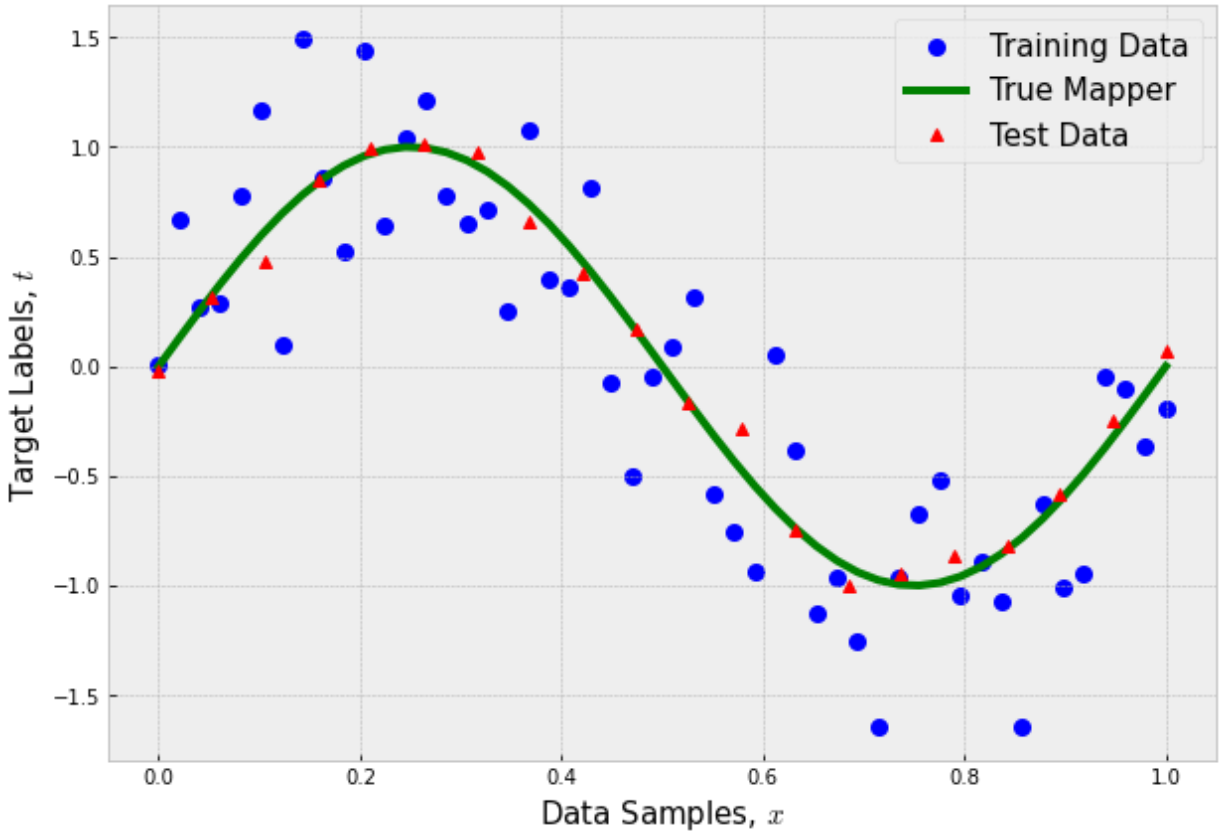
Lecture 5 - The Bias-Variance Trade-Off & The Curse of Dimensionality

In [3]:
```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline
plt.style.use('bmh')
```

In [4]:
```
def NoisySinusoidalData(N, a, b, sigma):
    '''Generates N data points in the range [a,b) sampled from a sin(2*pi*x)
    with additive zero-mean Gaussian random noise with standard deviation sigma'''

    # N input samples, evenly spaced numbers between [a,b) incrementing by 1/N
    x = np.linspace(a,b,N)

    # draw N sampled from a univariate Gaussian distribution with mean 0, sigma standa
    noise = np.random.normal(0,sigma,N)

    # desired values, noisy sinusoidal
    t = np.sin(2*np.pi*x) + noise

    return x, t
```

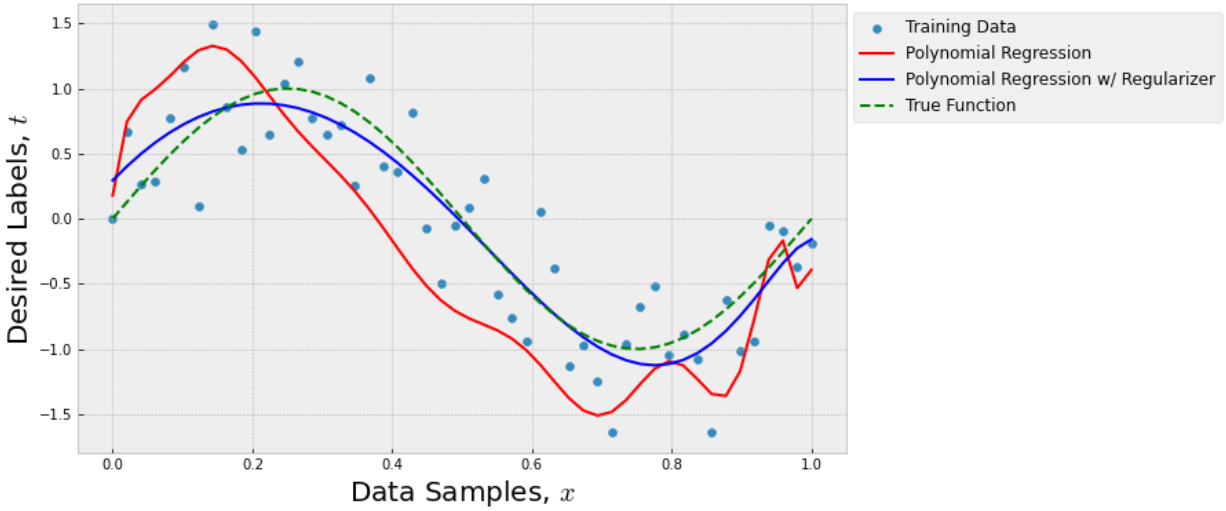
In [5]:
```
# Generate input samples and desired values
N_train = 50 # number of data samples for training
N_test = 20 # number of data samples for test

a, b = [0,1] # data samples interval

sigma_train = 0.4 # standard deviation of the zero-mean Gaussian noise -- training dat
sigma_test = 0.1 # standard deviation of the zero-mean Gaussian noise -- test data

x_train, t_train = NoisySinusoidalData(N_train, a, b, sigma_train) # Training Data - N
x_true, t_true = NoisySinusoidalData(N_train, a, b, 0) # True Sinusoidal - in practice
x_test, t_test = NoisySinusoidalData(N_test, a, b, sigma_test) # Test Data - Noisy sir

plt.figure(figsize=(10,7))
plt.scatter(x_train, t_train, c='b', linewidths=3, label = 'Training Data')
plt.plot(x_true, t_true, 'g', linewidth=4, label = 'True Mapper')
plt.plot(x_test, t_test, 'r^', label = 'Test Data')
plt.legend(fontsize=15)
plt.xlabel('Data Samples, $x$',size=15)
plt.ylabel('Target Labels, $t$',size=15);
```



In [6]:
```
def PolynomialRegression(x,t,M):
    '''Fit a polynomial of order M to the data input data x and desire values t'''
    # Compute feature matrix X with polynomial features
    X = np.array([x**m for m in range(M+1)]).T #computes the phi(x) = [x^0, x^1, ....,
    # Compute the solution for the parameters w
    w = np.linalg.inv(X.T@X)@X.T@t # Optimal set of parameters w
    # Compute model prediction
    y = X@w
    return w, y

def PolynomialRegression_reg(x,t,M,lam):
    # Compute feature matrix X with polynomial features
    X = np.array([x**m for m in range(M+1)]).T
    # Compute the solution for the parameters w
    w = np.linalg.inv(X.T@X + lam*np.eye(M+1))@X.T@t
    # Compute model prediction
    y = X@w
    return w, y
```

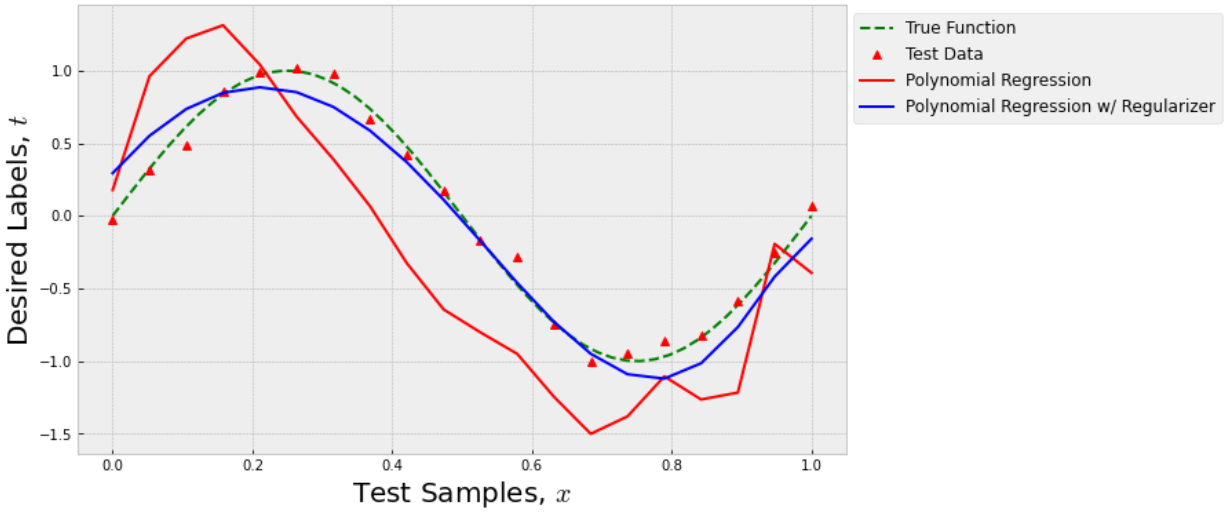
In [7]:
```
M = 20
lam = 0.001

w, y, = PolynomialRegression(x_train,t_train,M)
wreg, yreg = PolynomialRegression_reg(x_train,t_train,M,lam)

fig=plt.figure(figsize=(10,6))
plt.scatter(x_train,t_train, label='Training Data')
plt.plot(x_train,y,'r', label = 'Polynomial Regression')
plt.plot(x_train,yreg, 'b',label = 'Polynomial Regression w/ Regularizer')
plt.plot(x_true,t_true,'--g', label = 'True Function')
plt.legend(bbox_to_anchor=(1.5, 1),fontsize=12,ncol=1)
```

```
plt.xlabel('Data Samples, $x$', fontsize=20)
plt.ylabel('Desired Labels, $t$', fontsize=20);
```



In [8]:
```
def PolynomialRegression_test(x,M,w):
    # Feature matrix for test set
    X = np.array([x**m for m in range(M+1)]).T

    # Prediction for test set
    y = X@w

    return y
```

In [9]:
```
# Prediction for test set using non-regularized model
y_test = PolynomialRegression_test(x_test, M, w)

# Prediction for test set using regularized model
y_test_reg = PolynomialRegression_test(x_test, M, wreg)

# Plotting
fig=plt.figure(figsize=(10,6))
plt.plot(x_true, t_true, '--g', label = 'True Function')
plt.plot(x_test, t_test, 'r^', label = 'Test Data')
plt.plot(x_test,y_test,'r', label = 'Polynomial Regression')
plt.plot(x_test,y_test_reg, 'b',label = 'Polynomial Regression w/ Regularizer')
plt.legend(bbox_to_anchor=(1.5, 1),fontsize=12,ncol=1)
plt.xlabel('Test Samples, $x$', fontsize=20)
plt.ylabel('Desired Labels, $t$', fontsize=20);
```



Fine-tuning the Hyperparameters

The hyperparameters of ridge regression are the model order M and the regularizer coefficient λ.

How would you choose which value to use? -- Cross-validation.

Performance Measures

In order to determine if the model is able to **generalize** to a **validation set**, we need to determine a **performance measure**.

Which *quantitative* measure would you use to compare model performances?

Error Measures

Error measures are always a good start for **regression** tasks. Some examples include:

- **Mean Squared Error (MSE)** - best when the data does not have outliers. The MSE will penalize outliers heavily.
- **Mean Absolute Error (MAE)** - best when you suspect data has outliers.

In [10]:
```
# Residual error for Training data for polynomial regression without and with regulari
error_train = t_train - y
error_train_reg = t_train - yreg

# Residual error for Test data for polynomial regression without and with regularizer
error_test = t_test - y_test
error_test_reg = t_test - y_test_reg

# Error Measures
```

```
print('Mean Squared Error \n')
print('Training Set')
print('Without regularizer: ', np.mean(error_train**2))
print('With regularizer: ', np.mean(error_train_reg**2),'\n')
print('Test Set')
print('Without regularizer: ', np.mean(error_test**2))
print('With regularizer: ', np.mean(error_test_reg**2),'\n')
print('-------------------------------------------------------------')
print('Mean Absolute Error \n')
print('Training Set')
print('Without regularizer: ', np.mean(np.abs(error_train)))
print('With regularizer: ', np.mean(np.abs(error_train_reg)),'\n')
print('Test Set')
print('Without regularizer: ', np.mean(np.abs(error_test)))
print('With regularizer: ', np.mean(np.abs(error_test_reg)),'\n')
```

```
Mean Squared Error

Training Set
Without regularizer:  0.28247200516149473
With regularizer:  0.1317549195852023

Test Set
Without regularizer:  0.28236820436322385
With regularizer:  0.029780831998019407

-------------------------------------------------------------
Mean Absolute Error

Training Set
Without regularizer:  0.42180065341161815
With regularizer:  0.3043191481379674

Test Set
Without regularizer:  0.48669373264008453
With regularizer:  0.14653789245942633
```

Other error-based measures can be considered, e.g., normalized mean squared error (NMSE), normalized mean absolute error (NMAE), etc.

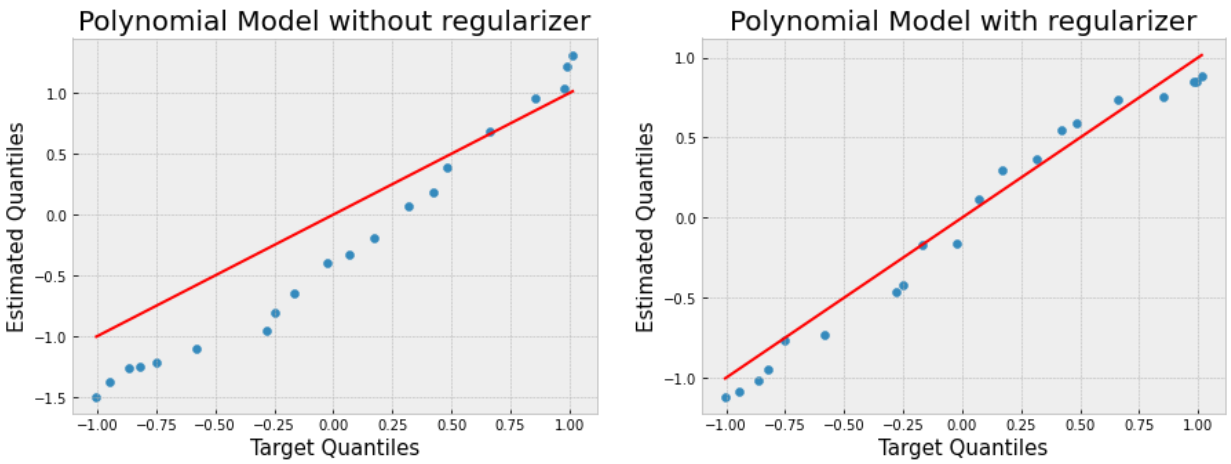
Q-Q Plot

We can also use the **Quantile-Quantile (Q-Q)** plot to assess qualitative measures of goodness-of-fit.

- The Q-Q plot help us assess if a set of data plausibly came from some theoretical distribution such as a Normal or exponential, or if two sets of samples were drawn from the same distribution. For example, if we run a statistical analysis that assumes our dependent variable is Normally distributed, we can use a Normal Q-Q plot to check that assumption. It's just a visual check, not an air-tight proof, so it is somewhat subjective. But it allows us to see at-a-glance if our assumption is plausible, and if not, how the assumption is violated and what data points contribute to the violation.

- What are *quantiles*? These are often referred to as *percentiles*. These are points in your data below which a certain proportion of your data fall which are captured in the Cumulative Distribution Function (CDF) of a random variable. For example, imagine the classic bell-curve standard Normal distribution with a mean of 0. The 0.5 quantile, or 50th percentile, is 0. Half the data lie below 0. That's the peak of the hump in the curve. The 0.95 quantile, or 95th percentile, is about 1.64. 95 percent of the data lie below 1.64.

In [11]:
```
base = np.linspace(min(t_test),max(t_test),100)

plt.figure(figsize=(15,5))
plt.subplot(1,2,1); plt.scatter(np.sort(t_test), np.sort(y_test))
#NOTE: the true values and predictions are sorted because we are
#inferring quantiles of the underlying probabilistic model from data samples
plt.plot(base,base,'r')
plt.xlabel('Target Quantiles', size=15)
plt.ylabel('Estimated Quantiles', size=15)
plt.title('Polynomial Model without regularizer',size=20)

plt.subplot(1,2,2); plt.scatter(np.sort(t_test), np.sort(y_test_reg))
#NOTE: the true values and predictions are sorted because we are
#inferring quantiles of the underlying probabilistic model from data samples
plt.plot(base,base,'r')
plt.xlabel('Target Quantiles', size=15)
plt.ylabel('Estimated Quantiles', size=15)
plt.title('Polynomial Model with regularizer',size=20);
```



And then take a summative **quantitative** measure, namely the **coefficient of determination**.

In [12]:
```
import scipy.stats as stats

print('Polynomial Regression Without Regularization - Test Set')
m, b, r, p, _ = stats.linregress(np.sort(t_test), np.sort(y_test))
print('Coefficient of Determination: ',r**2)
print('Slope: ',m)
print('Intercept: ',b)
print('p-value: ', p)
print('-------------------')
print('Polynomial Regression With Regularization - Test Set')
m, b, r, p, _ = stats.linregress(np.sort(t_test), np.sort(y_test_reg))
print('Coefficient of Determination: ',r**2)
print('Slope: ',m)
```

```
print('Intercept: ',b)
print('p-value: ', p)
```

```
Polynomial Regression Without Regularization - Test Set
Coefficient of Determination:  0.976116514746958
Slope:  1.3442139130520792
Intercept:  -0.27619474650396825
p-value:  4.741040362071496e-16
------------------
Polynomial Regression With Regularization - Test Set
Coefficient of Determination:  0.9797115271513868
Slope:  1.0444737529040595
Intercept:  -0.0589542991415779
p-value:  1.0902384572224697e-16
```

For most applications, we want a model with a coefficient of determination $r^2 \geq 99\%$.

