Observer Design, Kalman Filter

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- From the previous lecture we remember that we wanted the estimate of the state to have the same dynamics as the actual state.
- We also remember that we want to take into account the error between the estimated and measured output y

Assume the measurement is perfect: $\hat{\mathsf{y}} = \mathsf{y}$. Then we can propose observer as the following dynamical system:

$$\dot{\hat{\mathsf{x}}} = \mathsf{A}\hat{\mathsf{x}} + \mathsf{B}\mathsf{u} + \mathsf{L}(\mathsf{y} - \hat{\mathsf{y}})$$

Remember that state estimation error is $\epsilon = \hat{x} - x$ and the actual dynamics of the system is:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

Then, we can write the state estimation error dynamics:

$$\dot{\hat{x}} - \dot{x} = A\hat{x} - Ax + Bu - Bu + L(y - \hat{y})$$

or:

$$\dot{\epsilon} = A\epsilon + L(Cx - C\hat{x})$$

or:

$$\dot{\epsilon} = A\epsilon - LC\epsilon$$

Thus, with the proposed observer, state estimation error is:

$$\dot{\epsilon} = (\mathsf{A} - \mathsf{LC})\epsilon$$

From which immediately follows that the observer is *stable* (i.e., the state estimation error tends to zero), as long as the following matrix is negative-definite:

$$\mathsf{A} - \mathsf{LC} < 0$$

We need to find L.

Let us observe the key difference between observer design and controller design.

Controller design: find such K that:

$$\mathsf{A} - \mathsf{B}\mathsf{K} < 0$$

Observer design: find such L that:

$$\mathsf{A} - \mathsf{L}\mathsf{C} < 0$$

We have instruments for finding K, what about L?

Assume that C is identity I. Then the observer design problem becomes:

- find such L that $\mathsf{A} - \mathsf{L}\mathsf{I} < 0$.

Which is equivalent to the problem:

- find such L that $\mathsf{A} - \mathsf{B}\mathsf{L} < 0$, where $\mathsf{B} = \mathsf{I}$.

And in that formulation, it is equivalent to the controller design problem, which we know how to solve.

Observer Design

General case: design via Riccati eq.

In general, we can observe that if $\mathsf{A} - \mathsf{LC}$ is negative-definite, then $(\mathsf{A} - \mathsf{LC})^\top$ is negative-definite too (by definition of the negative-definiteness).

Therefore, we can solve the following *dual problem*:

- find such L that $\mathsf{A}^\top - \mathsf{C}^\top \mathsf{L}^\top < 0$.

The dual problem is *equivalent* to the control design problem. We can solve it by producing and solving algebraic Riccati equation, as in the LQR formulation. In pseudo-code it can be represented the following way:

$\mathsf{L}^\top$ = `lqr`$(\mathsf{A}^\top, \mathsf{C}^\top, \mathsf{Q}, \mathsf{R})$.
where Q and R are weight matrices, determining the "sensitivity" or "aggressiveness" of the observer.

Assume we have a discrete linear system with disturbance and noise:

$$\begin{cases} \mathsf{x}_{i+1} = \mathsf{A}\mathsf{x}_i + \mathsf{B}\mathsf{u}_i + \mathsf{w} \\ \mathsf{y}_i = \mathsf{C}\mathsf{x}_i + \mathsf{v} \end{cases}$$

where w and v are disturbance and noise, both are random processes. Covariance matrix of w is W, covariance matrix of v is V.

Our estimation $\hat{\mathsf{x}}$ is now done is two steps: after dynamics update, and after sensor update. We will use the following notation for it:

- $\hat{\mathsf{x}}_{i|i-1}$ is the i-th step estimate after dynamics update; also called an *a priori* estimate.
- $\hat{\mathsf{x}}_{i|i}$ is the estimate after sensor update; also called an *a posteriori* estimate.

Both estimates $\hat{x}_{i|i-1}$ and $\hat{x}_{i|i}$ have associated state estimation errors:

- $\epsilon_{i|i-1} = x_i - \hat{x}_{i|i-1}$
- $\epsilon_{i|i} = x_i - \hat{x}_{i|i}$

Those state estimation errors have their covariance matrices:

- $P_{i|i-1} = \text{cov}(\epsilon_{i|i-1})$,
- $P_{i|i} = \text{cov}(\epsilon_{i|i})$

We can consider the output estimation error $\varepsilon_i = y_i - C\hat{x}_{i|i-1}$ and its covariance matrix Y_i:

$$Y_i = \text{cov}(y_i - C\hat{x}_{i|i-1})$$

We know that $\hat{x}_{i|i-1}$ can be found as follows:

$$\hat{x}_{i|i-1} = A\hat{x}_{i-1|i-1} + Bu_i$$

where $\hat{x}_{i-1|i-1}$ is the estimate on the previous time step.

We will search for the update law for the a posteriori estimate $\hat{x}_{i|i}$ in the following form:

$$\hat{x}_{i|i} = \hat{x}_{i|i-1} + K_i(y_i - C\hat{x}_{i|i-1})$$

We need to find K_i, which is the Kalman filter gain.

We know or measure:

- A, B and C
- W and V
- y_i

We need to find:

- $\hat{x}_{i|i-1}$ and $\hat{x}_{i|i}$
- K_i
- Y_i, $P_{i|i-1}$ and $P_{i|i}$

There is a formula for the a priori estimation covariance update:

$$\mathsf{P}_{i|i-1} = \mathsf{A}\mathsf{P}_{i-1|i-1}\mathsf{A}^\top + \mathsf{W}$$

And a formula for the measurement covariance:

$$\mathsf{Y}_i = \mathsf{C}\mathsf{P}_{i|i-1}\mathsf{C}^\top + \mathsf{V}$$

With those two, we can find the Kalman filter gain:

$$\mathsf{K}_i = \mathsf{P}_{i|i-1}\mathsf{C}^\top\mathsf{Y}_i^{-1}$$

Now we can find the a posteriori covariance:

$$\mathsf{P}_{i|i} = (\mathsf{I} - \mathsf{K}_i\mathsf{C})\mathsf{P}_{i|i-1}$$

Lecture slides are available via Moodle.

You can help improve these slides at:
`https://github.com/SergeiSa/Linear-Control-Slides-Spring-2020`

Check Moodle for additional links, videos, textbook suggestions.