75

- $VV : \Lambda \to 2^{|V|}$
- $VV(x) = \{x\}$
- $VV((M)N) = VV(M) \cup VV(N)$
- $VV(\lambda x.M) = VV(M) - \{x\}$

a)

$$\begin{aligned}
VV(\lambda f.\lambda x.((f)(x)x)(f)(x)x) &= VV(\lambda x.((f)(x)x)(f)(x)x) - \{f\} \\
&= VV(((f)(x)x)(f)(x)x) - \{x\} - \{f\} \\
&= \{VV((f)(x)x) \cup VV((f)(x)x)\} - \{x, f\} \\
&= \{VV(f) \cup VV((x)x) \cup VV(f) \cup VV((x)x)\} - \{x, f\} \\
&= \{VV(f) \cup VV(x) \cup VV(x) \cup VV(f) \cup VV(x) \cup VV(x)\} - \{x, f\} \\
&= \{x, f\} - \{f, x\} \\
&= \{\}
\end{aligned}$$

$\Rightarrow$ Combinator

b)

$$\begin{aligned}
VV(\lambda f.(x)y) &= VV((x)y) - \{f\} \\
&= \{VV(x) \cup VV(y)\} - \{f\} \\
&= \{x, y\} - \{f\} \\
&= \{x, y\}
\end{aligned}$$

$\Rightarrow$ Geen combinator

c)

$$
\begin{aligned}
VV(\lambda g.(\lambda y.(x)y)\lambda z.z) &= VV((\lambda y.(x)y)\lambda z.z) - \{g\} \\
&= \{VV(\lambda y.(x)y) \cup VV(\lambda z.z)\} - \{g\} \\
&= \{\{VV((x)y) - \{y\}\} \cup \{VV(z) - \{z\}\}\} - \{g\} \\
&= \{\{VV(x) \cup VV(y) - \{y\}\} \cup \{\{z\} - \{z\}\}\} - \{g\} \\
&= \{\{\{x,y\} - \{y\}\} \cup \{\}\} - \{g\} \\
&= \{\{x\} \cup \{\}\} - \{g\} \\
&= \{x\}
\end{aligned}
$$

$\Rightarrow$ Geen combinator

Oefening 76

a) $[\lambda x.x/x]\lambda f.\lambda x.((f)(x)x)(f)(x)x$

$$
\begin{aligned}
&= \lambda f.[\lambda x.x/x]\lambda x.((f)(x)x)(f)(x)x && S5 \\
&= \lambda f.\lambda x.((f)(x)x)(f)(x) && S4
\end{aligned}
$$

b) $[\lambda x.(f)x/y]\lambda f.(x)y$

$$
\begin{aligned}
&= \lambda f'.[\lambda x.(f)x/y][f'/f](x)y && S6 \\
&= \lambda f'.[\lambda x.(f)x/y]([f'/f]x)[f'/f]y && S3 \\
&= \lambda f'.[\lambda x.(f)x/y](x)y && S2, S2 \\
&= \lambda f'.([\lambda x.(f)x/y]x)[\lambda x.(f)x/y]y && S3 \\
&= \lambda f'.(x)\lambda x.(f)x && S2, S1
\end{aligned}
$$

c) $[(y)(g)z/x]\lambda g.(\lambda y.(x)y)\lambda z.z$

$$
\begin{aligned}
&= \lambda g'.[(y)(g)z/x][g'/g](\lambda y.(x)y)\lambda z.z && S6 \\
&= \lambda g'.[(y)(g)z/x]([g'/g]\lambda y.(x)y)[g'/g]\lambda z.z && S3 \\
&= \lambda g'.[(y)(g)z/x](\lambda y.[g'/g](x)y)\lambda z.[g'/g]z && S5, S5 \\
&= \lambda g'.[(y)(g)z/x](\lambda y.([g'/g]x)[g'/g]y)\lambda z.z && S3, S2 \\
&= \lambda g'.[(y)(g)z/x](\lambda y.(x)y)\lambda z.z && S2, S2 \\
&= \lambda g'.([(y)(g)z/x]\lambda y.(x)y)[(y)(g)z/x]\lambda z.z && S3 \\
&= \lambda g'.(\lambda y'.[(y)(g)z/x][y'/y](x)y)\lambda z'.[(y)(g)z/x][z'/z]z && S6, S6 \\
&= \lambda g'.(\lambda y'.[(y)(g)z/x]([y'/y]x)[y'/y]y)\lambda z'.[(y)(g)z/x]z' && S3, S1 \\
&= \lambda g'.(\lambda y'.[(y)(g)z/x](x)y')\lambda z'.z' && S2, S1, S2 \\
&= \lambda g'.(\lambda y'.([(y)(g)z/x]x)[(y)(g)z/x]y')\lambda z'.z' && S3 \\
&= \lambda g'.(\lambda y'.((y)(g)z)y')\lambda z'.z' && S1, S2
\end{aligned}
$$