In this example, the exponential performs better than the polynomial model in the test set.

Other Performance Measures

Performance of an algorithm can be determined using a variety of statistical goodness-of-fit measures.

- For regression tasks this includes error-based measurements, hypothesis tests, Q-Q plots.
- For classification tasks this includes error rate, accuracy, ROC curves, performance-recall curves.

But it can also be in terms of:

1. Risk,
2. Training time,
3. Training storage/memory,
4. Testing time,
5. Testing storage/memory,
6. Interpretability, namely, whether the method allows knowledge extraction which can be checked and validated by experts, and
7. computational complexity.

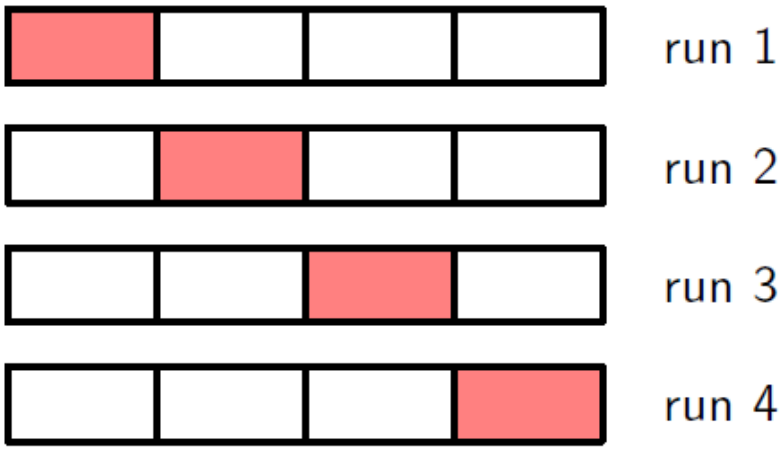
K-Fold Cross-Validation

The technique of k-fold cross-validation, illustrated below for the case of $k = 4$, involves taking the available data and partitioning it into k groups (in the simplest case these are of equal size). Then $k - 1$ of the groups are used to train a set of models that are then evaluated on the remaining group. This procedure is then repeated for all k possible choices for the held-out group, indicated in the picture below by the red blocks, and the performance scores from the runs are then averaged.

K is typically 10 or 30. As K increases, the percentage of training instances increases and we get more robust estimators, but the validation set becomes smaller. Furthermore, there is the cost of training the classifier K times, which increases as K is increased. As N increases, K can be smaller; if N is small, K should be large to allow large enough training sets.

In [2]:
```
from IPython.display import Image
Image('figures/Kfold CV.png',width=400)
```

Out[2]:



In [13]:
```
from sklearn.model_selection import KFold

KFold?
```

Goal: find the best value for the **hyperparameters** M (model order) and λ (regularization trade-off parameter).

Let's use **4-fold cross-validation** on this data:

In [14]:
```
k = 4 # number of folds

kf = KFold(n_splits=k,shuffle=True)
kf
```

Out[14]:
```
KFold(n_splits=4, random_state=None, shuffle=True)
```

In [15]:
```
# Split training feature matrix into training and validation sets

f=1
for train_index, validation_index in kf.split(x_train):
    print('Fold ', f)
    print('The training set has ', train_index.shape[0],' samples')
    print('Their index locations are: ', train_index)
    print('The validation set has ', validation_index.shape[0],' samples')
    print('Their index locations are: ', validation_index,'\n\n')
    f+=1
```

```
Fold  1
The training set has  37  samples
Their index locations are:  [ 1  2  3  4  8  9 10 11 12 13 14 15 17 18 20 21 23 25 26
28 29 30 31 32
 33 35 36 37 38 39 40 42 43 44 45 47 48]
The validation set has  13  samples
Their index locations are:  [ 0  5  6  7 16 19 22 24 27 34 41 46 49]


Fold  2
The training set has  37  samples
Their index locations are:  [ 0  1  3  4  5  6  7 10 11 12 15 16 17 18 19 21 22 24 25
26 27 28 29 31
 32 33 34 35 38 39 41 43 44 45 46 47 49]
The validation set has  13  samples
Their index locations are:  [ 2  8  9 13 14 20 23 30 36 37 40 42 48]


Fold  3
The training set has  38  samples
Their index locations are:  [ 0  1  2  3  5  6  7  8  9 11 13 14 16 19 20 22 23 24 25
27 28 30 32 33
 34 35 36 37 38 40 41 42 43 44 46 47 48 49]
The validation set has  12  samples
Their index locations are:  [ 4 10 12 15 17 18 21 26 29 31 39 45]


Fold  4
The training set has  38  samples
Their index locations are:  [ 0  2  4  5  6  7  8  9 10 12 13 14 15 16 17 18 19 20 21
22 23 24 26 27
 29 30 31 34 36 37 39 40 41 42 45 46 48 49]
The validation set has  12  samples
Their index locations are:  [ 1  3 11 25 28 32 33 35 38 43 44 47]
```

