

# Linear Algebra for LTI systems

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- Which states are possible?
- How to check if the state is possible?
- Check if affine system is stabilizable
- Criteria without orthonormal basis
- Checking if an arbitrary point can be stabilized
- All stabilizable points

# Which states are possible?

Consider discrete autonomous LTI system:

$$\mathbf{x}_{i+1} = \mathbf{A}\mathbf{x}_i \quad (1)$$

where  $\mathbf{x} \in \mathbb{R}^n$  is the state of the system, expressed in the basis  $\mathbf{O}$ .  
What possible values can  $\mathbf{x}_{i+1}$  attain?

From the (1) follows that all possible  $\mathbf{x}_{i+1}$  are in the *column space* of  $\mathbf{A}$ .

$$\mathbf{x}_{i+1} \in \mathcal{X} = \text{col}(\mathbf{A}) \subseteq \mathbb{R}^n$$

# How to check if the state is possible?

## Part 1

How can we tell if a particular value of  $\mathbf{x}_{i+1}$  is possible? More general, having an the equation  $\mathbf{h} = \mathbf{A}\mathbf{x}$ , how do check that for a particular  $\mathbf{h}$ ,  $\exists \mathbf{x}$ , s.t. the equality is satisfied?

Let  $\mathbf{P}$  be an orthonormal basis in the column space of  $\mathbf{A}$ :

$$\mathbf{P} = \text{orth}(\mathbf{A})$$

Columns of  $\mathbf{P}$  span the column space of  $\mathbf{A}$ , or in other words, the column spaces of  $\mathbf{A}$  and  $\mathbf{P}$  are the same:  $\mathcal{X} = \text{col}(\mathbf{A}) = \text{col}(\mathbf{P})$ .

# How to check if the state is possible?

## Part 2

Vector  $\mathbf{h}$  is expressed in the basis  $\mathbf{O}$ , which we can denote as  $\mathbf{h}^{\mathbf{O}}$ . Then, we can find coordinates of the projection of  $\mathbf{h}$  onto  $\mathcal{X}$  in the basis  $\mathbf{P}$ :

$$\mathbf{h}_p^{\mathbf{P}} = \mathbf{P}^{\top} \mathbf{h}^{\mathbf{O}}$$

In the basis  $\mathbf{O}$ , vector  $\mathbf{h}_p$  is given by the equation  $\mathbf{h}_p^{\mathbf{O}} = \mathbf{P} \mathbf{h}_p^{\mathbf{P}}$ :

$$\mathbf{h}_p^{\mathbf{O}} = \mathbf{P} \mathbf{P}^{\top} \mathbf{h}^{\mathbf{O}}$$

Notice, that if vector  $\mathbf{h}$  lies in the column space  $\mathcal{X}$ , its projection onto  $\mathcal{X}$ , namely  $\mathbf{h}_p^{\mathbf{O}}$ , should be equal to  $\mathbf{h}$ . Let us define projection residual  $\mathbf{e}$ :

$$\mathbf{e}^{\mathbf{O}} = \mathbf{h}_p^{\mathbf{O}} - \mathbf{h}^{\mathbf{O}}$$

Therefore we can formulate that  $\mathbf{h} \in \mathcal{X}$  iff  $\mathbf{e} = 0$ .

# How to check if the state is possible?

## Part 3

Now we can formulate the condition for  $\mathbf{h} \in \mathcal{X} = \text{col}(\mathbf{A})$ :

$$\mathbf{P}\mathbf{P}^\top \mathbf{h} - \mathbf{h} = 0$$

or:

$$(\mathbf{P}\mathbf{P}^\top - \mathbf{I})\mathbf{h} = 0$$

where  $\mathbf{P} = \text{orth}(\mathbf{A})$

# Check if affine system is stabilizable

## Part 1

Consider affine linear system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c} \quad (2)$$

Can it be stabilized? Let the control law be give affine:

$$\mathbf{u} = -\mathbf{K}\mathbf{x} - \mathbf{u}_0$$

We want to find such  $\mathbf{K}$  and  $\mathbf{u}_0$  that:

- dynamics of the system becomes  $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x}$ .
- $\mathbf{A} - \mathbf{BK} < 0$ .

