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# Introduction

This book outlines the most significant moments and achievements in the almost 20-year history of Polish algorithmic and programming contests organized or co-organized by staff and students of the Faculty of Mathematics, Informatics and Mechanics at the University of Warsaw. The history of competitions in Poland is all about the people—both organizers and contestants. The fact that Poland was chosen to host the International Olympiad in Informatics in 2005, and the 2012 World Finals of the ACM International Collegiate Programming Contest, says a lot about the quality of programming contests that have been held in Poland over the years.

Participants in Polish competitions have numerous international achievements on their résumés. The most important ones include: first place for Filip Wolski and Tomasz Kulczyński in the International Olympiad in Informatics in 2006 and 2007 respectively, as well as the two-time victory of the University of Warsaw in the World Finals of the ACM International Collegiate Programming Contest thanks to Tomasz Czajka, Andrzej Gąsienica-Samek and Krzysztof Onak in 2003 and Marek Cygan, Marcin Pilipczuk and Filip Wolski in 2007. Polish contestants have had their fair share of successes—including victories—in other competitions as well, such as TopCoder Open, Google Code Jam, Facebook Hacker Cup and Microsoft Imagine Cup. These achievements would not have been possible without the level of challenge presented by programming contests held in Poland, which have enabled talents to be identified and given adequate opportunity for development.

## Competitions in informatics

The role of competitions in selecting and educating particularly gifted high school and university students cannot be overestimated. These educational events require knowledge and skills greatly exceeding what is taught in schools and universities. A good competition is one that relates to the core of the discipline it concerns, and knowledge and skills acquired through participation therein are not volatile but constitute grounds for further field-related development. What is of even greater importance is for the competition to help participants develop skills that will be useful in their future professional lives: diligence, constant striving for success and self-development, self-discipline, hunger for knowledge, self-improvement, teamwork, honesty, ambition, eagerness to compete, dedication. Taking part in a good competition should pose an intellectual challenge to a young individual, whilst success should be a reason for pride and recognition.

Contest tasks say a lot about the quality of a competition itself. They should be original, engaging and of different levels of difficulty. Finding a solution should cause the contestant to feel great satisfaction, whereas being unable to solve a given task should encourage an individual to broaden their knowledge and develop new skills. This book contains the best tasks from algorithmic and programming competitions organized or co-organized by the University of Warsaw, together with their exemplary solutions. The selection of the tasks was undertaken by people who have played significant roles in the history of Polish algorithmic and programming competitions as their participants or organizers. All of the authors of the texts presented in this book are closely affiliated with the Faculty of Mathematics, Informatics and Mechanics at the University of Warsaw, whether as former or current students or as academic staff.

Each of the tasks presented and discussed in this book was used during one of the following events: Polish Olympiad in Informatics, Junior Polish Olympiad in Informatics, Polish Olympiad in Informatics Training Camp, Central European Olympiad in Informatics, Polish Collegiate Programming Contest and Algorithmic Engagements.

*

The **Polish Olympiad in Informatics** was established in 1993 by the Institute of Computer Science at the University of Wroclaw, then chaired by Professor Maciej Sysło. The main founders of the event included four staff members from the University of Warsaw: Piotr Chrząstowski-Wachtel, Jan Madey (director of the Institute of Informatics at that time), Wojciech Rytter and Stanisław Waligórski (who became the first chairman of the Main Committee of the Polish Olympiad in Informatics). Since the very beginning it was the University of Warsaw that facilitated the event, in terms of both scientific background and technical solutions. All authors of texts presented in this book were, or still are, affiliated with the Olympiad.

The Olympiad in Informatics is a competition for high school students. The first four winners of national programming competitions represent their countries at the International Olympiad in Informatics.

The Polish Olympiad in Informatics consists of three stages. The first stage is usually organized in October or November and attracts around a thousand contestants, who are presented with five tasks to be solved at home. Completed solutions are then sent for evaluation to the competition organizers over the Internet. About 400 contestants qualify for the second stage of the competition, which lasts for three days and is held in a few selected regional centers that cooperate closely with the universities that offer the best computer science programs in Poland. The purpose of the first day is to acquaint the participants with the rules and regulations of the contest. Over the following two days, participants are asked to solve two or three tasks on their own during a five-hour supervised session. The solutions are collected from all the regional centers in Poland and are evaluated in a common environment using the same checking methods. Around 80 authors of the best solutions will be selected for the Olympiad's finals. This third stage is organized in one place and lasts for five days: there are three competition days and two days for recreation and sightseeing. The finals are very similar to the second stage of the competition. The resulting four best contestants will represent Poland in international programming competitions, including the International Olympiad in Informatics.

To date, 19 meetings of the Polish Olympiad in Informatics have presented 15,000 students with a total of 300 tasks to solve.

*

The **International Olympiad in Informatics** takes place every summer and gathers together the best high school programmers in the world. The first International Olympiad in Informatics took place in 1989. Since then, Polish students have won 31 gold, 28 silver and 22 bronze medals.

Academic staff and students from the University of Warsaw have actively participated in organizing international programming competitions affiliated with the Olympiad in Informatics: the **International Olympiad in Informatics** itself (IOI 2005), three meetings of the **Central European Olympiad in Informatics** (CEOI 1997, 2004, 2011) and two instances of the **Baltic Olympiad in Informatics** (BOI 2001, 2008).

Each year, finalists in the Polish Olympiad in Informatics have the opportunity to participate in the **Polish Olympiad in Informatics Training Camp**, where they get to solve and thoroughly discuss algorithmic and programming tasks prepared mostly by older students from the University of Warsaw.

For the past six years, yet another event has been organized regularly, namely the **Junior Polish Olympiad in Informatics**—a competition for junior high school students. The idea for the Junior Olympiad came from the great teacher Ryszard Szubartowski, and the Institute of Informatics at the University of Warsaw was actively involved in launching and developing the contest.

*

Many finalists of the Olympiad in Informatics choose the University of Warsaw as the place to further their education in informatics. The University allows its students to develop their competitive drive by trying out for a spot in the team representing the University of Warsaw in team programming competitions, including the **ACM International Collegiate Programming Contest**. The ACM-ICPC is the oldest and most prestigious computer science competition in the world, and it is considered to be a world championship in team programming. Each team consists of three students representing a single university. Regionals are the first stage of the competition; a few dozen regional eliminations are organized on all inhabited continents. The best teams selected during the regionals (including winners and a few runner-up teams, depending on the strength of the region and the number of competing teams) advance to the finals.

Both regionals and finals have the same format: each team, composed of three members, is presented with one computer and given five hours in which to solve between eight and twelve tasks. The solutions proposed by the contestants are evaluated in real time, after which the teams receive post-evaluation feedback in the form of a short but clear communication: *accepted*; *run-time error*; *time limit exceeded*; *wrong answer*; *presentation error*. All tasks are evaluated individually and marked as accepted or rejected. The team that completes the most tasks successfully wins the competition. In the event that several teams complete the same number of tasks, they are ranked based on the total time spent solving the tasks, shorter times being better. However, each rejected submission of a completed task results in a 20-minute time penalty.

The history of the ACM-ICPC dates back to 1977 and is closely intertwined with the work of Bill Poucher, the originator and director of the competition. The contest was popularized among Polish students and academic staff by Professor Jan Madey, who formed the first team representing the University of Warsaw in the competition in 1994. The team was sent to Amsterdam to compete in the eliminations, and succeeded in getting through. As regional winners, these University of Warsaw students were invited to the finals in Nashville, where they came 11th. Ever since then, teams from the University of Warsaw have qualified for the ACM-ICPC World Finals every year, and have won the entire competition twice: in 2003 and 2007. In 2012 the University of Warsaw has the pleasure of hosting the finals of the ACM-ICPC.

It is worth mentioning that the University already has some experience in organizing large-scale events like this. Between 2001 and 2003 the University of Warsaw hosted the **Central European Regional Contest**—an event that serves as eliminations for ACM-ICPC. In 1998–2001 and 2011 the University organized the **Polish Collegiate Programming Contest**.

*

In 2001, during the course of preparations for the Baltic Olympiad in Informatics, a new programming competition, Algorithm Busters, was launched. In 2005 it changed its name to **Algorithmic Engagements**. Algorithmic Engagements is an individual competition for any person of any age and professional status who is willing to sign up for it. The only things that really matter are each contestant's knowledge and skills. The competition consists of two stages. The first is organized via the Internet, lasts for about a week, and is divided into five or six rounds. During each of the rounds, contestants are asked to solve between one and four tasks. The level of difficulty increases from one round to the next. The top 20 participants from the first stage are invited to regular finals that have a format similar to those of the ACM-ICPC (the only exception being that the contestants solve tasks on their own rather than in teams). The winner

receives the title of Algorithm Master of the Year. Algorithmic Engagements has become quite popular over the years; every year a few thousand people sign up for the event, and some of them even update their CVs with their competition achievements.

*

As one may imagine, over the course of many competitions we have organized in the past, hundreds of original programming tasks have been invented and properly described. In order to "keep them alive", we do our best to upload as many of them as possible to an educational portal called **MAIN** (to be found at http://main.edu.pl), which was created at the initiative of organizers of the Polish Olympiad in Informatics and is administered by academic staff and students of the University of Warsaw. MAIN's archive is integrated with the computer system of the Polish Olympiad in Informatics, called **SIO**, which enables solutions to tasks uploaded on the portal to be evaluated in real time, 24/7. Readers of this book can prepare their own solutions to tasks presented herein and then have them checked on the MAIN portal. Good luck!

**About the authors**

The authors of the texts presented in this book are closely affiliated with the University of Warsaw and have had a significant impact on the course of the history of algorithmic and programming competitions organized or co-organized by the University. Due to the nature of this book, we have limited ourselves solely to people who have experience in writing and solving competition tasks. The authors include lecturers and students, former contestants with international successes in programming competitions, educators and popularizers of informatics.

Each author was asked to choose the two tasks that they found the most interesting from competitions co-organized by the University of Warsaw, and to discuss their solutions. The choice was based on the level of difficulty of the tasks, the methods and techniques of solving them, their educational value or the particular author's personal memories of a specific task. Each task and description of its solution is preceded by a short biography of the author, illustrating the links between their careers and programming competitions as well as their scientific and private interests.



**Acknowledgements**

In the pages of this book we have only been able to honor a mere 30 individuals who have had a significant impact on the development of programming competitions in Poland—thereby contributing at the same time to the popularization of informatics and the education of many talented computer scientists. However, I would like to extend my thanks to everyone who has made it possible for this book to be published. I wish to thank the authors of the texts presented in this book, who managed to submit their extracts despite a very tight deadline. Special thanks should go to those who not only authored some of the texts in this book but also used their subject-matter expertise to edit the whole publication: Tomasz Idziaszek, Jakub Łącki and Jakub Radoszewski. Without their engagement and editorial hard work this book would have had no chance of being published. During the preparation of the English translation of the book we received help from my daughter Justyna Diks and my friend Jerzy Jaromczyk, assisted by Neil Moore. For the copy-editing of the English version, my special gratitude goes to Richard Hallas, who was supported in his work by Grzegorz Jakacki and Marcin Kubica. I am also extremely grateful to graphic studio Podpunkt for the layout design of this book and for patience with editors during the process of typesetting material for publication. While creating this book we received a lot of support and encouragement from Rafał Sikorski—a lawyer with a keen interest in informatics.

Having this book published as it is—both in Polish and English—and released in a limited run of 1000 copies in each language would not have been possible but for the financial support of the Ministry of Economy, PKO Bank Polski and Fundacja PKO Banku Polskiego. I would like to take this opportunity to express my appreciation to the management of these institutions for acknowledging the significant educational value of this book and supporting such educational undertakings.

To all readers of this book: enjoy the read and the satisfaction of discovering the secrets of algorithms!

*Krzysztof Diks*

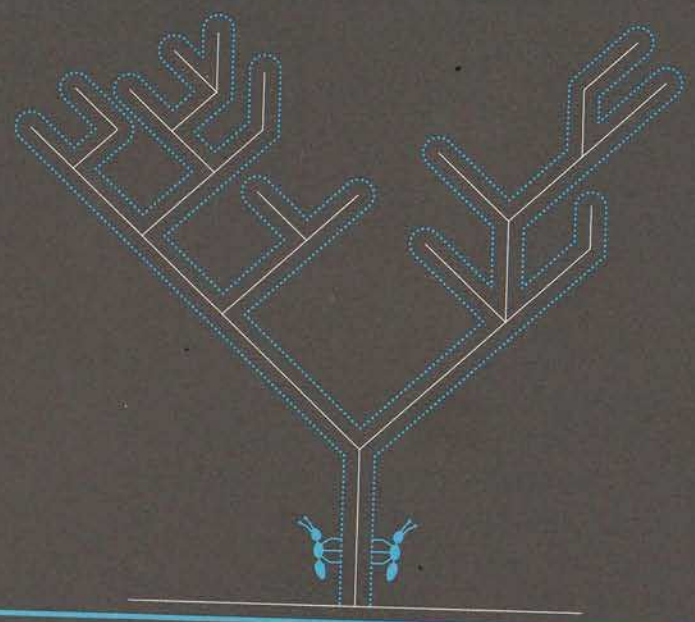
# The tasks >

**SZYMON ACEDAŃSKI**

Szymon Acedański is a Ph.D. student at the Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, and is interested in Natural Language Processing, especially morphosyntactic tagging and deep parsing. He cooperates with the Linguistic Engineering Group at the Polish Academy of Sciences, where he runs projects from the field. Szymon Acedański has won two gold medals at the Baltic Olympiad in Informatics. In 2004, his team won the ACM-ICPC Central European Regional Contest and represented the University of Warsaw at the ACM-ICPC World Finals in Shanghai, China. Szymon Acedański is the Technical Director of the Polish Olympiad in Informatics and a volunteer at the Polish Children's Fund, teaching at workshops for gifted high-school students every year. His hobbies include photography and, during warmer seasons of the year, sailing and traveling.

# / Ants

**Contest:** Polish Collegiate Programming Contest 2011
**Task author:** Szymon Acedański
**Memory:** 6 MB
http://main.edu.pl/en/archive/amppz/2011/drz



Computer geeks like trees. Ants also like trees. Therefore we are given a tree with two ants walking on it—the Left Ant and the Right Ant—as shown in the figure above. The ants walk along the path depicted by a dotted line. They start their journey at the lower end of the trunk, on opposite sides. The Left Ant needs 2 seconds to walk along a single edge of the tree if walking from the root (upwards), and 1 second if walking towards the root (downwards). The Right Ant is two times faster. When the two ants meet, they both turn around and start walking in the opposite direction. If either of the ants steps from the tree to the ground, it immediately starts to climb up the opposite side of the trunk. Apart from that, the ants are so tiny that they would not be visible even under a microscope (they are intentionally depicted much larger in the figure). Your task is to write a program that computes the moment when the ants turn around for the second time.



## Input

The first line of the input contains a single integer $t$ ($1 \le t \le 1000$) representing the number of test cases described in the input.

The description of each test case consists of two lines. The first line contains an even integer $n$ ($2 \le n \le 100{,}000{,}000$) specifying the number of edges in the tree. The second line holds a description of the tree. It is a string consisting of $\frac{n}{2}$ characters representing a $2n$-bit binary number written in hexadecimal format (using digits and small letters from $a$ to $f$). This number shows the Left Ant's path around the whole tree, assuming that the Right Ant stands still. The consecutive bits of this number (starting from the left) denote whether the Left Ant walks away from the root of the tree along the corresponding edge (bit = 1) or towards the root along this edge (bit = 0). The root has a trunk; that is, there is exactly one edge leading from the root of the tree.

The size of the input file does not exceed 50 MB, which is much more than the amount of memory available for your program.

## Output

Your program should output $t$ lines containing answers to the consecutive test cases. Each answer should represent the moment (in seconds) when the ants turn around for the second time, given as an irreducible fraction $p/q$ (without any white space around /), where $p$ and $q$ are positive integers. If the answer is integer then, obviously, $q = 1$.

## Example

For the input data:

1
28
fb1da30d1b7230

the correct result is:

282/5

**Explanation of the example:** The sample data corresponds to the figure above, and transforms to the following sequence of bits:

1111 1011 0001 1101 1010 0011 0000 1101 0001 1011 0111 0010 0011 0000

# / Solution

*Ants* is the only programming problem that I have successfully written with the aim of throwing something together quickly at the outset. All my earlier attempts at putting together a problem on request, with the expectation that it would be finished in a short time, had been futile. However, on that particular sunny afternoon, somewhere far away—thanks to the inspiring presence of my colleague Jakub Radoszewski, who announced that new problems were needed for the Polish Collegiate Programming Contest—I had this conversation with myself:

— Hmm, an exercise... Good. So a character is needed.
— Byteasar? Well... no, it's not for the Olympiad in Informatics... So perhaps some animal. Everyone likes animals.
— Let's choose from some typical candidates: termites; ants; fleas?
— Oh, ants! I recall a nice problem that I heard from Paweł Parys:

**Problem.** Ants are walking on a branch that extends horizontally from the trunk of a tree. All of the ants move at the same speed. Whenever two of them meet, they turn around and continue walking in opposite directions. Ants that reach the tree trunk also turn around and begin ascending the branch again. An ant that gets to the tip of the branch falls down. Given the initial positions of the ants and their speed, compute the time when the last of the ants falls down*.

— So, what if the ants do not fall down from the tree, and the task is to compute their positions on the branch?
— Well, nothing exciting. They would be going back and forth. So perhaps some of them could walk faster than others?
— No, no, too hard! I myself can only come up with a brute force solution. So what if, in addition, we reduce the number of ants to, say, two?
— Well, that's too easy. What if a branch is replaced with something more complex?
— Oh, a tree would be ideal. Moreover, a natural one, with the root pointing down.
— Well, a tree just slightly obscures the fact that the ants are moving in a circle.
— So, at least let me make the "up" segments harder to traverse.

* The reader is strongly encouraged to try to tackle this problem. It has a beautiful solution.



— Sure, but what would be the task? Perhaps finding the meeting point for the moving ants.
— Oh, and I know how to solve it without reading in the representation of the entire tree. It's sufficient to simulate the movement of one of the ants. Knowing the number of "up" and "down" segments still remaining in the tree for the ant, I can also compute at each point the time needed for the other ant to traverse this remaining portion of the tree... yes, that's it.

I was tempted to stop at this point and claim victory. But wait:

— What if, instead, the problem were to ask for the *second* meeting point of the ants to be computed?

I must admit that this question intrigued me. I did not expect that the described problem would have surprising properties, though. Specifically, I was surprised that the following observation holds true:

**Observation.** After the first meeting the Left Ant will always return to the root faster than the Right Ant.

It was the Right Ant that was supposed to move faster...

**Solution**

Dear reader, it is unlikely that the above story has brought you any closer to the solution! In fact, even the above observation seems somewhat suspicious. So I will try to explain the solution step by step. The input specifications are written so as to make it impossible to store the description of the entire tree in memory. The author's solution algorithm reads the description once without storing it. Here is a sketch of the solution:

1. We read in the size of the tree; it gives us the number of "up" and "down" segments.
2. While reading the description, we simulate the walk for the Left Ant until the first meeting with the Right Ant. For each edge, we can check if this is the place of the meeting since, knowing the distance traveled by the Left Ant, we know the distance the Right Ant would need to cover to the meeting point.
3. We continue reading in the input, this time simulating the walk of the Right Ant, until the Left Ant reaches the root.
4. From this point, we continue to simulate the walk of the Right Ant, until the second meeting point.

To see that the above sketch can be developed into a complete algorithm, let us take a look at a more formal description of the author's solution, written by Tomasz Idziaszek when he was preparing the problem for the contest.

We will use standard graph terminology for trees, with the exception that the root of the tree is at the bottom. A point on the tree can be described by a pair of real numbers $(a, k)$, where $a$ indicates the total number of edges traversed by the Left Ant from the root to this point (possibly traversing some edges in both directions, with those edges counted twice), and $k$ indicates the height of the point (the number of edges from the root to the point). If the ant is not at a vertex of the tree, $a$ and $k$ have non-integer parts indicating the portion of the ant's current edge that has been traversed, or that is below the ant, respectively. If it takes the Left Ant $t_g$ seconds to climb an "up" edge and $t_d$ seconds to descend along a "down" edge, then this ant reaches $(a, k)$ in

$$[a, k, t_g, t_d] := \frac{a+k}{2} t_g + \frac{a-k}{2} t_d$$

seconds. Therefore, the first meeting of the Left Ant with the Right Ant, $(a_1, k_1)$, happens when

$$[a_1, k_1, 2, 1] = [2n - a_1, k_1, 1, \tfrac{1}{2}]. \tag{1}$$

Again, we can find this point by simulating the walk of the Left Ant while reading the tree description edge by edge. When the Left Ant is at point $(a, k)$, which is the beginning of some edge, we can check if the meeting could happen on this edge itself. Let $b = 1$ if this is an "up" edge, and $b = -1$ otherwise. If the meeting is to happen after walking an $\varepsilon$ piece of the edge then (1) is satisfied for

$$a_1 = a + \varepsilon, \quad k_1 = k + b\varepsilon,$$

therefore

$$\varepsilon = \frac{6n - 9a - k}{9 + b}.$$

Thus, if $0 \le \varepsilon < 1$, then we have located the first meeting point. The point will be reached in

$$t_1 = [a_1, k_1, 2, 1] = \frac{3a_1 + k_1}{2}$$

seconds.



If we let the ants continue and return to the root, then the Left Ant would reach the root in $[a_1, -k_1, 2, 1] = t_1 - k_1$ seconds, and the Right Ant in

$$[2n - a_1, -k_1, 1, \tfrac{1}{2}] = t_1 - \frac{k_1}{2}$$

seconds. So we can see now that the Left Ant would be at the root before the Right Ant.

Now we can start looking for the second meeting point. We will do so by simulating the Right Ant's walk. After the Right Ant walks the distance $\varepsilon'$ on the edge starting at $(a', k')$, it reaches the second meeting point $(a_2, k_2)$, provided that

$$a_2 = a' + \varepsilon', \quad k_2 = k' + b'\varepsilon',$$

$$[a_1, -k_1, 2, 1] + [2n - a_2, k_2, 2, 1] = [a_2 - a_1, k_2 - k_1, 1, \tfrac{1}{2}],$$

thus

$$\varepsilon' = \frac{12n - 9(a' - a_1) + (k' - k_1)}{9 - b'}.$$

If $0 \le \varepsilon' < 1$, then we have found the second meeting point. The time to reach this point is

$$t_2 = [a_2 - a_1, k_2 - k_1, 1, \tfrac{1}{2}] = \frac{3(a_2 - a_1) + (k_2 - k_1)}{4}$$

seconds. Therefore the total time from the first to the second meeting of the ants equals

$$t_1 + t_2 = \frac{3(a_1 + a_2) + (k_1 + k_2)}{4}.$$

Note that $\varepsilon$ is a multiple of $\frac{1}{9+b}$, and simple algebraic manipulations reveal that $\varepsilon'$ is a multiple of $\frac{9-b}{(9+b)(9-b')}$. This implies that the numbers $a_1$, $a_2$, $k_1$, $k_2$ are multiples of $\frac{1}{800}$. Therefore, all calculations can be carried out using 64-bit integers representing values multiplied by 800.

All of this results in an algorithm whose every step consists of reading one consecutive edge from the input and executing a constant number of operations. The running time of the algorithm is thus $O(n)$ with constant memory.

**MARCIN ANDRYCHOWICZ**

Marcin Andrychowicz is a student at the Faculty of Mathematics, Informatics and Mechanics, University of Warsaw. In high school he won three gold medals at the International Olympiad in Informatics, ranked third at the TopCoder High School Tournament, and won Algorithmic Engagements in 2008. During his university years, he captained a team that won the ACM-ICPC Central European Regional Contest and earned a bronze medal at the ACM-ICPC World Finals in 2009.

Marcin Andrychowicz is interested in cryptographic protocols that are provably secure against physical attacks, and algorithmic trading. In his spare time, he enjoys mountain hiking, snowboarding and traveling.

# / Hyperclock

**Contest:** Polish Olympiad in Informatics Training Camp 2007
**Task author:** Jakub Pawlewicz
**Memory:** 32 MB
http://main.edu.pl/en/archive/ontak/2007/hip

Brainy Smurf concluded that days were too short, and, because of that, he would not manage to write all the volumes of *Quotations of Brainy Smurf*. He decided to visit Father Time and ask him to extend each day to 48 hours. Father Time didn't want to accept his request, but, after Brainy Smurf's importunate demands, eventually agreed. However, he imposed a condition: Brainy Smurf must first solve the Hyperclock puzzle.

The Hyperclock consists of $N$ clocks. Each clock has one hand and some numbers on its face. On the face of the $i$-th clock ($1 \le i \le N$) there are numbers from 1 to $k_i$. At the very beginning, all hands point to the number 1. The puzzle was explained by Father Time as follows:

*You have to perform a sequence of moves. In one move you can turn the hand of an arbitrary clock one position clockwise or counterclockwise. After the last move, the Hyperclock's hands must return to their initial positions. Moreover, each possible configuration of the Hyperclock must appear exactly once.*

Brainy Smurf has been tinkering with the Hyperclock for over an hour and is becoming increasingly convinced that, despite his wisdom, he will not manage to solve the puzzle. Help him!

## Input

The first row of the input contains one positive integer $N$. In each of the following $N$ rows there is one integer greater than 1: in the $(i+1)$-th row there is the number $k_i$. The number of all possible configurations (namely $k_1 k_2 \cdots k_N$) is not greater than $10^6$.

## Output

If there is no solution to the puzzle, output the single word NIE (Polish for *no*). Otherwise, $k_1 k_2 \cdots k_N$ rows should be output, describing the subsequent moves. Each row must contain two integers. The first number $i$ ($1 \le i \le N$) is the number of the clock whose hand is turned. The second number indicates a direction: 1 (for

clockwise) or −1 (for counterclockwise). If there are many possible solutions, you can output any of them.

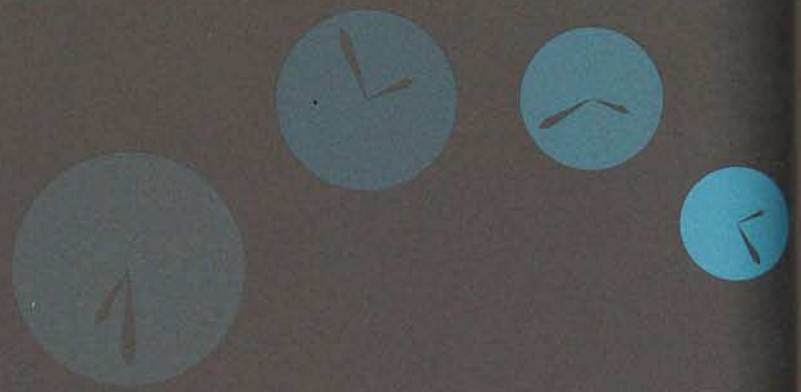
**Example**

For the input data:

3
2
2
3

the correct result is:

1 1
2 1
1 −1
3 1
1 1
2 −1
3 1
2 1
1 −1
2 −1
3 −1
3 −1

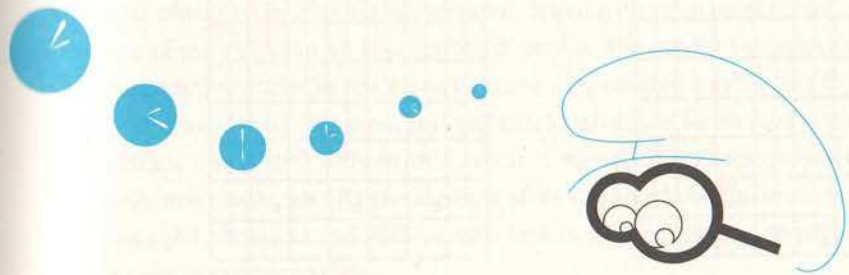




# / Solution

Let's start by expressing the task in mathematical terms. The Hyperclock's configuration can be described with a sequence of integers $x_1, x_2, \ldots, x_N$, indicating numbers being pointed to by individual clocks, where $x_i \in \{1, 2, \ldots, k_i\}$. In one step, we can turn one clock's hand one position clockwise or counterclockwise, so we can go from a configuration $x_1, x_2, \ldots, x_N$ to $y_1, y_2, \ldots, y_N$, if only there is an index $j$, such that $y_i = x_i$ for $i \neq j$ and $y_j = (x_j \pm 1) \bmod k_i$.

It is natural to interpret the Hyperclock as a graph, in which vertices correspond to possible configurations and edges to possible transitions in one move. How can we interpret the puzzle in terms of that graph? We are looking for a cycle going through each vertex exactly once, i.e. a *Hamiltonian cycle*. The problem of finding a Hamiltonian cycle in an arbitrary graph is an NP-complete problem. It means that it probably can't be solved in polynomial time. Our graph from the task has a specific form*, which allows it to be solved in linear time.



* Formally, this graph is a Cartesian product of simple cycles.

### Two clocks

Let's start by solving the task for the case where the Hyperclock consists of exactly two clocks. It is natural to visualize the situation as a $k_1 \times k_2$ grid. We can go from one field to another if they share a side; moreover, we can go from the last field of a row or column to the first one and *vice versa*.

Figure 1 presents the situation for $k_1 = 6$ and $k_2 = 9$ with the solution. The above idea can be easily generalized for situations where at least one of the sides has an even length. The following pseudocode corresponds to the picture, assuming that we start from the upper left corner.

```
Algorithm EvenSide(k1,k2)
  {we assume that 2 | k1}
  for i := 1 to k1/2 do
    for j := 1 to k2 − 1 do
      print 2 1 {down}
    print 1 1 {right}
    for j := 1 to k2 − 1 do
      print 2 −1 {up}
    print 1 1 {right}
```

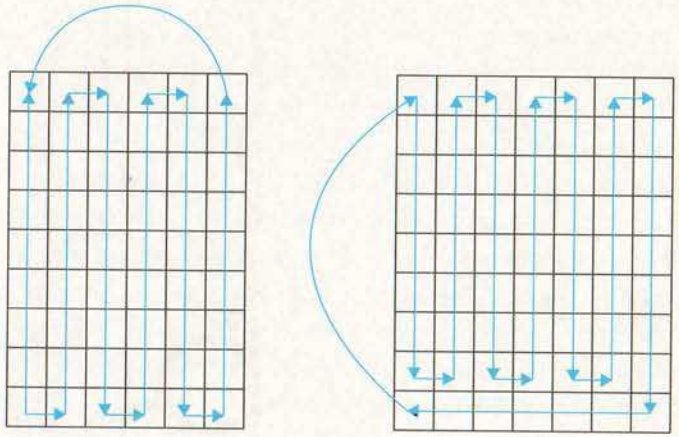


*Figure 1: Solution for $k_1 = 6$, $k_2 = 9$ (left) and for $k_1 = 7$, $k_2 = 9$ (right).*



Situations where both sides have odd lengths require only a minor modification of the above idea. In figure 1 there is a sample solution for $k_1 = 7$ and $k_2 = 9$. Again, the following pseudocode corresponds to the picture, assuming that we start from the upper left corner.

```
Algorithm OddSide(k1,k2)
  {we assume that 2 ∤ k1}
  for i := 1 to (k1 − 1)/2 do
    for j := 1 to k2 − 2 do
      print 2 1 {down}
    print 1 1 {right}
    for j := 1 to k2 − 2 do
      print 2 −1 {up}
    print 1 1 {right}
  for i := 1 to k2 − 1 do
    print 2 1 {down}
  for i := 1 to k1 − 1 do
    print 1 −1 {left}
  print 2 1 {down}
```

### More clocks

We already know how to cope with the problem for $N = 2$. It turns out that it is sufficient to solve the general version of the task. After finding a Hamiltonian cycle for the first two clocks, we can treat them as one clock with $k_1k_2$ states—it is enough to put the resulting Hamiltonian cycle on the face of the new clock!

The hand of the new clock points to a pair of numbers indicating the state of both clocks from which it was created. Moving its hand corresponds to moving one of the clocks in an appropriate direction. We can go between each pair of consecutive states in one move because they created a cycle (see figure 2).

Subsequently, the new, merged clock can again be merged with the third clock... and, continuing in this manner, we reach a situation where there is only one clock, which indicates the state of the whole Hyperclock. Then it is enough to turn its hand 360° and the task is solved. It means, in particular, that a solution always exists.

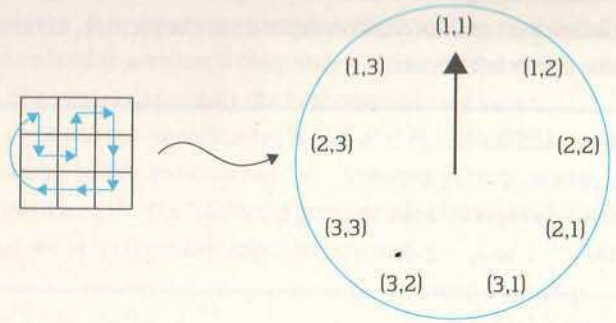



*Figure 2: Merging two clocks of length 3 into one.*

**The clocks' representation**

To implement the above idea, we need first to decide how we would store the created clocks. We don't really need to know the values on their faces—it is enough to realize what moves correspond to turning their hands clockwise. For instance, in the above example we need to know that moving from the first position to the second (from the state (1,1) to the state (1,2)) corresponds to turning the second clock clockwise (i.e. making a move 2 1) and so on. Therefore, a clock with $k$ states will be represented as an array of length $k$, where under the $i$-th index there is a pair $(a, b)$, such that $a \in \{1;2, \ldots, N\}$ and $b \in \{-1, 1\}$, indicating a move that has to be made to go from the $i$-th state to the $(i+1)$-th (or from the last one to the first).

This information is sufficient for turning the clock in both directions, because if a pair $(a, b)$ corresponds to the transition from the $i$-th state to the $(i+1)$-th, then a move from the $(i+1)$-th state to the $i$-th corresponds to the pair $(a, -b)$. Using the solution for two clocks, we can merge two clocks in time linear with the number of states in the clock being created. Let's estimate the total time of all $N-1$ merges. Let $M = k_1 k_2 \cdots k_N$. The whole solution works in time proportional to

$$k_1k_2 + k_1k_2k_3 + \cdots + M = M\left(\frac{1}{k_3k_4 \cdots k_N} + \frac{1}{k_4k_5 \cdots k_N} + \cdots + 1\right) \le$$
$$\le M\left(\frac{1}{2^{N-2}} + \frac{1}{2^{N-1}} + \cdots + 1\right) < 2M,$$

so it is linear with the input size.



# / Fishes

**Contest:** Algorithmic Engagements 2009
**Task author:** Eryk Kopczyński
**Memory:** 256 MB
http://main.edu.pl/en/archive/pa/2009/ryb

In an archipelago far away there lives a rare species of predator fish. These fish have a very regular rhythm of life. Each fish wakes up every morning at the same time of day and goes hunting. In the evening, it returns to the place from which it set off. It goes to sleep there at the same time every day, but it may wake up in a different place, as it can be moved a little by ocean currents.

Throughout the whole day, a fish sticks to the following rule: at every moment it has to be able to see where it was at the same time on the previous day; that is, exactly 24 hours earlier. Of course, a fish cannot see a point at the opposite side of any island.

Ichthyologists have been observing the fish in the archipelago for quite a long time, and every couple of days they have kept a record of one route taken by some fish. Unfortunately, after collecting a large amount of data, there has been an accident. Some of the data is now missing, and the rest is completely messed up. The scientists do not even know which fish traveled by each route they recorded. They have asked you for help. They are going to give you the messed up descriptions of the fishes' routes and ask you to tell them the minimum number of different fish that they have observed during their research.

**Input**

The input consists of two parts: a description of the archipelago and a description of the fishes' routes. The first line of the archipelago description contains two integers, $w$ and $h$ ($3 \le w \le 1000$, $3 \le h \le 1000$), separated by a single space. In each of the following $h$ lines there is a $w$-character-long string that describes a part of the archipelago. A character . represents ocean, whereas # represents land. There is water (. character) in all cells on the boundary of the map.

One point in the archipelago is visible from another if no segment connecting them has common points with the interior or boundary of any piece of land. The direction in which a fish is swimming does not matter here.

In the next line, there is an integer $n$ ($2 \le n \le 1000$) specifying the number of recorded fishes' routes. The following $2n$ lines contain descriptions of those

routes. In the first line of a route description, there are three integers, $x$, $y$, and $d$ ($1 \le x \le w$, $1 \le y \le h$, $2 \le d \le 10{,}000$), separated by single spaces. The numbers $x$ and $y$ are the coordinates of the place where a fish woke up (column and row) and $d$ is the length of the route. The second line is a string of $d$ characters, N, W, S or E, representing the direction of the fish. They stand for up, left, down and right respectively. It is guaranteed that the routes go only through cells containing water, that fish do not leave the fragment of archipelago described in the input, and that each route ends in the same cell from which it starts.

A fish can only swim horizontally or vertically; it moves along a broken line that connects the centers of the cells along its route. However, we do not know its speed. The fish can speed up or slow down so that it can always see the point it occupied exactly 24 hours earlier.

Ocean currents can move a sleeping fish by at most one cell up, down, left or right from the place where the fish went to sleep. You can assume that, for every two water cells in the archipelago, there exists some (hypothetical) fish route that goes through both of them.

**Output**

In the first line of the output your program should write an integer $k$ representing the smallest possible number of different fishes consistent with the data gathered by the scientists. Each of the following $k$ lines should contain a list of routes of a single fish. The fish *did not* necessarily need to swim along those routes *on consecutive days*, but on any two days of its life.

A fish can travel two routes *in two consecutive days*, if both routes start in the same cell or in neighboring cells (cells sharing an edge) and if the fish could swim along the second route, being able to see the point it occupied exactly 24 hours earlier at all times.

The routes are numbered from 1 to $n$, according to their order in the input. The numbers of routes in each line of the output should be written in increasing order, and the lines should be ordered in such a way that the first numbers in all lines form an increasing sequence.



**Example**

For the input data:

```
10  8
..........
........#..
.........#.
..........
...#......
......###.
......###.
..........
4
2  7  12
NNNEEESSSWWW
2  8  24
WNNNNNNNEEEEESSSSSSSWWWW
1  8  46
NNNNNNNEEEEESSSSSSSEEEENNNNNNNSSSSSSSWWWWWWWWW
1  8  32
NNNNNNNEEEEEEEESSSSSSSWWWWWWWWW
```

the correct result is:

```
2
1  2  3
4
```

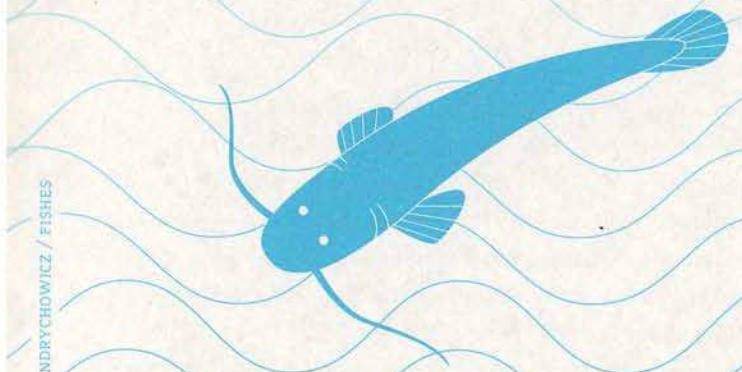
**Explanation of the example:** The first two routes could have been traveled by the same fish (even in two consecutive days). After a few days, the fish could have followed the third route. However, the last route must have been traveled by another fish. Unlike the first three routes, it goes around the big island.

# / Solution

The task *Fishes* is certainly not a standard one. It is the only task known to me that concerns topology. The fishes' behavior was described in the task statement in such a way that, after reading it, it is not intuitive when two routes could be traveled by one fish (not necessarily on consecutive days). It turns out that this condition can be expressed much more simply, and that it corresponds to a concept from topology. We will call such routes *equivalent*. The task is to compute the abstraction classes of routes' equivalence relation.

First notice that two routes with identical shapes and directions of swimming, differing only in the place where a fish woke up, are equivalent. This is because the fish could swim the same route every day with a constant velocity, starting each day from the next cell on the route (onto which it was moved during the night by the ocean's currents). Therefore, we can disregard the place where a fish wakes up and consider only the shapes of routes, which are directed loops.

Examining the example in figure 1, it can be guessed that two routes are equivalent when one can be continuously deformed into the other. In topology, such loops are called *homotopic*. The outline of the proof of this observation can be found at the end of the solution; knowledge of it is not necessary to understand the rest of the solution.





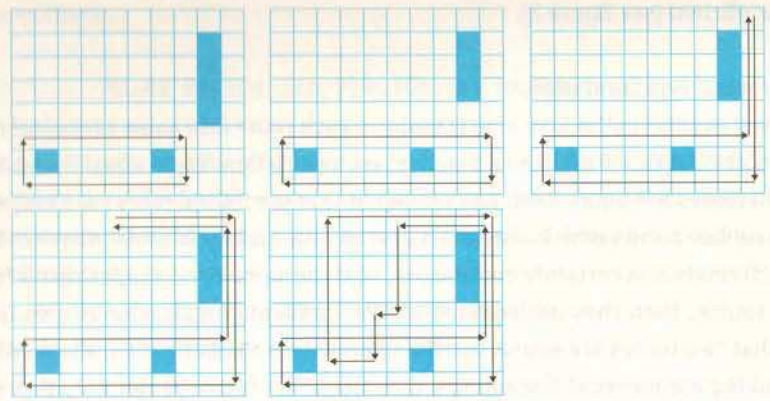


*Figure 1: An exemplary route that could be swum by one fish over consecutive days. The place where the fish wakes up is not marked, because it can be disregarded.*

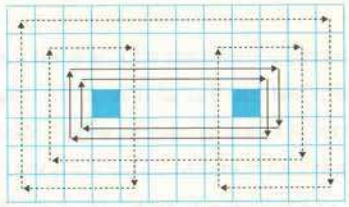


*Figure 2: Both routes circumnavigate both islands twice. Despite that, they are not equivalent.*

**The first idea**

Let us consider when two routes are equivalent. It can be seen from figure 1 that such routes can have completely different shapes. In an archipelago where there are no islands, all routes are equivalent. However, if there is one island, a route circumnavigating it is not equivalent to another that doesn't go around it. Moreover, a route circumnavigating it twice is not equivalent to a route that does so only once. The direction of circumnavigation (clockwise or counterclockwise) is relevant too. The above remarks may lead to the idea that counting, in some way, how many times each route circumnavigates each island permits

nonequivalent routes to be distinguished. However, it turns out that such testing is insufficient (see figure 2).

### The routes' representation

The key to solving the task is to transform each route into some canonical representation. We would like two routes' representations to be equal if and only if two routes are equivalent. Let's imagine that the fishes' routes are changed into rubber bands which shrink and wrap around any islands' shores. That transformation is certainly continuous, so if the same band shapes result from two routes, then they are equivalent. The converse implication is also true, i.e. that two routes are equivalent if and only if the rubber bands' shapes after shrinking are identical (including a direction). The reader may verify that fact by drawing arbitrary routes and finding the band's shapes. We assume that routes which don't circumnavigate any islands shrink to a microscopic size and disappear, so they are all equivalent.

How can we simulate the process of shrinking? Let's begin by imagining that on each horizontal continuous fragment of the ocean there is a laser beam, which registers the fact of its crossing by a fish and the fish's direction of movement (see figure 3).

We assume that a fish crosses a beam whenever it swims into the field with the beam from the south or swims out of it into the south. For each route we can compute which beams it crosses and in which directions. Let's observe that if a fish crosses a beam and then immediately goes back, we can disregard both intersections, getting the equivalent route. It may cause another pair of intersections to become adjacent, and we are able to delete them too.

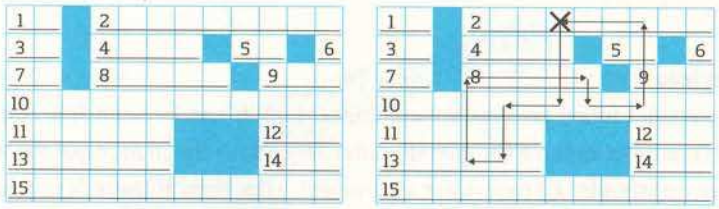



*Figure 3: Left: an archipelago with marked laser beams, which are additionally numbered with natural numbers. Right: an exemplary fish's route for which we find a sequence of intersections. A cell in which the fish wakes up is marked by a cross.*



For the example in figure 3, we would get the following sequence of intersections:

2S, 4S, 8S, 10S, 11S, 11N, 10N, 8N, 8S, 9N, 5N, 2N,

where N or S indicates that the fish swam, respectively, north or south. Erasing iteratively the adjacent intersections concerning the same beam, we get the following sequence of intersections (the pair that is deleted in each step is shown in bold):

1. 2S, 4S, 8S, 10S, **11S**, **11N**, 10N, 8N, 8S, 9N, 5N, 2N,
2. 2S, 4S, 8S, **10S**, **10N**, 8N, 8S, 9N, 5N, 2N,
3. 2S, 4S, **8S**, **8N**, 8S, 9N, 5N, 2N,
4. **2S**, 4S, 8S, 9N, 5N, **2N** (the route is cyclic in reality, so we assume that the first and last intersection are adjacent),
5. 4S, 8S, 9N, 5N.

Intersections to erase can be calculated in linear time during computation of the beams that are crossed by the fish. We can keep the encountered intersections on a stack. While facing a new intersection, we check whether it concerns the same beam as the one on top of the stack: if so, we delete both of them. At the end we erase the first and the last intersection as long as they concern the same beam. During removal, we can ignore the directions of intersections, because a fish cannot cross one beam twice in a row from the same side. We will call the sequence thus created a *representation* of the route. The directions are meaningful only while comparing the representations of two different routes, as they allow us to distinguish the routes circumnavigating the island clockwise and counterclockwise.

### Cyclic equivalence

It turns out that two routes are equivalent if and only if their representations are cyclically equivalent*. We leave that fact without a proof, but the reader may be convinced by analyzing a few examples. Representations may be shifted cyclically, because the cyclic shift of the route depends on the place where the fish wakes up, which, as we know, has no influence on the equivalence of the routes.

The task is to find the abstraction classes of the equivalence relation, which correspond to the abstraction classes of the cyclic equivalence relation of their representations. The simplest algorithm verifying whether two words are

* Two words are *cyclically equivalent* if one is a cyclic shift of the other. A *cyclic shift* of a word $a_1a_2...a_n$ is any word of form $a_ia_{i+1}...a_na_1a_2...a_{i-1}$.

cyclically equivalent works in time $O(d^2)$, where $d$ is the length of both words. We can compute abstraction classes, checking whether each pair of routes' representations are cyclically equivalent. It leads to a solution that works in time $O(wh + n^2d^2)$.

### Improving the complexity

Faster solutions may be obtained using more efficient algorithms for checking cyclic equivalence. It is not difficult to prove that two words $v$ and $w$ of the same length are cyclically equivalent if and only if $v$ is a subword of $ww$. Therefore, using any linear pattern-matching algorithm (e.g. the Knuth-Morris-Pratt algorithm) we can check whether two words are cyclically equivalent in linear time. It leads to a solution with time complexity $O(wh + n^2d)$.

However, we can try a different approach. Instead of checking cyclic equivalence for each pair of representations, we can transform each representation into some particular form, which we will call a *canonical representation*. It will be lexicographically maximum cyclic equivalent (LMCE), i.e. a cyclic equivalent of the representation that is the greatest in lexicographical order. For the aforementioned example, a canonical representation for 4S, 8S, 9N, 5N is 9N, 5N, 4S, 8S. It is worth mentioning that computing an LMCE can be reduced to finding a lexicographically maximum suffix in a word. It is easy to prove that a lexicographically maximum suffix in a word $vv\#$, where # is a new symbol less than all occurring in $v$, starts on a position indicating the beginning of the LMCE of $v$. The lexicographically maximum suffix in a word can be found using Duval's algorithm, which works in linear time.

Then two routes are equivalent if and only if their canonical representations are equal. We can compute a canonical representation for each route and then sort all the representations. A simple scan through the created array allows us to compute the abstraction classes. That kind of solution works in time $O(wh + nd \log n)$ if a linear-logarithmic sorting algorithm is applied. It can be improved to $O(wh + nd)$—that is, linear with the input size—by using bucket sorting or sorting hashes instead of the whole representations.

### Outline of a proof of the routes' homotopy theorem

This section is dedicated to the following theorem's outline of a proof:

***Theorem.*** Two routes are equivalent if and only if they are homotopic.

We will begin with some definitions from topology. Let $W \subseteq \mathbb{R}^2$ be an area covered by water.



***Definition 1.*** A *loop* (directed) in $W$ is any continuous function $p : [0,1] \to W$ such that $p(0) = p(1)$. Intuitively, $p(t)$ is a place where there was a fish at time $t$.

***Definition 2.*** Let $p$ and $q$ be loops in $W$. *Homotopy* between $p$ and $q$ is any continuous transformation $F : [0,1]^2 \to W$, such that the following holds for any $0 \le r, t \le 1$:

$$F(0,t) = p(t), \quad F(1,t) = q(t), \quad F(r,0) = F(r,1).$$

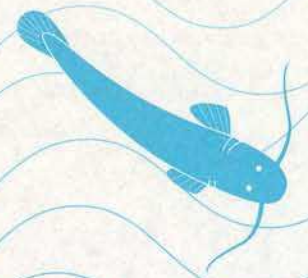
Two loops are called *homotopic* if there exists a homotopy between them.

***Lemma.*** The routes traveled by one fish in two consecutive days are homotopic.

*Proof.* Let $p$ and $q$ be such routes and $H : [0,1]^2 \to W$ is defined by a formula $H(r,t) = (1-r)p(t) + rq(t)$. The image of $H$ is contained in $W$, because the fish always sees the place where it was exactly 24 hours earlier, which means that the segment between $p(t)$ and $q(t)$ is contained in $W$ for any $t$. It is easy to verify that $H$ is then a homotopy between $p$ and $q$. □

A homotopy relation is an equivalence relation, so if two routes swum by one fish in two consecutive days are equivalent, then any two routes traveled by one fish are equivalent. This ends the proof of implication "rightwards".

Let's observe that the condition put on fishes' routes in two consecutive days means precisely that there exists a homotopy between them, which is linear with $r$. Therefore, two routes are equivalent if and only if there exists a piecewise linear homotopy between them (moreover, we demand that $H(r)$ is a correct route if $r$ is a point joining the linear segments). It turns out that the existence of an arbitrary homotopy implies the existence of a homotopy in the desired form, but we won't prove that fact here.



**PIOTR CHRZĄSTOWSKI**

Piotr Chrząstowski is an assistant professor at the Institute of Informatics, University of Warsaw. For years he has taught *Introduction to Programming* to first-year students of computer science. He likes to spot algorithmic problems where others might not even expect them, and then he uses them as the basis for tests and exams. The most challenging ones are presented to participants of the Polish Olympiad in Informatics. Piotr Chrząstowski has been a member of the Main Committee of the Polish Olympiad in Informatics, with a short break, ever since the event was launched.
His scientific research focuses on the theory of distributed computation, including, first and foremost, Petri nets, which he uses as a model to describe business processes. He is an author of many programming projects and programs, with extensive practical experience in combining computer science and business. Lately he has also been involved in the design of user interfaces.
Piotr Chrząstowski is a chairman of the jury of the EU Contest for Young Scientists. He enjoys playing chess, bridge and Go. One of his biggest passions is classical music.

# / Pilots

**Contest:** 17th Polish Olympiad in Informatics
**Task author:** Piotr Chrząstowski
**Memory:** 64 MB
http://main.edu.pl/en/archive/oi/17/pil

At the Bytean Training Center, the pilots prepare for missions that require extraordinary precision and control. One measure of a pilot's capability is the length of time he is able to fly along a desired route without deviating too much—simply put, whether he can fly steadily. This is not an easy task, as the simulator is so sensitive that it registers even the slightest movement of the yoke (a device used for piloting an aircraft). At each moment the simulator stores a single parameter describing the yoke's position. Before each training session a certain tolerance level $t$ is set. The pilot's task is to fly for as long as possible in such a way that all of the yoke's positional measurements during the flight differ by at most $t$. In other words, a fragment of the flight, starting at time $i$ and ending at time $j$, is within tolerance level $t$ if the positional measurements, starting with the $i$-th and ending with the $j$-th, form a sequence $a_i, \ldots, a_j$ such that, for all elements $a_k, a_l$ of this sequence, the inequality $|a_k - a_l| \leq t$ holds.

Your task is to write a program that, given a number $t$ and a sequence of yoke position measurements, determines the length of the longest portion of the flight that is within the tolerance level $t$.

### Input

In the first line of the input two integers are given, $t$ and $n$ ($0 \leq t \leq 2{,}000{,}000{,}000$, $1 \leq n \leq 3{,}000{,}000$), separated by a single space, specifying the tolerance level and the number of yoke position measurements taken. The second line gives those measurements, separated by single spaces. Each measurement is an integer within the interval from 1 to 2,000,000,000.

### Output

Your program should print a single integer to the output: the maximum length of a portion of the flight that is within the given tolerance level.

**Example**

For the input data:

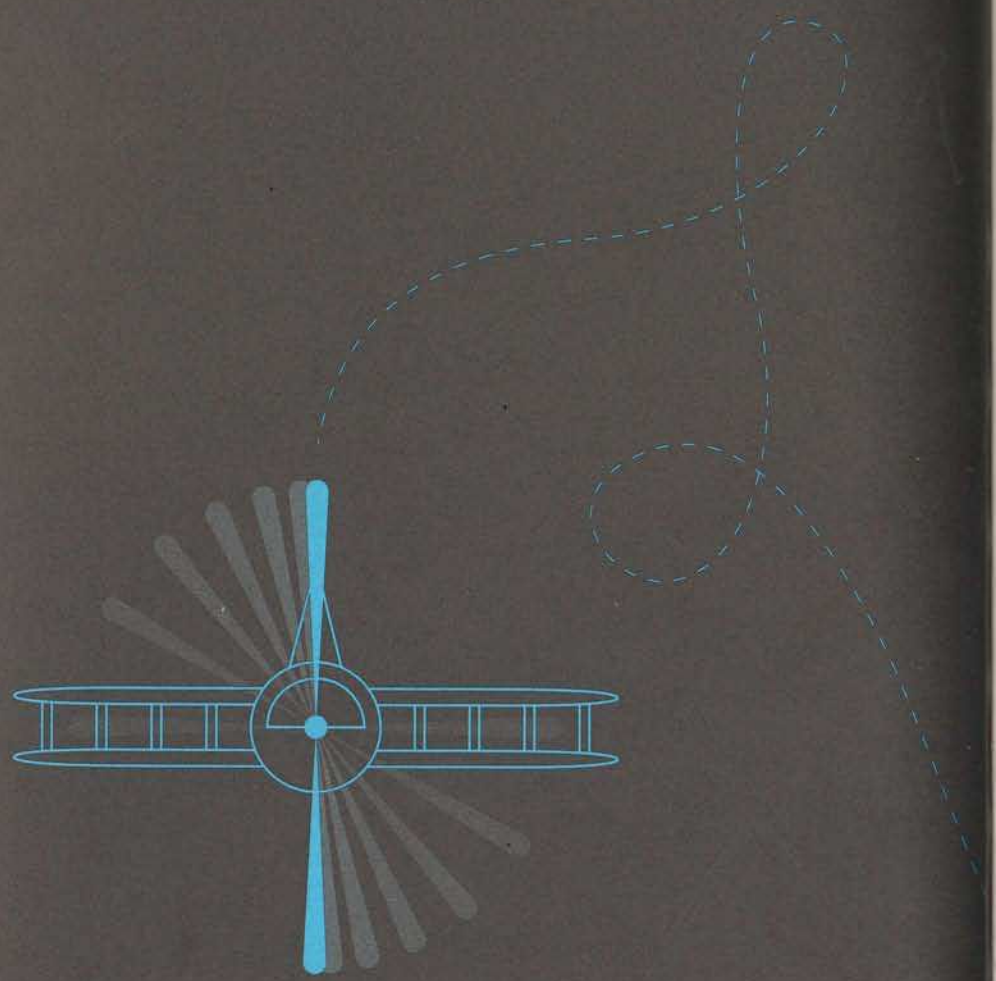
3 9

5 1 3 5 8 6 6 9 10

the correct result is:

4

**Explanation of the example:** There are two longest fragments, both of length 4: a sequence 5, 8, 6, 6 and a sequence 8, 6, 6, 9.





# / Solution

We begin by defining a few notions. Let the sequence of consecutive measurements be denoted by $a_1, a_2, \ldots, a_n$. We say that the segment $[i,j]$ is *t-stable* for a given $t$ if it satisfies the requirement of keeping the measurements within the range $t$, so for each pair of indices $i'$, $j'$, such that $i \le i' \le j' \le j$, we have $|a_{i'} - a_{j'}| \le t$. Our task is to determine the longest $t$-stable segment for a given constant $t$.

**Naïve $O(n^3)$ solution**

The first idea is to consider all pairs of indices $i$, $j$ and determine the one that corresponds to the longest $t$-stable segment.

```
record := 1 {the length of the longest t-stable segment so far}
for i := 1 to n do
    for j := i + 1 to n do
        if t-stable(i, j) then
            if j − i + 1 > record then
                record := j − i + 1
return record
```

How can we verify if a given segment $[i,j]$ is stable? Of course, we could consider all pairs $i'$, $j'$ such that $i \le i' \le j' \le j$, and verify if all values $a_{i'} - a_{j'}$ fit within the tolerance $t$. Such a solution would be too expensive: since there are quadratically many such pairs $(i',j')$ for each of the $O(n^2)$ segments $[i,j]$, its complexity would be $O(n^4)$. But a simple observation enables us to test the $t$-stability of each of the $[i,j]$ segments in linear time. To assure $t$-stability, one has to know only two values: the minimum and the maximum within the segment. If the difference between them does not exceed $t$, then such a segment is $t$-stable; otherwise it is not. We can do it easily using $2n - 2$ comparisons, and with some optimization even $\lceil 3n/2 \rceil - 2$ comparisons are enough. So, having found this idea, we have obtained a solution with a cost of $O(n^3)$, but we are still far away from the optimal solution.

**Improved $O(n^2)$ solution**

The first important remark is that extending a non-$t$-stable segment from any end cannot make it $t$-stable. In other words, if the segment $[i,j]$ is not $t$-stable, then no segment $[i',j']$ is $t$-stable for any $i' \le i$ or $j \le j'$. This means that if, for the fixed beginning of the segment $i$, we are moving the end of the segment $j$ forward, and recognize that for some $j$ the segment $[i,j]$ is not $t$-stable, then we can stop regarding $i$ as an interesting beginning: no further segments starting with $i$ will be $t$-stable, and hence the record will not be broken. Note that when we go forward with $j$, we can update the maximal and minimal values if only $a_j$ happens to be the biggest or the smallest of the considered values. This is the way to obtain a still-simple $O(n^2)$ algorithm.

```
i := 1
record := 1
while i ≤ n − record do
    {there is no need to consider greater i, because such i}
    {cannot be the beginning of the longest t-stable segment}
    min := a[i] {the smallest... }
    max := a[i] {and the biggest value in the range}
    j := i + 1
    stable := true
    while (j ≤ n) and stable do
        if a[j] > max then
            max := a[j]
        else if a[j] < min then
            min := a[j]
        if max − min > t then
            stable := false
        else
            j := j + 1
    if j − i > record then
        record := j − i
    {we need not check this condition at each loop iteration}
    {because we know the best j for given i}
return record
```



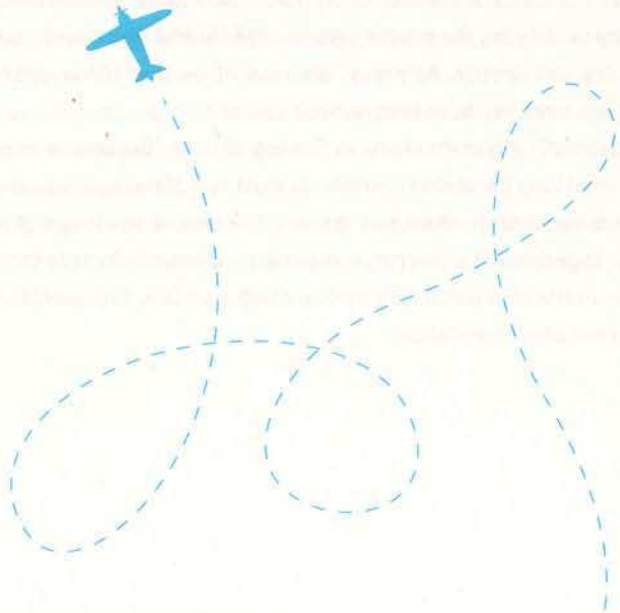
Note that we exit the loop with $j$ as the smallest index for which the segment starting at $i$ is not stable. The difference $j - i$ gives us the number of elements of the longest stable segment starting at $i$.

We have here one linear loop nested inside another, so the total cost of this solution is quadratic. Cutting the cost even further requires some more advanced techniques.

**Still better $O(n \log n)$ solution**

The main idea now is to increment two pointers representing the ends of a stable segment. If $a_j$ destroys the stability then we move $i$ forward; otherwise we increment $j$. In order to test the stability quickly, we should know the minimal and maximal values of the considered segment. To do it rapidly we will need a special technique to represent a multiset containing elements $a_i, \ldots, a_j$ in such a way that determining the minimal and maximal values is cheap. We need a multiset representation because the values can be repeated within one segment.

If we use balanced trees, like AVL, then with each value we will keep its multiplicity. The height of an AVL tree is logarithmic in the number of represented values. So, since the minimal and maximal values lie on the leftmost and the rightmost paths respectively, they can be determined in $O(\log n)$ time.



The pseudocode of this algorithm is quite simple.

```
i := 1
InitAVL(a[1]) {creates 1-element AVL tree containing a[1]}
record := 1
min := a[1]
max := a[1]
j := 2
while j ≤ n do
    InsertAVL(a[j])
    if a[j] > max then
        max := a[j]
    else if a[j] < min then
        min := a[j]
    while max − min > t do
        {if a[j] did not change min or max or if it did change, but}
        {without losing t-stability, then this loop will not be executed}
        DeleteAVL(a[i])
        Update(min, max)
        i := i + 1
    j := j + 1
    if j − i > record then
        record := j − i
return record
```

We can update the values *min* and *max* after deletion of $a[i]$ in a cleverer way by modifying the procedures *InsertAVL* and *DeleteAVL*, but let's not pursue this idea too deeply. Anyway, the cost of each of these operations is at most $O(\log n)$, even without this optimization.

The whole algorithm runs in $O(n \log n)$ time. The cost is so small because the number of loop iterations is linear: at least one of the indices $i$ and $j$ is incremented in each iteration (it does not matter that one of the loops is nested inside the other: together they can run at most $2n - 2$ times). In each loop iteration, every access to the multiset data structure requires $O(\log n)$ operations, thanks to the AVL trees implementation.



**Linear solutions**

The key observation is that we do not need to remember all the measurements from the segment $[i,j]$, but only the minimum and maximum ones. If we are able to do it quickly, the rest of the values will not be needed. We are going to present two solutions, each with a linear time cost.

**Linear implementation specific to this problem.** Let $p[j]$ for each $j$ denote the smallest index $i$ for which the $[i,j]$ segment is $t$-stable. We are looking for the longest $t$-stable segment, so if we know $p[j]$ for each $j$, then comparing the values $j - p[j] + 1$ will give us the result. Let $minval(i,j)$ and $maxval(i,j)$ be the smallest and the biggest values in the segment $[i,j]$.

We will find the solution by using inductive reasoning: a powerful method that is present in many efficient algorithms. In our case, we will consider consecutive elements $a_j$ for $j = 2,\ldots,n$ and design a method to advance from $j-1$ to $j$. Let's assume inductively that we know the value $p[j-1]$ and want to use this knowledge in determining $p[j]$. The first question is why this value should be different from $p[j-1]$. This can happen only when $a_j$ destroys stability, so either $a[j] - minval(p[j-1], j-1) > t$, or $maxval(p[j-1], j-1) - a[j] > t$. When none of these inequalities holds, then simply $p[j] = p[j-1]$.

The case in which the stability is broken, so $p[j] \neq p[j-1]$, requires finding the smallest index $i$ that makes the $[i,j]$ segment stable. Consider three cases:

- If $a[j-1] > a[j]$ then the danger comes only from the top. The new element $a[j]$ can be too small for some of the elements from the segment $[p[j-1], j-1]$. So we should find among these indices some $k$ such that none of the elements $a[k],\ldots,a[j-1]$ exceeds $a[j]$ by more than $t$. It is hence convenient to know the indices of the decreasing left-maximums in this segment, beginning with the greatest value in the whole segment. (A left-maximum in a segment is an element such that none of the elements with a bigger index is greater.)
- If $a[j-1] < a[j]$ then the danger comes only from the bottom. Analogously, we should have easy access to the increasing sequence of left-minimums. (A left-minimum in a segment is an element such that none of the elements with a bigger index is smaller.)
- If $a[j] = a[j-1]$ then we have nothing to do, except for verifying that the current segment is the longest (we have $p[j] = p[j-1]$, and the segment is now longer).

We see here that the key to success is the information about two specific sequences: the decreasing sequence of left-maximums and the increasing sequence of left-minimums for the segment $[p[j-1], j-1]$. In each of these cases we will memorize the indices of the stored values. We should also decide which of the possibly equal values will be stored. It is quite clear: if some left-maximum value $m$ is too big for a given $a[j]$, then all the occurrences of $m$ in the segment $[p[j-1], j-1]$ are equally bad, so we will be interested in the last occurrence of each such value. The first candidate for $p[j]$ will be the element whose index is greater by 1 than the last occurrence of this left-maximum. It can happen, however, that consecutive left-maximums (even all of them) are too big. In such a case, we should stop at any such leftmost left-maximum that does not violate $t$-stability (in the extreme case, the considered element $a[j]$ can become such a left-maximum, so we will always find one). An analogous situation concerns left-minimums: we need the indices of the latest occurrences of increasing values being left-minimums. For instance, for $t = 4$ and a 4-stable sequence

$$5, 4, 6, 8, 7, 7, 4, 6, 4, 5$$

the interesting decreasing values of left-maximums are 8, 7, 6, 5 with the corresponding indices being 4, 6, 8, 10, while the increasing values of left-minimums are 4, 5 with indices 9, 10. Clearly here we have $p[10] = 1$. If we extend the considered sequence with an 11th element, we must modify these two sequences. For instance, augmenting our sequence by $a[11] = 2$ we have $p[11] = 7$—the longest 4-stable segment begins with the value 4 at the 7th position. The updated sequence of left-maximums for this new 4-stable segment ending on the added 2 will be equal to 6, 5, 2, with indices 8, 10, 11, while the sequence of left-minimums will have just one element: 2 with the index 11.



Note that the actual value of $p[j]$ is not important except for the case where we increment $j$ without changing $p[j]$. Only then can the record be beaten.

Let us make it more formal. For each $j$ we define the decreasing sequence of left-maximums $UP_j$ in the following way. The first element of the sequence is the biggest value in the segment $[p[j], j]$. If there is more than one such value, we choose the last one. Consecutive elements of this sequence are the last occurrences of such elements $x$ that are smaller than the previous chosen elements, but which satisfy the property that up to the end of the segment (so up to $j$) there are no bigger values than $x$. The increasing sequence of left-minimums $DOWN_j$ is defined analogously, starting from the latest occurrence of the smallest element in the segment $[p[j], j]$.

Having defined these two sequences, we can easily determine the value $p[j]$. When we look at the first sequence $UP_j$, we see that all the values that are bigger than $a[j]$ by more than $t$ should be excluded, together with all the preceding elements. So a candidate for $p[j]$ will be the index bigger by 1 than the last element of $UP_j$, which destroys the $t$-stability. Of course, if even the first element of the sequence is not too big, then we stop this part of the investigation. In this case, $p[j]$ is possibly equal to $p[j-1]$. If, instead, the first element of the sequence exceeds $a[j]$ by more than $t$, then we delete from $UP_j$ all elements that are too big and stop the deletion on such an element whose value is bigger than $a[j]$ by at most $t$. We memorize the index of the last deleted element. The next index is a candidate for $p[j]$. Note that it may happen that all the elements of $UP_j$ will be deleted.

We perform the analogous procedure for the other sequence $DOWN_j$, determining another candidate for $p[j]$. Neither of them will object to a new segment's end $a[j]$, and both of them have indices greater by 1 than the last deleted elements. The value $p[j]$ (new value for $i$) is equal to the bigger index of these two candidates.

Now the time has come to update the values of *UP* and *DOWN* from the other end. The new value $a[j]$ must become the last value of these two sequences. In the first case we should delete all the values that are smaller than $a[j]$ and, if this value was not present in $UP_{j-1}$, augment the sequence by $a[j]$. If $a[j]$ was present in $UP_{j-1}$, then we need not augment the sequence with anything, but we must update the index of this value to $j$—this is its last occurrence now. Of course, we do the same with the other sequence. We delete all the greater values and add it at the end, or update the index if this value has been found there. Performing these actions will preserve the invariant of the loop: the sequences *UP* and *DOWN* are satisfying the definition, and the last occurrences of the values are memorized.

Now it is time to check whether the new element $a[j]$ has increased the longest $t$-stable segment so far. It is enough to check whether $j - i + 1$ is greater than the record.

Our algorithm now has three phases:

1. The updating of *UP* and *DOWN* from the beginning.
2. The updating of *UP* and *DOWN* from the end.
3. The updating of the record.

Both of our sequences can be maintained with the help of bi-queues, i.e. structures that allow the insertion and deletion of elements from both sides. The following operations will be needed (assume that the bi-queue elements will have two fields, *val* and *ind*, representing the value and the index of a segment element):

- Init($q$)—to initialize an empty bi-queue $q$,
- Empty($q$)—to verify the emptiness of $q$,
- First($q$)—to return the value of the first element of $q$,
- DeleteFirst($q$)—to delete the first element of $q$ and return its value,
- Last($q$)—to return the value of the last element of $q$,
- DeleteLast($q$)—to delete the last element of $q$ and return its value,
- InsertLast($q, x$)—to insert $x$ at the end of $q$.

Such a data structure can easily be implemented, for instance, with the help of a doubly-linked list or in an array representing a circular buffer. Each of these operations can be performed in constant time.