In [16]:

```python
# Set of values for lambda to explore
M_vals = range(1,21) # model order
lam_vals= np.arange(0.1,1.1,0.1) # set of values for lambda

for M in M_vals:
    for lam in lam_vals:

        print('M Value = ',M)
        print('Lambda Value = ',lam)

        # For each training/validation split
        f=1

        #initialize performance measures
        MSE_train_avg,MSE_val_avg = 0, 0

        for train_index, validation_index in kf.split(x_train):
            print('\nFold ',f)

            # Select training set using the indices found from kf.split
            x_train2, x_validation = x_train[train_index], x_train[validation_index]

            # Select validation set using the indices found from kf.split
```

```
            t_train2, t_validation = t_train[train_index], t_train[validation_index]

            # Training model with training set
            w, y_train = PolynomialRegression_reg(x_train2, t_train2, M, lam)

            # Evaluate trained model in validation set
            y_val = PolynomialRegression_test(x_validation, M, w)

            # Performance Measure
            MSE_train = np.mean((t_train2-y_train)**2)
            MSE_val   = np.mean((t_validation-y_val)**2)

            # Average performance measure
            MSE_train_avg = MSE_train_avg+MSE_train
            MSE_val_avg = MSE_val_avg+MSE_val
            print('MSE Training = ', MSE_train)
            print('MSE Validation = ', MSE_val)
            f+=1

        print('\nAverage Performance in Training = ', MSE_train_avg/k)
        print('Average Performance in Validation = ', MSE_val_avg/k)
        print('-----------------------------------------------------\n')
```

```
M Value =  1
Lambda Value =  0.1

Fold  1
MSE Training =  0.2727178531081548
MSE Validation =  0.3972233050420679

Fold  2
MSE Training =  0.2930397956997488
MSE Validation =  0.3331869404384707

Fold  3
MSE Training =  0.3324391842080548
MSE Validation =  0.2067703784973998

Fold  4
MSE Training =  0.2988131028754072
MSE Validation =  0.3086859706399158

Average Performance in Training =  0.29925248397284143
Average Performance in Validation =  0.31146664865446355
-----------------------------------------------------

M Value =  1
Lambda Value =  0.2

Fold  1
MSE Training =  0.31437952399997143
MSE Validation =  0.3597843407170561

Fold  2
MSE Training =  0.2980195887730816
MSE Validation =  0.3088695588570619

Fold  3
MSE Training =  0.2723345983158218
MSE Validation =  0.4198991129455964

Fold  4
MSE Training =  0.2983305094810528
MSE Validation =  0.3097898812719569

Average Performance in Training =  0.2957660551424819
Average Performance in Validation =  0.34958572344791783
-----------------------------------------------------

M Value =  1
Lambda Value =  0.30000000000000004

Fold  1
MSE Training =  0.29800667374431067
MSE Validation =  0.3520361403198661

Fold  2
MSE Training =  0.2753465303210166
MSE Validation =  0.4023473321837674

Fold  3
MSE Training =  0.31782272489230384
MSE Validation =  0.30628981240525793
```

```
Fold  4
MSE Training =  0.28513267961312827
MSE Validation =  0.4120859543577396

Average Performance in Training =  0.2940771521426898
Average Performance in Validation =  0.3681898098166577
----------------------------------------------------

M Value =  1
Lambda Value =  0.4

Fold  1
MSE Training =  0.30192716783883256
MSE Validation =  0.3464212449872568

Fold  2
MSE Training =  0.31820520492318827
MSE Validation =  0.29009261498030364

Fold  3
MSE Training =  0.3028822749011334
MSE Validation =  0.31269699739344337

Fold  4
MSE Training =  0.29241016778227513
MSE Validation =  0.37505484177771414

Average Performance in Training =  0.30385620386135737
Average Performance in Validation =  0.33106642478467946
----------------------------------------------------

M Value =  1
Lambda Value =  0.5

Fold  1
MSE Training =  0.24334622960336239
MSE Validation =  0.5198063998398965

Fold  2
MSE Training =  0.3251796561497074
MSE Validation =  0.2788300503466108

Fold  3
MSE Training =  0.34057178827526763
MSE Validation =  0.2152725338352515

Fold  4
MSE Training =  0.3201565432387752
MSE Validation =  0.28893108067301637

Average Performance in Training =  0.30731355431677815
Average Performance in Validation =  0.3257100161736938
----------------------------------------------------

M Value =  1
Lambda Value =  0.6

Fold  1
MSE Training =  0.3259116533230361
```

```
MSE Validation =  0.30970746146525335

Fold  2
MSE Training =  0.3215360716300433
MSE Validation =  0.2649046627930897

Fold  3
MSE Training =  0.30362150269774985
MSE Validation =  0.38269395840485393

Fold  4
MSE Training =  0.28945891630610304
MSE Validation =  0.38612656321676647

Average Performance in Training =  0.31013203598923306
Average Performance in Validation =  0.33585816146999087
-----------------------------------------------------

M Value =  1
Lambda Value =  0.7000000000000001

Fold  1
MSE Training =  0.3322996665855761
MSE Validation =  0.26834051095731887

Fold  2
MSE Training =  0.30104909770895355
MSE Validation =  0.4054228838692546

Fold  3
MSE Training =  0.327482368805288
MSE Validation =  0.2754143986395783

Fold  4
MSE Training =  0.28485254823452383
MSE Validation =  0.418631628851241

Average Performance in Training =  0.31142092033358537
Average Performance in Validation =  0.3419523555793482
-----------------------------------------------------

M Value =  1
Lambda Value =  0.8

Fold  1
MSE Training =  0.3143779373397095
MSE Validation =  0.33397585979221245

Fold  2
MSE Training =  0.29394433328017894
MSE Validation =  0.378300495634577

Fold  3
MSE Training =  0.35029291751180514
MSE Validation =  0.2459886854099171

Fold  4
MSE Training =  0.3152659589209198
MSE Validation =  0.36681502638583346
```

```
Average Performance in Training =  0.31847028676315337
Average Performance in Validation =  0.331270016805635
-----------------------------------------------------

M Value =  1
Lambda Value =  0.9

Fold  1
MSE Training =  0.34291768646974335
MSE Validation =  0.27383893700252787

Fold  2
MSE Training =  0.32987036369183453
MSE Validation =  0.30090647632865974

Fold  3
MSE Training =  0.31759071903929414
MSE Validation =  0.3520162959908886

Fold  4
MSE Training =  0.2872986930765136
MSE Validation =  0.5127363758919025

Average Performance in Training =  0.3194193655693464
Average Performance in Validation =  0.3598745213034946
-----------------------------------------------------

M Value =  1
Lambda Value =  1.0

Fold  1
MSE Training =  0.36779793444806597
MSE Validation =  0.23835758118531886

Fold  2
MSE Training =  0.3497047559285822
MSE Validation =  0.2529050312437558

Fold  3
MSE Training =  0.31865951958532385
MSE Validation =  0.41697212424198277

Fold  4
MSE Training =  0.2623514123659637
MSE Validation =  0.5029041306182951

Average Performance in Training =  0.3246284055819839
Average Performance in Validation =  0.35278471682233814
-----------------------------------------------------

M Value =  2
Lambda Value =  0.1

Fold  1
MSE Training =  0.30976563026159504
MSE Validation =  0.31871299448052426

Fold  2
MSE Training =  0.3059888792811903
MSE Validation =  0.33990496290062683
```

```
Fold  3
MSE Training =  0.25858942457215733
MSE Validation =  0.45394461370386274

Fold  4
MSE Training =  0.2954340437714538
MSE Validation =  0.3956130881517424

Average Performance in Training =  0.2924444944715991
Average Performance in Validation =  0.37704391480918903
-----------------------------------------------------

M Value =  2
Lambda Value =  0.2

Fold  1
MSE Training =  0.3094133665742835
MSE Validation =  0.2909244908308242

Fold  2
MSE Training =  0.3258242606606523
MSE Validation =  0.2738778005712423

Fold  3
MSE Training =  0.31318186078657023
MSE Validation =  0.3147689637782971

Fold  4
MSE Training =  0.24999713828244402
MSE Validation =  0.5121548273375537

Average Performance in Training =  0.2996041565759875
Average Performance in Validation =  0.3479315206294793
-----------------------------------------------------

M Value =  2
Lambda Value =  0.30000000000000004

Fold  1
MSE Training =  0.3398021144899639
MSE Validation =  0.21634881266064945

Fold  2
MSE Training =  0.3281417011034742
MSE Validation =  0.26225652906430913

Fold  3
MSE Training =  0.2422069443562262
MSE Validation =  0.5406695472773986

Fold  4
MSE Training =  0.3110347050674023
MSE Validation =  0.3082211917075507

Average Performance in Training =  0.30529636625426665
Average Performance in Validation =  0.331874020177477
-----------------------------------------------------

M Value =  2
```

```
Lambda Value =  0.4

Fold  1
MSE Training =  0.2896905659406241
MSE Validation =  0.4458939214966293

Fold  2
MSE Training =  0.31804256891966304
MSE Validation =  0.289449467037884

Fold  3
MSE Training =  0.28482846801222567
MSE Validation =  0.505877954146864

Fold  4
MSE Training =  0.30523918700987435
MSE Validation =  0.3418103884272154

Average Performance in Training =  0.29945019747059676
Average Performance in Validation =  0.3957579327771482
-----------------------------------------------------

M Value =  2
Lambda Value =  0.5

Fold  1
MSE Training =  0.31025261767575624
MSE Validation =  0.33434720878195373

Fold  2
MSE Training =  0.33235352027675347
MSE Validation =  0.2587026452152663

Fold  3
MSE Training =  0.28804948699393673
MSE Validation =  0.4101141866333984

Fold  4
MSE Training =  0.30830157621237175
MSE Validation =  0.3096597563610036

Average Performance in Training =  0.30973930028970453
Average Performance in Validation =  0.32820594924790547
-----------------------------------------------------

M Value =  2
Lambda Value =  0.6

Fold  1
MSE Training =  0.2574689272901728
MSE Validation =  0.47582998549142164

Fold  2
MSE Training =  0.364779300267668
MSE Validation =  0.17896761100558917

Fold  3
MSE Training =  0.2653662478704128
MSE Validation =  0.4821541777807581
```

```
Fold  4
MSE Training =  0.35164290908261386
MSE Validation =  0.23712759792070034

Average Performance in Training =  0.30981434612771686
Average Performance in Validation =  0.3435198430496173
----------------------------------------------------

M Value =  2
Lambda Value =  0.7000000000000001

Fold  1
MSE Training =  0.31650830574978156
MSE Validation =  0.32197598650994197

Fold  2
MSE Training =  0.27611652150660415
MSE Validation =  0.4848486428729515

Fold  3
MSE Training =  0.3532026494662371
MSE Validation =  0.23434294590211122

Fold  4
MSE Training =  0.2772276043581064
MSE Validation =  0.4779671601676582

Average Performance in Training =  0.30576377027018226
Average Performance in Validation =  0.37978368386316574
----------------------------------------------------

M Value =  2
Lambda Value =  0.8

Fold  1
MSE Training =  0.35863305133604423
MSE Validation =  0.22693472538620132

Fold  2
MSE Training =  0.2891803101057207
MSE Validation =  0.4558330674693376

Fold  3
MSE Training =  0.32156917101131866
MSE Validation =  0.3103162797129749

Fold  4
MSE Training =  0.2817109050048283
MSE Validation =  0.4219419740592212

Average Performance in Training =  0.312773359364478
Average Performance in Validation =  0.3537565116569337
----------------------------------------------------

M Value =  2
Lambda Value =  0.9

Fold  1
MSE Training =  0.3019231119433032
MSE Validation =  0.41225330531700366
```

```
Fold  2
MSE Training =  0.3294382191729552
MSE Validation =  0.27214549654506215

Fold  3
MSE Training =  0.31640656984739984
MSE Validation =  0.3198402670735296

Fold  4
MSE Training =  0.3166239549003457
MSE Validation =  0.34418950536581755

Average Performance in Training =  0.316097963966001
Average Performance in Validation =  0.3371071435753532
-----------------------------------------------------

M Value =  2
Lambda Value =  1.0

Fold  1
MSE Training =  0.28900331592761985
MSE Validation =  0.4659667619436216

Fold  2
MSE Training =  0.31696169468672636
MSE Validation =  0.32726949166539954

Fold  3
MSE Training =  0.33683540538315065
MSE Validation =  0.24405377217693178

Fold  4
MSE Training =  0.32538285728363625
MSE Validation =  0.32025826574927363

Average Performance in Training =  0.3170458183202832
Average Performance in Validation =  0.33938707288380665
-----------------------------------------------------

M Value =  3
Lambda Value =  0.1

Fold  1
MSE Training =  0.3062904383757532
MSE Validation =  0.21594904207802068

Fold  2
MSE Training =  0.2777903216044576
MSE Validation =  0.3000969916722291

Fold  3
MSE Training =  0.25309899870870795
MSE Validation =  0.3719601928524461

Fold  4
MSE Training =  0.2370433871940536
MSE Validation =  0.5396548287221222

Average Performance in Training =  0.2685557864707431
```

```
Average Performance in Validation =  0.35691526383120453
----------------------------------------------------

M Value =  3
Lambda Value =  0.2

Fold  1
MSE Training =  0.24209854069093806
MSE Validation =  0.4743823192187897

Fold  2
MSE Training =  0.28262561770443084
MSE Validation =  0.2962490189891588

Fold  3
MSE Training =  0.3083136326368721
MSE Validation =  0.3023140540892083

Fold  4
MSE Training =  0.29606180583275016
MSE Validation =  0.24369428988713748

Average Performance in Training =  0.2822748992162478
Average Performance in Validation =  0.32915992054607357
----------------------------------------------------

M Value =  3
Lambda Value =  0.30000000000000004

Fold  1
MSE Training =  0.312627539358706
MSE Validation =  0.2758678453801331

Fold  2
MSE Training =  0.2697801098570038
MSE Validation =  0.40485534566291226

Fold  3
MSE Training =  0.31354913037039084
MSE Validation =  0.2656334031665684

Fold  4
MSE Training =  0.2772027948380656
MSE Validation =  0.36330286093255465

Average Performance in Training =  0.2932898936060415
Average Performance in Validation =  0.3274148637855421
----------------------------------------------------

M Value =  3
Lambda Value =  0.4

Fold  1
MSE Training =  0.26306492981935276
MSE Validation =  0.4653064896908303

Fold  2
MSE Training =  0.30260268474454416
MSE Validation =  0.3034166872769268
```

```
Fold  3
MSE Training =  0.30823856787880244
MSE Validation =  0.2812802343206538

Fold  4
MSE Training =  0.32533445770189395
MSE Validation =  0.2580663028417037

Average Performance in Training =  0.29981016003614835
Average Performance in Validation =  0.32701742853252863
-----------------------------------------------------

M Value =  3
Lambda Value =  0.5

Fold  1
MSE Training =  0.30215804296619125
MSE Validation =  0.3337589310312221

Fold  2
MSE Training =  0.3325057133341782
MSE Validation =  0.25538546636694925

Fold  3
MSE Training =  0.26730796758299136
MSE Validation =  0.46655221129907604

Fold  4
MSE Training =  0.309954514382706
MSE Validation =  0.3452695915458441

Average Performance in Training =  0.3029815595665167
Average Performance in Validation =  0.3502415500607729
-----------------------------------------------------

M Value =  3
Lambda Value =  0.6

Fold  1
MSE Training =  0.28564303556981074
MSE Validation =  0.3848133476159222

Fold  2
MSE Training =  0.3264625862099343
MSE Validation =  0.2989754730056283

Fold  3
MSE Training =  0.3049143789175945
MSE Validation =  0.43319809854903446

Fold  4
MSE Training =  0.29998459536353383
MSE Validation =  0.3837142826661113

Average Performance in Training =  0.3042511490152183
Average Performance in Validation =  0.3751753004591741
-----------------------------------------------------

M Value =  3
Lambda Value =  0.7000000000000001
```

```
Fold  1
MSE Training =  0.2885475626960753
MSE Validation =  0.4598260269208855

Fold  2
MSE Training =  0.2878529572134797
MSE Validation =  0.38111491682504794

Fold  3
MSE Training =  0.33881759098534847
MSE Validation =  0.2911163957244787

Fold  4
MSE Training =  0.3215389065078379
MSE Validation =  0.276816098107352

Average Performance in Training =  0.30918925435068534
Average Performance in Validation =  0.352218359394441
-----------------------------------------------------

M Value =  3
Lambda Value =  0.8

Fold  1
MSE Training =  0.25426319880602716
MSE Validation =  0.6339367158790409

Fold  2
MSE Training =  0.34305971201062485
MSE Validation =  0.2665985132483256

Fold  3
MSE Training =  0.28896374382306006
MSE Validation =  0.4597529605709672

Fold  4
MSE Training =  0.3455286829286583
MSE Validation =  0.19640793587304642

Average Performance in Training =  0.3079538343920926
Average Performance in Validation =  0.389174031392845
-----------------------------------------------------

M Value =  3
Lambda Value =  0.9

Fold  1
MSE Training =  0.26852944579752075
MSE Validation =  0.5515420654539797

Fold  2
MSE Training =  0.29130999609815544
MSE Validation =  0.3802600077276734

Fold  3
MSE Training =  0.3414513875957866
MSE Validation =  0.27270356922404987

Fold  4
```

```
MSE Training =  0.36025168756902826
MSE Validation =  0.2344110741399533

Average Performance in Training =  0.31538562926512276
Average Performance in Validation =  0.359729179136414
-----------------------------------------------------

M Value =  3
Lambda Value =  1.0

Fold  1
MSE Training =  0.3284803614632877
MSE Validation =  0.35371762782049443

Fold  2
MSE Training =  0.2725280264531288
MSE Validation =  0.5104368539625761

Fold  3
MSE Training =  0.35394854768161216
MSE Validation =  0.21259268112977545

Fold  4
MSE Training =  0.31343660624803
MSE Validation =  0.4184513994517642

Average Performance in Training =  0.3170983854615147
Average Performance in Validation =  0.37379964059115256
-----------------------------------------------------

M Value =  4
Lambda Value =  0.1

Fold  1
MSE Training =  0.2598351950916166
MSE Validation =  0.18224411531228404

Fold  2
MSE Training =  0.2364841102484709
MSE Validation =  0.3102606653591335

Fold  3
MSE Training =  0.21460855878896326
MSE Validation =  0.3653905895728322

Fold  4
MSE Training =  0.2367504764318231
MSE Validation =  0.27901583044419714

Average Performance in Training =  0.23691958514021846
Average Performance in Validation =  0.28422780017211174
-----------------------------------------------------

M Value =  4
Lambda Value =  0.2

Fold  1
MSE Training =  0.2607490097965934
MSE Validation =  0.28853349678822426
```

```
Fold  2
MSE Training =  0.23292226346157757
MSE Validation =  0.3501191981835757

Fold  3
MSE Training =  0.2572079326332244
MSE Validation =  0.2696433836255509

Fold  4
MSE Training =  0.2843464683159227
MSE Validation =  0.18969249872119331

Average Performance in Training =  0.2588064185518295
Average Performance in Validation =  0.274497144329636
-----------------------------------------------------

M Value =  4
Lambda Value =  0.30000000000000004

Fold  1
MSE Training =  0.3256311784283992
MSE Validation =  0.11786038384316694

Fold  2
MSE Training =  0.27214734020905673
MSE Validation =  0.347695712391212

Fold  3
MSE Training =  0.2606503456269699
MSE Validation =  0.31799530456503433

Fold  4
MSE Training =  0.21229920740392527
MSE Validation =  0.48794773311126555

Average Performance in Training =  0.2676820179170878
Average Performance in Validation =  0.31787478347766973
-----------------------------------------------------

M Value =  4
Lambda Value =  0.4

Fold  1
MSE Training =  0.2881335278199773
MSE Validation =  0.28205781937738367

Fold  2
MSE Training =  0.2935170005923699
MSE Validation =  0.3068978933535664

Fold  3
MSE Training =  0.26415883936299983
MSE Validation =  0.3277466565903882

Fold  4
MSE Training =  0.2592899363569195
MSE Validation =  0.40959737270290497

Average Performance in Training =  0.27627482603306663
Average Performance in Validation =  0.3315749355060608
```

```
-----------------------------------------------------

M Value =  4
Lambda Value =  0.5

Fold  1
MSE Training =  0.2143587351192357
MSE Validation =  0.6411979261648114

Fold  2
MSE Training =  0.3165062742337456
MSE Validation =  0.24834347978888868

Fold  3
MSE Training =  0.31302250324479297
MSE Validation =  0.20783525211500276

Fold  4
MSE Training =  0.275683295443984
MSE Validation =  0.3711652954923002

Average Performance in Training =  0.2798927020104396
Average Performance in Validation =  0.36713548839025073
-----------------------------------------------------

M Value =  4
Lambda Value =  0.6

Fold  1
MSE Training =  0.29555875108688506
MSE Validation =  0.36079840735905816

Fold  2
MSE Training =  0.2770912581306022
MSE Validation =  0.414449203976499

Fold  3
MSE Training =  0.2844171583379835
MSE Validation =  0.30952434579207194

Fold  4
MSE Training =  0.31093392012275184
MSE Validation =  0.19470396760891603

Average Performance in Training =  0.2920002719195557
Average Performance in Validation =  0.31986898118413626
-----------------------------------------------------

M Value =  4
Lambda Value =  0.7000000000000001

Fold  1
MSE Training =  0.26979414418008696
MSE Validation =  0.43708469569493036

Fold  2
MSE Training =  0.2967375637009399
MSE Validation =  0.29197529584801235

Fold  3
```

```
MSE Training =  0.2806028314750097
MSE Validation =  0.4319276373040865

Fold  4
MSE Training =  0.35146193838650586
MSE Validation =  0.14999039189939745

Average Performance in Training =  0.2996491194356356
Average Performance in Validation =  0.3277445051866067
----------------------------------------------------

M Value =  4
Lambda Value =  0.8

Fold  1
MSE Training =  0.3432258720880999
MSE Validation =  0.15593411739646656

Fold  2
MSE Training =  0.30071486441767004
MSE Validation =  0.35401131087648563

Fold  3
MSE Training =  0.28591035307063534
MSE Validation =  0.39642549792633663

Fold  4
MSE Training =  0.2878588594830423
