Discrete Systems

by Sergei Savin

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Finite difference

Consider linear time-invariant autonomous system:

$$\dot{x} = Ax$$

The time derivative \dot{x} can be replaces with a finite difference:

$$\dot{ exttt{x}}pproxrac{1}{\Delta t}(exttt{x}(t+\Delta t)- exttt{x}(t))$$

Note that we could have also used other definitions of a finite difference:

$$\dot{\mathsf{x}} pprox rac{1}{\Delta t} (\mathsf{x}(t+0.5\Delta t) - \mathsf{x}(t-0.5\Delta t))$$

or

$$\dot{\mathsf{x}} pprox rac{1}{\Delta t} (\mathsf{x}(t) - \mathsf{x}(t - \Delta t))$$

Finite difference notation

We can introduce notation:

$$\begin{cases} x[0] = x(0) \\ x[1] = x(\Delta t) \\ x[2] = x(2\Delta t) \\ \dots \\ x[n] = x(n\Delta t) \end{cases}$$

We say that x[i] is the value of x at the time step i. Then the finite difference can be written, for example, as follows:

$$\dot{\mathsf{x}} pprox rac{1}{\Delta t} (\mathsf{x}[i+1] - \mathsf{x}[i])$$

Finite difference in an autonomous LTI

We can rewrite our original autonomous LTI as follows:

$$\frac{1}{\Delta t}(\mathsf{x}[i+1] - \mathsf{x}[i]) = \mathsf{A}\mathsf{x}[i]$$

Isolating x[i+1] on the left hand side, we get:

$$x[i+1] = (A\Delta t + I)x[i]$$

Or alternatively:

$$\frac{1}{\Delta t}(x[i+1]-x[i]) = Ax[i+1]$$

Isolating x[i+1] on the left hand side, we get:

$$x[i+1] = (I - A\Delta t)^{-1}x[i]$$

If we have dynamics in the form:

$$\dot{x} = Ax + Bu$$

we can do the same transformation as before:

$$\frac{1}{\Delta t}(x[i+1]-x[i]) = Ax[i] + Bu[i]$$

$$x[i+1] = (A\Delta t + I)x[i] + B\Delta tu[i]$$

Zero order hold

Defining discrete state space matrix $\bar{\mathsf{A}}$ and discrete control matrix $\bar{\mathsf{B}}$ as follows:

$$\bar{\mathsf{A}} = (\mathsf{A}\Delta t + \mathsf{I}) \; \bar{\mathsf{B}} = \mathsf{B}\Delta t$$

We get discrete dynamics:

$$x[i+1] = \bar{A}x[i] + \bar{B}u[i]$$

This way of defining discrete dynamics is called *Zero order hold*.

ZOH and other types of discretization

Zero order hold vs First order hold

Graphically, we can understand what zero order hold is, by comparing it to the first order hold:

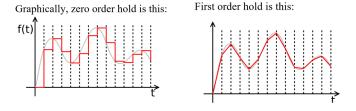


Figure: Different types of discretization

ZOH and other types of discretization

Precise discretization

Let the discrete state x[i] correspond to the moment of time continuous state x at the moment of time t_i , and x[i+1] - to the point of time t_{i+1} . Then, we can say that the discretization is *exact* if there is the following implication relation

$$x[i] = x(t_i) \to x[i+1] = x(t_{i+1})$$

or in other words, if state of both systems are identical for the discrete time points where the discrete states are defined.

We can compute exact discretization as follows:

$$ar{\mathsf{A}} = e^{\mathsf{A}\Delta t}$$
 $ar{\mathsf{B}} = \mathsf{B} \int_{t_0}^{t_0 + \Delta t} e^{\mathsf{A}s} ds$

Stability criterion

For continuous-time systems $\dot{x}=Ax$, the criterion is that the real parts of the eigenvalues of the matrix A are all negative.

For a discrete time system $x[i+1] = \bar{A}x[i]$, the criterion for the stability in the sense of Lyapunov is that all eigenvalues should be less than 1 by magnitude:

$$|\lambda_i(\bar{\mathsf{A}})| \leq 1, \ \forall i$$

if $|\lambda_i(\bar{A})| < 1$, $\forall i$, the system is asymptotically stable (or just stable).

Lecture slides are available via Moodle.

Check Moodle for additional links, videos, textbook suggestions.