Control Theory Homework 3

Selina Varouqa

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1 Variant

Name: Selina
Email: s.varouqa@innopolis.university
The variant for this assignment is C
Values: $M = 15.1; m = 1.2; l = 0.35$

2 Manipulator Form

Since our system is following the following dynamics:
$F = (M + m)\ddot{x} - mlcos(\theta)\ddot{\theta} + mlsin(\theta)\dot{\theta}^2$
$0 = -cos(\theta)\ddot{x} + l\ddot{\theta} - gsin(\theta)$
where (g = 9.81; M = 15.1; m = 1.2; l = 0.35) and the common structure is:

$F = a\ddot{x} + b\ddot{\theta} + c\dot{\theta}$
$0 = d\ddot{x} + e\ddot{\theta} + g\theta$

so from above we can see that: $a = M + m$, $b = mlcos(\theta)$, $c = mlsin(\theta)$, $d = -cos(\theta)$, $e = l$, and $g = gsin(\theta)$ and since our manipulator form should be like this:

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} c\dot{\theta} \\ g\theta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} F$$

we will have out manipulator form as this:

$$\begin{bmatrix} M + m & mlcos(\theta) \\ -cos(\theta) & l \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} mlsin(\theta)\dot{\theta}^2 \\ gsin(\theta) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} F$$

substituting the values of M, m, and l:

$$\begin{bmatrix} 16.2 & 0.42cos(\theta) \\ -cos(\theta) & 0.35 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0.42sin(\theta)\dot{\theta}^2 \\ 9.81sin(\theta) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} F$$

3 Dynamics in Control Affine Nonlinear Form

We use this manipulator form from first section:

$$\begin{bmatrix} 16.2 & 0.42cos(\theta) \\ -cos(\theta) & 0.35 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0.42sin(\theta)\dot{\theta}^2 \\ 9.81sin(\theta) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} F$$

we put it into this form:

$$\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix} = -\begin{bmatrix}16.2 & 0.42cos(\theta)\\ -cos(\theta) & 0.35\end{bmatrix}^{-1}\begin{bmatrix}0.42sin(\theta)\dot{\theta}^2\\ 9.81sin(\theta)\end{bmatrix} + \begin{bmatrix}16.2 & 0.42cos(\theta)\\ -cos(\theta) & 0.35\end{bmatrix}^{-1}\begin{bmatrix}1\\ 0\end{bmatrix}\text{F}$$

by calculating the inverse we get:

$$\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix} = \frac{-1}{5.67+0.42cos^2(\theta)}\begin{bmatrix}0.35 & -0.42cos(\theta)\\ cos(\theta) & 16.2\end{bmatrix}\begin{bmatrix}0.42sin(\theta)\dot{\theta}^2\\ 9.81sin(\theta)\end{bmatrix} + \frac{1}{5.67+0.42cos^2(\theta)}\begin{bmatrix}0.35 & -0.42cos(\theta)\\ cos(\theta) & 16.2\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}F$$

and then:

$$\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix} = \frac{-1}{5.67+0.42cos^2(\theta)}\begin{bmatrix}0.147sin(\theta)\dot{\theta}^2 - 4.1202cos(\theta)sin(\theta)\\ 0.42cos(\theta)sin(\theta)\dot{\theta}^2 + 158.922sin(\theta)\end{bmatrix} + \frac{1}{5.67+0.42cos^2(\theta)}\begin{bmatrix}0.35\\ cos(\theta)\end{bmatrix}F$$

and since $\dot{z} = f(z) = +g(z)u$ and $u = F$ then:

$$\dot{z} = \begin{bmatrix}\dot{x}\\ \dot{\theta}\\ \ddot{x}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{x}\\ \dot{\theta}\\ \frac{-0.147sin(\theta)\dot{\theta}^2+4.1202cos(\theta)sin(\theta)}{5.67+0.42cos^2(\theta)}\\ \frac{-0.42cos(\theta)sin(\theta)\dot{\theta}^2-158.922sin(\theta)}{5.67+0.42cos^2(\theta)}\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \frac{0.35}{5.67+0.42cos^2(\theta)}\\ \frac{cos(\theta)}{5.67+0.42cos^2(\theta)}\end{bmatrix}F$$

4 Linearization of Nonlinear Dynamics around Equilibrium Point

$$\bar{z} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}\ ;\ A = \frac{\partial f(z,u)}{\partial z}\bigg|_{\substack{z=z(0)=\bar{z}\\ u=u(0)=0}} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & \frac{mg}{M} & 0 & 0\\ 0 & \frac{(M+m)g}{M} & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0.77 & 0 & 0\\ 0 & 10.6 & 0 & 0\end{bmatrix}$$

$$B = \frac{\partial f(z,u)}{\partial u}\bigg|_{\substack{z=z(0)=\bar{z}\\ u=u(0)=0}} = \begin{bmatrix}0\\ 0\\ \frac{1}{M}\\ \frac{1}{Ml}\end{bmatrix} = \begin{bmatrix}0\\ 0\\ \frac{1}{15.1}\\ \frac{1}{5.3}\end{bmatrix}$$

so our final representation would be:

$$\partial\dot{z} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0.77 & 0 & 0\\ 0 & 10.6 & 0 & 0\end{bmatrix}\partial z + \begin{bmatrix}0\\ 0\\ \frac{1}{15.1}\\ \frac{1}{5.3}\end{bmatrix}\partial u$$

5 Stability of Linearized System

To check the stability of our system, we need to calculate eigenvalues of the matrix A, it is calculated in the file upload, seeing the results we can see that not all our eigenvalues are positive: [0. 0. 3.25576412 -3.25576412] which means that our system is unstable, which actually makes sense since it's an inverted pendulum.

6 Controllability of Linearized System

To check if the system is controllable, its controllability matrix has to have a full rank of the same dimension as A. The controllability matrix should is as follows: $C = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & 0.065 & 0 & 0.14476 \\ 0 & 0.188 & 0 & 1.9928 \\ 0.065 & 0 & 0.14476 & 0 \\ 0.188 & 0 & 1.9928 & 0 \end{bmatrix}$

It gives that the rank is 4, which is the same row dimension of A, which means that the system is controllable.