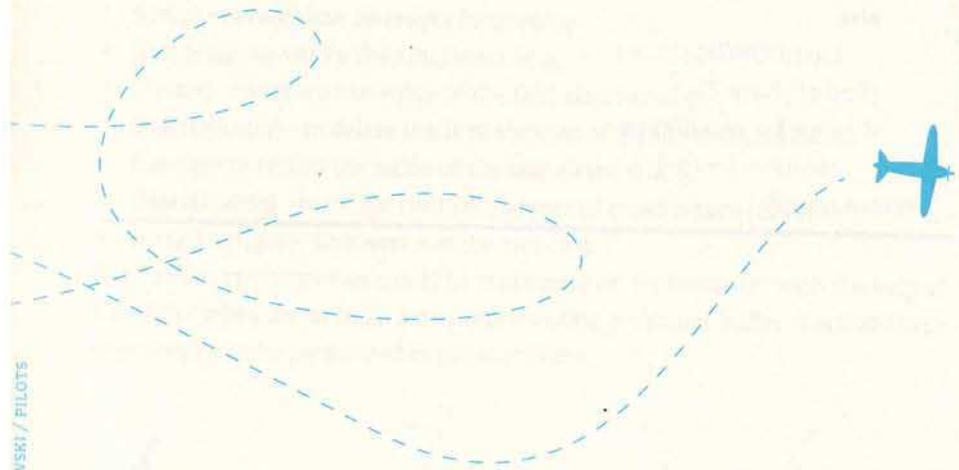
```
Init(UP)
Init(DOWN)
last_up := 0; last_down := 0
record := 1
for j := 1 to n do
    x.ind := j
    x.val := a[j]
    while (not Empty(UP)) and (First(UP).val − t > a[j]) do
        last_up := DeleteFirst(UP).ind
    while (not Empty(DOWN)) and (First(DOWN).val + t < a[j]) do
        last_down := DeleteFirst(DOWN).ind
    i := max(last_up + 1, last_down + 1)
    {End of phase 1}
    while (not Empty(UP)) and (Last(UP).val < a[j]) do
        DeleteLast(UP)
    if Empty(UP) or (Last(UP).val > a[j]) then
        InsertLast(UP, x)
    else
        Last(UP).ind := j
    while (not Empty(DOWN)) and (Last(DOWN).val > a[j]) do
        DeleteLast(DOWN)
    if Empty(DOWN) or (Last(DOWN).val < a[j]) then
        InsertLast(DOWN, x)
    else
        Last(DOWN).ind := j
    {End of phase 2}
    if j − i + 1 > record then
        record := j − i + 1
return record
```

Let's analyze the complexity of this solution. The first impression is that the complexity is not linear. Inside the outermost loop "for each $j$" we have two loops, each of them processing potentially linearly long sequences *UP* and *DOWN*. But the overall processing cannot take more than linear time, because the inner loops delete the elements, which had to be inserted previously. And at each loop iteration we can insert only one element for each $j$. So the total number of deletions is linear too.

**Range Minimum Query.** Another idea that results in a linear solution is the general RMQ algorithm. This allows us to find the minimal and maximal values in a given segment of an array quickly. After preprocessing, which is linear with respect to the sequence length, the answers are given in constant time for each desired segment. However, this algorithm is a bit tricky to implement.

### To sum up

This problem exposes a very typical case: we can improve the first idea we have by several orders of magnitude, but doing so requires quite a lot of effort and some knowledge. On this occasion we have achieved a tremendous decrease in complexity: from $O(n^3)$ (or even $O(n^4)$) to linear. Just as in real life, "cheap" does not usually mean "easy"; but it does pay off.





# / Skiers

**Contest:** 9th Polish Olympiad in Informatics
**Task author:** Piotr Chrząstowski
**Memory:** 32 MB
http://main.edu.pl/en/archive/oi/9/nar

On the south slope of Bytemount there are several ski tracks and one ski lift. All the tracks run from the top station of the ski lift to the bottom station. Every morning a group of ski lift workers examines the condition of the tracks. Together, they take the lift up to the top station, and then each of them skis down along a chosen track to the bottom station. Each worker skis down only once. The tracks of the workers may be partially the same. Each track examined by any of the workers always leads downwards.

The map of ski tracks consists of a network of clearings connected by cuttings in the forest. Each clearing lies at a different height. Any two clearings may be connected directly by at most one cutting. Skiing down from the top to the bottom station, one can choose a track to visit any one clearing (although probably not all of them in a single run). Ski tracks may cross only at clearings, and do not run through tunnels or over bridges.

### Task

Write a program that:

- reads from the input the map of ski tracks,
- computes the minimum number of workers needed to examine all the cuttings,
- writes the result to the output.

### Input

In the first line of the input there is one integer $n$ equal to the number of clearings ($2 \le n \le 5000$). The clearings are numbered from 1 to $n$.

Each of the successive $n - 1$ lines contains a sequence of integers separated by single spaces. The integers in the $(i + 1)$-th line specify which clearings the cuttings from clearing number $i$ lead down to. The first integer $k$ specifies the number of such clearings. The successive $k$ integers are the numbers of the clearings, ordered from west to east according to the arrangement of the cuttings

leading to them. The top station of the ski lift lies at clearing number 1 and the bottom station at clearing number $n$.

**Output**

In the first and only line of the output there should be exactly one integer: the minimum number of workers required to examine all the cuttings in the forest.

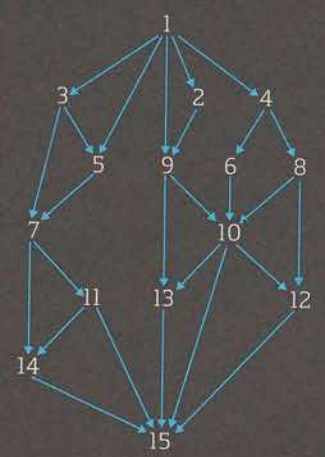
**Example**

For the input data:

```
15
5 3 5 9 2 4
1 9
2 7 5
2 6 8
1 7
1 10
2 14 11
2 10 12
2 13 10
3 13 15 12
2 14 15
1 15
1 15
1 15
```

the correct result is:

8






# / Solution

Note that we are dealing with a very special kind of graph. Nodes in the graph are the clearings, while edges are forest cuttings—each cutting connects a clearing with a different clearing. This is a planar graph, since it describes ski routes, which can intersect only at clearings: there are no tunnels or overpasses. Additionally, this graph has no cycles. Finally, the graph has two distinguished nodes: the upper and lower ski-lift stations. Each node can be reached from the top station, and from each node the bottom station can be reached. Let us call each such graph a *ski graph*.

For each ski graph we can define a dual graph as follows. The nodes of the dual graph are the edges of the original graph. There is an edge in the dual graph that connects $e_1$ with $e_2$ if the cutting $e_1$ exits at a clearing from which one can enter the cutting $e_2$. So the neighboring nodes in the dual graph are the edges of the original ski graph that lie one after another on the way from the top to the bottom of the hill. Let us denote the original ski graph by $N$ and the dual graph by $N_d$.

Consider now the relation $\leq$ in the dual graph $N_d$, defined between its nodes (cuttings) as follows: $e_1 \leq e_2$ if and only if there is a path in $N_d$ leading from $e_1$ to $e_2$. It is therefore a relation expressing the fact that one of the nodes is above the other one in a path from the top to the bottom. This relation has the following properties:

- it is reflexive, i.e. $\forall e \in E : e \leq e$;
- it is antisymmetric, i.e. $\forall e_1, e_2 \in E : e_1 \leq e_2 \wedge e_2 \leq e_1 \Rightarrow e_1 = e_2$;
- it is transitive, i.e. $\forall e_1, e_2, e_3 \in E : e_1 \leq e_2 \wedge e_2 \leq e_3 \Rightarrow e_1 \leq e_3$.

Each relation fulfilling these properties is called a *partial order* relation. Partial order relations play an important role in computing. They allow the classification of data and have been the subject of intensive research for many years.

We say that a set $A$ is partially ordered if a relation of partial order is defined in it. Such an ordered set is denoted as a pair $\langle A, \leq \rangle$. It is always important to know what order we are talking about, because the sets can be ordered in a variety of ways. Our set of forest cuttings has just been ordered by the relation $\leq$, as defined above.

If, between two elements of a partially ordered set, the relation $a \le b$ holds, we say that $a$ is not greater than $b$ and $b$ is not less than $a$. We will introduce two concepts that allow us to formulate the key theorem for the solution of our task.

In a given set $A$, partially ordered by the relation $\le$, the sequence $\{a_1, a_2, a_3, \ldots, a_n\}$ is called a *chain** if $a_1 \le a_2$, $a_2 \le a_3$, ..., $a_{n-1} \le a_n$. In our case, the chains are, for example, all possible segments of ski routes. We say that a chain $C$ is a *maximal chain* if you cannot extend it without losing the chain property. In our case, maximal chains will match routes that start at the top station and end at the bottom one. For a given chain $C = a_1, \ldots, a_n$, the $a_1$ element is called the *smallest* and $a_n$ the *biggest* element in $C$.

We define also the dual concept of an antichain. A set $B = \{b_1, \ldots, b_n\}$ is called an *antichain* if no two elements of this set are ordered by the relation $\le$. Similarly, antichain $B$ is called a *maximal antichain* if it cannot be extended without losing the property of being an antichain. In other words, $B$ is a maximal antichain if it is an antichain and, for each $x \in A \setminus B$, the set $B \cup \{x\}$ is not an antichain.

Let $A_1, \ldots, A_m$ be a family of subsets of $A$. We say that it *covers* the set $A$ if $\bigcup_{k=1}^{m} A_k = A$. Note that a given ski ride from top to bottom corresponds to selecting a maximal chain in the graph $N_d$. Our task is reduced therefore to finding such a family of maximal chains, whose elements cover the set of nodes of the dual graph of $N_d$ and which has the smallest possible number of elements.

We can make use of the following Dilworth theorem, discovered in 1950 and described in an excellent book by W. Marek and W. Lipski, *Combinatorial Analysis*.

**Theorem.** For any finite partially ordered set $\langle P, \le \rangle$ the maximum number of elements of an antichain is equal to the minimum number of chains covering the set $P$.

*Proof.* We follow the proof from the cited book. Assume that the maximum number of elements of an antichain in the set $P$ is $m$. Note first that you need at least $m$ chains to cover the set $P$. Every chain may contain at most one element of the antichain. We will show that $m$ chains are sufficient to cover the set $P$.

We will use induction on the number of elements of $P$. For $|P| = 1$ the theorem holds trivially. The only maximal antichain has one element, and this is the minimum number of chains that cover $P$.

* We consider only finite chains in this task. In set theory infinite chains are also considered, naturally extending the definition, but for our needs they are not necessary.



Suppose now that the thesis holds for all sets $P'$ with $k$ elements, where $k < n = |P|$. Consider any maximal chain $C$ in $P$ and any antichain $A$ of the largest possible cardinality $m$. Let $R(C)$ denote the set of elements of $C$. If we remove all elements of $C$ from the set $P$, and in the resulting set $P \setminus R(C)$ we restrict the relation $\le$ to these elements, then we have two cases.

In the first case the length of the longest antichain in the restricted graph, denoted by $A'$, is $m - 1$ and then, by the induction hypothesis, the set $P \setminus R(C)$ can be covered using $m - 1$ chains. Together with the chain $C$ we have found a cover that consists of $m$ chains.

In the second case the length of $A'$ is still equal to $m$ (it cannot be less than $m - 1$, because each chain can have at most one common element with each antichain, so while subtracting $R(C)$ we removed at most one element from $A$). Note that $A'$ is the maximal antichain not only in $P \setminus R(C)$ but also in $P$, and, in addition, $A' \cap C = \varnothing$. Define two sets:

$$B = \{x \in P : \exists\, a \in A' : a \le x\} \quad \text{and} \quad T = \{x \in P : \exists\, a \in A' : x \le a\}.$$

Intuitively, the set $B$ is the set of elements that do not lie above the antichain $A'$ and $T$ is the set of elements that do not lie below $A'$.

Consider the first element $c$ of $C$ (the highest). It does not belong to $B$, because otherwise the chain $C$ could be extended by some element $a \in A'$, $a < c$, contradicting the maximality of $C$. (In this case, $a < c$ implies that $a \le c$ and $a \ne c$.) Similarly, $T$ does not contain the maximal element of $C$. Thus, $|B| < |P|$ and $|T| < |P|$, and using the induction hypothesis we can see that there exist covers of these sets by chains:

$$B = B_1 \cup \cdots \cup B_m, \quad T = T_1 \cup \cdots \cup T_m,$$

because the antichain $A'$ in $P$ is also an antichain in each of these sets ($A' \subseteq B$ and $A' \subseteq T$).

Let $A' = \{a_1, \ldots, a_m\}$. Assume, without loss of generality, that $a_i \in B_i \cap T_i$ for each $1 \le i \le m$. Note that $P = B \cup T$, because otherwise having an element $x$ not comparable with any $a_i$ we could extend the antichain $A'$ by $x$, contradicting the assumption of its maximality. Therefore

$$P = (B_1 \cup T_1) \cup \cdots \cup (B_m \cup T_m)$$

is a decomposition of the set $P$ into $m$ chains. This completes the proof. □

We have therefore proven that the minimum number of ways you can go downhill to cover the whole graph is equal to the size of a longest antichain in the dual graph. Note that the Dilworth theorem does not require the chains to be maximal: we can always extend a non-maximal chain to a maximal one, allowing some overlapping of chains, and getting a correspondence with the minimum number of ski routes from the top to the bottom.

**Remark.** A clever general algorithm exists that, in a finite graph representing a partial order, finds a minimal set of chains covering the set of nodes. In this algorithm, we assume that the edges connect any two nodes that lie on the same path from the top to the bottom. We want to cover all the nodes with the minimum number of node-disjoint paths (the minimum path cover). Such cover will have as many elements as the minimum chain cover—we just have to make sure that the covering chains are disjoint. Finally, the minimum path cover of any acyclic directed graph can be found by a reduction to the problem of finding a maximum matching in a specific bipartite graph. In our case, however, such a solution would be quite inefficient. To obtain a better method, we will make use of the fact that the ski graphs are planar.

First, note that the cuttings and clearings divide the surface of the mountain into forest areas, visible to the naked eye when you look at the slope. Every two points of such an area can be connected by a polyline without crossing any of the ski routes. In figure 1, for instance, two such areas are surrounded by the cuttings connecting the nodes (sometimes uphill) 1, 2, 9, 10, 6, 4, 1 or 9, 10, 13, 9. We have 12 forest areas in the whole graph here, denoted by the letters A,B,...,L. Such areas in a planar graph are called *faces*. Now note that every maximal antichain of forest cuttings in $N_d$ consists of such cuttings, which represent a sequence of edges separating adjacent faces. Indeed, if some face were missing somewhere, then it would mean that we voluntarily gave up the maximality requirement of an antichain. Let us agree that we will therefore cross the edges from west to east. Each cutting goes at least a bit downhill, so we always know which end of a cutting is the western and which is the eastern. Such a sequence of crossed edges can easily be found in our sample graph: for instance, we can cut the graph "horizontally", crossing the edges 1-3, 1-5, 9-13, 9-10, 6-10, 8-10, 8-12. These edges form a maximal antichain, though in this graph there exist antichains that have more elements. The longest antichain will correspond to the longest horizontal cut in this graph, and its length will be greater by 1 than the number of crossed faces along this cut. How should we find this automatically?



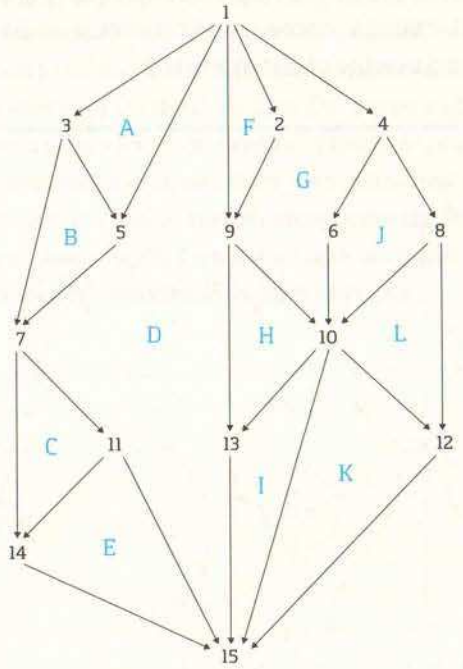



***Figure 1:*** *Example graph with associated labels of its faces.*

Let us start with the construction of the graph of faces $S$ for the original ski graph $N$. Let the nodes of $S$ be the faces of $N$, and let the edges connect these faces, which are separated by a cutting from the west to the east. Figure 2 shows such a graph of faces for the leading example. Graph nodes are labeled with symbols of the faces, and the edges of the graph are labeled by pairs of numbers representing the ends of the corresponding cuttings in the original graph $N$.

Note that in this graph of faces we have some freedom: the faces F and G could equally well be connected by the edge 2-9. In order not to complicate the description we have chosen any of the edges—certainly from two edges separating two faces, only one can be a member of any antichain, so the choice of any of them is arbitrary.

Now the problem reduces to determining the length of the longest path in this graph. The final result will be greater by 2, taking into account the external faces. This can be done, for example, using the following method. First we sort the faces topologically, storing them in an array one by one in this order. Next,

in a single pass, we can update the longest paths from some initial face. For our graph the maximum value will correspond to six equally good paths: BADFGJL, BADHGJL, BADHIKL, CEDFGJL, CEDHGJL and CEDHIKL, each having six edges. So, finally, the minimum number of test-skiers is 8.

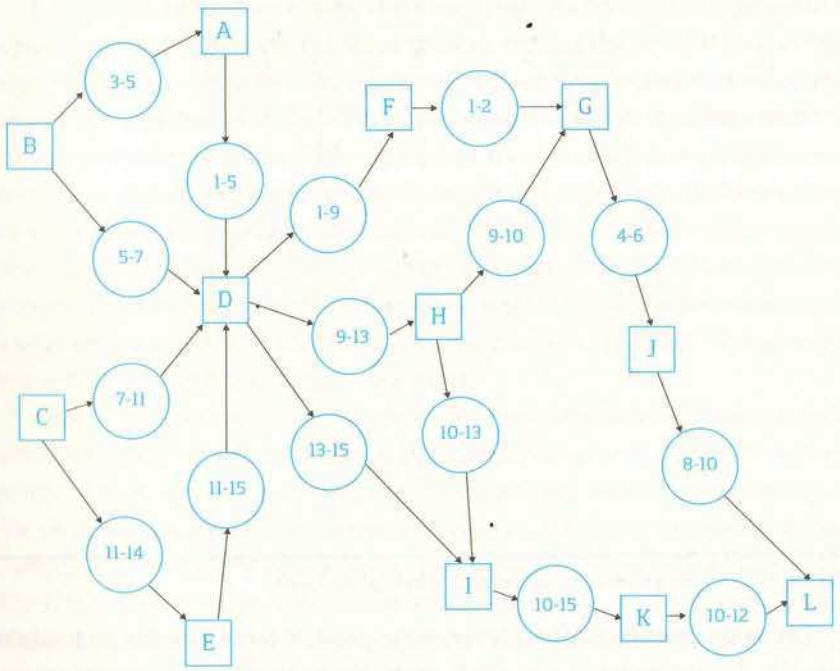



***Figure 2:*** *Graph of faces for the example ski graph.*

Consider the complexity of our algorithm. The graph of faces can be created in linear time with respect to the number of edges in $N$. Topological sorting also requires linear time. In planar graphs the number of edges and faces is linear with respect to the number of nodes (this follows from Euler's theorem), so the complexity of the overall algorithm is linear in the number of nodes of the original graph $N$.

You may, of course, complain that the solution of this task should not involve the proof of the non-trivial Dilworth theorem. It is enough to have "sufficiently good intuition" and "feel" it, and then program the described algorithm. This is true; but it is nevertheless the case that such solutions tend usually to be known, or invented, by the kinds of people who can carry out such proofs.

The idea for this task came into my mind when I stood in front of a map of an attractive ski resort and wondered how many times I would have to go up the mountain to try every part of the ski routes. Of course, a single person must go up as many times as the minimum number of test skiers ascend the ski lift in the morning. As you might imagine, there was more than one ski lift in the resort, but I thought better than to further complicate the task with multiple lifts. As it turned out, even for this simplified task with just one ski lift, only three contestants scored the maximum number of points.

**MAREK CYGAN**

Marek Cygan earned his Ph.D. at the Faculty of Mathematics, Informatics and Mechanics, University of Warsaw. His main research areas include different aspects of algorithmics, with special emphasis on approximation algorithms and parameterized complexity.

It was only after Marek Cygan had started his M.S. studies at the University of Warsaw that he scored his first successes in computer science competitions. Together with his teammates Marcin Pilipczuk and Filip Wolski (and earlier also with Piotr Stańczyk), Marek Cygan won the Polish Collegiate Programming Contest, ACM-ICPC Central European Regional Contest and ACM-ICPC World Finals in 2007. Individually, Marek Cygan was a champion of Google Code Jam 2005 and finished third in TopCoder Open 2009. As a multiple finalist of Google Code Jam, TopCoder Open and TopCoder Collegiate Challenge, he is currently ranked ninth in the TopCoder rankings.

Besides working and spending time with his family, Marek Cygan enjoys running, swimming and watching the financial markets.

# / Barricades

**Contest:** Algorithmic Engagements 2007
**Task author:** Marek Turski
**Memory:** 32 MB
http://main.edu.pl/en/archive/pa/2007/bar

Byteland is an island with a number of cities connected by two-way roads. The road network is designed in such a way that there is exactly one way of driving between any pair of cities (without turning back).

Unfortunately, hard times have come—Byteland is preparing for a war. Byteasar, the lead strategist of Byteland, is preparing a plan of defence of Byteland, which includes the creation of a *special security zone*. The zone will be created by blocking some of the existing roads in Byteland in such a way that it will not be possible to drive along these roads. In order to make the zone completely secure, the following conditions must be fulfilled:

- from each city inside the zone, it must be possible to drive to any other city inside that zone,
- it is not possible to get from a city outside the zone to a city inside the zone,
- the zone has to consist of exactly $k$ cities.

Many different solutions to the problem are being considered—for different values of $k$, it is necessary to determine the minimum number of roads to be blocked in order to obtain a special security zone of size $k$ (consisting of exactly $k$ cities). Help Byteasar—write a program that, for a specified value of $k$, calculates the required number of barricades.

## Task

Write a program that:

- reads from the input a description of the roads in Byteland and the set of queries (different values of $k$),
- for each query, determines the minimum possible number of barricades sufficient to construct a special security zone of the required size,
- writes the results to the output.

## Input

The first line of the input contains one integer $n$ ($1 \le n \le 3000$), representing the number of cities in Byteland. Cities are numbered $1, 2, \ldots, n$.

Each of the following $n-1$ lines of the input contains a pair of integers $a$, $b$ ($1 \le a, b \le n$). Each pair $a$, $b$ represents a direct road connecting cities $a$ and $b$. Each pair of cities is connected with at most one direct road.

In the following line of the input there is an integer $m$ ($1 \le m \le n$), representing the number of queries to process. Each of the following $m$ lines contains one integer $k_i$ ($1 \le k_i \le n$), representing query number $i$—the number of cities that must be inside the $i$-th special security zone.

**Output**

Your program should write to the output exactly $m$ numbers, each on a separate line. The number on the $i$-th line should be:

- $-1$, if creation of a special security zone of size $k_i$ is not possible,
- the minimum number of roads that must be blocked in order to construct a special security zone of size $k_i$ otherwise.

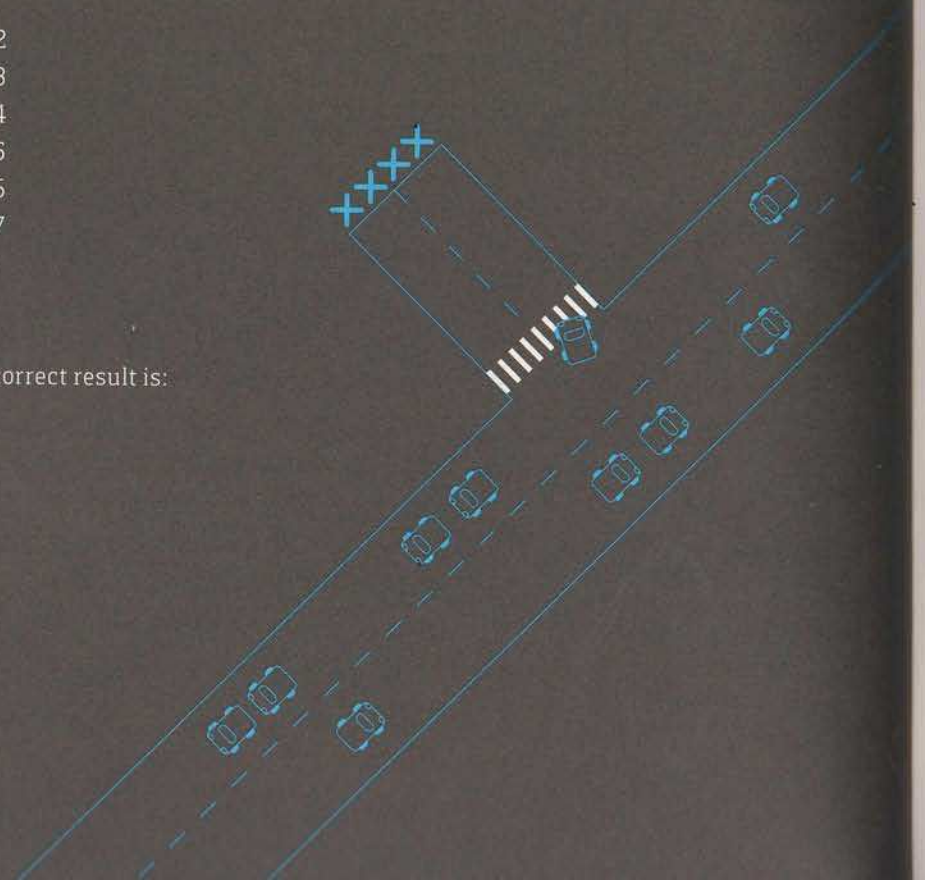
**Example**

For the input data:

```
7
1 2
1 3
2 4
2 5
3 6
3 7
2
2
3
```

the correct result is:

```
2
1
```





# / Solution

I encountered this problem as a finalist of Algorithmic Engagements 2007. Unfortunately, during the tournament I failed to solve the problem, which caused me to lose the competition.

Each seasoned contestant, after just a few moments, figures out a polynomial time algorithm for the problem. The time complexity of a standard dynamic programming routine can be easily upper-bounded by $O(n^3)$. However, one simple—and at the same time brilliant—observation proves the algorithm to run in $O(n^2)$ time. I find this trick very nice, and for this reason I would like to devote this chapter to *Barricades*.

**Algorithm**

Let us start by describing a standard solution using dynamic programming. For the sake of simplicity, let us root the input tree in an arbitrary vertex, and from now on assume that we are dealing with a rooted tree $T$. Consequently we use standard notation, such as parent and child relationships in the tree $T$. As usual, in a dynamic programming solution, the important part is to define exactly what our algorithm is going to compute.

For a vertex $v$ by $T_v$ denote the subtree of the tree $T$, rooted at $v$, whereas $|T_v|$ is the number of vertices in $T_v$. Moreover, for a vertex $v$ and an arbitrary integer $1 \le i \le |T_v|$, $t[v][i]$ is the minimum number of edges one has to remove from $T_v$ to make the connected component containing the vertex $v$ consist of exactly $i$ vertices. Having all the values $t[\cdot][\cdot]$ one can easily solve the problem, computing an answer for each query in $O(n)$ time. We leave the details as an exercise for the reader.

If $v$ is a leaf in the tree $T$, then the subtree rooted in $v$ has only one vertex, so $t[v][1] = 0$. Therefore, let us assume that the vertex $v$ has exactly $k$ children $u_1, \ldots, u_k$ in the tree $T$. Moreover, we assume that all the values $t[u_j][i]$, for $1 \le j \le k$ and $1 \le i \le |T_{u_j}|$, have already been computed. What remains is to merge the values that have already been obtained.

We compute an array analogous to $t$ for bigger and bigger subtrees rooted in $v$. First, we compute it just for the vertex $v$; next for $v$ and the subtree rooted in $u_1$; then for $v$ and subtrees rooted in $u_1$ and $u_2$, and so on. By $T_v^j$ let us denote the tree $T_v$ with subtrees rooted at $u_z$ removed, for each $j < z \le k$ (an example is depicted in figure 1). Let $t_j[v][i]$ be the value $t[v][i]$ obtained for the tree $T_v^j$. Observe that our goal is to compute $t[v][i] = t_k[v][i]$.

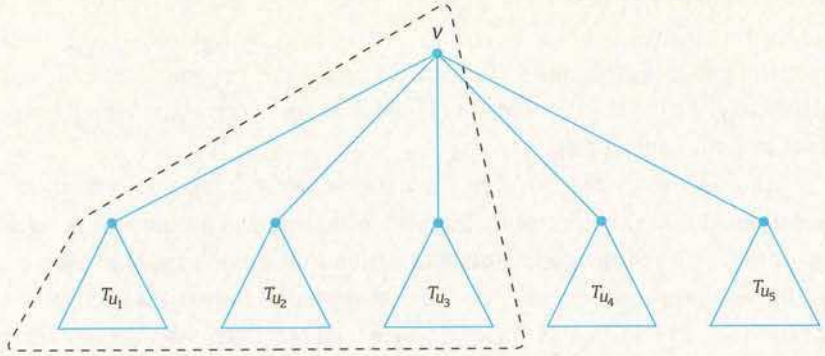



***Figure 1:*** *The area surrounded by the dashed frame contains the vertices of the tree $T_v^3$.*

We start with $j = 0$. Then $v$ is a leaf in $T_v^j$, so $t_j[v][1] = 0$. Now, we are going to compute values $t_j[v][i]$ for subsequent values of $j$ according to the following formula:

$$t_j[v][i] = \min\left(t_{j-1}[v][i] + 1, \min_{\substack{a+b=i \\ a,b\geq 1}} (t_{j-1}[v][a] + t[u_j][b])\right). \qquad (1)$$

The above formula considers two possibilities. We can erase edge $vu_j$, so that the connected component containing $v$ consists only of vertices from $T_v^{j-1}$. We can also leave the edge in the tree, in which case the connected component containing $v$ is made of $a$ vertices from $T_v^{j-1}$ and $b$ vertices from $T_{u_j}$.

**Complexity analysis**

Observe that the most time-consuming part of our algorithm relates to the formula (1). One can easily upper bound the number of operations needed to obtain all the values $t[\cdot][\cdot]$ by $O(n^3)$, since our algorithm fills $n-1$ arrays $t_j[v][\cdot]$ (one array per edge of $T$), each of which takes $O(n^2)$ running time. However, surprisingly, one can obtain a better upper bound without changing the algorithm. It suffices to investigate thoroughly the number of operations performed.

***Lemma.*** For a fixed vertex $v$, the number of operations needed to compute all the values $t[u][\cdot]$, where $u$ belongs to $T_v$, is $O(|T_v|^2)$.

*Proof.* We prove the lemma by induction over the depth of the tree $T_v$. If $v$ is a leaf, then the lemma clearly holds. Thus let us assume that the vertex $v$ has exactly $k$ children $u_1,\ldots,u_k$ and denote $a_j = |T_{u_j}|$. When performing the merge operation of the $j$-th child—that is, we compute $t_j[v][\cdot]$—we go through the values $t_{j-1}[v][a]$ and $t[u_j][b]$ for all possible $a$ and $b$. The value of $a$ can be chosen in $|T_v^{j-1}|$ ways, and $b$ in $|T_{u_j}|$ ways. This gives

$$O(|T_v^{j-1}|\cdot|T_{u_j}|) = O\big((1 + a_1 + a_2 + \cdots + a_{j-1})\cdot a_j\big)$$

operations. From the inductive assumption, computing all the values in the table $t$ for vertices below $v$ takes $O(\sum_{j=1}^{k} a_j^2)$ time. Summing up, the number of operations needed to compute the values $t[u][\cdot]$ for all the vertices $u$ belonging to $T_v$ is

$$O\Big(\sum_{i=1}^{k}\sum_{j=1}^{k} a_i a_j + \sum_{j=1}^{k} a_j + \sum_{j=1}^{k} a_j^2\Big) = O\Big((1 + \sum_{j=1}^{k} a_j)^2\Big) = O(|T_v|^2). \qquad \square$$

There is also a combinatorial interpretation of the proof of the lemma. Since attaching a single subtree involves $O(|T_v^{j-1}|\cdot|T_{u_j}|)$ operations, we can interpret this number as an amount of work proportional to the number of pairs of vertices of the tree $T$ that have $v$ as their lowest common ancestor. Since each pair of vertices is counted exactly once, we obtain $O(n^2)$ time complexity.

Observe that, using the lemma for the root of the tree $T$, we obtain an $O(n^2)$ upper bound on the total number of operations needed to compute all the values $t[\cdot][\cdot]$.

It is worth mentioning that, in order to obtain $O(n^2)$ time complexity, in the formula (1) we have to perform at most $O(|T_v^{j-1}|\cdot|T_{u_j}|)$ operations. In particular, we cannot allow ourselves to iterate over too wide a range of integer values $a$ and $b$, since even if each merge operation involves $\Omega(n\cdot|T_{u_j}|)$ complexity (or even $\Omega(|T_v|\cdot|T_{u_j}|)$), our algorithm will need $\Omega(n^3)$ worst case running time.


MAREK CYGAN / BARRICADES

# / Travel *Agency*

**Contest:** Algorithmic Engagements 2006
**Task author:** Marek Cygan
**Memory:** 32 MB
http://main.edu.pl/en/archive/pa/2006/biu

The inhabitants of Byteland have recently begun to travel more often than ever before. Enterprising Byteasar has had the idea of opening a travel agency. On the first day, $n$ customers came to the agency (we label them with integers ranging from 1 to $n$), each of them wanting to go on a trip. Unfortunately, every customer has individual requirements.

Your task is to help Byteasar in choosing the customers who should go on the trip so as to maximize Byteasar's profit.

Each customer tells Byteasar what value the trip has for him/her. Let $x_i$ be the value of the trip for the $i$-th customer ($-1{,}000{,}000 \le x_i \le 1{,}000{,}000$). If $x_i \ge 0$, then this customer pays $x_i$ bytean dollars for the trip, and if $x_i < 0$, then Byteasar has to pay the $i$-th customer $-x_i$ bytean dollars if this customer goes on the trip.

Apart from their financial requirements, customers also have social requirements. The $i$-th customer has $k_i$ ($0 \le k_i \le n-1$) such requirements; the $j$-th requirement of the $i$-th customer is represented by a pair of integers $(a_{ij}, b_{ij})$ ($1 \le a_{ij} \le n$, $a_{ij} \ne i$, $1 \le b_{ij} \le 1{,}000{,}000$). This requirement means that if the $i$-th customer goes on the trip, then the $a_{ij}$-th customer also has to go on the trip, or else the cost of the trip for the $i$-th customer has to be lowered by $b_{ij}$ bytean dollars. This might mean that, for some customers, the cost of the trip will decrease from positive to negative and Byteasar will have to pay some money to such customers, so they will go on the trip without some of their social requirements being satisfied.

Help Byteasar in choosing the customers who should go on the trip so as to maximize Byteasar's profit (the number of customers who can be taken on the trip is unlimited).

## Task

Tests in this task are open—all of them are available to contestants. The solution to the task should be a program that reads a single integer $k$ ($0 \le k \le 10$) from the input and writes the solution to the $k$-th test to the output.



In the first line of any output file there should be exactly one integer $k$ ($0 \le k \le n$): the number of customers who should be taken on the trip by Byteasar. If the number $k$ is positive, then the second line should contain $k$ integers, separated by single spaces and specifying the individual numbers of customers who should be taken on the trip. If there is more than one optimal solution, the output file should contain one of them. The numbers of the customers can be given in any order.

## Description of a single input file

In the first line there is exactly one integer $n$ ($1 \le n \le 1000$), specifying the number of customers of the travel agency. The next $n$ lines describe the customers. In the ($i$+1)-th line ($1 \le i \le n$) there are integers $x_i$ ($-1{,}000{,}000 \le x_i \le 1{,}000{,}000$) and $k_i$ ($0 \le k_i \le n-1$), and after them $k_i$ pairs of integers $a_{ij}, b_{ij}$ ($1 \le a_{ij} \le n$, $a_{ij} \ne i$, $1 \le b_{ij} \le 1{,}000{,}000$)—all numbers in the line are separated by single spaces. You may assume that every customer has at most one social requirement concerning any other customer.

## Example

For the input data:

```
4
5  0
6  2  1  10  3  1
-10  0
1  2  1  10  2  10
```

the correct result is:

```
3
1  2  4
```

**Explanation of the example:** If Byteasar chooses customers 1, 2, 4 he will have a profit of 11 bytean dollars, and this is an optimal solution.

# / Solution

I set the *Travel Agency* task for the Algorithmic Engagements 2006 competition. It is an example of an unusual application of the maximum flow and minimum cut theorem.

Even though it is possible to solve the problem in polynomial time, contestants were given access to input test data in order to encourage those unable to solve the problem to gain at least some points by investigating the special structure of graphs in the first test cases.

**Solution**

In a typical application of a reduction to the minimum cut problem, each solution of the original task can be transformed into a cut of exactly the same cost, and similarly each cut can be transformed into a solution of the original problem of exactly the same cost. In this sense, *Travel Agency* is a bit peculiar, since not all cuts correspond to solutions of the original problem. Nevertheless, this property holds for all *minimum* cuts, which are what we are about to analyze.

Let us assume, for simplicity, that there are no customers with $x_i = 0$. At the end of this section we will justify why we can make this assumption. For a given instance of the *Travel Agency* task, we create the following auxiliary directed graph $G = (V, A, c)$ with capacity defined for each arc by the function $c: A \to \mathbb{R}_+$. As the vertex set we take $V = \{s, t\} \cup K$, where $s$, $t$ are two special vertices, whereas the set $K = \{v_1, \ldots, v_n\}$ contains exactly $n$ vertices, one for each customer, where a vertex $v_i$ corresponds to the $i$-th customer. Since the trip value is either positive or negative for each customer, we partition the set $K$ into $K_+$ and $K_-$ depending on the sign of the value $x_i$ corresponding to a customer. We create the set of arcs as follows:

- For each customer $v_i \in K_+$ add to the set $A$ an arc $(s, v_i)$ of capacity $x_i$.
- For each customer $v_i \in K_-$ add to the set $A$ an arc $(v_i, t)$ of capacity $-x_i$ (recall that for $v_i \in K_-$ we have $x_i < 0$).
- For each social requirement $(a_{ij}, b_{ij})$ of an $i$-th customer, add to the set $A$ an arc $(v_i, v_{a_{ij}})$ of capacity $b_{ij}$.

A sample graph $G$ is depicted in figure 1.



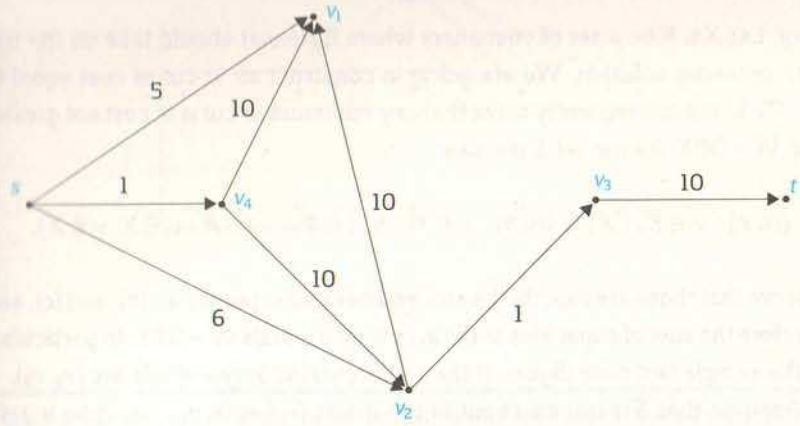



*Figure 1: Construction of the directed graph G for the sample test from the problem statement.*

A set of arcs $S \subseteq A$ is an $st$-cut if, after removing the arcs of $S$ from the graph, there is no path from the vertex $s$ to the vertex $t$. Our goal is to show that solving the *Travel Agency* problem is equivalent to finding a minimum $st$-cut in the graph $G$; that is, an $st$-cut with a minimal sum of capacities of the removed arcs.

Observe that Byteasar's problem is a maximization problem, whereas finding a minimum cut is obviously a minimization problem. Therefore we change his point of view. Let $W$ be the sum of values $x_i$ for customers from the set $K_+$. Clearly, $W$ is an upper bound on Byteasar's profit. By OPT we denote the maximum profit that Byteasar can have. Next, we consider how Byteasar's decisions affect his profit.

(a) If Byteasar does not choose a customer $v_i \in K_+$, he loses $x_i$ from the potential gain.

(b) If Byteasar chooses a customer $v_i \in K_-$, he loses $-x_i$, since he has to pay exactly $-x_i$ to the $i$-th customer.

(c) If Byteasar does not satisfy a social requirement $(a_{ij}, b_{ij})$, he has to lower the price for the $i$-th customer by $b_{ij}$ bytean dollars.

In the following lemmas the above decisions will correspond to removing arcs of each of the three types, enumerated when the set $A$ was defined.

**Lemma 1.** The cost of a minimum $st$-cut in the graph $G$ is not greater than $W - \mathrm{OPT}$.

*Proof.* Let $X \subseteq K$ be a set of customers whom Byteasar should take on the trip in an optimum solution. We are going to construct an $st$-cut of cost equal to $W - \mathrm{OPT}$, and consequently show that any minimum $st$-cut is of cost not greater than $W - \mathrm{OPT}$. As the set $S$ we take:

$$\{(s, v_i) : v_i \in K_+ \setminus X\} \cup \{(v_i, t) : v_i \in X \cap K_-\} \cup \{(v_i, v_j) \in A : v_i \in X,\ v_j \notin X\}.$$

Observe that those are exactly the arcs enumerated in points (a), (b), and (c), and therefore the sum of capacities of the arcs from $S$ equals $W - \mathrm{OPT}$. In particular, for the sample test case (figure 1) the set $S$ contains only a single arc $(v_2, v_3)$.

Assume that $S$ is not an $st$-cut in the graph $G$. Let $(s, u_1, \ldots, u_k, t)$ be a path from $s$ to $t$, which does not use any arc from the set $S$. By the definition of $G$, the arcs extending from $s$ have their endpoints in the set $K_+$, whereas all the arcs entering the vertex $t$ start in $K_-$. Therefore $u_1 \in K_+$ and $u_k \in K_-$. Since the arc $(s, u_1)$ does not belong to the set $S$, we infer that Byteasar takes the customer $u_1$ on the trip. Similarly, since $(u_k, t) \notin S$, we know that the customer $u_k$ was not taken on the trip. Consequently, there exists an integer $1 \le i < k$, such that the customer $u_i$ was on the trip and the customer $u_{i+1}$ stayed at home, and for this reason the social requirement corresponding to the arc $(u_i, u_{i+1})$ is not satisfied. Thus this arc belongs to the set $S$, which is a contradiction of the assumption that the considered path from $s$ to $t$ does not contain any arc from $S$. □

**Lemma 2.** The cost of a minimum $st$-cut in the graph $G$ is not smaller than $W - \mathrm{OPT}$.

*Proof.* Let $S \subseteq A$ be any minimum $st$-cut. We define a set of customers $X \subseteq K$ (depicted in figure 2), whom Byteasar should take on the trip, as follows. The set $X$ consists of

- each customer $v \in K_+$, for which the arc $(s, v)$ does not belong to the set $S$ and
- each customer $v \in K_-$, such that the arc $(v, t)$ belongs to the set $S$.

Let $c(S)$ be the sum of capacities of arcs from the set $S$. We want to show that the profit Byteasar achieves when taking customers from the set $X$ to the trip is at least $W - c(S)$, which consequently proves that $c(S) \ge W - \mathrm{OPT}$. In order to do that, we justify each unsatisfied social requirement by showing that its corresponding arc belongs to the set $S$.



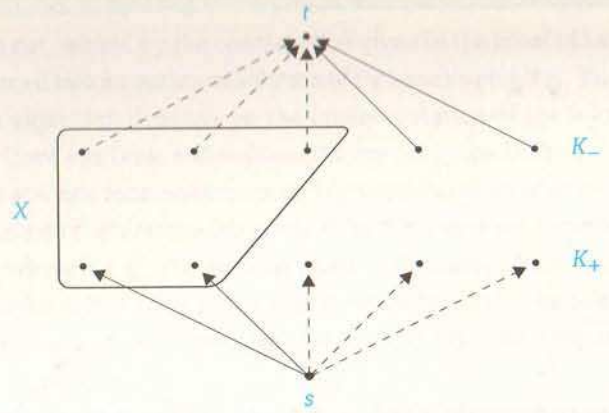



**Figure 2:** *A construction of the set of customers who should be taken on the trip based on an st-cut, the arcs of which are dashed. In the figure, arcs between vertices of the set $K_+ \cup K_-$ are not shown.*

Assume the contrary: that there exists a social requirement $(a_{ij}, b_{ij})$ of the $i$-th customer which is not satisfied, which means that the $i$-th customer belongs to the set $X$, while the customer $a_{ij}$ does not belong to the set $S$ and at the same time $(v_i, v_{a_{ij}}) \notin S$. Let $\alpha = v_i$ and $\beta = v_{a_{ij}}$ and consider the following cases depicted in figure 3. In each case, we show that in the graph $G \setminus S$ (that is, the graph $G$ with the edges from $S$ removed) there exists a path from $s$ to $t$, which contradicts the definition of a $st$-cut.

(i) $\alpha \in K_-$, $\beta \in K_-$. Since $\alpha \in X$, we have $(\alpha, t) \in S$, while from the minimality of the cut $S$ in the graph $G \setminus S$ there is a path from $s$ to $\alpha$, since otherwise removing the arc $(\alpha, t)$ would be pointless. However, $(\alpha, \beta) \notin S$, and consequently there exists a path from $s$ to $\beta$ in $G \setminus S$. By the definition of the set $X$ and by the assumption $\beta \notin X$, we infer that $(\beta, t) \notin S$, and thus there exists a path from $s$ to $t$ in the graph $G \setminus S$: a contradiction.

(ii) $\alpha \in K_-$, $\beta \in K_+$. Similarly, as in the previous case, we infer that there exists a path from $s$ to $\alpha$ in $G \setminus S$. Analogously, from the minimality of the cut $S$ it follows that there exists a path from $\beta$ to $t$, and consequently there exists a path from $s$ to $t$ in $G \setminus S$: a contradiction.

(iii) $\alpha \in K_+$, $\beta \in K_-$. In this case the arcs $(s, \alpha)$ and $(\beta, t)$ do not belong to the set $S$, and therefore in the graph $G \setminus S$ there exists a path from $s$ to $t$: a contradiction.

(iv) $\alpha \in K_+$, $\beta \in K_+$. By the definition of the set $X$, the arc $(s, \beta)$ does not belong to the set $S$. From the fact that $S$ is a minimum cut, we infer that from $\beta$ there exists a path to $t$ in the graph $G \setminus S$, and since $(s, \alpha) \notin S$, there is a path from $s$ to $t$ in the graph $G \setminus S$: a contradiction. □

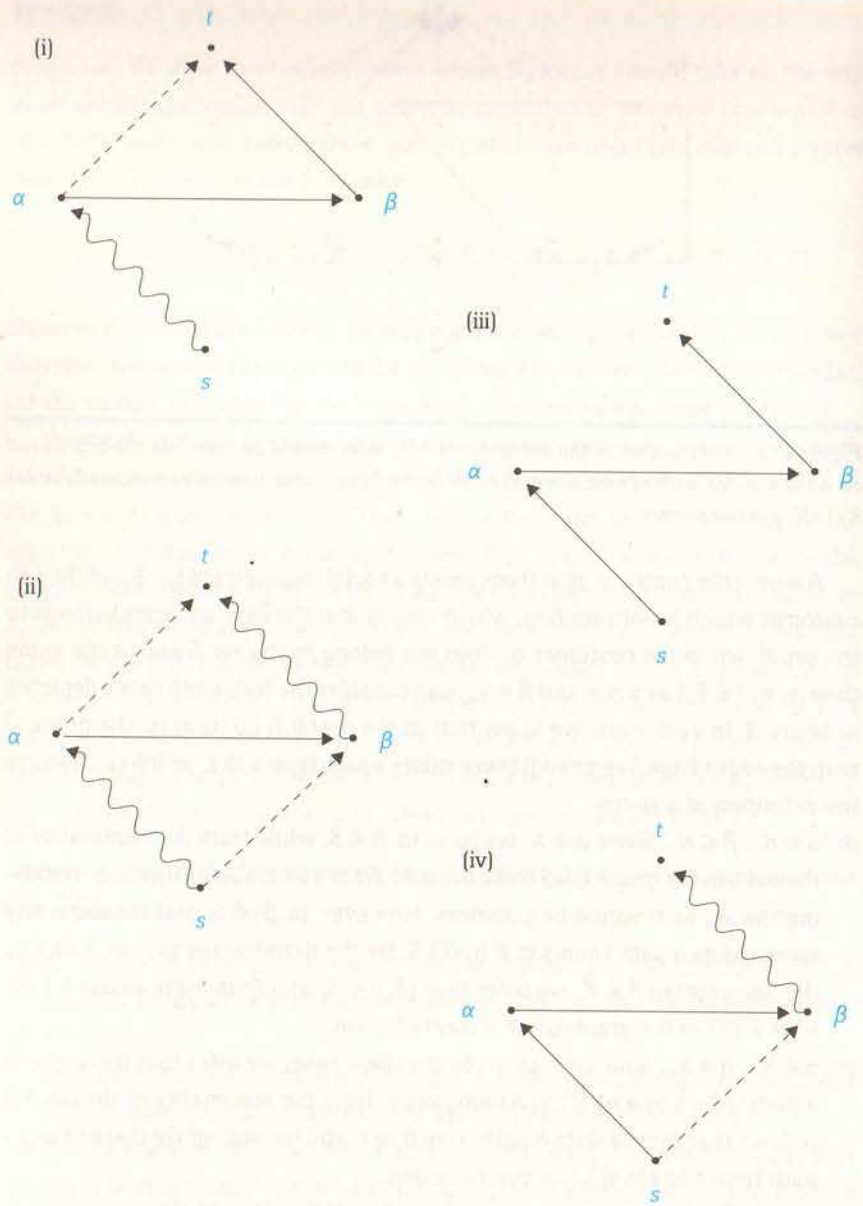



***Figure 3:*** *An illustration of the cases from the proof of Lemma 2. Dashed arcs belong to the set S, while the snake-shaped arcs denote paths that exist in the graph $G \setminus S$.*



By the maximum flow and minimum cut theorem we can effectively construct a minimum cut, which, by the construction given in the proof of Lemma 2, can be transformed into an optimum solution of Byteasar's problem. The complexity of such an algorithm depends on the implementation of the maximum flow algorithm. If we use Dinic's algorithm, the running time is $O(n^3)$.

There is still one outstanding issue. We should explain why we can assume that there are no customers with $x_i = 0$. If for all customers for which $x_i = 0$ we set $x_i := \varepsilon$, where $\varepsilon < \frac{1}{2n}$, the optimal profit of Byteasar changes by less than $\frac{1}{2}$ bytean dollars. It is easy to see that in order to obtain the solution for the original task, we can solve the modified one and take the computed group of customers on the trip.

Alternatively, we can define the set $K_+$ as the set of customers with nonnegative values $x_i$, and the only modification in the analysis would be an additional assumption in the proof of Lemma 2. Namely, if several minimum cuts exist, we consider a cut with the minimum number of removed arcs. The details, however, are left as an exercise for the reader.

**Epilogue**

I encountered an interesting application of the maximum flow and minimum cut theorem for the first time in the *Crazy Painter* problem set by Andrew Stankevich.

Furthermore, by a reduction to the minimum cut problem, one can solve the minimum weight vertex cover in bipartite graphs problem (e.g. the task *Intelligence Quotient* from the Polish Collegiate Programming Contest in 2011). An interesting task for the reader might be to give a reduction, showing that the minimum weight vertex cover in bipartite graphs is a special case of the *Travel Agency* problem. Hint: two independent sets of the graph are customers with positive and negative values of $x_i$.

**TOMASZ CZAJKA**

Tomasz Czajka is a graduate of the Faculty of Mathematics, Informatics and Mechanics, University of Warsaw and Purdue University. He has won two gold medals in the International Olympiad in Informatics and two silver medals in the International Mathematical Olympiad. He was a member of the team representing the University of Warsaw that won the ACM-ICPC World Finals for the first time in 2003. Later on, he won the TopCoder Collegiate Challenge, Top-Coder Open (three times), Google Code Jam Europe, Algorithmic Engagements and the online AI tournament, CodeCup (four times).

Tomasz Czajka currently works at the main Google campus in Mountain View, California, among other things preparing Google Code Jam competitions. In 2011 he became father to a son named Sebastian.

# / Mushrooms

**Contest:** Algorithmic Engagements 2010
**Task author:** Jakub Łącki
**Memory:** 32 MB
http://main.edu.pl/en/archive/pa/2010/grz

A heavy rain has come to the Bytean Forest. Byteasar—an eager mushroom hunter—has decided to take his chance and go mushrooming.

Byteasar knows a nice path leading through the forest, which contains several glades where lots of mushrooms may be found. Every two consecutive glades along the path are equidistant, and Byteasar needs 15 minutes to walk between any pair of adjacent glades. Having arrived at a glade, Byteasar—like any other self-respecting mushroom hunter—collects all the mushrooms in this glade in no time. It is a well-known fact that mushrooms spring up like—well, mushrooms!—and it takes them only half an hour to grow back after they are collected.

Knowing the numbers of mushrooms growing in the respective glades and the duration of Byteasar's walk, and assuming that he takes a route that maximizes his crops, find the number of mushrooms he will collect. Byteasar can finish his walk at any point on the path.

## Input

The first line of the input contains two integers, $n$ and $t$ ($1 \le n, t \le 1{,}000{,}000$), which specify the number of glades on the path and the duration of Byteasar's walk, in quarters. The second line contains a sequence of $n$ integers $a_i$ ($1 \le a_i \le 1{,}000{,}000$), representing the numbers of mushrooms in consecutive glades along the path. Byteasar starts his walk at the first of these glades. We assume that Byteasar can collect mushrooms both just before the first and just after the $t$-th quarter of mushrooming.

## Output

In the first and only line of the output, your program should write one integer: the maximum number of mushrooms that can be collected by Byteasar during his walk.

**Example**

For the input data:

5 4

3 4 3 5 1

the correct result is:

18

**Explanation of the example:** On Byteasar's optimal route, which takes him 60 minutes, he collects 18 mushrooms: 3 at time 0, 4 after 15 minutes, 3 after 30 minutes, 5 after 45 minutes and, finally, 3 after 60 minutes.



# / Solution

The problem *Mushrooms* appeared in the second online round of Algorithmic Engagements 2010, so it was one of the easier problems in that contest. Indeed, the optimal solution can be implemented in just a few lines of code. But coming up with it and proving that it is correct is not easy at all.

**Build a graph**

To start with, we should think about how to formalize the problem in the language of graph theory. We know that it takes 15 minutes to walk between adjacent glades, and that the mushrooms grow back within 30 minutes. But the problem does not require Byteasar to keep walking, or to walk at maximum speed. Might it sometimes be better to slow down or stop, in order to wait for mushrooms to grow back?

Fortunately, it is not that complicated. The key observation is that if Byteasar walks from glade $A$ to an adjacent glade $B$ then the mushrooms in $B$ have already reappeared. Why? Because even if Byteasar had collected mushrooms in $B$ before, it must have taken him at least 30 minutes to walk from $B$ to $A$ and back.

This means that, if we are moving, we can just as well walk at maximum speed. The only question remaining is whether, after we have collected mushrooms in a glade, it makes sense to stop and wait for 30 minutes until the mushrooms have grown back in the same place. The answer to that is also negative: it is not a good idea. After all, in 30 minutes we have enough time to walk to an adjacent place, collect the mushrooms there, and come back.

The only exception is when $n = 1$. Then, there is no adjacent spot to walk to, so we have no choice: we have to sit tight. This case has to be processed separately. Forgetting about simple edge cases like this is a common source of errors.

Let us summarize the findings in graph theory language. The glades are graph vertices, each vertex is assigned some *weight* (the number of mushrooms), and every two adjacent glades are connected with an edge. We are looking for a path starting in vertex 1, of length $t$ and maximum total weight.

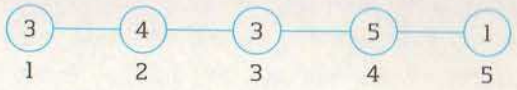



*Figure 1: The graph correspoding to the example from the problem statement.*

**The first algorithm**

Let $f_k(x)$ be the maximum total weight of a path of length $k$, starting in vertex $x$ (vertices are numbered from 1 to $n$). In other words, it is the maximum number of mushrooms that can be collected in $k$ quarters, starting at glade $x$. Of course, $f_0(x) = a_x$, and our goal is to find $f_t(1)$. We compute the values of $f_k$ for $k > 0$ using $f_{k-1}$ and a recursive formula.

**Algorithm** Mushrooms1$(n, t, a)$
  **if** $n = 1$ **then**
    **return** $(\lfloor \frac{t}{2} \rfloor + 1) \cdot a_1$
  **for** $x := 1$ **to** $n$ **do**
    $f_0(x) := a_x$
  **for** $k := 1$ **to** $t$ **do**
    $f_k(1) := a_1 + f_{k-1}(2)$
    $f_k(n) := a_n + f_{k-1}(n-1)$
    **for** $x := 2$ **to** $n-1$ **do**
      $f_k(x) := a_x + \max(f_{k-1}(x-1), f_{k-1}(x+1))$
  **return** $f_t(1)$

Notice that we do not have to store the whole two-dimensional table of values $f_k(x)$. When computing $f_k$ it is sufficient to store just two rows of the table: $f_k$ and $f_{k-1}$. Therefore the space required is $O(n)$. Time complexity, on the other hand, is $O(tn)$. That is too slow for the largest possible input ($n = t = 1{,}000{,}000$).

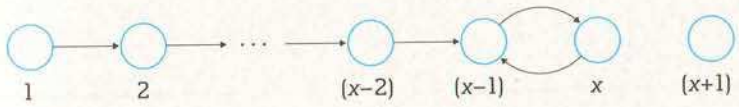
The above algorithm does not take advantage of the special form of our graph. With slight generalization, it would work for any graph. The given graph is quite simple: it is just one path. With a bit more work we can exploit that property to get a faster algorithm.



**Fast algorithm**

Let us imagine what the optimal walk looks like. A simple idea comes to mind: walk to the place with the greatest number of mushrooms in two adjacent glades, and then just walk back and forth between them. That would be the best strategy for a very large $t$, but unfortunately it is not optimal for a small $t$. For example, if $t = 7$, $n = 6$, $a = [4,4,1,1,5,5]$, the strategy would result in $4+4+1+1+5+5+5+5 = 30$ mushrooms. A better path is to walk back and forth between the first two glades, collecting $8 \cdot 4 = 32$ mushrooms. We collected a lot of mushrooms towards the end, but we lost many in trying to get to that fertile spot.

We do not give up just yet. It is not always best to go to the most fertile place, but maybe it is always a good idea to pick *some* two adjacent glades, go directly to them, and then walk back and forth between the two? That turns out to be true.




***Theorem.*** There is an optimal path of the form $1 \to 2 \to \ldots \to (x-2) \to (x-1) \to x \to (x-1) \to x \to (x-1) \to \ldots$

*Proof.* Suppose that the optimal path is not of that form. We are going to modify it so that it is, without reducing the number of mushrooms collected.

Byteasar visits, on his walk, some set of vertices $\{1,\ldots,z\}$. Find in that set two adjacent vertices $x-1$, $x$ with the largest total number of mushrooms; that is, $a_{x-1} + a_x \geq a_{y-1} + a_y$, for $y = 2,\ldots,z$.

Since the path is not of the desired form, it either ends in some vertex $y \notin \{x-1, x\}$ or Byteasar turns back at one such vertex.

If the path ends in a vertex $y \notin \{x-1, x\}$, and the last step is $(y' \to y)$, remove the last two vertices $y'$, $y$ from the path and replace them with $(x-1)$, $x$ by adding a cycle $(x-1) \to x \to (x-1)$ in an appropriate place.

Similarly, if the path turns back at some vertex $y \notin \{x-1, x\}$, i.e. Byteasar walks $(y' \to y \to y')$, remove these two steps, again adding a cycle $(x-1) \to x \to (x-1)$ instead.

In neither case do we decrease the total number of mushrooms collected, as $a_{x-1} + a_x \geq a_{y'} + a_y$. By repeating the procedure, we increase the number of times the vertices $x-1$ and $x$ are visited. After at most $t$ modifications this has to end; therefore we get an optimal path of the desired form. □

Thanks to the theorem we have a simple algorithm: just check the described path for every $x$ and pick the best one!

```
Algorithm Mushrooms2(n, t, a)
  if n = 1 then
    return (⌊t/2⌋ + 1) · a_1
  s := a_1 {Mushrooms on the direct path 1...x}
  r := 0 {The best result}
  for x := 2 to min(n, t + 1) do
    s := s + a_x
    r := max (r, s + ⌈(t−x)/2⌉ · a_{x−1} + ⌈(t−x)/2⌉ · a_x)
  return r
```

The time complexity of this algorithm is $O(\min(n, t))$, whereas the space complexity is constant.

# / Sweets

**Contest:** Algorithmic Engagements 2010
**Task author:** Jakub Łącki
**Memory:** 128 MB
http://main.edu.pl/en/archive/pa/2010/cuk

To celebrate Children's Day, a mother of three brothers—Albert, Denis and Boris—gives them loads of sweets. The treats are packed in $n$ boxes, and there are $a_i$ sweets in the $i$-th box. The brothers want to share the boxes fairly, so they agree to the following rules:

- an older brother may not receive fewer sweets in total than a younger brother (Albert is older than Denis, and Denis is older than Boris),
- the difference between the total number of sweets given to Albert and the total number of sweets given to Boris should be minimal.

### Input

The first line of the input contains an integer $n$ ($3 \leq n \leq 24$), specifying the number of boxes. The second line consists of $n$ positive integers $a_i$ ($1 \leq a_i \leq 1{,}000{,}000{,}000$), specifying the number of sweets in each box.

### Output

In the only line of output you should write one integer: the difference between the numbers of sweets given to Albert and to Boris.

### Example

For the input data:

4
5 4 7 6

the correct result is:

3

# / Solution

*Sweets* is one of the problems from the final round of Algorithmic Engagements 2010 in Zielona Góra, Poland. Its short problem statement and tiny input data may suggest that the problem is easy. Not so! The trivial algorithm, described below as Sweets1, is too slow.

The solution I implemented during the contest is Sweets2. It turned out to be fast enough. After the competition, a few people who had looked at my source code were disappointed that I had not implemented what I call Sweets3 here, and thus thought that it was just a slightly optimised version of the trivial algorithm, which was never supposed to have worked within the time limit.

I kept claiming that the solution was quite a bit faster than that. In the afternoon I described my algorithm to Eryk Kopczyński. We sat down together and analyzed its asymptotic complexity. It is not bad! The analysis follows below. Later, during the problem set discussion, Eryk described an even faster algorithm, Sweets3, that the organizers had in mind.

**The first try**

If there were only two brothers, this would be an optimization version of a classic problem called *Partition*. It is known to be NP-complete. Our problem can be treated as its generalization, and is also NP-complete. Hence, we should not hope for a polynomial solution.

On the other hand, since $n \le 24$, exponential algorithms are not out of the question. It would even seem that the easiest solution, which checks all possible ways of distributing the sweets, is good enough.

```
Algorithm Sweets1(a)
    r := ∞
    for aA, aD, aB form a partition of the multiset {a_1, ..., a_n} do
        A := ∑aA; D := ∑aD; B := ∑aB
        if A ≥ D ≥ B then
            r := min(r, A − B)
    return r
```

This results in time complexity $O(3^n n)$, since there are $3^n$ partitions of a set into three disjoint subsets. The running time can be reduced to $O(3^n)$, as the partial sums $A$, $D$, $B$ can easily be computed while generating the subsets.



$3^{24}$ is about $2.8 \cdot 10^{11}$, though. Assuming we could process a billion partitions per second, this gives 280 seconds. The time limit was 2 seconds, which means we have to think of something better.

**A better solution**

Let us return for a moment to the problem *Partition*, which consists in dividing the sweets between two brothers $A$ and $B$. We will show a known algorithm that is significantly faster than the trivial $O(2^n)$ solution.

The idea is as follows. Divide the boxes into two groups. In each group we will try all possible partitions and then merge the possibilities. But instead of trying to match every partition of the first group with every partition of the second group, which would result in the running time of $O(2^n)$, we will use a smarter approach. For each partition of the first group we can quickly find the best matching partition of the second group. In order to do that, first sort all possibilities in both groups by the difference in the number of sweets received by the two brothers.

```
Algorithm Partition(a)
    {Returns the optimal partition (A, B), A ≥ B of the multiset {a_1, ..., a_n}}
    K := multiset of all partitions (A_1, B_1) of the multiset {a_1, ..., a_⌊n/2⌋}
    L := multiset of all partitions (A_2, B_2) of the multiset {a_⌊n/2⌋+1, ..., a_n}
    {|K| = 2^⌊n/2⌋, |L| = 2^⌈n/2⌉}
    sort K decreasingly by A_1 − B_1
    sort L increasingly by A_2 − B_2
    r := ∞; i := 0; j := 0
    while i < |K| and j < |L| do
        (A_1, B_1) := K[i]
        (A_2, B_2) := L[j]
        d := A_1 + A_2 − B_1 − B_2
        if d ≥ 0 then
            if d < r then
                r := d
                (A, B) := (A_1 + A_2, B_1 + B_2)
            i := i + 1
        else
            j := j + 1
    return (A, B)
```

The running time of the Partition algorithm is dominated by sorting and is equal to $O(2^{n/2}\log(2^{n/2})) = O(\sqrt{2}^n n)$. A significant improvement!

Now we can directly use this algorithm for the *Sweets* problem. Note that at least one of the brothers has to receive not more than $\lfloor n/3 \rfloor$ boxes, and the remaining sweets should be split as evenly as possible. Let us check all such possibilities.

---

**Algorithm** Sweets2($a$)
  $r := \infty$
  **for** $aX \subseteq \{a_1, \ldots, a_n\}$, $|aX| \le n/3$ **do**
    $X := \sum aX$
    $(Y, Z) :=$ Partition($\{a_1, \ldots, a_n\} \setminus aX$)
    $r := \min(r, \max(X, Y, Z) - \min(X, Y, Z))$
  **return** $r$

---

We still have to compute the time complexity. This is not easy! Let us assume, for simplicity, that $n$ is divisible by 3. After all, we could always throw in an extra one or two empty boxes, which would not change the result.

There are $\binom{n}{k}$ subsets $aX$ of size $k$. Therefore, the time complexity is

$$T(n) = \sum_{k=0}^{n/3}\binom{n}{k} O(\sqrt{2}^{n-k}(n-k)) = O\Big(n\sum_{k=0}^{n/3}\binom{n}{k}\sqrt{2}^{n-k}\Big).$$

For $1 \le k \le n/3$,

$$\binom{n}{k-1} = \frac{n!}{(k-1)!(n-k+1)!} = \frac{k}{n-k+1}\binom{n}{k} \le \frac{n/3}{2n/3}\binom{n}{k} = \frac{1}{2}\binom{n}{k},$$

$$\binom{n}{k-1}\sqrt{2}^{n-(k-1)} \le \frac{1}{\sqrt{2}}\binom{n}{k}\sqrt{2}^{n-k}.$$

In other words, each term of the sum is not greater than $1/\sqrt{2}$ of the next one. Therefore

$$T(n) = O\Big(n\binom{n}{n/3}\sqrt{2}^{2n/3}\Big(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}^2} + \frac{1}{\sqrt{2}^3} + \cdots\Big)\Big) = O\Big(n\binom{n}{n/3}2^{n/3}\Big),$$

since the geometric series converges. We can estimate the binomial coefficient using Stirling's formula:

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\big(1 + o(1)\big) = \Theta(n^{n+1/2}e^{-n}),$$



$$\binom{n}{n/3} = \frac{n!}{(n/3)!(2n/3)!} = O\Big(\frac{n^{n+1/2}e^{-n}}{(n/3)^{n/3+1/2}e^{-n/3}(2n/3)^{2n/3+1/2}e^{-2n/3}}\Big) =$$
$$= O\Big(n^{-1/2}e^0 3^{n+1}2^{-(2n/3+1/2)}\Big) = O\Big(n^{-1/2}3^n 2^{-2n/3}\Big),$$

$$T(n) = n\binom{n}{n/3}2^{n/3} = O\Big(n^{1/2}3^n 2^{-n/3}\Big) = O\Big((3/\sqrt[3]{2})^n\sqrt{n}\Big) = O(2.39^n).$$

Finally, we arrive at the time complexity of algorithm Sweets2: $O(2.39^n)$.

**Author's solution**

In the previous algorithm we used the Partition procedure directly for two of the brothers. Instead of that, we can get an even faster algorithm by applying the idea of Partition directly to all three brothers. We split the boxes into two groups $K$ and $L$, partition both groups between the three brothers in every possible way, and try to quickly merge the partitions of $K$ with appropriate partitions of $L$.

Then, we perform the following transformation. For every partition $(A_1, D_1, B_1)$ of $K$ define a point on the plane, $(x_1, y_1) = (A_1 - D_1, D_1 - B_1)$. For every partition $(A_2, D_2, B_2)$ of $L$ define the point $(x_2, y_2) = (D_2 - A_2, B_2 - D_2)$.

We are looking for a solution $A = A_1 + A_2$, $D = D_1 + D_2$, $B = B_1 + B_2$ that satisfies $A \ge D$, $D \ge B$ and minimizes the value of $A - B$. It is easy to verify that these correspond to $x_1 \ge x_2$, $y_1 \ge y_2$, and we are trying to minimize $(x_1 - x_2) + (y_1 - y_2)$.

The problem has thus been reduced to a purely geometric one! Given two sets $K'$, $L'$ of points on a plane, find two points satisfying the above conditions.

