Optimal Control of LTI systems

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Let as define dynamics:

$$\dot{\mathsf{x}} = \mathsf{f}(\mathsf{x}, \mathsf{u})$$

with initial conditions $\mathsf{x}(0)$. Let J be an additive cost function:

$$J(\mathsf{x}(0), \mathsf{u}(t)) = \int_0^\infty g(\mathsf{x}, \mathsf{u}) dt$$

where $g(\mathsf{x}, \mathsf{u})$ is instantaneous cost.

Let J^* be the optimal (lowest possible) cost. In other words:

$$J^*(\mathsf{x}(0)) = \inf_{\mathsf{u}(t)} J(\mathsf{x}(0), \mathsf{u}(t))$$

Since $J^*(\mathsf{x}(0))$ does not depend on t, its full derivative is zero:

$$\frac{dJ^*(\mathsf{x}(0))}{dt} = 0$$

At the same time, we can expand the full derivative as follows:

$$\frac{dJ^*}{dt} = \frac{\partial J^*}{\partial \mathsf{x}}\dot{\mathsf{x}} + \frac{\partial J^*}{\partial t} = 0$$

Observe that $\frac{\partial J^*}{\partial t} = g(\mathsf{x}, \mathsf{u})$, and $\dot{\mathsf{x}} = \mathsf{f}(\mathsf{x}, \mathsf{u})$. Therefore:

$$\frac{\partial J^*}{\partial \mathsf{x}}\mathsf{f}(\mathsf{x}, \mathsf{u}) + g(\mathsf{x}, \mathsf{u}) = 0$$

With this, we can formulate Hamilton-Jacobi-Bellman equation (HJB):

$$\min_{\mathsf{u}} \ [g(\mathsf{x}, \mathsf{u}) + \frac{\partial J^*}{\partial \mathsf{x}} \mathsf{f}(\mathsf{x}, \mathsf{u})] = 0$$

And therefore:

$$u^* = \arg\min_{\mathsf{u}} \ [g(\mathsf{x}, \mathsf{u}) + \frac{\partial J^*}{\partial \mathsf{x}} \mathsf{f}(\mathsf{x}, \mathsf{u})]$$

For LTI, dynamics is:

$$\dot{x} = Ax + Bu$$

We can choose quadratic cost:

$$g(x, u) = x^\top Qx + u^\top Ru$$

Then HJB becomes:

$$\min_u \left[x^\top Qx + u^\top Ru + \frac{\partial J^*}{\partial x}(Ax + Bu)\right] = 0$$

where $Q = Q^\top \geq 0$ and $R = R^\top > 0$.

There is a theorem that says that for LTI with quadratic cost, J^* has the form:

$$J^* = \mathsf{x}^\top \mathsf{S} \mathsf{x}$$

where $\mathsf{S} = \mathsf{S}^\top \geq 0$.

Differentiating J^* with respect to x gives us:

$$\frac{\partial J^*}{\partial \mathsf{x}} = 2\mathsf{x}^\top \mathsf{S}$$

Then HJB becomes:

$$\min_{\mathsf{u}} \left[\mathsf{x}^\top \mathsf{Q} \mathsf{x} + \mathsf{u}^\top \mathsf{R} \mathsf{u} + 2\mathsf{x}^\top \mathsf{S}(\mathsf{A}\mathsf{x} + \mathsf{B}\mathsf{u})\right] = 0$$

Finding partial derivative of the HJB with respect to u and setting it to zero (as it is an extreme point) we get:

$$2\mathsf{u}^\top \mathsf{R} + 2\mathsf{x}^\top \mathsf{S}\mathsf{B} = 0$$

This expression can be transposed and u separated:

$$\mathsf{u} = -\mathsf{R}^{-1}\mathsf{B}^\top \mathsf{S}\mathsf{x}$$

This is the desired control law. We can see that it is *proportional*. We can re-write it as:

$$\mathsf{u} = -\mathsf{K}\mathsf{x}$$

where $\mathsf{K} = \mathsf{R}^{-1}\mathsf{B}^\top \mathsf{S}$ is the controller gain. This control law is called Linear Quadratic Regulator (LQR).

Substituting found control law into the HJB, we find:

$$\mathsf{x}^\top \mathsf{Q}\mathsf{x} + \mathsf{x}^\top \mathsf{S}\mathsf{B}\mathsf{R}^{-1}\mathsf{R}\mathsf{R}^{-1}\mathsf{B}^\top \mathsf{S}\mathsf{x} + 2\mathsf{x}^\top \mathsf{S}(\mathsf{A}\mathsf{x} - \mathsf{B}\mathsf{R}^{-1}\mathsf{B}^\top \mathsf{S}\mathsf{x}) = 0$$

After grouping the terms, we get:

$$\mathsf{x}^\top(\mathsf{Q} - \mathsf{S}\mathsf{B}\mathsf{R}^{-1}\mathsf{B}^\top \mathsf{S} + 2\mathsf{S}\mathsf{A})\mathsf{x} = 0$$

which would hold for all x iff:

$$\mathsf{Q} - \mathsf{S}\mathsf{B}\mathsf{R}^{-1}\mathsf{B}^\top \mathsf{S} + 2\mathsf{S}\mathsf{A} = 0$$

This is the Algebraic Riccati equation.

There are a number of ways to solve LQR:

- In MATLAB there is a function `[K,S,P] = lqr(A, B, Q, R)`, `where P=eig(A-B*K)`
- In Python, there is `S = scipy.linalg.solve_continuous_are(a, b, q, r)`
- In Drake (by MIT and Toyota Research) there is a function `(K,S) = LinearQuadraticRegulator(A,B,Q,R)`

Lecture slides are available via Moodle.

Check Moodle for additional links, videos, textbook suggestions.