MSE Validation =  0.41792102378912616

Average Performance in Training =  0.30442748726486185
Average Performance in Validation =  0.33107298749710373
----------------------------------------------------

M Value =  4
Lambda Value =  0.9

Fold  1
MSE Training =  0.3093256113373072
MSE Validation =  0.34010159251802385

Fold  2
MSE Training =  0.29197962824243523
MSE Validation =  0.3671134267826196

Fold  3
MSE Training =  0.2884242149815993
MSE Validation =  0.44989193715613274

Fold  4
MSE Training =  0.33057446738114943
MSE Validation =  0.305725470154866

Average Performance in Training =  0.3050759804856228
Average Performance in Validation =  0.36570810665291054
----------------------------------------------------

M Value =  4
Lambda Value =  1.0
```

```
Fold  1
MSE Training =  0.2681443845614297
MSE Validation =  0.48166017690429747

Fold  2
MSE Training =  0.3674932857443808
MSE Validation =  0.18541527868309174

Fold  3
MSE Training =  0.2875264910534805
MSE Validation =  0.4964571254626127

Fold  4
MSE Training =  0.3178809642684322
MSE Validation =  0.29361237633948417

Average Performance in Training =  0.31026128140693077
Average Performance in Validation =  0.36428623934737153
-----------------------------------------------------

M Value =  5
Lambda Value =  0.1

Fold  1
MSE Training =  0.18901714600465408
MSE Validation =  0.29406432550941525

Fold  2
MSE Training =  0.22983318095392946
MSE Validation =  0.1623779984646257

Fold  3
MSE Training =  0.2011836786929333
MSE Validation =  0.2714377675158665

Fold  4
MSE Training =  0.2226518511039601
MSE Validation =  0.18415523626425

Average Performance in Training =  0.21067146418886923
Average Performance in Validation =  0.22800883193853938
-----------------------------------------------------

M Value =  5
Lambda Value =  0.2

Fold  1
MSE Training =  0.24449750694992256
MSE Validation =  0.22922648948741683

Fold  2
MSE Training =  0.17614428208516403
MSE Validation =  0.40504912533873855

Fold  3
MSE Training =  0.26200449790149904
MSE Validation =  0.1557615453065683

Fold  4
MSE Training =  0.23968382641422503
```

```
MSE Validation =  0.2561380150872549

Average Performance in Training =  0.23058252833770268
Average Performance in Validation =  0.26154379380499465
-----------------------------------------------------

M Value =  5
Lambda Value =  0.30000000000000004

Fold  1
MSE Training =  0.25413721973919906
MSE Validation =  0.26570987496940723

Fold  2
MSE Training =  0.25981404949960496
MSE Validation =  0.18349383824180154

Fold  3
MSE Training =  0.23625472900414232
MSE Validation =  0.2820913177961064

Fold  4
MSE Training =  0.2306595232893356
MSE Validation =  0.34715184356674267

Average Performance in Training =  0.24521638038307048
Average Performance in Validation =  0.26961171864351446
-----------------------------------------------------

M Value =  5
Lambda Value =  0.4

Fold  1
MSE Training =  0.22142278030885826
MSE Validation =  0.37630724474782445

Fold  2
MSE Training =  0.2633152765784179
MSE Validation =  0.23028589840120176

Fold  3
MSE Training =  0.26457945673615174
MSE Validation =  0.28758888939946303

Fold  4
MSE Training =  0.268674257151282
MSE Validation =  0.27394204144216955

Average Performance in Training =  0.2544979426936775
Average Performance in Validation =  0.29203101849766466
-----------------------------------------------------

M Value =  5
Lambda Value =  0.5

Fold  1
MSE Training =  0.2707923757668225
MSE Validation =  0.2760202144213778

Fold  2
```

```
MSE Training =  0.23330322353676963
MSE Validation =  0.4495859623331765

Fold  3
MSE Training =  0.2680080533950438
MSE Validation =  0.3559858144916508

Fold  4
MSE Training =  0.2777050638019299
MSE Validation =  0.23014744317274957

Average Performance in Training =  0.26245217912514146
Average Performance in Validation =  0.32793485860473864
-----------------------------------------------------

M Value =  5
Lambda Value =  0.6

Fold  1
MSE Training =  0.27173053185208274
MSE Validation =  0.29418986213644127

Fold  2
MSE Training =  0.24485173479526604
MSE Validation =  0.5489044944951921

Fold  3
MSE Training =  0.3124320692603627
MSE Validation =  0.15578188479807914

Fold  4
MSE Training =  0.2424697828958599
MSE Validation =  0.3839124611578694

Average Performance in Training =  0.2678710297008928
Average Performance in Validation =  0.3456971756468955
-----------------------------------------------------

M Value =  5
Lambda Value =  0.7000000000000001

Fold  1
MSE Training =  0.31481818008092927
MSE Validation =  0.19544169515538284

Fold  2
MSE Training =  0.28657866219457767
MSE Validation =  0.2937250994292044

Fold  3
MSE Training =  0.2511921109797155
MSE Validation =  0.3903280890016918

Fold  4
MSE Training =  0.2774028454127556
MSE Validation =  0.3194323394104564

Average Performance in Training =  0.2824979496669945
Average Performance in Validation =  0.29973180574918384
-----------------------------------------------------
```

```
M Value =  5
Lambda Value =  0.8

Fold  1
MSE Training =  0.23519023026781632
MSE Validation =  0.40985170490804246

Fold  2
MSE Training =  0.3464285969928045
MSE Validation =  0.12059458815326114

Fold  3
MSE Training =  0.2508494450587909
MSE Validation =  0.4812773390283429

Fold  4
MSE Training =  0.31307943963210044
MSE Validation =  0.2279598618054874

Average Performance in Training =  0.28638692798787807
Average Performance in Validation =  0.3099208734737835
-----------------------------------------------------

M Value =  5
Lambda Value =  0.9

Fold  1
MSE Training =  0.28139420208889754
MSE Validation =  0.3709868371852423

Fold  2
MSE Training =  0.30945050047943334
MSE Validation =  0.2736331264799117

Fold  3
MSE Training =  0.31389395244851576
MSE Validation =  0.21316853307913475

Fold  4
MSE Training =  0.2697349364601373
MSE Validation =  0.41900117778213536

Average Performance in Training =  0.293618397869246
Average Performance in Validation =  0.319197418631606
-----------------------------------------------------

M Value =  5
Lambda Value =  1.0

Fold  1
MSE Training =  0.35296038855050305
MSE Validation =  0.13783152404173563

Fold  2
MSE Training =  0.2324789258640425
MSE Validation =  0.9006398881573373

Fold  3
MSE Training =  0.2851458457357926
```

```
MSE Validation =  0.3583703360490118

Fold  4
MSE Training =  0.28668367874731276
MSE Validation =  0.34257963716653833

Average Performance in Training =  0.2893172097244127
Average Performance in Validation =  0.4348553463536558
-----------------------------------------------------

M Value =  6
Lambda Value =  0.1

Fold  1
MSE Training =  0.20452804907563055
MSE Validation =  0.18268836275151415

Fold  2
MSE Training =  0.18186397369554097
MSE Validation =  0.24543173936445994

Fold  3
MSE Training =  0.16374261165479076
MSE Validation =  0.35687013373308235

Fold  4
MSE Training =  0.21076798030122082
MSE Validation =  0.13078186129929836

Average Performance in Training =  0.19022565368179578
Average Performance in Validation =  0.22894302428708868
-----------------------------------------------------

M Value =  6
Lambda Value =  0.2

Fold  1
MSE Training =  0.21856760096956004
MSE Validation =  0.2333099659457697

Fold  2
MSE Training =  0.21052638809172575
MSE Validation =  0.2725465329673954

Fold  3
MSE Training =  0.19127569890729118
MSE Validation =  0.31244788999971

Fold  4
MSE Training =  0.21724757854228519
MSE Validation =  0.21106234897863763

Average Performance in Training =  0.20940431662771553
Average Performance in Validation =  0.2573416844728782
-----------------------------------------------------

M Value =  6
Lambda Value =  0.30000000000000004

Fold  1
```

```
MSE Training =  0.2378379271224408
MSE Validation =  0.23102262598993467

Fold  2
MSE Training =  0.23713980030304294
MSE Validation =  0.19815312616448227

Fold  3
MSE Training =  0.1885343064884248
MSE Validation =  0.41124162589217755

Fold  4
MSE Training =  0.23542696559972984
MSE Validation =  0.2256439210392732

Average Performance in Training =  0.2247347498784096
Average Performance in Validation =  0.2665153247714669
-----------------------------------------------------

M Value =  6
Lambda Value =  0.4

Fold  1
MSE Training =  0.2339558778293713
MSE Validation =  0.2547064429970972

Fold  2
MSE Training =  0.20620107058533274
MSE Validation =  0.39134064697113274

Fold  3
MSE Training =  0.2563758969718778
MSE Validation =  0.1671045911987319

Fold  4
MSE Training =  0.24916919144163407
MSE Validation =  0.20983112194767983

Average Performance in Training =  0.23642550920705396
Average Performance in Validation =  0.25574570077866043
-----------------------------------------------------

M Value =  6
Lambda Value =  0.5

Fold  1
MSE Training =  0.24402221433143775
MSE Validation =  0.35826299862177174

Fold  2
MSE Training =  0.28789447509833854
MSE Validation =  0.12262477795042738

Fold  3
MSE Training =  0.21364204589384467
MSE Validation =  0.3239629892493337

Fold  4
MSE Training =  0.2315893841722071
MSE Validation =  0.41431016165468143
```

```
Average Performance in Training =  0.24428702987395703
Average Performance in Validation =  0.3047902318690536
-----------------------------------------------------

M Value =  6
Lambda Value =  0.6

Fold  1
MSE Training =  0.27369776096763915
MSE Validation =  0.24827885165793473

Fold  2
MSE Training =  0.22360522172612443
MSE Validation =  0.4814264813441141

Fold  3
MSE Training =  0.27978878673301505
MSE Validation =  0.1547590292749154

Fold  4
MSE Training =  0.2271148408543481
MSE Validation =  0.29007152042567363

Average Performance in Training =  0.2510516525702817
Average Performance in Validation =  0.29363397067565944
-----------------------------------------------------

M Value =  6
Lambda Value =  0.7000000000000001

Fold  1
MSE Training =  0.24916580563823848
MSE Validation =  0.31648728749986077

Fold  2
MSE Training =  0.2963519495399126
MSE Validation =  0.08937265010741856

Fold  3
MSE Training =  0.23105649612280443
MSE Validation =  0.4855798799855153

Fold  4
MSE Training =  0.2380065165860632
MSE Validation =  0.5616040227201267

Average Performance in Training =  0.2536451919717547
Average Performance in Validation =  0.36326096007823033
-----------------------------------------------------

M Value =  6
Lambda Value =  0.8

Fold  1
MSE Training =  0.2846491761157306
MSE Validation =  0.2973792975011373

Fold  2
MSE Training =  0.26083957130234553
```

```
MSE Validation =  0.36123200478222384

Fold  3
MSE Training =  0.2607494522906608
MSE Validation =  0.24461909548967775

Fold  4
MSE Training =  0.26800821596336016
MSE Validation =  0.2992695365139924

Average Performance in Training =  0.26856160391802425
Average Performance in Validation =  0.3006249835717578
-----------------------------------------------------

M Value =  6
Lambda Value =  0.9

Fold  1
MSE Training =  0.276889796059958
MSE Validation =  0.34579931480440246

Fold  2
MSE Training =  0.29462965128031016
MSE Validation =  0.2843655553340673

Fold  3
MSE Training =  0.2737087503152193
MSE Validation =  0.2741627238217852

Fold  4
MSE Training =  0.2564221982850611
MSE Validation =  0.35970663647971907

Average Performance in Training =  0.27541259898513715
Average Performance in Validation =  0.3160085576099935
-----------------------------------------------------

M Value =  6
Lambda Value =  1.0

Fold  1
MSE Training =  0.2865949957801585
MSE Validation =  0.22551882305959645

Fold  2
MSE Training =  0.2997915147787992
MSE Validation =  0.24340416358280248

Fold  3
MSE Training =  0.23897513041076232
MSE Validation =  0.5017287171939713

Fold  4
MSE Training =  0.30517113266097634
MSE Validation =  0.23602752032340626

Average Performance in Training =  0.28263319340767407
Average Performance in Validation =  0.30166980603994414
-----------------------------------------------------
```

```
M Value =  7
Lambda Value =  0.1

Fold  1
MSE Training =  0.16327944783304682
MSE Validation =  0.26619999497282454

Fold  2
MSE Training =  0.19029444775427035
MSE Validation =  0.17702770557526498

Fold  3
MSE Training =  0.20167733344534944
MSE Validation =  0.1420229466693379

Fold  4
MSE Training =  0.1787213651068394
MSE Validation =  0.20524696341738177

Average Performance in Training =  0.18349314853487647
Average Performance in Validation =  0.19762440265870232
-----------------------------------------------------

M Value =  7
Lambda Value =  0.2

Fold  1
MSE Training =  0.1789982187686279
MSE Validation =  0.3340707011095744

Fold  2
MSE Training =  0.2016743907480117
MSE Validation =  0.23988971285168376

Fold  3
MSE Training =  0.20926352457140585
MSE Validation =  0.16331194021492357

Fold  4
MSE Training =  0.1970026339189984
MSE Validation =  0.29797990849123307

Average Performance in Training =  0.19673469200176097
Average Performance in Validation =  0.2588130656668537
-----------------------------------------------------

M Value =  7
Lambda Value =  0.30000000000000004

Fold  1
MSE Training =  0.23844254953889826
MSE Validation =  0.12950568601699827

Fold  2
MSE Training =  0.17550369594105797
MSE Validation =  0.41789916673182326

Fold  3
MSE Training =  0.22539193203999852
MSE Validation =  0.17992846858088343
```

```
Fold  4
MSE Training =  0.20376244839918364
MSE Validation =  0.2366234909310024

Average Performance in Training =  0.2107751564797846
Average Performance in Validation =  0.24098920306517685
----------------------------------------------------

M Value =  7
Lambda Value =  0.4

Fold  1
MSE Training =  0.22765642574208478
MSE Validation =  0.2051690678570176

Fold  2
MSE Training =  0.20051686382000178
MSE Validation =  0.3261866317805702

Fold  3
MSE Training =  0.2201146400394788
MSE Validation =  0.2325513119168299

Fold  4
MSE Training =  0.23630917452682995
MSE Validation =  0.20978051572712175

Average Performance in Training =  0.22114927603209883
Average Performance in Validation =  0.24342188182038488
----------------------------------------------------

M Value =  7
Lambda Value =  0.5

Fold  1
MSE Training =  0.25367936440671385
MSE Validation =  0.20288624214315076

Fold  2
MSE Training =  0.2605682901754288
MSE Validation =  0.1363954373871395

Fold  3
MSE Training =  0.2109501010554721
MSE Validation =  0.3164672266332121

Fold  4
MSE Training =  0.20615839899773405
MSE Validation =  0.34860012389212086

Average Performance in Training =  0.23283903865883718
Average Performance in Validation =  0.2510872575139058
----------------------------------------------------

M Value =  7
Lambda Value =  0.6

Fold  1
MSE Training =  0.2330050467259841
```

```
MSE Validation =  0.3066088450328058

Fold  2
MSE Training =  0.2313813761279808
MSE Validation =  0.20114797413132035

Fold  3
MSE Training =  0.23303693017232363
MSE Validation =  0.28458425278194627

Fold  4
MSE Training =  0.23746242515551166
MSE Validation =  0.3267294565130629

Average Performance in Training =  0.23372144454545005
Average Performance in Validation =  0.27976763211478384
-----------------------------------------------------

M Value =  7
Lambda Value =  0.7000000000000001

Fold  1
MSE Training =  0.24746597270418136
MSE Validation =  0.330222737964485

Fold  2
MSE Training =  0.2679659390110226
MSE Validation =  0.15658478510859525

Fold  3
MSE Training =  0.24481826932585982
MSE Validation =  0.31217886794213523

Fold  4
MSE Training =  0.22002373275873469
MSE Validation =  0.33354206808458664

Average Performance in Training =  0.2450684784499496
Average Performance in Validation =  0.28313211477495054
-----------------------------------------------------

M Value =  7
Lambda Value =  0.8

Fold  1
MSE Training =  0.21604736710218286
MSE Validation =  0.3899617436279349

Fold  2
MSE Training =  0.2836443437064711
MSE Validation =  0.15790252578114192

Fold  3
MSE Training =  0.236873894551957
MSE Validation =  0.45923156541085075

Fold  4
MSE Training =  0.27488938718502187
MSE Validation =  0.21299737090200552
```

```
Average Performance in Training =  0.2528637481364082
Average Performance in Validation =  0.30502330143048323
-----------------------------------------------------

M Value =  7
Lambda Value =  0.9

Fold  1
MSE Training =  0.277009266402869
MSE Validation =  0.26659095107571407

Fold  2
MSE Training =  0.25720340580927104
MSE Validation =  0.2906478788074247

Fold  3
MSE Training =  0.23246288246376418
MSE Validation =  0.41505271721195564

Fold  4
MSE Training =  0.2768611607301441
MSE Validation =  0.21622943128442437

Average Performance in Training =  0.2608841788515121
Average Performance in Validation =  0.2971302445948797
-----------------------------------------------------

M Value =  7
Lambda Value =  1.0

Fold  1
MSE Training =  0.29401668288024185
MSE Validation =  0.2303160287202963

Fold  2
MSE Training =  0.2909459802448375
MSE Validation =  0.18251484537613266

Fold  3
MSE Training =  0.25069750568175686
MSE Validation =  0.3431122382657294

Fold  4
MSE Training =  0.24059905907626017
MSE Validation =  0.42472320771617683

Average Performance in Training =  0.26906480697077406
Average Performance in Validation =  0.2951665800195838
-----------------------------------------------------

M Value =  8
Lambda Value =  0.1

Fold  1
MSE Training =  0.19469308158972942
MSE Validation =  0.15985934524540088

Fold  2
MSE Training =  0.1513829201943113
MSE Validation =  0.32102766041916275
```

```
Fold  3
MSE Training =  0.18916058066619756
MSE Validation =  0.1670864733668366

Fold  4
MSE Training =  0.18406035721896277
MSE Validation =  0.15867895409707833

Average Performance in Training =  0.17982423491730026
Average Performance in Validation =  0.20166310828211964
-----------------------------------------------------

M Value =  8
Lambda Value =  0.2

Fold  1
MSE Training =  0.21425141440786635
MSE Validation =  0.17042440370305775

Fold  2
MSE Training =  0.1806784607836949
MSE Validation =  0.2538236239071144

Fold  3
MSE Training =  0.18863062739875566
MSE Validation =  0.23232434354677325

Fold  4
MSE Training =  0.18574331708757494
MSE Validation =  0.30072524184754934

Average Performance in Training =  0.19232595491947296
Average Performance in Validation =  0.23932440325112367
-----------------------------------------------------

M Value =  8
Lambda Value =  0.30000000000000004

Fold  1
MSE Training =  0.2396008590620436
MSE Validation =  0.10280240358574057

Fold  2
MSE Training =  0.17360116458779787
MSE Validation =  0.37778639528316543

Fold  3
MSE Training =  0.21524722797874038
MSE Validation =  0.1440560208677573

Fold  4
MSE Training =  0.17838801313051408
MSE Validation =  0.31662612206156754

Average Performance in Training =  0.201709316189774
Average Performance in Validation =  0.2353177354495577
-----------------------------------------------------

M Value =  8
```

```
Lambda Value =  0.4

Fold  1
MSE Training =  0.21273333316982648
MSE Validation =  0.20994709652087792

Fold  2
MSE Training =  0.1919573075440187
MSE Validation =  0.3507222183374195

Fold  3
MSE Training =  0.20817924854010494
MSE Validation =  0.26581418442625065

Fold  4
MSE Training =  0.23180508458954535
MSE Validation =  0.16641691782094561

Average Performance in Training =  0.2111687434608739
Average Performance in Validation =  0.2482251042763734
-----------------------------------------------------

M Value =  8
Lambda Value =  0.5

Fold  1
MSE Training =  0.2285985311009764
MSE Validation =  0.24135479481206737

Fold  2
MSE Training =  0.20084590698311455
MSE Validation =  0.2847475070896854

Fold  3
MSE Training =  0.24541142328636764
MSE Validation =  0.15901866185094718

Fold  4
MSE Training =  0.2096109453064271
MSE Validation =  0.25945897745037433

Average Performance in Training =  0.22111670166922143
Average Performance in Validation =  0.23614498530076855
-----------------------------------------------------

M Value =  8
Lambda Value =  0.6

Fold  1
MSE Training =  0.22765741423218416
MSE Validation =  0.22193775665240187

Fold  2
MSE Training =  0.24640412243880472
MSE Validation =  0.3364474641725502

Fold  3
MSE Training =  0.22041264270202277
MSE Validation =  0.32198007221063885
```

```
Fold  4
MSE Training =  0.2138631720623084
MSE Validation =  0.2524411740429094

Average Performance in Training =  0.22708433785883
Average Performance in Validation =  0.2832016167696251
-----------------------------------------------------

M Value =  8
Lambda Value =  0.7000000000000001

Fold  1
MSE Training =  0.22697271673723407
MSE Validation =  0.2667372182408617

Fold  2
MSE Training =  0.27883443635776317
MSE Validation =  0.1362452637719963

Fold  3
MSE Training =  0.18862455026092628
MSE Validation =  0.47732110558717733

Fold  4
MSE Training =  0.23182474556991972
MSE Validation =  0.3185048285144621

Average Performance in Training =  0.2315641122314608
Average Performance in Validation =  0.29970210402862435
-----------------------------------------------------

M Value =  8
Lambda Value =  0.8

Fold  1
MSE Training =  0.21622907290286605
MSE Validation =  0.3915478198837754

Fold  2
MSE Training =  0.2486227491338374
MSE Validation =  0.20138847756497463

Fold  3
MSE Training =  0.2536129072281956
MSE Validation =  0.31948920166819184

Fold  4
MSE Training =  0.24481514602233598
MSE Validation =  0.2116889812060241

Average Performance in Training =  0.24081996882180878
Average Performance in Validation =  0.2810286200807415
-----------------------------------------------------

M Value =  8
Lambda Value =  0.9

Fold  1
MSE Training =  0.2364605145768387
MSE Validation =  0.33252500621124803
```

```
Fold  2
MSE Training =  0.27496072081953743