We can solve the problem using a sweep line algorithm. Sort all points by the $y$ coordinate (and then by $x$) and process them in order. Whenever we find a point of $L'$, we put it in a set $S$. When we find a point of $K'$, we find in $S$ the point $(x, y)$ with no larger $x$ coordinate and maximal sum $x + y$. This way, for each point of $K'$ we quickly find the "closest" point of $L'$, which allows us to find the optimal solution efficiently.

It remains to describe how to represent the set $S$. We store the points of $S$ in a balanced binary search tree (a red-black tree, for instance), sorted increasingly by $x$. Additionally, before we insert a point $(x, y)$ into the tree, we always check whether there is another point $(x', y')$ in there such that $x' \ge x$ and $x' + y' \le x + y$. If so, the point $(x', y')$ will no longer be needed in our algorithm, since the new point is "better". Hence, we delete all such points from the tree.

This way, our tree will always be sorted increasingly, both by $x$ and $x+y$. Thus, given a point $(x,y)$, we can quickly find in the tree the point $(x',y')$ with the largest $x' \le x$, and at the same time it will also be the point with the largest $x'+y'$.

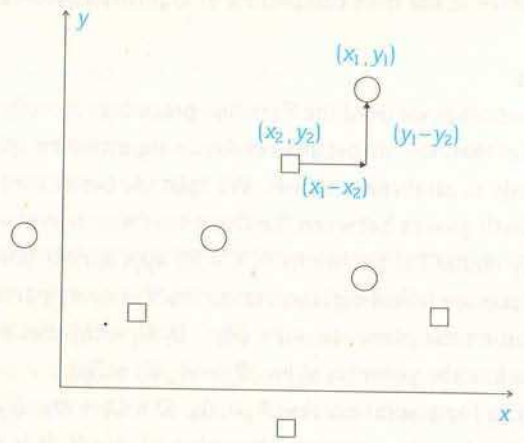



***Figure 1:*** *The circles are elements of $K'$, while the squares are elements of $L'$.*

**Algorithm** Sweets3($a$)

$K :=$ multiset of all partitions $(A_1, D_1, B_1)$ of the multiset $\{a_1, \ldots, a_{\lfloor n/2 \rfloor}\}$
$L :=$ multiset of all partitions $(A_2, D_2, B_2)$ of the multiset $\{a_{\lfloor n/2 \rfloor+1}, \ldots, a_n\}$
$\{|K| = 3^{\lfloor n/2 \rfloor}, |L| = 3^{\lceil n/2 \rceil}\}$
$K' :=$ multiset $\{(A_1 - D_1, D_1 - B_1) : (A_1, D_1, B_1) \in K\}$
$L' :=$ multiset $\{(D_2 - A_2, B_2 - D_2) : (A_2, D_2, B_2) \in L\}$
sort $K' \cup L'$ increasingly by $y$, then by $x$
$r := \infty$
$S := \varnothing$
**for** $(x, y) \in$ sorted $K' \cup L'$ **do**
  $\{S$ is sorted by both $x$ and $(x+y)\}$
  **if** $(x,y)$ comes from $K'$ **then**
    $(x', y') :=$ the point of $S$, such that $x'$ is largest among $x' \le x$
    **if** such a point exists **then**
      $r := \min(r, (x+y) - (x'+y'))$
  **else**
    **while** $S$ contains $(x', y')$, such that $x' \ge x$ and $x'+y' \le x+y$ **do**
      $S := S \setminus \{(x', y')\}$
    $S := S \cup \{(x,y)\}$
**return** $r$

The sizes of $K'$ and $L'$ are $O(3^{n/2}) = O(\sqrt{3}^n)$. All operations on $S$ work in logarithmic time and every point is added to and deleted from $S$ at most once. Therefore the time complexity of the whole algorithm is

$$O(\sqrt{3}^n \log \sqrt{3}^n) = O(\sqrt{3}^n n) = O(1.74^n).$$

**KRZYSZTOF DIKS**

Krzysztof Diks is a professor at the Institute of Informatics, University of Warsaw. He specializes in algorithmics and teaches Algorithms and data structures. Krzysztof Diks has been involved in the Polish Olympiad in Informatics since 1994, chairing its Main Committee since 1999. In 2005 he was the chairman of the Organizing Committee of the International Olympiad in Informatics that took place in Poland. He is the originator and co-organizer of the most popular algorithmic competition in Poland, Algorithmic Engagements. He is an author of many competition tasks and a keen promoter of computer science.

Since 1999, together with Professor Jan Madey, Krzysztof Diks has been mentoring teams representing the University of Warsaw in various programming competitions. In 2003 and 2007 he was a co-coach of world champions at the ACM-ICPC.

# / Coding *of Permutations*

**Contest:** 2nd Polish Olympiad in Informatics
**Task author:** Krzysztof Diks
**Memory:** 32 MB
http://main.edu.pl/en/archive/oi/2/kod

Every permutation $A = (a_1,\ldots,a_n)$ of numbers $1,\ldots,n$ can be coded by a sequence $B = (b_1,\ldots,b_n)$ in which $b_i$ equals the number of all $a_j$ such that $j < i$ and $a_j > a_i$, for $i = 1,\ldots,n$.

**Example**

The sequence $B = (0, 0, 1, 0, 2, 0, 4)$ is the code of the permutation $A = (1, 5, 2, 6, 4, 7, 3)$.

**Task**

Write a program that:

- reads from the input the length $n$ and the successive elements of the sequence $B$,
- examines whether the sequence is a code of some permutation of the numbers $1,\ldots,n$,
- if so, finds that permutation and writes it to the output, or otherwise writes to the output one word: NIE (Polish for *no*).

**Input**

In the first line of the input there is a positive integer $n$ ($n \leq 30{,}000$), representing the number of elements in the sequence $B$. Each of the following $n$ lines contains the successive elements of the sequence $B$ as nonnegative integers not greater than 30,000.

**Output**

The output should contain:

- on each of $n$ consecutive lines, one element of the permutation $A$, whose code is the sequence $B$ supplied in the input,
- one word, NIE, if the sequence $B$ is not a code of any permutation.

**Examples**

For the input data:

7
0
0
1
0
2
0
4

the correct result is:

1
5
2
6
4
7
3

whereas for the input:

4
0
2
0
0

the correct output is:

NIE



# / Solution

I would like to present two problems that I prepared for the Polish Olympiad in Informatics. At first glance, both of these problems may appear rather simple. Indeed, their straightforward solutions are easily found even by beginners. Most probably, however, such *ad hoc* approaches will run unacceptably slowly: the true challenge is in designing fast, efficient algorithms.

Both of the problems exemplify my philosophy of what constitutes good design for exercises in algorithmics. In proposing a new algorithmic task, my goal is to prepare a problem that is not only original and novel, but that also imparts important principles such as the need for proper algorithm design techniques and appropriate data structures to achieve efficiency. Additionally, a thoughtful solution to a well-designed problem should have the potential to become a useful pattern for tackling a range of tasks.

The problem *Coding of Permutations* demonstrates such characteristics. Although quite a typical problem by 2012 standards, this task embraces pedagogical values that were critical to teaching algorithmics in 1994, when access to educational resources (such as textbooks and web-based contest servers) was rather limited compared to what is available today. These values remain important even now.

The problem *Rooks* shares similar pedagogical values with *Coding of Permutations* and adds another important aspect: it encourages experimentation and playing with props, such as a chessboard or a sheet of paper.

The year 1994 was particularly important to me. In 1994, I started my adventures in popularizing algorithmics with Polish high school students; it was the beginning of my involvement with the Olympiad in Informatics. Working with algorithmically gifted youths has been very rewarding to me, and their successes have brought me tremendous enjoyment. I was very fortunate to have an opportunity to work with the future winners of the International Olympiad in Informatics and champions of the World Finals of the ACM International Collegiate Programming Contest. Watching the successful careers of past participants of programming competitions, who have become top researchers and employees of the world-leading IT companies, has brought me a lot of joy. And everything started with solving problems such as the two described below.

### An abstract point of view

A typical beginner's mistake is to look at algorithm design from the outset in implementational rather than abstract terms. Such an approach often immediately eliminates an array of potential solutions. It is much more productive to start thinking about the problem and its solution using abstract constructs (such as a set or a sequence) and abstract operations on such objects (e.g. "add an element to a set", "test if a given object is a member of a set", "return the $i$-th element in a sequence"). Expressing the algorithm in abstract terms opens a variety of options for selecting the best implementation for the problem at hand. Here, the "best implementation" is the one that results in the best running-time performance.

Let us try the approach advocated above on *Coding of Permutations*. Note that, for every $n$-element permutation $A$, the elements of its code satisfy $0 \le b_i < i$ for $i = 1, 2, \ldots, n$. This implies that there are at most $n!$ different codes, since $b_i$ can have at most $i$ different values. On the other hand, one can show that different $n$-element permutations $A'$ and $A''$ have different codes. To this end, consider the largest index $j$ such that $a'_j$ and $a''_j$ are different. Without loss of generality we can assume that $a'_j < a''_j$. Thus $b'_j > b''_j$, because all elements that are larger than $a''_j$ and are to the left of $a''_j$ in the permutation $A''$ are to the left of $a'_j$ in the permutation $A'$. Additionally, $a''_j$ is located to the left of $a'_j$. Since there are $n!$ permutations, the number of codes is also $n!$. Therefore, every sequence $B$ that satisfies the condition

$$0 \le b_i < i \quad \text{for all } i = 1, 2, \ldots, n \qquad (*)$$

is a (unique) code for some permutation.

It is not difficult to check in linear time whether a given sequence satisfies the condition (*). Thus, we will henceforth assume that $B$ encodes some permutation, and our goal is to "decode" this permutation.

Let us start with the observation that $b_n$ unambiguously identifies an element in the set of *candidates* $\{n, n-1, \ldots, 1\}$ that is the last in the permutation $A$. Indeed, $b_n$ indicates that, in the permutation $A$, to the left of $a_n$ there are exactly $b_n$ elements larger than $a_n$. Since $a_n$ is at the last position in $A$, it means that $a_n$ is equal to the $(b_n + 1)$-th element in $\{n, n-1, \ldots, 1\}$, counting from the largest. After removing $a_n$ from the set of candidates, $a_{n-1}$ is equal to the $(b_{n-1} + 1)$-th largest element in the updated set of candidates $\{n, n-1, \ldots, 1\} \setminus \{a_n\}$. Continuing, $a_{n-2}$ is equal to the $(b_{n-2} + 1)$-th largest element in $\{n, n-1, \ldots, 1\} \setminus \{a_n, a_{n-1}\}$, etc.



Below is pseudocode for the entire abstract-level algorithm.

```
Algorithm DecodePermutation((b_1, ..., b_n))
    Candidates := {n, n − 1, ..., 1}
    for i := n downto 1 do
        k := b_i + 1
        e := k-th largest element of Candidates
        a_i := e
        Candidates := Candidates \ {e}
    return (a_1, ..., a_n)
```

Once we have this abstract algorithm, we can start thinking about its implementation. Both the code $B$ and permutation $A$ will be represented by $n$-element arrays, $B[1..n]$ and $A[1..n]$, respectively. Crucial to the solution, however, is the right choice of implementation for the set *Candidates*. What do we know about this set? At the beginning, *Candidates* includes all of the numbers $1, 2, \ldots, n$. In each iteration of the loop, we find an element $e$ that is the $k$-th largest in *Candidates*, where $k$ is a number between 1 and the current cardinality of *Candidates*. Next, $e$ is removed from *Candidates*. In our solution, we will combine these two operations into a single operation *FindAndDelete*($k$). It is clear that the running-time complexity of the algorithm will depend on the representation of the set *Candidates* and the implementation of the operation *FindAndDelete*($k$).

### Implementation

**Approach 1—an array.** In this approach, we represent the set *Candidates* in an $n$-element array *Cand*[1..$n$] of 0s and 1s. *Cand*[$e$] is equal to 1 if $e$ is in *Candidates*, and 0 otherwise. With this choice of representation, *FindAndDelete* can be simply implemented as a scan of the array *Cand* from $n$ down to 1 to find the $k$-th 1 value. Deleting the found element is accomplished by assigning 0 to its corresponding position in *Cand*.

```
Function: FindAndDelete(k)
    count := 0; e := n + 1
    while count < k do
        e := e − 1
        if Cand[e] = 1 then
            count := count + 1
    Cand[e] := 0
    return e
```

It is easy to see that finding the element $e$ with *FindAndDelete* requires $n - e + 1$ positions to be scanned in the array *Cand*. Since ultimately each of the elements in *Cand* is returned by *FindAndDelete*, a total of $\sum_{e=1}^{n}(n - e + 1) = n(n + 1)/2$ positions will be inspected to decode the permutation. Unfortunately, with this approach, the cost of decoding is always of order $n^2$, independent of the code $B$.

**Approach 2—a list.** The quadratic performance of the previous approach to searching for the $k$-th element is caused by scanning both the elements that are still in the set *Candidates* (1s in *Cand*) and those that were already removed from this set (0s in *Cand*). To alleviate this problem, we can represent the set of candidates as a decreasing list of its elements. In such a representation, finding the $k$-element in the set corresponds to finding the $k$-th element of the list, and requires only $k$ elements in the set *Candidates* to be scanned—the first $k$ elements in the list. An element can be deleted in constant time. We assume that the list is constructed with fields *next*[$e$] pointing to the next element in the list, with *next*[$e$] = 0 if $e$ is the last element in the list. Access to the list is given by a pointer *Cand* to $s$ (a sentinel) with *next*[$s$] pointing to the first element in *Candidates*. Thus *FindAndDelete* can be implemented as follows:

```
Function FindAndDelete(k)
    count := 1; f := Cand
    while count < k do
        f := next[f]
        count := count + 1
    e := next[f]; next[f] := next[e]
    return e
```



In this implementation, the cost of decoding depends on the code $B$. The total number of times elements of the set *Candidates* are scanned is $\sum_{i=1}^{n}(b_i + 1)$, which has a minimum of $n$ when all $b_i$ are equal to 0, and a maximum of $n(n + 1)/2$ if $B = (0, 1, 2, \ldots, n - 1)$. Thus, with the list representation, the running time of decoding varies from linear to quadratic. What is the average cost? It depends, of course, on the distribution of data. If we assume that $B$ is a code of a random permutation, and each permutation is equally likely (that is, with probability $1/n!$), then it is possible to show that the average number of elements scanned in the changing list *Cand* is $n(n + 2)/4$.

Can we solve our problem in better time?

**Approach 3—binary search and trees.** Let us return to the representation of the *Candidate* set as a 0-1 array *Cand*[1..$n$]. Assume now that we have an "oracle" function *HowManyOnes*($l, p$) that returns the number of 1s in the subarray *Cand*[$l$..$p$]. Now, the $k$-th 1 from the right end of the array can be found with a binary search. Divide *Cand*[$l$..$p$] into two equal (up to one element) parts *Cand*[$l$..$s$] and *Cand*[$s + 1$..$p$], where $s = \lfloor (l + p) / 2 \rfloor$. If *HowManyOnes*($s + 1, p$) $\geq k$, then the search continues for the $k$-th 1 from the right in the subarray *Cand*[$s + 1$..$p$]. If, on the other hand, *HowManyOnes*($s + 1, p$) $< k$, then we continue the search for 1 from the position $k -$ *HowManyOnes*($s + 1, p$) from the right in *Cand*[$l$..$s$], since all 1s in *Cand*[$s + 1$..$p$] are excluded from the search. The algorithm is presented as a recursive function *FindAndDelete*($k, l, p$) that finds and deletes the $k$-th 1 from the right in the subarray *Cand*[$l$..$p$]. The $k$-th 1 from the right in the entire array (that is, the $k$-th largest element) can be found by calling *FindAndDelete*($k$, 1, $n$).

```
Function FindAndDelete(k, l, p)
    if l = p then
        Cand[l] := 0
        return l
    else
        s := ⌊(l + p)/2⌋
        if HowManyOnes(s + 1, p) ≥ k then
            return FindAndDelete(k, s + 1, p)
        else
            return FindAndDelete(k − HowManyOnes(s + 1, p), l, s)
```

If we could execute the function *HowManyOnes* in constant time then the total cost of *FindAndDelete*$(k, 1, n)$ would be $O(\log n)$, because in each step of the recursive search the size of the subarrays is halved (plus/minus one element). But where do we get the values of *HowManyOnes*? Actually, they can be memorized. Assume that we have an array *HowMany*$[1..n]$ such that, for $l < p$ which are possible arguments for *FindAndDelete*, *HowMany*$[s]$ = *HowManyOnes*$(s+1, p)$, where $s = \lfloor (l+p)/2 \rfloor$. Then, calls to *HowManyOnes*$(s+1, p)$ in *FindAndDelete* can be replaced with calls to *HowMany*$[s]$. And how about deleting an element $e$ from the *Candidate* set? That is also simple. Whenever the search continues in the right subarray *Cand*$[s+1, p]$, we know that a 1 will be removed from this subarray. Consequently, *HowMany*$[s]$ should be decreased by 1. The above ideas are summarized in the following function:

```
Function FindAndDelete(k, l, p)
    if l = p then
        Cand[l] := 0
        return l
    else
        s := ⌊(l + p)/2⌋
        if HowMany[s] ≥ k then
            HowMany[s] := HowMany[s] − 1
            return FindAndDelete(k, s + 1, p)
        else
            return FindAndDelete(k − HowMany[s], l, s)
```

We still have to show how to initialize the array *HowMany*. Since at the beginning *Cand* contains only 1s, the task is not difficult. A natural recursive function to initialize *HowMany* is given below:

```
Function InitHowMany(l, p)
    if l < p then
        s := ⌊(l + p)/2⌋
        HowMany[s] := p − s
        InitHowMany(l, s)
        InitHowMany(s + 1, p)
```



We leave the proof that *InitHowMany*$(1, n)$ has linear cost as an exercise for the reader. The interested reader should have noticed that, in the last approach, the array *Cand* is not needed, except perhaps for clarity of presentation.

The problem presented above is one of the easiest problems in this book. It is not hard to notice the presence of inversions in this problem, which represent the difficulty of the insertion sort algorithm. An inversion in a sequence is a pair of numbers in the sequence that are not in order. The cost of the insertion sort algorithm is proportional to the length of the sequence and the number of inversions within it.

Despite being simple, the problem allowed us to demonstrate a strategy for tackling algorithmic problems to obtain efficient solutions (efficient running time). The strategy is to start with the development of an algorithm in abstract terms and then select the best implementation for the abstract functions and objects. Additionally, the problem can be used to introduce and illustrate a variety of data structures (arrays, lists, trees), search algorithms (linear, binary), recursion and the concept of the running-time complexity (worst-case and average).

**Reversing the problem**

Finally, let us discuss efficient solutions for the reversed problem: given a permutation $A$, find its code $B$. A natural approach is to scan the permutation $A$ from left to right and count, for each of its elements $a_i$, the number of elements in $A$ larger than $a_i$ and to the left of $a_i$ in $A$. A straightforward implementation of this approach has $O(n^2)$ running time. A better implementation could use balanced binary trees to obtain running time $O(n \log n)$. Below we will show yet another implementation with running time $O(n \log n)$. The solution, based on an idea proposed by Professor Wojciech Rytter, does not use any complex data structures.

Occasionally, it is easier to solve and prove the correctness of a generalized version of a problem then to do so for the original, more specific problem. Let us consider the following, more general task.

***Task.*** An array $A[1..n]$ of integers from the interval 0 to $n-1$, not necessarily distinct, is given. Find an array $B[1..n]$ such that $B[j]$ is the number of elements in $A$ that are larger than $A[j]$ and are to the left of $A[j]$ in $A$; that is,

$$B[j] = \left|\{1 \le k < j : A[k] > A[j]\}\right|.$$

Note that if $A$ is a permutation of $\{1,\ldots,n\}$ then, after subtracting 1 from each of the $A[j]$ and finding $B$, we obtain the code for the permutation $A$.

We will solve the above problem using the following observation:

***Observation.*** For any nonnegative distinct integers $a < b$ the following holds true: $\lfloor a/2 \rfloor = \lfloor b/2 \rfloor$ if and only if $a = 2k$ and $b = 2k + 1$, for some integer $k \geq 0$.

What does this observation imply? If we divide (integer division) each $A[j]$ by 2, then the number of pairs $(A[l], A[p])$ such that $A[l] > A[p]$ and $l < p$ will not change, not counting pairs that, before the division operation, were $A[l] = 2k + 1$ and $A[p] = 2k$, for some integer $k \geq 0$. This observation can be used in our solution.

While the array $A$ contains at least one element larger than 0, for each even $A[j]$ compute the number of elements (they are odd) that are larger by 1 than this element and are to its left. Add this number to the total number of elements larger than the element at position $j$ in the original array $A$ and located to its left (in positions with indices smaller than $j$). Next, divide (integer division) each element in $A$ by 2 and repeat the above process. The program below summarizes the above ideas. The array $Odd[0..n]$ stores the count of odd numbers that are present in the (changing) array $A$ during the scanning of this array from left to right.



```
Algorithm EncodePermutation(A)
    for i := 1 to n do
        B[i] := 0
    HowManyZeros := 0
    while HowManyZeros < n do
        for i := 0 to n do
            Odd[i] := 0
        HowManyZeros := 0
        for i := 1 to n do
            if 2 ∤ A[i] then
                Odd[A[i]] := Odd[A[i]] + 1
            else
                B[i] := B[i] + Odd[A[i] + 1]
            A[i] := A[i] div 2
            if A[i] = 0 then
                HowManyZeros := HowManyZeros + 1
    return B
```

Each iteration of the **while** loop takes $O(n)$ time. Since the number of iterations is at most $\lceil \log(n + 1) \rceil$, the resulting running time of the entire algorithm is $O(n \log n)$.

# / Rooks

**Contest:** 3rd Polish Olympiad in Informatics
**Task author:** Krzysztof Diks
**Memory:** 32 MB
http://main.edu.pl/en/archive/oi/3/wie

On a chessboard of size $n \times n$ we place $n$ rooks. The arrangement of the rooks should satisfy the following rules:

- For each $i = 1,\ldots,n$, the $i$-th rook can be placed within a rectangle specified by two pairs of coordinates: $(a_i, b_i)$, $(c_i, d_i)$, where $(a_i, b_i)$ are the coordinates of the square in the upper left corner of the rectangle (row, column), $(c_i, d_i)$ are the coordinates of the square in the lower right corner of the rectangle, $1 \le a_i \le c_i \le n$ and $1 \le b_i \le d_i \le n$. The square in the upper left corner has coordinates $(1, 1)$; the square in the lower right corner has coordinates $(n, n)$.
- No two rooks can attack each other, i.e. they cannot appear in the same row or the same column.

## Task

Write a program that:

- reads from the input the size of the chessboard $n$ and for each $i = 1,\ldots,n$ the coordinates of the rectangle in which the $i$-th rook can be placed,
- verifies whether the rooks may be placed in appropriate rectangles such that no two of them may attack each other and, if so, finds one such arrangement,
- writes to the output one arrangement of rooks that satisfies the given conditions, or the single word NIE (Polish for *no*) if no such arrangement exists.

## Input

In the first line of the input there is a positive integer $n$ ($1 \le n \le 3000$). In each of the following $n$ lines appear four positive integers not greater than $n$, separated by single spaces. The numbers in the $i$-th line are the coordinates of the rectangle in which the $i$-th rook may be placed ($a_i$, $b_i$, $c_i$ and $d_i$ respectively).



## Output

The output should comprise either one word, NIE, or a series of $n$ lines on each of which are written two integers separated by a single space. The numbers in the $i$-th line should specify the position of the $i$-th rook (row, column). Each such position should be within the rectangle specified by the coordinates in the $(i + 1)$-th line of the input. Pay attention to the fact that the positions of the rooks should be written in the same order as the coordinates of the rectangles are specified in the input.

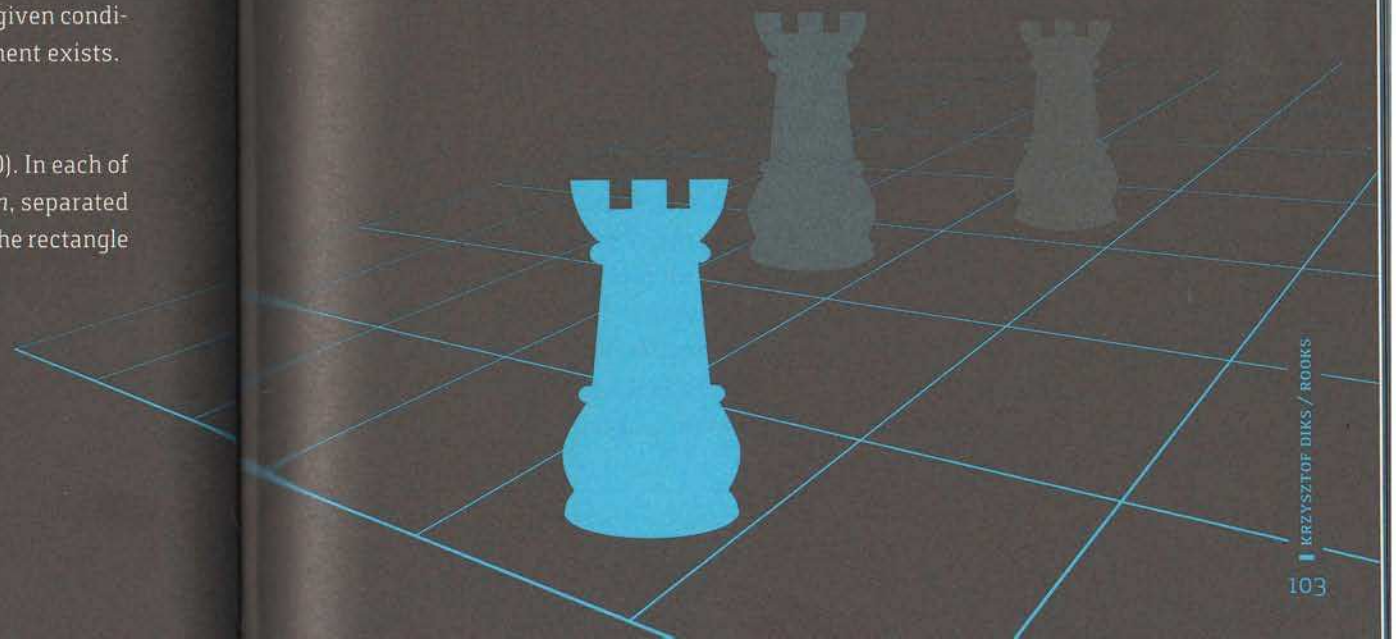
## Example

For the input data:

```
4
1 1 1 1
1 3 2 4
3 1 4 2
2 2 4 4
```

the correct result is:

```
1 1
2 3
3 2
4 4
```



# / Solution

Although the problem is formulated in two dimensions, we will show that it can be reduced to one dimension. Note that if a desired arrangement of rooks exists—call it an *admissible* arrangement—then there is at most one rook in each row and, similarly, at most one in each column. Let $W_i$ denote the $i$-th rook. Indices of the rows where $W_i$ can be placed belong to the closed interval $[a_i..c_i]$ of integers, and indices of columns where $W_i$ can be placed are in the interval $[b_i..d_i]$. Let $[w_1, w_2, \ldots, w_n]$ be an arbitrary permutation of the row indices (numbers $1,\ldots,n$) such that, for each $i$, $w_i \in [a_i..c_i]$; that is, an admissible choice of rows for the rooks. Similarly, let $[k_1,k_2,\ldots,k_n]$ be an arbitrary permutation of the column indices (numbers $1,\ldots,n$) such that $k_i \in [b_i..d_i]$; that is, an admissible choice of columns for the rooks. Then a sequence of pairs $[(w_1,k_1),(w_2,k_2),\ldots,(w_n,k_n)]$ is an admissible arrangement of the rooks on a chessboard. Moreover, every admissible arrangement of the rooks on a chessboard is a combination of an admissible choice for rows and an admissible choice for columns. Therefore, it is sufficient to solve the following problem:

***Problem.*** A positive integer $n$ and $n$ intervals $J_i = [l_i..r_i]$, $1 \le l_i \le r_i \le n$, of integer numbers are given. The task is to check whether there is a permutation $[p_1, p_2, \ldots, p_n]$ of $1, 2, \ldots, n$ such that each $p_i \in [l_i..r_i]$. The number $p_i$ will be called a *representative of the interval* $J_i$. In other words, we are asking if there is a sequence of different representatives for the intervals $J_i$. Provide one such sequence if any exists.

Instead of assigning representatives to intervals, let us reverse the assignment. Each representative $1, 2, \ldots, n$ will be assigned an interval to represent. What interval can safely be assigned to 1, and what does "safely" mean? The answer is simple. The number 1 can be assigned to an interval such that the remaining intervals can be assigned to the representatives $2, \ldots, n$. Firstly, the interval has to contain 1. There may be many such intervals, however, so which one to choose? It is not hard to see that the safest choice is to select the shortest interval; that is, the one with the smallest right endpoint. Why? Assume that $J = [1..r]$ is the shortest interval containing the element 1 and, in some admissible assignment of intervals to the representatives, $J$ is assigned to $x > 1$. Now, let $J' = [1..r']$ be an interval assigned to 1. Since $J$ is the shortest, $r' \ge r$. After swapping the assignments for $J$ and $J'$, we obtain an admissible assignment with $J$ assigned to 1.



Having an interval for the representative 1, we turn to 2. To 2 we assign the shortest interval containing 2, not counting the one selected earlier for 1. We repeat this process for 3, 4 and so on, selecting from the intervals not previously assigned.

This process can be formalized as follows:

```
Algorithm ArrangeRooks
  S := ∅
  for i := 1 to n do
    P := a set of intervals with the left endpoint in i
    S := S ∪ P
    if S = ∅ then
      print NO; exit
    else
      I := an interval in S with the smallest right endpoint
      set i as the representative for I
      S := S \ {I}
```

It remains to be decided how to implement this algorithm. In essence, there are two issues to address: how to compute the set $P$ quickly, and how to implement the dynamic set $S$ and operations on it.

Computing $P$ is not a complex task. It is sufficient to sort all the intervals with respect to the left endpoints and group together intervals with the same left endpoint. Since the endpoints are integers from $[1..n]$, such sorting can be done in $O(n)$ time.

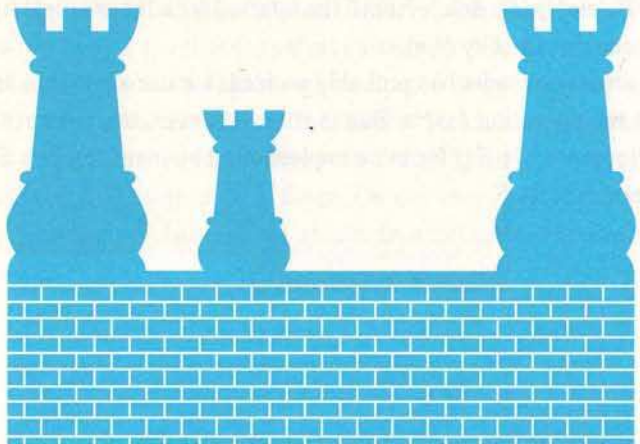
Now let us take a look at the set $S$. It consists of intervals with integer endpoints from $[1..n]$. The following operations are performed on $S$:

- *Init*($S$): initialize $S$ as an empty set,
- *Insert*($S, I$): insert interval $I$ to $S$,
- *Min*($S$): return a pointer to the interval in $S$ with the smallest right endpoint,
- *DeleteMin*($S$): delete from $S$ the interval with the smallest right endpoint (the one returned by *Min*).

An attentive reader has probably noticed that our algorithm does not explicitly call the operation *Insert*. This is true; however, the abstract operation in the assignment $S := S \cup P$ can be implemented by inserting into $S$ the consecutive intervals from $P$.

A set $S$ with the operations *Init*, *Insert*, *Min* and *DeleteMin* is nothing more than a priority queue organized with respect to the right endpoints of the intervals. Such a priority queue can either be implemented directly with a heap (a data structure used, for example, in the heapsort algorithm) or using an implementation from STL. The running time of the *Insert* and *DeleteMin* operations in such implementations is $O(\log k)$, where $k$ is the number of elements in the priority queue; and the operation *Min* requires constant time. Consequently, the worst-case running time of the above implementation of the algorithm for finding interval representatives is $O(n \log n)$. Since we reduced our original problem to two independent tasks for intervals, we have obtained a solution for the *Rooks* problem with $O(n \log n)$ running time.

Finally, the *Rooks* problem offers yet another advantage: the task can be viewed as a good introduction to the highly interesting problem of finding representatives for families of finite sets that are not necessarily intervals of integers. This general problem—known as a *system of distinct representatives*—can be solved, for example, with graph matching algorithms; but that's another story.



**ANDRZEJ GĄSIENICA-SAMEK**

Andrzej Gąsienica-Samek graduated from the Faculty of Mathematics, Informatics and Mechanics, University of Warsaw with a degree in informatics. He is a three-time winner of the Polish Olympiad in Informatics, and won one silver and three gold medals at the International Olympiad in Informatics.

In 2003 Andrzej Gąsienica-Samek and his team ranked first in the ACM-ICPC World Finals in Beverly Hills. Two years earlier, at the ACM-ICPC World Finals in Vancouver in 2001, he and his team had achieved sixth place.

For five years Andrzej Gąsienica-Samek worked at Comarch SA, one of the largest Polish IT business solutions providers, where he held positions of department director and R&D center director. During his time there he was responsible for managing departments covering R&D and software deployment in the public administration sector. Since 2008 he has been CEO at Atinea, a company that deals primarily with tailor-made IT solutions and currently employs 30 programmers. Its core activity encompasses R&D in the field of flexible database systems.

Andrzej Gąsienica-Samek is interested in digital intelligence. Even though he claims that the current state of scientific knowledge in this sphere has little to do with actual intelligence, his long-term goal is to create a program that would be capable of "thinking" on its own.

# Monkeys

**Contest:** 10th Polish Olympiad in Informatics
**Task author:** Andrzej Gąsienica-Samek
**Memory:** 32 MB
http://main.edu.pl/en/archive/oi/10/mal

There are $n$ monkeys on a tree, numbered from 1 to $n$. Monkey number 1 clutches a branch by its tail. The remaining monkeys are held by other monkeys, or hold on to other monkeys, or both. Each monkey can use two hands and can hold at most one other monkey in each hand (gripping its tail). Starting from moment 0, every second, one monkey releases the grip of one hand. This may cause some monkeys to fall down to the ground, where they can continue releasing their grips (the time taken to fall is negligibly small).

### Task

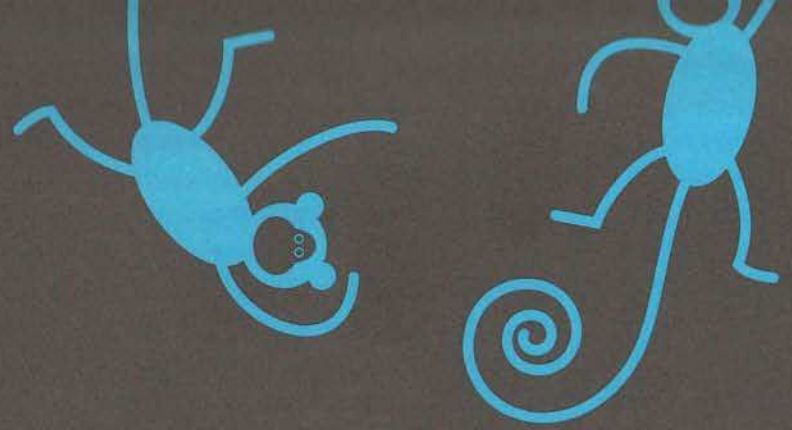
Write a program that:

- reads from the input the description of how the monkeys hold together, and in what order they release their grips,
- for each monkey, computes the time when it falls down to the ground,
- writes the result to the output.

### Input

The first line of the input consists of two positive integers, $n$ and $m$ ($1 \le n \le 200{,}000$, $1 \le m \le 400{,}000$). The number $n$ specifies the number of monkeys and the number $m$ specifies the time (in seconds) when we observe the monkeys. The next $n$ lines contain the description of the initial situation. In the $(k+1)$-th line ($1 \le k \le n$) there are two integers specifying the numbers of the monkeys that are held by monkey number $k$. The first integer is the number of the monkey that is held by the left hand; the second is the number of the monkey held by the right hand. A value of $-1$ indicates that the monkey's hand is free. The following $m$ lines record the result of the observation of the monkeys. In the $i$-th of those lines ($1 \le i \le m$) there are two integers. The first is the number of the monkey and the second is the number of the hand (1 = left, 2 = right) with which the monkey releases its grip in the moment $i-1$.





### Output

Your program should write to the output exactly $n$ integers, one per line. The number in the $i$-th line should specify the moment the $i$-th monkey fell down to the ground, or should be $-1$ if the monkey did not fall to the ground during the observation.

### Example

For the input data:

```
3  2
-1  3
3  -1
1  2
1  2
3  1
```

the correct result is:

```
-1
1
1
```

# / Solution

The problem *Monkeys* was inspired by the mechanism of *garbage collection* that was made popular by the introduction of the Java programming language in the late nineties. Garbage collection frees memory that is no longer referenced by the running program.

Many programmers confuse memory management with resource management. The fundamental difference between these two mechanisms is in the requirements: unlike the release of memory, which may happen with a delay unknown to and unpredictable by a programmer, files or network connections must be released at precisely determined points.

A good understanding of the problem *Monkeys* helps to explain why the mechanism of garbage collection is not efficient for general resource management. In addition, the problem throws light on the misconception that garbage collection solves the issue of memory leaks.

**Monkeys and garbage collectors**

Let us try to model garbage collection using the terminology of object-oriented languages. Specifically, we have:

- a stack of activation records,
- objects,
- references (to objects), which can be object fields or variables allocated on the stack.

A program has access to only those objects that are referenced through a path of references that originates in the stack. Other objects are no longer accessible to the program and should be released (for example, see objects *D* and *E* in figure 1).

What is the connection between the above model and the *Monkeys* problem? The monkeys represent objects, the stack corresponds to the tree on which monkey 1 is hanging, and references correspond to the hands with which monkeys hold the tails of other monkeys.

The problem *Monkeys* asks for the exact moment when a group of monkeys falls. This question is analogous to asking when an object becomes unavailable to a program. In a similar way to knowing the exact moment when each of the monkeys drops down, we would like to release memory as soon as it becomes unreferenced. This would allow us to apply garbage collection to general resource management.



**Garbage collection and resource management**

Attempts to employ a garbage collector in general resource management lead immediately to efficiency issues, since finding which objects may be released is not a simple task. This stems from the fact that a change to a single reference may potentially allow us to release an arbitrarily large number of other objects. For example, if a stack frame for some function contains a list object that, in turn, contains a thousand other objects, then upon a return from this function the only reference to the list will be lost and the list object, as well as all the objects in this list, should be released.

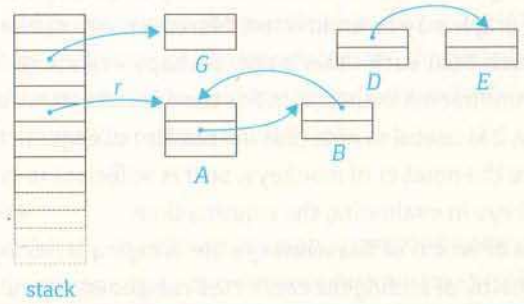



***Figure 1:*** *A schematic view of garbage collection. Rectangles represent objects; arrows represent references.*

The first idea that comes to mind is to release the object as soon as the reference to the object is popped from the stack, or if this reference is modified and starts pointing to another object. Of course, such a strategy would be effective for the aforementioned example of the list, but in general more than one reference can point to the same object.

Refining the above idea, we could associate with each object a *reference counter* to store the number of references pointing to that object, and update the counter each time references change. Thanks to reference counters, we would be able to release memory bound to objects as soon as the last reference disappeared, and a similar approach would also work for resource management. In fact, this strategy is used by some programming languages; for example, Perl.

Unfortunately, chains of references may form cycles. Object *A* may refer to some object *B* that, in turn, has a reference to *A*. With reference counters, we would never release objects *A* and *B* (there are references to both of them) even if they become unavailable to the program (in the example from figure 1, this would happen after reference *r* had been popped from the stack).

Identification of objects that are no longer available to the program requires all the objects and references to be checked. Such a process is costly and is therefore performed by garbage collectors only rarely, which leads to delays in releasing memory.

Let us go back to the problem *Monkeys* to see how much time it takes to identify monkeys that fall.

**Monkeys and graph theory**

The monkeys from our problem can be modeled as a graph, with the monkeys represented as vertices and the hands that hold tails as edges. Note that it does not make any difference whether a monkey is holding a tail or having its tail held, so the graph may be undirected. Moreover, one can imagine that two monkeys can each hold each other's tail, perhaps even with both hands, so the graph is an undirected multigraph. For the sake of estimating the running time complexity, it is useful to note that the number of edges in this graph does not exceed twice the number of monkeys, so it is sufficient to consider just the number of monkeys in evaluating the running time.

The question of which of the monkeys are hanging is, using graph theory terminology, a matter of finding the connected component that includes monkey 1. Less formally, starting with monkey 1, we want to visit all monkeys reachable through hands holding tails. This can simply be accomplished with a depth-first search: that is, recursively traversing the hands and memorizing all the visited monkeys.

The running time complexity of such a traversal is proportional to the size of the connected component containing monkey 1. This number is $O(n)$ since, as we noted above, the number of hands is proportional to $n$. Unfortunately, this operation is repeated each time any monkey releases its grip on a tail, resulting in an $O(n^2)$ cost. From the description of the input size, it is clear that an $O(n^2)$ solution is unacceptable for the *Monkeys* problem.

**Returning to a garbage collector**

In view of the above discussion, it makes sense to analyze how garbage collectors do their job, knowing that detecting objects unreachable from the program is so expensive. Firstly, garbage collection does not happen after each change of references, which may unfortunately result in substantial delays in releasing memory. Therefore, if the object has methods such as *close* or *dispose*, the programmer should always use them before losing the last reference to the object. Otherwise, nondeterministic results, depending on the garbage collector, are possible.



Secondly, developers of garbage collectors are focused more on efficiency for cases that can occur in practice than on the asymptotic complexity. Therefore, there are scenarios where the time taken by garbage collection is hard to predict. However, a similar comment applies to the *malloc* and *free* (*new* and *delete*) operations, where attaining good asymptotic efficiency is also not the primary concern.

On the other hand, taking a theoretical rather than empirical view, it is valid to ask about the best running time complexity for garbage collection. Is good efficiency at all possible? It appears that garbage collection can be accomplished in linear time with respect to the size of allocated memory. A simple argument to this end is that, assuming the availability of twice as much memory as is needed for processing and performing garbage collection after space is fully exhausted, the cost of garbage collection can be amortized into the cost of allocation. More interesting problems related to garbage collection can be found by browsing information about *real-time garbage collection*.

**A better solution**

Reasons that inhibit the use of memory management for general resource management have been presented in the above sections. In the problem *Monkeys*, we are asked to identify the exact moments when monkeys fall, and we know that a $O(n^2)$ solution is unacceptable. Let us then take another look at the assumptions of *Monkeys*, with an eye on what makes the problem different from memory management.

The first observation is that monkeys have two hands, unlike the multiple reference fields that are possible for objects. Even so, each object could be split into several pairs (in a similar way as in the programming language Lisp) and not much would change in the conditions for garbage collection. Similarly, we can imagine collapsing two monkeys into one monkey with two tails and three hands, which does not change the problem either.

The second observation is that, during the process, after releasing its grip, a monkey never catches any other tail with its hands. This observation is more interesting than the previous one. If we stop considering falling monkeys, and instead start thinking in terms of a graph from which edges are being removed, then the scenario becomes that of a graph that is slowly decomposing into smaller connected components. Our concern is with the connected component that includes vertex 1. It is computationally costly to find what part of this connected component becomes detached after removing an edge, as it requires all vertices connected to 1 to be visited. They are visited every time an edge is removed, and that contributes to the cost.

However, if instead of splitting the graph we try to build it, the process becomes much more efficient. If all the vertices in the connected component containing vertex 1 are properly marked, we can easily detect if a newly added edge attaches anything new to this component. In constant time, we can check whether both ends of the edge belong to the connected component of vertex 1. If so, nothing else needs to be done. Otherwise, in time proportional to the number of vertices being attached, we mark them as members of the connected component of vertex 1. There is no need to revisit vertices already belonging to this component. Consequently, the operation of adding an edge takes time proportional to the number of attached vertices.

Now, if the entire process is executed backwards—that is, monkeys on the ground are consecutively attached to the tree—then each step takes time proportional to the number of monkeys attached in that step. In the result, this simple solution has $O(n)$ running time, the best possible in view of the input size.

It is important to note that a similar trick is not applicable to memory management, which is much less deterministic than the sequence of events with monkeys. For that reason, programmers should always remember to call explicit *close* or *dispose* methods on objects that provide them. The garbage collector has no way to predict the precise moment when the last reference will disappear. Similarly, we must be careful with a single individual monkey that carries the entire group of monkeys on its tail. In such a case, not even a garbage collector would protect us from memory leaks.

**TOMASZ IDZIASZEK**

Tomasz Idziaszek is a research assistant at the Faculty of Mathematics, Informatics and Mechanics, University of Warsaw. His research focuses on automata theory, with emphasis on connections between logic and tree automata. He has been involved in programming competitions for eleven years. He is a finalist of several individual contests (TopCoder, Google Code Jam, Algorithmic Engagements) as well as team competitions (with Szymon Acedański and Jacek Jurewicz he won the ACM-ICPC Central European Regional Contest in 2004 and represented the University of Warsaw at the ACM-ICPC World Finals). Currently he is more involved in organizing events such as the Polish Olympiad in Informatics, Polish Olympiad in Informatics Training Camp, Polish Collegiate Programming Contest and Algorithmic Engagements (the latter giving him the opportunity to write algorithmic problems for the most challenging round of the contest, which is what he enjoys best).
Moreover, Tomasz Idziaszek is an author and editor of computer science articles in the Polish popular science monthly, *Delta*. In his spare time he enjoys playing the piano.

# / Plotter

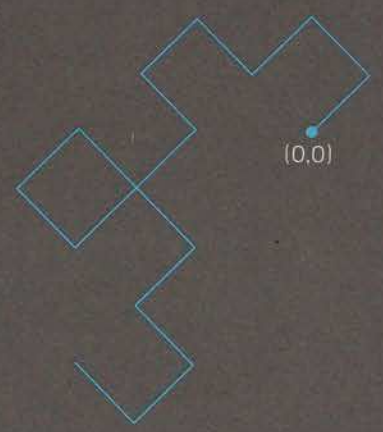
**Contest:** Algorithmic Engagements 2011
**Task author:** Tomasz Idziaszek
**Memory:** 32 MB
http://main.edu.pl/en/archive/pa/2011/plo

To test the functionality of his newly-purchased plotter, Byteasar decided to plot a few bytecurves. A bytecurve of order $n$ consists of $2^n$ segments of length $\sqrt{2}$ each. The first of them connects the points with coordinates $(0,0)$ and $(1,1)$. A bytecurve of order $n$ can be described with a word $L_n$ of length $2^n - 1$ based on a two-letter alphabet {**L**, **R**}. The word's $i$-th letter indicates that, after the $i$-th segment has been printed, the pen turns and moves orthogonally to the left (letter **L**) or to the right (letter **R**) to print the next segment. In other words, the pen changes its direction to the left or to the right by 90°.

$L_1$ consists of a single letter **L** (left turn) and $L_2$ consists of three letters **LLR**, indicating two left turns followed by one right turn. Inductively, $L_n$ is generated from $L_{n-1}$ in the following way: separate all letters in $L_{n-1}$ with spaces and place an additional space in front and at the end of $L_{n-1}$. Then, in the newly-created spaces, place alternating letters **L** and **R**, beginning with **L**. For example, $L_3$ is generated as follows:

$$L_2 = \mathbf{LLR} \rightarrow _\mathbf{L}_\mathbf{L}_\mathbf{R}_ \rightarrow \underline{\mathbf{L}}\mathbf{L}\underline{\mathbf{R}}\mathbf{L}\underline{\mathbf{L}}\mathbf{R}\underline{\mathbf{R}} = L_3.$$

Similarly, $L_4$ = **LLRLLRRLLLRRLRR** (see the figure below).




It takes exactly one second to draw one segment of the bytecurve. While waiting for the plotter to complete printing, Byteasar ponders the following question: given a point $(x, y)$ on the plane, when will the pen be positioned at that point? For example, for the bytecurve of order 4 in the above figure, the pen will be at point $(-3, -1)$ seven seconds after the beginning of plotting, and again eleven seconds after the beginning of plotting. Your task is to answer Byteasar's question.

## Input

The first line of the input contains two integers, $n$ and $m$ ($1 \le n, m \le 2000$), where $n$ indicates that the bytecurve is described by $L_n$ and $m$ indicates that there are $m$ query points. The next $m$ lines each contain two integers, $x_i$ and $y_i$ ($-10^9 \le x_i, y_i \le 10^9$), indicating the coordinates of the $i$-th query point. Note that the coordinates in any particular input line are not guaranteed to represent a point on the bytecurve. Moreover, no point appears more than once in the input.

## Output

The output should consist of $m$ lines containing answers to the consecutive queries. Each answer consists of a nonnegative integer $k_i$, specifying the number of visits the pen makes to the query point $(x_i, y_i)$; followed by $k_i$ nonnegative integers indicating the times of those visits in increasing order, in seconds since the beginning of plotting. Numbers should be separated by single spaces.

## Example

For the input data:

```
4  3
-3  -1
1  1
-1  0
```

the correct result is:

```
2  7  11
1  1
0
```

# / Solution

The bytecurve described in the problem is in fact a fractal, known to a wider audience as a *dragon curve*. It was first studied in 1966 by two physicists from NASA, John Heighway and William Harter, but it was popularized by Martin Gardner in his magazine column *Mathematical Games* and by writer Michael Crichton in his novel *Jurassic Park*.

Figure 1 illustrates how the dragon curve earns its name by showing a few curves of low orders. It can be proven that the dragon curve does not intersect itself (though it can touch itself); therefore the plotter's pen will visit every point at most two times. We denote by $S_n$ the dragon curve of order $n$. We also assume that the first segment of the curve joins points (0,0) and (1,0); thus the points from the input have to be transformed using the formula

$$(x',y') = \left(\frac{x+y}{2}, \frac{y-x}{2}\right).$$

The dragon curve has many equivalent definitions; in the problem statement we encounter just one of them. Another definition states that the word $L_n$ describing the curve satisfies the following recurrence relation:

$$L_1 = \mathbf{L}, \quad L_n = L_{n-1}\,\mathbf{L}\,r(L_{n-1}), \tag{1}$$

where $r$ denotes an operation that reverses the order of letters in the word and simultaneously interchanges letters **L** and **R**; e.g. $r(L_2)=$ **LRR**. We leave the proof of equivalence of both definitions as an exercise for the reader.

Let us denote by $(x_n, y_n)$ the last point on the curve of order $n$. It is easy to show the recurrence that calculates it:

$$x_1 = y_1 = 1, \quad x_n = x_{n-1} - y_{n-1}, \quad y_n = x_{n-1} + y_{n-1}.$$

The formula (1) draws our attention to a very important property of the dragon curve, namely that the curve of order $n$ is obtained by joining two curves of order $n-1$. Speaking more precisely: the curve $S_n$ consists of a curve $S_{n-1}$ and its copy, fixed at point $(x_n, y_n)$ and rotated by 90° clockwise (see again figure 1).



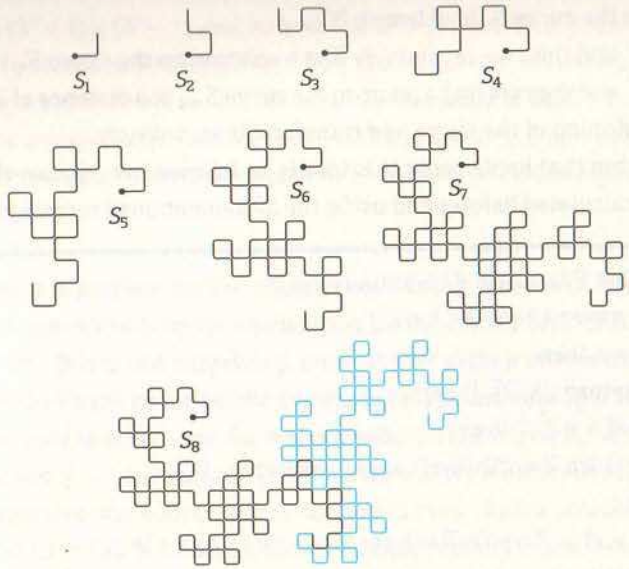



*Figure 1: The family of dragon curves; $S_n$ is the curve of order n. For $S_8$ we have marked out its two component $S_7$ curves.*

**Why do we like fractals?**

Writing a program that generates a picture of a fractal is an ideal task for a beginner programmer. Although the rules of generating fractals are simple, they lead to complicated patterns. With relatively little programming effort one can achieve quite ambitious visual effects. Moreover, the self-similarity of fractals allows us to exercise one of the most fundamental programming concepts—recursion.

It is no wonder that authors of programming contest problem sets like to look to the world of fractals for inspiration. One of the popular kinds of tasks is generating space-filling curves, such as the Peano curve, Hilbert curve or their variants. The size of such curves grows exponentially in the order of the curve, therefore the task is usually stated as follows: for the curve of order $n$ and a number $k$, calculate the coordinates of a point in the plane that lies at a distance of $k$ from the start of the curve. It is expected that the program will run in time complexity $O(n)$.

Usually, such tasks are not conceptually difficult; for example, if we wish to solve such a problem for the dragon curve $S_n$, we need to consider two cases (recall that the curve $S_n$ is of length $2^n$):

(a) $k < 2^{n-1}$ and then we recursively find a solution on the curve $S_{n-1}$,

(b) $k \geq 2^{n-1}$ and then we find a point on the curve $S_{n-1}$ at a distance of $2^n - k$ from the beginning of the curve and transform it accordingly.

The algorithm that implements this idea is as follows (we assume that values $x_i, y_i$ were calculated beforehand using the aforementioned recursion):

```
Algorithm PointOnTheDragonCurve(k, n)
    {we assume that 0 ≤ k ≤ 2^n}
    if n = 1 then
        return (⌈k/2⌉, ⌊k/2⌋)
    else if k < 2^(n-1) then
        return PointOnTheDragonCurve(k, n − 1)
    else
        (x, y) := PointOnTheDragonCurve(2^n − k, n − 1)
        return (x_n + y, y_n − x)
```

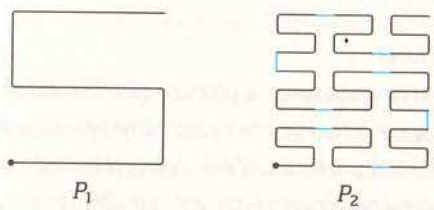



*Figure 2: A Peano curve of order n is constructed by joining nine curves of order $n-1$.*

**Converse problem**

The task can be stated in reverse: given a point in the plane $(x, y)$ that lies on the curve, we are asked at what distance from the start of the curve it lies. Before we show how to solve this problem for the dragon curve, we look at the solution for another fractal: the Peano curve (also known as a second Hilbert curve; see figure 2).

This task appeared in the ACM-ICPC World Finals in 2003 (see the task entitled *Riding the Bus*). Its solution is fairly easy: a Peano curve $P_n$ of order $n$ fits on a grid of size $(3^n - 1) \times (3^n - 1)$, and its length is $9^n - 1$. Considering values $\lfloor x/3^{n-1} \rfloor$ and $\lfloor y/3^{n-1} \rfloor$, we are able to decide to which of nine smaller curves $P_{n-1}$ the point $(x, y)$ belongs. If it belongs to the $i$-th curve (in the order of their appearance in $P_n$) then the answer is the sum of $i \cdot 9^{n-1}$ and a solution to a subtask for curve $P_{n-1}$ and accordingly rotated point $(x \bmod 3^{n-1}, y \bmod 3^{n-1})$. The algorithm runs in time $O(n)$.

The converse problem for the dragon curve is the essence of the task *Plotter*. The task proved to be quite challenging for the contestants of Algorithmic Engagements. This is not surprising, since at first sight it seems that the idea we used for the Peano curve will be useless here. We lack an important piece of knowledge: how to determine, for a given point on the curve $S_n$, which of two smaller curves $S_{n-1}$ it belongs to. Therefore we will try a different approach: let us call a recursive procedure for both smaller curves. Such a solution will run, of course, in time $O(2^n)$. Observe, however, that in many cases, when we look for a point on the wrong curve of smaller order, after a few recursive descents our point will lie so far from the currently examined part of the curve that we will be able to detect it easily and prune the search.

Let us see how this idea works in practice. Denote by $d(x, y, n)$ a function that returns a set of distances at which the point $(x, y)$ lies from the start of the curve $S_n$ (the given point can lie at no more than two distances from the start of the curve). The function will make two recursive calls. Its value is a sum of two sets:

$$d(x, y, n) = d(x, y, n-1) \cup \{2^n - k : k \in d(y_n - y, x - x_n, n-1)\}.$$

For $S_1$ we have

$$d(x, y, 1) = \begin{cases} \{x + y\} & \text{for } 0 \leq y \leq x \leq 1, \\ \varnothing & \text{otherwise.} \end{cases}$$

When can we prune the search? We cannot easily check *whether* a point belongs to the curve $S_n$, but for many points we can say that they are, without doubt, *not* on the curve. Indeed, consider a bounding box $R_n$, i.e. the smallest rectangle with sides parallel to the axes of the coordinate system that contains all points of the curve $S_n$. Every time the function $d(x, y, n)$ is called with a point

$(x, y)$ that lies outside the bounding box $R_n$, we can immediately return $\varnothing$. Let $r_n$, $t_n$, $l_n$, $b_n$ denote the distances of point $(0,0)$ from the sides of the rectangle $R_n$ (right, upper, left and lower side respectively). We can calculate them recursively:

$$
\begin{aligned}
r_1 = t_1 = 1, \quad l_1 = b_1 = 0,\\
r_n = \max(r_{n-1}, t_{n-1} + x_n),\\
t_n = \max(t_{n-1}, l_{n-1} + y_n),\\
l_n = \max(l_{n-1}, b_{n-1} - x_n),\\
b_n = \max(b_{n-1}, r_{n-1} - y_n).
\end{aligned}
$$

We still need to analyze how the above optimization changes the running time of the algorithm. Empirical tests suggest that the effect is quite significant. This is the reason why some contestants implemented similar ideas but were unable to show why their solutions worked so quickly. We shall now show that the above optimization improves the time complexity of the algorithm from exponential to linear!

Let us draw the curve $S_n$ and mark out all bounding boxes of $2^{n-m}$ curves $S_m$ that form the curve $S_n$. Let us estimate how many, at most, drawn bounding boxes can enclose point $(x, y)$. This number will reveal how many times recursive calls to $d(\cdot, \cdot, m)$ will not be pruned. If this number is a constant then, consequently, at every level of recursion we will have a constant number of calls; thus the whole algorithm will run in time $O(n)$.

It can be proven by induction that $x_m$ and $y_m$ are divisible by $M = 2^{\lfloor m/2 \rfloor}$, thus every curve $S_m$ begins at one of the points from the set $Z_M = \{(Mi, Mj) : i, j \in \mathbb{Z}\}$. It can be also shown that

$$r_m + l_m, t_m + b_m < 2^{\lfloor m/2 \rfloor + 1},$$

therefore every drawn bounding box is enclosed in a square of side $2M - 1$. That means that bounding boxes that contain a point $(x, y)$ cover in total at most 16 points from the set $Z_M$. At each of these points, at most four curves $S_m$ can begin; thus $(x, y)$ belongs to no more than $16 \cdot 4$ bounding boxes. Therefore our algorithm does indeed run in linear time.



# / Termites

**Contest:** Algorithmic Engagements 2010
**Task author:** Tomasz Idziaszek
**Memory:** 256 MB
http://main.edu.pl/en/archive/pa/2010/ter

Two termites are eating an old wooden fence. It consists of $n$ planks of possibly different heights. Termites have already eaten some of them, and they thought that they should make their meal more interesting. They have decided to play a game and take turns to eat the planks, one by one. During one turn, a termite may only choose to eat a plank which is next to a plank that has already been consumed. Assuming that each termite chooses the planks in such a way that, after the whole game, the sum of the heights of all the planks eaten by that termite is as large as possible, compute the amount of wood that each of them will have eaten.

## Input

In the first line of the input there is an integer $n$ ($1 \le n \le 1{,}000{,}000$), specifying the number of planks in the fence. The second line contains a sequence of $n$ integers $l_i$ ($0 \le l_i \le 1{,}000{,}000{,}000$), describing the heights of the consecutive planks. If $l_i = 0$, then the corresponding plank has already been eaten. Plank number $i$ (for $1 < i < n$) is adjacent to planks $i - 1$ and $i + 1$. The only plank adjacent to plank number 1 is plank number 2, and the only plank next to plank number $n$ is plank number $n - 1$. At least one number $l_i$ is equal to zero.

## Output

You should write two integers to the output. The first one should be the sum of the heights of the planks that will be consumed by the termite that starts the game, whereas the second should specify the amount of wood that will be eaten by the other termite.

## Example

For the input data:

```
8
1 2 0 3 7 4 0 9
```

the correct result is:

```
17 9
```

**Explanation of the example:** The fence consisted of eight planks, of which two have already been eaten. The first termite, on its first turn, can choose between planks of height 2, 3, 4 or 9. During an optimal game, in consecutive turns, the termites will eat planks of height 9, 2, 1, 4, 7 and 3.

# / Solution

If anyone asked me which of my own problems I like the most, I would answer *Termites* without hesitation. It's not every day that one is lucky enough to come up with a problem with a simple, neat statement that causes disbelief among experienced contestants, who are certain that the author has made a mistake in the constraints of the input, since the problem apparently cannot be solved effectively.

Yet *Termites* does have a solution: a solution, moreover, that is short and quite elegant. My experience with this problem proves that everybody learns throughout their whole lives, and even now there are algorithmic pearls that are unknown to us, still waiting to be discovered.

A considerable amount of time passed between the original conception of this task and its presentation, in its current form, at the weekend round of Algorithmic Engagements in 2010. Its history began in 2005 at the International Olympiad in Informatics, which was held in Poland in the city of Nowy Sącz. During this event, Jakub Radoszewski shared with me a problem that he could not solve effectively.

***Problem 1.*** We have $n$ stacks of coins. Every coin is either a golden ducat or a worthless penny. There are two players in the game who collect coins alternately. In one move, a player can grab one coin from the top of any stack. We are to determine what the outcome of the game will be if both players play optimally in an attempt to maximize the number of collected ducats.

As we will see later, assumptions that occur in the statement of *Termites*—that the heights of the planks are positive and that at most two sides of the fence could be inaccessible—are of little importance. Therefore, the above problem is a simplified version of *Termites*. Since the problem looked interesting to me, I decided to attack it. Unfortunately, over the next few years I was not able to tackle it, except for a few hypotheses, which alone did not draw me any nearer to the solution (e.g. that it does not pay off to take a penny when one could take a ducat), and several not-so-clever tricks that decrease the exponent in an obvious exponential solution based on dynamic programming.

However, five years later, when motivated by the approaching Algorithmic Engagements, I sat down with this task and decided to work for as long as I could to make appreciable progress. My persistence was rewarded: a polynomial algorithm was developed in a week, and I spent the following month proving its correctness. What is more, I noticed that the algorithm is general enough that it also works in the case where we can grab coins from both ends of a stack. This observation was a little surprising to me, since I had encountered a similar problem before...

**From mathematical folklore: Coins in a Row**

This problem is surely known to many readers. I ask for a little more patience: for now, we are only presenting new problem statements, but soon we will start to solve them.

***Problem 2 (Coins in a Row).*** There are $n$ coins in a row, each with a value. There are two players in the game, Annie and Bertie, who alternately take one coin from the left or right end of the row. Annie makes the first move. What will be the outcome of the game (in terms of the sum of collected values) when both children play optimally?

As we can see, this problem is also a simplified version of *Termites* in which we have only one part of the fence, accessible from both sides. Let us try to solve this variant before we move on to the general case.

Firstly, let us make certain that the greedy strategy—which consists of always taking the most valuable available coin—does not work. Indeed, consider the following sequence of coins:

5 7 10 6

If Annie takes 6 in her first move, then the sum of values of coins collected by her, reduced by the sum accumulated by Bertie, will be equal to $6 - 10 + 7 - 5 = -2$. On the other hand, grabbing 5 gives Annie an advantage over Bertie of $5 - 7 + 10 - 6 = 2$. In the following discussion we use the term *value of the game* to denote the difference between the sum of values collected by Annie and the sum accumulated by Bertie when they play optimally. It is easy to see that Annie's goal is to maximize the value of the game, while Bertie tries to minimize it.

So, the naïve greedy strategy does not work. It turns out, however, that in the case where the number of coins is even, there is a very simple strategy that the first player can use to ensure that he does not lose (i.e. the first player will get at least the same value of coins as the opponent). Moreover, the existence of this strategy does not depend on the values of the coins. Now, for even $n$, Annie can be sure to take all the coins that are in the odd positions in the row,

or all the coins in the even positions. Thus she can choose whichever variant yields the bigger sum. Let us look at the example:

$$\underline{5}\ 2\ \underline{1}\ 4\ \underline{9}\ 8$$

In this case, the sum of the coins' values in the odd positions (underlined) is $s_1 = 5 + 1 + 9 = 15$, and the sum of the even positions (not underlined) is $s_2 = 2 + 4 + 8 = 14$. Annie could employ the strategy of taking coins from non-underlined positions, which gives her an advantage of $s_1 - s_2 = 1$. The strategy exists because the following invariant holds: before each of Annie's moves, exactly one end of the row has an underlined coin, and before each of Bertie's moves, both ends have non-underlined coins.

Therefore, we see that, for even $n$, Annie never loses. Let us observe, however, that the aforementioned strategy is not necessarily optimal. Indeed, in the above example Annie could get an advantage of 3. The optimal strategy is to take 5 in the first move and then play greedily. This leads to the optimal play of sum $5 - 8 + 9 - 4 + 2 - 1 = 3$. Thus the value of the game in this case is 3.

So what do we have to do if we want to compute the value of the game? We can calculate it in time $O(n^2)$ using dynamic programming. It is sufficient to calculate the value of the game for every continuous subsequence of coins. If we denote by $d[i,j]$ Annie's advantage for a sequence of coins in positions $i, i+1, \ldots, j$ $(1 \le i \le j \le n)$, then the values in the array $d$ could be computed by using the following recursive formula:

$$d[i,j] = \begin{cases} a[i] & \text{for } i = j, \\ \max(a[i] - d[i+1,j], a[j] - d[i,j-1]) & \text{for } i < j. \end{cases}$$

I first encountered this problem at the beginning of my experience with programming competitions. It was a classic example that illustrated the dynamic programming method, and the above $O(n^2)$ algorithm was presented as its exemplary solution. I wondered at the time whether a faster solution could be found, but nobody I talked to about the problem knew a better one. Nobody claimed that a better solution did not exist, though...



**An optimal solution for Coins in a Row**

Finally, after my many years of uncertainty, it turned out that a better algorithm could indeed be developed. We show now how to solve this problem in the optimal time complexity of $O(n)$.

For a given row of coins $G$ we denote by val($G$) the value of the game. Assume, for simplicity, that in the row $G$ there is exactly one coin of the greatest value $M$. Since both players try to maximize the sum of collected values, it is clear that both of them would like to grab the coin $M$. Let us consider who has the chance to do it.

It turns out that, although the naïve greedy strategy does not work in general, in situations of extreme greediness it is correct:

***Theorem 1 (Greedy Move Principle).*** Let the coin of the greatest value $M$ be on the end of a row $G$. Let $G'$ be the row after removing this coin. Then val($G$) = $M$ – val($G'$).

Thus, every time we can take the biggest value, it is profitable to do so—such a move will be optimal. Moreover, we could strengthen this theorem: this will be the only optimal move.

However, most of the time we will not be lucky enough to utilize the Greedy Move Principle directly. Thus, we need to consider what to do when the greatest value $M$ lies in the middle of the row. At some point, one of the players (say Bertie) during their move will uncover the value $M$. Then, from the strong Greedy Move Principle, we know that $M$ must be taken immediately by Annie. Observe, though, that in these two moves Bertie gains fewer points than Annie. The reason why he would make such a move in the first place is a desire to take the coin that is uncovered after taking $M$. This rationale lies behind the key theorem, which will allow us to calculate the value of the game:

***Theorem 2 (Fusion Principle).*** Let $x$, $M$, $y$ be adjacent coins (in this order) in the row $G$, satisfying the condition $x, y \le M$. Let $G'$ be the row in which these three coins have been replaced by one coin of value $x - M + y$. Then val($G$) = val($G'$).

Observe that the statement of the above theorem is much stronger than the presented insight. We do not assume that $M$ is the greatest value in the row, but only that it is greater than the values of adjacent coins. The action of replacing coins $x$, $M$, $y$ by one coin of value $x - M + y$ corresponds to the situation where two outermost coins are taken by one player and the middle one is collected by the opponent.

The Fusion Principle leads us to the following algorithm: as long as we can find three adjacent coins in the row that satisfy the premises of the theorem, we

replace them by one coin. That will shorten the sequence and will not change the value of the game. Let us look at an example in which the theorem is applied three times:

$$\underbrace{5\ 7\ 10\ \underbrace{6\ 9\ 9}_{6}}_{3}\ 1\ \underbrace{7\ 11\ 8}_{4}$$

$$\downarrow$$

$$5\ 3\ 1\ 4$$

If we cannot make fusions any more, we know that the coins' values form a bitonic sequence, i.e. a sequence which is decreasing to some point and then increasing. In particular, it has the nice property that no matter how we take coins from the sequence, the current greatest value will be at one end. Thus we can make use of the Greedy Move Principle, which means that calculating the value of the game for such a sequence is very easy: if $a_1,\ldots,a_m$ are the values in increasing order from the bitonic sequence $G$, then the value of the game is equal to

$$\mathrm{val}(G) = \sum_{i=1}^{m} (-1)^{i+1} a_i.$$

In the above example it would be $5 - 4 + 3 - 1 = 3$.