# Check if affine system is stabilizable

## Part 2

In order for the  $\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c} \\ \mathbf{u} = -\mathbf{K}\mathbf{x} - \mathbf{u}_0 \end{cases}$  to become  $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}$ , the following equality needs to hold:

$$\mathbf{B}\mathbf{u}_0 = \mathbf{c}$$

In other words,  $\mathbf{c}$  needs to be in the column space of  $\mathbf{B}$ . And we know the conditions we need to check:

$$(\mathbf{P}\mathbf{P}^\top - \mathbf{I})\mathbf{c} = \mathbf{0},$$

where  $\mathbf{P} = \text{orth}(\mathbf{B})$



# Criteria without orthonormal basis

Finding an orthonormal basis might be excessive for this task, and there is a more direct way of checking if for a given  $\mathbf{h}$ ,  $\exists \mathbf{x}$ , s.t.  $\mathbf{h} = \mathbf{A}\mathbf{x}$ . If there exists  $\mathbf{x}^*$  such that  $\mathbf{h} = \mathbf{A}\mathbf{x}^*$ , then it can be found as:

$$\mathbf{x}^* = \operatorname{argmin} \|\mathbf{h} - \mathbf{A}\mathbf{x}\|$$

Solution to this *linear least squares problem* is a psuedo inverse:

$$\mathbf{x}^* = \mathbf{A}^+ \mathbf{h}$$

Therefore, projection residual equation can be written as:

$$\mathbf{e} = \mathbf{A}\mathbf{x}^* - \mathbf{h}$$

Same as before, original question then comes down to proving that  $\mathbf{e} = 0$ :

$$\mathbf{A}\mathbf{A}^+ \mathbf{h} - \mathbf{h} = 0$$

or

$$(\mathbf{A}\mathbf{A}^+ - \mathbf{I})\mathbf{h} = 0$$

# Checking if an arbitrary point can be stabilized

Consider a linear system:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{u} = -\mathbf{K}(\mathbf{x} - \mathbf{x}_0) - \mathbf{u}_0 \end{cases} \quad (3)$$

Is point  $\mathbf{x}_0$  stabilizable? What this question means, is can we find such  $\mathbf{K}$  and  $\mathbf{u}_0$  that the system converges to  $\mathbf{x}_0$ ?

Assume we are at the point  $\mathbf{x}_0$ . Then  $\mathbf{K}(\mathbf{x} - \mathbf{x}_0) = 0$ . Can we make sure we stay at this point? If not, it is not stabilized.

This comes down to solving equation:

$$\mathbf{A}\mathbf{x}_0 - \mathbf{B}\mathbf{u}_0 = 0$$

In other words, we need to find if  $\mathbf{A}\mathbf{x}_0$  is in the column space of  $\mathbf{B}$ . Here is the criterion:

$$(\mathbf{B}\mathbf{B}^+ - \mathbf{I})(\mathbf{A}\mathbf{x}_0) = 0$$

# All stabilizable points

For the system from the previous example we can easily find all points that are stabilizable. For that we consider equation:

$$\mathbf{A}\mathbf{x}_0 - \mathbf{B}\mathbf{u}_0 = 0$$

but this time we make both  $\mathbf{x}_0$  and  $\mathbf{u}_0$  our variables. This system has a nontrivial solution if matrix  $[\mathbf{A}, -\mathbf{B}]$  has a nontrivial *null space*.

Let  $\mathbf{z} = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_0 \end{bmatrix}$ . And let  $\mathbf{N}$  be a basis in the null space of  $[\mathbf{A}, -\mathbf{B}]$ :

$$\mathbf{N} = \text{null}([\mathbf{A}, -\mathbf{B}])$$

Then any combination of its columns produces a vector  $\mathbf{z}$ , which is stabilizable:

$$\begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_0 \end{bmatrix} = \mathbf{z} = \mathbf{N}\mathbf{r}$$

where  $\mathbf{r}$  is a random vector.

Lecture slides are available via Moodle.

You can help improve these slides at:

<https://github.com/SergeiSa/Linear-Control-Slides-Spring-2020>

Check Moodle for additional links, videos, textbook suggestions.