MSE Validation =  0.19000894398467288

Fold  3
MSE Training =  0.2372784842633411
MSE Validation =  0.24573312265106217

Fold  4
MSE Training =  0.2466997049952305
MSE Validation =  0.3180575805080022

Average Performance in Training =  0.24884985616373692
Average Performance in Validation =  0.2715811633387463
-----------------------------------------------------

M Value =  8
Lambda Value =  1.0

Fold  1
MSE Training =  0.2049778859832691
MSE Validation =  0.5088350685828603

Fold  2
MSE Training =  0.27479689122903367
MSE Validation =  0.2152978181615503

Fold  3
MSE Training =  0.2684022809505198
MSE Validation =  0.2009264585596855

Fold  4
MSE Training =  0.2668008721620795
MSE Validation =  0.21983567811499946

Average Performance in Training =  0.2537444825812255
Average Performance in Validation =  0.2862237558547739
-----------------------------------------------------

M Value =  9
Lambda Value =  0.1

Fold  1
MSE Training =  0.19067724192131022
MSE Validation =  0.18116571236232099

Fold  2
MSE Training =  0.17246211496567074
MSE Validation =  0.22278977135023828

Fold  3
MSE Training =  0.18789047918879573
MSE Validation =  0.1757511845254228

Fold  4
MSE Training =  0.16269176417000109
MSE Validation =  0.25566917682408047

Average Performance in Training =  0.17843040006144445
```

```
Average Performance in Validation =  0.20884396126551563
--------------------------------------------------

M Value =  9
Lambda Value =  0.2

Fold  1
MSE Training =  0.21685712970685528
MSE Validation =  0.14707827277232782

Fold  2
MSE Training =  0.17186502140915336
MSE Validation =  0.25604401339212013

Fold  3
MSE Training =  0.17711585376522893
MSE Validation =  0.2708850254963821

Fold  4
MSE Training =  0.18893837331249552
MSE Validation =  0.21175383356964197

Average Performance in Training =  0.18869409454843325
Average Performance in Validation =  0.221440286307618
--------------------------------------------------

M Value =  9
Lambda Value =  0.30000000000000004

Fold  1
MSE Training =  0.21101252902734763
MSE Validation =  0.1649955388774629

Fold  2
MSE Training =  0.22750210279342145
MSE Validation =  0.12740739114931796

Fold  3
MSE Training =  0.198222548409998
MSE Validation =  0.21691503761070527

Fold  4
MSE Training =  0.16552301958541651
MSE Validation =  0.3299832758452994

Average Performance in Training =  0.2005650499540459
Average Performance in Validation =  0.2098253108706964
--------------------------------------------------

M Value =  9
Lambda Value =  0.4

Fold  1
MSE Training =  0.23058283677380423
MSE Validation =  0.1552983608548749

Fold  2
MSE Training =  0.20443862183246267
MSE Validation =  0.22679327610333536
```

```
Fold  3
MSE Training =  0.2186699258941824
MSE Validation =  0.16815508155337996

Fold  4
MSE Training =  0.17756309532624856
MSE Validation =  0.3280853451369746

Average Performance in Training =  0.20781361995667447
Average Performance in Validation =  0.21958301591214122
----------------------------------------------------

M Value =  9
Lambda Value =  0.5

Fold  1
MSE Training =  0.21828273125142283
MSE Validation =  0.23869336688003037

Fold  2
MSE Training =  0.19354675484902112
MSE Validation =  0.26629417259433574

Fold  3
MSE Training =  0.22724395737201233
MSE Validation =  0.1830397659109918

Fold  4
MSE Training =  0.21094420199526537
MSE Validation =  0.25306347234445353

Average Performance in Training =  0.21250441136693043
Average Performance in Validation =  0.23527269443245286
----------------------------------------------------

M Value =  9
Lambda Value =  0.6

Fold  1
MSE Training =  0.21348378160714382
MSE Validation =  0.20476341675259802

Fold  2
MSE Training =  0.2510580291595097
MSE Validation =  0.1918857440003356

Fold  3
MSE Training =  0.19834736082570284
MSE Validation =  0.34719070290720394

Fold  4
MSE Training =  0.20163521802014278
MSE Validation =  0.39220574730550667

Average Performance in Training =  0.2161310974031248
Average Performance in Validation =  0.28401140274141107
----------------------------------------------------

M Value =  9
Lambda Value =  0.7000000000000001
```

```
Fold  1
MSE Training =  0.2407480911862275
MSE Validation =  0.27607107791344115

Fold  2
MSE Training =  0.2281577506248719
MSE Validation =  0.22552573174967275

Fold  3
MSE Training =  0.23800506754133743
MSE Validation =  0.1755841031806776

Fold  4
MSE Training =  0.20610459836281766
MSE Validation =  0.305363991932707

Average Performance in Training =  0.22825387692881363
Average Performance in Validation =  0.24563622619412465
-----------------------------------------------------

M Value =  9
Lambda Value =  0.8

Fold  1
MSE Training =  0.2610587147967429
MSE Validation =  0.17416261480204243

Fold  2
MSE Training =  0.24153556762110212
MSE Validation =  0.33775157180859405

Fold  3
MSE Training =  0.21857263239296984
MSE Validation =  0.21399731220056198

Fold  4
MSE Training =  0.19195123705497416
MSE Validation =  0.44660340399559445

Average Performance in Training =  0.2282795379664473
Average Performance in Validation =  0.29312872570169823
-----------------------------------------------------

M Value =  9
Lambda Value =  0.9

Fold  1
MSE Training =  0.22062727191561782
MSE Validation =  0.30407664810384566

Fold  2
MSE Training =  0.2618148153837427
MSE Validation =  0.19465451907191866

Fold  3
MSE Training =  0.242222449999056
MSE Validation =  0.23701978877485586

Fold  4
```

```
MSE Training =  0.22750454642783144
MSE Validation =  0.399837700250593

Average Performance in Training =  0.238042270931562
Average Performance in Validation =  0.2838971640503033
-----------------------------------------------------

M Value =  9
Lambda Value =  1.0

Fold  1
MSE Training =  0.29522103422103774
MSE Validation =  0.12126444132914588

Fold  2
MSE Training =  0.23622920462654878
MSE Validation =  0.2906815659290208

Fold  3
MSE Training =  0.239663058843281
MSE Validation =  0.30943837240702515

Fold  4
MSE Training =  0.2169923276553049
MSE Validation =  0.32355079725710695

Average Performance in Training =  0.2470264063365431
Average Performance in Validation =  0.2612337942305747
-----------------------------------------------------

M Value =  10
Lambda Value =  0.1

Fold  1
MSE Training =  0.19224729587161385
MSE Validation =  0.17838989410888162

Fold  2
MSE Training =  0.1832847408094917
MSE Validation =  0.1817175970589

Fold  3
MSE Training =  0.18159167437791512
MSE Validation =  0.22579311672012933

Fold  4
MSE Training =  0.15432382550933088
MSE Validation =  0.2958619743691571

Average Performance in Training =  0.1778618841420879
Average Performance in Validation =  0.220440645564267
-----------------------------------------------------

M Value =  10
Lambda Value =  0.2

Fold  1
MSE Training =  0.19914775173259813
MSE Validation =  0.16960218088715534
```

```
Fold  2
MSE Training =  0.1798678252170785
MSE Validation =  0.25119250882038247

Fold  3
MSE Training =  0.2001871587638244
MSE Validation =  0.17070076576656912

Fold  4
MSE Training =  0.1788810110582765
MSE Validation =  0.2656473258298138

Average Performance in Training =  0.18952093669294437
Average Performance in Validation =  0.21428569532598019
-----------------------------------------------------

M Value =  10
Lambda Value =  0.30000000000000004

Fold  1
MSE Training =  0.19885841110233954
MSE Validation =  0.2015184784108245

Fold  2
MSE Training =  0.169514709426397
MSE Validation =  0.30954967018110024

Fold  3
MSE Training =  0.23900482412501278
MSE Validation =  0.05280298489999568

Fold  4
MSE Training =  0.17911786970284385
MSE Validation =  0.2753448267605013

Average Performance in Training =  0.19662395358914828
Average Performance in Validation =  0.20980399006310543
-----------------------------------------------------

M Value =  10
Lambda Value =  0.4

Fold  1
MSE Training =  0.1983737722063176
MSE Validation =  0.27530210984141096

Fold  2
MSE Training =  0.223752093638682
MSE Validation =  0.15610662741989062

Fold  3
MSE Training =  0.20251388690956337
MSE Validation =  0.2505192909128404

Fold  4
MSE Training =  0.17881648555930593
MSE Validation =  0.28651336483867745

Average Performance in Training =  0.20086405957846723
Average Performance in Validation =  0.24211034825320488
```

```
-----------------------------------------------------

M Value =  10
Lambda Value =  0.5

Fold  1
MSE Training =  0.19533670509731232
MSE Validation =  0.2575700769947987

Fold  2
MSE Training =  0.22421027522382253
MSE Validation =  0.20199981038708628

Fold  3
MSE Training =  0.2050703694399847
MSE Validation =  0.21510282409166578

Fold  4
MSE Training =  0.209282441494187
MSE Validation =  0.2659964828559109

Average Performance in Training =  0.20847494781382664
Average Performance in Validation =  0.2351672985823654
-----------------------------------------------------

M Value =  10
Lambda Value =  0.6

Fold  1
MSE Training =  0.22630151355912023
MSE Validation =  0.17113257504000015

Fold  2
MSE Training =  0.21794917182374052
MSE Validation =  0.19438542797857974

Fold  3
MSE Training =  0.17942739352472045
MSE Validation =  0.4221574041307607

Fold  4
MSE Training =  0.23687063354872887
MSE Validation =  0.2572217466211802

Average Performance in Training =  0.2151371781140775
Average Performance in Validation =  0.2612242884426302
-----------------------------------------------------

M Value =  10
Lambda Value =  0.7000000000000001

Fold  1
MSE Training =  0.20437211366836996
MSE Validation =  0.3769214230469347

Fold  2
MSE Training =  0.24050637787140866
MSE Validation =  0.15333174543840936

Fold  3
```

```
MSE Training =  0.2570508246513983
MSE Validation =  0.08148310297798313

Fold  4
MSE Training =  0.17005608560515548
MSE Validation =  0.4857692967077536

Average Performance in Training =  0.2179963504490831
Average Performance in Validation =  0.2743763920427702
-----------------------------------------------------

M Value =  10
Lambda Value =  0.8

Fold  1
MSE Training =  0.21676393985888945
MSE Validation =  0.26815950120684817

Fold  2
MSE Training =  0.2539075463509569
MSE Validation =  0.17047678618358558

Fold  3
MSE Training =  0.22620094531603255
MSE Validation =  0.2818032270579787

Fold  4
MSE Training =  0.21632893293284694
MSE Validation =  0.28068934332044126

Average Performance in Training =  0.22830034111468145
Average Performance in Validation =  0.25028221444221344
-----------------------------------------------------

M Value =  10
Lambda Value =  0.9

Fold  1
MSE Training =  0.22094844256183166
MSE Validation =  0.28647701470709147

Fold  2
MSE Training =  0.252813560307433
MSE Validation =  0.17218000499931993

Fold  3
MSE Training =  0.2169600339694007
MSE Validation =  0.35800666023779204

Fold  4
MSE Training =  0.23585606675148538
MSE Validation =  0.3032994890134482

Average Performance in Training =  0.2316445258975377
Average Performance in Validation =  0.2799907922394129
-----------------------------------------------------

M Value =  10
Lambda Value =  1.0
```

```
Fold  1
MSE Training =  0.25116698931114173
MSE Validation =  0.20235295294514444

Fold  2
MSE Training =  0.2367772609000289
MSE Validation =  0.24684635762773185

Fold  3
MSE Training =  0.2522086068609075
MSE Validation =  0.20225682661731378

Fold  4
MSE Training =  0.22076542423981055
MSE Validation =  0.3680416313044134

Average Performance in Training =  0.24022957032797215
Average Performance in Validation =  0.2548744421236509
-----------------------------------------------------

M Value =  11
Lambda Value =  0.1

Fold  1
MSE Training =  0.1475224471486918
MSE Validation =  0.3067587354996984

Fold  2
MSE Training =  0.20583676359513264
MSE Validation =  0.13405766706133235

Fold  3
MSE Training =  0.188420246305382
MSE Validation =  0.20546108199426646

Fold  4
MSE Training =  0.1821954804546395
MSE Validation =  0.1848521448455482

Average Performance in Training =  0.1809937343759615
Average Performance in Validation =  0.20778240735021136
-----------------------------------------------------

M Value =  11
Lambda Value =  0.2

Fold  1
MSE Training =  0.19391184410701515
MSE Validation =  0.20642069894637327

Fold  2
MSE Training =  0.174869156853575
MSE Validation =  0.25560942402566017

Fold  3
MSE Training =  0.1777902023317535
MSE Validation =  0.2569894393296203

Fold  4
MSE Training =  0.2017004517447681
```

```
MSE Validation =  0.182803756046609

Average Performance in Training =  0.18706791375927795
Average Performance in Validation =  0.2254558295870657
----------------------------------------------------

M Value =  11
Lambda Value =  0.30000000000000004

Fold  1
MSE Training =  0.19910962282981198
MSE Validation =  0.19789851715143303

Fold  2
MSE Training =  0.23592719760887126
MSE Validation =  0.08581758461863265

Fold  3
MSE Training =  0.16611811588937014
MSE Validation =  0.3199810124743789

Fold  4
MSE Training =  0.17935202539409348
MSE Validation =  0.270495999457244

Average Performance in Training =  0.1951267404305367
Average Performance in Validation =  0.21854827842542213
----------------------------------------------------

M Value =  11
Lambda Value =  0.4

Fold  1
MSE Training =  0.17280885820319802
MSE Validation =  0.35088602248491685

Fold  2
MSE Training =  0.19544412418573584
MSE Validation =  0.2120843022281582

Fold  3
MSE Training =  0.21796162319036055
MSE Validation =  0.13789085913739338

Fold  4
MSE Training =  0.2087508659066101
MSE Validation =  0.21718457083550277

Average Performance in Training =  0.19874136787147614
Average Performance in Validation =  0.22951143867149282
----------------------------------------------------

M Value =  11
Lambda Value =  0.5

Fold  1
MSE Training =  0.21145176734032203
MSE Validation =  0.23731324703648407

Fold  2
```

```
MSE Training =  0.22752340789025133
MSE Validation =  0.1818632113826298

Fold  3
MSE Training =  0.19851146612457504
MSE Validation =  0.270562288905984

Fold  4
MSE Training =  0.17336378310218808
MSE Validation =  0.326515384328765

Average Performance in Training =  0.20271260611433412
Average Performance in Validation =  0.25406353291346573
----------------------------------------------------

M Value =  11
Lambda Value =  0.6

Fold  1
MSE Training =  0.2186308855597814
MSE Validation =  0.1728197779710422

Fold  2
MSE Training =  0.22388970016757703
MSE Validation =  0.2592496622764892

Fold  3
MSE Training =  0.2057542858625969
MSE Validation =  0.2728898772595377

Fold  4
MSE Training =  0.19134475318640126
MSE Validation =  0.34352332784459033

Average Performance in Training =  0.20990490619408914
Average Performance in Validation =  0.2621206613379149
----------------------------------------------------

M Value =  11
Lambda Value =  0.7000000000000001

Fold  1
MSE Training =  0.2053094742925283
MSE Validation =  0.2755082153733789

Fold  2
MSE Training =  0.2286351833003876
MSE Validation =  0.21720011949553764

Fold  3
MSE Training =  0.21542525965734272
MSE Validation =  0.280344124720599

Fold  4
MSE Training =  0.2196635825257977
MSE Validation =  0.25168805731308763

Average Performance in Training =  0.21725837494401407
Average Performance in Validation =  0.2561851292256508
----------------------------------------------------
```

```
M Value =  11
Lambda Value =  0.8

Fold  1
MSE Training =  0.2335864256280337
MSE Validation =  0.23048294889022264

Fold  2
MSE Training =  0.19971893106858365
MSE Validation =  0.3849038859507857

Fold  3
MSE Training =  0.20713754971903456
MSE Validation =  0.28190666772400524

Fold  4
MSE Training =  0.24600254256061405
MSE Validation =  0.13313598682067268

Average Performance in Training =  0.2216113622440665
Average Performance in Validation =  0.25760737234642156
-----------------------------------------------------

M Value =  11
Lambda Value =  0.9

Fold  1
MSE Training =  0.23733643331553939
MSE Validation =  0.21074786687038466

Fold  2
MSE Training =  0.20166508949152046
MSE Validation =  0.3127892316984938

Fold  3
MSE Training =  0.23797509601433747
MSE Validation =  0.23183066313894365

Fold  4
MSE Training =  0.24033837330706737
MSE Validation =  0.2136489102896875

Average Performance in Training =  0.22932874803211617
Average Performance in Validation =  0.2422541679993774
-----------------------------------------------------

M Value =  11
Lambda Value =  1.0

Fold  1
MSE Training =  0.20613495116567432
MSE Validation =  0.36020851293223355

Fold  2
MSE Training =  0.25969447837593385
MSE Validation =  0.21483236226718888

Fold  3
MSE Training =  0.23651089596421557
```

```
MSE Validation =  0.18215689455365447

Fold  4
MSE Training =  0.23044362680049477
MSE Validation =  0.2998387095225071

Average Performance in Training =  0.23319598807657962
Average Performance in Validation =  0.264259119818896
-----------------------------------------------------

M Value =  12
Lambda Value =  0.1

Fold  1
MSE Training =  0.1657916621576033
MSE Validation =  0.28573988437392067

Fold  2
MSE Training =  0.1955070428844015
MSE Validation =  0.16563001729138227

Fold  3
MSE Training =  0.19821403506228027
MSE Validation =  0.14874156050963022

Fold  4
MSE Training =  0.14910715162130975
MSE Validation =  0.3912072663782775

Average Performance in Training =  0.1771549729313987
Average Performance in Validation =  0.24782968213830267
-----------------------------------------------------

M Value =  12
Lambda Value =  0.2

Fold  1
MSE Training =  0.21228075396408028
MSE Validation =  0.1707791035488503

Fold  2
MSE Training =  0.16528054726122474
MSE Validation =  0.3228990793684925

Fold  3
MSE Training =  0.18132324841355826
MSE Validation =  0.21927061296095796

Fold  4
MSE Training =  0.17939061362954964
MSE Validation =  0.2937169265444886

Average Performance in Training =  0.18456879081710323
Average Performance in Validation =  0.2516664306056973
-----------------------------------------------------

M Value =  12
Lambda Value =  0.30000000000000004

Fold  1
```

```
MSE Training =  0.18519298554093422
MSE Validation =  0.2318976902277821

Fold  2
MSE Training =  0.1901471030052991
MSE Validation =  0.24821397985534166

Fold  3
MSE Training =  0.20422963364233357
MSE Validation =  0.18279724905604214

Fold  4
MSE Training =  0.19660054000689967
MSE Validation =  0.24555749429779394

Average Performance in Training =  0.19404256554886665
Average Performance in Validation =  0.22711660335923994
-----------------------------------------------------

M Value =  12
Lambda Value =  0.4

Fold  1
MSE Training =  0.18975707443146428
MSE Validation =  0.24085135348841083

Fold  2
MSE Training =  0.2269442818464211
MSE Validation =  0.13182768506516662

Fold  3
MSE Training =  0.18418253524490247
MSE Validation =  0.31247073552776444

Fold  4
MSE Training =  0.19650477054900406
MSE Validation =  0.23591635705807792

Average Performance in Training =  0.19934716551794798
Average Performance in Validation =  0.23026653278485495
-----------------------------------------------------

M Value =  12
Lambda Value =  0.5

Fold  1
MSE Training =  0.2036427310449912
MSE Validation =  0.23738111543266568

Fold  2
MSE Training =  0.22321395538145544
MSE Validation =  0.1512118858351844

Fold  3
MSE Training =  0.20090514092289435
MSE Validation =  0.276783176565822

Fold  4
MSE Training =  0.1916320893440011
MSE Validation =  0.25204768842815334
```

```
Average Performance in Training =  0.20484847917333554
Average Performance in Validation =  0.22935596656545637
-----------------------------------------------------

M Value =  12
Lambda Value =  0.6

Fold  1
MSE Training =  0.20717222066341745
MSE Validation =  0.26499676621126517

Fold  2
MSE Training =  0.1985622989950336
MSE Validation =  0.30233425882848375

Fold  3
MSE Training =  0.2251163836919801
MSE Validation =  0.21028580451282683

Fold  4
MSE Training =  0.2026138563280082
MSE Validation =  0.20827969634156143

Average Performance in Training =  0.20836618991960984
Average Performance in Validation =  0.2464741314735343
-----------------------------------------------------

M Value =  12
Lambda Value =  0.7000000000000001

Fold  1
MSE Training =  0.2088459337502322
MSE Validation =  0.22926519595574787

Fold  2
MSE Training =  0.21065891162564432
MSE Validation =  0.2488288439298529

Fold  3
MSE Training =  0.22484044889756225
MSE Validation =  0.24350419241921506

Fold  4
MSE Training =  0.2010856014304834
MSE Validation =  0.31726703582853505

Average Performance in Training =  0.21135772392598054
Average Performance in Validation =  0.2597163170333377
-----------------------------------------------------

M Value =  12
Lambda Value =  0.8

Fold  1
MSE Training =  0.218220020471139
MSE Validation =  0.2835127699176148

Fold  2
MSE Training =  0.2429055138712892
```

```
MSE Validation =  0.1626839404421214

Fold  3
MSE Training =  0.21081111995987703
MSE Validation =  0.30470520830015624

Fold  4
MSE Training =  0.20673528955230974
MSE Validation =  0.2572104061020612

Average Performance in Training =  0.21966798596365372
Average Performance in Validation =  0.2520280811904884
----------------------------------------------------

M Value =  12
Lambda Value =  0.9

Fold  1
MSE Training =  0.20675965528247078
MSE Validation =  0.2830136364893678

Fold  2
MSE Training =  0.23479389458638109
MSE Validation =  0.2721151512693911

Fold  3
MSE Training =  0.21642824452714127
MSE Validation =  0.22384351072974185

Fold  4
MSE Training =  0.24499618966775283
MSE Validation =  0.20479466928761325

Average Performance in Training =  0.2257444960159365
Average Performance in Validation =  0.2459417419440285