The last step is to implement the algorithm effectively. We can perform fusions in an arbitrary order. One approach is to put subsequent values on a stack and check whether a fusion can be made for the topmost three elements on the stack. This leads to an algorithm running in $O(n)$ time. Determining the increasing order of values boils down to merging two sorted sequences that comprise a bitonic sequence; thus it can also be done in linear time. The algorithm in pseudocode is as follows:



```
Algorithm CoinsInARow((a_1, ..., a_n))
    m := 0
    s := ∅
    for i := 1 to n do
        m := m + 1
        s[m] := a_i
        while m ≥ 3 and s[m − 2] ≤ s[m − 1] and s[m − 1] ≥ s[m] do
            s[m − 2] := s[m − 2] − s[m − 1] + s[m]
            m := m − 2
    value := 0
    sign := 1
    i := 1; j := m
    while i ≤ j do
        if s[i] ≥ s[j] then
            value := value + sign · s[i]
            i := i + 1
        else
            value := value + sign · s[j]
            j := j − 1
        sign := −sign
    return value
```

**Solution for an arbitrary number of rows of coins**

Let us try to generalize Problem 2 and think about a solution for the case where there is more than one row of coins. Using dynamic programming, it is easy to come up with a solution that works in time complexity $O(kn^k)$, where $k$ is a doubled number of rows. Obviously this solution is not acceptable.

It turns out, however, that it is easy to modify our optimal algorithm for the single row of coins in order to enable it to cope with an arbitrary number of sequences of coins. This is because the Greedy Move Principle and the Fusion Principle still work in the general case, i.e. their statements are valid even after changing the word "row" to "collection of rows".

Therefore, we are very close to the full solution. We only need to consider the case of rows in which we can only grab coins from one end. Observe that if the greatest value $M$ is at the inaccessible end of the row, and it is adjacent to some value $x$, then it would be fruitless for a player to take $x$, since then the opponent

would grab $M$ (from the strong Greedy Move Principle) and the sequence will be emptied. Thus the following theorem is true:

***Theorem 3 (Worthless Move Principle).*** Let $M$, $x$ be adjacent coins in the collection of rows of coins $G$, where the value $M$ is at the inaccessible end of sequence and $x \leq M$. Let $n$ be the total number of coins in $G$. Let $G'$ be the collection of rows after removing these two coins. Then $\mathrm{val}(G) = \mathrm{val}(G') + (-1)^n(x - M)$.

Again, this theorem is also true in the case where $M$ is simply not smaller than $x$. Which player is forced to make a worthless move depends on the parity of the number of coins $n$—for even $n$ it is Annie.

Ultimately, the algorithm is as follows. For as long as we can, we apply the Fusion Principle, which causes every sequence of coins to become unimodal. Then, for as long as we can, we apply the Worthless Move Principle. This means that we will be able to apply the Greedy Move Principle until the end of the game.

This algorithm can be implemented with time complexity $O(n \log n)$ if we use sorting to obtain the order of coins in the second phase of algorithm, or $O(n \log k)$, where $k$ is the number of accessible ends, if we use an algorithm that merges $k$ sorted lists.

**Practice and theory**

At this point, readers will certainly have realized that the generalized version of the *Coins in a Row* problem, after adding an entomological setting, is indeed the task *Termites*.

Practical readers can implement the above algorithm and check for themselves that it works. However, to fully justify its correctness, we still need to prove three theorems. Unfortunately, though their proofs are not very difficult, they are quite long and technical. Therefore we have decided not to include the proofs in the book. Any reader who is interested in seeing the formal proofs can find them on the author's website*.

* http://www.mimuw.edu.pl/~idziaszek/termity

**GRZEGORZ JAKACKI**

Grzegorz Jakacki graduated from the Faculty of Mathematics, Informatics and Mechanics, University of Warsaw. Over the years he has taken part in diverse software development projects in Poland, the United States and China. In 2001 he was one of the original authors of the System Verilog Assertions, a protocol description language that became the industry standard for VLSI design tools. In 2005, working on a massive hiring campaign in Beijing, China, he invented a way of automating the screening of software developers based on methods derived from programming competitions. This experience eventually led him to establishing his own start-up business, which subsequently gained worldwide recognition under the name Codility.com. Grzegorz Jakacki was a bronze medalist in the International Olympiad in Informatics in 1993, and has also co-organized multiple programming contests. He occasionally teaches classes in advanced programming at the University of Warsaw. He is a founding member of the Warsaw Hackerspace.

# / Window

**Contest:** 5th Polish Olympiad in Informatics
**Task author:** Wojciech Guzicki
**Memory:** 32 MB
http://main.edu.pl/en/archive/oi/5/okn

We have a polygon, chosen in the Cartesian coordinate system, whose sides are all parallel to the axes of the coordinates. Every two consecutive sides are perpendicular, and the coordinates of every vertex are integers. We also have a window, which is a rectangle whose sides are parallel to the axes of the coordinates. The interior of the polygon (but not its periphery) is colored blue. What is the number of separate blue fragments of the polygon that can be seen through the window?

We assume that the boundary of the polygon is a simple polygonal chain; i.e. every vertex of the polygon belongs to exactly two (consecutive) sides, and every other point of the boundary belongs to exactly one side.

**Example**

Look at the figure below:

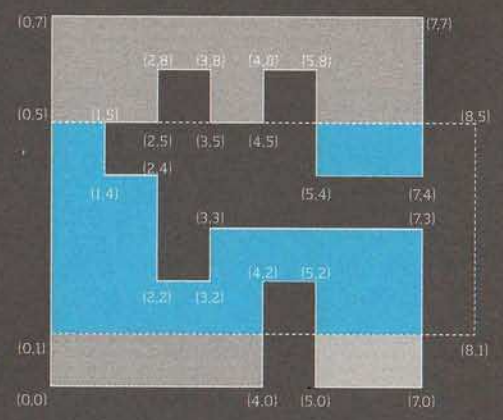



*Figure: There are two separate fragments of the polygon that can be seen through the window.*



**Task**

Write a program that:

- reads descriptions of a window and a polygon from the input,
- computes the number of separate blue fragments of the polygon that can be seen through the window,
- writes the result to the output.

**Input**

In the first line of the input there are four integers, $x_1, y_1, x_2, y_2$ ($1 \le x_1, y_1, x_2, y_2 \le 10{,}000$), separated by single spaces. The numbers $x_1, y_1$ are the coordinates of the top-left corner of the window. The numbers $x_2, y_2$ are the coordinates of the bottom-right corner of the window.

The next line of the input contains one integer $n$ ($4 \le n \le 5000$), which equals the number of vertices of the polygon. In the following $n$ lines, the coordinates of the polygon's vertices are given in counterclockwise direction; i.e. the interior of the polygon is on the left side of its periphery when we move along the sides of the polygon according to the given order. Each line contains two integers $x$, $y$ separated by a single space ($0 \le x \le 10{,}000$, $0 \le y \le 10{,}000$). The numbers in the $(i+2)$-th line ($1 \le i \le n$) are the coordinates of the $i$-th vertex of the polygon.

**Output**

In the first and only line of the output there should be one integer—the number of separate blue fragments of the polygon that can be seen through the window.

**Example**

For the input data (split into two columns):

```
0 5 8 1
24
0 0
4 0
4 2
5 2
5 0
7 0
7 3
3 3
3 2
2 2
2 4
1 4
1 5
2 5
2 6
3 6
3 5
4 5
4 6
5 6
5 4
7 4
7 7
0 7
```

the correct result is:

2

# / Solution

I learned about this interesting task by Wojciech Guzicki back in 1998 when I was preparing its model solution and test data for the Polish Olympiad in Informatics. This seemingly typical problem in computational geometry suggests a pretty mundane solution using a standard technique—a sweep line. It turns out, however, that a much easier solution exists, which skips the complexities of advanced data structures suggested by the canonical approach.

**Solution**

Imagine that the polygon occurs in the real world and that it is large—as large as the mysterious Nazca Lines—and the window is as large as a soccer field. Due to these enormous dimensions, we cannot keep the whole image in sight (except if looking from an airplane), but we are still able to count the fragments by following the boundaries of the polygon and the window.

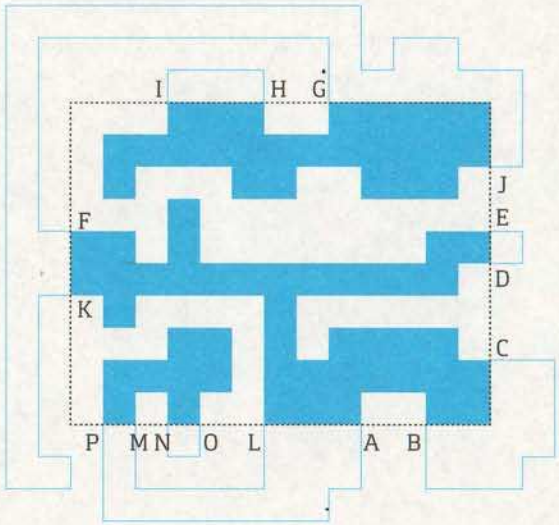



***Figure 1:*** *A polygon and a window. Three fragments are visible.*



If the boundary of the polygon intersects the boundary of the window, then by alternating between the consecutive boundaries of the polygon and the window we can count the fragments. Each polygon fragment consists of the fragments of window and polygon boundaries (see figure 1). In order to count the polygon fragments we shall traverse each fragment counterclockwise along its border.

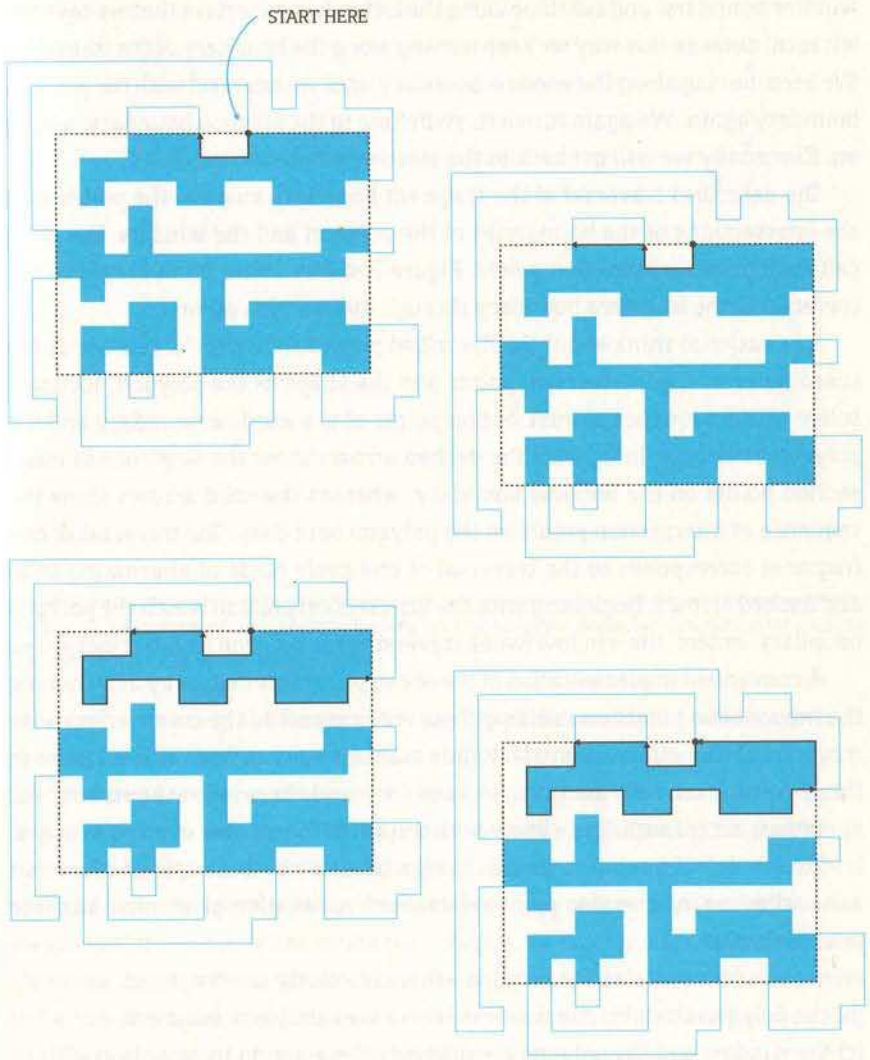



***Figure 2:*** *Subsequent steps of fragment traversal.*

We begin with an arbitrary polygon vertex external to the window and we follow the polygon boundary counterclockwise, up to the intersection with the window boundary (it is an intersection point where the polygon edge "enters" the window). This is when the traversal of one visible fragment begins. We continue to move along the polygon boundary until we find the spot where the polygon boundary crosses the window boundary and leaves the interior of the window. At this intersection point we turn left, switch from the polygon boundary to the window boundary, and continue along the latter. It is important that we take the left turn, because this way we keep moving along the boundary of the fragment. We keep moving along the window boundary until we intersect with the polygon boundary again. We again turn left, switching to the polygon boundary, and so on. Eventually we will get back to the starting point.

The described traversal of the fragment boundary stops at the points that are intersections of the boundaries of the polygon and the window. We shall call such points *intersection points*. Figure 2 shows subsequent stages in the traversal of the fragment boundary through intersection points.

It is easier to think about the described procedure if one forgets about the coordinates of the intersection points and the shape of the polygon, focusing solely on the sequence of intersection points of the window boundary and the polygon boundary. In figure 3 the dashed arrows show the sequence of intersection points on the window boundary, whereas the solid arrows show the sequence of intersection points on the polygon boundary. The traversal of one fragment corresponds to the traversal of one cycle made of alternating solid and dashed arrows, beginning with the intersection point in which the polygon boundary "enters" the window (when traversing the polygon counterclockwise).

A convenient implementation of the above procedure begins by determining the intersection points and sorting them with respect to the counterclockwise traversal of the window boundary. This makes it easy to find the next point in the traversal, on both the polygon boundary and the window boundary. All operations except sorting are linear with respect to the number of polygon edges.

To complete the solution we still have to take care of three special cases that arise when no intersection point exists. Such a situation gives raise to three possible cases:

(a) the window contains the polygon—there is exactly one fragment, or
(b) the polygon contains the window—there is exactly one fragment, or
(c) the window and the polygon are disjoint—there are no fragments at all.



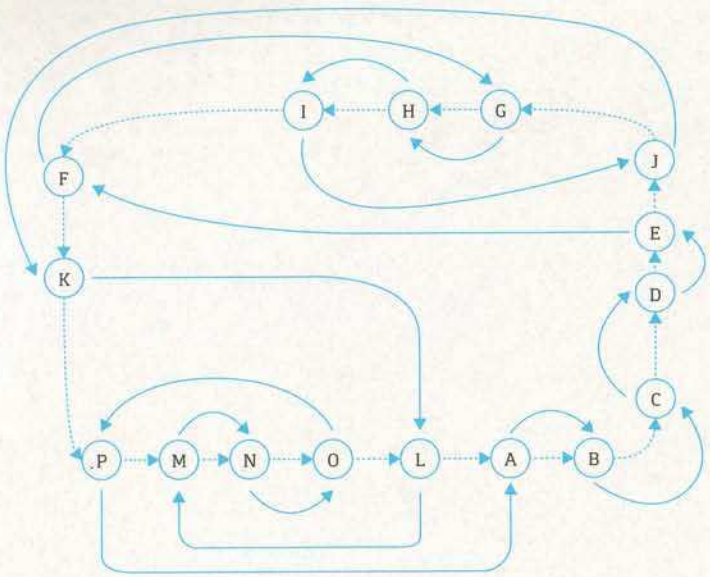



***Figure 3:*** *Sequence of intersection points on the polygon boundary (solid lines) and on the window boundary (dashed lines).*

If any vertex (e.g. the first one) of the polygon lies inside the window (which is easy to determine), then we are in case (a). Otherwise, we need to check whether any window vertex (e.g. the lower left corner) lies inside the polygon. This is a bit more complex. We draw an arbitrary ray (half-line) originating in the said window vertex (it is most convenient to use a vertical or horizontal ray) and count how many times it stabs the polygon boundary. An odd number means that the window lies inside the polygon, so exactly one fragment exists. However, cases when a whole side of the polygon is contained in the ray need to be handled more carefully; we can also avoid such situations by choosing as the starting point of the ray any window point of non-integer coordinates.

# / Altars

**Contest:** 6th Polish Olympiad in Informatics
**Task author:** Grzegorz Jakacki
**Memory:** 32 MB
http://main.edu.pl/en/archive/oi/6/olt

According to Chinese folklore, evil spirits can only move in a straight line. This is of great importance when temples are built. The temples are constructed on rectangular planes with sides parallel to the north-south or east-west directions. No two rectangles have common points. An entrance is situated in the middle of one of four walls, and its width is equal to half of the length of the wall. An altar appears in the center of the temple, where the diagonals of the rectangle intersect. If an evil spirit appears at this point, the temple will be profaned. This may happen only if there exists a ray that runs from an altar, through an entrance, to infinity and neither intersects nor touches the walls of any temple (on a plane parallel to the plane of a construction area); i.e. if one can draw, in a construction area, a line that starts at the altar and runs to infinity without touching any wall.

**Task**

Write a program that:

- reads descriptions of the temples from the input,
- verifies which temples could be profaned,
- writes their numbers to the output.

**Input**

In the first line of the input appears one integer $n$, the number of temples ($1 \le n \le 1000$).

In each of the following $n$ lines there is a description of one temple (the $i$-th line contains a description of the $i$-th temple). The description of a temple consists of four nonnegative integers, not greater than 8000, and a letter: E, W, S or N. The first two numbers are the coordinates of the temple's northwestern corner, and the next two numbers are the coordinates of the opposite southeastern corner. In order to specify the coordinates of a point, first we give its geographical longitude, which increases from the west to the east, and then its latitude, which increases from the south to the north. The fifth element of the description indicates the wall with the entrance (E = eastern, W = western, S = southern, N = northern). The elements of the temple's description are separated by single spaces.

**Output**

In the consecutive lines of its output, your program should write, in ascending order, the numbers of any temples that may be profaned by an evil spirit. Each number is placed on a separate line. If there are no such temples, you should write one word: BRAK (Polish for *lack*).

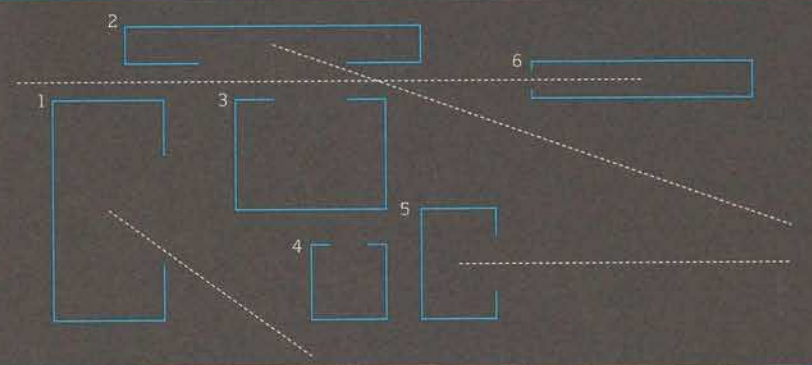
**Example**

For the input data:

```
6
1  7  4  1  E
3  9  11  8  S
6  7  10  4  N
8  3  10  1  N
11  4  13  1  E
14  8  20  7  W
```

the correct result is:

```
1
2
5
6
```




**Explanation of the example:** The picture shows the temples described in the sample input. The dashed lines show the possible routes of evil spirits.

# / Solution

In southeastern China, near the border with Burma, Laos and Vietnam, lies the Yunnan province, famous for its exquisite tea plantations. In the very heart of Yunnan lies Kunming, the City of Eternal Spring, as it is called for its thriving vegetation and mild climate. On the outskirts of Kunming sits the Purple Bamboo Temple (Qiongzhu Si), where the idea for the task *Altars* popped into my head.

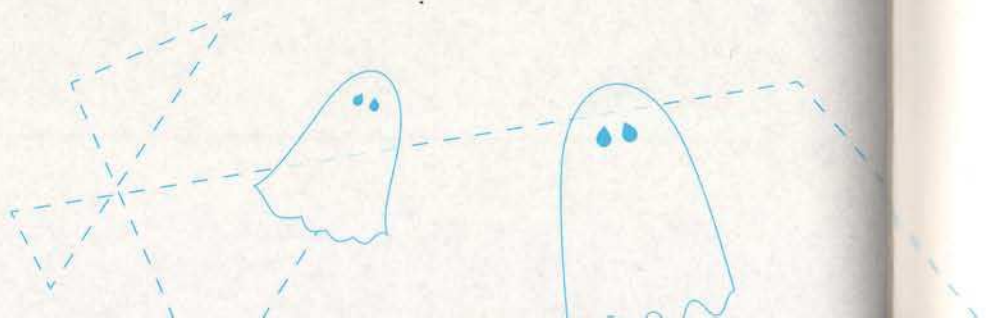
The Chinese believe that *influences* (bad influences in particular) propagate only along straight lines. *Spirit Walls* are traditionally built in front of entrances to the temples to protect the insides by blocking straight-line entry. For similar reasons, many houses are accessible only through winding passages.

One may imagine, if standing at the temple's altar, i.e. in the center of the rectangle circumscribed by the temple's walls, that the specifications of the temple may allow it to become desecrated if, in one's field of vision, there is a gap revealing the outside world, not just the other surrounding temples. The model solution is based on this analogy.

**The first solution**

Imagine a ray (half-line), pinned in the altar, that scans the field of vision like an airport radar. The scan begins at one door frame and ends at the other. Observing the count of walls that intersect the scanning ray, we notice that moving the ray onto a new wall increases the count by one and leaving the wall behind decreases the count by one. A fragment of the field of vision in which the count is equal to zero is a gap to the outside world.

The ray scanning and wall counting can be implemented by sorting all walls' vertices on their angular coordinates in the polar coordinate system originating at the altar point of the temple in question. Sorting must be implemented in such a way as to preserve the correspondence between both endpoints of the wall.





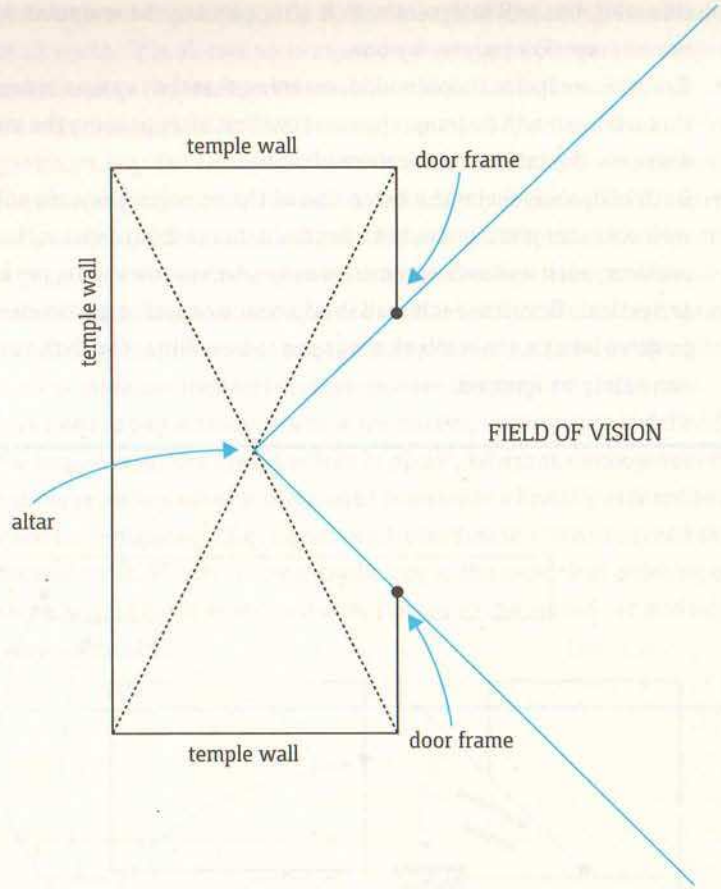



***Figure 1:*** *Layout of a temple.*

Imagine that we rotate the scanning ray such that it is always pinned to the altar point of the selected temple, and such that it stops at the endpoint of each wall (with the proper order of the wall endpoints being determined by the angular sorting). Each position of the rotating ray divides the plane into two open semiplanes*: "downwind", i.e. the semiplane being left behind as a result of the rotation, and "upwind", i.e. the one which is being entered by the rotating ray. Visiting the sorted wall endpoints, we face one of three cases:

* An open semiplane is a semiplane excluding its boundaries.

- The other endpoint is upwind, meaning that the ray did not intersect with this wall, but will intersect with it after passing the endpoint. We increase the intersection counter by one.
- The other endpoint is downwind, meaning that the ray was intersecting with this wall, but will no longer intersect with it after passing the endpoint. We decrease the intersection counter by one.
- Both endpoints lie on the extension of the sweeping ray, meaning that the wall obscures just a point, not a segment, in the field of vision. In the original problem, such a situation can arise only when the sweeping ray is horizontal or vertical. Because each wall is adjacent to another perpendicular wall of positive length, the walls that happen to be collinear with the sweeping ray can safely be ignored.

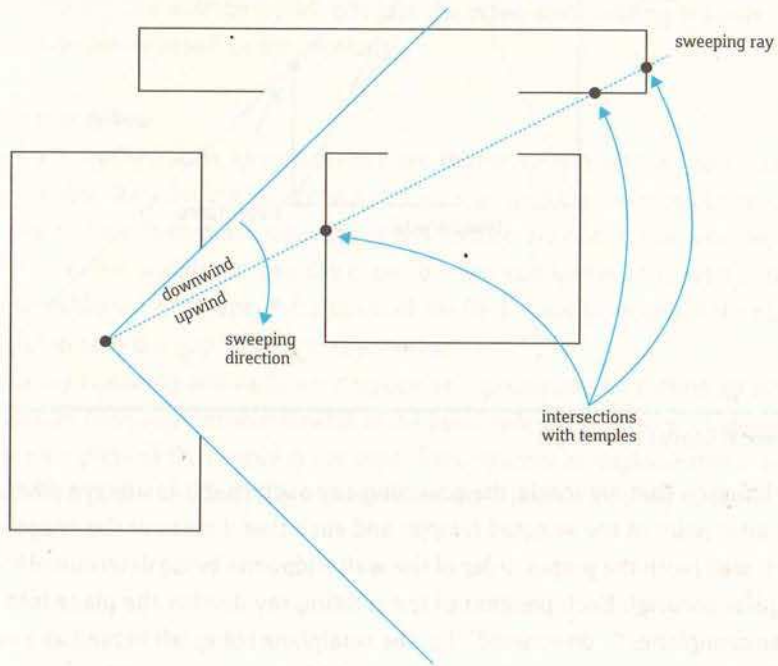



***Figure 2:*** *The sweeping ray and its intersections with the temples' walls.*



Three issues are still outstanding. First, we need to start properly. In order to do that, we can set the sweeping ray to go through the starting door frame and traverse all walls. This allows us to count how many walls are intersected by the ray and initialize the intersection counter. But there's a simpler way.

The initial set of walls (being just line segments) can be transformed by intersecting with an angular section of the plane determined by the angle with a vertex at the altar and rays going through door frames (see figure 3). This way, we ignore irrelevant segments outside the field of vision and also the irrelevant fragments of segments that are visible only partially. This operation contributes to a reduction of the sorting time, and we also guarantee that the sweeping ray anchored at the door frame does not intersect any of the (transformed) walls. This enables us to initialize the intersection counter to zero.

Second, we need to pay attention to how the sorting relationship is defined. Logically, the sequence of the swept points is cyclic, whereas employment of angular coordinates (or any other projection of points into a linearly ordered set) causes the need for "wrapping". E.g. a point with coordinate 1° may succeed the point with coordinate 359°, which is contradictory to the canonical ordering of numbers. An easy solution is to include door frames in the sorted set and rely on them as loop sentinels.

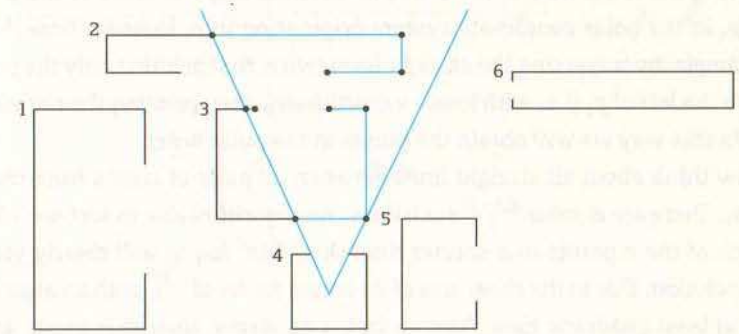



***Figure 3:*** *Set of walls restricted to the intersection with the field of vision.*

Third, we have to deal with angular coordinates carefully. Translating Cartesian coordinates (as given in the input data) into polar coordinates involves a danger of rounding errors, due to the use of floating-point arithmetic. Luckily the computation of the sign of appropriate vector products can entirely replace the need for trigonometry.

The algorithm shown above determines whether one particular temple can be desecrated. In order to determine this property for all the temples, one needs to repeat this procedure $n$ times. The algorithm's dominant operation for one temple is sorting (running time $O(n \log n)$), thus the total running time is $O(n^2 \log n)$. Such an approach seems extravagant, because substantial amounts of computation are performed for each temple, but their results are not used when processing the following temples. This can be remedied.

**How to perform angular sorting**

The algorithm's main operation is the angular sorting of the set of points with respect to one of them. There exists an algorithm that performs this step in $O(n^2)$ total time. It is interesting and difficult, and we will summarize it briefly.

Think about the set of points $p_1, \ldots, p_n$. Let us choose one of them. It does not matter which one, so let us take $p_1$. Run a straight line through $p_1$ so that it gets connected with all other points $p_2, \ldots, p_n$ (see figure 4). Now, for each of those remaining points, there is one straight line. If we know how to sort these straight lines with respect to their slopes (i.e. the $a$ in the equation $y = ax + b$ describing the line), we would be able to infer the angular order of the points $p_2, \ldots, p_n$ in the polar coordinate system originating in $p_1$ in linear time. How? For example, by traversing the straight lines twice, first printing only the points lying to the left of $p_1$ (i.e. with lower $x$ coordinate), then printing the remaining ones. In this way we will obtain the points in the polar order.

Now think about all straight lines between all pairs of points from the set $p_1, \ldots, p_n$. There are at most $\frac{n(n-1)}{2}$ such lines. An algorithm able to sort $n-1$ lines for each of the $n$ points in a shorter time than $\Theta(n^2 \log n)$ will clearly yield a better solution. Due to the sheer size of its output (order of $n^2$), such an algorithm needs at least quadratic time. And, in fact, a quadratic algorithm exists and is based on an extremely interesting concept.



**Dualization.** Imagine a geometric transformation $T$, which maps a point $p$ into a straight line $T(p)$, and a straight line $l$ into a point $T(l)$ as follows:

$$p = (a, b) \rightarrow T(p) : y = ax - b$$
$$l : y = kx + d \rightarrow T(l) = (k, -d).$$

Thus a point $(a, b)$ is mapped into a straight line $y = ax - b$. The straight line $y = kx + d$ is mapped into a point $(k, -d)$. Both domain and co-domain of the above transformation are the two-dimensional plane $\mathbb{R}^2$. We shall call the domain a primary space and the co-domain a dual space. The transformation just described has an interesting property. A point $p$ lies on the straight line $l$ if, and only if, the straight line $T(p)$ intersects the point $T(l)$.

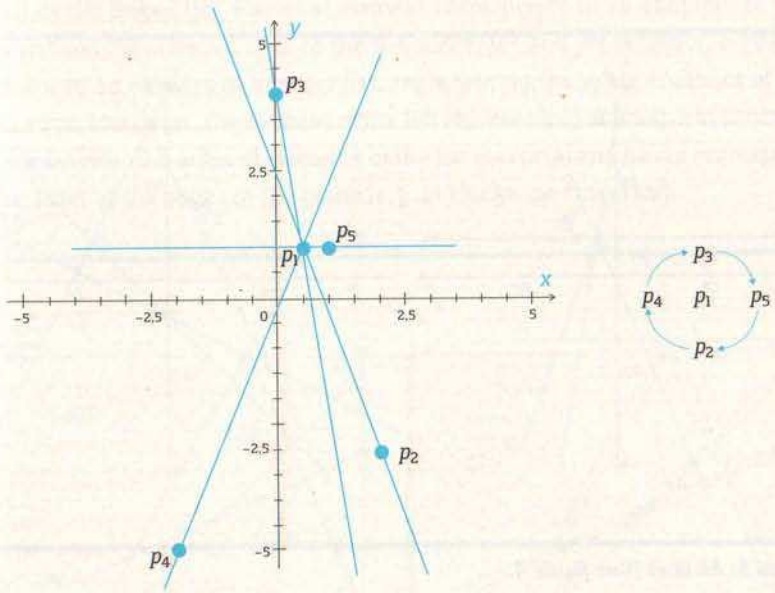



*Figure 4: Straight lines from $p_1$ and the angular order of points with respect to $p_1$.*

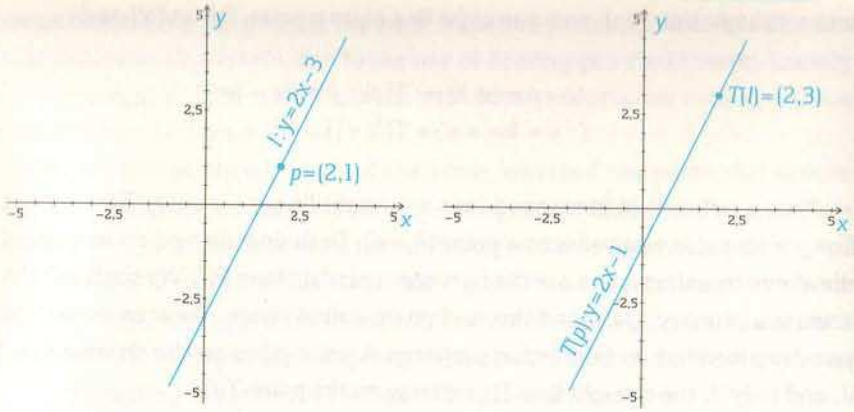



***Figure 5:*** *Primary space (on the left) and its corresponding dual space (on the right).*

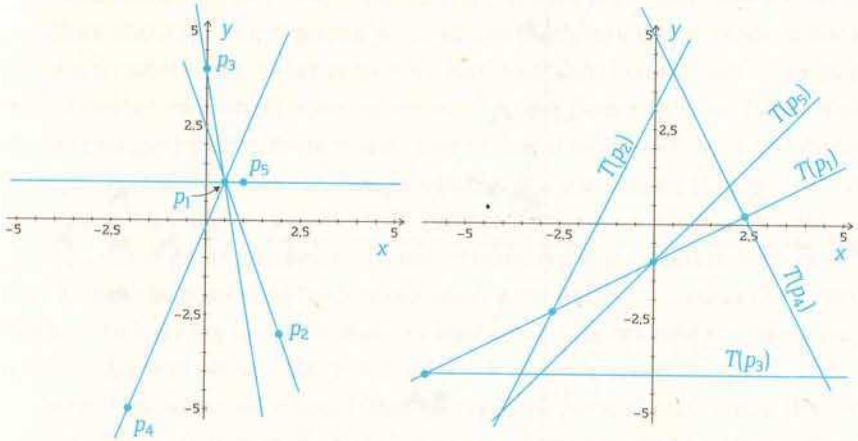



***Figure 6:*** *All lines from figure 4.*

The set of points $p_1,\ldots,p_n$ and the set of straight lines running through the pairs of these points can be mapped into the set of straight lines $T(p_1),\ldots,T(p_n)$. In the dual space, the straight lines connecting $p_1$ with other points correspond to the intersections of the straight line $T(p_1)$ with the straight lines $T(p_2),\ldots,T(p_n)$. However, there is a small exception. If some point $p_i$ lies over or under $p_1$, in the dual space, the lines $T(p_1)$ and $T(p_i)$ are parallel. We will handle this in a moment.



We have arrived at the following problem: $n$ straight lines in the plane are given. For each of them, find the order of the intersection with the remaining straight lines. It follows from the definition of the dual space that no straight line will be vertical. From the ordering of the intersections of the line $T(p)$ in the dual space we can recreate the ordering of the straight lines intersecting the point $p$ in the primary space and, further, the angular ordering of the remaining points with respect to $p$. The same procedure will work for any point $p_1,\ldots,p_n$. Should there be any points above and under $p_1$, they must be considered separately. After sorting all other points in the angular order with respect to $p_1$, we need to insert the points lying above or under $p_1$ into this ordering. This can easily be done in linear time. From now on we assume that no such points exist.

**Finding line intersections.** Let us take a look at figure 7, showing the data structure representing the set of endpoint-connected segments. To be precise, this is a set of segments such that the segments intersect only at endpoints, and each endpoint belongs to at least two segments. Each endpoint is represented with a cyclic linked list. Each list element corresponds to an endpoint of one segment and, besides pointers to the list successor and predecessor, contains a pointer to an element of another list, representing the other endpoint of the same edge. Moreover, the element of the list representing an endpoint contains its coordinates. The order of elements in the list is crucial and has to correspond to the order of the edges in the plane (e.g. in clockwise traversal).

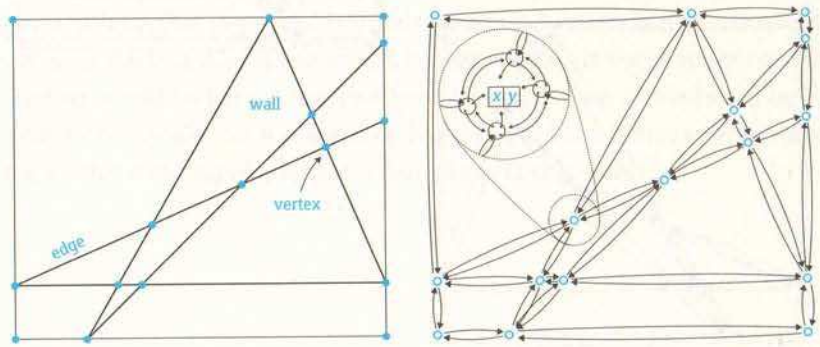



***Figure 7:*** *Data structure (on the right) representing a set of segments (on the left). The set of segments was built by drawing a rectangular frame around the lines from figure 6.*

This data structure is a variant of a so-called *doubly-connected edge list*. It enables the quick insertion of new edges into an existing set. Notice also that, thanks to the pointers to adjacent edges and to the proper ordering of the elements in the cyclic linked lists corresponding to endpoints, one can easily traverse the boundaries of faces made up of the edges. Figure 8 shows several steps in the process of adding line segments to the set.

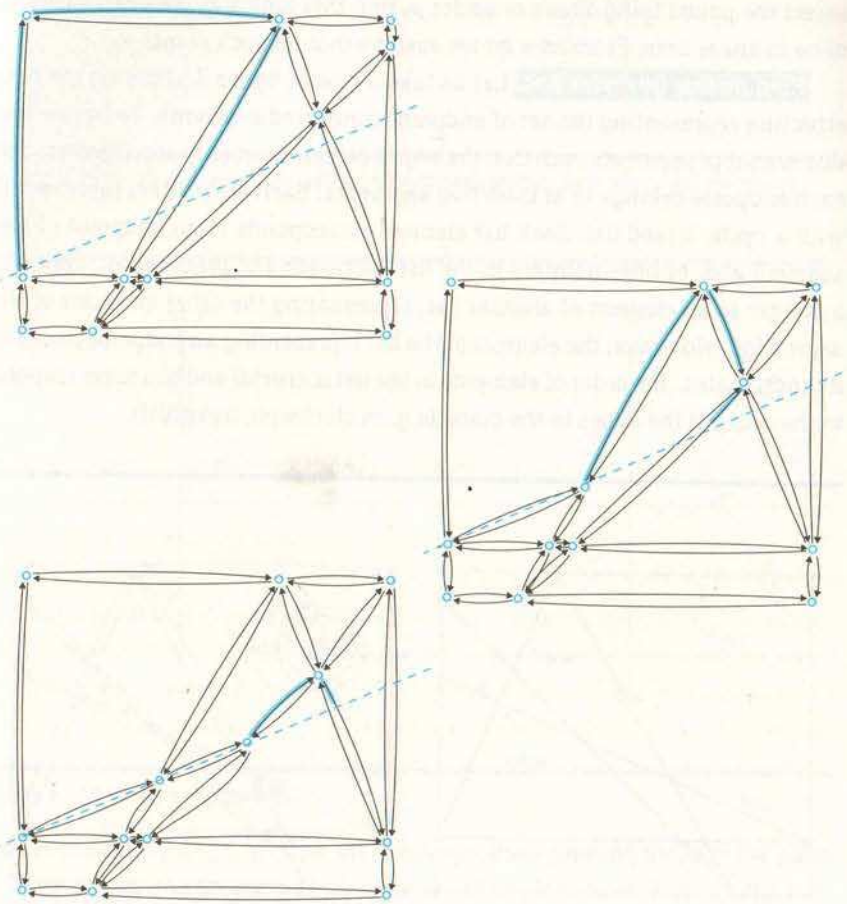


***Figure 8:*** *Adding a new line to the data structure.*



The operations on this data structure seem quite straightforward. We are interested in how many of them may need to be performed (with respect to $n$), because this is crucial to the time complexity of this phase of the algorithm. It is clear that individual new faces may have many edges (the order of $n$). It turns out, however, that the *total* number of edges that need to be visited in order to add one straight line to the data structure is also of the order of $n$. We omit the proof, but it is not excessively complex and is based on the observation that visited edges can be mapped onto the intersection points of the straight line being added in such a way that there are at most 10 edges per one intersection point. To learn more, research the *Zone Theorem*.

The above finding is crucial for the analysis of the algorithm's time complexity. Observe that the structure can be built in $O(n^2)$ time. Traversal of the subsequent segments of one straight line can be realized in $O(n)$ time. Observe that if we dualize the set of points $p_1,\ldots,p_n$ into the set of lines $T(p_1),\ldots,T(p_n)$ and contain these lines within an envelope*, we will arrive at a set of segments similar to the one in figure 7. The algorithm for building the doubly-connected edge list described above can determine the intersection points for all lines in $O(n^2)$ total time. Now, if we wish to determine the angular order of all the points $p_1,\ldots,p_n$ with respect to one chosen point (e.g. $p_1$), we can read the ordering of the intersection of the straight line $T(p_1)$ with the remaining straight lines. The indices of the remaining straight lines will determine the ordering of the points $p_2,\ldots,p_n$ in the angular order with respect to $p_1$, which has already been discussed above.

The wholesale angular sorting of the set of $n$ points speeds up our initial algorithm from $O(n^2 \log n)$ to $O(n^2)$. This speed-up does not matter much in the solution of the task *Altars*, because its implementation is very complex and the speed-up would be hardly observable on the input data sets allowed by the problem description. Such a speed-up has merit in more serious applications, such as the rendering of graphics or integrated circuit designs.

* A rectangular envelope containing all intersection points of $n$ straight lines can be determined as follows. Sort (in $O(n \log n)$) all lines with respect to their slopes and consider only the intersection points of the lines that have adjacent slopes (including the pair consisting of the first and last lines). All other intersection points lie inside the convex hull circumscribed around these.

**TOMASZ KOCIUMAKA**

Tomasz Kociumaka is a student of a double degree program in Computer Science and Mathematics at the Faculty of Mathematics, Informatics and Mechanics, University of Warsaw. He started to take part in algorithmic competitions during his sophomore year at high school. A year later he won the Polish Olympiad in Informatics and earned a gold medal at the International Olympiad in Informatics. During his university years, his team was a runner-up in the Polish Collegiate Programming Contest. He has been successful in mathematical competitions as well: he earned a silver medal at the International Mathematical Olympiad, and later on celebrated—twice—a first place at its counterpart for university students, the International Mathematical Competition. Recently, he has been engaged in research into text algorithms. His greatest achievement in this area is designing a linear-time algorithm for finding seeds of a string. This problem was used at Algorithmic Engagements 2009 as the task *Quasi-template*. Tomasz Kociumaka also works as a jury member of the Polish Olympiad in Informatics, Central European Olympiad in Informatics (in 2011) and the Polish Olympiad in Informatics Training Camp. He also organizes yearly algorithmic workshops of the Polish Children's Fund. As a high school student, he was a participant in the Fund's program. He likes spending his free time hiking in the mountains.

# / Axes *of Symmetry*

**Contest:** 14th Polish Olympiad in Informatics
**Task author:** Piotr Stańczyk
**Memory:** 32 MB
http://main.edu.pl/en/archive/oi/14/osi

Little Johnny—a well-respected young mathematician—has a younger sister, Justina. Johnny likes his sister very much and he gladly helps her with her homework, but, like most scientific minds, he does object to solving the same problems repeatedly. Unfortunately, Justina is a very diligent pupil, and so she asks Johnny to review her assignments many times, for the sake of certainty. One sunny Friday, just before the famous Long May Weekend*, the math teacher set many exercises consisting of finding the axes of symmetry of various geometric figures. Justina is most likely to spend a considerable amount of time solving these tasks. Little Johnny had arranged himself a trip to the seaside long before; nevertheless, he feels obliged to help his little sister. Soon, he has found a solution—it would be best to write a program that would ease the checking of Justina's solutions. Since Johnny is a mathematician rather than a computer scientist, and you are his best friend, it falls to you to write it.

## Task

Write a program that:

- reads the descriptions of the polygons from the input,
- determines the number of axes of symmetry for each one of them,
- writes the result to the output.

## Input

In the first line of the input there is one integer $t$ ($1 \le t \le 10$), specifying the number of polygons for which the number of axes of symmetry is to be determined. Next, $t$ descriptions of the polygons follow. The first line of each description contains one integer $n$ ($3 \le n \le 100{,}000$) specifying the number of

* In Poland, there is an accumulation of public and national holidays at the beginning of May. They are: May 1—May Day, the International Workers' Day; May 2—the Day of the Polish Flag; May 3—the Anniversary of the Establishment of the First Constitution of Poland (dating back to the year 1795). Though the middle, quite recent, feast is not actually a holiday, it is, however, customary to make it a day off from work too. Now imagine that May 1 is on a Monday, or May 3 on a Friday, and you will get a feeling for what the Long May Weekend is like!

vertices of the polygon. In each of the following $n$ lines there are two integers, $x$ and $y$ ($-100{,}000{,}000 \le x, y \le 100{,}000{,}000$), representing the coordinates of subsequent vertices of the polygon. The polygons need not be convex, but they have no self-intersections; any two sides have at most one common point: their common endpoint, if they actually share it. Furthermore, no pair of consecutive sides is parallel.

**Output**

Your program should output exactly $t$ lines, with the $k$-th line containing a single integer—the number of axes of symmetry of the $k$-th polygon.

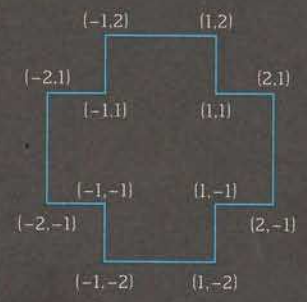
**Example**

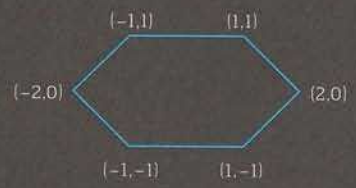
For the input data:

```
2
12
1  -1
2  -1
2  1
1  1
1  2
-1  2
-1  1
-2  1
-2  -1
-1  -1
-1  -2
1  -2
6
-1  1
-2  0
-1  -1
1  -1
2  0
1  1
```

the correct result is:

```
4
2
```






# / Solution

The vast majority of algorithmic problems can be classified according to the area they concern. As such, we may easily distinguish graph problems, text problems, geometry problems, number theory problems, game theory problems etc. Usually, a problem is quickly classified in this way—generally as early as during the initial reading of the task statement. Very often, the solution uses techniques and algorithms specific to the problem's area, sometimes with the aid of data structures that are fundamental to the whole area of algorithms.

Far fewer tasks have solutions that are based on reductions to problems in different areas. Such solutions tend to be simultaneously tricky and easy to code. As we will see in a moment, this is the case in *Axes of Symmetry*, which is definitely a geometrical problem but which turns out to have a solution that utilizes text algorithms.

**Problem analysis—a load of geometry**

Let us find out when a line is a polygon's axis of symmetry. First, notice that such a line must cross the perimeter of the polygon. This is intuitively clear—if the line does not cross the border, the whole polygon needs to lie on the same side of the line*, which is impossible for an axis of symmetry. Hence let us consider only lines crossing the polygon's border.

Let us conduct a thought experiment. Imagine we have two robots, which are represented as points on the plane. Assume that they can follow polygonal paths with a constant velocity, which is common to both of them. Let us put the robots at a single starting position at a crossing point of the line and the polygon's perimeter. Now, make them move along the perimeter in opposite directions until they reach the starting point again. The polygon has no self-intersections, so their movement is well defined. What is more, both robots finish at the same time and they pass each other exactly once, in the middle of their way.

Robots follow the whole perimeter, so if their positions remain symmetric with respect to the line, then this line is an axis of symmetry of the perimeter, and consequently of the whole polygon. Clearly, the opposite implication is also true (see figure 1). Note that the symmetry axis crosses the perimeter at exactly two points: the starting point and the point where the robots passed each other.

* We do not provide rigorous proofs of intuitively clear geometrical statements like this. In order to do so, we would need to build a formal setting with appropriate definitions and axioms. This way, descriptions would be harder to read and key ideas would not be made easier to understand.

How can we benefit from this discussion to check whether a given line is an axis of symmetry? We know that it is enough to find out whether the robots' positions stay symmetric with respect to the line. In order to invent a way of doing so, imagine that the robots cannot follow the perimeter on their own, but need us to give them a sequence of commands. Assume that each robot is oriented; that is, it has a "forward" direction. The commands are of two types:

- rotate by an angle of $180° - \alpha$ (denoted as $\alpha$),
- move forward by $d$ units (denoted as $d$).

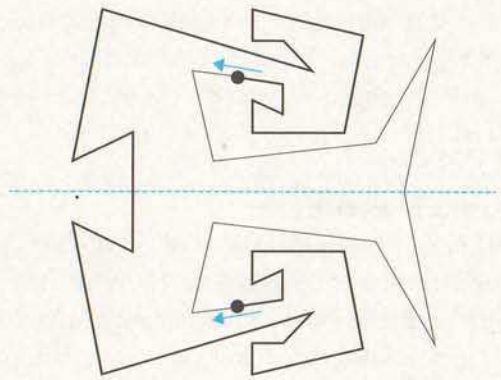


***Figure 1:*** *If a line is an axis of symmetry, the robots' positions are always symmetric with respect to that line.*

Additionally, assume that the robot that goes around the polygon clockwise rotates clockwise, and the other one counterclockwise. Moreover, at the starting point, let the robots be oriented in opposite directions perpendicular to the line under investigation. The sequence of commands we give, if necessary, begins with an appropriate rotation to the starting directions. Then, moving forward to the next vertex alternates with the rotation at that vertex.

If both robots are given the same sequence of commands, their positions remain symmetrical with respect to the line. What is more, if we never ask them to go forward by 0 units or rotate by 0°, the opposite implication is also true. The commands we give using the polygon's description to make the robots follow the perimeter satisfy this condition; that is because, as noted in the problem statement, no two consecutive sides are parallel.



The observations we have already made allow us to limit the set of lines that are worth considering as candidates for axes of symmetry. First of all, the initial rotations must be equal for both robots. Hence, the starting point may either be a vertex, with the line being the bisector of the corresponding angle, or a point of an edge, with the line being perpendicular to that edge. In the latter case, the point must be the edge's midpoint and the line the bisector of the edge, since the first movements forward must be equal.

In this way we have limited the set of candidates to $2n$ lines: $n$ for either case. Note that actual axes of symmetry (and possibly some other lines) are counted twice here: once for each crossing point with the perimeter.

The only thing we need to do for each candidate is to compare the sequence of commands. Note that we can omit the initial ones, which we used to prove that the axis of symmetry is a bisector of an angle or of an edge. This way, we just compare sequences consisting alternately of edge lengths and angle values (see figure 2).

### The solution—a shift to text algorithms

How can we compare such sequences of commands? If we proceed in a naïve way, we obtain an $O(n^2)$-time algorithm. However, the sequences we are comparing are very regular, and this makes it possible to design a much faster algorithm.

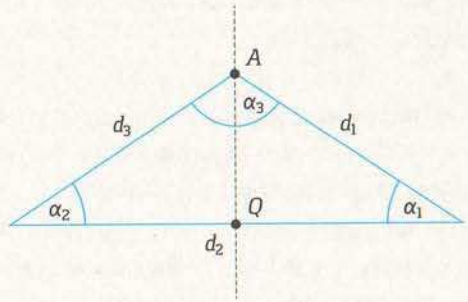



***Figure 2:*** *When A is the starting point we need to compare* $(d_1,\alpha_1,d_2,\alpha_2,d_3)$ *with* $(d_3,\alpha_2,d_2,\alpha_1,d_1)$. *If it is Q, the sequences are* $(\alpha_2,d_3,\alpha_3,d_1,\alpha_1)$ *and* $(\alpha_1,d_1,\alpha_3,d_3,\alpha_2)$.

Consider a cyclic sequence of consecutive edge lengths alternating with the values of angles between them:

$$(d_1, \alpha_1, d_2, \alpha_2, \ldots, d_n, \alpha_n).$$

Each check we make can now be described as starting from some position $i$ in this sequence and, for each $j = 1, 2, \ldots, 2n-1$, comparing the values at positions $i+j$ and $i-j$. The candidate is an axis of symmetry if and only if, for a corresponding $i$ and for each $j$, the values are equal. Hence, the final result is half of the number of such positions in the cyclic sequence.

In other words, we want to find out for which $i$ a contiguous subsequence of length $4n-1$, with its midpoint in $i$, is a *palindrome*—i.e. the same whether read backwards or forwards. We can solve the problem for the standard linear (i.e. not cyclic) string by unwinding the cycle three times and considering just palindromes with midpoints among the middle $2n$ positions. For example, for point $Q$ of figure 2, we check whether, in the sequence below, the first underlined word read forwards is equal to the second one read backwards:

$$d_1, \alpha_1, d_2, \underline{\alpha_2, d_3, \alpha_3, d_1, \alpha_1}, d_2, \underline{\alpha_2, d_3, \alpha_3, d_1, \alpha_1}, d_2, \alpha_2, d_3, \alpha_3.$$

Again, the final answer is half of the number of palindromes of length $4n-1$ centered at such positions.

In this way, we have reduced the geometrical problem of finding the axes of symmetry to a text problem: in a word of length $6n$, find the midpoints of the palindromes of length $4n-1$.

In order to solve this task, as well as most of the problems related to palindromes, we may use the classical text algorithm named after Glenn Manacher. Given a word, for each position this algorithm finds the maximum length of the palindrome centered at this position. For example, for *abaaaba* it outputs a sequence 1, 3, 1, 7, 1, 3, 1. Recall that Manacher algorithm works in linear time.

In our problem we just need to identify the positions for which the values computed by Manacher algorithm are at least $4n-1$, since whenever a position is the center of an odd palindrome of given length, it is the center of the palindromes of all smaller odd lengths.



# / Leonardo *Numbers*

**Contest:** Polish Olympiad in Informatics Training Camp 2008
**Task author:** Tomasz Kulczyński
**Memory:** 32 MB
http://main.edu.pl/en/archive/ontak/2008/leo

The famous Fibonacci numbers are not the only discovery of Leonardo of Pisa (also known as Fibonacci); he is also known for a similar sequence called the Leonardo numbers. Let us define the sequence of Leonardo numbers $(L_i)$ as $L_0 = L_1 = 1$ and $L_{i+1} = L_i + L_{i-1} + 1$ for $i \geq 1$. Today, 800 years after Leonardo Fibonacci's death, we would like to find the value of the following expression:

$$L_0^k + L_1^k + L_2^k + \cdots + L_n^k.$$

**Task**

Write a program that:

- reads from the input the integers $n$ and $k$,
- evaluates the expression,
- writes the last nine digits of the result to the output.

**Input**

The first and only line of input contains two positive integers, $n$ and $k$ ($1 \leq k \leq 13$, $n$ fits in a 64-bit unsigned integer type).

**Output**

The only line of output should contain the last nine digits of the decimal representation of the result of the expression.

**Example**

For the input data:

3 2

the correct result is:

000000036

**Explanation of the example:** The expression corresponding to this example is $1^2 + 1^2 + 3^2 + 5^2 = 36$.

# / Solution

This task appeared at the Polish Olympiad in Informatics Training Camp in 2008. I was a participant at this camp, but at the time I had only been interested in algorithmics for about a year, and the problem was too hard for me. When I learned the solution, I found its techniques very interesting. I started analysing those techniques, and by combining them with some observations I had made during the contest, I managed to create a solution that was not only faster but also easier to code than the original. Moreover, when I came back to the problem in order to write this text more than three years later, I was able to design an even better algorithm. The character of number-theory problems like this makes me believe that an even faster algorithm might be devised; possibly even a not very complicated one.

**Initial remarks on residues.** We are to compute only the last nine digits of the result, i.e. its residue modulo $M = 10^9$. The formula for the sum of powers of Leonardo numbers involves only addition, multiplication and exponentiation. Those operations—and hence also determining the value of Leonardo numbers—can be done on residues modulo $M$. Addition and multiplication of such residues can easily be implemented using 64-bit integer variables.

In the following text, we usually operate on residues modulo $M$. For brevity, we will use standard notations for operations both on integers and on residues. In particular, $L_n$ often denotes not the Leonardo number itself, but its remainder in division by $M$. In terms of abstract algebra, we perform most of the calculations in the $\mathbb{Z}_M$ ring.

### How to compute the Leonardo numbers

In this task we need to deal with Leonardo numbers that are really far along the sequence. The only assumption we make about $n$ is that it fits into a 64-bit unsigned integer variable, i.e. $n < 2^{64}$. This makes even determining the value of $L_n$ a nontrivial problem. The simple algorithm, using the recursive formula and written so that it doesn't compute the same number multiple times, has $\Theta(n)$ time complexity. That is far from acceptable. As a warm-up, let us significantly improve this running time.

**Fibonacci numbers.** We begin by solving the same problem for the famous relative of the Leonardo numbers—the Fibonacci numbers, defined recursively as follows:

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1}.$$



The Fibonacci numbers are broadly known and well studied. In particular, in many books and other sources one can find the closed-form formula for the $n$-th Fibonacci number:

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}},$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$. Unfortunately, $\varphi$ and $\psi$ are not integers (not even rational numbers). Although this is not a problem when we are to find the approximate values of Fibonacci numbers (or their leading digits; floating point variables can be used), this issue makes the formula useless when we are to find the remainders of Fibonacci numbers (in particular, their trailing digits).

The formula that proves most useful for our purpose is as follows. It uses the basic notions of linear algebra, i.e. matrices and vectors:

$$\begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$$

This clearly gives a different closed-form expression of Fibonacci numbers:

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

To compute $F_n$, one can just use the formula we have obtained; that is, compute the $n$-th power of the appropriate $2 \times 2$ matrix and multiply it by a vector. The key part is exponentiation—it involves $n$, which might be really large. To perform this operation quickly, one can use a method called *exponentiation by squaring*, which allows the $n$-th power of a matrix to be found using $O(\log n)$ matrix multiplications. (This technique is much more general: it is commonly used to compute powers in the $\mathbb{Z}_M$ ring and in many other circumstances.) This method is based on the following simple observation:

$$A^{2n} = (A^2)^n,$$
$$A^{2n+1} = A \cdot (A^2)^n.$$

Direct application of these equalities leads to a recursive algorithm. To improve its practical performance (reduce the constant factor in time cost) and the memory consumption (to constant), the algorithm is usually implemented in the iterative form:

```
Algorithm Power(A, n)
  P := I {identity matrix}
  while n > 0 do
    if n mod 2 = 1 then
      P := P · A
    A := A · A
    n := n div 2
  return P
```

This algorithm gives an $O(\log n)$ time and constant space algorithm for computing $F_n$ in $\mathbb{Z}_M$.

**Leonardo numbers.** The recursive formula for Leonardo numbers is

$$L_0 = L_1 = 1, \quad L_{n+1} = L_n + L_{n-1} + 1, \tag{1}$$

which is slightly more complicated. In particular, this formula cannot be encoded in a $2 \times 2$ matrix, as we have done before. Speaking in terms of linear algebra, $L_{n+1}$ is not a linear combination of $L_n$ and $L_{n-1}$*.

This means that we need to use the matrix exponentiation technique in a more subtle way. The key observation is that $L_{n+1}$ is a linear function of $L_n$, $L_{n-1}$ and 1 (a constant sequence). Hence, we can use a larger (but still constant-sized) matrix. This leads to the following recursive formula:

$$\begin{bmatrix} L_1 \\ L_0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} L_{n+1} \\ L_n \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} L_n \\ L_{n-1} \\ 1 \end{bmatrix}$$

Again we get an $O(\log n)$ time and constant-space solution.

### The original solution by the problem's author

We have learned how to find $L_n$ in $O(\log n)$ time. However, our task is harder: we are to compute a sum $\sum_{i=0}^{n} L_i^k$. Let us start by solving it for $k = 1$ and $k = 2$. This will give us some insight for the general case.

* When writing of linear combinations, we abuse the notation—so, for example, by $L_{n+1}$ we mean a sequence whose $n$-th term is $L_{n+1}$. If we were formulating this statement without the notion of linear algebra on sequences, we would need to say that $L_{n+1}$ cannot be written as a linear combination of $L_n$ and $L_{n-1}$ with coefficients independent of $n$.



**Sum of Leonardo numbers.** The current task is to find the $n$-th term of the sequence $S_{1,n} = \sum_{i=0}^{n} L_i$. Directly from the definition one gets

$$S_{1,0} = L_0 = 1, \quad S_{1,n} = S_{1,n-1} + L_n.$$

Writing the recursive formula in matrix form led us to the effective algorithms, so let us try it again. Clearly, $S_{1,n}$ can be expressed as a linear combination of the $S_{1,n-1}$ term and $L_n$. This is the reason why we extend the previous matrix:

$$\begin{bmatrix} S_{1,0} \\ L_1 \\ L_0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} S_{1,n} \\ L_{n+1} \\ L_n \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_{1,n-1} \\ L_n \\ L_{n-1} \\ 1 \end{bmatrix}$$

This matrix allows us simultaneously to find both Leonardo numbers and their sums. Note that this is the first time in this solution where two coordinates of the left-hand side vectors are not merely rewritten coordinates of the right-hand side vector. The only assumption we need in order to use the matrix exponentiation technique is that a vector $[S_{1,n}, L_{n+1}, L_n, 1]$ can be expressed as a linear function of $[S_{1,n-1}, L_n, L_{n-1}, 1]$ (with the linear function independent of $n$).

Hence, we have devised an algorithm that performs $O(\log n)$ multiplications of $4 \times 4$ matrices, which gives $O(\log n)$ time and constant space complexity. Thus, let us go a step further.

**Sum of the Leonardo numbers' squares.** Now we are to determine the values $S_{2,n} = \sum_{i=0}^{n} L_i^2$. The recursive formula is

$$S_{2,0} = L_0^2 = 1, \quad S_{2,n} = S_{2,n-1} + L_n^2.$$

This time the problem is not so easy, $L_n^2$ should be an element of the vector, and it is not $L_n^2$ but $L_n$ for which we have a linear recursive formula. However, the formula we have gives a more complicated but still linear recursive formula for $L_{n+1}^2$:

$$L_{n+1}^2 = (L_n + L_{n-1} + 1)^2 = L_n^2 + L_{n-1}^2 + 2L_nL_{n-1} + 2L_n + 2L_{n-1} + 1.$$

The "new" term here is $L_nL_{n-1}$. Hence we need one more formula:

$$L_{n+1}L_n = (L_n + L_{n-1} + 1)L_n = L_n^2 + L_nL_{n-1} + L_n.$$

Summing up, the matrix recursive formula can be written as follows:

$$\begin{bmatrix} S_{2,0} \\ L_1^2 \\ L_1L_0 \\ L_0^2 \\ L_1 \\ L_0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} S_{2,n} \\ L_{n+1}^2 \\ L_{n+1}L_n \\ L_n^2 \\ L_{n+1} \\ L_n \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_{2,n-1} \\ L_n^2 \\ L_nL_{n-1} \\ L_{n-1}^2 \\ L_n \\ L_{n-1} \\ 1 \end{bmatrix}$$

Once again we get a solution of time complexity $O(\log n)$, but this time the constant hidden in the $O$ notation is quite large.

**The general case.** Having obtained some experience by analyzing the special cases of $k = 1$ and $k = 2$, we are finally ready to tackle the problem in its general form. We still want to solve it by matrix exponentiation. The elements of this matrix, and even its size, will depend on $k$. Hence, this time our program will need to compute this matrix before it exponentiates it.

For the sequence $S_{k,n} = \sum_{i=0}^{n} L_i^k$ we have a trivial formula:

$$S_{k,0} = L_0^k = 1, \quad S_{k,n} = S_{k,n-1} + L_n^k.$$

The linear recursive formula that lets us find $L_{n+1}^k = (L_n + L_{n-1} + 1)^k$ is a consequence of Newton's trinomial formula, and is a rather complicated one:

$$L_{n+1}^k = (L_n + L_{n-1} + 1)^k = \sum_{r=0}^{k} \sum_{s=0}^{k-r} \binom{k}{r,s,k-r-s} L_n^r L_{n-1}^s 1^{k-r-s}.$$

The coefficients that occur above are the trinomial coefficients defined as

$$\binom{p}{r,s,t} = \frac{p!}{r!\,s!\,t!} \quad \text{for } r,\ s,\ t \geq 0 \text{ and } p = r + s + t. \tag{2}$$

Despite being complicated, this formula allows us to express $L_{n+1}^k$ as a linear function of $L_n^r L_{n-1}^s 1^{k-r-s} = L_n^r L_{n-1}^s$, where $r$, $s$ and $k - r - s$ are all nonnegative integers. Fortunately (as we will show in a moment), for any such $r$, $s$ the term $L_{n+1}^r L_n^s$ is also expressible as the linear function of the same terms.

This means that the vector that we will compute is

$$[S_{k,n}, L_{n+1}^k, L_{n+1}^{k-1}L_n^1, \ldots, L_{n+1}^1 L_n^{k-1}, L_n^k, L_{n+1}^{k-1}, \ldots, L_n^{k-1}, \ldots, L_{n+1}^1, L_n^1, 1].$$



Its length is $1 + \binom{k+2}{2} = \Theta(k^2)$. This is also the size of the square matrix whose $n$-th power we will compute. The time complexity of multiplying two square matrices of this size can in practice be bounded by $O(k^6)$. For theoretical considerations, one could use a better bound of $O(k^{2\omega})$ to express this cost*.

What remains to complete the solution is the first phase of the algorithm— we need to compute the elements of the matrix. For $k \leq 2$ we did this manually; now we need an algorithm. From Newton's trinomial formula we get:

$$L_{n+1}^p L_n^q = (L_n + L_{n-1} + 1)^p L_n^q = \sum_{r=0}^{p} \sum_{s=0}^{p-r} \binom{p}{r,s,p-r-s} L_n^{q+r} L_{n-1}^s.$$

This means that the first row of the matrix is 1, 1, 0, ..., 0, while—due to the above formula—the terms of all other rows are zeros or trinomial coefficients. If we know the values of the trinomial coefficients, the matrix can be computed in $O(k^4)$ time, which is good since the number of entries in the matrix is also $O(k^4)$. (The details of this purely technical problem are left as an exercise.)

The definition of trinomial coefficients stated in (2) is unfortunately not appropriate for computation in the $\mathbb{Z}_M$ ring. That is because it involves division, which is not well defined in this ring. Thus, we need a better definition. To simplify the formulas, let us extend the definition and assume that $\binom{p}{r,s,t} = 0$ if $p \neq r + s + t$ or there is a negative number among $r$, $s$, $t$. In this notation it is easy to verify that for $p \geq 1$:

$$\binom{p}{r,s,t} = \binom{p-1}{r-1,s,t} + \binom{p-1}{r,s-1,t} + \binom{p-1}{r,s,t-1}$$

Obviously the border case is $\binom{0}{0,0,0} = 1$.

Hence, all the trinomial coefficients we need (i.e. those with $p \leq k$) can be found in $O(k^3)$ time, so the whole matrix can be found in $O(k^4)$. Unfortunately, the procedure is technically complicated.

This concludes the description of the algorithm. Its overall time complexity is $O(k^{2\omega} \log n) = O(k^6 \log n)$ and it consumes $O(k^4)$ memory. For $k \leq 13$ and $\log n \leq 64$, and with a small constant hidden in $O$ notation (matrices have size $\frac{k^2}{2} + O(k)$), this solution is fast enough to earn the perfect score.

* The naïve algorithm multiplying two square matrices of size $m \times m$ works in $\Theta(m^3)$. A number $\omega$ is defined as the smallest real number such that two such matrices can be multiplied in $O(m^\omega)$. The algorithms used in practice reach $O(m^{\log_2 7}) = O(m^{2.808})$. Coppersmith and Winograd devised a very complicated algorithm in 1990, which achieves $O(m^{2.3755})$. At the end of 2011, this long-standing record was beaten: Virginia Vassilevska Williams published a paper with an algorithm that uses $O(m^{2.3727})$ time. In our case, this leads to $O(k^{4.7452})$ time complexity.

**A faster and simpler solution**

Analysing the previous solution, most readers are probably not fond of the huge sums that appear there because of the trinomial formula. If it were binomial—that is, if we computed the power of the sum of two terms—the formulas would be much simpler.

This would be the case if we somehow got rid of the constant term in the recursive formula (1) for Leonardo numbers. Here is how we can do that.

Let us start with the following lemma (which we state in a generalized version in order to give a better insight into the technique we use). Its proof is very simple and we leave it as an exercise for the reader.

***Lemma 1.*** Let $x_n$ be the sequence defined by the recursive formula of the following form:

$$x_n = a_1x_{n-1} + a_2x_{n-2} + \cdots + a_lx_{n-l} + b,$$

where $a_1 + a_2 + \cdots + a_l \neq 1$. Then the sequence

$$y_n = x_n + \frac{b}{a_1 + \cdots + a_l - 1}$$

satisfies the formula

$$y_n = a_1y_{n-1} + a_2y_{n-2} + \cdots + a_ly_{n-l}.$$

In our case, $x_n$ is $L_n$, so from (1) we get $l = 2$, $a_1 = a_2 = b = 1$. Hence we obtain a sequence $K_n = L_n + 1$, which can be defined by

$$K_0 = 2, \quad K_1 = 2, \quad K_{n+1} = K_n + K_{n-1}.$$

An attentive reader has probably already noticed that

$$K_n = 2F_{n+1}, \tag{3}$$

but this is not important for us—the crucial thing is that we have a recursive formula without the constant term.