Oefening 79

$$\begin{aligned}
(succ)c_4 &\equiv (\lambda n.\lambda f.\lambda x.(f)((n)f)x)c_4 \\
&=_\beta \lambda f.\lambda x.(f)((c_4)f)x \\
&\equiv \lambda f.\lambda x.(f)((\lambda a.\lambda b.(a)(a)(a)(a)b)f)x \\
&=_\beta \lambda f.\lambda x.(f)(\lambda b.(f)(f)(f)(f)b)x \\
&=_\beta \lambda f.\lambda x.(f)(f)(f)(f)(f)x \\
&\equiv c_5
\end{aligned}$$

$$\begin{aligned}
((plus)c_2)c_2 &\equiv ((\lambda n.\lambda m.\lambda f.\lambda x.((n)f)((m)f)x)c_2)c_2 \\
&=_\beta (\lambda m.\lambda f.\lambda x.((c_2)f)((m)f)x)c_2 \\
&=_\beta \lambda f.\lambda x.((c_2)f)((c_2)f)x \\
&\equiv \lambda f.\lambda x.((c_2)f)((\lambda a.\lambda b.(a)(a)b)f)x \\
&=_\beta \lambda f.\lambda x.((c_2)f)(\lambda b.(f)(f)b)x \\
&=_\beta \lambda f.\lambda x.((c_2)f)(f)(f)x \\
&\equiv \lambda f.\lambda x.((\lambda a.\lambda b.(a)(a)b)f)(f)(f)x \\
&=_\beta \lambda f.\lambda x.(\lambda b.(f)(f)b)(f)(f)x \\
&=_\beta \lambda f.\lambda x.(f)(f)(f)(f)x \\
&\equiv c_4
\end{aligned}$$

$$\begin{aligned}
(((cons)c_1)c_2)car &\equiv (((\lambda a.\lambda d.\lambda z.((z)a)d)c_1)c_2)car \\
&=_\beta ((\lambda d.\lambda z.((z)c_1)d)c_2)car \\
&=_\beta (\lambda z.((z)c_1)c_2)car \\
&=_\beta ((car)c_1)c_2 \\
&\equiv ((\lambda a.\lambda b.a)c_1)c_2 \\
&=_\beta (\lambda b.c_1)c_2 \\
&=_\beta c_2
\end{aligned}$$

$$\begin{aligned}
((plus)c_3)c_2 &\equiv ((\lambda n.\lambda m.\lambda f.(n)(m)f)c_3)c_2 \\
&=_\beta (\lambda m.\lambda f.(c_3)(m)f)c_2 \\
&=_\beta \lambda f.(c_3)(c_2)f \\
&\equiv \lambda f.(c_3)(\lambda a.\lambda b.(a)(a)b)f \\
&=_\beta \lambda f.(c_3)\lambda b.(f)(f)b \\
&\equiv \lambda f.(\lambda d.\lambda e.(d)(d)(d)e)\lambda b.(f)(f)b \\
&=_\beta \lambda f.\lambda e.(\lambda b.(f)(f)b)(\lambda b.(f)(f)b)(\lambda b.(f)(f)b)e \\
&=_\beta \lambda f.\lambda e.(\lambda b.(f)(f)b)(\lambda b.(f)(f)b)(f)(f)e \\
&=_\beta \lambda f.\lambda e.(\lambda b.(f)(f)b)(f)(f)(f)(f)e \\
&=_\beta \lambda f.\lambda e.(f)(f)(f)(f)(f)(f)e \\
&\equiv c_6
\end{aligned}$$

Oefening 80

1. (a) $and \equiv \lambda a.\lambda b.(((if)a)b)a$

(b) $or \equiv \lambda a.\lambda b.(((if)a)a)b$
(c) $not \equiv \lambda a.(((if)a)false)true$

2. $pow \equiv \lambda n.\lambda m.(n)m$

Oefening 81

1. Let op: $a = b \Leftrightarrow a - b = 0$, maar in λ-calculus is $10 - 200 = 0$, maar $200 - 10 \neq 0$! Dus $a = b \Leftrightarrow a - b = 0 \wedge b - a = 0$

