Optimal Control of LTI systems

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Hamilton-Jacobi-Bellman equation
Cost and optimal cost

Let as define dynamics:

$$\dot{\mathsf{x}} = \mathsf{f}(\mathsf{x}, \mathsf{u})$$

with initial conditions $\mathsf{x}(0)$. Let J be an additive cost function:

$$J(\mathsf{x}(0), \mathsf{u}(t)) = \int_0^\infty g(\mathsf{x}, \mathsf{u}) dt$$

where $g(\mathsf{x}, \mathsf{u})$ is instantaneous cost.

Let J^* be the optimal (lowest possible) cost. In other words:

$$J^*(\mathsf{x}(0)) = \inf_{\mathsf{u}(t)} J(\mathsf{x}(0), \mathsf{u}(t))$$

Since $J^*(\mathsf{x}(0))$ does not depend on t, its full derivative is zero:

$$\frac{dJ^*(\mathsf{x}(0))}{dt} = 0$$

At the same time, we can expand the full derivative as follows:

$$\frac{dJ^*}{dt} = \frac{\partial J^*}{\partial \mathsf{x}}\dot{\mathsf{x}} + \frac{\partial J^*}{\partial t} = 0$$

Observe that $\frac{\partial J^*}{\partial t} = g(\mathsf{x}, \mathsf{u})$, and $\dot{\mathsf{x}} = f(\mathsf{x}, \mathsf{u})$. Therefore:

$$\frac{\partial J^*}{\partial \mathsf{x}}f(\mathsf{x}, \mathsf{u}) + g(\mathsf{x}, \mathsf{u}) = 0$$

With this, we can formulate Hamilton-Jacobi-Bellman equation (HJB):

$$\min_u \left[g(\mathsf{x}, \mathsf{u}) + \frac{\partial J^*}{\partial \mathsf{x}} f(\mathsf{x}, \mathsf{u}) \right] = 0$$

And therefore:

$$u^* = \arg\min_u \left[g(\mathsf{x}, \mathsf{u}) + \frac{\partial J^*}{\partial \mathsf{x}} f(\mathsf{x}, \mathsf{u}) \right]$$

For LTI, dynamics is:

$$\dot{x} = Ax + Bu$$

We can choose quadratic cost:

$$g(x, u) = x^\top Q x + u^\top R u$$

Then HJB becomes:

$$\min_u \left[x^\top Q x + u^\top R u + \frac{\partial J^*}{\partial x}(Ax + Bu) \right] = 0$$

where $Q = Q^\top \geq 0$ and $R = R^\top > 0$.

There is a theorem that says that for LTI with quadratic cost, J^* has the form:

$$J^* = x^\top S x$$

where $S = S^\top$.

Then HJB becomes:

$$\min_u \ [x^\top Q x + u^\top R u + 2x^\top S(Ax + Bu)] = 0$$

Finding partial derivative of the HJB with respect to u and setting it to zero (as it is an extreme point) we get:

$$2u^\top R + 2x^\top SB = 0$$

This expression can be transposed and u separated:

$$u = -R^{-1}B^\top Sx$$

This is the desired control law. We can see that it is *proportional*. We can re-write it as:

$$u = -Kx$$

where $K = R^{-1}B^\top S$ is the controller gain. This control law is called Linear Quadratic Regulator (LQR).

Substituting found control law into the HJB, we find:

$$x^\top Qx + x^\top SBR^{-1}RR^{-1}B^\top Sx + 2x^\top S(Ax - BR^{-1}B^\top Sx) = 0$$

After grouping the terms, we get:

$$x^\top(Q - SBR^{-1}B^\top S + 2SA)x = 0$$

which would hold for all x iff:

$$Q - SBR^{-1}B^\top S + 2SA = 0$$

This is the Algebraic Riccati equation.

There are a number of ways to solve LQR:

- In MATLAB there is a function [K,S] = lqr(A, B, Q, R)
- In Python, there is S = scipy.linalg.solve_continuous_are(a, b, q, r)
- In Drake (by MIT and Toyota Research) there is a function (K,S) = LinearQuadraticRegulator(A,B,Q,R)

Lecture slides are available via Moodle.

Check Moodle for additional links, videos, textbook suggestions.