Our situation is unfortunately not as good at it may seem. In order to find $S_{k,n} = \sum_{i=0}^{n} L_i^k$ we need to use the formula



$$S_{k,n} = \sum_{i=0}^{n} L_i^k = \sum_{i=0}^{n} (K_i - 1)^k = \sum_{i=0}^{n} \sum_{j=0}^{k} \binom{k}{j} K_i^j (-1)^{k-j} =$$

$$= \sum_{j=0}^{k} \left( \binom{k}{j} (-1)^{k-j} \left( \sum_{i=0}^{n} K_i^j \right) \right).$$

Let us therefore define $T_{j,n} = \sum_{i=0}^{n} K_i^j$. In order to compute $S_{k,n}$ it is enough to determine $T_{j,n}$ for all $0 \le j \le k$ and spend $O(k^2)$ time on computing the binomial coefficients (for example, by generating an appropriate part of the Pascal triangle).

**Computing $T_{k,n}$.** Let us start in the same way as we did for $S_{k,n}$. Again we have a trivial recursion:

$$T_{k,0} = K_0^k = 2^k, \quad T_{k,n} = T_{k,n-1} + K_n^k.$$

Moreover,

$$K_{n+1}^k = (K_n + K_{n-1})^k = \sum_{r=0}^{k} \binom{k}{r} K_n^r K_{n-1}^{k-r}.$$

Note that this time we obtain a linear function of just $k + 1$ terms. What is more,

$$K_{n+1}^p K_n^{k-p} = (K_n + K_{n-1})^p K_n^{k-p} = \sum_{r=0}^{p} \binom{p}{r} K_n^{k-p+r} K_{n-1}^{p-r},$$

so the vector

$$v_{n+1} = [T_{k,n}, K_{n+1}^k, K_{n+1}^{k-1}K_n, \ldots, K_{n+1}K_n^{k-1}, K_n^k]$$

can be expressed as a linear function of the similar vector $v_n$.

This time, the matrix in the recursive formula is simple enough to be written down below. Its size is $(k+2) \times (k+2)$ and its main part is the Pascal triangle.

$$\begin{bmatrix} T_{k,0} \\ K_1^k \\ K_1^{k-1}K_0 \\ \vdots \\ K_1K_0^{k-1} \\ K_0^k \end{bmatrix} = \begin{bmatrix} 2^k \\ 2^k \\ 2^k \\ \vdots \\ 2^k \\ 2^k \end{bmatrix} \quad \begin{bmatrix} T_{k,n} \\ K_{n+1}^k \\ K_{n+1}^{k-1}K_n \\ \vdots \\ K_{n+1}K_n^{k-1} \\ K_n^k \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & k & \cdots & k & 1 \\ 0 & 1 & k-1 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} T_{k,n-1} \\ K_n^k \\ K_n^{k-1}K_{n-1} \\ \vdots \\ K_nK_{n-1}^{k-1} \\ K_{n-1}^k \end{bmatrix}$$

Of course, this matrix can be computed in $O(k^2)$, which is the number of its entries. The values of $T_{k,n}$ can thus be found in $O(k^\omega \log n) = O(k^3 \log n)$ time using $O(k^2)$ space. Therefore we get a new algorithm for our main problem. Its time complexity is $O(k^{\omega+1} \log n) = O(k^4 \log n)$ and it uses $O(k^2)$ space. Thus we have created a solution that is simultaneously simpler, faster and less memory-consuming than the previous one.

**Even faster algorithm**

It turns out that the algorithm we have just presented is still not the best that we can do. That is rather typical for such problems of computational number theory. The time complexity depends heavily on the formulas we use to build an algorithm. Often, some interesting and unexpected ones lead to fast, highly nontrivial, and sometimes even simple solutions. In a way, this is what happened in the previous section. In order to improve this result, we need to use more complicated mathematical tricks.

Our main aim is very simple: obtain a new, better recursive formula for $k$-th powers of Leonardo numbers.

**Fibonacci numbers once more.** We noticed in (3) that $K_n^k = 2^k F_{n+1}^k$. Moreover, we have seen the closed-form for Fibonacci numbers

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}, \tag{4}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$. Note that $\varphi$ and $\psi$ are the roots of the polynomial $x^2 - x - 1$. This in particular means that $\varphi\psi = -1$ and $\psi^2 = \varphi^{-2}$. Thus

$$5^{\frac{k}{2}} F_n^k = (\varphi^n - \psi^n)^k = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} (\varphi^i \psi^{k-i})^n.$$

Hence, the sequence $F_n^k$ turns out to be a linear combination of sequences $\{(\varphi^i \psi^{k-i})^n : 0 \le i \le k\}$. Similarly, using (3) we conclude that $K_n^k$ is such a combination as well. Since

$$L_n^k = (K_n - 1)^k = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} K_n^i,$$

the sequence $L_n^k$ is a linear combination of $K_n^i$ for $0 \le i \le k$. Combining these two facts, we find that $L_n^k$ is a linear combination of sequences

$$\{(\varphi^i \psi^{j-i})^n : 0 \le i \le j \le k\}.$$



**The main trick.** The equality $\varphi\psi = -1$ turns out to have deep and useful implications.

***Lemma 2.*** All elements of the set $\{\varphi^i \psi^{j-i} : 0 \le i \le j \le k\}$ are roots of the polynomial

$$P(x) = \prod_{t=-k}^{k} (x^2 - \varphi^{2t}). \tag{5}$$

*Proof.* Let us consider a fixed element $x_{ij} = \varphi^i \psi^{j-i}$ from the set. We have

$$x_{ij}^2 = (\varphi^i \psi^{j-i})^2 = \varphi^{2i} \psi^{2(j-i)} = \varphi^{2i-2j+2i} = \varphi^{2(2i-j)}.$$

Moreover, $-k \le 2i - j \le k$, so for $t = 2i - j$ in the product we get $x_{ij}^2 - \varphi^{2t} = \varphi^{2(2i-j)} - \varphi^{2(2i-j)} = 0$. □

Let us write the polynomial $P(x)$ in the power basis:

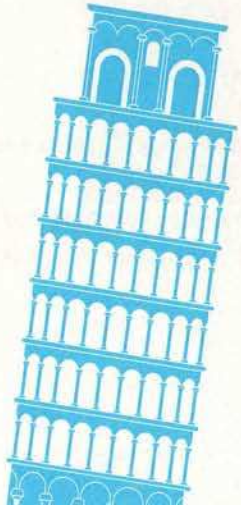
$$P(x) = x^{4k+2} + p_{2k} x^{4k} + p_{2k-1} x^{4k-2} + \cdots + p_1 x^2 + p_0.$$

Clearly, if $P(x) = 0$ then also $P(x) x^{n-4k-2} = 0$, which means that

$$x^n = -p_{2k} x^{n-2} - p_{2k-1} x^{n-4} + \cdots - p_1 x^{n-4k} - p_0 x^{n-4k-2}.$$

Values $x_{ij} = \varphi^i \psi^{j-i}$ are all roots of $P$, which by the above reasoning means that the geometrical progressions $a_n = (\varphi^i \psi^{j-i})^n$ all satisfy the following recursive formula:

$$a_n = -p_{2k} a_{n-2} - p_{2k-1} a_{n-4} + \cdots - p_1 a_{n-4k} - p_0 a_{n-4k-2}. \tag{6}$$



Being a linear combination of such geometric progressions, $L_n^k$ must also satisfy this formula.

This is very good news for us, since we can solve our problem by naïvely computing the first few elements of $L_n^k$ and then using the following recursive formula:

$$\begin{bmatrix} S_{k,4k+1} \\ L_{4k+2}^k \\ L_{4k+1}^k \\ L_{4k}^k \\ \vdots \\ L_1^k \end{bmatrix} = \cdots \begin{bmatrix} S_{k,n+4k+1} \\ L_{n+4k+2}^k \\ L_{n+4k+1}^k \\ L_{n+4k}^k \\ \vdots \\ L_{n+1}^k \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -p_{2k} & \cdots & 0 & -p_0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} S_{k,n+4k} \\ L_{n+4k+1}^k \\ L_{n+4k}^k \\ L_{n+4k-1}^k \\ \vdots \\ L_n^k \end{bmatrix}$$

Note that in order to compute $a_n$ using the formula (6), we need to know the previous values up to $a_{n-4k-2}$, which means that the size of our matrix is $(4k+3) \times (4k+3)$.

The only thing that remains is to find the $p_i$ values. What is more, these numbers need to be integers, so we can perform the computations in $\mathbb{Z}_M$. Note that from (4) we get

$$5F_t^2 = (\varphi^t - \psi^t)^2 = \varphi^{2t} - 2\varphi^t\psi^t + \psi^{2t} = \varphi^{2t} - 2(-1)^t + \varphi^{-2t},$$

so $\varphi^{2t} + \varphi^{-2t} = 5F_t^2 + 2(-1)^t$.

Let us in (5) join the factors for $t$ of the same absolute value:

$$(x^2 - \varphi^{2t})(x^2 - \varphi^{-2t}) = x^4 - (\varphi^{2t} + \varphi^{-2t})x^2 + 1 = x^4 - (5F_t^2 + 2(-1)^t)x^2 + 1.$$

Finally, we get

$$P(x) = (x^2 - 1)\prod_{t=1}^{k}\left(x^4 - (5F_t^2 + 2(-1)^t)x^2 + 1\right).$$

This means that the coefficients of this polynomial are integers and we can find them (or rather their residues modulo $M$) in $O(k^2)$ time.

The matrix we exponentiate is of $\Theta(k)$ size, so we can conclude that we have invented a solution that works in $O(k^\omega \log n) = O(k^3 \log n)$ time using $O(k^2)$ space.

**ERYK KOPCZYŃSKI**

Eryk Kopczyński learned to program in BASIC at the age of six, back in the days of Amstrad CPC computers. In 1994 he was a finalist of the first Polish Olympiad in Informatics. He also received gold medals at the International Mathematical Olympiad and the International Olympiad in Informatics. He has represented the University of Warsaw in ACM-ICPC World Finals twice, in 2000 and 2002. Later on, he completed his Ph.D. degree and is currently working at the Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, doing research in theoretical computer science.

Eryk Kopczyński still actively participates in international online programming contests, such as TopCoder (he won TopCoder Open in 2005 and is currently ranked 18th) and Google Code Jam. He is also an author of programming tasks and a coach at programming competitions.

Eryk Kopczyński enjoys biking, mountain trekking, music and all kinds of games.

# / Ritual

**Contest:** Polish Olympiad in Informatics Training Camp 2007
**Task author:** Eryk Kopczyński
**Memory:** 32 MB
http://main.edu.pl/en/archive/ontak/2007/ryt

BAB (Bytean Algorithmic Bureau) is a private agency, employing a team of computer scientists, that helps wealthy people solve their confidential problems with the aid of powerful computer systems. Recently, BAB was invited to meet the Grand Master of the Bytoo Worship. On the way to the Bytoo Temple the BAB team managed to collect some information about the Worship.

*The Bytoo Worship started when the Prophet received the great silicon slab with two numbers written on it: the Great Sacred Number and the Small Sacred Number. [...] As a part of the ritual, each morning, priests write the Great Sacred Number on the inner side of the wall of the Temple. The priests erase some of its digits during the day, according to a secret rule, and leave the remaining digits on the wall for the night. Next morning, the Great Sacred Number is written on the wall again and the ritual continues.*

The Grand Master of the Bytoo Worship gave BAB a preliminary test: they were to figure out the secret rule of erasing digits. On the first evening, BAB noticed that the remaining digits formed a palindromic number: the first digit is the same as the last one, the second digit is the same as the last but one, and so on. That was, however, only a part of the secret rule. After a few days, BAB used Euclid's algorithm to discover the second part of the rule: the number that remains after the digits are erased must be divisible by the number of days in the Bytoo week, which is equal to the Small Sacred Number, 666. The remaining number may have leading and trailing zeros, but it must have at least one digit (the priests cannot erase all digits at a time).

The Grand Master congratulated BAB on successfully completing the test and then revealed their real mission. Each day, the priests must erase a different configuration of digits. The end of the world will occur on the first day when the priests are unable to find any new configuration. The Grand Master would like to know which day of the week will be the last day (priests erased the first configuration of digits on the first day of the Bytoo week). He asked BAB to write a program to compute this information.



**Task**

Write a program that:

- reads the Great Sacred Number from the input,
- computes the day of the week after which the world will end (assuming the perfect accuracy of the priests),
- writes the result to the output.

**Input**

The first line of the input contains one integer $G$ ($1 \le G < 666^{36}$), the Great Sacred Number.

**Output**

The first and only line of the output should contain one number from the set {1, 2,..., 666}, specifying the day of the week after which the world will end.

**Example**

For the input data:
6666666
the correct result is:
42

**Explanation of the example:** There are 35 ways to erase four digits of the Great Sacred Number so that the number 666 remains on the wall and seven ways to erase one digit leaving 666666 on the wall. Obviously, 666 and 666666 satisfy both requirements of the ritual.

# / Solution

This problem appeared in the Polish Olympiad in Informatics Training Camp in 2007. My idea was to train the participants in dynamic programming algorithms by combining their two relatively standard applications.

However, this combination turned out not to be as easy and straightforward as it seemed. There are lots of subtleties here. The problem features two cult numbers: 666 and 42, which suggests that the author cared about style and wanted to use interesting numbers, instead of mundane powers of 10 and 2. However, this was not the only reason: the number 666 was not chosen arbitrarily; we had to take advantage of one of its special properties. Otherwise, one would obtain a solution with significantly worse time and space complexities, which would not run within the time and space limits imposed.

As far as I remember, these subtleties meant that none of the participants was able to solve the problem during the contest at the training camp. Moreover, I arrived at the camp later, after the problem had been used, and found that the solution presented by the trainers was the suboptimal one (they did not take advantage of the properties of 666). I think that the idea of using a special property of an innocent number is quite original and interesting. With hindsight I regret that this problem was used at a training camp, as it would probably have been appropriate for a more serious competition.

**Solution**

Let's first consider the problem in the case where we do not care about divisibility, and only have to compute the number of ways of erasing digits so that the remaining digits form a palindrome. Experienced coders quickly discover that this can be done with dynamic programming.

Let $l_1 l_2 \ldots l_N$ be the digits of the Great Sacred Number. First, note that each subpalindrome of length greater than 2, having $l_a$ and $l_b$ as its first and last digits, consists of these two digits, two sequences of erased digits $\cancel{l_{a+1} \ldots l_{c-1}}$ and $\cancel{l_{d+1} \ldots l_{b-1}}$, and a *heart* (lying between the two erased sequences), which is a subpalindrome with $l_c$ and $l_d$ as its first and last digits.

For each pair $(a, b)$, where $a \le b$, let $P_{a,b}$ denote the number of palindromes obtained by erasing some digits in the sequence $l_a l_{a+1} \ldots l_b$, except for $l_a$ and $l_b$, which have to remain. From the property described above we find that $P_{a,b}$ can be calculated from the values $P_{c,d}$ for $(c, d)$ such that $a < c \le d < b$:

$$P_{a,b} = \begin{cases} 1 & \text{for } a = b, \\ 0 & \text{for } l_a \ne l_b, \\ 1 + \sum\limits_{c,d:a<c\le d<b} P_{c,d} & \text{for } a \ne b \text{ and } l_a = l_b. \end{cases}$$

The formula above leads to an algorithm with complexity $O(N^4)$. For each pair $(a, b)$ we have to test all the positions of the heart $(c, d)$, so we have four nested loops. We start with pairs where $a = b$ and then proceed to greater and greater values of $b - a$.

How can we improve the running time of this algorithm? Maybe we should redefine $P_{a,b}$ and no longer force $l_a$ and $l_b$ to remain? In this case, we would get the following formula: $P_{a,b} = P_{a+1,b} + P_{a,b-1} + c_{a,b}$, where $c_{a,b}$ equals 0 for $l_a \ne l_b$ and $1 + P_{a+1,b-1}$ for $l_a = l_b$. The complexity is reduced to only $O(N^2)$!

Unfortunately, this solution is not correct. For example, consider 161. The palindrome $\cancel{161}$ is counted twice: both by extending $\cancel{16}$ and $\cancel{61}$. This is because, when erasing digits surrounding the heart of our palindrome, the formula allows us to choose arbitrarily whether we extend to the left or to the right. Thus, if we consider 61111611116, its subpalindrome 666 is counted $\binom{8}{4} = 70$ times, since there are 70 ways of ordering four erasures of 1s to the left, and four erasures of 1s to the right.

How do we fix this? We simply force digits to the right of the heart to be erased before the digits to the left are erased. Let $P_{a,b}$ be the number of subpalindromes of $l_a \ldots l_b$ that do not have to include $l_a$ and $l_b$, and let $Q_{a,b}$ be the number of subpalindromes that include $l_a$. We extend the palindrome in the following way. Suppose that we have a subpalindrome with digits $l_a$, $l_c$, $l_d$ and $l_b$ (all of which have not been erased); all digits between $l_{a+1}$ and $l_{c-1}$, and between $l_{d+1}$ and $l_{b-1}$, are erased. Then:

- The heart $l_c \ldots l_d$ is counted in $Q_{c,d}$.
- We extend it to the right, thus we count $l_c \ldots l_d \cancel{l_{d+1} \ldots l_{b-1}}$ in $Q_{c,b-1}$.
- The same subpalindrome will be counted in $P_{c,b-1}$.
- We extend it to the left, thus $\cancel{l_{a+1} \ldots l_{c-1}} l_c \ldots l_d \cancel{l_{d+1} \ldots l_{b-1}}$ will be counted in $P_{a+1,b-1}$.
- We extend it by $l_a$ and $l_b$, and count the resulting palindrome in $Q_{a,b}$.

The following formulae describe this process:

$$Q_{a,b} = \begin{cases} 1 & \text{for } a = b, \\ Q_{a,b-1} + P_{a+1,b-1} + 1 & \text{for } a \neq b \text{ and } l_a = l_b, \\ Q_{a,b-1} & \text{for } a \neq b \text{ and } l_a \neq l_b, \end{cases}$$

$$P_{a,b} = Q_{a,b} + P_{a+1,b}.$$

The formulae above force the erased digits to be added first to the right from the heart, then from the left. After we compute all these values, the total number of palindromes is stored in $P_{1,N}$.

Now, how should we take divisibility into account? Again, let's assume first that we only want to know how many subnumbers of the Great Sacred Number are divisible by $M$. This can also be done with dynamic programming. Let $V_{a,d}$ be the number of subnumbers of $l_1 \ldots l_a$ that are congruent to $d$ modulo $M$. We can express $V_{a,d}$ with a recursive formula:

$$V_{a,d} = V_{a-1,d} + \sum_{c:10c+l_a \equiv d} V_{a-1,c}.$$

The first component describes subnumbers where $l_a$ is erased, and the second describes subnumbers where it remains. At the beginning we set $V_{0,0} = 1$ and $V_{0,d} = 0$ for $d > 0$. By the formula, $V_{N,0}$ is equal to the number of all subnumbers of the Great Sacred Number that are divisible by $M$ (including the one in which we have erased all the digits). Of course, we don't calculate the sum in the formula above by trying all the possible values $c$ for each $d$ (which would lead to $O(NM^2)$ time complexity). Instead, for each value of $a$ we try each value of $c$, and we add $V_{a-1,c}$ to respective $V_{a,d}$ (namely, for $d = (10c + l_a) \bmod M$).

Now we have to combine the two solutions. Note that the key in the second solution was to classify the subnumbers of $l_1 \ldots l_a$ with respect to their remainders modulo $M$. How about simply applying this to our palindrome counting algorithm? Let $P_{a,b,d}$ be the number of subpalindromes of $l_a \ldots l_b$ congruent to $d$, and $Q_{a,b,d}$ be the number of those that include $l_a$.

This will not be enough, though. Consider again the subpalindrome starting with $l_a$ and ending with $l_b$, whose heart ranges from $l_{a'}$ to $l_{b'}$. Assume that the heart divided by $M$ leaves a remainder $c$. What is the big palindrome congruent to? We do not have enough information, since the answer is $(l_a(10^{i+1} + 1) + 10c) \bmod M$, where $i$ is the length of the heart, which we do not know.



Thus, we need to classify our palindromes again, with respect to their lengths: let $P_{a,b,d,i}$ be the number of subpalindromes of $l_a \ldots l_b$ of length $i$ and congruent to $d$. We classify $Q$-type palindromes in a similar way. This leads to an algorithm of complexity $O(N^3M)$. Unfortunately, this complexity is too high...

How can we fix this? We need to use the fact that $M$ is equal to 666. What are the remainders of dividing $10^{i+1} + 1$ by 666? For $i = 0, 1, 2$ we get 11, 101, and 335, respectively. However, for $i = 3$ we get 11 again, for $i = 4$ we get 101, and so on. Therefore, it is enough to consider a coarser classification: let $P_{a,b,d,i}$, for $i = 0, 1, 2$, be the number of subpalindromes of $l_a \ldots l_b$ congruent to $d$, whose length is congruent to $i$ modulo 3.

This leads to an algorithm of complexity $O(N^3)$, and the result is obtained by summing $P_{1,N,0,i}$ over all three values of $i$. Be aware that the problem contains a small but nasty trap: if the total number of palindromes is divisible by 666, we should answer 666 instead of 0.

We should also be careful with space usage. We cannot just memorize all the values of $P_{a,b,d,i}$. However, to calculate $P_{a,b,d,i}$, where $b - a = L$, we only need to know values for which $b - a \in \{L - 2, L - 1, L\}$. Thus, it is enough to remember only three arrays $P_{a,d,i}$ corresponding to three consecutive values of $b - a$, and similarly only three arrays $Q_{a,d,i}$. We encourage the reader to figure out the technical details.

# / Questions

**Contest:** Algorithmic Engagements 2008
**Task author:** Eryk Kopczyński
**Memory:** 32 MB
http://main.edu.pl/en/archive/pa/2008/pyt

One day, king Bytour invented a game for his sons. He described the game to his counsellor, sorcerer Bytelean, in the following way:

*I ordered my three sons (numbered with integers 1, 2, 3) to stand in a row and I placed a gold or a silver crown upon the head of each of them. Son number 1 could then see the crowns of sons 2 and 3, and son number 2 could see the third son's crown. Each of the sons knew that there were at most two gold crowns in total. After that, I asked son number 1 if he knew his crown's color. He answered that he did not know. After that, I asked son number 2 if he knew his crown's color. He also answered that he did not know.*

At this moment, Bytelean interrupted Bytour and told him that he already knew what crown prince 3 wore. Bytour asked him how he knew that. He answered as follows:

*If prince 1 saw two gold crowns, then he would know that he had received a silver one (because there are at most two gold crowns). He answered* NO, *so this situation could not have taken place. Now, if prince 2 saw a gold crown on the third prince's head, then he would know that he had received a silver one—in the opposite case, prince 1 would have answered* YES. *But he did not know that. Therefore, it is a certainty that prince 3 had a silver crown.*

Your task is to implement a (quite) general simulator of such situations. Facts about which the king can ask the princes and the sorcerer (in the preceding situation it was the crown's type) are encoded as a sequence of variables. Some of them are in some way related to the previous variables; for the remaining ones, only the ranges of their values are known. A more detailed description of the problem that is to be solved is contained in the *Input* section.

## Task

Write a program that:

- reads a description of a situation from the input,
- computes the answers that the sorcerer should give,
- writes the result to the output.



## Input

The first line of the input contains three integers, $P$, $V$ and $A$, separated by single spaces. $P$ denotes the number of princes (numbered from 1 to $P$), $V$ denotes the number of variables (numbered from 1 to $V$), and $A$ denotes the number of actions. You can assume that the following inequalities hold: $1 \le P \le 10$, $1 \le V \le 600$, $1 \le A \le 600$.

The following $V$ lines describe variables $v_1, v_2, \ldots, v_V$. Each of these lines is of the form "$Z_i\ A_i\ B_i$" (this notation contains single spaces), where $Z_i$ is one of the characters =, +, −, ∗, /, %, > and $A_i$ and $B_i$ are integers. Depending on the value of $Z_i$ specified in the description of $v_i$, different kinds of facts about the variables are provided. The meanings of the possible $Z_i$ characters are as follows:

| | |
|---|---|
| $= A_i\ B_i$ | $v_i$ is an integer between $A_i$ and $B_i$ ($-1{,}000{,}000 \le A_i \le B_i \le 1{,}000{,}000$) |
| $+ A_i\ B_i$ | $v_i = v_{A_i} + v_{B_i}$ (in this and all subsequent cases, $1 \le A_i, B_i < i$) |
| $- A_i\ B_i$ | $v_i = v_{A_i} - v_{B_i}$ |
| $* A_i\ B_i$ | $v_i = v_{A_i} \cdot v_{B_i}$ |
| $/ A_i\ B_i$ | $v_i = v_{A_i} / v_{B_i}$ (integer part of the division) |
| $\% A_i\ B_i$ | $v_i = v_{A_i} \bmod v_{B_i}$ (the remainder) |
| $> A_i\ B_i$ | $v_i = 1$ if $v_{A_i} > v_{B_i}$; $v_i = 0$ otherwise. |

This information is provided at the beginning of the game to all princes and also to the sorcerer. The following $A$ lines describe the actions. The following actions can occur (the descriptions of actions contain single spaces):

- **S** *g n*
  The value of $v_n$ is revealed to son number $g$. The fact of the revelation of this value to the son is revealed to all princes and to the sorcerer (this does not mean that they are provided with the value itself).
- **T** *g n*
  The king asks son number $g$ whether he knows the value of $v_n$. He is answered YES, and passes this answer to the sorcerer. (The other sons will hear the answer only when action **A** is performed—consequently, a few of them can answer questions "simultaneously" and the answers of some of them will not be used by others.)
- **N** *g n*
  Just as in the previous action, except that the king receives the answer NO.
- **X** *g n*
  Just as in the previous action, except that the king does not pass the answer to the sorcerer, instead asking him to guess the answer. The sorcerer answers YES, NO or I DON'T KNOW (the answer I DON'T KNOW means that the sorcerer

is not sure whether the son knew the answer to the king's question). During this action the king *does not inform* the sorcerer about the actual answer of prince $g$. The king also *does not pass* the sorcerer's answer to the princes.

→ **A** 0 0

All sons are provided with the answers given by all the other sons to all of the king's previously asked questions (answers given during actions **T**, **N** and **X**). During this action the king also *does not inform* the sorcerer about the answers that he received in actions of type **X**.

→ **M** $w$ $n$

The king tells the sorcerer that variable $v_n$ has value $w$.

→ **Q** 0 $n$

The king asks the sorcerer what possible values—according to his current knowledge—$v_n$ may have. The king *does not pass* the sorcerer's answer to the princes.

It should be assumed that both the princes and the sorcerer perform perfect reasoning—meaning that, at every moment, they are able to deduce all the facts that are implied by the constraints revealed by the king at the beginning of the game and by what has happened in the game so far. Additionally, each of them knows that all of them perform perfect reasoning.

### Constraints

The following constraints hold: $1 \le P \le 10$, $1 \le V \le 600$, $1 \le A \le 600$. Additionally, the number of all possibilities (i.e. the product of values $B_i - A_i + 1$ over all variables of type "=") does not exceed 600. In every theoretically possible valuation of variables, the absolute value of each variable is not greater than 1.000.000, and, in the case of operations "/" and "%", $v_{A_i}$ is nonnegative and $v_{B_i}$ is positive. Variable $v_X$ may appear in the definition of $v_Y$ only if $X < Y$.

### Output

For every action of type **Q**, exactly one line should be written out, containing a sequence of all possible values of $v_n$, from the smallest to the largest, separated by single spaces. For every action of type **X**, one line containing TAK (Polish for *yes*), NIE (Polish for *no*) or NIE WIEM (Polish for *I don't know*) should be written out. The answers should be written to the output in the order of appearance of the actions (in the input) to which they apply; the order is independent of the types of these actions.



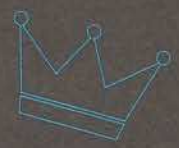
### Example

For the input data:

| | | | |
|---|---|---|---|
| 3 | 7 | 16 | 3 princes, 7 variables, 16 actions. |
| = | 0 | 1 | $v_1$: crown color of son 1 (0 = silver, 1 = gold). |
| = | 0 | 1 | $v_2$: crown color of son 2. |
| = | 0 | 1 | $v_3$: crown color of son 3. |
| + | 1 | 2 | $v_4 = v_1 + v_2$. |
| + | 3 | 4 | $v_5 = v_3 + v_4 = v_1 + v_2 + v_3$, which is, in other words, equal to the total number of gold crowns. |
| = | 3 | 3 | $v_6 = 3$. |
| > | 6 | 5 | $v_7$ determines whether the number of gold crowns is fewer than 3. |
| **S** | 1 | 7 | All sons know that there are fewer than 3 gold crowns. |
| **S** | 2 | 7 | |
| **S** | 3 | 7 | |
| **M** | 1 | 7 | The sorcerer also knows that. |
| **S** | 1 | 3 | Prince 1 can see the crown of prince 3. |
| **S** | 2 | 3 | Prince 2 can see the crown of prince 3. |
| **S** | 1 | 2 | Prince 1 can see the crown of prince 2. |
| **X** | 3 | 3 | Does (according to the sorcerer) son 3 know his crown's color? (no). |
| **X** | 1 | 1 | Does (according to the sorcerer) son 1 know his crown's color? (the sorcerer does not know that). |
| **N** | 1 | 1 | Son 1 does not know his crown's color. |
| **A** | 0 | 0 | All sons are informed that 1 and 3 do not know their crown's colors. |
| **N** | 2 | 2 | Son 2 answers that he still does not know his crown's color. |
| **X** | 3 | 3 | The sorcerer knows that son 3 still does not know his crown's color (he did not hear 2's answer). |
| **A** | 0 | 0 | |
| **X** | 3 | 3 | Now son 3 knows. |
| **Q** | 0 | 3 | The sorcerer also knows that prince 3 has crown of type 0. |

the correct result is:

NIE
NIE WIEM
NIE
TAK
0



# / Solution

There are several logic puzzles similar to the one given to the sorcerer by the king; they are known as *induction puzzles*. One of the most interesting and well known is the one in which the king puts gold and silver crowns on the heads of his $n$ sons (everyone knows that somebody has a gold crown, and every son sees the crowns on the heads of the other sons), and then on each day he asks all his sons whether each knows the color of the crown on his own head. After some days, it turns out that all the sons know their colors! This puzzle appears in many versions. One of them is about wives who kill their husbands after deducing that they are cheating on them (each wife initially knows whether each man other than her husband cheats, and that there is a husband who cheats in the town)—this is called *Josephine's Problem*. Another variant is about people on an island who do not know the color of their own eyes (this version has appeared in an online comic series, *xkcd*). There are also several other similar puzzles, such as:

- Two logicians both receive a nonnegative integer, and they know that their numbers differ by 1. We ask them alternately whether they know their numbers. After a number of questions, both of them know.
- "My neighbor claimed to have three daughters. When I asked him about their ages, he said that the product of their ages was 36. This was not enough information for me, so he told me that the sum of their ages was equal to the number of his house. Despite knowing this number, I still could not calculate their ages. Thus, he said that his oldest daughter had a birthday yesterday. This was sufficient for me to solve the puzzle." The reader can solve it too, even though he does not know the sum.
- There are children from four families playing in a backyard. The number of children in each family is different. Doctor Watson tells Sherlock Holmes that the total number of children is less than 18, and reveals to him the product of the numbers of children in each family. This is not enough for Sherlock to deduce the number of children in every family. He has to ask Watson about the number of children in the smallest family, and this provides sufficient information. The reader can now deduce the answer too.

**Solution**

The game can be briefly summarized as follows: one of several possible situations has happened, and we do not know which one. In this solution the possible situations will be called *worlds*. From the task statement it follows that there are $M \leq 600$ worlds.

We label the possible worlds with integers from 1 to $M$ and calculate the values of all variables for each world. Let $val[v][n]$ be the value of variable number $n$ in world $v$. There's nothing interesting here.

Now we come across the main difficulty in this problem: how should we represent the knowledge available to the sons and the sorcerer? I think that this is the biggest challenge, because if we solve it right, the implementation is quite straightforward.

Imagine an observer who is watching the game. He knows which worlds are possible and the value of each variable in each world. In addition, he can observe that some variable or an answer to a question is revealed to the sons, although he is not given the value or the answer. While the observer does not know the contents of the information given to the sons, he can still simulate what would happen in every possible world. Such an observer has the knowledge that is common to all the sons. He can also tell what each son would do in any particular world.

Note that the sorcerer is independent of the sons. The information that he receives does not influence the sons in any way; nor does the observer benefit from it. In particular, the observer can ignore actions **M** (a value is revealed to the sorcerer) and **Q** (the king asks the sorcerer). Moreover, actions **N**, **T** and **X** (a son answers a question) differ only in what is told to the sorcerer, which means that they are the same from the observer's point of view. This observation allows us to remove the sorcerer from the game for the time being and focus on modeling the world from the observer's point of view.

A son can get to know information of two kinds: the value of a variable (action **S**) and the answer of another son (action **T**, **N** or **X**, followed by an announcement **A**). Note that, from the observer's point of view, none of these actions restricts the set of possible worlds: the son we are considering would get consistent information, possibly different for each world. The observer

knows, however, what would happen in any particular world. If we assume that we are in world $w$ and a son gets to know $v_n$, then afterwards he will only consider the worlds in which the value of $v_n$ is the same as in $w$. A similar thing happens when a son gets to know another prince's answer to a question. We can model this with a relation $\sim_g$, which captures "indiscernibility" for son $g$. We say that the two worlds $v$ and $w$ indiscernible ($v \sim_g w$) if, in world number $v$, son $g$ thinks that $w$ is potentially possible. Note that $\sim_g$ is an equivalence relation.

Initially, $\sim_g$ is a full relation: it has only one equivalence class containing all the worlds—son $g$ knows nothing and thinks that all the worlds are possible, no matter which one is the actual world. However, actions of type **S** and **A** break the equivalence classes into smaller ones.

Let us consider how the sons reason—from this we will know how to update the relation $\sim_g$. For example, consider $n$ sons, each of whom has a silver or a gold crown and knows the types of crowns of other sons. The relation $\sim_g$ is as follows: for each son, there are $2^{n-1}$ equivalence classes, each consisting of two worlds differing in the type of crown on the head of son $g$.

Now, every son learns whether somebody wears a gold crown. As a result, the equivalence class *all other sons have a silver crown* breaks into two, because son $g$ is now able to deduce the type of the crown on his own head.

Then, each son is asked whether he knows the type of the crown on his head. A son answers *yes* only if he sees only silver crowns.

After **A** each son gets to know the answers of other sons. Imagine we are a prince $g$ who sees only one gold crown. Before we discover the answers, two worlds $v$ and $w$ are possible from our point of view. However, the answers allow us to deduce the type of the crown on our head: if we had a silver crown, the owner of the gold crown would say *yes*; otherwise he would say *no*. In other words, the answer given by the prince wearing the gold crown allows us to differentiate worlds $v$ and $w$. Thus, the new knowledge breaks down equivalence classes *we see one gold crown* into two, $v \not\sim_g w$. Equivalence classes with more gold crowns remain untouched.

Again, we ask each son about the color of his crown—now, the answer is *yes* if we see at most one gold crown. After action **A**, just as before, equivalence classes with two gold crowns in sight are split. And so on: after each iteration of questioning and broadcast, more and more equivalence classes are broken.



Let us try to generalize this. When some son $g$ discovers, say, an answer of some other son to a question, we iterate through all equivalence classes of $\sim_g$. For a class consisting of worlds $w_1,\ldots,w_k$ we check what answer the son would receive in each world. Then we split the equivalence class into two smaller ones, one consisting of worlds where the answer would be *yes* and the other of worlds with the answer *no*.

Now, how should we implement this? We can do it as follows. We have arrays of Boolean values $valid[g][v][w]$. We set $valid[g][v][w]$ true if $v \sim_g w$. This leads to a solution that runs in $O(MV + APM^2)$ time.

We can also do it more effectively: instead of an array, we enumerate the equivalence classes in $class[g][v]$. Thus, $v \sim_g w$ if and only if $class[g][v] = class[g][w]$. The simplest way is to assume that $class[g][v]$ is a representative of the given class, i.e. $class[g][class[g][v]]$ always equals $class[g][v]$.

How might we implement questions (**X** $g$ $n$)? First, let $R$ be the set of class representatives, i.e. all values of $v$ such that $class[g][v]$ equals $v$. For each representative $v \in R$ we examine the value of $val[v][n]$. We now want to find the equivalence classes in which the value of variable $n$ is unambiguous—in these worlds son $g$ knows the value of $n$. Let us denote this set by $W$. Formally speaking, it is the set of representatives $v \in R$, such that for all $w$ satisfying $v \sim_g w$ we have $val[v][n] = val[w][n]$. The set $W$ can easily be calculated in $O(M)$—we simply examine all worlds $w$ and check whether $val[w][n]$ is equal to $val[class[g][w]][n]$. In world $v$ the son answers *yes* if $class[g][v] \in W$; otherwise he answers *no*. We write down the answers in $answer[v][a]$, where $v$ is the world index and $a$ is the action index.

Actions **S** $g$ $n$ are handled similarly. For each world $v$ we define a pair $(class[g][v], val[v][n])$ and sort these pairs lexicographically in $O(M \log M)$. After that, we are able to break classes into smaller ones in such a way that, in each resulting class, the values $val[v][n]$ are equal. Actions **A** are executed likewise, except that we use $answer[v][a]$ for all values of $a$ corresponding to new answers, instead of $val[v][n]$.

Now we know how to simulate the sons' knowledge. It is quite straightforward to simulate the sorcerer's knowledge: we simply consider $S$, the set of *all* worlds that are consistent with the sorcerer's knowledge. Initially, $S = \{1,\ldots,M\}$, but the set may lose some of its members after each action of type **M**, **T** or **N**. The king's queries are also implemented quite easily: we simply check the answer for each element of $S$ and output all possible answers.

**Epilogue**

I think that the problem is interesting for two reasons. First, it is based on well-known puzzles. Although induction puzzles are very interesting, they appear only rarely in programming contests. Second, we have to implement a very simple programming language, which is also rather uncommon. The language is very simple, but even so, each of the puzzles we mentioned in the introduction can be implemented in it, as a special case of our problem (of course, we need to limit the numbers in the first puzzle). And indeed, all these puzzles appeared in the test data used to grade the contestants.

**MARCIN KUBICA**

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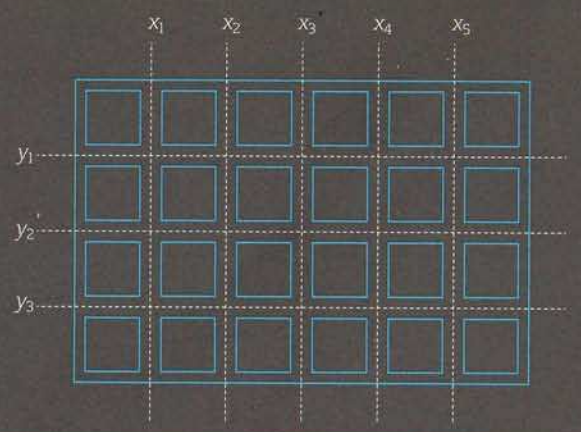
He is a scientific secretary of the Polish Olympiad in Informatics. In the past he has been involved in organizing numerous international programming contests such as the International Olympiad in Informatics (2005), Central European Olympiad in Informatics (2004, 2011) and Baltic Olympiad in Informatics (2001, 2008). Marcin Kubica is a former member of the International Scientific Committee of the International Olympiad in Informatics, from the years 2003-2006 and 2007-2010.

# / Chocolate

**Contest:** 10th Polish Olympiad in Informatics
**Task author:** Marcin Kubica
**Memory:** 32 MB
http://main.edu.pl/en/archive/oi/10/cze

We are given a bar of chocolate composed of $m \times n$ square pieces. The chocolate should be broken into single squares. Parts of the chocolate may be broken along vertical and horizontal lines, as indicated by the dashed lines in the picture. A single break of a part of the chocolate along a chosen vertical or horizontal line divides that part into two smaller pieces. Each break of a part of the chocolate is charged a cost, expressed by a positive integer. This cost does not depend on the size of the part that is being broken but only depends on the line the break goes along. Let us denote the costs of breaking along consecutive vertical lines with $x_1, x_2, \ldots, x_{m-1}$ and along horizontal lines with $y_1, y_2, \ldots, y_{n-1}$. The cost of breaking the whole bar into single squares is the sum of the successive breaks. The objective is to compute the minimum cost of breaking the whole bar of chocolate into single squares.




For example, if we break the chocolate presented in the picture first along the horizontal lines, and then break each resulting part along the vertical lines, then the cost of the whole operation will be $y_1 + y_2 + y_3 + 4 \cdot (x_1 + x_2 + x_3 + x_4 + x_5)$.



**Task**

Write a program that:

- reads the numbers $x_1, x_2, \ldots, x_{m-1}$ and $y_1, y_2, \ldots, y_{n-1}$,
- computes the minimum cost of breaking the whole chocolate bar into single squares,
- writes the result to the output.

**Input**

In the first line of the input there are two positive integers, $m$ and $n$ ($2 \le m, n \le 1000$), separated by a single space. In the successive $m-1$ lines there are numbers $x_1, x_2, \ldots, x_{m-1}$, one per line ($1 \le x_i \le 1000$). In the successive $n-1$ lines there are numbers $y_1, y_2, \ldots, y_{n-1}$, one per line ($1 \le y_i \le 1000$).

**Output**

In the first and only line of output your program should write one integer—the minimum cost of breaking the whole chocolate into single squares.

**Example**

For the input data:

```
6 4
2
1
3
1
4
4
1
2
```

the correct result is:

```
42
```

# / Solution

Oral examinations are exhausting, especially for the examiners. The majority of students who decide to take oral exams (to correct results from a prior written test) fall into two groups. The first group consists of those students who have to take the oral exam, or otherwise they would fail. The other group comprises very ambitious students: those who are determined to get the highest possible mark. Although both groups demand different kinds of questions, the examiner usually runs out of ideas for new, original questions quite quickly.

So it was, once, during an oral exam for *Introduction to Programming*. The exam was slowly coming to its end when another student arrived. He was not satisfied with *A*– and wanted to improve it to *A*. I decided to give him a task on dynamic programming, in which one had to calculate something for each rectangular fragment of a given rectangle. A very good example of a rectangle, and a delicious one too, is a chocolate bar. But what can one do with a chocolate bar? Of course it can be eaten, after dividing it into single pieces. The question of how many times one has to break the chocolate to divide it into single pieces is too easy—with each break, the number of pieces increases by one and the final number of pieces is known (of course, provided that no-one is eating them in the meantime!). The complexity of the problem had to be increased somehow. Breaking a piece of chocolate into smaller pieces can be associated with some cost, depending on the place where it is broken. A possible goal is to minimize the total cost. That is how the task *Chocolate* emerged.

**Solution**

When the student started describing a greedy solution to this problem, I switched on and started listening eagerly. Would he perhaps not deserve an *A* after all? The student had to prove the correctness of the greedy algorithm. This was not easy; cases and sub-cases kept multiplying all the time. Finally, we came to a common conclusion: that the correctness of the algorithm can be proven. Here is how the proof goes.

The basic observation is that it is better first to break the chocolate along the lines with the highest cost, so that the number of breaks along such lines is minimal. First, let us note that the total number of breaks does not depend on the way we break the chocolate into pieces, and always equals $n \cdot m - 1$. This fact is probably well known to all chocolate connoisseurs.



If the number of breaks does not depend on the order of breaking, then we should break the chocolate in such a way that the number of high-cost breaks is as small as possible. Intuitively, the chocolate should be broken along lines in order of their decreasing costs. The following lemma proves this intuition.

***Lemma 1.*** There exists an optimal solution in which the costs of consecutive breaks are non-increasing.

*Proof.* Let us assume that the lemma is false, and let

$$(p_1, p_2, \ldots, p_{n \cdot m-1})$$

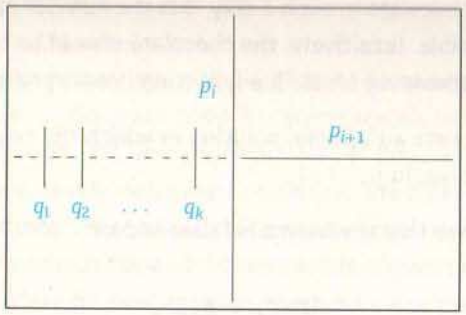
be an optimal sequence of breaks with a maximal number of inversions* of the breaks' costs. There exist two consecutive breaks, $p_i$ and $p_{i+1}$, such that the cost of $p_{i+1}$ is larger than the cost of $p_i$. If there are several such indices $i$, then let us choose the largest one. The costs of breaks $p_{i+1}, p_{i+2}, \ldots, p_{n \cdot m-1}$ form a non-increasing sequence. There are a few possible cases:

1. Break $p_{i+1}$ is located outside a piece of chocolate broken by $p_i$. In this case we can swap breaks $p_i$ and $p_{i+1}$, preserving the optimality of the solution. However, the sequence of costs of breaks

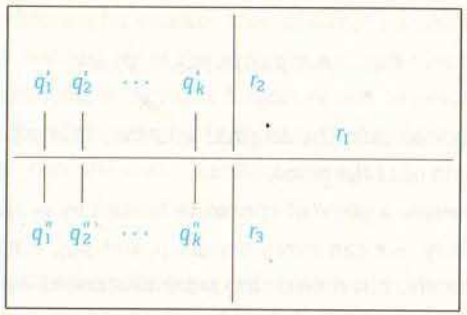
$$(p_1, \ldots, p_{i-1}, p_{i+1}, p_i, p_{i+2}, \ldots, p_{n \cdot m-1})$$

   has more inversions than the original solution. This is not possible; hence such a case could not take place.
2. Break $p_{i+1}$ lies inside a piece of chocolate broken by $p_i$ and both breaks are parallel. Similarly, we can swap breaks $p_i$ and $p_{i+1}$, obtaining an optimal sequence of breaks, but containing more inversions—a contradiction, so such a case is not possible either.
3. Break $p_{i+1}$ lies inside a piece of chocolate broken by $p_i$ and both breaks are orthogonal to each other. This situation is shown in the figure on the following page. Among the breaks to follow, there must be one or more breaks along a dashed line. The number of such breaks depends on the number of breaks $q_1, \ldots, q_k$ crossing the dashed line.

* The number of inversions of a sequence $(x_1, x_2, \ldots, x_k)$ is the number of pairs $(i, j)$ such that $i < j$ and $x_i > x_j$.




Let us consider a slightly modified way of breaking the chocolate. Let us replace breaks $p_i$, $p_{i+1}$ and breaks along the dashed line with breaks $r_1$, $r_2$ and $r_3$, as shown in the figure below. Additionally, let us replace breaks $q_1,\ldots,q_k$ with breaks $q'_1, q''_1, \ldots, q'_k, q''_k$.




How much does the total cost of breaking change? Each of the breaks $p_i, q_1, \ldots, q_k$ is replaced by two breaks of the same cost, and $k+2$ breaks with the same cost as $p_{i+1}$ are replaced by one such break. Note that the costs of breaks $q_1,\ldots,q_k$ are not larger than the cost of $p_{i+1}$. Since the cost of $p_{i+1}$ is larger than the cost of $p_i$, we obtain a solution that's better than the optimal one. This is not possible, so such a case could not happen.

The assumption that the lemma is false leads to a contradiction. Hence, the lemma is true. □



We have shown that there exists an optimal solution in which the costs of consecutive breaks form a non-increasing sequence. But does it suffice to order the costs of the breaks in such a way in order to obtain an optimal solution? The following lemma shows that it does.

***Lemma 2.*** All ways of breaking the chocolate, such that the costs of consecutive breaks form a non-increasing sequence, have the same total cost.

*Proof.* The proof goes by induction over the size of the chocolate bar. If the chocolate bar consists of just one piece, then there exists only one possible (empty) way of breaking, so the lemma holds.

Let us assume that the chocolate bar is larger. Consider all the lines of breaking with maximal cost. We start by breaking along these lines. Let us assume that there are $k$ such vertical and $l$ such horizontal lines. They divide the chocolate bar into $(k+1)\cdot(l+1)$ smaller rectangular chocolate bars.

Let us treat these smaller bars as single pieces of chocolate for a moment. This corresponds to a $(k+1)\times(l+1)$ chocolate bar with all the breaking lines having the same cost. Hence, each way of breaking such a chocolate bar has the same total cost and requires $(k+1)\cdot(l+1)-1$ breaks.

Observe that each way of breaking the whole chocolate bar, with non-increasing costs of consecutive breaks, consists of breaks with maximal costs augmented with ways of breaking the remaining smaller chocolate bars. By inductive assumption, for each of the remaining smaller chocolate bars, each way of breaking it with non-increasing costs of consecutive breaks has the same total cost. Hence, each way of breaking the whole chocolate bar, with non-increasing costs of consecutive breaks, has the same total cost. □

The following theorem is a direct consequence of Lemmas 1 and 2:

***Theorem 3.*** Breaking the chocolate by breaking along the whole chocolate bar and along lines of non-increasing costs gives an optimal way of breaking the chocolate bar.

Based on this theorem, the task can be solved using a greedy approach. First, we sort the lines of breaking according to their costs, remembering which are vertical and which horizontal. Then we break the chocolate bar (across the whole bar) along lines of non-increasing costs. We do not have to track exactly the state of the chocolate bar. It is sufficient just to remember the number of vertical and horizontal lines along which the chocolate bar has been broken. The cost of breaking across the whole chocolate bar equals the cost of breaking

along the given line multiplied by the number of orthogonal breaks (plus one) done so far. Here is pseudocode for such a solution:

```
Algorithm Chocolate((x_1, ..., x_{m-1}), (y_1, ..., y_{n-1}))
    lines := ((x_1, true), ..., (x_{m-1}, true), (y_1, false), ..., (y_{n-1}, false))
    sort lines in non-increasing order, according to the first elements of the pairs
    cost := 0
    vertical := 1; horizontal := 1
    for (w, d) in lines do
        if d then
            cost := cost + horizontal · w
            vertical := vertical + 1
        else
            cost := cost + vertical · w
            horizontal := horizontal + 1
    return cost
```

The cost of sorting dominates the time complexity of this algorithm. In the task presented at the Olympiad, costs of breaking were integers between 1 and 1000. This allows us to apply counting sort and achieve $\Theta(n+m)$ time complexity. Without this restriction, the time complexity of the solution is $\Theta((n+m)\log(n+m))$.

The presented solution is much simpler that the proof of its correctness. Also, the greedy strategy is intuitive and straightforward. No wonder, then, that many contestants have applied such a strategy without proving its correctness. The task turned out to be relatively easy—71% of contestants received the maximum number of points, and proved themselves to be smarter than the author during the described oral exam! Moreover, the aforementioned student who had to justify the correctness of the greedy strategy did indeed deserve his improved final mark of *A*.

# / Fibonacci *Sums*

**Contest:** 12th Polish Olympiad in Informatics
**Task author:** Marcin Kubica
**Memory:** 32 MB
http://main.edu.pl/en/archive/oi/12/sum

The Fibonacci numbers form an integer sequence defined in the following way: $Fib_0 = 1$, $Fib_1 = 1$, $Fib_i = Fib_{i-2} + Fib_{i-1}$ (for $i \geq 2$). The first few numbers in this sequence are: 1, 1, 2, 3, 5, 8,...

The great computer scientist Byteasar is constructing an unusual computer in which numbers are represented in a *Fibonacci system*: i.e. a bit string $(b_1, b_2, \ldots, b_n)$ denotes the number $b_1 \cdot Fib_1 + b_2 \cdot Fib_2 + \cdots + b_n \cdot Fib_n$. (Note that we do not use $Fib_0$.) Unfortunately, such a representation is ambiguous: i.e. the same number can have different representations. The number 42, for instance, can be written as: (0, 0, 0, 0, 1, 0, 0, 1), (0, 0, 0, 0, 1, 1, 1, 0) or (1, 1, 0, 1, 0, 1, 1). For this very reason, Byteasar has limited himself to using only representations that satisfy the following conditions:

- if $n > 1$ then $b_n = 1$, i.e. the representation of a number does not contain leading zeros,
- if $b_i = 1$ then $b_{i+1} = 0$ (for $i = 1, \ldots, n-1$), i.e. the representation of a number does not contain two (or more) consecutive 1s.

The construction of the computer has proved more demanding than Byteasar supposed. He has difficulties in implementing addition. Help him!

## Task

Write a program that:

- reads from the input the representations of two positive integers,
- calculates and writes to the output the representation of their sum.

## Input

The input contains the Fibonacci representations (satisfying the aforementioned conditions) of two positive integers $x$ and $y$—one in the first line, the other in the second. Each of these representations is in the form of a sequence of nonnegative integers, separated by single spaces. The first number in the line specifies the length of the representation $n$ ($1 \leq n \leq 1{,}000{,}000$). It is followed by $n$ 0s and/or 1s.

**Output**

In the first and only line of the output your program should write the Fibonacci representation (satisfying the aforementioned conditions) of the sum $x + y$. The representation should be in the form of a sequence of nonnegative integers, separated by single spaces, as described in the *Input* section. The first number in the line specifies the length of the representation $n$. It is followed by $n$ 0s and/or 1s.

**Example**

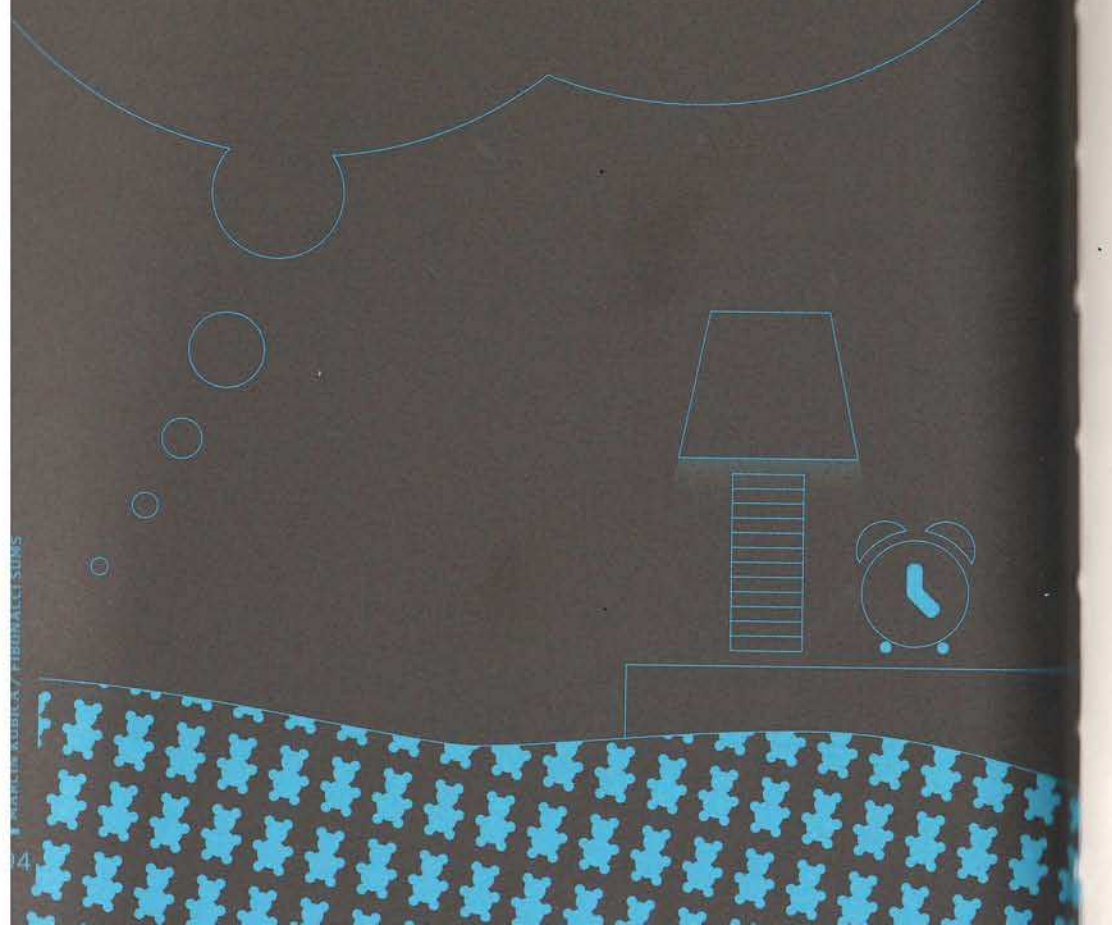
For the input data:

4 0 1 0 1

5 0 1 0 0 1

the correct result is:

6 1 0 1 0 0 1





# / Solution

When you work late at night and think up new things, they often do not seem the same in the morning. This is what happened one night before a written test on *Introduction to Programming*. Fighting against sleepiness, I had decided to contrive an easy task. What came to my mind was the addition of integers represented in Zeckendorf's system.

It turned out that this task surprised me in several ways. The first surprise was that no student managed to finish the test before the time ran out. The second surprise was that even the best students (including former champions in the International Olympiad in Informatics) complained about the difficulty of the task. But the biggest surprise came when the students inquired about the linear-time solution of this task, and I just realized that the problem was not as easy as it had seemed the night before. After a week or two, I thought I could solve the task. Unfortunately, the more inquisitive students disabused me. After two months Jacek Chrząszcz, with whom I gave an *Introduction to Programming* tutorial, helped me in finding a linear-time solution to this task. It was time to turn an initial failure into a success, and the task was used in the second stage of the 12th Polish Olympiad in Informatics.

But that was not the end of the surprises. During the preparation of this text, Tomasz Idziaszek noticed that the solution I presented could be simplified. This resulted in a simpler algorithm and a more elegant proof of its correctness than those presented in the *Blue Book*, containing analyses of tasks from the Olympiad*.

**Zeckendorf's system**

Before we see the linear-time solution of this task, let us have a look at Zeckendorf's system. It appeared in the task statement under the name of the *Fibonacci system*. This system is a mathematical curiosity and has no practical applications; the reason is that the implementation of arithmetical operations is more complicated and less efficient than in the case of the classical binary system.

* Unfortunately, the *Blue Book* is available only in Polish.

First, let us make sure that this system is an unambiguous and correct positional system; that is, each positive integer has exactly one representation in this system. Let us denote by $(b_k b_{k-1} \ldots b_1)_{Fib}$ the sum $\sum_{i=1}^{k} b_i \cdot Fib_i$. Let $x$ be any nonnegative integer. We can transform it into Zeckendorf's system in the following way:

1. If $x \leq 1$, then its representation consists of just one digit: $x_{Fib} = x$.
2. If $x > 1$, then let $Fib_k$ be the largest Fibonacci number not exceeding $x$. The representation of $x$ consists of $k$ digits and the most significant digit is $b_k = \lfloor x / Fib_k \rfloor$. The remaining digits $b_{k-1}, \ldots, b_1$ form a representation of $x - b_k \cdot Fib_k$, completed by some leading 0s if necessary.

Let us check that this algorithm is correct: that is that, for each positive integer, it gives a result satisfying the conditions just stated.

***Lemma 1.*** For any positive integer $x$, the above algorithm produces a sequence of $k$ digits 0 and 1, such that:

(a) $(b_k b_{k-1} \ldots b_1)_{Fib} = x$,
(b) $b_k = 1$,
(c) if $b_i = 1$, then $b_{i+1} = 0$ (for $i = 1, \ldots, k-1$).

*Proof.* Clearly, the representation produced by the algorithm satisfies condition (a). Proof of the other two conditions goes by induction over $x$. For $x \leq 2$ one can easily check that the lemma holds.

Assume that $x > 2$ and that the lemma holds for values smaller than $x$. The largest Fibonacci number not exceeding $x$ is $Fib_k$, so:

$$Fib_k \leq x < Fib_{k+1} = Fib_{k-1} + Fib_k,$$

and:

$$x - Fib_k < Fib_{k-1}.$$

Hence $b_k \leq 1$ (since $Fib_{k-1} < Fib_k$). From the construction of the algorithm, we know that $b_k \geq 1$. Thus we obtain condition (b). The above inequalities also imply that $b_{k-1} = 0$ and the representation of $x - Fib_k$ consists of at most $k-2$ digits. By the induction assumption we find that the sequence of digits $b_k, \ldots, b_1$ satisfies condition (c). Moreover, all the generated digits are 0s and 1s. □

Representations satisfying the conditions of Lemma 1 are called *correct*. Now we will show that representations in Zeckendorf's system are unambiguous.

***Lemma 2 (Zeckendorf).*** For each positive integer $x$, there exists exactly one correct representation of $x$ in Zeckendorf's system.

*Proof.* Let $x$ be any positive integer such that $x < Fib_{k+1}$. Its representation can have at most $k$ digits (that is, $b_i = 0$ for $i > k$). Let us count the number of correct representations of length at most $k$, denoted as $f_k$. Since $1_{Fib}$ is the only correct one-digit representation, we have $f_0 = 0$ and $f_1 = 1$.

To count the number of correct representations of length at most $k$ (for $k \geq 2$), we should add the number of correct representations of length $k$ to the number of correct representations of length at most $k-1$. If a representation $b_k, \ldots, b_1$ starts with $b_k = 1$ then, by condition (c), $b_{k-1} = 0$ and $b_{k-2}, \ldots, b_1$ can be any correct representation of length at most $k-2$ extended with an appropriate number of leading 0s or a sequence of 0s. Hence, we obtain:

$$f_k = f_{k-1} + f_{k-2} + 1.$$

It can be proven by simple induction that:

$$f_k = Fib_{k+1} - 1.$$

Indeed, for $k \leq 1$ it can be easily checked. If the above equation is true for all indices smaller than $k$, then:

$$f_k = f_{k-1} + f_{k-2} + 1 = Fib_k - 1 + Fib_{k-1} - 1 + 1 = Fib_k + Fib_{k-1} - 1 = Fib_{k+1} - 1.$$

Since every positive integer smaller than $Fib_{k+1}$ (and there are $Fib_{k+1} - 1$ such integers) has a correct representation of length at most $k$ (and there are also $Fib_{k+1} - 1$ such representations), each such integer has an unambiguous representation. □

**Simple addition algorithm**

The simplest way to add numbers in Zeckendorf's system is to add Fibonacci numbers corresponding to 1s in the representation of one number to representation of the other number. To add $Fib_i$ to representation of $x$, first we increase the $i$-th digit by 1. As a result we can obtain:

- a correct representation,
- a representation containing two or three neighboring 1s,
- a representation containing 2 surrounded by zeroes.

If the resulting representation is not correct, we apply the following transformations.

**Elimination of a single 2.** If the result contains a digit 2, we can eliminate it by applying the following identities, starting from the most significant digits:

$$x0200y_{Fib} = x1001y_{Fib} \tag{1}$$

$$x0201y_{Fib} = x1002y_{Fib} \tag{2}$$

$$x020_{Fib} = x101_{Fib} \tag{3}$$

$$x02_{Fib} = x10_{Fib} \tag{4}$$

Note that in the identity (2), digit 2 does not disappear, but is shifted to the right. However, applying the above identities from most to least significant digits, it is eliminated in one of the following steps. It is worth noting that a newly created 2 is also surrounded by 0s (or is the last digit of the representation), since 1 at this position before transformation was also surrounded by 0s (or was the last digit, respectively). The whole process requires linear time. Finally, we obtain a representation that does not contain any 2s, but may contain some neighboring 1s.

**Elimination of neighboring 1s.** We can eliminate neighboring 1s by applying the following identities, starting from the least significant digits:

$$x0\underbrace{11\ldots11}_{2l\text{ digits}}y_{Fib} = x\underbrace{10\ldots10}_{2l\text{ digits}}0y_{Fib}$$

$$x0\underbrace{11\ldots11}_{2l+1\text{ digits}}y_{Fib} = x\underbrace{10\ldots10}_{2l\text{ digits}}01y_{Fib}$$

The whole transformation takes linear time.

So, the time-cost of addition of $Fib_i$ and a number represented in Zeckendorf's system is also linear. Hence, the addition of two integers has a quadratic overall time complexity.



**Efficient addition algorithm**

An efficient algorithm for addition in Zeckendorf's system is more complicated. We start by adding corresponding digits of the two summands. As a result, we obtain a representation that:

- may contain 0s, 1s and 2s,
- permits 1s to be placed next to each other,
- allows 2s to be surrounded only by 0s.

Again, such a representation can be transformed to a correct one in two steps. In the first step we eliminate all 2s in linear time. In the second step, we can eliminate neighboring 1s as presented in the previous section.

**Elimination of 2s.** First, let us transform the representation in such a way that it satisfies the following condition:

to the left of each 2 there is 0,
to the right of each 2 there is 1 or a sequence $02y_{Fib}$. (∗)

It turns out that this can be achieved by applying identities (1)-(4) and:

$$x012y_{Fib} = x101y_{Fib} \tag{5}$$

starting from the most significant digits.

To prove the correctness of such a transformation, first let us note that initially each 2 is surrounded by 0s, so the first part of (∗) is satisfied. The only identity that can possibly violate it is (1), but that would mean that $y$ starts with a 2 and it will be eliminated in the next step by (5).

Let us focus on the most significant 2, which does not satisfy the second part of condition (∗). It is followed by $00y_{Fib}$, $01y_{Fib}$, $0_{Fib}$ or is the least significant digit of the representation. Each of these corresponds to one of the identities (1)-(4). Note that identities (1), (3) and (4) remove 2s, and (2) shifts 2 to the right. In the latter case, digit 2 will either be transformed in the next step, or $y$ starts with 1 and this 2 does not violate condition (∗).

Additionally, none of the applied identities violates the second part of the condition (∗), because if $x$ ends with 2, then it can be followed by sequences of the form $02y'_{Fib}$ or $1y'_{Fib}$. The only case left to be considered is when we apply (1) and $y$ starts with 2. But then, this 2 will be eliminated in the very next step by applying identity (5).

Now we are ready to eliminate all the remaining 2s. Scanning the representation from the least significant to the most significant digits, we apply the following identity:

$$x021y_{Fib} = x110y_{Fib} \tag{6}$$

Condition (∗) guarantees that we can apply identity (6) to the least significant 2 in the representation. Moreover, this identity preserves condition (∗)—argumentation for the case where $x$ ends with 2 is the same as above.

So, finally, all the 2s are eliminated. What remains, in obtaining a correct representation, is to apply the elimination of neighboring 1s, described above. Hence, the algorithm for the addition of two integers in Zeckendorf's system works in linear time.

**TOMASZ KULCZYŃSKI**

Tomasz Kulczyński is currently doing his M.S. degree in computer science at the Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, and participates in all kinds of programming contests. He won the International Olympiad in Informatics in 2007 and the Baltic Olympiad in Informatics in 2006. He was awarded over ten medals of different colors in Polish and international olympiads for high school students, both in computer science and mathematics. He finished in top places in TopCoder competitions (he is a five-time finalist of these competitions) as well as Google Code Jam 2006 and Facebook Hacker Cup 2011 and 2012. Together with his team, Tomasz Kulczyński has won the Polish Collegiate Programming Contest three times and the ACM-ICPC Central European Regional Contest twice. He also represented the University of Warsaw in the ACM-ICPC World Finals in 2011 and 2012 (in a team with Jakub Pachocki and Wojciech Śmietanka).

Additionally, Tomasz Kulczyński is passionate about billiards, solves lots of puzzles, and enjoys running and playing soccer.

# / The Search

**Contest:** 16th Polish Olympiad in Informatics
**Task author:** Tomasz Kulczyński
**Memory:** 32 MB
http://main.edu.pl/en/archive/oi/16/pos

Young Bytie and Byteen are in deep trouble. The evil sorcerer Bitter, knowing they are head over heels in love, has kidnapped Byteen for ransom. What Bitter did not know is that, while Bytie is determined to bring Byteen back, he is not well off and thus cannot afford the ransom. Having few options, Bytie has challenged Bitter for Byteen's return. Bitter despises all fights, but burning ambition made him propose the following solution. If the boy guesses the floor on which the girl is imprisoned, the sorcerer will let her free.

There are many floors in the sorcerer's tower—they are numbered from 1 to $n$. Bytie can only obtain information by asking Bitter questions of the form: "Is Byteen above/below floor number $x$?" Of course, Bytie can choose the number $x$ arbitrarily, as well as either of the words "above" or "below". Bitter promises to answer such queries truthfully, but asks for a reward. The sorcerer charges $a$ bytean dollars for every question with the answer YES and $b$ bytean dollars for every question with the answer NO. He also refuses to answer questions of any form other than that mentioned.

Bytie wonders what questions he should ask. Obvously, Bytie would like to spend as little as possible. To be precise: if, at any point during the conversation, it is possible to deduce that, from then on—irrespective of Bitter's answers—Bytie can guess Byteen's location by spending no more than $K$ bytean dollars, then Bytie wants to spend not more than $K$ bytean dollars in the rest of the conversation. Help Bytie find the right solution!

## Communication

You are to implement a program that solves Bytie's problem using a supplied library (which simulates the evil sorcerer Bitter).

The library supplies three functions:

- `void inicjuj(int *n, int *a, int *b)` (Polish for *init*)—returns the number of floors $n$ and the costs $a$ and $b$. It should be called once, at the very beginning of the program's execution.
- `int pytaj(char c, int x)` (Polish for *ask*)—character $c$ denotes the type of question (W for above or N for below), while $x$ is the floor number. Returns the Boolean answer to the query (0 = false, 1 = true). Your program may call this function arbitrarily many times.
- `void odpowiedz(int w)` (Polish for *answer*)—with this function you should declare the floor on which Byteen is located. It should be called exactly once. The call will terminate your program's execution.

Your program cannot open any files; nor can it use input and output.

## Constraints

You can assume that $1 \le n \le 10^9$ and $1 \le a, b \le 10{,}000$.