----------------------------------------------------

M Value =  12
Lambda Value =  1.0

Fold  1
MSE Training =  0.24593042334585208
MSE Validation =  0.23638056693542417

Fold  2
MSE Training =  0.2513230735035237
MSE Validation =  0.19832295533813166

Fold  3
MSE Training =  0.18545513712840037
MSE Validation =  0.40774115027218033

Fold  4
MSE Training =  0.23893482455038464
MSE Validation =  0.21528200628857375

Average Performance in Training =  0.2304108646320402
Average Performance in Validation =  0.26443166970857745
----------------------------------------------------
```

```
M Value =  13
Lambda Value =  0.1

Fold  1
MSE Training =  0.17557240977674207
MSE Validation =  0.23588829693825317

Fold  2
MSE Training =  0.17538247153752798
MSE Validation =  0.24972333683640785

Fold  3
MSE Training =  0.1693645207267478
MSE Validation =  0.24342578667652745

Fold  4
MSE Training =  0.2034903986922835
MSE Validation =  0.1147228625616973

Average Performance in Training =  0.18095245018332531
Average Performance in Validation =  0.21094007075322144
-----------------------------------------------------

M Value =  13
Lambda Value =  0.2

Fold  1
MSE Training =  0.21635091408457835
MSE Validation =  0.11958568244442398

Fold  2
MSE Training =  0.16119751459107529
MSE Validation =  0.313583080577011

Fold  3
MSE Training =  0.1693431022499539
MSE Validation =  0.3001261794480532

Fold  4
MSE Training =  0.20170858123973243
MSE Validation =  0.21832567218292678

Average Performance in Training =  0.187150028041335
Average Performance in Validation =  0.23790515366310372
-----------------------------------------------------

M Value =  13
Lambda Value =  0.30000000000000004

Fold  1
MSE Training =  0.19917325072133824
MSE Validation =  0.2474576185779913

Fold  2
MSE Training =  0.19494192923040782
MSE Validation =  0.22992293521730878

Fold  3
MSE Training =  0.18894754513567025
MSE Validation =  0.25212854368856075
```

```
Fold  4
MSE Training =  0.16633826079086342
MSE Validation =  0.4029599831945965

Average Performance in Training =  0.18735024646956994
Average Performance in Validation =  0.28311727016961435
----------------------------------------------------

M Value =  13
Lambda Value =  0.4

Fold  1
MSE Training =  0.17861951810785784
MSE Validation =  0.322789146219332

Fold  2
MSE Training =  0.15789872366066154
MSE Validation =  0.35993553107356036

Fold  3
MSE Training =  0.22928433109676397
MSE Validation =  0.10406999434661392

Fold  4
MSE Training =  0.22312826179738518
MSE Validation =  0.15942473261312862

Average Performance in Training =  0.19723270866566714
Average Performance in Validation =  0.2365548510631587
----------------------------------------------------

M Value =  13
Lambda Value =  0.5

Fold  1
MSE Training =  0.19538514809993457
MSE Validation =  0.2505419827389537

Fold  2
MSE Training =  0.2139672127181831
MSE Validation =  0.26605681731892666

Fold  3
MSE Training =  0.19010997427799947
MSE Validation =  0.24249266868614025

Fold  4
MSE Training =  0.20994116934750276
MSE Validation =  0.23223008156428104

Average Performance in Training =  0.20235087611090496
Average Performance in Validation =  0.2478303875770754
----------------------------------------------------

M Value =  13
Lambda Value =  0.6

Fold  1
MSE Training =  0.21247365069920335
```

```
MSE Validation =  0.21012991821278465

Fold  2
MSE Training =  0.19762016149263575
MSE Validation =  0.23870317362539936

Fold  3
MSE Training =  0.1924818396264189
MSE Validation =  0.30168693585087647

Fold  4
MSE Training =  0.23656590854087936
MSE Validation =  0.16746972898897436

Average Performance in Training =  0.20978539008978433
Average Performance in Validation =  0.2294974391695087
-----------------------------------------------------

M Value =  13
Lambda Value =  0.7000000000000001

Fold  1
MSE Training =  0.19790454261635754
MSE Validation =  0.24317174737101513

Fold  2
MSE Training =  0.2057735573732392
MSE Validation =  0.2628189369159567

Fold  3
MSE Training =  0.1975568120162691
MSE Validation =  0.3450847610394661

Fold  4
MSE Training =  0.25195007006941583
MSE Validation =  0.15280624115293967

Average Performance in Training =  0.21329624551882043
Average Performance in Validation =  0.2509704216198444
-----------------------------------------------------

M Value =  13
Lambda Value =  0.8

Fold  1
MSE Training =  0.2123936020956189
MSE Validation =  0.23480852931035687

Fold  2
MSE Training =  0.23544727073744368
MSE Validation =  0.23537016113361003

Fold  3
MSE Training =  0.18924540535584522
MSE Validation =  0.36994964902767546

Fold  4
MSE Training =  0.24149967676946432
MSE Validation =  0.13354025459443614
```

```
Average Performance in Training =  0.21964648873959303
Average Performance in Validation =  0.24341714851651963
-----------------------------------------------------

M Value =  13
Lambda Value =  0.9

Fold  1
MSE Training =  0.20727120324297732
MSE Validation =  0.33291796333194434

Fold  2
MSE Training =  0.21567311265649197
MSE Validation =  0.24543270246382326

Fold  3
MSE Training =  0.22808841622860196
MSE Validation =  0.24197706323755463

Fold  4
MSE Training =  0.24450527450261073
MSE Validation =  0.16356470582153687

Average Performance in Training =  0.2238845016576705
Average Performance in Validation =  0.24597310871371475
-----------------------------------------------------

M Value =  13
Lambda Value =  1.0

Fold  1
MSE Training =  0.2068712500020529
MSE Validation =  0.2654130330562211

Fold  2
MSE Training =  0.22187860523530323
MSE Validation =  0.3890818962421711

Fold  3
MSE Training =  0.24156048403178146
MSE Validation =  0.2036547852360563

Fold  4
MSE Training =  0.24600151237666898
MSE Validation =  0.1774467683656085

Average Performance in Training =  0.22907796291145163
Average Performance in Validation =  0.25889912072501425
-----------------------------------------------------

M Value =  14
Lambda Value =  0.1

Fold  1
MSE Training =  0.17185129363920848
MSE Validation =  0.28434470460815525

Fold  2
MSE Training =  0.19107760939461338
MSE Validation =  0.16564889693596285
```

```
Fold  3
MSE Training =  0.17038902288236688
MSE Validation =  0.22735652005164694

Fold  4
MSE Training =  0.17820007514521818
MSE Validation =  0.244233136726234

Average Performance in Training =  0.17787950026535174
Average Performance in Validation =  0.23039581458049976
-----------------------------------------------------

M Value =  14
Lambda Value =  0.2

Fold  1
MSE Training =  0.18722862356277328
MSE Validation =  0.2295981927715164

Fold  2
MSE Training =  0.19304585545670008
MSE Validation =  0.20707341627614703

Fold  3
MSE Training =  0.16393263869584052
MSE Validation =  0.3495411011732393

Fold  4
MSE Training =  0.20877839251983937
MSE Validation =  0.16260380432462254

Average Performance in Training =  0.1882463775587883
Average Performance in Validation =  0.2372041286363813
-----------------------------------------------------

M Value =  14
Lambda Value =  0.30000000000000004

Fold  1
MSE Training =  0.21485306111221436
MSE Validation =  0.1553885027320175

Fold  2
MSE Training =  0.16837225831331804
MSE Validation =  0.36618630518551504

Fold  3
MSE Training =  0.1830591019152225
MSE Validation =  0.31701286904656206

Fold  4
MSE Training =  0.19739248277726953
MSE Validation =  0.21692815977913546

Average Performance in Training =  0.19091922602950612
Average Performance in Validation =  0.2638789591858075
-----------------------------------------------------

M Value =  14
```

```
Lambda Value =  0.4

Fold  1
MSE Training =  0.18621392999762815
MSE Validation =  0.2792461548771451

Fold  2
MSE Training =  0.19312887630384284
MSE Validation =  0.2904354123839124

Fold  3
MSE Training =  0.22498526748416714
MSE Validation =  0.13684210773826314

Fold  4
MSE Training =  0.1840623639768586
MSE Validation =  0.2760250915966501

Average Performance in Training =  0.19709760944062418
Average Performance in Validation =  0.2456371916489927
----------------------------------------------------

M Value =  14
Lambda Value =  0.5

Fold  1
MSE Training =  0.17562739159479862
MSE Validation =  0.3103779598637212

Fold  2
MSE Training =  0.20577958749606576
MSE Validation =  0.2174135226834238

Fold  3
MSE Training =  0.21001496164880948
MSE Validation =  0.28311664747605875

Fold  4
MSE Training =  0.22332675250007353
MSE Validation =  0.15290353664518183

Average Performance in Training =  0.20368717330993685
Average Performance in Validation =  0.2409529166670964
----------------------------------------------------

M Value =  14
Lambda Value =  0.6

Fold  1
MSE Training =  0.20285876135843886
MSE Validation =  0.22054536028241417

Fold  2
MSE Training =  0.19483174917611754
MSE Validation =  0.2814276320068363

Fold  3
MSE Training =  0.23345515680303863
MSE Validation =  0.14006509957900096
```

```
Fold  4
MSE Training =  0.19624237382052856
MSE Validation =  0.3162106614940769

Average Performance in Training =  0.2068470102895309
Average Performance in Validation =  0.23956218834058207
-----------------------------------------------------

M Value =  14
Lambda Value =  0.7000000000000001

Fold  1
MSE Training =  0.22182912219338252
MSE Validation =  0.17728263472466096

Fold  2
MSE Training =  0.21224599290856078
MSE Validation =  0.22031530022451057

Fold  3
MSE Training =  0.173195546123646
MSE Validation =  0.5686403150193932

Fold  4
MSE Training =  0.22298409167695843
MSE Validation =  0.22301767548953236

Average Performance in Training =  0.20756368822563692
Average Performance in Validation =  0.29731398136452425
-----------------------------------------------------

M Value =  14
Lambda Value =  0.8

Fold  1
MSE Training =  0.24490508050543197
MSE Validation =  0.14122989094134214

Fold  2
MSE Training =  0.22083755617016274
MSE Validation =  0.2780043901647734

Fold  3
MSE Training =  0.18702112461365156
MSE Validation =  0.35476138211320757

Fold  4
MSE Training =  0.21801072692475595
MSE Validation =  0.24112402446542594

Average Performance in Training =  0.21769362205350057
Average Performance in Validation =  0.25377992192118726
-----------------------------------------------------

M Value =  14
Lambda Value =  0.9

Fold  1
MSE Training =  0.21705468933438649
MSE Validation =  0.231204424719661
```

```
Fold  2
MSE Training =  0.23043670171259387
MSE Validation =  0.25354699270611125

Fold  3
MSE Training =  0.2113868251121982
MSE Validation =  0.3032830404261933

Fold  4
MSE Training =  0.23572958608680672
MSE Validation =  0.1852125919379233

Average Performance in Training =  0.2236519505614963
Average Performance in Validation =  0.2433117624474722
-----------------------------------------------------

M Value =  14
Lambda Value =  1.0

Fold  1
MSE Training =  0.21319224577976023
MSE Validation =  0.2982378241973382

Fold  2
MSE Training =  0.22911110680341867
MSE Validation =  0.2018262256619236

Fold  3
MSE Training =  0.22657205796341498
MSE Validation =  0.31335412063835805

Fold  4
MSE Training =  0.2437155800016962
MSE Validation =  0.2145025637997594

Average Performance in Training =  0.22814774763707252
Average Performance in Validation =  0.2569801835743448
-----------------------------------------------------

M Value =  15
Lambda Value =  0.1

Fold  1
MSE Training =  0.20670415585680224
MSE Validation =  0.12204013209874626

Fold  2
MSE Training =  0.16809714163209083
MSE Validation =  0.26647707225528927

Fold  3
MSE Training =  0.17598921358086353
MSE Validation =  0.2236945481824998

Fold  4
MSE Training =  0.17205365912082 36
MSE Validation =  0.238305203810766

Average Performance in Training =  0.18071104254764506
```

```
Average Performance in Validation =  0.21262923908682535
----------------------------------------------------

M Value =  15
Lambda Value =  0.2

Fold  1
MSE Training =  0.17437635288846803
MSE Validation =  0.2466648145990757

Fold  2
MSE Training =  0.22198379022079107
MSE Validation =  0.10433060629641223

Fold  3
MSE Training =  0.1633660595299458
MSE Validation =  0.36688217630414816

Fold  4
MSE Training =  0.20000200308229638
MSE Validation =  0.19672190622429656

Average Performance in Training =  0.18993205143037534
Average Performance in Validation =  0.22864987585598315
----------------------------------------------------

M Value =  15
Lambda Value =  0.30000000000000004

Fold  1
MSE Training =  0.17406114549606277
MSE Validation =  0.29825058899767265

Fold  2
MSE Training =  0.19834522385935713
MSE Validation =  0.24568185768070133

Fold  3
MSE Training =  0.21193801297731477
MSE Validation =  0.16350984413575617

Fold  4
MSE Training =  0.19286598004509273
MSE Validation =  0.3075564266432969

Average Performance in Training =  0.19430259059445684
Average Performance in Validation =  0.2537496793643568
----------------------------------------------------

M Value =  15
Lambda Value =  0.4

Fold  1
MSE Training =  0.2368799857749841
MSE Validation =  0.11204872353398845

Fold  2
MSE Training =  0.17553426315607396
MSE Validation =  0.3670946293469535
```

```
Fold  3
MSE Training =  0.19172883511935646
MSE Validation =  0.23394065226388597

Fold  4
MSE Training =  0.18737635323104065
MSE Validation =  0.3098020250839703

Average Performance in Training =  0.19787985932036378
Average Performance in Validation =  0.2557215075571996
-----------------------------------------------------

M Value =  15
Lambda Value =  0.5

Fold  1
MSE Training =  0.23782794194143086
MSE Validation =  0.14322968073848957

Fold  2
MSE Training =  0.22703072375797118
MSE Validation =  0.14679350732561738

Fold  3
MSE Training =  0.1886888826321084
MSE Validation =  0.26339692119734665

Fold  4
MSE Training =  0.1591224005943742
MSE Validation =  0.44644999272851865

Average Performance in Training =  0.20316748723147116
Average Performance in Validation =  0.24996752549749307
-----------------------------------------------------

M Value =  15
Lambda Value =  0.6

Fold  1
MSE Training =  0.21653884443980134
MSE Validation =  0.24004086574969155

Fold  2
MSE Training =  0.21430547270328998
MSE Validation =  0.2285476890762141

Fold  3
MSE Training =  0.22692580804170892
MSE Validation =  0.1801865827133022

Fold  4
MSE Training =  0.18387761424508714
MSE Validation =  0.3035067035166548

Average Performance in Training =  0.21041193485747184
Average Performance in Validation =  0.23807046026396567
-----------------------------------------------------

M Value =  15
Lambda Value =  0.7000000000000001
```

```
Fold  1
MSE Training =  0.19513789476869173
MSE Validation =  0.3469151751500761

Fold  2
MSE Training =  0.21116529496572473
MSE Validation =  0.26003483549772494

Fold  3
MSE Training =  0.22581219376732842
MSE Validation =  0.1699877262794447

Fold  4
MSE Training =  0.22060705150678134
MSE Validation =  0.1824540906469592

Average Performance in Training =  0.21318060875213157
Average Performance in Validation =  0.2398479568935512
-----------------------------------------------------

M Value =  15
Lambda Value =  0.8

Fold  1
MSE Training =  0.23992357538037767
MSE Validation =  0.15961460446245843

Fold  2
MSE Training =  0.20861811248512646
MSE Validation =  0.3420064136821243

Fold  3
MSE Training =  0.23135842661158296
MSE Validation =  0.13208981048687576

Fold  4
MSE Training =  0.16981791085789 73
MSE Validation =  0.4480875534284263

Average Performance in Training =  0.2124295063337461
Average Performance in Validation =  0.27044959551497116
-----------------------------------------------------

M Value =  15
Lambda Value =  0.9

Fold  1
MSE Training =  0.2113982065319526
MSE Validation =  0.33384690168637854

Fold  2
MSE Training =  0.23020405515528022
MSE Validation =  0.23863910366093785

Fold  3
MSE Training =  0.2218771373942079
MSE Validation =  0.2668130172008971

Fold  4
```

```
MSE Training =  0.21681454338921247
MSE Validation =  0.24421450130762523

Average Performance in Training =  0.2200734856176633
Average Performance in Validation =  0.27087838096395966
-----------------------------------------------------

M Value =  15
Lambda Value =  1.0

Fold  1
MSE Training =  0.23538006556852434
MSE Validation =  0.20481146962941393

Fold  2
MSE Training =  0.23333023027124655
MSE Validation =  0.23610511158182457

Fold  3
MSE Training =  0.2383877135467732
MSE Validation =  0.19385838533612088

Fold  4
MSE Training =  0.20463066441436206
MSE Validation =  0.34731393890497064

Average Performance in Training =  0.22793216845022654
Average Performance in Validation =  0.24552222636308252
-----------------------------------------------------

M Value =  16
Lambda Value =  0.1

Fold  1
MSE Training =  0.18496796036872898
MSE Validation =  0.1966466416738581

Fold  2
MSE Training =  0.19317491198428444
MSE Validation =  0.18151005235397527

Fold  3
MSE Training =  0.17728818523136206
MSE Validation =  0.23466048085552402

Fold  4
MSE Training =  0.16645420864639604
MSE Validation =  0.2480866438106573

Average Performance in Training =  0.18047131655769288
Average Performance in Validation =  0.21522595467350367
-----------------------------------------------------

M Value =  16
Lambda Value =  0.2

Fold  1
MSE Training =  0.21392386688844037
MSE Validation =  0.1383532520775711
```

```
Fold  2
MSE Training =  0.19118353192495652
MSE Validation =  0.20383355714603915

Fold  3
MSE Training =  0.15404612240904328
MSE Validation =  0.3784660213950124

Fold  4
MSE Training =  0.2078990625036139
MSE Validation =  0.16920946938313278

Average Performance in Training =  0.1917631459315135
Average Performance in Validation =  0.22246557500043884
-----------------------------------------------------

M Value =  16
Lambda Value =  0.30000000000000004

Fold  1
MSE Training =  0.20101630890778593
MSE Validation =  0.18003374739254535

Fold  2
MSE Training =  0.21272992278253974
MSE Validation =  0.18271862181210258

Fold  3
MSE Training =  0.17928530299594217
MSE Validation =  0.30582592311401646

Fold  4
MSE Training =  0.19845087952403806
MSE Validation =  0.21761959766330133

Average Performance in Training =  0.19787060355257646
Average Performance in Validation =  0.22154947249549142
-----------------------------------------------------

M Value =  16
Lambda Value =  0.4

Fold  1
MSE Training =  0.18307867618291057
MSE Validation =  0.30222308758343247

Fold  2
MSE Training =  0.18354628081269525
MSE Validation =  0.3420813236580387

Fold  3
MSE Training =  0.18937289003989716
MSE Validation =  0.24628580479926962

Fold  4
MSE Training =  0.24151537820293048
MSE Validation =  0.07867753280376753

Average Performance in Training =  0.19937830630960834
Average Performance in Validation =  0.2423169372111271
```

```
-----------------------------------------------------

M Value =  16
Lambda Value =  0.5

Fold  1
MSE Training =  0.20149861121354684
MSE Validation =  0.2668368655599881

Fold  2
MSE Training =  0.220759957679836
MSE Validation =  0.22424358250037854

Fold  3
MSE Training =  0.2079222638682559
MSE Validation =  0.20781537966631572

Fold  4
MSE Training =  0.1852233528329755
MSE Validation =  0.26880065669929926

Average Performance in Training =  0.20385104639865356
Average Performance in Validation =  0.24192412110649542
-----------------------------------------------------

M Value =  16
Lambda Value =  0.6

Fold  1
MSE Training =  0.2272294064195271
MSE Validation =  0.17402046690256914

Fold  2
MSE Training =  0.20256392122021394
MSE Validation =  0.32221234064328785

Fold  3
MSE Training =  0.20508924134366624
MSE Validation =  0.21332540824111526

Fold  4
MSE Training =  0.20497386147709767
MSE Validation =  0.23481899342809545

Average Performance in Training =  0.20996410761512624
Average Performance in Validation =  0.2360943023037669
-----------------------------------------------------

M Value =  16
Lambda Value =  0.7000000000000001

Fold  1
MSE Training =  0.23255434873037795
MSE Validation =  0.15660630676340548

Fold  2
MSE Training =  0.21037574573477402
MSE Validation =  0.25997417083335406

Fold  3
```

```
MSE Training =  0.20607520671031798
MSE Validation =  0.3145032699350469

Fold  4
MSE Training =  0.20619490389400785
MSE Validation =  0.229910287345912

Average Performance in Training =  0.21380005126736945
Average Performance in Validation =  0.2402485087194296
-----------------------------------------------------

M Value =  16
Lambda Value =  0.8

Fold  1
MSE Training =  0.19183365786021708
MSE Validation =  0.3758929031445435

Fold  2
MSE Training =  0.21691885068835248
MSE Validation =  0.282058779664249

Fold  3
MSE Training =  0.2508750007155141
MSE Validation =  0.12394574013934569

Fold  4
MSE Training =  0.2052010108245467
MSE Validation =  0.26298754204512526

Average Performance in Training =  0.2162071300221576
Average Performance in Validation =  0.26122124124831586
-----------------------------------------------------

M Value =  16
Lambda Value =  0.9

Fold  1
MSE Training =  0.1908462153025498
MSE Validation =  0.3044614630282506

Fold  2
MSE Training =  0.24804727874967877
MSE Validation =  0.17550433285538958

Fold  3
MSE Training =  0.23927608520003416
MSE Validation =  0.18065762366059793

Fold  4
MSE Training =  0.20610407417719961
MSE Validation =  0.3519817174795295

Average Performance in Training =  0.2210684133573656
Average Performance in Validation =  0.2531512842559419
-----------------------------------------------------

M Value =  16
Lambda Value =  1.0
```

```
Fold  1
MSE Training =  0.2369544542146619
MSE Validation =  0.2751450333330859

Fold  2
MSE Training =  0.24044794003468578
MSE Validation =  0.188983416470058

Fold  3
MSE Training =  0.23167778351525029