   $=\equiv \lambda a.\lambda b.((and)(iszero)((minus)a)b)(iszero)((minus)b)a$

2. $a < b \Leftrightarrow a - b = 0 \wedge a \neq b$

   $<\equiv \lambda a.\lambda b.((and)(iszero)((minus)a)b)(not)((=)a)b$

3. F c_n keer op A toepassen:

   $repeat \equiv \lambda n.\lambda f.\lambda a.((n)f)a$

Oefening 83

De regeltjes:

1. Schrijf de λ-expressie "gewoon" neer
2. Als er recursie is
   (a) Vervang $\equiv$ door $=_\beta$
   (b) Plaats de recursie buiten haakjes
   (c) Definieer de λ-expressie als een fixpunt

En dan nu de oefening:

1. $fib \equiv \lambda n.(((if)((<)n)c_2)c_1)((plus)(fib)(pred)n)(fib)(pred)(pred)n$

   Maar dit is een lang woord door recursief, dus!

   $fib =_\beta (\lambda f.\lambda n.(((if)((<)n)c_2)c_1)((plus)(f)(pred)n)(f)(pred)(pred)n)fib$

   We nemen nu FIB gelijk aan $\lambda f.\lambda n.(((if)((<)n)c_2)c_1)((plus)(f)(pred)n)(f)(pred)(pred)n$

   $fib \equiv (Y)FIB$

2. $ggd \equiv \lambda a.\lambda b.(((if)((=)a)b)a)(((if)((<)a)b)((ggd)((minus)b)a)a)((ggd)((minus)a)b)b$

   Maar dit is een lang woord door recursief, dus!

   $ggd =_\beta (\lambda g.\lambda a.\lambda b.(((if)((=)a)b)a)(((if)((<)a)b)((g)((minus)b)a)a)((g)((minus)a)b)b)ggd$

   We nemen nu GGD gelijk aan $\lambda g.\lambda a.\lambda b.(((if)((=)a)b)a)(((if)((<)a)b)((g)((minus)b)a)a)((g)((minus)a)b)b$

   $ggd \equiv (Y)GGD$

Oefening 84

1. $F \equiv \lambda m.(m)m$, geen recursie dus OK
2. $F \equiv \lambda m.(m)F$, recursie dus
   (a) $F =_\beta \lambda m.(m)F$
   (b) $F =_\beta (\lambda f.\lambda m.(m)f)F$
   (c) $F \equiv (Y)\lambda f.\lambda m.(m)f$

3. $F \equiv \lambda m.F$, recursie dus
   (a) $F =_\beta \lambda m.F$
   (b) $F =_\beta (\lambda f.\lambda m.f)F$
   (c) $F \equiv (Y)\lambda f.\lambda m.f$
4. $F \equiv \lambda m.\lambda n.((m)(n)m)n$, geen recusrie dus OK

Een noot bij de Y-combinator

Neem oefening 83.2 "Maak een λ-expressie F zodanig dat voor elke λ-expressie M geldt: $(F)M =_\beta (M)F$"
$F \equiv \lambda m.F$, recursie dus

1. $F \equiv \lambda m.(m)F$, recursie dus
   (a) $F =_\beta \lambda m.(m)F$
   (b) $F =_\beta (\lambda f.\lambda m.(m)f)F$
   (c) $F \equiv (Y)\lambda f.\lambda m.(m)f$

Is de $(F)M$ wel nu de $(M)F$? Laten we D nemen voor $\lambda f.\lambda m.(m)f$, dus dan is $F \equiv (Y)D$

$$\begin{aligned}
(F)M &\equiv ((Y)D)M \\
&\equiv ((\lambda f.(\lambda x.(f)(x)x)\lambda x.(f)(x)x)D)M \\
&=_\beta ((\lambda x.(D)(x)x)\lambda x.(D)(x)x)M \\
&=_\beta (D)(\lambda x.(D)(x)x)\lambda x.(D)(x)x)M \\
&=_\beta (D)((\lambda f.\lambda x.(f)(x)x)\lambda x.(f)(x)x)D)M \\
&\equiv (D)((Y)D)M \\
&\equiv (\lambda f.\lambda m.(m)f)(Y)D)M \\
&=_\beta (\lambda m.(m)(Y)D)M \\
&=_\beta (M)(Y)D \\
&\equiv (M)F
\end{aligned}$$