## Exemplary execution of a program

| FUNCTION CALL | VALUES RETURNED AND EXPLANATION |
|---|---|
| `inicjuj(&n,&a,&b);` | From now on, $n = 5$, $a = 1$, $b = 2$. |
| `pytaj('W',3);` | The answer is 0. You ask if Byteen is above the third floor. You receive the answer NO. You pay 2 bytean dollars. |
| `pytaj('N',2);` | The answer is 0. You ask if Byteen is below the second floor. You receive the answer NO, for which you pay 2 more bytean dollars. |
| `pytaj('W',2);` | The answer is 1. You ask if Byteen is above the second floor. You receive the answer YES, for which you pay 1 bytean dollar. |
| `odpowiedz(3);` | You guess that Byteen is on the third floor. Your guess is correct. You spend 5 bytean dollars in total. |

The above interaction is correct but not optimal. Hence, the program would not receive any points for this test. In particular, for $n = 5$, $a = 1$, $b = 2$, a well-written program can ask questions in such a way that it never pays more than 4 bytean dollars.

# Solution

I decided to write about this particular task—which I invented—because of a very smart trick used in the model solution. The idea is to reverse the question from the task description. This task was created during the International Olympiad in Informatics in 2008, as Jakub Radoszewski and I coached the Polish team. Our long discussions during warm Egyptian nights resulted in a few tasks, one of which is *The Search*.

**Similarity to binary search**

The description of this task reminds us of binary search. This association is natural, as binary search is the optimal way to solve the task if the cost for each question is equal to 1. It is also optimal when $a = b$, with the cost of finding Byteen equal to $a \cdot \lceil \log n \rceil$. In the general case, binary search has a pessimistic cost of $\max(a, b) \cdot \lceil \log n \rceil$. We can clearly see that, for $a \neq b$ and large values of $n$, this is not optimal. One should make queries about a point depending on $a$ and $b$, rather than the center of the interval, to compensate for the difference between the two query costs. It is not easy to determine this point, which is why we will use dynamic programming for this purpose.

**First solution**

The first solution tries to solve the problem directly by computing, for each positive integer $k$, the minimal worst-case cost of finding Byteen in an interval consisting of $k$ floors. Clearly, this cost does not depend on where the floors are located within the whole tower but only on their number $k$. From now on we will denote this cost by $t[k]$.

We will compute the values $t[k]$ in order of increasing $k$ via dynamic programming. Obviously, $t[1] = 0$. If $k > 1$, we must pose at least one question to find Byteen. Assume that the first question asks if Byteen is located below the floor $p + 1$ among those available ($1 \leq p < k$). If Bitter's answer is positive, Bytie will spend $a$ for asking the question and will continue his search with an interval consisting of the first $p$ floors. In the opposite case, Bytie will spend $b$ and the new interval's length will be $k - p$. In both cases, we used all the information we could obtain from a single answer from Bitter to draw a conclusion about Byteen's location.



Assume that, in the rest of the conversation, Bytie will ask questions in a way that minimizes the total cost. Then, in the case of a positive answer to the first question, Bytie will spend $a + t[p]$ in total; otherwise, his total cost will be $b + t[k - p]$. The pessimistic cost is the maximum of the two values.

We should now realize that the questions:

- "Is Byteen located above floor $p$?"
- "Is Byteen located below floor $k - p + 1$?"

do not differ from each other, taking into consideration the lengths of intervals that we get after Bitter's answer (equal to $k - p$ and $p$) and the costs of the first question ($a$ and $b$ respectively). That is why it is sufficient to consider one of the types of questions. To compute $t[k]$ we only have to choose the value of $p$ such that the cost of finding Byteen with the first question, as stated above, is minimal. We have:

$$t[k] = \min_{1 \leq p < k} \{\max(a + t[p],\ b + t[k - p])\}.$$

We will also remember the value of $p$ that minimizes the cost and call it $p[k]$, so that:

$$t[k] = \max(a + t[p[k]],\ b + t[k - p[k]]).$$

This way we can find all the values of $t[k]$ and $p[k]$ for $k = 1, 2, \ldots, n$ in $O(n^2)$ time with space usage of $O(n)$. After this precomputation the process of asking questions is easy to implement: at each step, if we have an interval that consists of $k$ floors (with $k > 1$), the next question Bytie should ask is: "Is Byteen located below floor $p[k] + 1$ in this interval?" The total time of the conversation phase is $O(n)$.

**Model solution**

Let us reverse the problem. Instead of finding the optimal costs of locating Byteen in an interval of floors, for each cost $c$ let us find the maximum number of floors in an interval in which Byteen can be located at the cost of at most $c$. We will denote this number of floors by $Q[c]$.

First, we observe that $Q[0] = 1$. For simplicity we assume that for $c < 0$, we have $Q[c] = -\infty$. From now on we consider $c > 0$. Assume that we have already

limited the search to an interval of $Q[c]$ floors and want to ask the next question. The answer we get splits the interval into two shorter intervals. Now we need to find Byteen in one of them with a maximum cost of $c-a$ or $c-b$ respectively. Those two intervals should then have lengths of at most $Q[c-a]$ and $Q[c-b]$ respectively. This leads to the formula

$$Q[c] = \max(Q[c-a] + Q[c-b], 1)$$

from which $Q[c]$ can be computed via dynamic programming.

To ask the right question, we have to find the smallest cost $w$ such that $Q[w-1] < n \le Q[w]$. This is the cost that we need to spend in a pessimistic case. Let us denote $n_0 = n$, $w_0 = w$. Each question will be asked so as to split the interval of length

$$n_i \le Q[w_i] = Q[w_i - a] + Q[w_i - b]$$

into two intervals of length $Q[w_i - a]$ (which we will consider if the answer is positive) and $n_i - Q[w_i - a] \le Q[w_i - b]$ (in the opposite case). Then we either spend $a$ and search in an interval of length $n_{i+1} = Q[w_i - a]$ or spend $b$ and search in an interval of length $n_{i+1} \le Q[w_i - b]$. In both cases we will have a total cost of at most $w_i$.

However, we need to remember that, in the case of a negative answer, $w' = w_i - b$ is not necessarily the smallest $w'$ such that $Q[w'] \ge n_{i+1}$. It is possible that $w_{i+1}$ is actually smaller:

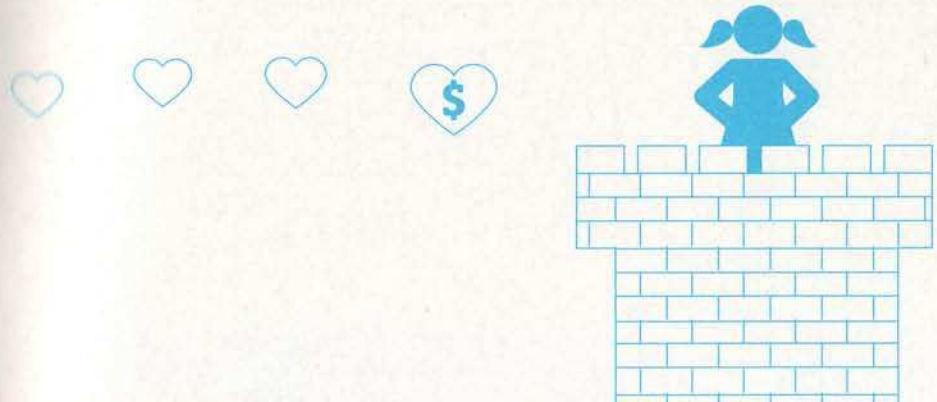
$$w_{i+1} = \min\{j : j \le w' \wedge Q[j] \ge n_{i+1}\}.$$

The necessity to remember about this comes from the requirement to save as much as we can at each step.

This solution has time complexity $O(w)$. First, for each $i \le w$ we have to compute $Q[i]$, in order of increasing $i$, to find $w$ itself. Then we have to go through the values once more while asking questions in order of decreasing $i$.

```
Algorithm Search
    init(n, a, b) {inicjuj}
    Q[0] := 1
    w := 0
    while Q[w] < n do
        w := w + 1
        if w < max(a, b) then
            Q[w] := 1
        else
            Q[w] := Q[w − a] + Q[w − b]
    down := 1; up := n
    while down < up do
        while Q[w − 1] ≥ up − down + 1 do
            w := w − 1
        if ask('N', down + Q[w − a]) then {pytaj}
            up := down + Q[w − a] − 1
            w := w − a
        else
            down := down + Q[w − a]
            w := w − b
    answer(down) {odpowiedz}
```

To estimate the complexity of this solution properly we still need to know how big $w$ can get. We know that $w$ is the cost of the optimal strategy, so it is not greater than the cost of any strategy and, in particular, not greater than the cost of binary search. We have already stated that this cost does not exceed $\max(a,b) \cdot \lceil \log n \rceil$. This estimate is not a bad one, because for $a = b$ we have exactly $w = \max(a,b) \cdot \lceil \log n \rceil$. The time and space complexity is therefore $O(\log n \cdot \max(a,b))$.



**Results**

The tables below show the values of $t$, $p$ and $Q$ from both solutions for $n = 10$ and several different pairs of $a$, $b$.

→ $a = 1$, $b = 2$:

| $i =$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $t[i] =$ | 0 | 0 | 2 | 3 | 4 | 4 | 5 | 5 | 5 | 6 | 6 |
| $p[i] =$ | - | - | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 4 | 5 |

| $i =$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $Q[i] =$ | 1 | 1 | 2 | 3 | 5 | 8 | 13 |

→ $a = 3$, $b = 10$:

| $i =$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $t[i] =$ | 0 | 0 | 10 | 13 | 16 | 19 | 20 | 22 | 23 | 23 | 25 |
| $p[i] =$ | - | - | 1 | 2 | 3 | 4 | 4 | 5 | 5 | 6 | 7 |

| $i =$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $Q[i] =$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| $i =$ | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| $Q[i] =$ | 3 | 3 | 3 | 4 | 4 | 4 | 5 | 6 | 6 | 7 | 9 | 9 | 10 |

→ $a = 5$, $b = 7$:

| $i =$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $t[i] =$ | 0 | 0 | 7 | 12 | 14 | 17 | 19 | 19 | 21 | 22 | 24 |
| $p[i] =$ | - | - | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 |

| $i =$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $Q[i] =$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 3 |
| $i =$ | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | |
| $Q[i] =$ | 3 | 4 | 4 | 4 | 5 | 5 | 7 | 7 | 8 | 9 | 9 | 12 | |

→ $a = 6$, $b = 6$:

| $i =$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $t[i] =$ | 0 | 0 | 6 | 12 | 12 | 18 | 18 | 18 | 18 | 24 | 24 |
| $p[i] =$ | - | - | 1 | 1 | 2 | 1 | 2 | 3 | 4 | 1 | 2 |

| $i =$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $Q[i] =$ | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 4 |
| $i =$ | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | |
| $Q[i] =$ | 4 | 4 | 4 | 4 | 4 | 8 | 8 | 8 | 8 | 8 | 8 | 16 | |



# / Hashing

**Contest:** Polish Olympiad in Informatics Training Camp 2008
**Task author:** Jakub Pawlewicz
**Memory:** 32 MB
http://main.edu.pl/en/archive/ontak/2008/has

Participants in the POI Training Camp have put Byteasar in a difficult position. Instead of turning in solutions based on a beautiful and natural application of the Karp-Miller-Rosenberg algorithm, they submitted solutions based on hashing. In principle, the hashing-based approach is incorrect due to the possibility of hash function collisions. Byteasar has decided to construct test cases that will invalidate such submitted solutions. To this end, he has to find a pair of words with the same hash value.

A word is defined as a non-empty string consisting of at most 20 lowercase letters of the English alphabet. The value of the hash function $h$ on a word is a number from the set $\{0, 1, \ldots, n-1\}$. To a very good approximation, the probability that two different words $x$ and $y$ have the same hash value is $\frac{1}{n}$. Your task is the following: given a hash function with the above properties, demonstrate two different words $x$ and $y$ with the same hash value.

## Hash function

The hash function used in this problem is available through a C/C++ library and takes two arguments: $n$, indicating the range of values returned by hash(n,x), and $x$, indicating the input word (non-empty, at most 20 characters long).

This function has the following prototype:

```
unsigned long long hash(unsigned long long n, const char *x);
```

## Task

Write a program that:

→ reads in $n$ from the input,
→ finds two words $x$ and $y$ such that hash(n,x) = hash(n,y),
→ outputs the answer.

## Input

The input describes one input instance and consists of one line, containing a positive integer $n$.

**Output**

The output consists of two lines, each containing exactly one string of at most 20 lowercase English letters, such that hash returns the same value for both words.

**Example**

For the input data:

10000

the correct result is:

bgvhfqbfenb

xeonduwaxi

**Tests**

Test cases that will be used to test your submission are made available in advance. In effect, your solution may consist of a program with hardcoded answers that are returned after the program reads and identifies the input $n$.

**Comments and remarks**

Note that the task *is not* to find the hash function or its values. Additionally, hash values are unspecified for values of $n$ that are not in the available test files and for strings that do not satisfy the specifications.

# / Solution

The task *Hashing* appeared in the Polish Olympiad in Informatics Training Camp in 2008. I had the pleasure of being a jury member at that camp, and this task drew my attention at once. The problem seems quite natural: it is well known that algorithms that use hashing are inaccurate, so why not try to find sample data for which such an algorithm fails?

The participants of the camp had many different ideas as to how to solve the problem. Some of them tried to decompile binary files of the library containing the *hash* method and use its implementation details to solve the inputs one by one. Of course, that was not the intended approach.

We will try to solve the task without any additional assumptions about how the hash function is implemented. First of all, it is easy to observe that it is sufficient to check any $n + 1$ different strings, because some two of them will have the same hash values (by the pigeonhole principle). This yields a simple solution with $O(n)$ complexity, which is not satisfactory—but in practice we can expect it to run much faster, i.e. it is likely to find the requested pair long before all the $n + 1$ strings have been checked. The only information we have about the hash function is a probabilistic fact, so it naturally comes to mind to choose subsequent strings randomly until a collision is found. The following pseudocode implements this simple approach.

```
S := ∅
while true do
    a := random_word()
    x := hash(n, a)
    if (x, b) ∈ S for some b ≠ a then
        return (a, b)
    S := S ∪ {(x, a)}
```

Denote by $w$ the number of different strings that we need to examine before we find the result. The above algorithm performs $O(w \log w)$ operations and uses $O(w)$ memory. We still have to think about how big $w$ can get.

**Lemma.** The probability of a collision within $k = \lceil\sqrt{n}\rceil + 1$ random strings is greater than $\frac{1}{3}$.

*Proof.* We will try to find an upper estimate for the probability of the lack of collisions. Certainly, the first string we draw does not create any collisions. The second string has a different hash value with the probability $\frac{n-1}{n}$. Similarly, the $i$-th string drawn does not create a collision with any previous string with probability $\frac{n+1-i}{n}$. The probability that there will be no collisions at all equals:

$$P = \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-k+1}{n}.$$

Using a well known inequality $1 + x \le e^x$ we estimate that value as follows:

$$P = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) \le e^{-\frac{1}{n}} e^{-\frac{2}{n}} \cdots e^{-\frac{k-1}{n}} = e^{-\frac{k(k-1)}{2n}} \le e^{-\frac{1}{2}} < \frac{2}{3}. \qquad \square$$

From the lemma it follows with high probability that $w$ is of the order of $O(\sqrt{n})$. E.g. $w > 12 \cdot (\lceil\sqrt{n}\rceil + 1)$ with a probability of less than $1 - \left(\frac{2}{3}\right)^{12} < 0{,}01$. Therefore we can expect that the above solution will perform $O(\sqrt{n} \log n)$ operations with a memory usage of $O(\sqrt{n})$.

The expected running time of the solution seems satisfactory, but the space complexity of $O(\sqrt{n})$ for $n$ up to $10^{18}$ might be problematic. We will now try to reduce the space complexity. For this, we must avoid storing strings that we have already drawn. Let us start by selecting a new function $g(x)$, which maps integers from $H = \{0, 1, \ldots, n-1\}$ to different strings. This function can be chosen arbitrarily. It is important, though, for us to be able to compute its values quickly. Let $f$ be the composition of $g$ and *hash*: $f(x) = hash(n, g(x))$. We know that $f$ maps $H$ to $H$ and has the following property: $f(x) = f(y)$ with $x \ne y$ if and only if $g(x)$ and $g(y)$ are different strings having the same hash value. Now the task reduces to finding two distinct integers $x$ and $y$ such that $f(x) = f(y)$. Note that we can assume that, for each $x$, $f(x)$ is a uniformly random value from the set $H$, because *hash* has that property and $g$ was chosen independently of the function *hash*. Because of that, for every string $s_0$ and $x_0 = hash(n, s_0)$, the sequence

$$x_0, \quad x_1 = f(x_0), \quad x_2 = f(f(x_0)), \ldots, x_i = f^{(i)}(x_0), \ldots \qquad (1)$$

behaves as a random sequence of values from $H$. As before, we expect a collision among the first $O(\sqrt{n})$ elements of the sequence. Depending on the value of $x_0$, the sequence will have one of the forms depicted in figure 1.

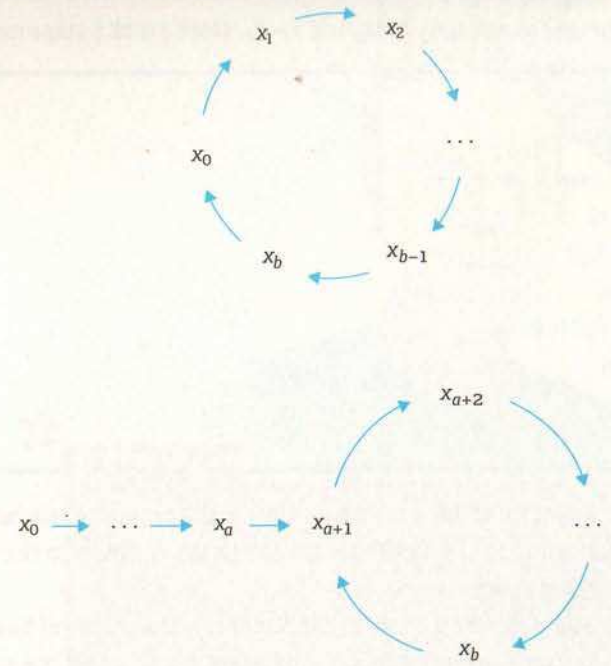



***Figure 1:*** *Possible forms of the sequence (1).*

In the first case, we should output $s_0$ and $g(x_b)$, and in the second case the output is $g(x_a)$ and $g(x_b)$. To avoid considering both cases, we can introduce $x_{-1} \notin H$ such that $g(x_{-1}) = s_0$ and $f(x_{-1}) = x_0$. Then in both cases the answer is $g(x_a)$, $g(x_b)$ (with $a = -1$ in the first case). From now on, we only have to find indexes $a$ and $b$. How can this be done without storing the sequence $(x_i)$?

Let $d$ be the length of the cycle, $d = b - a$. We first take two pointers, $p$ and $q$, and make them both point to $x_0$. We then take consecutive steps, and within each step $p$ goes one element forward and $q$ goes two elements forward (which is equivalent to the assignments $p := f(p)$ and $q := f(f(q))$). We keep performing such steps until $p$ and $q$ become equal. This will happen at some point, since, when both pointers are on the cycle, the distance between them increases by one (modulo $d$) with each step. Let $k$ be the total number of steps. We know that $q$ has made a full cycle several times more than $p$.

In the second run, we introduce two new pointers $r$ and $s$, initially pointing to $x_{-1}$. At first, we apply, $k$ times, the assignment $s := f(s)$. Next, we move both pointers one step forward ($r := f(r)$, $s := f(s)$) until we reach a moment when $f(r) = f(s)$. Then we have exactly $r = x_a$ and $s = x_b$, since $s$ took $k$ steps more than $r$.

```
p, q := x_0
r, s := x_-1
while p ≠ q do
    p := f(p)
    q := f(f(q))
    s := f(s)
while f(r) ≠ f(s) do
    r := f(r)
    s := f(s)
return (g(r), g(s))
```

The time complexity of this solution is linear in the number of elements of $(x_i)$ that we inspect before the first collision is found, that is, $O(\sqrt{n})$ in the expected case, with constant space usage.

In the whole analysis we assumed that computing the values of *hash*, $g$ and $f$ would each take constant time. This makes some sense, as all the strings for which these functions are applied are relatively short.

**JAKUB ŁĄCKI**

Jakub Łącki is a Ph.D. student at the Institute of Informatics, University of Warsaw. His research focuses on dynamic graph algorithms and algorithms on planar graphs. He has won the Polish Collegiate Programming Contest twice, and has been a runner-up in the ACM-ICPC Central European Regional Contest two times as well. Since 2006 he has co-organized an annual Polish Olympiad in Informatics Training Camp, and since 2009 has also been responsible for the selection of tasks for Algorithmic Engagements. He participated in the organization of the Baltic Olympiad in Informatics in 2008 and the Central European Olympiad in Informatics in 2011.

Jakub Łącki is a passionate traveler. In his free time he plays the guitar or soccer.

# / Afternoon *Tea*

**Contest:** Polish Collegiate Programming Contest 2011
**Task author:** Tomasz Idziaszek
**Memory:** 32 MB
http://main.edu.pl/en/archive/amppz/2011/her

During his visit to the Bytic Islands, Byteasar really enjoyed the national beverage of Byteans: that is, tea with milk. This drink is always prepared in a strictly determined manner, which is as follows. Firstly the teacup is filled with tea mixed half and half with milk. Then an $n$-letter ceremonial word consisting of the letters **H** and **M** is chosen. Now, for $i = 1, 2, \ldots, n$, the following action is performed: if the $i$-th letter of the ceremonial word is **H**, one should drink half of the teacup, add tea until the teacup is full, and stir. On the other hand, if the $i$-th letter of the word is **M**, one should perform a similar action, but milk should be added instead of tea. After such action is performed for each letter of the ceremonial word, the remaining liquid is disposed of.

Each time Byteasar performs the ceremony, he wonders which of the ingredients he has drunk more of: tea or milk. Help Byteasar to answer this question.

**Input**

The first line of input holds an integer $n$ ($1 \le n \le 100{,}000$). The second line contains an $n$-letter word consisting of the letters **H** and **M**; this is the ceremonial word used by Byteasar.

**Output**

Your program should output a single letter **H** if Byteasar has drunk more tea than milk; a single letter **M** if he has drunk more milk than tea; or the word **HM** if he has drunk equal amounts of tea and milk.

**Example**

| For the input data: | the correct result is: |
|---|---|
| 5<br>HMHHM | H |

**Explanation of the example:** Byteasar has drunk $1\frac{37}{64}$ teacups of tea and $\frac{59}{64}$ teacups of milk in total.



# / Solution

It was a few weeks before the Polish Collegiate Programming Contest that Tomasz Idziaszek told me about his idea for the task. It did not seem new to me. When I was in elementary or perhaps junior school, during a math lesson I tackled a similar problem in which a boy was drinking a mixture of two beverages and filling his glass. That version was much easier, however: the boy filled the glass only three times and drank everything at the end. It is a bit embarrassing to confess that I laboriously added up fractions that became more and more complicated after every turn, and calculated how much the boy drank every time. I did not realize that it was sufficient just to keep track of the beverage that he pours into his glass...

I did not do much better with the task about Byteasar either! This time I knew that counting what happens at each step is probably not the best idea. Also, it was obvious that simulating the process by using the floating-point types (e.g. *double*) that are embedded in the majority of programming languages would not be appropriate: their precision is far too limited for manipulating the fractions that emerge in this task.

So how can we approach this task? It gets a bit easier if we change our perspective a little. Instead of calculating the amount of tea and milk that were poured into the glass, let us consider the amount that Byteasar does not drink. In other words, we focus on calculating what remains in the cup after the ceremony. Since we know how many cups of each beverage were poured into the cup during the ceremony, this information is enough to answer the question. Moreover, in order to avoid working with fractions, we assume that the volume of the cup is $2^{n+1}$ milliliters and that it is initially filled with $2^n$ milliliters of milk and the same amount of tea.

Let us focus on the case when, in the first turn, Byteasar pours in $2^n$ milliliters of tea. How much of this tea will remain in the cup until the end of the ceremony? In the second turn, Byteasar drinks half of this tea, so there are $2^{n-1}$ milliliters remaining. After the third turn, $2^{n-2}$ milliliters remain, and so on. After $n$ turns—that is, $n-1$ dilutions—in the cup there will be 2 milliliters of tea that come from the first turn.

The liquid poured in the second turn is diluted $n-2$ times, so it determines 4 milliliters of the final contents of the cup. In general, from the $i$-th portion, $2^i$ milliliters last until the end. Apart from that, there is also 1 milliliter of tea and 1 milliliter of milk, which were both poured at the very beginning.

For the sample ceremonial word **HMHHM**, at the end there are $1+2^1+2^3+2^4$ milliliters of tea and $1+2^2+2^5$ milliliters of milk in the cup.

This allows us to express the final contents of the cup as a sum of powers of 2. We then have to add all these numbers, which in some programming languages requires us to implement big-integer arithmetic. Similarly, we can count the amount of tea and milk that Byteasar pours into the cup. At the end, it is sufficient to perform two subtractions to discover how much of each beverage Byteasar has drunk.

This allows us to reach the solution. However, it turns out that we do not have to perform complicated calculations. Recall that we only have to determine which beverage Byteasar has drunk more of, so we do not have to calculate the exact numbers.

**A better solution**

From the description, assume that Byteasar does not perform the last pouring, so the cup is half empty after the ceremony. This does not change anything, as Byteasar does not drink what he pours at the end.

Let us denote by $t_p$ and $t_c$ the amount of tea poured into the cup and the amount that remained in the end. Similarly, let $m_p$ and $m_c$ denote equivalent values with regard to milk. We know that Byteasar has drunk $t_p - t_c$ milliliters of tea and $m_p - m_c$ milliliters of milk. At the end, the cup is half full, so $t_c + m_c = 2^n$.

We first focus on the case where there are more letters of one kind than the other in the ceremonial word. It turns out that in such a situation Byteasar drinks more of the beverage that was poured into the cup more often. Assume that he poured more tea, so $t_p$ is larger than $m_p$ by at least half of the cup ($2^n$ milliliters), so $t_p \geq m_p + 2^n$. To compute the amount drunk by Byteasar, we have to subtract the final contents of the cup: that is, $t_c$ and $m_c$. The cup, however, is half empty, and we know that it contains at least 1 milliliter of each beverage (coming from the first pouring). In particular, there are at most $2^n - 1$ milliliters of tea remaining in the cup ($t_c \leq 2^n - 1$). Hence, the amount of tea drunk by Byteasar is bounded as follows:

$$t_p - t_c \geq t_p - 2^n + 1 \geq m_p + 2^n - 2^n + 1 = m_p + 1.$$

This means that the amount of tea Byteasar has drunk is greater than the total amount of milk he has poured into the cup. Consequently, when the ceremonial word contains more **H**s than **M**s, Byteasar drinks more tea. Analogously, if the word contains more **M**s, Byteasar drinks more milk.



It remains to consider the case where the numbers of **H**s and **M**s are equal, so the cup is filled with the same amount of each of the beverages. In particular, this happens when $n = 1$. This is a special case, in which Byteasar drinks only half of the cup containing tea mixed half and half with milk. So he drinks an equal amount of each beverage.

Let us now assume that $n > 1$. If the amounts of milk and tea poured into the cup are equal, Byteasar drinks more of the beverage whose remaining amount in the cup is *smaller*. We know that half of the final cup contents comes from the next-to-last pouring (the last that we have not ignored). Assume that Byteasar poured milk then. In addition, in the cup there is 1 milliliter of milk poured at the very beginning. We infer that there has to be more milk than tea in the cup, so Byteasar has drunk more tea.

Therefore, the entire algorithm boils down to three simple cases. If $n = 1$, Byteasar drinks an equal amount of both beverages. If one letter is more common than the other in the ceremonial word (not considering its last letter), then Byteasar drinks more of the beverage represented by the more common letter. If the two letters appear in the same quantities, then we know that Byteasar drinks less of the beverage that he poured in the next-to-last turn.

# / Kangaroos

**Contest:** Algorithmic Engagements 2011
**Task authors:** Jakub Łącki, Jakub Radoszewski
**Memory:** 32 MB
http://main.edu.pl/en/archive/pa/2011/kan

Byteasar, who is interested in wildlife photography, is planning a trip to Australia. He has started preparing his photographic equipment for taking pictures of kangaroos, and has to decide what items to pack. His equipment includes an extensive collection of photographic lenses with varying characteristics for different angles of view and focusing abilities. The best quality of pictures of animals, such as kangaroos, is achieved when the distance between the camera and the animal is within the optimal range of the lens.

Byteasar's trip passes through a sequence of access points for wildlife observation and photography. The Australian guides gave Byteasar a detailed description of these observation points—in particular, the range of distances at which kangaroos can be expected.

Obviously, not all the lenses from Byteasar's large collection will fit in his luggage, so some decisions must be made. Since Byteasar wishes to keep his lens-changes to a minimum, he wants to find—for each of the lenses—the longest contiguous sequence of observation points on the trip for which that lens is suitable. A lens is suitable for a given observation point if there exists a distance from the range of expected distances that lies within the optimal range of that lens.

## Input

The first line of the input contains two integers, $n$ and $m$ ($1 \le n \le 50{,}000$, $1 \le m \le 200{,}000$), where $n$ is the number of observation points on the trip and $m$ is the number of lenses in Byteasar's collection.

The next $n$ lines describe the access points for wildlife observation and photography on Byteasar's trip. Each of these lines contains two integers, $a_i$ and $b_i$ ($1 \le a_i \le b_i \le 10^9$), indicating that at the $i$-th observation point kangaroos can appear at a distance of $a_i$ to $b_i$ bytean feet, inclusive.

The next $m$ lines describe the lenses. Each of these lines contains two integers, $c_j$ and $d_j$ ($1 \le c_j \le d_j \le 10^9$), indicating that the $j$-th lens works best for kangaroos that are at a distance from $c_j$ to $d_j$ bytean feet from the camera, inclusive.



## Output

The output consists of $m$ lines containing exactly one integer each. The $j$-th output line indicates the number of observation points in the longest *contiguous* segment of the trip where the $j$-th lens can be used by Byteasar. The lenses are numbered according to their order in the input file.

## Example

For the input data:

```
3 3
2 5
1 3
6 6
3 5
1 10
7 9
```

the correct result is:

```
2
3
0
```

# / Solution

"You are given two sequences of intervals. Find the longest subsequence of consecutive intervals in the first sequence, which..."—it seems we have another task about intervals to deal with. We need to sort something, and perhaps use an appropriate data structure, then iterate through the intervals in the right order and we are done. I believe that a simple task statement implies a simple solution, and this task seemed to match this rule perfectly. It was supposed to be used in the final round of Algorithmic Engagements in 2010; however, it was not, and the reason was quite serious: we could not solve it!

Only in the following year, when preparing the successive competition, did I manage to find an algorithm that was better than a naïve one. The task was then prepared for the contest by Adam Karczmarz, and he found the best solution known to us. Our approaches differ significantly, and I think that both deserve to be described here. The former is quite general, and uses a technique that can come in handy in other situations. The latter, on the other hand, leads to a faster algorithm and, while it is quite complex, is built from very simple blocks.

**The first approach**

We are given two sequences of closed intervals:

$$[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n] \text{ and } [c_1, d_1], [c_2, d_2], \ldots, [c_m, d_m].$$

As in the task statement, the first of them will be called the sequence of observation points, and the second, the sequence of lenses. For simplicity we treat each lens and observation point as an interval. Because of that, we will use some rather odd (but convenient) terminology, such as "the intersection of a lens with an observation point". What we mean in such cases is obviously the intersection of the intervals corresponding to the lens and observation point. The whole task can be summarized in a single sentence. For each lens, find the longest subsequence of consecutive observation points that intersect with the lens.

We first note that the most natural approach works in $O(mn)$ time. In this approach, for each of the $m$ lenses $[c_i, d_i]$ we find the longest subsequence of consecutive observation points that intersect with the lens. This can of course be done in $O(n)$ time.



We now present a solution that is even slower than that, but we later show how to speed it up significantly. We use a data structure that represents a subset of integers from 1 to $n$. The data structure allows us to add and remove its elements. Moreover, it can report the longest sequence of consecutive integers that it contains. For example, if the structure contains numbers 3, 4, 7, 11, 12, 13, 25, 26, then the sequences of consecutive integers are (3,4), (7), (11,12,13), (25,26), and the longest among them has length 3. It's not hard to guess that we will be inserting indices of the observation points into this structure.

**Finding sequences of consecutive integers.** How can we implement this data structure? There are many possibilities. One of them is to use an appropriately tuned segment tree. Another is to use any data structure that represents a sorted set of objects and implements the basic set of operations. We choose the second option, as it is a bit simpler to describe*.

Let $A$ denote the subset of integers from 1 to $n$, which is to be stored in the data structure. We represent $A$ as a collection of sequences of consecutive integers; for example, set $\{1,3,4,5,7,8,12\}$ is represented by $[1,1]$, $[3,5]$, $[7,8]$, $[12,12]$. These intervals will be stored in sorted order in a data structure $F_A$. We can sort them by their left or right ends—this makes no difference, as the intervals are disjoint. In addition, we have a second data structure $D_A$ that stores the set (or, rather, a multiset, since we can have many copies of the same element) of the lengths of intervals describing set $A$. In our example, the lengths are 1, 3, 2 and 1.

Querying for the longest subsequence of consecutive integers in $A$ boils down to finding the maximum in $D_A$.

When we add an integer $k$ to $A$, we add interval $[k,k]$ to $F_A$ and find the interval that is just before and after it in the sorted order. If we find an interval $[r, k-1]$ or $[k+1, s]$, then the appropriate intervals have to be merged—it is best just to delete them and insert an interval formed by merging them together.

When a given element, say $k$, is deleted from $A$, we do a similar thing: find an interval $[r,s]$ in $F_A$ that contains $k$ and delete it, and then insert intervals $[r, k-1]$ and $[k+1, s]$ into $F_A$, provided that they are nonempty.

Observe that, in all these operations, only at most three intervals are involved, which allows us to update $D_A$ easily. Each time we insert or delete an interval to/from $F_A$, we need to add or remove its length to/from $D_A$. Using an efficient implementation of sorted dictionaries, each operation on both structures $F_A$ and $D_A$ can be carried out in $O(\log n)$ time.

* And it can be implemented using the set template from STL.

**The general idea.** Let us go back to the task. Denote by $P_i$ the data structure we have just described, filled with indices of the observation points that intersect with lens $[c_i, d_i]$. It takes $O(n)$ operations on the data structure to build it, so the required time is $O(n \log n)$. In order to get the solution of the entire task, we need to know $P_i$ for all $i$ from 1 to $m$, which, with the straightforward approach, takes $O(nm \log n)$ time. This does not look impressive, especially if we recall that the naïve solution is faster. However, we will show that this new approach can be speeded up considerably.

The high-level idea is as follows. Assume that after obtaining $P_j$, we would like to compute $P_i$. In addition, let us assume that the set represented by $P_i$ differs only a little from the set represented by $P_j$. Thus $P_i$ can be easily computed—we can take $P_j$ and, after a small number of add or delete operations, get $P_i$.

This would work nicely if only $P_i$ and $P_j$ were similar to each other. How can we guarantee that? Observe that the structures $P_i$ can be computed in any order, as long as we iterate through all of them. As we will show in a moment, for *some* order of constructing the data structures, the total number of differences between consecutive structures is not too big.

But before we show the above property, let us think how to find the observation points that should be erased from $P_j$ or added to it, in order to obtain $P_i$. Denote by $[c, d]$ the lens represented by $P_j$; that is, assume that $P_j$ contains the indices of observation points that intersect with $[c, d]$. On that basis we now want to compute the $P_i$ that represents the lens $[c', d']$. For presentation, let us assume that $c < c'$ and $d < d'$.

Suppose that $c' \le d$, so the intervals intersect. We first build a data structure representing $[c, d']$. We move the right end of our interval to $d'$ and find all observation points that we encounter on our way; that is, the ones that should be added to our data structure. It should be clear that these are exactly the observation points whose left end $a$ satisfies $d < a \le d'$. Next, we move the left end $c$ right to $c'$. We now have to get rid of all the observation points that no longer intersect with our lens, namely the ones whose right end $b$ satisfies $c \le b < c'$. Analogous reasoning can be applied to $d < c'$; that is, when the intervals are disjoint. In this case, the intervals fully contained in $(d, c')$ will first be added to the data structure, only to be deleted in a moment.

All this can easily be generalized to other cases of the mutual relations between $c$ and $c'$ and between $d$ and $d'$. To sum up, it is sufficient to iterate through all intervals whose right end is contained between $c$ and $c'$, and whose left end is contained between $d$ and $d'$.



**Transforming the input data.** How many intervals have the aforementioned property? It is difficult to say, as it depends on the input data. But we can cope with that. Observe that the input data can be freely modified, as long as the smaller/greater relationships between the numbers are preserved. We transform the input data so that it will be easier to bound the number of differences between data structures $P_i$. The outcome will be that, in order to obtain a data structure representing $[c', d']$ from a data structure representing $[c, d]$, at most $|c - c'| + |d - d'|$ insertions or deletions will have to be made. We start by subtracting $\frac{1}{2}$ from every $c_i$ and adding $\frac{1}{2}$ to every $d_i$. This means that we extend the intervals a bit. It is easy to see that the operation does not influence the intervals' intersections. Second, we "compress" all coordinates: that is, we renumber all $a_i$ and $b_i$, maintaining their mutual relations, so that they become distinct integers from 1 to $2n$. For example, the set of intervals $[1,5]$, $[1,7]$, $[3,10]$ can be transformed into $[1,4]$, $[2,5]$, $[3,6]$. In order to achieve this, all $a_i$ and $b_i$ have to be sorted (together) and then the number from the $i$-th position is replaced by $i$.

Since we have changed the ends of intervals $[a_i, b_i]$, we also have to move $c_i$ and $d_i$, to restore their original relations to $a_i$ and $b_i$. Note that we maintain the fractional parts of $c_i$ and $d_i$, which are equal to $\frac{1}{2}$.

All these transformations can be implemented in $O((n+m) \log n)$ total time. We leave the details to the reader.

After performing all the above transformations, for every integer $k$ from 1 to $2n$ there is exactly one interval $[a_i, b_i]$, whose end is $k$. Hence, if we have built a data structure that represents $[c, d]$ and we now want to compute the data structure for $[c', d']$ ($c$, $c'$, $d$, $d'$ now denote the transformed values), then the number of intervals $[a, b]$ that satisfy $d < a \le d'$ is at most $d' - d$. Similarly, there are at most $c' - c$ intervals satisfying $c \le b < c'$. This can be generalized to other cases of mutual relations between $c$ and $c'$ and between $d$ and $d'$. In the end, the outcome is that building a data structure for interval $[c', d']$, given a data structure for interval $[c, d]$, requires at most $|c - c'| + |d - d'|$ insert or delete operations.

**Geometric problem.** The observation made in the previous paragraph gives us a simple geometric interpretation of our problem. For each interval $[c_i, d_i]$ we draw a point $(c_i, d_i)$ in the plane. Our goal is to draw a broken line that visits all these points and is as short as possible. Additionally, since we measure the distance according to the Manhattan distance—that is, the distance between $(x, y)$ and $(x', y')$ is equal to $|x - x'| + |y - y'|$—the broken line has to consist solely of horizontal and vertical segments. If we find a broken line of length $l$,

then the consecutive points it visits will correspond to the order in which we should construct data structures $P_i$. We know that in this case we will need to perform at most $n$ operations to build the initial data structure, and then at most $l$ operations to build all structures, basing them on the previous ones. All the structures are computed in $O((n + l) \log n)$ time in this way. It remains for us to show that we can always find a sufficiently short broken line.

We will solve a slightly more general problem. Let us assume that we are given $m$ points in a square of size $n \times n$. Our points lie in a square that is a bit bigger than $2n \times 2n$, but this does not matter from the asymptotic notation's point of view. We also do not use other properties of the points, although we know that, for example, $c_i \leq d_i$.

The first idea is to try to find the shortest broken line possible. This, however, probably cannot be done in polynomial time, as this problem is NP-complete. But this is not a big issue—our problem has a nice property that if the optimal length of the broken line is $l_{opt}$, and we find an algorithm that always finds broken lines of length, say, at most $10l_{opt}$, then the overall running time complexity will be the same. All in all, since the running time is $O((n + l) \log n)$, increasing $l$ ten times does not increase the asymptotic complexity.

It turns out that we can indeed find the broken line that has this nice property. We will describe it briefly, as the approach we take in the end will be different. It is sufficient to connect all $m$ points into a minimum spanning tree (in the plane with Manhattan distance this can be done in $O(m \log m)$ time), root the tree arbitrarily and construct the broken line by connecting the consecutive nodes visited during a preorder traversal of the tree. It can be shown that the broken line constructed in this way is at most twice as long as the optimal one.

Thus we obtain a rather unusual algorithm. We first solve an optimization problem and then, the better the solution we find, the faster the second phase will be. We have only given a high-level description of the algorithm because there is also a simpler way of finding a good broken line. While it can sometimes give much worse results, both approaches work similarly well in the pessimistic case.



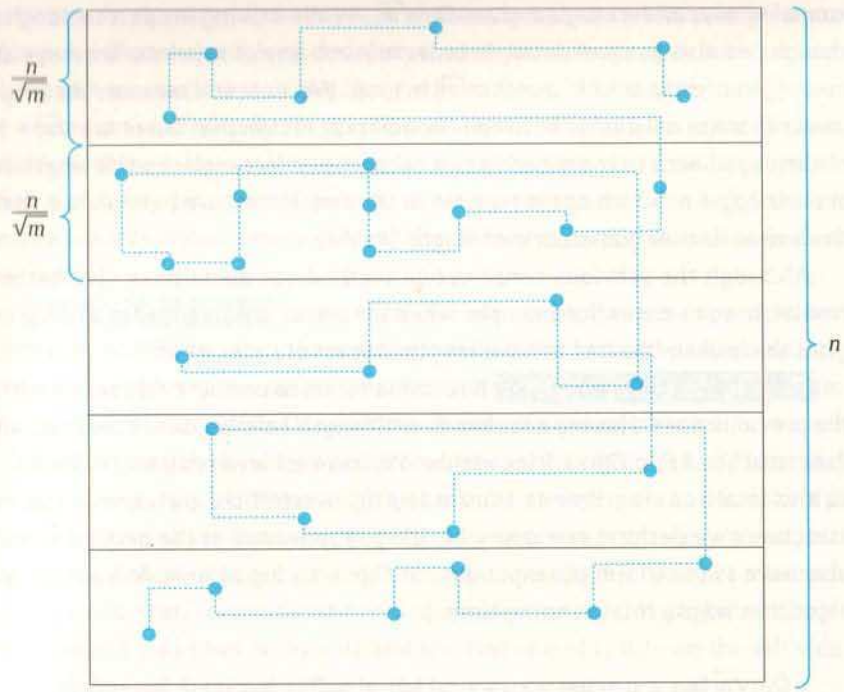



*Figure 1: Finding a broken line of length $O(n\sqrt{m})$.*

Let us divide our $n \times n$ square into $\sqrt{m}$ rectangles of size $\frac{n}{\sqrt{m}} \times n$, as shown in figure 1. The broken line is constructed as follows. Visit all points in the bottommost rectangle in increasing order of $x$ coordinate. Then go to the point with the smallest $x$ coordinate in the rectangle above, and go through all points in this rectangle in a similar way; and so on. We stop after visiting all the points.

Inside each rectangle the broken line goes only right, up or down. The total length of the segments going right inside one rectangle is at most $n$, which, after summing over all rectangles, gives $O(n\sqrt{m})$. While moving inside a rectangle, though, we also go up or down. In order to reach any of $m$ points, we move at most $\frac{n}{\sqrt{m}}$ vertically, so also use $O(n\sqrt{m})$ in total. We must still measure the fragments that are used to go between the adjacent rectangles. There are $\sqrt{m}-1$ of them, and each is contained in two consecutive rectangles, so its length is at most $2\frac{n}{\sqrt{m}}+n$, which again sums up to $O(n\sqrt{m})$. Hence, we have shown that the broken line we construct is of length $O(n\sqrt{m})$.

Although the previous construction method can sometimes give better results, in some cases (for example, when the points are arranged in a uniform grid) the broken line has to have length at least of order $n\sqrt{m}$.

**Putting the pieces together.** It remains for us to combine this result with the previous ones. Having a broken line of length $l$ allows us to construct all data structures $P_i$ in $O((n+l)\log n)$ time. We have achieved that for $l = O(n\sqrt{m})$, so the total running time is $O(n\sqrt{m}\log n)$. In addition, for each of the $m$ structures we perform one query in $O(\log n)$ time, and at the beginning we also have to transform the input data in $O((n+m)\log n)$ time. We obtain an algorithm whose total running time is

$$O(n\sqrt{m}\log n + m\log n + (n+m)\log n) = O(m\log n + n\log n\sqrt{m}).$$

The technique we have used is quite general, and can also be used in other problems, especially if we have to answer queries about intervals. One example is the task *Powerful Array* from the Codeforces contest in 2011.

Our approach is better than the $O(nm)$ solution by about $O(\frac{\sqrt{m}}{\log n})$. While this is a significant improvement from a theoretical point of view, for the sizes of input data we often deal with, the numerator is only a few times greater than the denominator, so the gain is moderate. However, the problem can be solved even more efficiently by using a completely different method.

### The second, more efficient solution

This time we take a different approach. We break the problem into two subproblems that we will solve independently.

Assume that the input data have already been transformed, as described in the previous solution.



Let us color every $t$-th observation point blue: that is, we mark points $[a_t, b_t]$, $[a_{2t}, b_{2t}]$,...; there are $\lfloor \frac{n}{t} \rfloor$ of them in total. Now, for each lens, we first find the longest subsequence of consecutive observation points that includes some blue point (first subproblem) and then the longest subsequence without any blue point (second subproblem). As it will turn out, the first subproblem can be solved more efficiently for greater values of $t$, whereas the more efficient solutions of the second problem exist for smaller values of $t$. For an appropriate choice of $t$, we will obtain solutions to both problems that can be combined into an algorithm that is more efficient that the one resulting from the previous approach.

**The first subproblem.** In this case we use a very natural approach. For each lens $[c_i, d_i]$ and every blue point $j$ we find the longest subsequence of consecutive observation points that contains the $j$-th point and only intervals that intersect $[c_i, d_i]$. We will call it the solution anchored at point $j$.

We first compute how many consecutive intervals to the left of $j$ (starting from $j$) intersect $[c_i, d_i]$. Then we do the same for the intervals to the right of $j$. These two steps are analogous, so we now focus on the latter. For a given $j$ we want to find the first index $j' > j$, such that interval $[a_{j'}, b_{j'}]$ does not intersect $[c_i, d_i]$. To do that, it is sufficient to find the first interval $[a_p, b_p]$ that lies to the right of $[c_i, d_i]$ (that is, $a_p > d_i$) and the first one—$[a_q, b_q]$—on the left side of $[c_i, d_i]$ ($b_q < c_i$). It is then the case that $j'$ is a minimum of $p$ and $q$.

In the end, for a fixed blue point in which a solution is anchored, we solve the following problem.

***Problem 1.*** You are given an array of integers $a_1,\ldots,a_n$, where $1 \le a_i \le 2n$, and a sequence of integers $c_1,\ldots,c_m$. For each $c_i$ find the first index $j$, such that $a_j > c_i$.

We can tackle this easily. We first compute an array $p$, where $p[i]$ is the first position of number $i$ in array $a$ (or $\infty$ if $i$ is not in array $a$). This allows us to compute array $r$, in which $r[i]$ is the first index in array $a$, such that $a[r[i]]$ is greater than $i$. It is sufficient to set $r[i]$ equal to $\min(p[i+1], p[i+2],\ldots,p[2n])$. This can of course be done in linear time, since $r[i]$ can be computed using $r[i+1]$; namely $r[i] = \min(r[i+1], p[i+1])$. Having array $r$, we can immediately find the answer for each $c_i$. Thus, solving this problem requires $O(n+m)$ time.

We use it four times for each blue point, to find a greater/smaller number to the left/right of the given blue point. The overall running time for a single blue point can be bounded by $O(n+m)$, so the entire first subproblem can be solved in $O(\frac{n}{t}(n+m))$ time.

**The second subproblem.** It remains for us to solve the second subproblem. This time, for each lens we are looking for a solution that is not anchored at any blue point. If the value of $t$ is small, there are not too many candidate intervals to consider. The blue points divide the sequence of observation points into $\lfloor \frac{n}{t} \rfloor + 1$ intervals of length smaller than $t$. In an interval of length $t$ we can choose a subinterval in $O(t^2)$ ways; thus in this subproblem the solution for each lens is chosen from among $O(\frac{n}{t} \cdot t^2) = O(nt)$ candidates.

In this case we assume that $nt$ is rather small. All candidates have lengths from 1 to $t-1$. Let us divide them into groups based on their lengths. We will process these groups in order of decreasing lengths of intervals. If we find an interval which is good for some lens, then we immediately have the answer for that lens, so we can remove the lens from further consideration. This is because we check the longer candidates first.

Let us give some more details. Again we will use...

**Geometric interpretation.** Consider a sequence of observation points $[a_r, b_r]$, $[a_{r+1}, b_{r+1}], \ldots, [a_s, b_s]$. This is a good sequence for lens $[c, d]$, if every observation point intersects the lens, so for each $r \le i \le s$ we have $b_i \ge c$ and $a_i \le d$. This can be equivalently rewritten as

$$\min_{r \le i \le s} b_i \ge c \quad \text{and} \quad \max_{r \le i \le s} a_i \le d.$$

This allows us to draw our problem in the plane. We assign to the aforementioned sequence of observation points the point

$$(\min_{r \le i \le s} b_i, \max_{r \le i \le s} a_i).$$

On the other hand, for each lens $[c_i, d_i]$ we draw a point $(c_i, d_i)$. We can now see that the sequence of observation points represented with a point $(B, A)$ is good for lens $(c, d)$ if $B \ge c$ and $A \le d$, so $(B, A)$ lies to the right and down of $(c, d)$. Thanks to that, solving this part boils down to the following problem.

***Problem 2.*** In the plane, there are $g$ green points $(b_j, a_j)$ and $m$ red points $(c_i, d_i)$. All the points have integer coordinates from 1 to $2n$. The goal is to find all red points $(c_i, d_i)$ such that there exists some green point $(b_j, a_j)$ lying to the right and down of $(c_i, d_i)$; that is, $b_j \ge c_i$ and $a_j \le d_i$.



We begin by getting rid of all useless green points. If, for some green point, there exists another green point that lies to the right and down of it, then the first point can be deleted. This can be achieved with a simple algorithm. First, inside each group of points of equal $x$ coordinate we leave only the point with the smallest $y$ coordinate—it is clear that the other points are not useful. Then we sort all green points *decreasingly* with respect to the $x$ coordinate and go through them one by one. We delete all the points whose $y$ coordinate is not smaller than the smallest $y$ coordinate of the points we have already visited. This process eliminates all useless points. As the coordinates are at most $2n$, this step can be implemented to run in $O(n + g)$ time. After that, at most $2n$ useful points remain, since each of them has to have a different $x$ coordinate. We encourage the reader to show the correctness of this algorithm.

Although getting rid of the useless points may seem redundant, it will allow us to solve Problem 2 more effectively. We insert all red points into $2n$ stacks $S_1, \ldots, S_{2n}$. In stack $S_i$ we have points whose $x$ coordinate is equal to $i$. Inside each stack we sort the points increasingly: that is, on the top we put the point with the greatest $y$ coordinate. It is easy to see why we choose this particular ordering: if some green point is good for some point in the stack, then it is also good for all higher points in the stack. This means that we can process points in every stack starting from the top.

The remaining part is simple. Let $(b_1, a_1)$, $(b_2, a_2), \ldots$ denote the set of green useful points sorted decreasingly by their $x$ coordinate. Since they are all useful, we also know that they are sorted decreasingly by their $y$ coordinate. We now find all red points in stacks $S_{b_2+1}, \ldots, S_{b_1}$, for which point $(b_1, a_1)$ is good. These points are removed from the stacks. Then we consider the point $(b_2, a_2)$ and stacks $S_{b_3+1}, \ldots, S_{b_2}$ and so on. Each time we check a point on a stack, we either delete this point or move to the next stack. Thus, if the number of red points we find equals $r$, this phase of the algorithm runs in $O(r + n)$ time.

This gives us a solution to Problem 2. This solution has now to be run for each group of intervals of equal length. Observe that it is sufficient to sort all red points and insert them into the stacks only once, at the beginning.

The ingredients of the solution to this subproblem are as follows:

- Generate $O(nt)$ candidate intervals.
- Convert the candidates to points in the plane in $O(n \log n + nt)$ time.
- Insert $m$ lenses into stacks and sort them within each stack, in total time of $O(n + m \log m)$.
- Delete useless points in $O(nt)$ time.

→ Run the algorithm that solves Problem 2 $t-1$ times. In each iteration we use $O(r+n)$ time, where $r$ is the number of lenses for which we want the answer. Hence, the sum of these numbers over all $t-1$ runs of the algorithm is at most $m$ and the total time is $O(m+nt)$.

This gives an overall running time of $O(n \log n + m \log m + nt)$.

**Merging the subproblems.** We are almost done. Our algorithm first modifies the input in $O((n+m) \log n)$ time, then solves the first subproblem in $O(\frac{n}{t}(n+m))$ and the second subproblem in $O(n \log n + m \log m + nt)$ time. It remains for us to find the optimal value for $t$.

We would like to minimize $\frac{n}{t}(n+m) + nt$, which is equivalent to minimizing $\frac{n+m}{t} + t$. It is easy to see that the smallest value is attained for $t = \sqrt{n+m}$. The algorithm works within the following time bound:

$$O((n+m) \log n + n\sqrt{n+m} + n \log n + m \log m + n\sqrt{n+m}) =$$
$$= O(n\sqrt{n+m} + m(\log m + \log n)).$$

It takes some time to realize it, but this is indeed better complexity than in the previous solution. In particular, if we assume that—as in the greatest possible input data—$m > n$, then the running time can be rewritten as $O(n\sqrt{m} + m \log m)$.

Some of the contestants obtained slightly poorer solutions to one of the subproblems. For example, if we solved the first subproblem using sorting or some data structure in $O(\frac{n}{t}(n+m) \log n)$, then $t$ would have to be chosen to minimize $\frac{n}{t}(n+m) \log n + nt$ or, equivalently, $t + \frac{(n+m)\log n}{t}$. The best choice of $t$ is then $\sqrt{(n+m)\log n}$ and the algorithm would then run slower by a factor of $\sqrt{\log n}$, in time

$$O(n\sqrt{(n+m) \log n} + m(\log m + \log n)).$$

I guess every computer scientist will occasionally wonder why a particular algorithm has a strange running time, like the fastest known algorithm for counting inversions in a permutation, which runs in $O(n\sqrt{\log n})$ time. The reason is often similar to that presented above. The algorithm consists of two components with different running times, and we decide how to divide the work to be performed between these pieces. We find the optimal way of splitting the work, and it turns out that the running times meet in a rather unexpected place.

**KRZYSZTOF ONAK**

Krzysztof Onak is currently a Simons Postdoctoral Fellow at Carnegie Mellon University. He earned his Ph.D. degree in computer science at the Massachusetts Institute of Technology in 2010, following a M.S. degree at the University of Warsaw in 2005. He was a member of the team from the University of Warsaw that won the 2003 ACM-ICPC World Finals. Krzysztof Onak's main research interests are focused on sublinear algorithms. In his spare time, he enjoys trying to convince others that he has mastered the saber at a higher level than they have.

# / Sums

**Contest:** 10th Polish Olympiad in Informatics
**Task author:** Krzysztof Onak
**Memory:** 32 MB
http://main.edu.pl/en/archive/oi/10/sum

We are given a set of positive integers $A$. Consider a set of nonnegative integers $A'$, such that a number $x$ belongs to $A'$ if and only if $x$ is a sum of some elements from $A$ (the elements may be repeated). For example, if $A = \{2, 5, 7\}$, then sample numbers belonging to the set $A'$ are: 0 (the sum of 0 elements), 2, 4 ($2 + 2$) and 12 ($5 + 7$ or $7 + 5$ or $2 + 2 + 2 + 2 + 2 + 2$); and the following do not belong to $A'$: 1 and 3.

## Task

Write a program that:

- reads from the input the description of the set $A$ and the sequence of numbers $b_i$,
- for each number $b_i$ determines whether it belongs to the set $A'$,
- writes the result to the output.

## Input

In the first line there is one integer $n$: the number of elements of the set $A$ ($1 \le n \le 5000$). The following $n$ lines contain the elements of the set $A$, one per line. In the $(i + 1)$-th line there is one positive integer $a_i$ ($1 \le a_i \le 50{,}000$). $A = \{a_1, a_2, \ldots, a_n\}$, $a_1 < a_2 < \cdots < a_n$.

In the $(n + 2)$-th line there is one integer $k$ ($1 \le k \le 10{,}000$). Each of the following $k$ lines contains one integer in the range 0 to 1,000,000,000: they are respectively the numbers $b_1, b_2, \ldots, b_k$.

## Output

The output should consist of $k$ lines. The $i$-th line should contain the word TAK (Polish for *yes*) if $b_i$ belongs to $A'$, or the word NIE (Polish for *no*) otherwise.

**Example**

For the input data:

3
2
5
7
6
0
1
4
12
3
2

the correct result is:

TAK
NIE
TAK
TAK
NIE
TAK



# Solution

The problem statement may suggest that a solution is based on number theory. However, we reduce the problem to one of the most fundamental graph problems: finding distances in a weighted graph. I believe that this is why many people enjoyed this problem and why I was asked to describe it for this volume. Nonetheless, if you know a number theory solution or any other solution that is more efficient than the one described here, I would love to hear about it.

**Reduction to a graph problem**

Let us start with a simple observation. Fix an arbitrary number $a \in A$. If an integer $y$ belongs to $A'$, then so do $y + a$, $y + 2a$, $y + 3a$, and so on. Our goal is to answer queries of the form "Does $b$ belong to $A'$?" If we knew the smallest number $t \in A'$ such that $b \equiv t \pmod{a}$, then we could determine whether $b$ belongs to $A'$ just by comparing it to $t$. It holds that $b \in A'$ if and only if $b \geq t$. Note that if such a number $t$ does not exist, we may assume that $t = \infty$ and check the membership of $b$ in $A'$ in the same way.

Therefore, to be able to reply to all queries, it suffices that for each $i \in \{0, 1, \ldots, a-1\}$, we compute the smallest number $t_i \in A'$ such that $t_i \equiv i \pmod{a}$. If a specific $t_i$ does not exist, we set $t_i = \infty$. Note that $t_0 = 0$.

We now construct a weighted directed graph $G$ on $a$ vertices $v_i$ $(0 \leq i \leq a-1)$. Each vertex corresponds to a different remainder modulo $a$. For each vertex $v_i$ and each $x \in A$, we add a directed edge from $v_i$ to $v_{(i+x) \bmod a}$ of length $x$. The edge expresses the fact that if we add $x$ to a number with $i$ as a remainder modulo $a$, then we obtain a number with the remainder $(i+x) \bmod a$. See figure 1 for an example of such a graph $G$. We show that the numbers $t_i$ can be determined using specific distances in $G$. The claim follows due to an equivalence between paths starting at $v_0$ and sequences of numbers in $A$.

***Claim.*** Let $d_i$ $(0 \leq i \leq a-1)$ be the distance from $v_0$ to $v_i$ in $G$ (or $\infty$ if there is no path from $v_0$ to $v_i$). For each $i \in \{0, 1, \ldots, a-1\}$, $d_i = t_i$.

*Proof.* Consider an arbitrary path starting at $v_0$. Let $a'_1, \ldots, a'_k$ be the lengths of consecutive edges on the path and let $l = a'_1 + \cdots + a'_k$ be the total length of the path. It easily follows by induction that the $j$-th vertex on the path $(1 \leq j \leq k+1)$ is $v_{r_j}$, where $r_j = (\sum_{q=1}^{j-1} a'_q) \bmod a$. Therefore, the final vertex of the path is $v_{l \bmod a}$. We use this fact below.

Fix an $i \in \{0, \ldots, a-1\}$. By definition, we can express each $t_i < \infty$ as a sum $a'_1 + \cdots + a'_k$, where each $a_j$ $(1 \le j \le k)$ belongs to $A$. Consider now a path starting at $v_0$ that takes $k$ edges, where the $j$-th edge on the path $(1 \le j \le k)$ has length $a'_j$. It follows from the definition of $G$ that this path exists. The final vertex of the path is $v_i$. This proves that $d_i \le t_i$, because the length of this path is $t_i$.

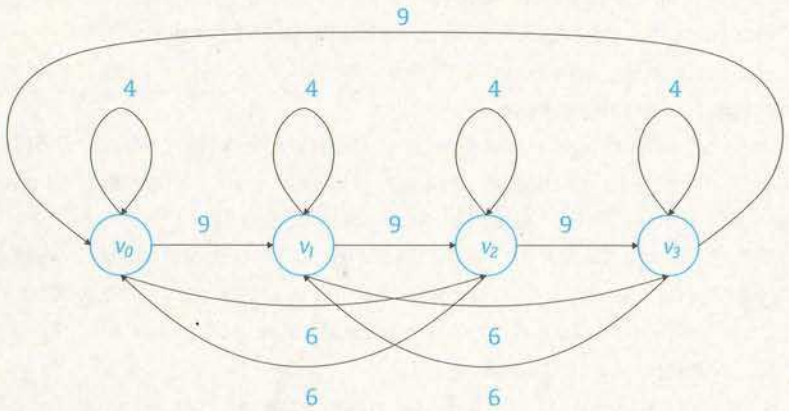



***Figure 1:*** *A sample graph for $A = \{4, 6, 9\}$ and $a = 4$.*

On the other hand, consider the shortest path from $v_0$ to $v_i$, assuming that it exists. Recall that $d_i$ is the length of this path. Let $a'_1, \ldots, a'_k$ be the lengths of consecutive edges taken by the path. By the construction of $G$, all values of $a'_j$ belong to $A$, so $d_i \in A'$. Moreover, since $v_i$ is the final vertex, $d_i \equiv i \pmod{a}$, so $d_i \ge t_i$.

If one of $d_i$ and $t_i$ is finite, the above reasoning shows that the other number is also finite and equal. Otherwise, $d_i = t_i = \infty$, i.e. the equality holds as well. □

**Shortest paths and time complexity**

We now know that it is sufficient to compute distances from $v_0$ to all values of $v_i$ in order to answer all queries $b_i$ efficiently. This can be done using the Dijkstra algorithm. The Dijkstra algorithm is described in almost every textbook on algorithms. Depending on the selection of data structures in the implementation of the Dijkstra algorithm, we may obtain different running times. The most naïve implementation gives a running time of $O(a_1^2 + a_1 \cdot |A|)$, where $a_1$ is the smallest number in $A$. If $|A|$ is much smaller than $a_1$ then it may make sense to use the binary heap as a priority queue in the Dijkstra algorithm, in which case we obtain a running time of $O(a_1 \cdot |A| \cdot \log a_1)$. Note also that we do not have to construct $G$ explicitly, which would require $\Omega(a_1 \cdot |A|)$ space. Instead, it is sufficient to remember $A$, as all edges incident to a specific vertex are determined by $A$. Therefore, computing all distances $d_i$ via the Dijkstra algorithm requires only $O(a_1 + |A|)$ space.

Finally, recall that once we have computed all the values $t_i = d_i$, we can answer each query $b$ in constant time. Namely, $b$ belongs to $A'$ if and only if $b \ge t_{b \bmod a}$.

# / Superknight

**Contest:** 9th Polish Olympiad in Informatics
**Task author:** Krzysztof Onak
**Memory:** 32 MB
http://main.edu.pl/en/archive/oi/9/sup

On an infinite chequered board there is a superknight which can make different moves. Each kind of move is described by two integers: the first says how many columns (to the right if the number is positive or to the left if the number is negative) and the second says how many rows (forward if the number is positive or backward if the number is negative) the knight traverses when making such a move.

## Task

Write a program that:

- reads from the input sets of data describing different superknights,
- determines for each superknight whether it is able to reach any square of the board by making allowed moves,
- writes the result to the output.

## Input

In the first line of the input there is one integer $k$ specifying the number of sets of data ($1 \le k \le 100$). It is followed by $k$ sets of data. In the first line of each set there is an integer $n$ representing the number of kinds of move that the superknight can make ($1 \le n \le 100$). Each of the following $n$ lines of the set contains two integers, $p$ and $q$ ($-100 \le p, q \le 100$), separated by a single space. These numbers describe one kind of move.

## Output

The output should consist of $k$ lines. The $i$-th line should contain one word, TAK (Polish for *yes*), if the superknight described by the $i$-th set of data is able to reach any square on the board, or the word NIE (Polish for *no*) otherwise.

## Example

For the input data:

```
2
3
1  0
0  1
-2  -1
5
3  4
-3  -6
2  -2
5  6
-1  4
```

the correct result is:

```
TAK
NIE
```

# / Solution

I created this problem when I was a freshman at the University of Warsaw. I did not know at that point that there is a whole body of work related to bases of vectors with integer coordinates. Any reader interested in this kind of topic should search online, for instance, for the famous Lenstra-Lenstra-Lovász algorithm.

How does one come up with problems like this? I remember that the idea popped into my head while I was simultaneously watching a ski jumping competition and solving some linear algebra homework. A ski jumper and a chess knight are described in Polish using the same word, "skoczek", which literally means a "jumper".

That competition was won by Adam Małysz, one of the greatest ski jumpers ever. Two years later, before attending the ACM-ICPC World Finals with Tomek Czajka and Andrzej Gąsienica-Samek, I watched the Nordic World Ski Championships, at which Adam Małysz won gold medals in both individual ski jumping contests. Watching the grace and ease with which he won both contests, I started wondering whether we could perform just as well in the ICPC finals. And we did!

**Overview**

We treat allowed moves as two-dimensional vectors $(v_x, v_y)$. Let $S$ be their set. We say that $S$ is *complete* if each pair of integers $(x, y)$ can be expressed as a sum of a finite sequence $v_1, \ldots, v_k$ of vectors in $S$, where each vector in $S$ can appear multiple times in the sequence. The goal of the problem is to determine whether $S$ is complete.

Our algorithm consists of two stages. First, we determine whether the set of moves suffices to reach both the positive and negative half-axes of the Cartesian coordinate system. More precisely, we say that $S$ is *axis-complete* if there exist positive integers $a$, $b$, $c$, and $d$ such that vectors $(a, 0)$, $(-b, 0)$, $(0, c)$, and $(0, -d)$ can be obtained by adding vectors in $S$. Obviously, if $S$ is not axis-complete, then it is not complete either.

Note that, in general, if the superknight can make a move $v = (v_x, v_y)$, it does not imply that there is a sequence of moves that results in moving the superknight by the vector $-v = (-v_x, -v_y)$. The simplest example is $S = \{(0, 1)\}$, where it is not possible to move the superknight by $(0, -1)$. Formally, we say that $S$ is *reversible* if, for each vector $v \in S$, $-v$ can be expressed as a sum of a finite sequence of vectors from $S$. We show later that if $S$ is axis-complete, then it is reversible as well.

For any set $S$ of vectors, we write $S^*$ to denote the set of all vectors that can be obtained by adding and subtracting vectors in $S$. More formally, for a finite $S = \{v_1, \ldots, v_n\}$, we have

$$S^* = \left\{ \sum_{i=1}^{n} a_i v_i : a_1, \ldots, a_n \in \mathbb{Z} \right\},$$

where $\mathbb{Z}$ is the set of all integers. Note that if $S$ is reversible, then $S^*$ is exactly the set of vectors that can be obtained by adding vectors in $S$, because every subtraction can be simulated by a sequence of moves reversing a specific move. Therefore, a reversible $S$ is complete if and only if $S^* = \mathbb{Z}^2$.

In the second stage, our algorithm tests whether $S^* = \mathbb{Z}^2$. It follows from our reasoning that this is sufficient to test completeness of axis-complete sets $S$, i.e. those that have passed the first stage. We show that for any set $S$, it is possible to compute two vectors $v_1$ and $v_2$ such that $\{v_1, v_2\}^* = S^*$. In particular, we give a procedure that takes three vectors $v_1$, $v_2$, and $w$ and outputs two vectors $v_1'$ and $v_2'$ such that $\{v_1, v_2, w\}^* = \{v_1', v_2'\}^*$. Moreover, the final vectors $v_1$ and $v_2$ are such that it is easy to check whether $\{v_1, v_2\}^* = \mathbb{Z}^2$.

This finishes the sketch of our algorithm. We will fill in all missing details in the next three sections.

```
Algorithm IsComplete(S)
    if S is not axis-complete then
        return false
    else if S* = Z^2 then
        return true
    else
        return false
```

**Testing axis-completeness**

In this section we give an algorithm that determines whether a set $S$ of vectors is axis-complete. Without loss of generality, we only show how to determine whether it is possible to obtain a vector $(a, 0)$ as a sum of a sequence of vectors in $S$ for some $a > 0$. The other three cases can be reduced to this by rotating the set of vectors.

We assign each vector of $S$ to one of the following three sets, based on the sign of the vector's second coordinate:

$$S_0 = S \cap \mathbb{Z} \times \{0\},$$
$$S_+ = S \cap \mathbb{Z} \times \{y : y > 0\},$$
$$S_- = S \cap \mathbb{Z} \times \{y : y < 0\}.$$

In other words, $v = (v_x, v_y) \in S$ belongs to $S_0$, $S_+$ or $S_-$ if $v_y$ is zero, positive or negative, respectively.

Note that if $S_0$ contains a vector $(a,0)$ for some $a > 0$, we are trivially done. Otherwise, observe that if a vector $(a,0)$ for some $a > 0$ can be obtained as a sum of vectors in $S$, then we can also obtain a vector of this form as a sum of vectors only in $S_- \cup S_+$. In this case, if at least one of $S_-$ and $S_+$ is empty, we are also trivially done, since by adding a nonzero number of vectors in $S_-$ or $S_+$, we can only obtain a vector with a nonzero second coordinate. Summarizing, this means that for the rest of this section we can focus on showing an algorithm for determining if a vector of the form $(a,0)$ for some $a > 0$ can be obtained as a sum of vectors in $S_- \cup S_+$, where both $S_-$ and $S_+$ are nonempty.