MSE Validation =  0.19782043135928032

Fold  4
MSE Training =  0.1947424231307442
MSE Validation =  0.36509721084828767

Average Performance in Training =  0.22595565022383554
Average Performance in Validation =  0.25676152300267796
-----------------------------------------------------

M Value =  17
Lambda Value =  0.1

Fold  1
MSE Training =  0.18057255136339556
MSE Validation =  0.19237363612971387

Fold  2
MSE Training =  0.1747023969605392
MSE Validation =  0.233389331199913

Fold  3
MSE Training =  0.19691227174184642
MSE Validation =  0.14804338118411906

Fold  4
MSE Training =  0.1591884840085378
MSE Validation =  0.3074094571021094

Average Performance in Training =  0.17784392601857976
Average Performance in Validation =  0.2203039514039638
-----------------------------------------------------

M Value =  17
Lambda Value =  0.2

Fold  1
MSE Training =  0.19329593240013743
MSE Validation =  0.21466298139763007

Fold  2
MSE Training =  0.17485046985803437
MSE Validation =  0.375126728557034

Fold  3
MSE Training =  0.17129248700242664
MSE Validation =  0.2748590260343209

Fold  4
MSE Training =  0.20413806635057685
```

```
MSE Validation =  0.1991188392238453

Average Performance in Training =  0.18589423890279383
Average Performance in Validation =  0.26594189380320754
-----------------------------------------------------

M Value =  17
Lambda Value =  0.30000000000000004

Fold  1
MSE Training =  0.19638021760353017
MSE Validation =  0.2539764260886358

Fold  2
MSE Training =  0.21927929369618465
MSE Validation =  0.12685762951396792

Fold  3
MSE Training =  0.20061416347924046
MSE Validation =  0.3398882173920544

Fold  4
MSE Training =  0.16557583296059014
MSE Validation =  0.32328774322996096

Average Performance in Training =  0.19546237693488633
Average Performance in Validation =  0.26100250405615477
-----------------------------------------------------

M Value =  17
Lambda Value =  0.4

Fold  1
MSE Training =  0.18256639124782612
MSE Validation =  0.27633938349508774

Fold  2
MSE Training =  0.2232079846329793
MSE Validation =  0.162440326784569

Fold  3
MSE Training =  0.20083407437494394
MSE Validation =  0.24135456990014117

Fold  4
MSE Training =  0.20786236372484826
MSE Validation =  0.1996875895745075

Average Performance in Training =  0.2036177034951494
Average Performance in Validation =  0.21995546743857636
-----------------------------------------------------

M Value =  17
Lambda Value =  0.5

Fold  1
MSE Training =  0.2251210271413811
MSE Validation =  0.17536924927485234

Fold  2
```

```
MSE Training =  0.19347238720088428
MSE Validation =  0.3174655531258716

Fold  3
MSE Training =  0.20473029695059583
MSE Validation =  0.20727094938316226

Fold  4
MSE Training =  0.20142066766125435
MSE Validation =  0.24881260588808765

Average Performance in Training =  0.2061860947385289
Average Performance in Validation =  0.23722958941799346
--------------------------------------------------

M Value =  17
Lambda Value =  0.6

Fold  1
MSE Training =  0.18403295384474386
MSE Validation =  0.36806135762496933

Fold  2
MSE Training =  0.23365973534111545
MSE Validation =  0.13968045943256882

Fold  3
MSE Training =  0.19954875001938954
MSE Validation =  0.27536729362872964

Fold  4
MSE Training =  0.22848847156863003
MSE Validation =  0.15241115106781555

Average Performance in Training =  0.21143247769346973
Average Performance in Validation =  0.23388006543852083
--------------------------------------------------

M Value =  17
Lambda Value =  0.7000000000000001

Fold  1
MSE Training =  0.23731295759209586
MSE Validation =  0.14471805430685103

Fold  2
MSE Training =  0.21787491827369307
MSE Validation =  0.23984692253375017

Fold  3
MSE Training =  0.18353915415783154
MSE Validation =  0.3116682521394511

Fold  4
MSE Training =  0.21815417391375003
MSE Validation =  0.2664740411561746

Average Performance in Training =  0.2142203009843426
Average Performance in Validation =  0.24067681753405673
--------------------------------------------------
```

```
M Value =  17
Lambda Value =  0.8

Fold  1
MSE Training =  0.2098045984248458
MSE Validation =  0.21861651713002836

Fold  2
MSE Training =  0.2342512833100877
MSE Validation =  0.1837179585553455

Fold  3
MSE Training =  0.18384936318371847
MSE Validation =  0.4381257890006372

Fold  4
MSE Training =  0.24730490793465165
MSE Validation =  0.13113874712933465

Average Performance in Training =  0.2188025382133259
Average Performance in Validation =  0.24289975295383642
-----------------------------------------------------

M Value =  17
Lambda Value =  0.9

Fold  1
MSE Training =  0.20085807751580478
MSE Validation =  0.32292164387514816

Fold  2
MSE Training =  0.2355380509943282
MSE Validation =  0.1926670827282598

Fold  3
MSE Training =  0.22887781473316168
MSE Validation =  0.21378235314937122

Fold  4
MSE Training =  0.22099656432317719
MSE Validation =  0.26466352814384597

Average Performance in Training =  0.22156762689161796
Average Performance in Validation =  0.24850865197415628
-----------------------------------------------------

M Value =  17
Lambda Value =  1.0

Fold  1
MSE Training =  0.21652404334801362
MSE Validation =  0.335722389746688

Fold  2
MSE Training =  0.1987346012259632
MSE Validation =  0.3255482022061541

Fold  3
MSE Training =  0.22050983165980842
```

```
MSE Validation =  0.2735503704706355

Fold  4
MSE Training =  0.2591212163677255
MSE Validation =  0.11241376567012135

Average Performance in Training =  0.22372242315037766
Average Performance in Validation =  0.26180868202339974
-----------------------------------------------------

M Value =  18
Lambda Value =  0.1

Fold  1
MSE Training =  0.1667857453911121
MSE Validation =  0.26213380121959734

Fold  2
MSE Training =  0.1671260546612572
MSE Validation =  0.23778735283616692

Fold  3
MSE Training =  0.18728407251742962
MSE Validation =  0.16599596533593866

Fold  4
MSE Training =  0.1844696417004333
MSE Validation =  0.22680332415006832

Average Performance in Training =  0.17641637856755807
Average Performance in Validation =  0.22318011088544282
-----------------------------------------------------

M Value =  18
Lambda Value =  0.2

Fold  1
MSE Training =  0.18199907356928038
MSE Validation =  0.26234264017395476

Fold  2
MSE Training =  0.20577685777056437
MSE Validation =  0.1925484377066335

Fold  3
MSE Training =  0.1767006808060589
MSE Validation =  0.3137442824124205

Fold  4
MSE Training =  0.18069202841244175
MSE Validation =  0.24590385066375886

Average Performance in Training =  0.18629216013958638
Average Performance in Validation =  0.2536348027391919
-----------------------------------------------------

M Value =  18
Lambda Value =  0.30000000000000004

Fold  1
```

```
MSE Training =  0.19382603456745057
MSE Validation =  0.3346161341588208

Fold  2
MSE Training =  0.17612774323728198
MSE Validation =  0.29783340788359847

Fold  3
MSE Training =  0.20464267234869415
MSE Validation =  0.2165400839804911

Fold  4
MSE Training =  0.2046739716800406
MSE Validation =  0.18953119244764383

Average Performance in Training =  0.1948176054583668
Average Performance in Validation =  0.2596302046176385
-----------------------------------------------------

M Value =  18
Lambda Value =  0.4

Fold  1
MSE Training =  0.19429032529575446
MSE Validation =  0.24812888929728028

Fold  2
MSE Training =  0.17553198533828274
MSE Validation =  0.4182373823445704

Fold  3
MSE Training =  0.22753373762868184
MSE Validation =  0.11910696136057908

Fold  4
MSE Training =  0.20017835707643808
MSE Validation =  0.2206910451455927

Average Performance in Training =  0.19938360133478927
Average Performance in Validation =  0.2515410695370056
-----------------------------------------------------

M Value =  18
Lambda Value =  0.5

Fold  1
MSE Training =  0.17805889237685982
MSE Validation =  0.3387019803054694

Fold  2
MSE Training =  0.25635301236096314
MSE Validation =  0.07235035709727765

Fold  3
MSE Training =  0.18118601058540357
MSE Validation =  0.3494366159882301

Fold  4
MSE Training =  0.192887814270981
MSE Validation =  0.32609672299207015
```

```
Average Performance in Training =  0.20212143239855188
Average Performance in Validation =  0.2716464190957618
-----------------------------------------------------

M Value =  18
Lambda Value =  0.6

Fold  1
MSE Training =  0.1720551579276532
MSE Validation =  0.30880076450170496

Fold  2
MSE Training =  0.2125156239315481
MSE Validation =  0.27050009239585904

Fold  3
MSE Training =  0.2188916313023433
MSE Validation =  0.21852625681883253

Fold  4
MSE Training =  0.23273950036169483
MSE Validation =  0.17425499195515917

Average Performance in Training =  0.20905047838080987
Average Performance in Validation =  0.2430205264178889
-----------------------------------------------------

M Value =  18
Lambda Value =  0.7000000000000001

Fold  1
MSE Training =  0.21487616029603945
MSE Validation =  0.22387578918287926

Fold  2
MSE Training =  0.20041437313198082
MSE Validation =  0.28358422843078335

Fold  3
MSE Training =  0.22575882950665316
MSE Validation =  0.2090182358976802

Fold  4
MSE Training =  0.20830346361453247
MSE Validation =  0.28298588822568255

Average Performance in Training =  0.21233820663730146
Average Performance in Validation =  0.24986603543425634
-----------------------------------------------------

M Value =  18
Lambda Value =  0.8

Fold  1
MSE Training =  0.22270240092317134
MSE Validation =  0.2937686916797753

Fold  2
MSE Training =  0.2159950402815894
```

```
MSE Validation =  0.28181377341689634

Fold  3
MSE Training =  0.22430681725865773
MSE Validation =  0.19376785069819238

Fold  4
MSE Training =  0.19963462689021028
MSE Validation =  0.2900916709800832

Average Performance in Training =  0.2156597213384072
Average Performance in Validation =  0.2648604966937368
----------------------------------------------------

M Value =  18
Lambda Value =  0.9

Fold  1
MSE Training =  0.22648331218203527
MSE Validation =  0.23285606740169948

Fold  2
MSE Training =  0.20326837830443292
MSE Validation =  0.32344925734942803

Fold  3
MSE Training =  0.2366615783550948
MSE Validation =  0.21528557327411194

Fold  4
MSE Training =  0.209671551926027
MSE Validation =  0.28843672081336297

Average Performance in Training =  0.2190212051918975
Average Performance in Validation =  0.2650069047096506
----------------------------------------------------

M Value =  18
Lambda Value =  1.0

Fold  1
MSE Training =  0.22255419158954487
MSE Validation =  0.2733617089678555

Fold  2
MSE Training =  0.21598855682149537
MSE Validation =  0.28126056539140987

Fold  3
MSE Training =  0.24746277504348974
MSE Validation =  0.1608734788894162

Fold  4
MSE Training =  0.21743542376491187
MSE Validation =  0.3018107078298627

Average Performance in Training =  0.22586023680486045
Average Performance in Validation =  0.25432661526963607
----------------------------------------------------
```

```
M Value =  19
Lambda Value =  0.1

Fold  1
MSE Training =  0.1652332547941668
MSE Validation =  0.25556840889665433

Fold  2
MSE Training =  0.1876489914098894
MSE Validation =  0.16289468866310897

Fold  3
MSE Training =  0.17316702247792123
MSE Validation =  0.3877051400910236

Fold  4
MSE Training =  0.17430052579533697
MSE Validation =  0.23130961192600155

Average Performance in Training =  0.1750874486193286
Average Performance in Validation =  0.2593694623941971
----------------------------------------------------

M Value =  19
Lambda Value =  0.2

Fold  1
MSE Training =  0.19689668162399993
MSE Validation =  0.19605768043161126

Fold  2
MSE Training =  0.1445463664440696
MSE Validation =  0.4606975309618679

Fold  3
MSE Training =  0.19133946519345113
MSE Validation =  0.21076776104651174

Fold  4
MSE Training =  0.20942776548300612
MSE Validation =  0.13724579989860927

Average Performance in Training =  0.1855525696861317
Average Performance in Validation =  0.25119219308465
----------------------------------------------------

M Value =  19
Lambda Value =  0.30000000000000004

Fold  1
MSE Training =  0.22546871649563752
MSE Validation =  0.12162404488519772

Fold  2
MSE Training =  0.18233759209699116
MSE Validation =  0.27892735379332073

Fold  3
MSE Training =  0.20020506548439138
MSE Validation =  0.23976128099518168
```

```
Fold  4
MSE Training =  0.18206835352873474
MSE Validation =  0.27313078578304106

Average Performance in Training =  0.1975199319014387
Average Performance in Validation =  0.22836086636418532
----------------------------------------------------

M Value =  19
Lambda Value =  0.4

Fold  1
MSE Training =  0.17973027314738338
MSE Validation =  0.4003021229143211

Fold  2
MSE Training =  0.18751315886963707
MSE Validation =  0.24961622354445342

Fold  3
MSE Training =  0.22243215013880635
MSE Validation =  0.17751824866487945

Fold  4
MSE Training =  0.20841218178849183
MSE Validation =  0.22304628019583742

Average Performance in Training =  0.19952194098607967
Average Performance in Validation =  0.26262071882987287
----------------------------------------------------

M Value =  19
Lambda Value =  0.5

Fold  1
MSE Training =  0.21034614845007604
MSE Validation =  0.2446384746326077

Fold  2
MSE Training =  0.18619994611059595
MSE Validation =  0.27766188485357135

Fold  3
MSE Training =  0.2180206852567001
MSE Validation =  0.19993790535778314

Fold  4
MSE Training =  0.2118557248968458
MSE Validation =  0.22567359731759085

Average Performance in Training =  0.2066056261785545
Average Performance in Validation =  0.23697796554038825
----------------------------------------------------

M Value =  19
Lambda Value =  0.6

Fold  1
MSE Training =  0.23166571189304258
```

```
MSE Validation =  0.19254327295430546

Fold  2
MSE Training =  0.19625532406852428
MSE Validation =  0.2534299252686617

Fold  3
MSE Training =  0.20112470340827276
MSE Validation =  0.3397792087437643

Fold  4
MSE Training =  0.21090356057361243
MSE Validation =  0.20333081620343807

Average Performance in Training =  0.20998732498586303
Average Performance in Validation =  0.24727080579254238
-----------------------------------------------------

M Value =  19
Lambda Value =  0.7000000000000001

Fold  1
MSE Training =  0.22658028716350848
MSE Validation =  0.2670797943681841

Fold  2
MSE Training =  0.19946440715551345
MSE Validation =  0.3164773737760412

Fold  3
MSE Training =  0.1863419504435384
MSE Validation =  0.38562628220804024

Fold  4
MSE Training =  0.2256519087446641
MSE Validation =  0.17081635902782388

Average Performance in Training =  0.2095096383768061
Average Performance in Validation =  0.2849999523450224
-----------------------------------------------------

M Value =  19
Lambda Value =  0.8

Fold  1
MSE Training =  0.19075977819402842
MSE Validation =  0.45145721950152706

Fold  2
MSE Training =  0.21758308845600854
MSE Validation =  0.2027879025202937

Fold  3
MSE Training =  0.22958346614083514
MSE Validation =  0.21127428712186372

Fold  4
MSE Training =  0.22668979941292458
MSE Validation =  0.19420445409348877
```

```
Average Performance in Training =  0.21615403305094916
Average Performance in Validation =  0.2649309658092933
-----------------------------------------------------

M Value =  19
Lambda Value =  0.9

Fold  1
MSE Training =  0.24029378596022438
MSE Validation =  0.15725046734610862

Fold  2
MSE Training =  0.22436404879519917
MSE Validation =  0.250728644958539

Fold  3
MSE Training =  0.21103266392349984
MSE Validation =  0.32831608014751795

Fold  4
MSE Training =  0.20616803232652944
MSE Validation =  0.3251043981986475

Average Performance in Training =  0.22046463275136322
Average Performance in Validation =  0.2653498976627033
-----------------------------------------------------

M Value =  19
Lambda Value =  1.0

Fold  1
MSE Training =  0.21671611133352897
MSE Validation =  0.32155703639330846

Fold  2
MSE Training =  0.21187034260304494
MSE Validation =  0.2995370516883676

Fold  3
MSE Training =  0.24025306996430798
MSE Validation =  0.18421603417975022

Fold  4
MSE Training =  0.23669750063345452
MSE Validation =  0.17650313082644764

Average Performance in Training =  0.2263842561335841
Average Performance in Validation =  0.24545331327196845
-----------------------------------------------------

M Value =  20
Lambda Value =  0.1

Fold  1
MSE Training =  0.17577111284582175
MSE Validation =  0.17541786277402904

Fold  2
MSE Training =  0.17542512016480966
MSE Validation =  0.24749113837846717
```

```
Fold  3
MSE Training =  0.15913759632534913
MSE Validation =  0.30780304585894164

Fold  4
MSE Training =  0.18361579193813454
MSE Validation =  0.20624486212292656

Average Performance in Training =  0.17348740531852874
Average Performance in Validation =  0.23423922728359112
-----------------------------------------------------

M Value =  20
Lambda Value =  0.2

Fold  1
MSE Training =  0.18083683953449714
MSE Validation =  0.2562104862359575

Fold  2
MSE Training =  0.21615668949823813
MSE Validation =  0.11698916804184775

Fold  3
MSE Training =  0.1846968155567929
MSE Validation =  0.26733425274744316

Fold  4
MSE Training =  0.1745234281745171
MSE Validation =  0.2612270436871692

Average Performance in Training =  0.1890534431910113
Average Performance in Validation =  0.2254402376781044
-----------------------------------------------------

M Value =  20
Lambda Value =  0.30000000000000004

Fold  1
MSE Training =  0.16094474796314245
MSE Validation =  0.37001167599076684

Fold  2
MSE Training =  0.2297919244693955
MSE Validation =  0.11547996696391236

Fold  3
MSE Training =  0.21073942898873926
MSE Validation =  0.19482524658970432

Fold  4
MSE Training =  0.14896148406760187
MSE Validation =  0.47509608404175524

Average Performance in Training =  0.1876093963722198
Average Performance in Validation =  0.2888532433965347
-----------------------------------------------------

M Value =  20
```

```
Lambda Value =  0.4

Fold  1
MSE Training =  0.17463592044866091
MSE Validation =  0.31173066057735027

Fold  2
MSE Training =  0.20073129251932004
MSE Validation =  0.23305887352909962

Fold  3
MSE Training =  0.210361986262267
MSE Validation =  0.17760349196004144

Fold  4
MSE Training =  0.22587244525791328
MSE Validation =  0.18573821751170103

Average Performance in Training =  0.2029004111220403
Average Performance in Validation =  0.2270328108945481
-----------------------------------------------------

M Value =  20
Lambda Value =  0.5

Fold  1
MSE Training =  0.17595441493448202
MSE Validation =  0.3248624101045253

Fold  2
MSE Training =  0.2269660256340155
MSE Validation =  0.1541089977305381

Fold  3
MSE Training =  0.23324714382946007
MSE Validation =  0.13972131164425147

Fold  4
MSE Training =  0.19644343316111915
MSE Validation =  0.2827336185828153

Average Performance in Training =  0.20815275438976918
Average Performance in Validation =  0.22535658451553256
-----------------------------------------------------

M Value =  20
Lambda Value =  0.6

Fold  1
MSE Training =  0.23214233765526343
MSE Validation =  0.18943654627810427

Fold  2
MSE Training =  0.20685269780947652
MSE Validation =  0.25554865714347247

Fold  3
MSE Training =  0.20902323822523394
MSE Validation =  0.2460672748680951
```

```
Fold  4
MSE Training =  0.19769918130467104
MSE Validation =  0.2801984678155496

Average Performance in Training =  0.21142936374866123
Average Performance in Validation =  0.24281273652630536
-----------------------------------------------------

M Value =  20
Lambda Value =  0.7000000000000001

Fold  1
MSE Training =  0.18972149963638887
MSE Validation =  0.4205884521539805

Fold  2
MSE Training =  0.23703228224109066
MSE Validation =  0.1759958944707174

Fold  3
MSE Training =  0.22964181569364306
MSE Validation =  0.1939575046104832

Fold  4
MSE Training =  0.1856983790133967
MSE Validation =  0.32712521907259806

Average Performance in Training =  0.21052349414612984
Average Performance in Validation =  0.2794167675769448
-----------------------------------------------------

M Value =  20
Lambda Value =  0.8

Fold  1
MSE Training =  0.19272550734647828
MSE Validation =  0.3404401752727641

Fold  2
MSE Training =  0.22844537155226813
MSE Validation =  0.1778225927815342

Fold  3
MSE Training =  0.19531735944661718
MSE Validation =  0.37688560149729106

Fold  4
MSE Training =  0.2537415939730127
MSE Validation =  0.12054601871282472

Average Performance in Training =  0.2175574580795941
Average Performance in Validation =  0.2539235970661035
-----------------------------------------------------

M Value =  20
Lambda Value =  0.9

Fold  1
MSE Training =  0.22768204954427965
MSE Validation =  0.24246350074611714
```

```
Fold  2
MSE Training =  0.22298040835030078
MSE Validation =  0.25578419237185535

Fold  3
MSE Training =  0.24962655535060085
MSE Validation =  0.16470026386396877

Fold  4
MSE Training =  0.19137857218823354
MSE Validation =  0.33345956443536956

Average Performance in Training =  0.2229168963583537
Average Performance in Validation =  0.24910188035432768
-----------------------------------------------------

M Value =  20
Lambda Value =  1.0

Fold  1
MSE Training =  0.21416936607796527
MSE Validation =  0.3253651565684894

Fold  2
MSE Training =  0.21383405494130142
MSE Validation =  0.2888062116445434

Fold  3
MSE Training =  0.22208244308264175
MSE Validation =  0.3067685150888737

Fold  4
MSE Training =  0.24335222129229744
MSE Validation =  0.1981842683876496

Average Performance in Training =  0.22335952134855147
Average Performance in Validation =  0.279781037922389
-----------------------------------------------------
```

