

# Optimal Control of LTI systems

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# Hamilton-Jacobi-Bellman equation

## Cost and optimal cost

Let us define dynamics:

$$\dot{x} = f(x, u)$$

with initial conditions  $x(0)$ . Let  $J$  be an additive cost function:

$$J(x(0), u(t)) = \int_0^\infty g(x, u) dt$$

where  $g(x, u)$  is instantaneous cost.

Let  $J^*$  be the optimal (lowest possible) cost. In other words:

$$J^*(x(0)) = \inf_{u(t)} J(x(0), u(t))$$

# Hamilton-Jacobi-Bellman equation

## Differentiating optimal cost

Since  $J^*(x(0))$  does not depend on  $t$ , its full derivative is zero:

$$\frac{dJ^*(x(0))}{dt} = 0$$

At the same time, we can expand the full derivative as follows:

$$\frac{dJ^*}{dt} = \frac{\partial J^*}{\partial x} \dot{x} + \frac{\partial J^*}{\partial t} = 0$$

Observe that  $\frac{\partial J^*}{\partial t} = g(x, u)$ , and  $\dot{x} = f(x, u)$ . Therefore:

$$\frac{\partial J^*}{\partial x} f(x, u) + g(x, u) = 0$$

# Hamilton-Jacobi-Bellman equation

HJB

With this, we can formulate Hamilton-Jacobi-Bellman equation (HJB):

$$\min_u \left[ g(x, u) + \frac{\partial J^*}{\partial x} f(x, u) \right] = 0$$

And therefore:

$$u^* = \arg \min_u \left[ g(x, u) + \frac{\partial J^*}{\partial x} f(x, u) \right]$$

For LTI, dynamics is:

$$\dot{x} = Ax + Bu$$

We can choose quadratic cost:

$$g(x, u) = x^T Q x + u^T R u$$

Then HJB becomes:

$$\min_u [x^T Q x + u^T R u + \frac{\partial J^*}{\partial x} (Ax + Bu)] = 0$$

where  $Q = Q^T \geq 0$  and  $R = R^T > 0$ .

# Algebraic Riccati

## HJB for LTI, part 2

There is a theorem that says that for LTI with quadratic cost,  $J^*$  has the form:

$$J^* = \mathbf{x}^\top \mathbf{S} \mathbf{x}$$

where  $\mathbf{S} = \mathbf{S}^\top \geq 0$ .

Differentiating  $J^*$  with respect to  $\mathbf{x}$  gives us:

$$\frac{\partial J^*}{\partial \mathbf{x}} = 2\mathbf{x}^\top \mathbf{S}$$

Then HJB becomes:

$$\min_{\mathbf{u}} [\mathbf{x}^\top \mathbf{Q} \mathbf{x} + \mathbf{u}^\top \mathbf{R} \mathbf{u} + 2\mathbf{x}^\top \mathbf{S} (\mathbf{A} \mathbf{x} + \mathbf{B} \mathbf{u})] = 0$$

Finding partial derivative of the HJB with respect to  $u$  and setting it to zero (as it is an extreme point) we get:

$$2u^T R + 2x^T S B = 0$$

This expression can be transposed and  $u$  separated:

$$u = -R^{-1} B^T S x$$

This is the desired control law. We can see that it is *proportional*. We can re-write it as:

$$u = -Kx$$

where  $K = R^{-1} B^T S$  is the controller gain. This control law is called Linear Quadratic Regulator (LQR).



Substituting found control law into the HJB, we find:

$$\mathbf{x}^\top \mathbf{Q} \mathbf{x} + \mathbf{x}^\top \mathbf{S} \mathbf{B} \mathbf{R}^{-1} \mathbf{R} \mathbf{R}^{-1} \mathbf{B}^\top \mathbf{S} \mathbf{x} + 2 \mathbf{x}^\top \mathbf{S} (\mathbf{A} \mathbf{x} - \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^\top \mathbf{S} \mathbf{x}) = 0$$

After grouping the terms, we get:

$$\mathbf{x}^\top (\mathbf{Q} - \mathbf{S} \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^\top \mathbf{S} + 2 \mathbf{S} \mathbf{A}) \mathbf{x} = 0$$

which would hold for all  $\mathbf{x}$  iff:

$$\mathbf{Q} - \mathbf{S} \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^\top \mathbf{S} + 2 \mathbf{S} \mathbf{A} = 0$$

This is the Algebraic Riccati equation.

# Algebraic Riccati

## Numerical methods

There are a number of ways to solve LQR:

- In MATLAB there is a function  $[K,S,P] = \text{lqr}(A, B, Q, R)$ , where  $P = \text{eig}(A - B*K)$
- In Python, there is  $S = \text{scipy.linalg.solve\_continuous\_are}(a, b, q, r)$
- In Drake (by MIT and Toyota Research) there is a function  $(K,S) = \text{LinearQuadraticRegulator}(A,B,Q,R)$

Lecture slides are available via Moodle.

Check Moodle for additional links, videos, textbook suggestions.