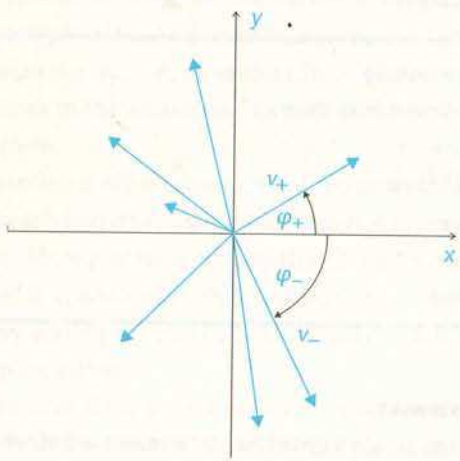



***Figure 1:*** *Angles $\varphi_+$ and $\varphi_-$*



Let $\varphi_-$ be the minimum angle that a vector in $S_-$ makes with the positive half of the $x$-axis. Let $v_- = (x_-, y_-)$ be any of the vectors in $S_-$ with this property. We analogously define $\varphi_+$ and $v_+ = (x_+, y_+)$ for $S_+$. See figure 1 for an example. We prove a lemma that relates the question that we want to answer to the sum of $\varphi_-$ and $\varphi_+$.

***Lemma 1.*** The following three conditions are equivalent:

- $\varphi_- + \varphi_+ < 180°$,
- there is a positive integer $a$ such that the vector $(a,0)$ can be obtained as a sum of vectors in $S_- \cup S_+$,
- $-x_+ \cdot y_- + x_- \cdot y_+ > 0$.

*Proof.* We say that a vector $v$ is a *nonnegative combination* of vectors $v_1, \ldots, v_k$ if $v = \sum_{i=1}^{k} a_i v_i$, for some nonnegative real numbers $a_i$. Let

$$w = (-y_-) \cdot v_+ + y_+ \cdot v_- = (-x_+ \cdot y_- + x_- \cdot y_+, 0).$$

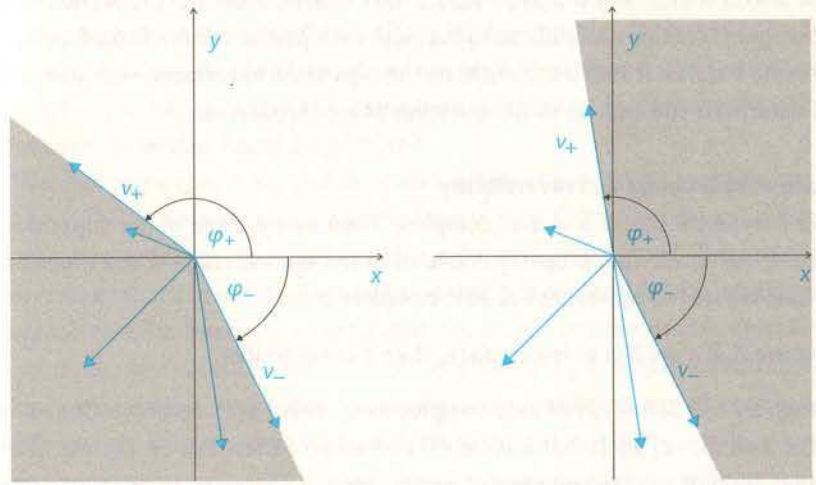



***Figure 2:*** *The case that $\varphi_- + \varphi_+ \geq 180°$ (left) and $\varphi_- + \varphi_+ < 180°$ (right). The gray area depicts nonnegative combinations of vectors in S.*

Consider the case that $\varphi_- + \varphi_+ \geq 180°$ (see figure 2). In this case, no point on the positive half of the $x$-axis is a nonnegative combination of vectors in $S_- \cup S_+$. All nonnegative combinations lie within the cone limited by $v_-$ and $v_+$, spanning clockwise from $v_-$ to $v_+$. It is therefore not possible to reach the positive half of the $x$-axis by adding vectors in $S_- \cup S_+$. Also, since $w$ is a nonnegative combination of $v_-$ and $v_+$, it must be the case that $-x_+ \cdot y_- + x_- \cdot y_+ \leq 0$.

Consider now the case that $\varphi_- + \varphi_+ < 180°$. Note that $w$ is a nonnegative combination of $v_-$ and $v_+$ with positive integer coefficients, which means that $w$ can be expressed as a sum of a finite sequence consisting of $v_-$ and $v_+$. Since the angle between $v_-$ and $v_+$ is not 180°, these vectors are not collinear, and $w \neq (0,0)$. Furthermore, all nonnegative combinations of $v_-$ and $v_+$ lie within the cone limited by $v_-$ and $v_+$, spanning clockwise from $v_+$ to $v_-$. This means that no point on the negative half of the $x$-axis is a nonnegative combination of $v_-$ and $v_+$. This proves that $-x_+ \cdot y_- + x_- \cdot y_+$, the first coordinate of $w$, is positive, and finishes the proof. □

Both $v_+$ and $v_-$ can easily be determined in linear time. Without loss of generality, since the cases are symmetric, let us focus on determining $v_+$. Consider two vectors $u = (x_u, y_u)$ and $w = (x_w, y_w)$ in $S_+$. The angle between $u$ and the positive half of the $x$-axis is smaller than the angle between $w$ and the positive half of the $x$-axis if and only if $x_u/y_u > x_w/y_w$. This follows from the properties of the cotangent function and suffices to compare each pair of vectors in $S_+$. Finally, as Lemma 1 states, it suffices to evaluate the sign of the expression $-x_+ \cdot y_- + x_- \cdot y_+$ to determine the answer to the question being considered.

### Axis-completeness and reversibility

We now show that if $S$ is axis-complete, then every move of the superknight can be reversed. This property is useful in the second stage of our algorithm, where we test completeness of axis-complete sets.

***Lemma 2.*** If a set $S$ is axis-complete, then it is reversible.

*Proof.* By the definition of axis-completeness, there are vectors $(a,0)$, $(-b,0)$, $(0,c)$, and $(0,-d)$ such that $a,b,c,d > 0$ and each of them can be expressed as a summation of a finite sequence of vectors from $S$. Consider an arbitrary vector $v = (v_x, v_y)$ in $S$. It is sufficient to show that $-v$ can be expressed as a summation of a finite number of vectors from $S$. If $v = (0,0)$, the claim is trivially true. We therefore assume for the rest of the proof that $v \neq (0,0)$.



Consider a vector $(w_x, 0)$, where $w_x$ equals $a$ if $v_x < 0$, and $-b$ otherwise. Similarly, consider $(0, w_y)$, where $w_y$ equals $c$ if $v_y < 0$, and $-d$ otherwise. Note that $(w_x, 0)$ and $(0, w_y)$ are two of the four vectors that we obtained from the definition of axis-completeness. Observe that

$$\begin{aligned} &|w_x \cdot w_y| \cdot v + |w_y \cdot v_x| \cdot (w_x, 0) + |w_x \cdot v_y| \cdot (0, w_y) = \\ &\quad = \big(|w_x \cdot w_y| \cdot v_x + |w_y \cdot v_x| \cdot w_x,\ |w_x \cdot w_y| \cdot v_y + |w_x \cdot v_y| \cdot w_y\big) = (0,0), \end{aligned}$$

where the second inequality follows from the fact that $v_x$ and $w_x$ do not have the same sign, and nor do $v_y$ and $w_y$. Note that since both $w_x$ and $w_y$ are nonzero, $|w_x \cdot w_y| \geq 1$ and therefore

$$\big(|w_x \cdot w_y| - 1\big) \cdot v + |w_y \cdot v_x| \cdot (w_x, 0) + |w_x \cdot v_y| \cdot (0, w_y) = -v$$

can be obtained by adding a finite number of vectors $v$, $(w_x, 0)$, and $(0, w_y)$. □

### Determining whether $S^* = \mathbb{Z}^2$

We write $\mathrm{GCD}(x,y)$ to denote the greatest common divisor of two integers $x$ and $y$. If at least one of $x$ and $y$ is nonzero, then $\mathrm{GCD}(x,y)$ is the largest positive integer that divides both $x$ and $y$. We assume that $\mathrm{GCD}(0,0) = 0$. In this section, we use Euclid's well-known algorithm for computing the greatest common divisor. The running time of the algorithm is $O(\log M)$ if $|x|$ and $|y|$ are bounded by $M$. Moreover, the algorithm can be modified to compute integers $t_x$ and $t_y$ such that $\mathrm{GCD}(x,y) = t_x \cdot x + t_y \cdot y$, where $|t_x|$ and $|t_y|$ are bounded by $M$. (This modification is known as *extended Euclid's algorithm*.)

We now show that it is possible to give at most two vectors $v_1$ and $v_2$ such that $\{v_1, v_2\}^* = S^*$. Our vectors are of the form $v_1 = (x_1, 0)$ and $v_2 = (x_2, y_2)$, where $x_1$ and $y_2$ are nonnegative. Moreover, if $y_2 = 0$ then $x_2 = 0$ as well. This requirement ensures that if the vectors are collinear then one of them is $(0,0)$. Intuitively, $v_1$ represents the minimum horizontal shift, the second coordinate of $v_2$ describes the minimum vertical shift, and the first coordinate of $v_2$ describes an additional horizontal deviation, which may be impossible to completely remove using $v_1$.

We now show how for two vectors of this form, and any vector $w = (x_3, y_3)$, we compute two vectors $v_1' = (x_1', 0)$ and $v_2' = (x_2', y_2')$ such that $\{v_1, v_2, w\}^* = \{v_1', v_2'\}^*$. Furthermore, $v_1'$ and $v_2'$ are of the same form as $v_1$ and $v_2$, and can be thought of as updated $v_1$ and $v_2$. Without loss of generality, we assume that $y_3 \geq 0$. If $y_3 < 0$ then $w$ could be replaced by $-w$.

We set $x_1$ as follows:

$$x_1' = \begin{cases} \mathrm{GCD}(x_1, x_3) & \text{if } y_3 = 0, \\ x_1 & \text{if } y_2 = 0 \text{ and } y_3 \neq 0, \\ \mathrm{GCD}(x_1, \frac{y_3}{\mathrm{GCD}(y_2,y_3)} \cdot x_2 - \frac{y_2}{\mathrm{GCD}(y_2,y_3)} \cdot x_3) & \text{if } y_2 \cdot y_3 \neq 0. \end{cases}$$

Note that $v_1' \in \{v_1, v_2, w\}^*$. We prove the following claim which states that $v_1'$ indeed represents the minimum horizontal shift.

***Lemma 3.*** It holds that $\{v_1'\}^* = \{v_1, v_2, w\}^* \cap \mathbb{Z} \times \{0\}$.

*Proof.* If $y_3 = 0$ then any vector of the form $(a, 0) \in \{v_1, v_2, w\}^*$ can be expressed as $\alpha \cdot v_1 + \beta \cdot w$, where $\alpha, \beta \in \mathbb{Z}$, because $v_2$ is either $(0,0)$ or is the only vector with a nonzero second coordinate. Therefore, $\{v_1, v_2, w\}^* \cap \mathbb{Z} \times \{0\}$ is the set of vectors of the form $\alpha \cdot (\mathrm{GCD}(x_1, x_3), 0) = \alpha \cdot v_1'$, for $\alpha \in \mathbb{Z}$.

Consider now the case that $y_3 \neq 0$ and $y_2 = 0$. It must also be the case that $v_2 = (0,0)$. Therefore, all vectors of the form $(a, 0) \in \{v_1, v_2, w\}^*$ can be expressed as $\alpha \cdot v_1$, where $\alpha \in \mathbb{Z}$.

The last remaining case is $y_2 \cdot y_3 \neq 0$. Due to the earlier assumptions, it in fact holds that $y_2, y_3 > 0$. A vector $\alpha \cdot v_1 + \beta \cdot v_2 + \gamma \cdot v_3$ is of the form $(a, 0)$ if and only if $\beta \cdot y_2 + \gamma \cdot y_3 = 0$. For a given integer $\beta$, a corresponding integer $\gamma$ exists if and only if $y_3$ divides $\beta \cdot y_2$. This is equivalent to $\beta \cdot y_2$ being a multiple of the least common multiple of $y_2$ and $y_3$. This means that the vectors of the form $(a, 0)$ in $\{v_1, v_2, w\}^*$ are exactly the set of vectors of the form

$$\alpha \cdot v_1 + \beta \cdot \left(\frac{y_3}{\mathrm{GCD}(y_2,y_3)} \cdot v_2 - \frac{y_2}{\mathrm{GCD}(y_2,y_3)} \cdot v_3\right) \quad \text{for } \alpha, \beta \in \mathbb{Z},$$

because the least common multiple of $y_2$ and $y_3$ equals $\frac{y_2 \cdot y_3}{\mathrm{GCD}(y_2,y_3)}$. Equivalently, these vectors can be expressed as

$$\alpha \cdot (x_1, 0) + \beta \cdot \left(\frac{y_3}{\mathrm{GCD}(y_2,y_3)} \cdot x_2 - \frac{y_2}{\mathrm{GCD}(y_2,y_3)} \cdot x_3, 0\right) \quad \text{for } \alpha, \beta \in \mathbb{Z},$$

which proves that they are of the form

$$\alpha \cdot \left(\mathrm{GCD}\left(x_1, \frac{y_3}{\mathrm{GCD}(y_2,y_3)} \cdot x_2 - \frac{y_2}{\mathrm{GCD}(y_2,y_3)} \cdot x_3\right), 0\right) \quad \text{for } \alpha \in \mathbb{Z}.$$

This finishes the proof of the lemma. □



We now set $v_2' = (x_2', y_2')$. Let $t_2$ and $t_3$ be integers computed by extended Euclid's algorithm such that $t_2 \cdot y_2 + t_3 \cdot y_3 = \mathrm{GCD}(y_2, y_3)$. Let $v_2'' = t_2 \cdot v_2 + t_3 \cdot w$. Let $x_2''$ and $y_2''$ be the first and second coordinate of $v_2''$, respectively. We set $y_2' = y_2''$ and

$$x_2' = \begin{cases} x_2'' & \text{if } x_1' = 0, \\ \min\{x_2'' + \alpha \cdot x_1' : \alpha \in \mathbb{Z} \text{ i } x_2'' + \alpha \cdot x_1' \geq 0\} & \text{otherwise.} \end{cases}$$

Note that computing the minimum in the above definition can be implemented in constant time, using the modulo operator. Note also that $v_2' = v_2'' + \alpha \cdot v_1'$ for some $\alpha \in \mathbb{Z}$, which implies that $v_2' \in \{v_1, v_2, w\}^*$. It is not hard to verify that $v_1'$ and $v_2'$ have the desired form.

***Lemma 4.*** It holds that $\{v_1', v_2'\}^* = \{v_1, v_2, w\}^*$.

*Proof.* It holds that $\{v_1', v_2'\}^* \subseteq \{v_1, v_2, w\}^*$, because $v_1'$ and $v_2'$ were constructed in such a way that it is clear that $v_1'$ and $v_2'$ belong to $\{v_1, v_2, w\}^*$. It remains to prove the other inclusion. Consider any vector $u = (x_u, y_u)$ in $\{v_1, v_2, w\}^*$. Clearly, $y_u$ has to be a multiple of the greatest common divisor of the second coordinates of $v_1$, $v_2$, and $w$. More precisely, it has to be a multiple of $y_2' = \mathrm{GCD}(y_2, y_3)$. Let $\alpha \in \mathbb{Z}$ be such that $y_u = \alpha \cdot y_2'$. For some integer $q$, it holds that $u = (q, 0) + \alpha \cdot v_2'$. Since both $u$ and $\alpha \cdot v_2'$ belong to $\{v_1, v_2, w\}^*$, so does $(q, 0)$. It therefore follows from Lemma 3 that $(q, 0) \in \{v_1'\}^* \subseteq \{v_1', v_2'\}^*$. Finally, since both $(q, 0)$ and $\alpha \cdot v_2'$ belong to $\{v_1', v_2'\}^*$, so does $u$. □

Our algorithm for determining whether $S^* = \mathbb{Z}^2$ starts by initiating two variables $v_1$ and $v_2$ to $(0,0)$. Then, for each vector $w \in S$, it updates the vectors $v_1$ and $v_2$ as described above. We know that $\{v_1, v_2\}^*$ for the updated $v_1$ and $v_2$ equals $\{v_1, v_2, w\}^*$ for the old $v_1$ and $v_2$. The final vectors $v_1$ and $v_2$ are such that $\{v_1, v_2\}^* = S^*$, which trivially follows by induction. Due to how our procedure updates the vectors, $\{v_1, v_2\}^* = \mathbb{Z}^2$ if and only if $v_1 = (1,0)$ and $v_2 = (0,1)$.

**Time complexity**

The first part of the algorithm, determining whether the set is axis-complete, runs in $O(n)$ time. Let $M$ be an upper bound on the absolute value of all coordinates in $S$. It can easily be proved by induction that in the second part of the algorithm, both $|x_1|$ and $|x_2|$ are bounded by $M^2$, and $|y_2|$ is bounded by $M$. Therefore, processing each vector from the input takes $O(\log M)$ time, since we run Euclid's algorithm at most a constant number of times and use only a constant number of additional arithmetic operations. Hence the total running time of the algorithm is $O(n \log M)$.

**JAKUB PACHOCKI**

Jakub Pachocki is a student at the Faculty of Mathematics, Informatics and Mechanics, University of Warsaw. In high school he was a silver medalist at the International Olympiad in Informatics and a winner of the Baltic Olympiad in Informatics. As a university student he won both the Polish Collegiate Programming Contest and the ACM-ICPC Central European Regional Contest twice, and represented the University of Warsaw at the ACM-ICPC World Finals in 2011 and 2012. In 2011 he ranked third in the world finals of Google Code Jam and fourth at the Facebook Hacker Cup. In the same year Jakub Pachocki won two Polish programming competitions: Algorithmic Engagements and Wielka Przesmycka (University of Wroclaw Programming Contest). Coming up with challenging algorithmic tasks is one of his hobbies, along with reading interesting books (fantasy, fiction, science and popular science books), writing and swimming.

# Ice Skates

**Contest:** 16th Polish Olympiad in Informatics
**Task author:** Jakub Radoszewski
**Memory:** 32 MB
http://main.edu.pl/en/archive/oi/16/lyz

Byteasar runs a skating club. Its members meet on a regular basis and train together, and they always use the club's ice skates. The skate sizes are (by convention) numbered from 1 to $n$. Naturally, each club member has a particular foot size, but there's more to it than that! Skaters have skate size tolerance factor $d$: a skater with foot size $r$ can wear skates with sizes from $r$ up to $r + d$. It should be noted, though, that *no skater ever wears two skates of different sizes simultaneously.*

To supply the club, Byteasar bought $k$ pairs of ice skates of each size, i.e. from 1 to $n$. As time passes, new people join the club, just as some established members leave it. Byteasar worries whether he will have enough skates of appropriate sizes for every member at each training session.

We assume that initially the club has no members at all. Byteasar will give you a sequence of $m$ events of the following form: $x$ members with foot size $r$ join or leave the club. Right after each such event, Byteasar wants to know whether he has enough skates of appropriate size for every club member. He asks you to write a program that will check this for him.

## Input

The first line of the input contains four integers, $n$, $m$, $k$ and $d$ ($1 \le n \le 200{,}000$, $1 \le m \le 500{,}000$, $1 \le k \le 10^9$, $0 \le d < n$), separated by single spaces, that specify, respectively: maximum skate size, number of events, number of skate pairs of each size that Byteasar initially bought, and the skate size tolerance factor. The following $m$ lines contain the sequence of $m$ events, one per line. The $(i + 1)$-th line (for $1 \le i \le m$) holds two integers: $r_i$ and $x_i$ ($1 \le r_i \le n - d$, $-10^9 \le x_i \le 10^9$), separated by a single space. If $x_i \ge 0$, it means that $x_i$ new members, each with foot size $r_i$, have just joined the club. And if $x_i < 0$, it means that $x_i$ members, each with foot size $r_i$, have just left the club. You may assume the sequence is sensible; i.e. someone who never joined the club cannot leave it.

**Output**

Your program should write $m$ lines to the output. The $i$-th line (for $1 \le i \le m$) should contain either the word TAK (Polish for *yes*) or the word NIE (Polish for *no*), depending on whether Byteasar has skates of appropriate size for every club member right after the $i$-th event.

**Example**

For the input data:

```
4 4 2 1
1 3
2 3
3 3
2 −1
```

the correct result is:

```
TAK
TAK
NIE
TAK
```

**Explanation of the example:** After all events from the input sequence have taken place, there are three club members who can wear skates of size 1 or 2, two members who can wear size 2 or 3, and three members who can wear size 3 or 4. With such a list of members, two pairs of ice skates in each of the sizes 1, 2, 3, and 4 suffice:

- two members get skates of size 1,
- the skates of size 2 are given to the member who can wear size 1 or 2, and another one who can wear size 2 or 3,
- the skates of size 3 are given to the member who can wear size 2 or 3, and another one who can wear size 3 or 4,
- the remaining two members receive skates of size 4.



# / Solution

I had the pleasure of battling with the task about ice skating during the second stage of the 16th Polish Olympiad in Informatics. The duel did not last overly long, since—thanks to positive currents—I managed to solve the task in under forty minutes from the start of the competition. Nevertheless, *Ice Skates* sank into my memory as an exceptionally elegant fusion of two apparently entirely independent algorithmic problems.

My first (and successful, at that!) approach to the task was very direct. Namely, let us consider when ice skates can be assigned to all members of the club. This problem has a natural graph interpretation: we may construct a bipartite graph, with the club members on its left side and the ice skates on its right side, and draw edges between every skater and the ice skates that he is allowed to use. For now, let us try *not* to think about the various special properties of the graph (originating from the simple rules that describe the tolerance of the skaters). In our task, we ask whether there exists a one-sided perfect matching in this bipartite graph. It is natural to suspect that this problem is well known. Indeed, the criterion that determines the existence of such a matching is known as Hall's theorem.

***Hall's theorem.*** Assume a bipartite graph is given, with its edges connecting the vertices from a set $U$ on the left side and a set $V$ on the right side. There is a way to assign to each vertex from $U$ a different, neighboring vertex from $V$ if and only if for every $X \subseteq U$ the vertices from the set $X$ are connected to at least $|X|$ vertices from $V$.

If we denote by $L(X)$ the set of vertices from $V$ connected to any vertex from $X$, the above theorem dictates that for every $X$ the inequality $|X| \ge |L(X)|$ must hold. In other words, we can assign ice skates to all members of Byteasar's club if and only if there exists no subset of skaters such that there are fewer ice skates that fit *any* of those skaters than the cardinality of the set. This gives us the first solution to the task: each time we want to answer one of the queries, we can go through all subsets of the skaters and check whether any of them fail to fulfill the condition from Hall's theorem. The complexity of such a solution is, however, far from satisfactory, since the time needed to answer a single query depends exponentially on $n$.

Do we actually have to go through all subsets of the skaters? Assume that at some point the condition from Hall's theorem is not fulfilled by the set $X \subseteq U$.

If $X$ could be represented as the sum of two nonempty disjoint sets $X_1, X_2$ such that additionally $L(X_1) \cap L(X_2) = \emptyset$, then it is easy to verify that one of the sets $X_1$, $X_2$ would fail to fulfill the condition of Hall's theorem. Therefore, we only need to consider sets $X$ that cannot be divided in this way (which we shall refer to as *indivisible* sets from now on). Suppose, in turn, that such an indivisible set $X$ is strictly contained in some set $Y \subseteq U$, such that $L(X) = L(Y)$. Then $Y$ is also indivisible and, naturally, $L(Y) = L(X) < |X| < |Y|$; therefore $Y$ also fails to fulfill the condition of Hall's theorem.

The right time has now come to use the special properties of our graph. Let $U$ be the set of club members and $V$ be the set of ice skates. It follows from the above observations that if there exists a subset of the skaters that does not fulfill the condition of Hall's theorem, there also exist such numbers $l, r$ $(1 \le l \le r \le n - d)$ for which the set of *all* skaters of foot size in the interval $[l, r]$ is indivisible and also fails to fulfill this condition! Moreover, we may assume that there exist skaters with foot size $l$ and $r$ (otherwise we could simply pick a smaller interval).

Determining whether Hall's condition holds for such simply defined sets turns out to be much easier than in the general case. Let us observe that the total number of ice skates that fit skaters whose foot size lies within the interval $[l, r]$, picked as above, is equal to $k(r - l + 1 + d)$ (here we use the fact that the set of these skaters is indivisible). If we denote by $a_i$ the number of skaters with foot size $i$, the total number of club members with foot size in the given interval will be equal to $a_l + a_{l+1} + \cdots + a_r$. Hall's condition is therefore not fulfilled if $(a_l - k) + (a_{l+1} - k) + \cdots + (a_r - k) > kd$. If we denote $b_i := a_i - k$, then there exists a subset of club members that does not fulfill Hall's condition if and only if the greatest possible sum of a contiguous subsequence of the sequence $b_1, \ldots, b_{n-d}$ exceeds $kd$. Therefore, if we can find a data structure to hold the sequence $b$, that will allow for fast modification of one element of the sequence and answer a query about the maximum sum of a contiguous subsequence, we will obtain a complete solution of the original task.

This structure turns out to be the segment tree—something of a classic in the Polish Olympiad in Informatics. The idea is relatively simple: to construct a tree that operates on a sequence of numbers, we create a root that aggregates information about the whole sequence, and in its children we create analogous trees corresponding to the left and right halves of our sequence. Note that such a structure contains only logarithmically many vertices that aggregate information concerning any single element of the sequence; therefore, if we can compute the information for a given node of the tree in constant time (assuming the information for its children has already been computed), we will obtain a structure that updates itself after the modification of any element in logarithmic time.

Is the information about the maximum sum of a contiguous subsequence of the sequence corresponding to the given node something we can compute in constant time, based on the data for the children of that node? Unfortunately... not by itself. It turns out, however, that it is sufficient to keep—apart from the function $s(u)$ (the maximum sum of a contiguous subsequence for a node $u$ of our tree)—three auxiliary functions:

- $pref(u)$, denoting the maximum sum of a prefix of the sequence corresponding to the node $u$,
- $suf(u)$, denoting the maximum sum of a suffix of the sequence corresponding to the node $u$,
- $sum(u)$, denoting the sum of all elements of the sequence corresponding to the node $u$.

Then we obtain a set that's "closed" with respect to the possibility of being quickly computed based on information kept in the children. Indeed, if $t$ is a node of the tree, and $u$ and $v$ are its children, it holds that:

$$\begin{aligned} s(t) &= \max\big(s(u), s(v), suf(u) + pref(v)\big), \\ pref(t) &= \max\big(pref(u), sum(u) + pref(v)\big), \\ suf(t) &= \max\big(suf(v), suf(u) + sum(v)\big), \\ sum(t) &= sum(u) + sum(v). \end{aligned}$$

Our solution will therefore, on every change to the set of members of the club, update the computed values of the functions $s$, $pref$, $suf$, $sum$ in $O(\log n)$ vertices of the tree and print TAK ("yes") if and only if the value of $s$ in the root of the tree does not exceed $kd$. Interestingly, the described algorithm has a surprisingly concise implementation!

If the reader has found pleasure in searching for a way to quickly compute information regarding matchings in a specific graph, I also recommend reading the following tasks: *Balloons* from the final round of Algorithmic Engagements 2008, *Hotel* from the 18th Central European Olympiad in Informatics, and *Permutation* from Algorithmic Engagements 2009. The statements of all these problems are available on the MAIN website.

# / Termites 2

**Contest:** Algorithmic Engagements 2010
**Task author:** Jakub Pachocki
**Memory:** 128 MB
http://main.edu.pl/en/archive/pa/2010/ted

Do you remember the two friendly plank-eating termites that never grow tired of mental exercise? (See the task *Termites* on page 123.) After eating away most of the fences in Byteland, they have acquired an appetite for trees.

A *tree* is a connected, undirected graph with $n$ vertices and $n-1$ edges. Our two termites quickly got bored with monotonously eating the vertices—to make their meal more interesting, they invented another game. They agreed on an ordering $(e_1, e_2, \ldots, e_{n-1})$ of the edges of the tree they are going to eat. The game will last for at most $n-1$ rounds, with exactly one termite making a move in each round. The players move alternately (the starting player plays in round 1, the second in round 2, the first again in round 3, and so on). In the $k$-th round the playing termite must choose an uneaten end of edge $e_k$ and eat it. If both ends of $e_k$ are eaten before the termite makes its move, the game ends with its loss. If the game does not end in $n-1$ rounds, it is tied.

We assume that the termites (who are, after all, experienced in this kind of game) never make an error, and that the termite with a winning strategy wants to win in the earliest possible round, while its opponent tries to delay its own defeat for as long as it can. Your task is to determine, for a given tree and the ordering of its edges set by the termites, the round in which the game will end.

### Input

In the first line of the input there is an integer $n$ ($2 \le n \le 500{,}000$), specifying the number of vertices in the tree. In the next $n-1$ lines are given the tree's edges in the order set by the termites. In the $i$-th of these lines there are two integers $u_i$ and $v_i$ ($1 \le u_i, v_i \le n$), specifying the indices of the ends of edge $e_i$.

### Output

In the first and only line of the output exactly one integer should be written— the number of the round in which the game will end, or $-1$ if the game will end in a draw.



### Example

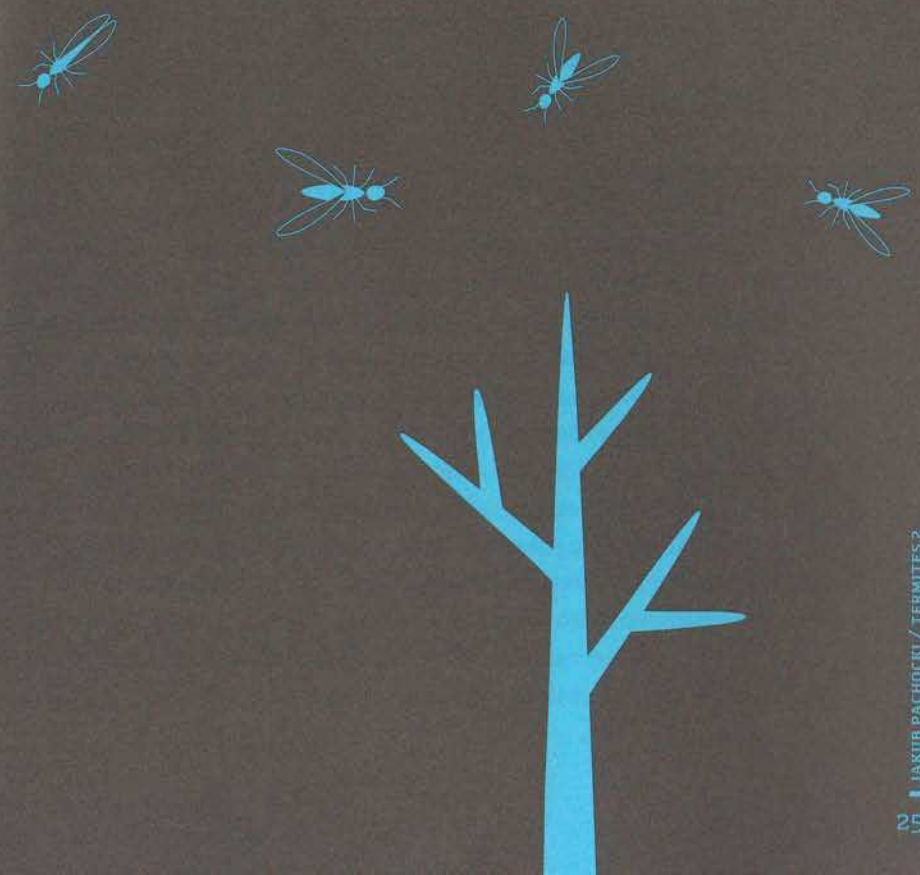
For the input data:

```
5
2 3
1 2
4 5
3 4
```

the correct result is:

```
4
```

**Explanation of the example:** If, in the first round, the starting termite eats vertex 3, and in the third round it eats vertex 4, its opponent will not have any valid moves in the fourth round, regardless of its play in the second round.



# / Solution

*Termites 2* is one of my favorite, and simultaneously one of the hardest, of my tasks. It is different from most of my other tasks in that it was *not* created in a sudden flash of inspiration, but rather as the result of a few hours of strenuously attempting to create an interesting problem.

Reading the brilliant task *Termites* by Tomasz Idziaszek awoke in me a desire to become a problem setter myself for the great Algorithmic Engagements contest. I entertained this idea for a few days until finally, on a sunny May morning, I got up at dawn and started searching for good topics.

My initial idea seemed really interesting at first. The problem was as follows: we have a board with $m$ rows and $n$ columns (many great algorithmic problems take place on a board!); in each of its cells a real number is inscribed. Our goal is to place rooks on this board in such a way that no two rooks attack each other, and the sum of values of the fields on which the rooks stand is maximized. My intuition suggested that this task must have a solution that would probably be interesting and nontrivial. How great was my disappointment when, a while later, I realized that the solution—which does indeed exist—is already commonly known, as the Hungarian algorithm.

What was there to do? I decided to stick with the idea of placing rooks on a board; however, to make the task a little more difficult and move away from the class of problems solved by classical algorithms, I decided to add the element of a game. What if we had two players placing rooks alternately, both of them trying to maximize the sum of values of the cells on which they placed rooks, with that total being decreased by the sum of values of the cells containing rooks placed by their opponent? This sounded really interesting. However, as often happens with interesting-sounding problems, I hit a wall of a different kind—namely, I was unable to solve the task as stated! I therefore tried to make it easier by getting rid of the optimization element—let us forget about the values inscribed in the cells; let some of them be blocked and the others free, and have the players simply place rooks in free cells alternately, until one of them cannot make a valid move (is unable to place a rook where no other rook will attack it). Unfortunately, despite the inviting, natural appearance of the problem, I was still unable to take even a single step towards solving it.

All the indications were that I had made the task too hard when introducing the game element. I therefore tried to introduce a further simplification



which, I believe, was the most important of my ideas during my work on this task; the path from the stated task to *Termites 2* was actually quite straightforward. The idea is as follows: let the set of moves that the players can make depend on the current round. In the $i$-th round, let the player be able to place a rook only in the $i$-th row. This leads to a number of interesting consequences:

- The first player will only place rooks in odd rows and the second player only in even rows.
- It is no longer necessary to worry about the rooks being able to attack to the left and to the right (since we already know that no two will be placed in the same row anyway).
- For the state of the game at any given moment, the exact placement of rooks on the board does not matter; the only important concerns are the round number and the set of columns that are already occupied. This leads in a simple way to a solution based on dynamic programming, with an $O(2^n mn)$ running time. It is not particularly interesting, but it is a visible sign of progress: finally, we have obtained a task that we can solve with a nontrivial algorithm!

Unfortunately there was still no polynomial algorithm to solve the problem in sight. I therefore focused on the aforementioned solution that works in time exponentially dependent on the number of columns. I wondered what would happen if we bounded the number of free cells in each row—let us say that their count would not exceed $r$. I devoted a lot of time to this idea, but I was unable to find a solution. In spite of this, I decided to follow the punch and bound the value of $r$.

Values of $r$ such as 4 or 5 did not bode well. Things seemed to get interesting at $r = 3$. If there were at most three free cells in each row, a player would have seemingly very few options in each move—if more than one of the cells in which he could place a rook was taken, he would have at most one possible move. Despite long deliberation, I was still unable to reach an interesting solution.

I was therefore left with the last value of $r = 2$ (for $r = 1$ the players never have any choice; the game would therefore never be any more interesting than the War card game!). In the presented problem, the difference between $r = 3$ and $r = 2$ is diametrical, quite like the difference between the problems 3-SAT and 2-SAT. Let us assume that in every row there are precisely two free cells—rows in which there are fewer free cells do not seem to bring anything interesting to the game. Then we can model our board as an undirected graph

in which the columns are vertices and the rows are edges between them. For example, the board:

```
.**..
**...
...**
..**.
```

corresponds to a graph with five vertices and four edges, of which the first edge connects vertices 2 and 3, the second 1 and 2, the third 4 and 5, and the fourth 3 and 4. The players alternately take the next edge in turn and cross out one of its undeleted ends; the first player to be unable to make a valid move loses.

As can be seen, setting $r = 2$ has nearly brought me to the final formulation of the task—at this point, the only important difference was that, in the graphs on which the game was played, I admitted the presence of cycles. Interested in the graph model of the board and the lack of an obvious solution, I set out to search for a polynomial algorithm to solve the problem. I started with a rather obvious observation—if, after adding the first $k$ edges, in the graph there appears a connected component in which there are more edges than vertices, the game has to end with one player winning during the first $k$ moves (this follows from the fact that we had to cross out at least one vertex for every edge of the selected component). We may therefore only consider graphs in which, in every connected component, there are at most as many edges as vertices.

Let us look at the possible states of the game after $k$ moves. The edges considered thus far determine some number of connected components in the graph. It is apparent that the moves made on separate connected components were independent; let us therefore think about the possible state of a single connected component. If it contains $n$ vertices, it must contain at least $n - 1$ edges (otherwise it would not be connected). Since for every edge some vertex had to be crossed out, in the connected component there is at most one undeleted vertex remaining—there are therefore at most $n$ possible states!



Let us look at the possible endings to the game. We can separate them into two kinds:

1. A player makes a move on an edge connecting vertices $u$ and $v$ belonging to two previously separate connected components; the opponent has, however, previously forced the deletion of both the vertex $u$ in the first of those components and the vertex $v$ in the second.
2. A player makes a move on an edge connecting vertices $u$ and $v$ belonging to the same connected component (which must be a tree in the graph formed by previously considered edges). The opponent has, however, previously forced the deletion of both vertices $u$ and $v$ in that component.

To determine whether a player $X$ can avoid losing in the given round, we therefore have to be able to effectively answer the corresponding questions:

1. Can player $X$ ensure that the vertex $u$ is not crossed out? Can player $X$ ensure that the vertex $v$ is not crossed out? (Let us note that these questions are completely independent of each other if $u$ and $v$ have been in different connected components so far!)
2. Can player $X$ ensure that at least one of the vertices $u$, $v$ is not crossed out (without stating which one)?

For each connected component we may therefore keep all the information of the form "player $X$ can ensure that the only undeleted vertex is the vertex $v$" and "player $X$ can ensure that the only undeleted vertex is one of the vertices $u$, $v$; however it may be the opponent who chooses one of them specifically". Following this lead, I actually obtained a polynomial solution to the problem. I did, however, ask myself the question: does the presence of cycles in the graph, introducing the clearly more conceptually complex second case in the above considerations, really add enough value to the task? I decided that it does not— it seemed to me that this is another example of a task in which the solution for a graph with a single cycle (a connected graph with an equal number of edges and vertices) is just a more complicated version of the solution for a tree. Thus I reached the final version of the problem—adding the story about termites was now natural, considering my initial inspiration for creating the task!

Let us make sure that we can actually effectively maintain information of the kind "player $X$ can ensure that the only undeleted vertex in a given component is the vertex $v$" while considering subsequent edges. Let us denote by $A(S)$ the set of vertices that Player 1 can ensure will remain undeleted in the component $S$, and by $B(S)$ the set of vertices that Player 2 can ensure will remain undeleted in the component $S$. Suppose we know that the next move is to be made by Player 1 on the edge $(u, v)$, at which $u$ in the graph formed by previously considered edges

belongs to the component $T_1$, while $v$ belongs to $T_2$ ($T_1 \neq T_2$). Let us denote by $T$ the component formed after adding the edge $(u, v)$ (so $T = T_1 \cup T_2$). After a moment of thought it is not hard to come to the conclusion that:

$$\begin{cases} A(T) = A(T_1) \cup A(T_2),\ B(T) = \varnothing, & \text{if } u \in A(T_1) \text{ and } v \in A(T_2), \\ A(T) = A(T_2),\ B(T) = B(T_2) \setminus \{v\}, & \text{if } u \in A(T_1) \text{ and } v \notin A(T_2), \\ A(T) = A(T_1),\ B(T) = B(T_1) \setminus \{u\}, & \text{if } u \notin A(T_1) \text{ and } v \in A(T_2). \end{cases}$$

Of course, if $u \notin A(T_1)$ and $v \notin A(T_2)$, then Player 2 can force the defeat of Player 1 in the current move.

A simple implementation of the above observations, storing the sets $A$ and $B$ in balanced binary search trees, allows us to obtain a solution that works in $O(n \log^2 n)$ time (where $n$ is the number of vertices of the tree). After a more careful look at the nature of these sets, it is possible to obtain solutions that work in complexities $O(n \log n)$, $O(n \log^* n)$ and, finally, a very simple and elegant solution in $O(n)$ time—the finding of which I strongly encourage the reader to attempt!

**PAWEŁ PARYS**

Paweł Parys is a research assistant at the Institute of Informatics, University of Warsaw, where he has recently earned his Ph.D. degree. His miscellaneous scientific interests are connected with automata theory, including mainly algorithms for searching within XML documents and higher-order pushdown automata.

He started taking part in computer science competitions in high school. He is a two-time winner of the Polish Olympiad in Informatics and a runner-up of the International Olympiad in Informatics. During his university years, Paweł Parys and his team enjoyed three victories at the Polish Collegiate Programming Contest and two victories at the ACM-ICPC Central European Regional Contest. At the ACM-ICPC World Finals his team ranked tenth in 2004 and seventh in 2006. As an individual contestant, he was a multiple finalist of Google Code Jam, TopCoder Open and TopCoder Collegiate Challenge. He has also participated successfully in mathematical competitions.

Paweł Parys enjoys traveling, especially when it involves biking and mountain hiking.

# / Cave

**Contest:** 11th Polish Olympiad in Informatics
**Task author:** Paweł Parys
**Memory:** 16 MB
http://main.edu.pl/en/archive/oi/11/jas

There is a cave in Byteland. It consists of $n$ chambers and corridors connecting them. The corridors are arranged in such a way that there is a unique path between each pair of chambers. In one of these chambers Hansel has hidden a treasure, but he won't say which chamber it is. Gretel wants to know where the treasure is, so she asks Hansel about various chambers. When she guesses right, Hansel tells her she is right, and when she guesses wrong, he tells her which way from this chamber leads to the treasure.

## Task

Write a program that:

- reads from the input the description of the cave,
- finds the minimum number of questions Gretel has to ask, in the worst case, in order to learn in which chamber the treasure is hidden,
- writes the result to the output.

## Input

In the first line of the input there is one positive integer $n$ ($1 \le n \le 50{,}000$), specifying the number of chambers in the cave. The chambers are numbered from 1 to $n$. In the following $n - 1$ lines the corridors linking the chambers are described, one per line. Each of these lines contains a pair of distinct positive integers, $a$ and $b$ ($1 \le a, b \le n$), separated by a single space, indicating that there is a corridor between chambers $a$ and $b$.

## Output

Your program should write one integer to the output, stating the minimum number of questions Gretel has to ask in the worst case (i.e. we assume Gretel asks the questions in the best possible way, but the treasure is hidden in the chamber that requires the largest number of questions).



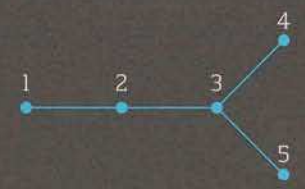
## Example

For the input data:

```
5
1 2
2 3
4 3
5 3
```

the correct result is:

```
2
```



# / Solution

This problem turned out to be one of the most difficult in the history of the Polish Olympiad in Informatics. The best student's solution received just 52 points (out of 100), and only a few contestants achieved more than 10 points. This was a great surprise to me (as the author of the problem). My feeling was that it would be a relatively standard problem; the line of reasoning described in the following text appeared to me completely natural—in fact, the only approach possible—and every one of its steps not too difficult. But the contestants, in the limited time available during the competition, proved unable to deal with this problem.

After the contest, I wrote a paper together with Krzysztof Onak based on this problem. The problem is very natural: a generalization of binary search into trees. We have added another variant, where the questions are asked in terms of edges instead of nodes, and a few other small extensions. Our paper was accepted by an American conference: IEEE Symposium on Foundations of Computer Science (FOCS).

**Strategy functions**

So, how should this problem be solved? Of course, the plan of the cave is a tree. But how can we describe a strategy for asking questions? We have to determine only the minimum number of questions to ask, but without considering a strategy for asking questions, we will probably not manage to do that. So let us think about how we might ask questions in our problem. When asking about some node of the cave, we split the cave into some number of parts (connected components), which arise when this node is removed. As our answer we get one of these parts, and in it we have to find the treasure. It makes no sense to ask any of the subsequent questions about anything outside this part, because the answer to such a question would not give us any new information; afterwards we will be asking questions only about nodes inside this part. Nevertheless, we have to consider all of these parts, because any answer is possible. So we can think about this as follows. In the tree, we remove one selected node. In the next step we again choose one node, in each of the resulting parts (connected components), and we remove these nodes. We repeat this until the parts contain only one node each. In the problem, our goal is to minimize the number of such steps.

Based on this observation, we will be describing a strategy using a function $f$ which assigns a nonnegative integer to each tree node. This number will tell us in which step, counting from the end, the node will be removed. Such a function has to satisfy the following condition:



for every two nodes $u$, $v$ such that $f(u) = f(v)$,
on the path between $u$ and $v$ there is a node $w$ such that $f(w) > f(u)$. (*)

This condition corresponds to the requirement that if we remove two nodes in the same step, some of the nodes between them would have had to be removed earlier. Such a function will be called a *strategy function*. We see that every reasonable strategy of asking questions (i.e. such that we do not ask questions that would not provide us with any new information) can be described using a strategy function, where the number of questions asked (for the worst sequence of answers) is equal to the maximum value of the function. For example, in figure 1 we present an optimal strategy function for the tree from the problem statement, and the corresponding decision tree. But of course this correspondence also goes in the other direction: having a strategy function $f$, it is easy to give a strategy that requires no more than $\max_v f(v)$ questions. We just ask about the node $v$ in which $f(v)$ is maximal. As an answer, we get some part of the tree. Again we ask about the node $v$ in which $f(v)$ is maximal in this part of the tree, and so on. In the current part of the tree, we always have exactly one node with maximal $f(v)$; this follows from the condition (*) above. Either we will hit the node with the treasure at some point or, at the end, we will get a tree consisting of one node and we will know that the treasure is in that node. To conclude, it is sufficient to find a strategy function $f$ for which $\max_v f(v)$ is as small as possible.

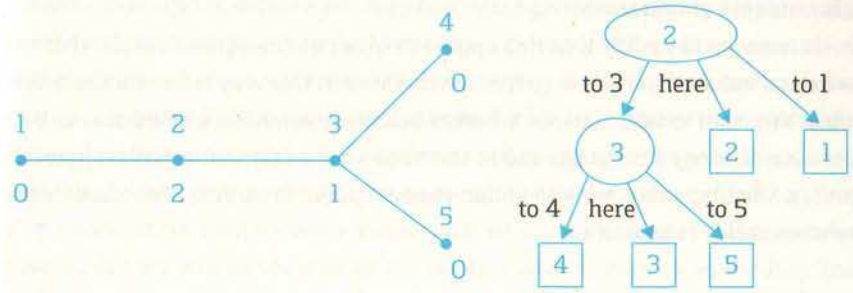



***Figure 1:*** *Left: an optimal strategy function for the tree from the problem statement. Node numbers appear above the nodes; below them appear the values of the strategy function. Right: the corresponding decision tree. Inside the nodes appear Gretel's questions; on the edges, Hansel's answers.*

### Idea of the solution

How should we find the best strategy function? A naïve approach is to begin by placing the greatest value of the function (the first question), and then consecutively smaller and smaller values. We see that the greatest value has to be assigned approximately in the middle of the cave. But it is not clear in which sense it should be the middle. We see also that it is better to choose a node that has more neighbors, since then we will divide the tree into a greater number of smaller parts. We can start in a node from which the distance to all other nodes is minimal; or in such a way that the emerging parts have the smallest number of nodes. But it is easy to ascertain that any such easy approach will not succeed. The problem is that our "size" of a tree should not be equal to the number of nodes or a diameter, but to the required number of questions; the first question should be asked in such a way that the "sizes" of the resulting parts, defined in this way, are minimal. We could start from smaller parts of the tree and, based on their results, analyze the larger ones; but the total number of parts of the tree might be very big, so we cannot process all of them in a reasonable time.

Another possible approach is to process the tree starting from the leaves and going into its center. In order to simplify the argumentation, let us choose any node of the tree as the root. This is done simply to fix the order in which we will traverse the nodes of the tree while calculating the solution: we will be considering the tree starting from the leaves and going towards the root; of course the choice of the root will not have any impact on the result. It turns out that the optimal strategy function can be obtained through a greedy approach: to every node, going from leaves to root, we assign the smallest possible strategy function value that does not conflict with the values already assigned in the descendants of that node.

It remains to justify that this approach gives us the correct result: that the maximal value of the strategy function created in this way is the smallest possible. We want to say: if, in the subtrees below a given node $v$, there are the best possible strategy functions, and to the node $v$ we assign the smallest possible non-conflicting value, we will obtain the best possible strategy function for the whole subtree rooted in $v$.

### Visibility sequences

It is easy to see that, in such an approach, a best possible strategy function is not only one whose maximal value is the smallest possible. A situation where the maximal value is at the root of a subtree is much better than a situation where this value is somewhere deeper. In the first case, above the subtree we can assign arbitrary values different from this maximal one, which is not true in the second case. We ascertain that all values "visible" from the root of a subtree are important. The value $f(v)$ of a given node $v$ is called *visible* in a given subtree if all nodes $w$ on the path from $v$ to the root (excluding $v$, but including the root) satisfy $f(w) < f(v)$. In other words, any greater value does not cover the value $f(v)$ when we are looking from the subtree's root. The *visibility sequence* of a given subtree is the sequence of all values visible in it, in descending order. Notice that every value visible in a subtree can be visible in only one node (otherwise the condition (∗) would not be satisfied for the two nodes in which this value is visible). We see also that the arrangement of the other values is unimportant: if the visibility sequence remains the same, the values in the subtree can be rearranged arbitrarily, and it will not cause any trouble from the point of view of the rest of the tree.

We want to prove that, when we greedily assign the smallest possible values of the strategy function in order from leaves to the root, in each subtree we will obtain the best possible visibility sequence. So which visibility sequence is the best possible (i.e. allowing the greatest capabilities in the rest of the tree)? First of all, the first (largest) value of the sequence counts, as this is the maximum of the strategy function in the subtree. But, as we have already said, the remaining values of the sequence also count. It turns out that visibility sequences have to be compared lexicographically: a lexicographically smaller sequence is a better one. Indeed, let us consider a tree $T$ with $v$ as its root, and its subtree $T_w$ with $w$ as its root (where $w$ is a child of $v$). Assume that for some strategy function $f$ the visibility sequence in $T_w$ is $F_w$, and in $T$ it is $F_v$. Say that in the subtree $T_w$ someone managed to define a strategy function $g$ giving a visibility sequence $G_w$, lexicographically smaller than $F_w$. How can we extend $g$ to the whole tree, so that we will obtain a function whose visibility sequence is lexicographically smaller that $F_v$ or equal to $F_v$? It is very easy. Let $a$ be the largest number that appears in $F_w$ but does not appear in $G_w$ (all greater numbers appear in none of them, or in both). If $f(v) > a$, on the rest of the tree we can leave the values of $f$. All values of $g$ visible in the subtree were visible also when there were $f$ (values greater than $a$), or they will be covered by $f(v)$ (values smaller than $a$), so the function defined in this way will be a strategy function. Its visibility sequence in $T$ is $F_v$, as for $f$. On the other hand, if $f(v) < a$, we assign value $a$ to the root, and on the rest of the tree we leave the values of $f$. It is easy to check that the function defined in this way is a strategy function. Its visibility sequence in $T$ begins in

the same way as $F_v$, but numbers smaller than $a$ stop being visible. So in this case we also obtain a visibility sequence which is lexicographically smaller than $F_v$, or equal to $F_v$, whereas the case $f(v) = a$ is impossible, as condition (∗) would not be satisfied for $v$ and the node of $T_w$ in which $a$ is visible.

As a consequence of the above we see that, if we want to obtain the lexicographically smallest visibility sequence for the whole tree, it is enough to choose for each subtree (among the subtrees rooted in the children of the root) a strategy function that gives the lexicographically smallest visibility sequence for this subtree, and then to choose some value in the root. It is not difficult to see that the smaller value will be assigned to the root, the lexicographically smaller will be the visibility sequence for the whole tree. Of course, the same reasoning can be applied by induction for all smaller subtrees. This proves that our greedy approach is correct.

**Algorithm**

We have not yet said how to write a procedure that assigns to a node $v$ the smallest possible value of strategy function, which does not conflict with the values already assigned in its descendants. Here again the visibility sequences are useful. As we have already said, for the whole tree rooted in a child of $v$, it is enough to know its visibility sequence. Let $F_1, F_2, \ldots, F_k$ be the visibility sequences for the subtrees rooted in the children of $v$. Condition (∗) for pairs containing nodes from one subtree is satisfied by assumption. We have to ensure that condition (∗) is satisfied for pairs of nodes from different subtrees. If some number $a$ appears in two sequences $F_i$, $F_j$, we have to put in $v$ something greater than $a$. Additionally, we have to ensure condition (∗) for pairs consisting of $v$ and a node from one of the subtrees. Thus we cannot place in $v$ a value which is already present in some sequence $F_i$. In other words: we look for the maximal number $a$ appearing in two sequences $F_i$, $F_j$ (assuming $a = -1$ if such a number does not exist); to node $v$ we assign the smallest number greater than $a$ which does not appear in any $F_i$. Moreover, in order to perform the same operation in the parent of $v$, we want to compute the visibility sequence for the whole subtree rooted in $v$: it contains $f(v)$ and those values from each $F_i$ which are greater than $f(v)$.

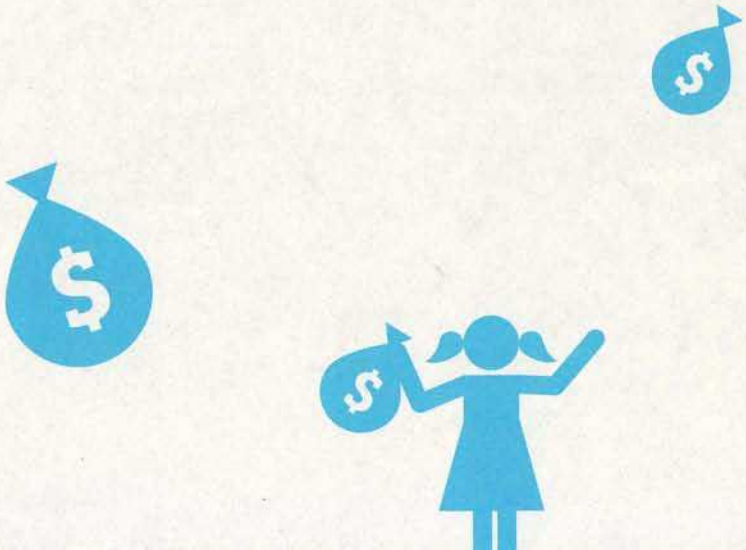


The whole algorithm can be written using the pseudocode below.

```
Function Compute(v : node)
    {let w_1, w_2, ..., w_k—children of v}
    for i := 1 to k do
        F_i := Compute(w_i)
    a := (the greatest number appearing in two F_i
          or −1 if such a number does not exist)
    b := (the smallest number greater than a not appearing in any F_i)
    return {b} ∪ {c ∈ F_i : c > b, 1 ≤ i ≤ k}

Algorithm HowManyQuestions()
    choose arbitrary node v as the root
    output the greatest element of Compute(v)
```

Notice that in every tree of size $n$ it is sufficient to ask $\lfloor \log n \rfloor$ questions. It is true because in every tree of size $n$ there exists a node dividing it into parts from which each has at most $\frac{n}{2}$ nodes; in such a node we ask the first question, and we continue the same way. Thus the visibility sequences processed by the algorithm will have length at most $\lfloor \log n \rfloor$. It is easy to implement the above algorithm so that it works in time $O(n \log n)$. As a curiosity, let us add that the solution can also be implemented in linear time, representing the visibility sequences as integers.



# / Shuffle

**Contest:** 10th Polish Olympiad in Informatics
**Task author:** Paweł Parys
**Memory:** 32 MB
http://main.edu.pl/en/archive/oi/10/tas

Byteasar has a pack of $n$ cards that he likes to shuffle. The positions of the cards are numbered from 1 to $n$. Byteasar has acquired such skill in shuffling that he obtains the same arrangement every time—i.e. a card from the $k$-th position ($1 \le k \le n$) always goes to the same $a_k$-th position. We denote by $b_k$ the position of the $k$-th card after Byteasar has repeated the shuffling $l$ times.

### Task

Write a program that:

- reads from the input the numbers $n$ and $l$ and the sequence of numbers $(b_k)$,
- determines the sequence of numbers $(a_k)$,
- writes that sequence to the output.

### Input

In the first line of the input there are two positive integers, $n$ and $l$ ($1 \le n, l \le 1{,}000{,}000$). In the consecutive $n$ lines there are successive elements of the sequence $(b_k)$, one per line. In the $(k+1)$-th line there is a positive integer $b_k$ ($1 \le b_k \le n$) specifying the final position of the card from the $k$-th position.

### Output

Your program should write to the output $n$ integers: successive elements of the sequence $(a_k)$, one per line. In the $k$-th line there should be one number $a_k$: the position of the card from the $k$-th position after a single shuffle. You may assume that, for the test data, the desired sequence $(a_k)$ always exists. If there are multiple such sequences, your program should write any of them.

### Example

| For the input data: | the correct result is: | or: |
|---|---|---|
| 5 2 | 1 | 2 |
| 1 | 2 | 1 |
| 2 | 4 | 4 |
| 5 | 5 | 5 |
| 3 | 3 | 3 |
| 4 | | |



# / Solution

At first glance, the problem appears to be easy: some transformation of permutations is involved, and that's all. However, the truth is that one has to analyze the situation carefully. During the contest, only two people (in fact, the winners in the first two places) succeeded in doing that; all the other contestants received no more than 30 points out of 100. Thus, the problem turned out to be quite difficult.

How to solve this problem? Before we come to that, let us introduce some concepts that will be helpful in understanding the solution. Instead of shuffling, we will be talking about permutations. A permutation is a function from set $\{1,2,\ldots,n\}$ to itself, in which every value is taken exactly once. A permutation can be composed which corresponds to repeating a shuffle. Instead of composing $l$ times, we can refer to taking the $l$-th power. In the problem, we are looking for a permutation whose $l$-th power gives the permutation from the input. Notice that taking powers of a permutation is very similar to taking powers of a number; in particular, $f^{a+b}$ is equal to $f^a$ composed with $f^b$, and $f^{ab}$ is equal to $(f^a)^b$.

### Powers of a permutation

A cycle in a permutation is a sequence of elements $a_1, a_2, \ldots, a_k$ such that

$$f(a_1) = a_2,\ f(a_2) = a_3,\ \ldots,\ f(a_{k-1}) = f(a_k),\ f(a_k) = a_1.$$

As one can easily deduce, in order to solve this problem it is essential to consider cycles of our permutations. One can see that a cycle of the initial permutation in its $l$-th power is transformed either to a single cycle or to a few cycles, all of the same length. More precisely, a cycle of length $k$ in the $l$-th power is transformed to $\mathrm{GCD}(k,l)$ cycles of length $\frac{k}{\mathrm{GCD}(k,l)}$ (where GCD denotes the greatest common divisor).

In order to see what can happen, let us consider an example. Fix $l = 6$. A cycle of length 4 disintegrates in the $l$-th power into two cycles of length 2. Also, a cycle of length 12 disintegrates into cycles of length 2; this time we have six of them. But a cycle of length 2 disintegrates into two cycles of length 1. Thus, first: it can happen that a cycle of length $k'$ can be obtained in the $l$-th power (for fixed $l$) from cycles of greater length, and not from cycles of length $k'$. Second: the same final effect can be obtained as a power of different permutations.

We want to reverse the power operation. Thus, we want to deduce, given $k'$ and $l$, the possible values of $k$ such that cycles of length $k$ can give cycles of

length $k'$ in the $l$-th power. In other words: for which $k$ the equation $k' = \frac{k}{\mathrm{GCD}(k,l)}$ is satisfied. Let us decompose $k'$ and $l$ into prime factors:

$$k' = p_1^{\alpha_1} \cdots p_a^{\alpha_a} r_1^{\gamma_1} \cdots r_c^{\gamma_c}, \quad l = q_1^{\beta_1} \cdots q_b^{\beta_b} r_1^{\delta_1} \cdots r_c^{\delta_c},$$

where all $\alpha_i, \beta_i, \gamma_i, \delta_i$ are positive, and all $p_i, q_i, r_i$ are distinct. No prime number $p_i$ can appear in $\mathrm{GCD}(k,l)$; thus it has to appear in $k$ in the same power as in $k'$. On the other hand, each $q_i$ and $r_i$ appears in $\mathrm{GCD}(k,l)$ provided that it appears in $k$; its exponent in $\mathrm{GCD}(k,l)$ will be the smaller of its exponents in $k$ and $l$. Thus, $q_i$ can appear in $k$ any number of times between 0 and $\beta_i$, and $r_i$ has to appear exactly $\gamma_i + \delta_i$ times. To conclude,

$$k = p_1^{\alpha_1} \cdots p_a^{\alpha_a} q_1^{\varepsilon_1} \cdots q_b^{\varepsilon_b} r_1^{\gamma_1+\delta_1} \cdots r_c^{\gamma_c+\delta_c}, \qquad \text{where } 0 \le \varepsilon_i \le \beta_i \text{ for } 1 \le i \le b.$$

Notice that all such numbers $k$ are multiplicities of

$$k_{\min}(k') = p_1^{\alpha_1} \cdots p_a^{\alpha_a} r_1^{\gamma_1+\delta_1} \cdots r_c^{\gamma_c+\delta_c}.$$

Consequently, if there exists a permutation whose $l$-th power gives the permutation on input, there exists also such permutation that all cycles of length $k'$ in the input permutation arise from cycles of length $k_{\min}(k')$ in this permutation (instead of one cycle of some larger length $k$ we can take multiple cycles of length $k_{\min}(k')$).

**Algorithm**

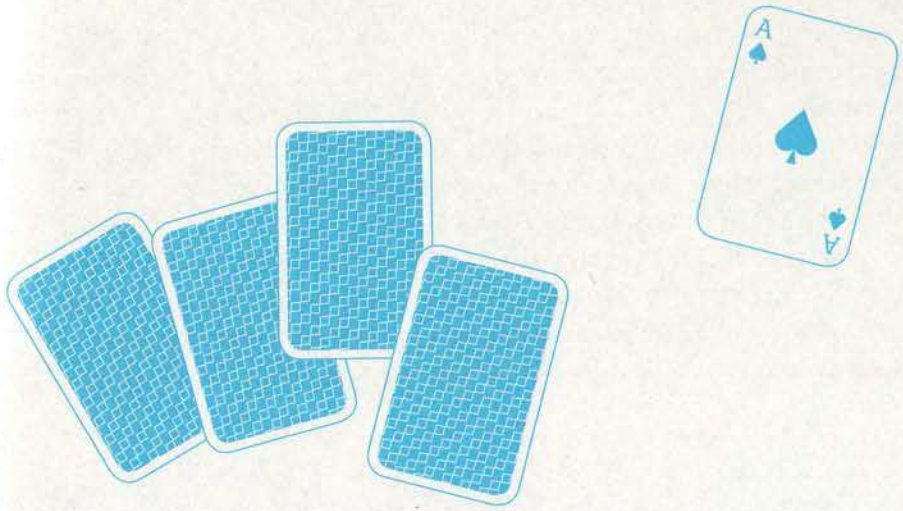
After all these observations, we come to the question of how to find the original permutation. Concentrate first on a special case where the length $k'$ of a cycle (in the permutation on input) is coprime with $l$. Then $k_{\min}(k') = k$, so every cycle of this length originates from one cycle in the original permutation. From the basic facts of number theory we know that in such a situation there exists a natural number $p$ such that $p \cdot l \equiv 1 \pmod{k'}$ (such a number is called the *inverse*). It is easy to see that for $a \equiv b \pmod{k'}$ the $a$-th and the $b$-th power of a cycle of length $k'$ are equal. Thus, taking the $p$-th power of our cycle, we obtain what we want: a cycle whose $l$-th power gives the input cycle.

Next consider the general case. Take $\frac{k_{\min}(k')}{k'}$ cycles of length $k'$. We want to find a cycle of length $k_{\min}(k')$ whose $l$-th power disintegrates into these cycles. We see that the $\frac{k_{\min}(k')}{k'}$-th power already gives us cycles of length $k'$ (obtained in such a way that we read every $\frac{k_{\min}(k')}{k'}$-th element of the original cycle). Then for $m = \frac{l \cdot k'}{k_{\min}(k')}$ the $m$-th power of these cycles is taken. But we know that $k'$ and $m$ are coprime (this can be easily seen from the decomposition into prime numbers). Thus we first find cycles whose $m$-th power gives our cycles of length $k'$ from the input. Then we "interlace" cycles obtained in this way. This means that we first take one element from each cycle, then we take the second element from each cycle, and so on.

To conclude, we proceed as follows:

1. We find all cycles of the permutation, and calculate how many cycles of each length we have.
2. Until we have processed all cycles, we do the following:
   (a) We take any group of $\frac{k_{\min}(k')}{k'}$ not yet processed cycles of some length $k'$.
   (b) We compute $p$ such that $m \cdot p \equiv 1 \pmod{k'}$, where $m = \frac{l \cdot k'}{k_{\min}(k')}$.
   (c) We take the $p$-th power of each of these cycles.
   (d) We interlace the resulting cycles.

Some details remain. The computation of the inverse of $m$ modulo $k'$ can be effectively implemented using the Euclidean algorithm (and in other situations it should be done this way). But here it is completely sufficient to check all numbers between 1 and $k'-1$; the complexity of the whole program remains the same. And how to compute $k_{\min}(k')$? We will be repeatedly dividing $l$ by $\mathrm{GCD}(k',l)$ (in a loop) as long as $\mathrm{GCD}(k',l) > 1$. When we get $\mathrm{GCD}(k',l) = 1$, in $l$ we will have $q_1^{\beta_1} \cdots q_b^{\beta_b}$ (for the decomposition of $l$ and $k'$ as above); having this, it is then easy to deduce the value of $k_{\min}(k')$. Operations such as computing the length of a cycle, taking a power of a cycle, and "interlacing" cycles can easily be implemented in such a way that the whole program will work in time linear in $n$.



**JAKUB PAWLEWICZ**

Jakub Pawlewicz works at the Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, where in 2009 he earned his Ph.D. in algorithmic methods for solving games. He is the author of an optimal program for playing Yahtzee. His scientific research is focused on methods of artificial intelligence in games (his main area of interest) and algorithmic number theory.

From 1994 to 1996 Jakub Pawlewicz won three medals at the International Olympiad in Informatics: bronze, silver and gold. In 1998, together with his team, he was ranked ninth at the ACM-ICPC World Finals in Atlanta. This was the first time in history that a team representing the University of Warsaw made it to the top ten of the best teams in the world.

# / Game of Tokens

**Contest:** Polish Olympiad in Informatics Training Camp 2010
**Task author:** Jakub Pawlewicz
**Memory:** 32 MB
http://main.edu.pl/en/archive/ontak/2010/zet

The *Game of Tokens* is a two-player game that is played with $n$ tokens, labeled with the integers from 0 to $n - 1$, on an $n \times n$ square grid filled with numbers, one per cell. The tokens and the grid are visible to both players.

The players take turns picking up tokens from the table, one per move, until all of the tokens are removed. Then the score for each player is computed as the sum of numbers in the cells captured by the player, where the cell in the $i$-th row and $j$-th column is captured if and only if the player picked up the tokens with labels $i$ and $j$ (or, if $i = j$, the single token $i$). The objective of the game is to maximize one's score advantage over the other player (and, if that is not possible, to minimize one's disadvantage). Your task is to design and implement a program that will play *Game of Tokens*.

## Programming interface

Your solution should implement the following functions.

- void startgame(int *n*)—this function will be called exactly once, before either player's first move. The argument $n$ indicates the number of tokens and the size of the grid ($1 \le n \le 40$).
- void opponentmove(int *token*)—this function is called when the opponent picks up a token, and should update your idea of the state of the game; the argument token is the label of the token ($0 \le token \le n - 1$) that the opponent chose.
- int play()—this function should return your move, i.e. the label of the token that your implementation decides to take in your current move.

The functions play and opponentmove will be called in turns, with the first call indicating who makes the first move. In your implementation you may assume that the opponent's moves are correct and, in particular, that no token is picked up more than once.

The function valueat will be provided; it allows the state of any grid cell to be discovered, and may be called at any point in the program as many times as needed.

- int valueat(int *row*, int *col*)—this function returns the number placed in row *row* and column *col* of the grid ($0 \le row, col \le n-1$). Numbers in the grid are nonnegative integers not larger than $10^6$.

Your solution (functions) must neither produce any output nor terminate the execution of the program. The program will terminate automatically after the last move.

**Scoring**

Your submission will gain a point for each match played in an optimal way: that is, maximizing the value of your score minus the opponent's score. This means that points may be gained even if the match is lost, provided that your program made optimal moves against the opponent and thereby minimized the loss.

**Example**

For $n = 3$ and the grid

2 8 1
3 5 5
3 2 7

a possible sequence of function calls when running a program that plays the *Game of Tokens* is:

```
-> startgame(3)
    <- valueat(0,0) = 2
    <- valueat(0,1) = 8
    <- valueat(0,2) = 1
    <- valueat(1,0) = 3
    <- valueat(1,1) = 5
    <- valueat(1,2) = 5
    <- valueat(2,0) = 3
    <- valueat(2,1) = 2
    <- valueat(2,2) = 7
-> opponentmove(1)
-> play() = 2
-> opponentmove(0)
```

**Explanation of the example:** As required, startgame is called first. In this example implementation, it calls valueat for each cell to return the numbers in the grid. The first move is made by the opponent, who picks up token 1 (call to opponentmove(1)). The player responded by picking up token 2. In the last move, the opponent takes token 0 (the only remaining token) and the game terminates. Both players made optimal moves, and the game ended with the opponent scoring 11 more points than the player.



# / Solution

The task's constraint ($n \le 40$) suggests that one can try some brute-force methods, such as a game-tree search or some variant of the meet-in-the-middle technique. However, it turns out that the token game can be reduced to another very simple game with an obvious strategy. To obtain that reduction, we take a closer look into the game result.