Discussion

- We select the set of hyperparameters $\{M, \lambda\}$ with the best performance score (e.g. smallest MSE or largest r^2).
- If the value of the hyperparameter falls at the edge of provided range of values, it is appropriate to expand the range of values for further exploration.
- Other performance measures can be used, for example, the coefficient of determination of the Q-Q plot for regression tasks, or accuracy score for classification tasks.

No Free Lunch Theorem The No Free Lunch Theorem states that there is no single learning algorithm that in any domain always induces the most accurate learner. The usual approach

is to try many and choose the one that performs the best on a separate validation set. For any learning algorithm, there is a dataset where it is very accurate and another dataset where it is very poor. When we say that a learning algorithm is good, we only quantify how well its inductive bias matches the properties of the data.

The Bias-Variance Trade-Off

Regardless of the form and number of the basis functions, we still face the problem of **overfitting**. In fact, this is true for any ML model.

The problem of searching for *optimal* model complexity has been extensively studied; in fact, we are *simply* searching for a level of complexity that fits the data *well* yet *not too well*. This phenomenon can be summarized by the **bias-variance trade-off** of complexity.

The Bias-Variance Trade-Off The bias-variance trade-off is a way of analyzing a learning algorithm's expected generalization error with respect to a particular problem as a sum of three terms, the bias, variance, and a quantity called the irreducible error, resulting from noise in the problem itself.

Consider the objective function:

$$J(\mathbf{x}, \mathbf{w}) = \frac{1}{2}\|\mathbf{t} - \mathbf{y}\|_2^2$$

where $\mathbf{y} = f(\mathbf{w}, \phi(\mathbf{x}))$ and the function $f(x)$ is a linear function of the form $\mathbf{y} = f(\mathbf{w}, \phi(\mathbf{x})) = \sum_{j=0}^{M} w_j \phi_j(\mathbf{x})$.

For some dataset D, we can make a model prediction $\mathbf{y}$ and compute the objective function value. Hence, we can treat its values as random variable and take the expected value as a function of the dataset D:

$$\begin{aligned} E\left[J(\mathbf{x}, \mathbf{w})\right] &= E\left[\frac{1}{2}\|\mathbf{t} - \mathbf{y}\|_2^2\right] \\ &= E\left[(\mathbf{t} - \mathbf{y})^2\right] \end{aligned}$$

where $\mathbf{y}$ is the model and $\mathbf{t}$ is the desired response.

From the experimental design section, we learned that this quantity depends on the particular data set D. And so, we can take its average over the ensemble of data sets.

- In practice we really only have access to $E_D[\mathbf{y}]$
- But $\mathbf{y}$ here, is simply representing the model *if* we had an *infinite* amount of data and could effectively represent it.

If we add and subtract the quantity $E_D[\mathbf{y}]$, we obtain:

$$\begin{aligned} E\left[(\mathbf{t}-\mathbf{y})^2\right] &= E\left[(\mathbf{y}-\mathbf{t})^2\right] \\ &= E\left[(\mathbf{y}-E_D[\mathbf{y}]+E_D[\mathbf{y}]-\mathbf{t})^2\right] \\ &= E\left[(\mathbf{y}-E_D[\mathbf{y}])^2\right]+E\left[(E_D[\mathbf{y}]-\mathbf{t})^2\right]+E\left[2(\mathbf{y}-E_D[\mathbf{y}])(E_D[\mathbf{y}]-\mathbf{t})\right] \\ &= \text{variance } + \text{bias }^2 + \text{irreducible error} \end{aligned}$$

The Curse of Dimensionality

The Curse of Dimensionality illustrates various phenomena that arise when we work with high-dimensional data spaces that would not otherwise occur in lower-dimensional settings (such as the 3-dimensional space).

> ## Our intuitions do not hold in high-dimensional spaces 🤔

Let's illustrate this with two examples.

Example 1: Volume of a Crust

Consider two embedded spheres, S_1 and S_2, where sphere S_1 has radius r, sphere S_2 has radius $r-\epsilon$, with $0<\epsilon<r$.

Let's calculate the ratio between the volume of the crust and the volume of the outer sphere S_1:

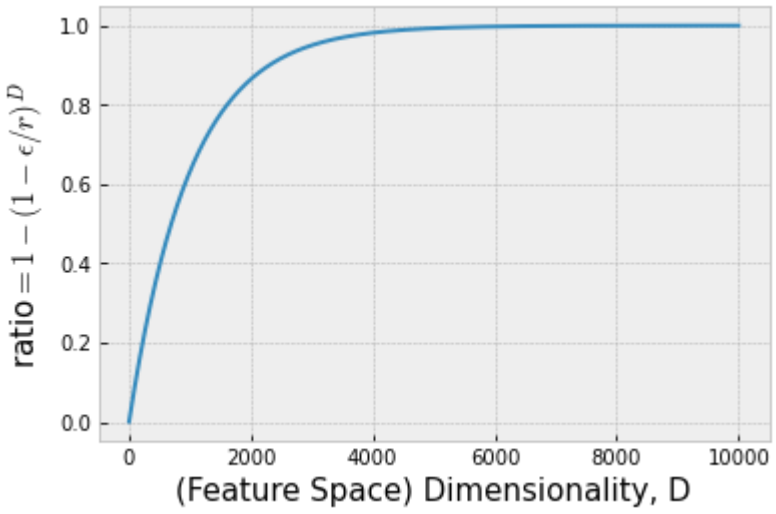
$$\text{ratio} = \frac{V_{crust}}{V_{S_1}} = \frac{V_{S_1}-V_{S_2}}{V_{S_1}}$$

- The D-dimensional volume of a sphere of radius r in D-dimensional space is: $V = \frac{r^D \pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}+1\right)}$, then

$$
\begin{aligned}
\text{ratio} &= \frac{V_{S_1} - V_{S_2}}{V_{S_1}} \\
&= 1 - \frac{V_{S_2}}{V_{S_1}} \\
&= 1 - \frac{\frac{(r-\epsilon)^D \pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}+1\right)}}{\frac{r^D \pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}+1\right)}} \\
&= 1 - \frac{(r-\epsilon)^D}{r^D} \\
&= 1 - \frac{r^D\left(1-\frac{\epsilon}{r}\right)^D}{r^D} \\
&= 1 - \left(1-\frac{\epsilon}{r}\right)^D
\end{aligned}
$$

- For a fixed value for ϵ, a fixed radius r and $\epsilon < r$, what happens as D increases?

In [17]:
```
# Crust volume between spheres with epsilon different radii and increasing dimensional

r = 1 # radius of outer sphere
eps = 0.001 # epsilon value
D = range(1,10000) # dimensionality
RatioVol = [1-(1-eps/r)**d for d in D] # ratio of the volume as a function of dimensio
plt.plot(D, RatioVol) # plotting results
plt.ylabel('ratio$=1 - (1 - \epsilon/r)^D$',size=15)
plt.xlabel('(Feature Space) Dimensionality, D',size=15);
```



> 🤯 **Let's see another example!**

Example 2: Unit Porcupine

Consider the unit porcupine which is represented as a unit hypersphere inscribed within a unit hypercube.

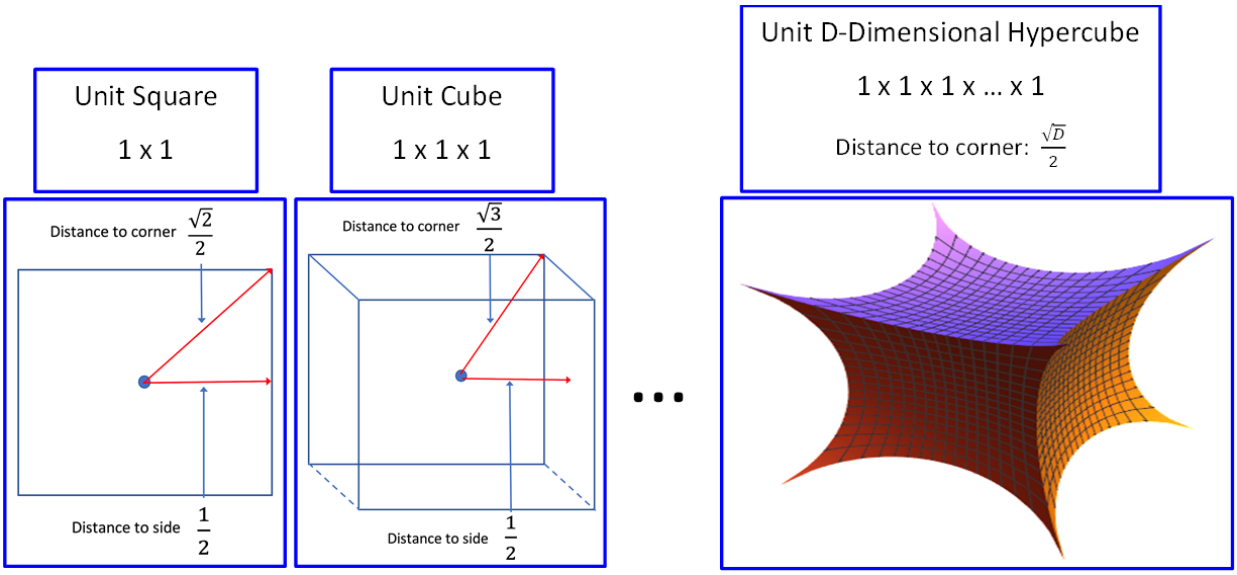
Recall that the D-dimensional volume of a D-dimensional cube with radius r is $(2r)^D$.

- What happens to the ratio between the volume of the sphere and the volume of the cube as dimensionality D increases?

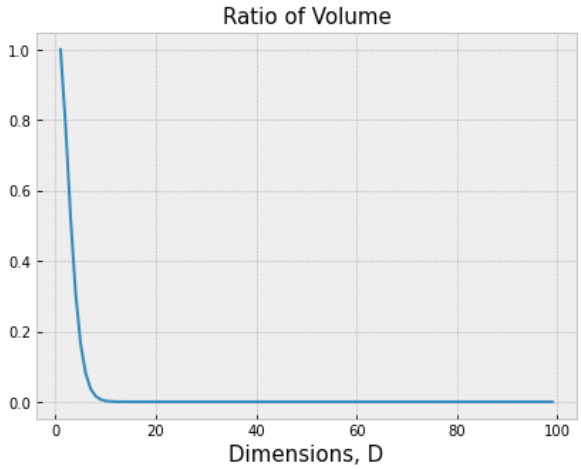
$$\begin{aligned}\frac{V(\text{sphere})}{V(\text{cube})} &= \frac{\frac{r^D \pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}+1\right)}}{(2r)^D} \\ &= \frac{r^D \pi^{\frac{D}{2}}}{(2r)^D \Gamma\left(\frac{D}{2}+1\right)} \\ &= \frac{\pi^{\frac{D}{2}}}{2^D \Gamma\left(\frac{D}{2}+1\right)}\end{aligned}$$

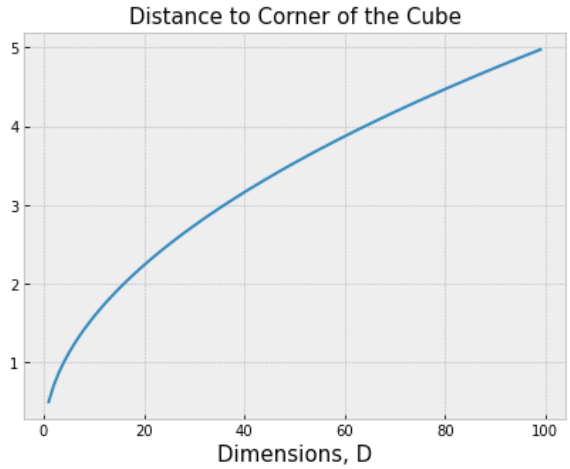
In [1]:
```
from IPython.display import Image
Image('figures/hypercube.png',width=1000)
```

Out[1]:



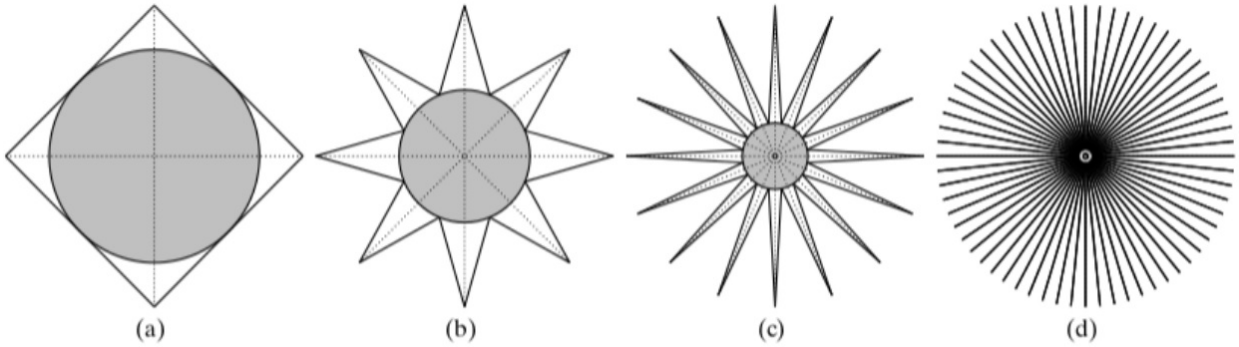
In [18]:
```
# The Unit Porcupine (the unit hyper-sphere inscribed within the unit hyper-cube)

import math

D = range(1,100) # dimensionality, D
V = [np.pi**(i/2)/(2**i*math.gamma(i/2 + 1)) for i in D] # ratio as a function of dime
dist_to_Corner = [math.sqrt(d)/2 for d in D] # distance

# plotting
plt.figure(figsize=(15,5))
plt.subplot(1,2,1); plt.plot(D, V)
plt.title('Ratio of Volume',size=15)
plt.xlabel('Dimensions, D',size=15)

plt.subplot(1,2,2)
plt.plot(D, dist_to_Corner)
plt.title('Distance to Corner of the Cube',size=15)
plt.xlabel('Dimensions, D',size=15);
```





In [19]:
```
Image('figures/Unit Porcupine.jpg',width=800)
```

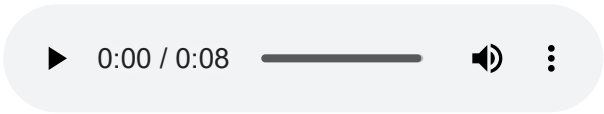
Out[19]:



> **If the Curse of Dimensionality could talk, I imagine it would sound like this...**

In [2]:
```
from IPython.display import Audio
Audio("EvilLaugh.mp4", autoplay=False)
```

Out[2]:



Discussions

In the unit porcupine example, for higher-dimensionality D, all of the volume will reside in the "corners". So, distances between neighboring points is extremely large. The notion of **similarity** as measure by distances becomes a challenge.

As the dimensionality of the feature space increases, we need exponentially more data in order to explain a highly increasing volume.

We need to be careful choosing a model as that choice "injects" what we want the data to look like or follow a specific behavior. For example, in higher-dimensions, the tails of a Gaussian density function will become highly dense.

Always employ the **Occam's Razor** principle: the simplest model that works for our data is usually the most appropriate and sufficient. Model simplicity can mean different things, but we can consider a model to be complex if it has too many hyperparameter values to configure.

When we are in a high-dimensional input space (such as images), much of that space is empty. The input data can be represented in only a few *degrees of freedom* of variability. We say that the data in *embedded* in a **manifold** of equal dimensionality as the degrees of freedom (which is drastically smaller than the input space dimensionality). We will study a few dimensionality reduction and manifold learning techniques later on in this course.

And again, intuitions or assumptions do not always hold in higher dimensions.
