

# Discrete Systems

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# Discretization

## Finite difference

Consider linear time-invariant autonomous system:

$$\dot{x} = Ax$$

The time derivative  $\dot{x}$  can be replaced with a finite difference:

$$\dot{x} \approx \frac{1}{\Delta t}(x(t + \Delta t) - x(t))$$

Note that we could have also used other definitions of a finite difference:

$$\dot{x} \approx \frac{1}{\Delta t}(x(t + 0.5\Delta t) - x(t - 0.5\Delta t))$$

or

$$\dot{x} \approx \frac{1}{\Delta t}(x(t) - x(t - \Delta t))$$

# Discretization

## Finite difference notation

We can introduce notation:

$$\begin{cases} x[0] = x(0) \\ x[1] = x(\Delta t) \\ x[2] = x(2\Delta t) \\ \dots \\ x[n] = x(n\Delta t) \end{cases}$$

We say that  $x[i]$  is the value of  $x$  at the time step  $i$ . Then the finite difference can be written, for example, as follows:

$$\dot{x} \approx \frac{1}{\Delta t} (x[i+1] - x[i])$$

# Discretization

## Finite difference in an autonomous LTI

We can rewrite our original autonomous LTI as follows:

$$\frac{1}{\Delta t}(\mathbf{x}[i+1] - \mathbf{x}[i]) = \mathbf{A}\mathbf{x}[i]$$

Isolating  $\mathbf{x}[i+1]$  on the left hand side, we get:

$$\mathbf{x}[i+1] = (\mathbf{A}\Delta t + \mathbf{I})\mathbf{x}[i]$$

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Or alternatively:

$$\frac{1}{\Delta t}(\mathbf{x}[i+1] - \mathbf{x}[i]) = \mathbf{A}\mathbf{x}[i+1]$$

Isolating  $\mathbf{x}[i+1]$  on the left hand side, we get:

$$\mathbf{x}[i+1] = (\mathbf{I} - \mathbf{A}\Delta t)^{-1}\mathbf{x}[i]$$

If we have a non-autonomous dynamics in the form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

we can do the same transformation as before:

$$\frac{1}{\Delta t}(\mathbf{x}[i+1] - \mathbf{x}[i]) = \mathbf{A}\mathbf{x}[i] + \mathbf{B}\mathbf{u}[i]$$

$$\mathbf{x}[i+1] = (\mathbf{A}\Delta t + \mathbf{I})\mathbf{x}[i] + \mathbf{B}\Delta t\mathbf{u}[i]$$

# Discretization

## Zero order hold

Defining *discrete state space matrix*  $\bar{A}$  and *discrete control matrix*  $\bar{B}$  as follows:

$$\bar{A} = (A\Delta t + I), \quad \bar{B} = B\Delta t$$

We get discrete dynamics:

$$x[i+1] = \bar{A}x[i] + \bar{B}u[i]$$

This way of defining discrete dynamics is called *Zero order hold*.

# ZOH and other types of discretization

## Zero order hold vs First order hold

Graphically, we can understand what zero order hold is, by comparing it to the first order hold:

Graphically, zero order hold is this:



First order hold is this:



Figure: Different types of discretization



# ZOH and other types of discretization

## Precise discretization

Let the discrete state  $x[i]$  correspond to the moment of time continuous state  $x$  at the moment of time  $t_i$ , and  $x[i + 1]$  - to the point of time  $t_{i+1}$ . Then, we can say that the discretization is *exact* if there is the following implication relation

$$x[i] = x(t_i) \rightarrow x[i + 1] = x(t_{i+1})$$

or in other words, if state of both systems are identical for the discrete time points where the discrete states are defined.

We can compute exact discretization as follows:

$$\bar{A} = e^{A\Delta t}$$
$$\bar{B} = B \int_{t_0}^{t_0 + \Delta t} e^{As} ds$$

# Stability criterion

For continuous-time systems  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ , the criterion is that the real parts of the eigenvalues of the matrix  $\mathbf{A}$  are all negative.

For a discrete time system  $\mathbf{x}[i+1] = \bar{\mathbf{A}}\mathbf{x}[i]$ , the criterion for the stability in the sense of Lyapunov is that all eigenvalues should be less than 1 by magnitude:

$$|\lambda_i(\bar{\mathbf{A}})| \leq 1, \forall i$$

if  $|\lambda_i(\bar{\mathbf{A}})| < 1, \forall i$ , the system is asymptotically stable (or just stable).

Lecture slides are available via Moodle.

Check Moodle for additional links, videos, textbook suggestions.