Some notations have to be introduced. We represent a rectangular board with numbers by a matrix:

$$A = \begin{bmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,n-1} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,0} & a_{n-1,1} & \cdots & a_{n-1,n-1} \end{bmatrix}$$

Let $T = \{0,1,\ldots,n-1\}$ denote the set of all tokens. Let $T_i$, for $i = 1,2$, denote tokens collected by the $i$-th player. The pair of sets $T_1$ and $T_2$ is a partition of the set $T$: $T_1 \cup T_2 = T$ and $T_1 \cap T_2 = \emptyset$. The advantage of the first player over the second player is determined by the formula

$$Res = \sum_{i,j \in T_1} a_{i,j} - \sum_{i,j \in T_2} a_{i,j}.$$

Let us try to express the value of *Res*—the game result—by other means. Let us investigate the influence of a player taking token number $i$. Firstly, the opponent certainly gets no points for numbers from the $i$-th row and $i$-th column. Moreover, the benefit from the field $a_{i,i}$ is doubled—besides denying the opponent the field's points, the player is also able to claim them.

Let us denote the sum of numbers of the $i$-th row of the matrix $A$ as $r_i$ and the sum of numbers of the $i$-th column as $c_i$:

$$r_i = \sum_{j \in T} a_{i,j}, \quad c_i = \sum_{j \in T} a_{j,i}.$$

We can try to use these symbols to express the game result. Let us count points according to the above reasoning. Assume that, for taking the $i$-th token,

the player gets $r_i + c_i$ points, because this action strips from the opponent the opportunity to get points for numbers from the $i$-th row and $i$-th column, and at the same time acquires the points for field $a_{i,i}$, which is counted twice in the sum $r_i + c_i$. This counting method gives the following total advantage for the first player.

$$Res' = \sum_{i \in T_1} (r_i + c_i) - \sum_{i \in T_2} (r_i + c_i).$$

How does this method compare to the previous one? Consider the element $a_{i,j}$. If both tokens $i$ and $j$ belong to the first player then $a_{i,j}$ is counted twice in the first sum. But if both tokes belong to the opponent then $a_{i,j}$ is subtracted twice because it is in the second sum. In the third case, if tokens $i$ and $j$ were split between players then $a_{i,j}$ is added in the first sum and subtracted in the second sum.

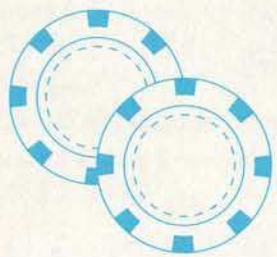
Therefore, we see that the advantage counted using the new method is twice the advantage computed with the regular rules. Thus, if we assign to token $i$ the value

$$v_i = r_i + c_i,$$

then the doubled game result will be

$$2Res = \sum_{i \in T_1} v_i - \sum_{i \in T_2} v_i.$$

In this way, the token game reduces to a game in which each token has a fixed value $v_i$ and the total number of points obtained by a player is the sum of possessed tokens. In such a game, a greedy strategy is optimal: always take the token with the largest value.





# / Straight *Lines*

**Contest:** Algorithmic Engagements 2006
**Task author:** Jakub Pawlewicz
**Memory:** 32 MB
http://main.edu.pl/en/archive/pa/2006/pro

We are given six integers, $A_1$, $B_1$, $C_1$, $A_2$, $B_2$, $C_2$, such that $A_1B_2 \neq A_2B_1$. These numbers are coefficients in the equations of two intersecting lines:

$$l_1 : A_1x + B_1y + C_1 = 0,$$
$$l_2 : A_2x + B_2y + C_2 = 0.$$

The lines divide the plane into four parts. We represent each part by any point with integer coordinates belonging to this part (but not belonging to any of the lines $l_1$, $l_2$). You are given a point $(a, b)$ with integer coordinates representing one part. Find a point with integer coordinates $(c, d)$ representing the same part, such that its distance from the point of intersection of lines $l_1$ and $l_2$ is the smallest possible.

## Task

Write a program that:

- reads the equations of lines $l_1$ and $l_2$ and a point representing one part of the plane,
- finds the point with integer coordinates representing the given part which is closest to the point of intersection of lines $l_1$ and $l_2$,
- writes the answer to the output.

## Input

The first line of the input contains three numbers, $A_1$, $B_1$, $C_1$, specifying the coefficients of $l_1$'s equation. The second line contains three numbers, $A_2$, $B_2$, $C_2$, specifying the coefficients of $l_2$'s equation. It is guaranteed that $A_1B_2 \neq A_2B_1$. The third and final line contains two integers, $a$, $b$, specifying the coordinates of a point representing one part of the plane. Point $(a, b)$ does not belong to any of lines $l_1$, $l_2$. Every number $x$ from the input falls within the range $-2{,}100{,}000{,}000 < x < 2{,}100{,}000{,}000$.

**Output**

Your program should write to the output two numbers, $c$, $d$, separated by a single space: the coordinates of a point $(c, d)$ representing the given part, closest to the point of intersection of lines $l_1$ and $l_2$. If there are multiple such points, your program should output only one of them.

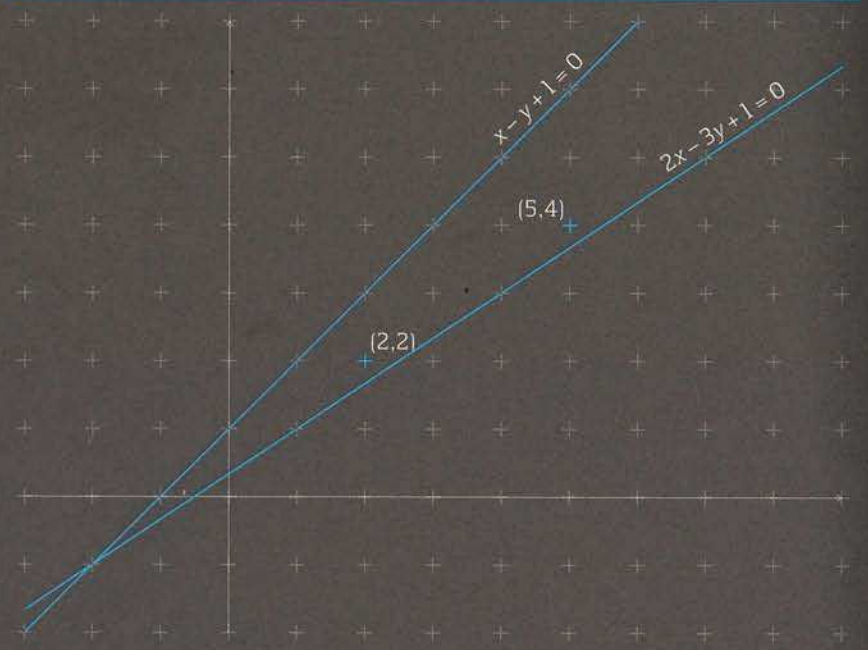
**Example**

For the input data:

1 −1 1
2 −3 1
5 4

the correct result is:

2 2






# / Solution

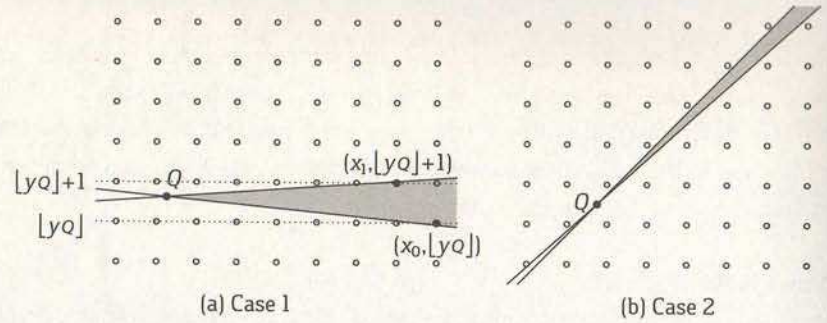
In order to simplify the task analysis we introduce some simple notations. Let $Q = (x_Q, y_Q)$ be the intersection of the lines $l_1$ and $l_2$. These lines divide the plane into four parts. We call each such part a *quarterplane* and we refer to the part pointed out in the task as the *examined quarterplane*. We say that a point with integer coordinates is a *lattice point*.

**Where is the problem?**

At the first glance the task seems to be easy. It is sufficient to scan lattice points inside the examined quarterplane and output the coordinates of the one closest to $Q$. How should such points be scanned? The simplest way is to focus only on points surrounding $Q$, hoping that there will be a point belonging to the examined quarterplane. We can take into account only those points at which the difference of each coordinate from the corresponding coordinate of $Q$ is bounded by a constant. For example, we can set this constant to 5; then we need to check only about 100 lattice points. We describe this as a *brute force* method.

Unfortunately, there are instances in which the closest lattice point to $Q$, belonging to the examined quarterplane, is far, far away. Nevertheless, a brute force method reduces our task to a case in which the examined quarterplane is very "narrow". With respect to 90° rotation, two cases are possible (see figure 1).

In the first case (figure 1a) we can find a solution: it is either $(x_0, \lfloor y_Q \rfloor)$ or $(x_1, \lfloor y_Q \rfloor + 1)$, where $x_0$ and $x_1$ are the smallest integers whose points belong to the examined quarterplane. The second case (figure 1b) is still problematic, and we need more sophisticated tools to solve it.




(a) Case 1 (b) Case 2

***Figure 1:*** *Narrow quarterplanes immune to a brute force method.*

**Narrowing the search space of lattice points**

We present a method that enables us to iterate efficiently over lattice points inside many typical geometric objects such as our quarterplane. We start by introducing some notations and definitions.

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{Z}$ the set of integers. For a set $A \subseteq \mathbb{R}$ we denote as $A_+$ the set $\{a \in A \mid a > 0\}$. We have already introduced an informal definition of a quarterplane; now we provide a more formal one.

***Definition 1.*** For a point $O \in \mathbb{R}^2$ and vectors $\vec{u}, \vec{v} \in \mathbb{R}^2$ such that $\vec{u}$ is not parallel to $\vec{v}$ we define a *quarterplane* $Q[O, \vec{u}, \vec{v}]$ as the set of points $\{O + \alpha\vec{u} + \beta\vec{v} \mid \alpha, \beta \in \mathbb{R}_+\}$ (see figure 2).

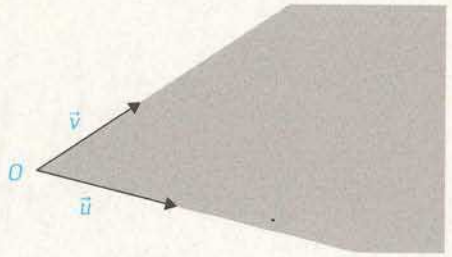



***Figure 2:*** *Quarterplane $Q[O, \vec{u}, \vec{v}]$.*



We are interested in lattice points, though, and for those we need quarter-lattices.

***Definition 2.*** For a point $O \in \mathbb{Z}^2$ and vectors $\vec{u}, \vec{v} \in \mathbb{Z}^2$ such that $\vec{u}$ is not parallel to $\vec{v}$ we define a *quarterlattice* $\mathcal{L}[O, \vec{u}, \vec{v}]$ as the set of points $\{O + \alpha\vec{u} + \beta\vec{v} \mid \alpha, \beta \in \mathbb{Z}_+\}$ (see figure 3).

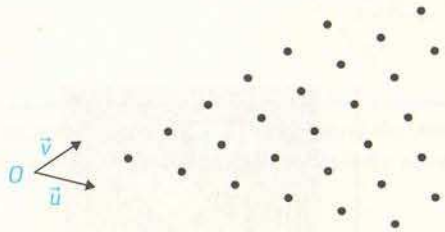



***Figure 3:*** *Quarterlattice $\mathcal{L}[O, \vec{u}, \vec{v}]$.*

Quarterlattice $\mathcal{L}[O, \vec{u}, \vec{v}]$ represents a set of lattice points inside a quarterplane $Q[O, \vec{u}, \vec{v}]$ thus:

$$\mathcal{L}[O, \vec{u}, \vec{v}] \subseteq Q[O, \vec{u}, \vec{v}] \cap \mathbb{Z}^2.$$

If one wants to iterate over all lattice points inside a quarterplane, then in the above relation we would rather have an equation sign.

***Definition 3.*** We say that a quarterlattice is *full* if the following equation holds:

$$\mathcal{L}[O, \vec{u}, \vec{v}] = Q[O, \vec{u}, \vec{v}] \cap \mathbb{Z}^2.$$

A full quarterlattice is a convenient way of representing all lattice points inside some quarterplane. An ability to construct full quarterlattices is key to a solution.

**Constructing full quarterlattices.** Assume that $O$ is some fixed lattice point. The most basic full quarterlattice is $\mathcal{L}[O, (1,0), (0,1)]$. In order to create other quarterlattices the following fact is sufficient:

**Fact 4.** If quarterlattice $\mathcal{L}[O,\vec{u},\vec{v}]$ is full then quarterlattices $\mathcal{L}[O,\vec{u},\vec{u}+\vec{v}]$ and $\mathcal{L}[O,\vec{u}+\vec{v},\vec{v}]$ are also full.

*Proof.* Any lattice point $P = O + \alpha\vec{u} + \beta\vec{v}$, $\alpha, \beta \in \mathbb{Z}_+$, belonging to quarterplane $Q[O,\vec{u},\vec{v}]$, falls into one of the following sets (see figure 4):
(i) $Q[O,\vec{u},\vec{u}+\vec{v}]$ if $\alpha > \beta$,
(ii) $Q[O,\vec{u}+\vec{v},\vec{v}]$ if $\alpha < \beta$,
(iii) a ray starting at $O$ going off in direction $\vec{u}+\vec{v}$ if $\alpha = \beta$.

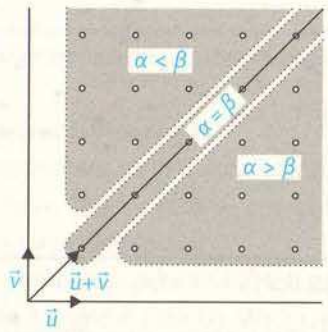



**Figure 4:** *Three parts of a quarterlattice.*

In case (i) point $P$ belongs to quarterlattice $\mathcal{L}[O,\vec{u},\vec{u}+\vec{v}]$ since $P = O + (\alpha-\beta)\,\vec{u} + \beta(\vec{u}+\vec{v})$, whereas in case (ii) point $P$ belongs to quarterlattice $\mathcal{L}[O,\vec{u},\vec{u}+\vec{v}]$ since $P = O + \alpha(\vec{u}+\vec{v}) + (\beta-\alpha)\vec{v}$. □

Applying Fact 4 many times, we reach the following conclusion:

**Corollary 5.** If quarterlattice $\mathcal{L}[O,\vec{u},\vec{v}]$ is full then quarterlattices $\mathcal{L}[O,\vec{u},k\vec{u}+\vec{v}]$ and $\mathcal{L}[O,\vec{u}+k\vec{v},\vec{v}]$ are also full for every $k \in \mathbb{Z}_+$.

**Goal.** Our goal is to build a quarterlattice such that there will be a clear way to output a small set of candidates for a solution. We build more and more narrow quarterlattices by Corollary 5. Quarterlattice $\mathcal{L}[O,\vec{u},\vec{v}]$ represents some new coordinate system with origin at $O$ and axes set by $\vec{u}$ and $\vec{v}$. It means that any point $(x,y)$ is represented by a point $(\alpha,\beta)$ such that $(x,y) = O + \alpha\vec{u} + \beta\vec{v}$. As we construct increasingly narrow quarterlattices, the examined quarterplane becomes increasingly wide from the point of view of the new coordinate system (see figure 5).



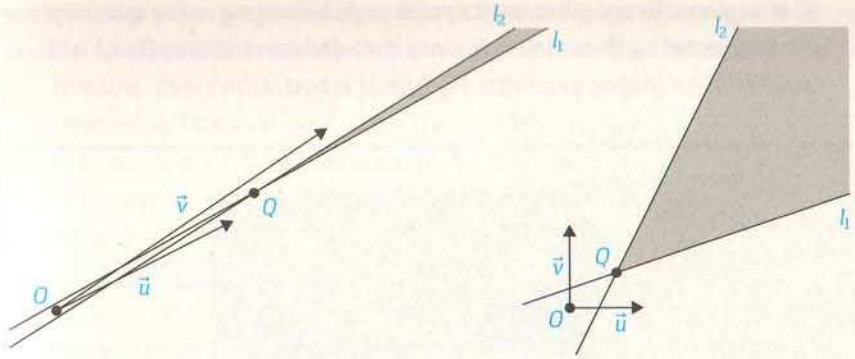



**Figure 5:** *A narrow quarterlattice $\mathcal{L}[O,\vec{u},\vec{v}]$ comprising the examined quarterplane (on the left) and a look at the examined quarterplane from the point of view of $\mathcal{L}[O,\vec{u},\vec{v}]$ (on the right).*

It is possible to find a quarterlattice $\mathcal{L}[O,\vec{u},\vec{v}]$ such that, from the point of view of this quarterlattice, the following conditions hold:
(i) The entire examined quarterplane between lines $l_1$ and $l_2$ is inside quarterplane $Q[O,\vec{u},\vec{v}]$.
(ii) Coordinates $(\alpha_Q,\beta_Q)$ of point $Q$ are close to the origin; that is, $0 \le \alpha_Q,\beta_Q < 1$.
(iii) The angle between the $X$ axis and line $l_1$ is less than 45°.
(iv) The angle between the $Y$ axis and line $l_2$ is less than 45°.
In such a situation—thanks to the observation below—it is easy to give candidates for the required point.

**Observation 6.** Let $\alpha_1,\beta_1,\alpha_2,\beta_2 \in \mathbb{Z}_+$ be such that $\alpha_1 \le \alpha_2$ and $\beta_1 \le \beta_2$. Then the distance of point $P_1 = O + \alpha_1\vec{u} + \beta_1\vec{v}$ from point $Q$ is not greater than the distance of point $P_2 = O + \alpha_2\vec{u} + \beta_2\vec{v}$ from point $Q$.

*Proof.* In vectors $\vec{u}$ and $\vec{v}$ both coordinates are nonnegative, thus each coordinate of vector $\overrightarrow{QP_1}$ is not greater than the corresponding coordinate of vector $\overrightarrow{QP_2}$; hence the requested inequality between Euclidean distances follows. □

We examine candidates from the point of view of the coordinate system of quarterlattice $\mathcal{L}[O,\vec{u},\vec{v}]$. If point $(1,1)$ belongs to the examined quarterplane then by Observation 6 it is the only candidate; thus this is the required point. Otherwise there are two possible cases:
1. Point $(1,1)$ lies on line $l_1$ or underneath (see figure 6). Then it is sufficient to consider only two candidates: $(k_1,k_1)$ and $(k_2,k_2+1)$, where $k_1,k_2 \in \mathbb{Z}_+$

are the smallest numbers such that given points belong to the examined quarterplane. Every other lattice point $(\alpha,\beta)$ belonging to the quarterplane can be rejected by Observation 6, since if $\alpha \geq \beta$ then candidate $(k_1,k_1)$ is closer to $Q$ and if $\alpha < \beta$ then candidate $(k_2,k_2+1)$ is better.

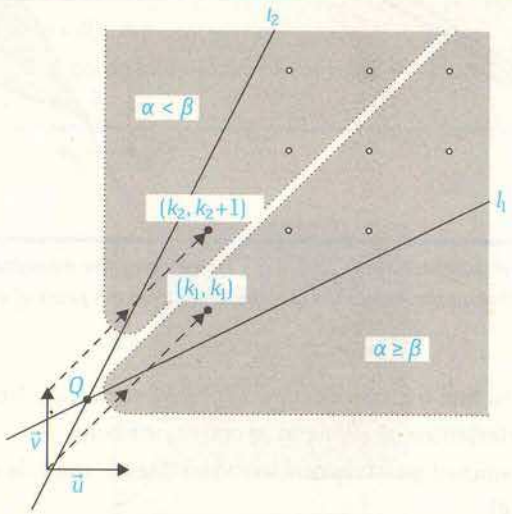



***Figure 6:*** *Point* $(1,1)$ *is below line* $l_1$.

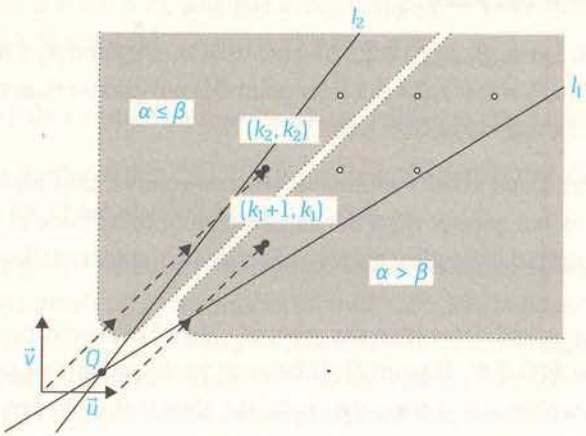



***Figure 7:*** *Point* $(1,1)$ *is above line* $l_2$.



2. Point $(1,1)$ lies on line $l_2$ or above it (see figure 7). Then it is sufficient to consider only two candidates: $(k_1+1,k_1)$ and $(k_2,k_2)$, where $k_1,k_2 \in \mathbb{Z}_+$ are the smallest numbers such that given points belong to the examined quarterplane. Every other lattice point $(\alpha,\beta)$ belonging to the quarterplane can be rejected by Observation 6, since if $\alpha > \beta$ then candidate $(k_1+1,k_1)$ is closer to $Q$, and if $\alpha \leq \beta$ then candidate $(k_2,k_2)$ is better.

In both cases we get two candidates, and it is sufficient to check which one is closer to $Q$.

**Reaching the goal.** Let us use another, more convenient equation $bx - ay = c$ to represent straight lines. Vector $\overrightarrow{(a,b)}$ is parallel to such a line, and we call it a *direction vector*. Of course, it is easy to express the lines from the task using the introduced representation:

$$l_1 : b_1x - a_1y = c_1,$$
$$l_2 : b_2x - a_2y = c_2,$$

Moreover, because we are solving the case from figure 1b, we can further assume that coefficients $a_1,b_1,a_2,b_2$ are nonnegative.

To check whether the angle between a line $bx - ay = c$ and the $X$ or $Y$ axis is smaller than 45°, it is enough to test direction vector $\overrightarrow{(a,b)}$. Thus we can restate conditions (iii) and (iv) from the *Goal* section:

(iii) $a_1 > b_1$,

(iv) $a_2 < b_2$.

Let us find how the equation of a line changes after altering a coordinate system. Assume a line $l$ can be expressed by equation $bx - ay = c$ in the coordinate system for quarterlattice $\mathcal{L}[O,\vec{u},\vec{v}]$. What does the line look like in the coordinate system for quarterlattice $\mathcal{L}[O,\vec{u},\vec{u}+\vec{v}]$? Let $P = (\alpha,\beta)$ be an arbitrary point in the system of the new quarterlattice. The coordinates of $P$ in the previous system are $(\alpha+\beta, \beta)$, because $O + \alpha\vec{u} + \beta(\vec{u}+\vec{v}) = O + (\alpha+\beta)\vec{u} + \beta\vec{v}$. Therefore, if point $P$ belongs to line $l$, then $b(\alpha+\beta) - a\beta = c$ must hold, and we conclude that the equation of the line after altering the coordinates is $bx - (a-b)y = c$. Similarly, the equation of the line in the coordinate system for quarterlattice $\mathcal{L}[O,\vec{u}+\vec{v},\vec{v}]$ is $(b-a)x - ay = c$. Repeating the above reasoning $k$ times leads to the following property:

***Fact 7.*** If the equation of a line is $bx - ay = c$ in the coordinate system for quarterlattice $\mathcal{L}[O,\vec{u},\vec{v}]$, then the equation of this line is

$$bx - (a - kb)y = c$$

in the coordinate system for quarterlattice $\mathcal{L}[O, \vec{u}, k\vec{u} + \vec{v}]$, and

$$(b - ka)x - ay = c$$

in the coordinate system for quarterlattice $\mathcal{L}[O, \vec{u} + k\vec{v}, \vec{v}]$, where $k \in \mathbb{Z}_+$.

This fact suggests how to refine a quarterlattice if conditions (iii) or (iv) are not fulfilled. Namely, in such case we have

1. $a_2 \geq b_2$ and $a_1 \geq b_1$ or
2. $a_1 \leq b_1$ and $a_2 \leq b_2$.

There are no other cases, since vector $\overrightarrow{(a_1, b_1)}$ is below $\overrightarrow{(a_2, b_2)}$. From this mutual relation of the vectors we also conclude that $a_1 > 0$ and $b_2 > 0$.

Suppose that the first case is true. Let $k_2 = \lfloor a_2/b_2 \rfloor$ and assume that we move to quarterlattice $\mathcal{L}[O, \vec{u}, k_2\vec{u} + \vec{v}]$. Then, coefficients $a_1$ and $a_2$ of both lines are altered. Let $a'_1$ and $a'_2$ be the altered coefficients:

$$a'_1 = a_1 - k_2 b_1,$$
$$a'_2 = a_2 - k_2 b_2.$$

Then $a'_2 = a_2 \bmod b_2$ and so $a'_2 < b_2$ holds. It means that moving to the new quarterlattice makes condition (iv) true. If $b_1 = 0$ then, of course, $a'_1 > b_1$, which makes (iii) also true. Therefore, let us assume that $b_1 > 0$ and test what the relation is between $a'_1$ and $b_1$. Let $k_1 = \lfloor a_1/b_1 \rfloor$. Since $\overrightarrow{(a_1, b_1)}$ is below $\overrightarrow{(a_2, b_2)}$, we have $k_2 \leq k_1$. If $k_2 = k_1$ then $a'_1 = a_1 \bmod b_1$ and $a'_1 > b_1$. For $k_2 < k_1$ there is $a'_1 \leq b_1$ where the equality $a'_1 = b_1$ holds only if $k_2 + 1 = k_1$ and $a_1 = k_1 b_1$.

So, after altering direction coefficients there is a hope that $a'_1 > b_1$, and this means that (iii) is fulfilled and the goal is reached. Otherwise we have $a'_1 \leq b_1$, but then we are in the second case in which we deduct analogously. In general, it looks like the Euclidean algorithm applied simultaneously to pairs $(a_1, b_1)$ and $(a_2, b_2)$. As soon as the algorithm "diverges", i.e. $k_1 \neq k_2$, we reach our conditions.

The above procedure allows us to find vectors $\vec{u}$ and $\vec{v}$ of the searched quarterlattice in logarithmic time. We still need to find point $O$. For this it is enough to find out the coordinates of the intersection of the lines in the new coordinate system and round them down to the nearest integer. The entire discussion is summarized in the pseudocode below.



```
ū := (1,0); v̄ := (0,1)
while not (a1 > b1 and a2 < b2) do
    if a2 ≥ b2 then
        [The angle between X axis and each line is not greater than 45°]
        k := ⌊a2/b2⌋
        v̄ := kū + v̄
        a1 := a1 − kb1; a2 := a2 − kb2
    else [a1 ≤ b1]
        [The angle between Y axis and each line is not greater than 45°]
        k := ⌊b1/a1⌋
        ū := ū + kv̄
        b1 := b1 − ka1; b2 := b2 − ka2
(αQ, βQ) := intersection of b1x − a1y = c1 and b2x − a2y = c2
O := ⌊αQ⌋ū + ⌊βQ⌋v̄
```

**Summary of the solution**

The entire solution appears as follows.

1. Use a brute force method to find a solution or to reduce the problem to the case of a narrow quarterplane.
2. Apply the method of narrowing the search space of lattice points to obtain quarterlattice $\mathcal{L}[O, \vec{u}, \vec{v}]$ in which the examined quarterplane would be "wide enough".
3. Determine candidates (one or two) in quarterlattice $\mathcal{L}[O, \vec{u}, \vec{v}]$.
4. Convert back coordinates of candidates to the initial coordinate system and choose that which is closest to the point $Q$.

The key point in the presented solution is the ability to deal with quarterlattices. The method of narrowing the search space of lattice points delivers a convenient and efficient way to iterate over lattice points inside a quarterplane. However, it has many more applications. For example—after some modifications—one can use it to iterate over all points inside some geometric objects such as triangles or even arbitrary polygons with rational coordinates. This method is especially useful with narrow objects. We leave it to the reader to find other problems where it can be applied.

**MARCIN PILIPCZUK**

Marcin Pilipczuk's interests used to focus mostly on mathematics, occasionally also touching on computer science and physics. In high school he won gold medals in two international olympiads: one in mathematics and one in physics. Computer science became his main area of interest during his M.S. studies, and this change led to a championship in the ACM-ICPC World Finals in 2007, which he won for the University of Warsaw in a team with Marek Cygan and Filip Wolski. For many years he has taught programming courses for high school students, and has occasionally authored programming tasks for Polish programming competitions. Currently, as a Ph.D. student at the Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Marcin Pilipczuk conducts research in parameterized complexity and moderately-exponential algorithms. At times he also engages in graph theory and some aspects of calculus.
As the father of two young children, he does not have any free time. Before, he used to play board games, hike in the mountains and read a lot of books, mostly fantasy.

# / Guilds

**Contest:** 17th Polish Olympiad in Informatics
**Task author:** Marcin Pilipczuk
**Memory:** 64 MB
http://main.edu.pl/en/archive/oi/17/gil

King Byteasar faces a serious matter. Two competing trade organizations, The Tailors Guild and The Sewers Guild, asked, at the same time, for permission to open offices in each town of the kingdom.

There are $n$ towns in Byteland. Some of them are connected with bidirectional roads. Each of the guilds postulates that every town should:

- have an office of the guild, or
- be directly connected to another town that does.

The king, however, suspects foul play. He fears that if there is just a single town holding the offices of both guilds, it could lead to a clothing cartel. For this reason he seeks your help.

## Input

Two integers, $n$ and $m$ ($1 \le n \le 200{,}000$, $0 \le m \le 500{,}000$), are given in the first line of the input. These specify, respectively, the number of towns and roads in Byteland. The towns are numbered from 1 to $n$. Then the roads are given as follows: the $(i + 1)$-th line of the input describes the $i$-th road; it holds the numbers $a_i$ and $b_i$ ($1 \le a_i, b_i \le n$, $a_i \ne b_i$), specifying that the $i$-th road connects the towns $a_i$ and $b_i$. Every pair of towns is (directly) connected by at most one road. The roads do not cross—meeting only in towns—but may lead through tunnels and overpasses.

## Output

Your program should write one word in the first line of the output: TAK (Polish for *yes*) if the offices can be placed in towns according to these rules, or NIE (Polish for *no*) otherwise. If the answer is TAK, then the following $n$ lines should give an exemplary placement of the offices. Thus the $(i + 1)$-th line should hold:

- the letter K if there should be an office of The Tailors Guild in town $i$, or
- the letter S if there should be an office of The Sewers Guild in town $i$, or
- the letter N if there should be no office in town $i$.

**Example**

For the input data:

7 8
1 2
3 4
5 4
6 4
7 4
5 6
5 7
6 7

the correct result is:

TAK
K
S
K
S
K
S
N

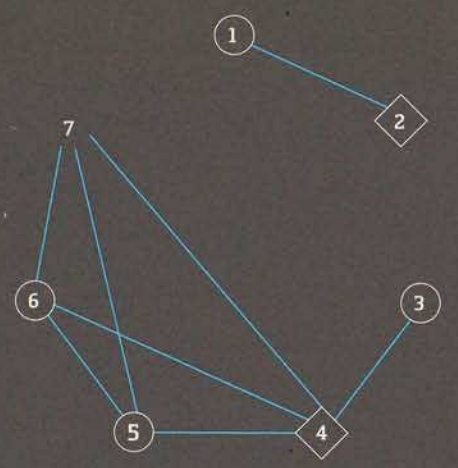



***Figure.*** *The towns in which an office of The Tailors Guild should open are marked with circles, whereas those in which an office of The Sewers Guild should open are marked with rhombi.*



# / Solution

I would like to show you my favorite type of contest problems—those in which there are myriads of solutions.

In the great majority of problems you encounter at a programming contest, you are supposed to find an *optimal* solution—the single optimal one, or one from a very narrow choice of optimal solutions. In such cases, all algorithms are usually extremely similar to the jury solution—sometimes they are a bit slower, sometimes they are smarter in some way, but they rarely use techniques unknown to the author of the problem.

However, there are other problems that do not follow this scheme. In such problems we are given some conditions relating to the required solution that make finding a solution fairly nontrivial, but that nevertheless leave 'lots of freedom' in its construction. Usually there is no requirement for the solution to be optimal. When solving such problems, the contestants—always very creative—often invent weird methods and techniques, surprising the author and the jury with their originality.

Let us proceed to a few examples. I would like to present two interesting problems with "lots of freedom". It is worth mentioning that in both problems—after a correct way to construct a solution has been discovered—the implementation is extremely easy, making these problems even more enjoyable.

**Guilds**

*Guilds* was one of the simplest problems in the first stage of the Polish Olympiad in Informatics some time ago. Let us translate the problem statement into the language of graph theory: given a graph $G = (V, E)$ with $n$ vertices and $m$ edges, we are to find two *disjoint* sets of vertices $K, S \subseteq V$ (towns where the offices of the tailors and sewers guilds will be open), so that both sets are in some sense *dense* in the vertex set $V$. Such sets $K$ and $S$ that satisfy the conditions of the problem statement are called, in the literature, *dominating sets*: formally, a set $X \subseteq V$ is a *dominating set* in a graph $G = (V, E)$ if each vertex of $G$ either belongs to $X$ or has at least one neighbor in the set $X$. Thus, in the problem *Guilds*, we are to find two disjoint dominating sets in a given graph $G$.

Can we always succeed? It is quite easy to see that the answer is no: if there is a town in Byteland without any road going out of it (i.e. an isolated vertex in the graph $G$), there is no solution—in this vertex an office of only one guild can be placed, so the second guild will not service (dominate) this vertex. If we

spend a few more minutes drawing examples on a sheet of paper, we should convince ourselves that the isolated vertices are probably the only obstacle for king Byteasar. Let us try to find an algorithm that places the offices of the guilds in Byteland, provided that there are no isolated vertices.

Let us start with a few simplifications. First, clearly the connected components of $G$ can be considered independently. Second, any superfluous edge in the graph only *helps* us in the constructions: if we suspect that we can find a solution in any graph without isolated vertices, then the hardest case would be to find it in a tree (connected acyclic graph). Third, we do not gain anything by leaving a town without any office. Thus, we can assume that we will build an office in each town; in other words, we are to *partition* the vertex set $V$ into two dominating sets $K$ and $S$.

Let us assume for a moment that $G$ is a tree, and let $v$ be its arbitrary leaf (a vertex of degree one). Without loss of generality, we can assume that $v \in K$; that is, we construct an office of the tailors guild in the vertex $v$. Then, in the unique neighbor of $v$—denote it by $u$—we need to place an office of the sewers guild (i.e. take $u \in S$), as otherwise $S$ would not dominate the vertex $v$. By the same argument, if $u$ has any other neighbors that are leaves, all these neighbors must belong to $K$, as otherwise they would not be dominated by the set $K$. So maybe let us assign all neighbors of $u$ to the set $K$—then they would be dominated by both $S$ (due to the vertex $u$) and $K$. And assign neighbors of neighbors of $u$ again to set $S$, and so on.

Let us summarize the aforementioned algorithm: for each connected component of $G$ we choose its arbitrary spanning tree, we root it in an arbitrary vertex, and we assign the vertices of the tree to the sets $K$ and $S$ in an alternating fashion—say, we assign vertices at an even distance from the root to $K$ and at an odd distance to $S$. It is easy to see that this construction is correct: if we look only at the edges of the chosen spanning tree, each vertex has only neighbors with offices of the other guild than the vertex itself. Thus, if no connected component consists only of one vertex, both $K$ and $S$ are dominating sets in $G$.

Observe that if we choose the spanning tree in a smart way—e.g. we use the tree of a depth-first search (DFS)—our algorithm is very easy to implement, because we do not need to store the tree explicitly in memory. Below we present pseudocode for such a solution.



```
Algorithm Guilds(V, E)
  for all v ∈ V do
    guild[v] := 0
    {0—not yet assigned, 1—tailors guild, 2—sewers guild}
  for all v ∈ V do
    if guild[v] = 0 then
      DFS(v, 1)  {in the root we place an office of the tailors guild}

Function DFS(v, g)
  guild[v] := g
  for all u neighbor of v do
    if guild[u] = 0 then
      DFS(u, 3 − g)
      {in the children of v we place offices of the second guild}
```

The above solution was the one I had in mind when proposing the problem. As expected, the contestants turned out to be very creative and, among many inventive and obscure solutions, I found the following two both to be simpler than mine.

**First solution by creative contestants.** Initially, each vertex of the graph has no office assigned to it. We go over all edges of the graph in an arbitrary order. When considering an edge $uv$, if we did not assign any office to at least one of the vertices $u$ or $v$, we assign offices to both of these vertices in such a way that the offices in $u$ and $v$ are of different guilds (note that this can always be done, regardless of whether one of these vertices already has an office assigned to it, or both are unassigned). If both $u$ and $v$ have their offices already assigned, we do nothing.

To see that this solution is correct, take an arbitrary vertex $v$ and consider the moment when we assign an office to the vertex $v$. It happens when we consider some edge coincident with $v$, say $uv$. At this moment, we assign the offices in such a way that $u$ and $v$ receive offices of different guilds—and therefore $v$ is dominated by both guilds.

This solution is extremely simple to implement, as shown below.

```
Algorithm Guilds2(V,E)
  for all v ∈ V do
    guild[v] := 0
    {0—not yet assigned, 1—tailors guild, 2—sewers guild}
  for all uv ∈ E do
    if guild[u] = 0 and guild[v] = 0 then
      guild[u] := 1; guild[v] := 2
    else if guild[u] = 0 then
      guild[u] := 3 − guild[v]
    else if guild[v] = 0 then
      guild[v] := 3 − guild[u]
```

**Second solution by creative contestants.** Again, initially each vertex has no office assigned to it. We go over the vertices of the graph in an arbitrary order. If we find a vertex that has no office assigned to it as yet, we place an office of the tailors guild in it, and, moreover, we place offices of the sewers guild in all neighboring vertices that have no office assigned to them.

To show the correctness of this solution, first notice that no two vertices with an office of the tailors guild are adjacent, since whenever we place an office of the tailors guild somewhere, we immediately pass all the unassigned neighbors to the sewers guild. In graph theory, a set of vertices with no two vertices connected by an edge is called an *independent set*. Note that the set of vertices of the graph where the offices of the tailors guild were placed (i.e. the set $K$) is not only an independent set, but an *inclusion-wise maximal* independent set: we cannot extend it by any vertex without closing already-opened offices of the tailors guild. Summing up, in the language of graph theory, the above solution constructs an inclusion-wise maximal independent set, places offices of the tailors guild in the vertices of the independent set, and places offices of the sewers guild in the remaining vertices.

Let us now show that, in a graph without isolated vertices, any inclusion-wise maximal independent set and its complement are both dominating sets. For the first part, it is sufficient to observe that any vertex of the graph is either in the chosen inclusion-wise maximal independent set or has a neighbor in it, since otherwise we could extend the independent set with the considered vertex. For the second part, we need to use the assumption that the graph does not contain isolated vertices: any vertex of the graph that is chosen to be in the independent set has at least one neighbor; this neighbor cannot be in the independent set, and thus dominates the considered vertex.

Again, the pseudocode below shows that this solution is very easy to implement.

```
Algorithm Guilds3(V,E)
  for all v ∈ V do
    guild[v] := 0
    {0—not yet assigned, 1—tailors guild, 2—sewers guild}
  for all v ∈ V do
    if guild[v] = 0 then
      guild[v] := 1
      for all u neighbor of v do
        if guild[u] = 0 then
          guild[u] := 2
```

The presented algorithms for the problem *Guilds* are obviously not the only ones, but in my opinion they are the simplest. It is worth noting that *Guilds* was the simplest problem in the first stage of the 17th Polish Olympiad in Informatics, and almost half of the contestants received perfect scores for their solutions.

### A short note on the theoretical background

Let us now consider the problem *Guilds* from a broader perspective. We will see that behind a very simple problem we can find a nice piece of theoretical computer science, lying somewhere between algorithmics and complexity theory.

We have already established that our problem, in the language of graph theory, boils down to one of finding two disjoint dominating sets in a given graph. What happens if we demand more pairwise disjoint sets? In other words, what happens if, in Byteland, there are not only tailors and sewers guilds, but also silkworm breeders or Gore-Tex manufacturers?

It turns out that the problem with three guilds is already NP-complete—which means that we should not expect an algorithm that solves the problem in polynomial time in the size of the graph. For a given graph $G$, the largest natural number $k$, such that in $G$ we can construct offices of $k$ guilds that satisfy the conditions from the problem statement, is called the *domatic number* of the graph $G$. As deciding if the domatic number of a graph is at least 3 is already hard, computing the exact value of the domatic number is even harder.

The situation here is thus almost the same as for the case of the *chromatic number*—the minimum number of colors needed to color the vertices of the graph such that no two adjacent vertices receive the same color (here we partition the set of vertices of the graph into independent sets). It is easy to see that two colors suffice if and only if the graph $G$ is bipartite (which is easy to check). It was proven that deciding whether three colors suffice is NP-complete even if we know that the input graph is planar and each vertex has degree exactly 4. Both the chromatic and domatic number can be computed in time $O(2^n p(n))$ (where $p(n)$ is some polynomial) by a non-trivial application of the inclusion-exclusion principle. The question of whether the constant 2 in the base of the exponent can be improved is now a major open problem.

Let us go back for a moment to the concept of a dominating set. It turns out that finding a minimum dominating set is a very hard task, with many applications: it models the problem of finding the cheapest way to place sensors (or resources) in a network such that each node is controlled by a sensor (or has quick access to a resource, respectively). Let us consider a decision version of this problem, where, beside the graph $G$, we are given an integer $k$ and we are to decide whether the graph $G$ admits a dominating set of size at most $k$. This problem is also NP-complete, and similar in terms of computing domatic and chromatic numbers.

In order to consider the difficulty of the problem of finding a minimum dominating set from another perspective, let us recall a similar problem of finding a minimum vertex cover of a graph. A set of vertices $X \subseteq V$ is called a *vertex cover* of the graph $G = (V, E)$ if each edge of $G$ has at least one endpoint in the set $X$. Intuitively, a vertex cover of a graph is a set of vertices that dominates each *edge*, whereas a dominating set has to dominate each *vertex*.

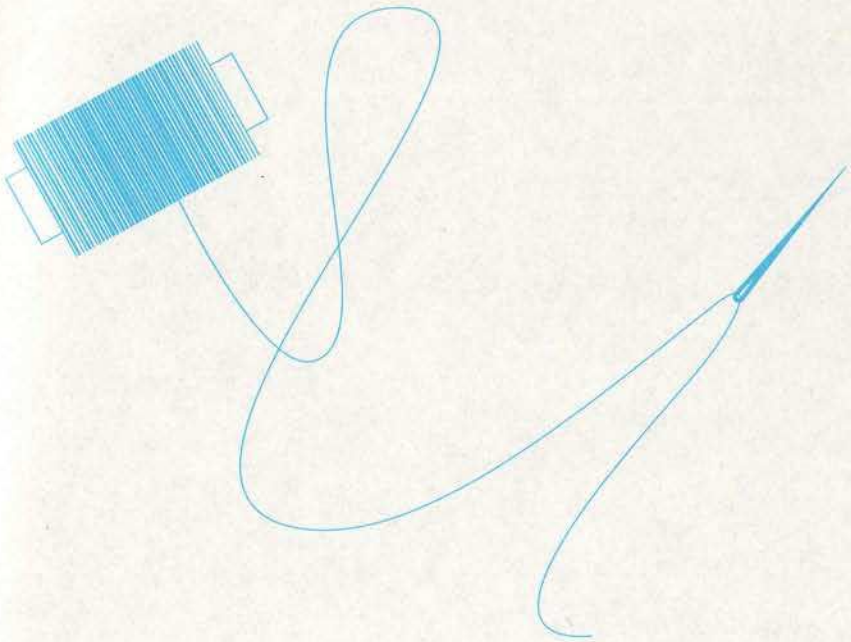
Note that we can decide whether a given graph $G$ with $n$ vertices has a vertex cover or dominating set of size at most $k$ by checking all possible answers. Such a solution works in time roughly $\binom{n}{k}$, which, for $k$ being much smaller than $n$, is of order roughly $n^k$.

However, in the vertex cover problem we can use a much smarter approach. Consider the following branching algorithm: as long as $G$ contains at least one edge, take an arbitrary edge, say $uv$, and branch into two sub-cases: in one case we take $u$ to the constructed vertex cover, and in the second case we take $v$. Whenever we take a vertex to the constructed vertex cover, we delete it and all its incident edges, and decrease $k$ by one. This algorithm finds a solution if, and only if, in one of the sub-cases we find a solution before $k$ becomes negative. A careful implementation of this algorithm runs in $O(2^k(n+m))$ time and, thus,



is very efficient for small values of $k$; say $k \sim 15$. By contrast, for $k \sim 15$ the brute-force algorithm with time complexity $n^k$ is very slow.

Can we do a similar trick for the dominating set problem? It turns out that the answer is probably no: it was shown that the problem of finding a minimum dominating set is hard for a class of problems called $W[2]$. Without getting into the technical details of the definitions, it means that the existence of an algorithm that solves this problem in time similar to the algorithm for the vertex cover problem described in the previous paragraph would imply the existence of such algorithms for a large class of other problems—and this is considered unlikely.



# / Reconstruction *of Byteland*

**Contest:** Algorithmic Engagements 2008
**Task author:** Marcin Pilipczuk
**Memory:** 32 MB
http://main.edu.pl/en/archive/pa/2008/odb

Everyone get to work! Byteland needs to be reconstructed after a devastating war!

You are responsible for assigning postal codes to Bytean cities. Each city should receive one postal code—a positive integer not greater than $10^9$. Different cities should be assigned different postal codes.

The Bytean Postal Service is organized in quite a strange way: a letter can be sent from city $A$ to city $B$ only if the postal codes of the two cities have a common divisor greater than 1. Obviously, one of your objectives is that it should be possible to send letters directly from every city to every other city after the postal codes are assigned.

Additionally, the new anti-corruption law requires that, for each set of cities containing more than half of the Bytean cities, postal codes assigned to cities within this set must not have any common divisor greater than one.

**Task**

Write a program that:

- reads the number of cities of Byteland from the input,
- assigns a postal code to each Bytean city,
- writes the assignment to the output.

**Input**

The first and only line of input contains one integer $n$ ($4 \le n \le 100$) specifying the number of cities of Byteland.

**Output**

The output should consist of exactly $n$ lines. The $i$-th line should contain one positive integer not greater than $10^9$—the postal code of the $i$-th Bytean city. You can assume that for each possible input there exists a valid assignment of postal codes to the cities. If there are multiple correct solutions, your program should output any one of them.

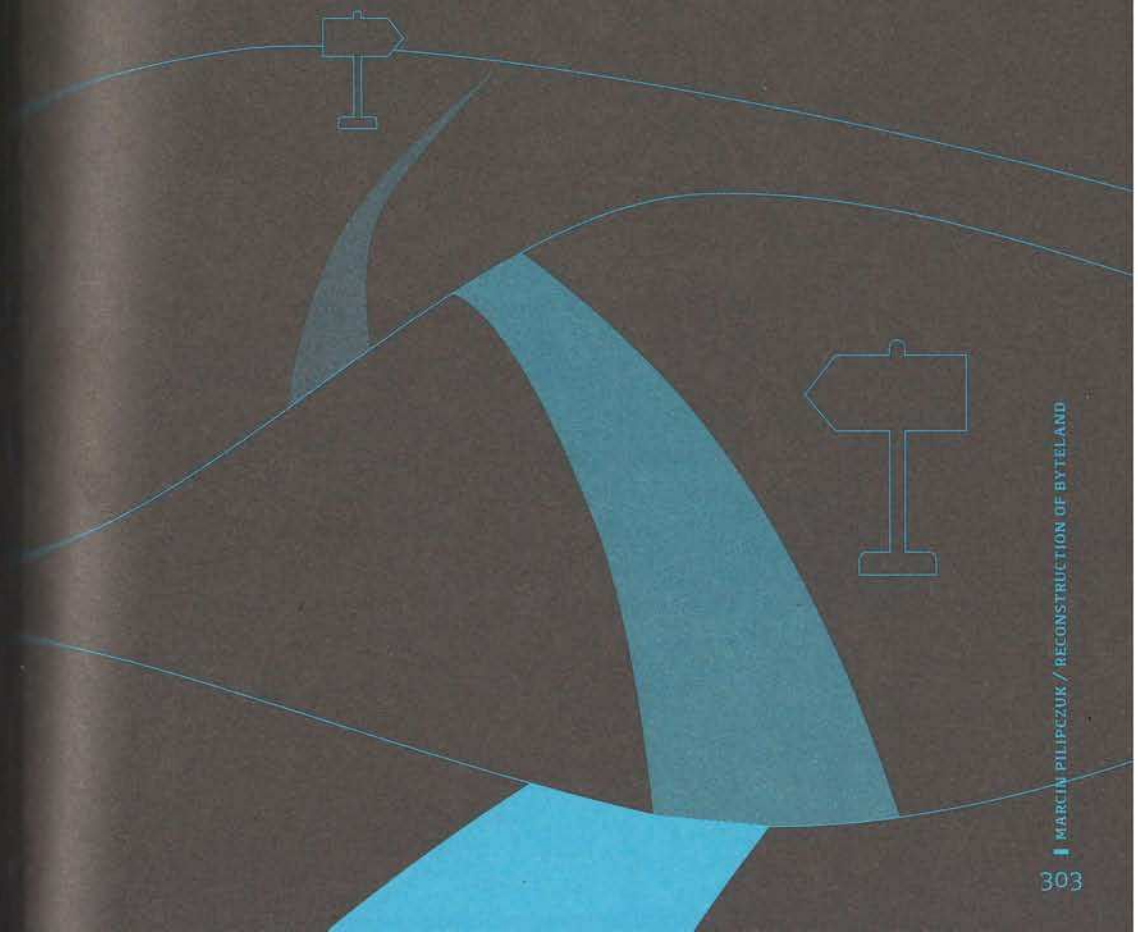
**Example**

For the input data:

5

the correct result is:

714
2090
4485
29029
215441



# / Solution

The second problem from the line of 'lots of freedom' problems that I would like to present here is *Reconstruction of Byteland*. Although this problem does not have such a deep and interesting theoretical background as was shown for the *Guilds* problem, it has a handful of possible solutions and you can show your creativity while solving it.

After reading the problem statement, it is quite natural to look at the postal codes—the numbers assigned to the cities—from the point of view of their factorization into prime numbers. Each pair of postal codes needs to have a common prime factor; however, no set of more than half of the cities may have such a factor. Moreover, the cities need to receive pairwise different postal codes.

Let us try the following solution: take $\binom{n}{2}$ prime numbers and assign a different prime number to each pair of cities; the postal code of a city is the product of all numbers assigned to pairs that contain this city. This way, the postal codes of each pair of cities have a common divisor, and this does not hold for any trio of cities—let alone half of all the cities. Moreover, it is easy to see that the postal codes generated in this way are pairwise different.

Unfortunately, in Byteland there could be 100 cities, so we need 4950 prime numbers. The 4950th smallest prime number is 48049, and our postal codes are products of 99 such numbers. Thus, the postal codes generated in this way are very large: much larger than the required upper bound of $10^9$. Precise calculations show that this technique works for up to six cities without violating the upper bound.

What can be improved? Note that we have fulfilled the anti-corruption law very strictly—too strictly: not only can any set of more than half of all the cities not have a common divisor, but no trio of cities can have such a divisor either. Let us now try to be less strict: let us partition the cities into five roughly equal parts, and assign to each pair of parts one of the $\binom{5}{2} = 10$ smallest prime numbers. The postal code of a city is a product of four prime numbers assigned to pairs of parts in which the city in question is contained.

Is this a correct solution? Unfortunately, it requires two corrections. The first problem is easy to notice: the postal codes assigned to the cities of each part are equal. Luckily, this is easy to fix: take the next 100 prime numbers and multiply the code of each city by a different prime number. Let us check whether we will fit into the $10^9$ upper bound: the tenth prime number is 29, the hundred-and-tenth is 601, and we have $29^4 \cdot 601 < 10^9$.



The second error is much more subtle: this solution does not work for $n = 7$. In this case, the parts into which we divide the set of all cities have the sizes 2, 2, 1, 1 and 1. The common divisor of the two parts with two cities each will be a common divisor of the postal codes of four cities, which is more than half of all the cities. This unfortunate case can be solved in a few ways. One is to partition the cities into six groups. One can also solve this case manually and hard-code the solution.

This ends the description of my solution of the problem *Reconstruction of Byteland*. Naturally, this is only one of many possible algorithms for this problem. I conclude with a fun fact: the bounds in the problem statement—100 cites and postal codes not larger than $10^9$—did not originate from the technical difficulties of solutions—the algorithm described above generalizes to a larger number of cities easily with only a slight increase in the upper bound on the numbers—but are required by the program that verifies the solutions. Note that this program needs to factorize all postal codes, check whether each pair of codes has a common divisor and verify that no divisor appears in more than half of the cities. This is much more work than is expected from a solution to this problem!

**MICHAŁ PILIPCZUK**

Michał Pilipczuk completed a double degree program in Computer Science and Mathematics at the Faculty of Mathematics, Informatics and Mechanics, University of Warsaw. Currently he is a Ph.D. student at the University of Bergen, Norway. His scientific research focuses primarily on parameterized complexity and kernelization. However, his broader interests include combinatorics and especially its application in computer science.

During his university years, Michał Pilipczuk participated in several programming competitions, including the ACM-ICPC, where he won a silver medal in 2010 at the World Finals in Harbin (together with Karol Kurach and Krzysztof Pawłowski).

For the past few years, Michał Pilipczuk has been involved in the Polish Mathematical Olympiad, where he was mostly responsible for the selection of problems. He was the deputy leader of the Polish team at the International Mathematical Olympiad three times. During the holiday season you can usually find him in the mountains or some other wilderness area.

# / Army *Training*

**Contest:** Algorithmic Engagements 2010
**Task author:** Michał Pilipczuk
**Memory:** 64 MB
http://main.edu.pl/en/archive/pa/2010/pol

The Byteland Army plans to conduct the greatest military exercise in its history this weekend. The exercise will take place on the training ground in Northern Bytetown. The officers of the Byteland Army know this training ground perfectly; they do not, however, know the assignments. This is why they have requested your help, recruit!

Your superiors know exactly where the strategic spots on the training ground are located. During the exercise they will be asked multiple times to capture various areas of the training ground. One of the most crucial decisions will be the allocation of forces and determining the strength needed to capture a particular area—this strength should be proportional to the number of strategic spots in the attacked area. Your task is to determine for each area, represented as a polygon with vertices at the strategic spots, the number of strategic spots lying strictly inside the interior of the area.

## Input

In the first line of the input two integers, $n$ and $m$, are given ($3 \le n \le 1000$, $1 \le m \le 100{,}000$), specifying respectively the number of strategic spots on the training ground and the number of queries. The strategic spots are numbered from 1 to $n$.

The next $n$ lines give descriptions of the strategic spots. In the $i$-th line two integers, $x_i$ and $y_i$ ($-10^9 \le x_i, y_i \le 10^9$), are given, specifying the coordinates of the $i$-th strategic spot. No three strategic spots are collinear.

The next $m$ rows contain the descriptions of the $m$ queries. Each description begins with an integer $k_j$ ($3 \le k_j \le n$), specifying the number of vertices of the polygon. It is followed by $k_j$ different integers from the interval $[1,n]$, which specify the identifiers of strategic spots that are subsequent vertices of the polygon. All the given polygons will be simple (i.e. without self-intersections), and their vertices are specified in clockwise order. The sum of all the numbers $k_j$ does not exceed 1,000,000.

**Output**

Your program should write $m$ lines to the output. The $j$-th line should contain one integer, being the number of strategic spots in the interior of the polygon described in the $j$-th query.

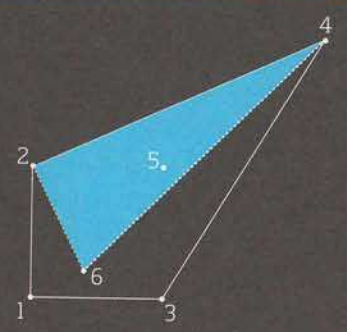
**Example**

For the input data:

```
6 4
0 0
0 5
5 0
11 10
5 5
2 1
4 1 2 4 3
4 1 2 5 3
3 6 2 4
3 1 2 6
```

the correct result is:

```
2
1
1
0
```




**Explanation of the example:** The points in the figure depict strategic spots, while the numbers next to the circles are their identifiers. The figure shows the areas from the first (continuous lines) and third (dashed lines, filled in blue) queries.



# / Solution

**From the area of a polygon to the number of points inside**

Assume that we are given a polygon $P = A_1A_2A_3...A_k$, where the points $A_i$ are listed in clockwise order. There are several methods of evaluating the area of $P$, based on the idea of expressing this value as the sum of areas of simpler figures. Probably the best known one uses trapezoids: we traverse the consecutive sides of the polygon in a clockwise order and, for each side, add or subtract the area of the trapezoid below it, depending on whether we are increasing or decreasing the $x$-coordinate along that side. It is not hard to verify that all the parts of the plane are added in this sum exactly the same number of times as subtracted, except for the polygon $P$ itself, which is added exactly one more time than it's subtracted (see figure 1a).

In the solution of our problem it will be more convenient to use a similar technique, but based on triangles with a common vertex (figure 1b). We consecutively add or subtract the areas of triangles $OA_iA_{i+1}$ (the indices behave in a cyclic manner), depending on whether the orientation of the triangle is clockwise or counterclockwise ($O$ is the center of the coordinate system). In this way we express $\mu(A_1A_2A_3...A_k)$ as $\sum_{i=1}^{k}\mu(OA_iA_{i+1})$, where $\mu$ is the signed area of a polygon, i.e. taken with a positive sign if its orientation is clockwise, and with a negative sign otherwise. As we know how to find the areas of triangles easily, we can compute the area of $P$ in linear time. Note that the point $O$ can be chosen arbitrarily; from now on we take $O = A_1$.

In our problem we would like to count the number of given points from $S$ (strategic objects) inside the polygon instead of its area. However, we could simply treat the function $\mu$ in our formulas not as the area of a figure, but the number of points in the interior—everything should still work! In this way, if we are able to construct a data structure that can answer queries of the form, "How many points from $S$ are inside the triangle with vertices $X, Y, Z \in S$?" then by computing the same sum as before we should get the number of points inside the polygon. Are we sure? Well, in order to prove it formally, we need to show that every point from $S$ lying inside the polygon is counted exactly once in the sum, whereas all the points outside the polygon, including its vertices, are counted zero times.

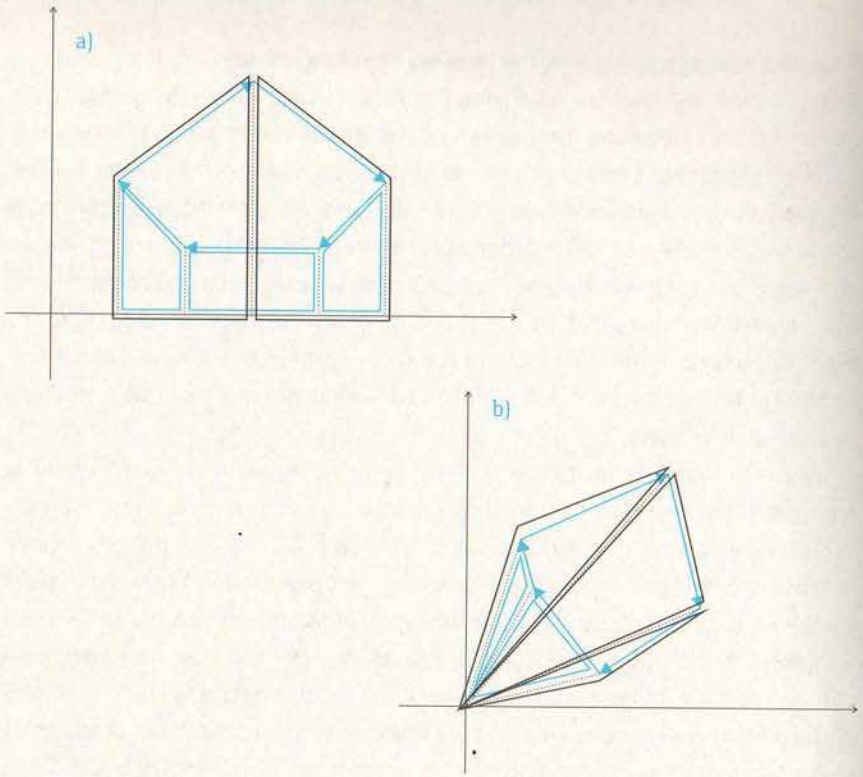



***Figure 1:*** *Computing the area of a polygon using trapezoids and triangles.*

Let us consider some $X \in S$ and look at the ray $\overrightarrow{OX}$. Imagine a stickman that starts a journey at $X$ and moves along the ray in the opposite direction to $O$. If $X$ is inside $P$, then the stickman begins inside $P$, and after getting sufficiently far he will always be outside $P$, so he has to leave $P$ exactly one time more than he enters $P$. In the sum, the point $X$ is counted positively for every side through which the stickman leaves $P$, and negatively for every side through which he enters $P$. This means that each $X$ lying inside $P$ will be added exactly one time more than it will be subtracted (see figure 2a). Let us note that the stickman will never travel via a vertex of $P$; this would mean that three points from $S-O$, $X$ and the considered vertex—were collinear.

Similarly, it can be proved that every $X$ lying outside $P$ is added and subtracted the same number of times as the stickman starts outside $P$ and also



finishes outside $P$ (see figure 2c). However, there is a problem for $X$ being a vertex of $P$, i.e. $X = A_i$ for some $i$. If, directly after leaving $X$, the stickman finds himself outside $P$, then the point $X$ is indeed added and subtracted the same number of times (see figure 2b). However, if we in fact enter $P$ at $X$, then the point $X$ is counted exactly once (see figure 2d), when it should be counted zero times. Fortunately, this is not a big problem: every such vertex that is counted one extra time can easily be recognized from the mutual positions of incident sides. This means that, by using an additional scan through the perimeter of $P$, we can compute this additional contribution to the sum.

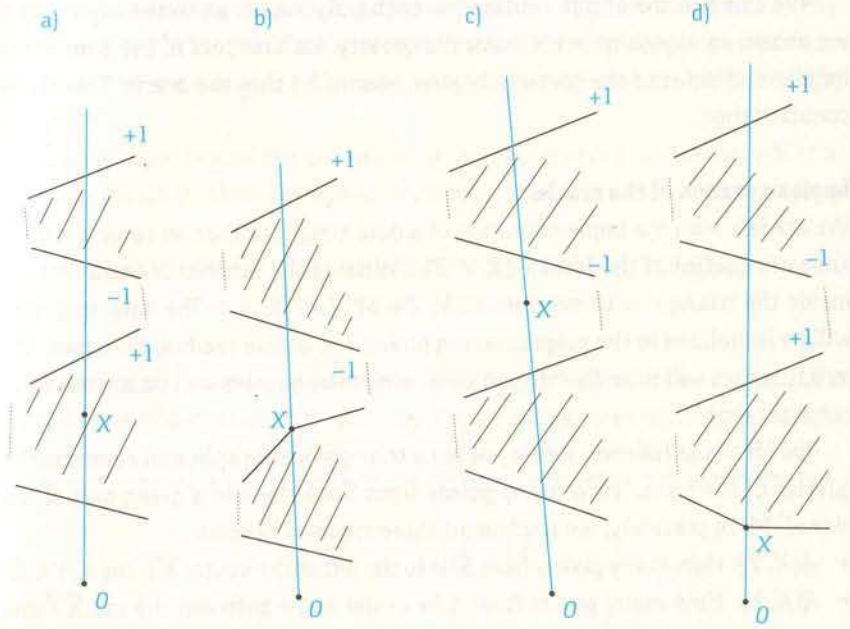



***Figure 2:*** *Cases of alignment of point X with respect to P.*

Using a subroutine, which for any three points $X, Y, Z \in S$ that form a triangle oriented clockwise computes the number of points from $S$ lying in its interior, we can easily implement the whole algorithm. This subroutine will be called the *oracle*.

```
Algorithm ArmyTraining(P = A1A2A3...Ak, oracle T)
    result := 0
    for i := 2 to k − 1 do
        if triangle A1AiAi+1 is oriented clockwise then
            result := result + T(A1, Ai, Ai+1)
        else
            result := result − T(A1, Ai+1, Ai)
    for i := 2 to k do
        if ray A1Ai enters the interior of P in Ai then
            result := result − 1
    return result
```

We can run the above routine for each polygon given in the input. Thus we obtain an algorithm with linear complexity with respect to the sum of the numbers of sides of the given polygons, assuming that the oracle $T$ works in constant time.

**Implementation of the oracle**

We are left with the implementation of a data structure that, in constant time, answers queries of the form $T(X, Y, Z)$: "What is the number of points from $S$ inside the triangle with vertices $X, Y, Z \in S$?" Let $|S| = n$. The data structure will be initialized in the preprocessing phase, i.e. before reading the input. The initialization will take $O(n^2 \log n)$ time, while the queries will be answered in constant time.

The idea is as follows: a query about a triangle will be split into even simpler queries of the form, "How many points from $S$ are there in a given part of the plane?" More precisely, we implement three types of queries:

- $A(X, Y)$: How many points from $S$ lie to the left of the vector $\overrightarrow{XY}$, for $X, Y \in S$?
- $R(X, Y)$: How many points from $S$ lie in the angle between the ray $\overrightarrow{XY}$ and the horizontal ray with origin in $X$ and directed to the right, for $X, Y \in S$?
- $B(X, Y, Z)$: How many points from $S$ lie in the angle between the rays $\overrightarrow{XZ}$ and $\overrightarrow{XY}$, for $X, Y, Z \in S$?

In the queries $R$ and $B$ we consider the angle between the rays constructed by clockwise traversal from the first to the second. In the results we omit the points lying on the selected lines and/or rays. Also, we assume that $X \neq Y \neq Z \neq X$. See figure 3 for an illustration.

Let us note that the query $B(X, Y, Z)$ can be trivially implemented via the query $R(X, Y)$. We have $B(X, Y, Z) = R(X, Z) - R(X, Y) - 1$ if $R(X, Z) > R(X, Y)$, and $B(X, Y, Z) = (n-2) + R(X, Z) - R(X, Y)$ otherwise. The answers to queries $A$ and $R$ we will simply precompute in two-dimensional arrays of size $O(n^2)$.

For each point $X \in S$ we sort all the remaining points $Y \in S$ with respect to the angle between the ray $\overrightarrow{XY}$ and the horizontal ray with $X$ as the origin, directed to the right. Let us denote the list of sorted $S \setminus \{X\}$ as $L_X$. The values $R(X, Y)$ can be computed very easily: if $Y$ has index $i$ in the list $L_X$ (indices begin with zero), then $R(X,Y) = i$. Therefore, we can simply scan through the list $L_X$ and complete one row of the array of values of $R$.

Memorizing the answers for $A$ queries is only slightly more difficult. We traverse the list $L_X$ using a pointer $Y$, while maintaining also a second pointer $Y'$ that points to the farthest (in the angular distance) point on $L_X$ that still counts in the query $A(X, Y)$, or on $Y$ if no such point exists. When incrementing the pointer $Y$ we simply push $Y'$ as far as possible. Now we can observe that $A(X, Y) = B(X, Y, Y') + 1$ if $Y \neq Y'$ and $A(X,Y) = 0$ otherwise, so we can memorize the answer for the query $A(X,Y)$ using precomputed answers for queries $R(X,Y)$ and $R(X,Y')$.

Having implemented the queries $A$, $B$, we can proceed to the query $T$. If $X$, $Y$, $Z$ form a triangle that is oriented clockwise, then

$$2(n-3) + T(X, Y, Z) = B(X, Z, Y) + B(Z, Y, X) + B(Y, X, Z) + \\ + A(X, Y) + A(Y, Z) + A(Z, X).$$

This formula can easily be verified by considering how many times each point in $S$ is counted on both sides of the equality. Figure 4 illustrates the right-hand side.

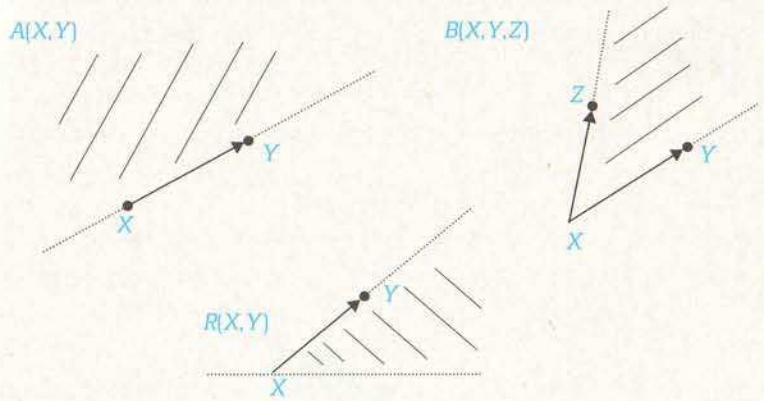



***Figure 3:*** *Queries A, R, B.*

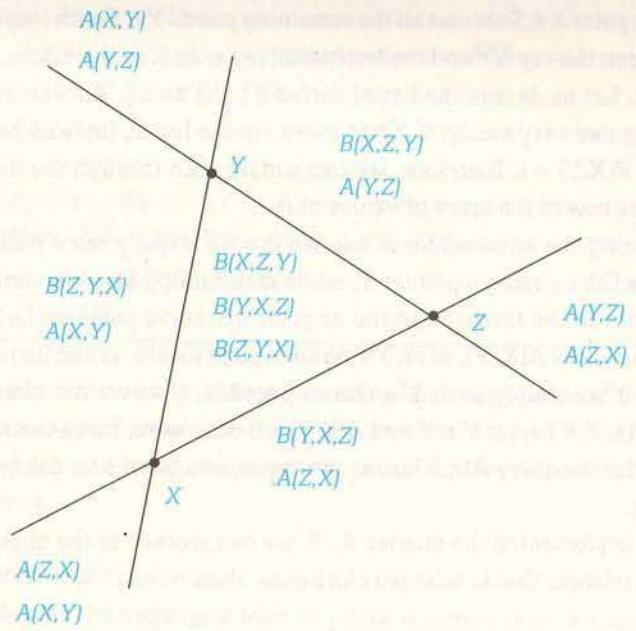
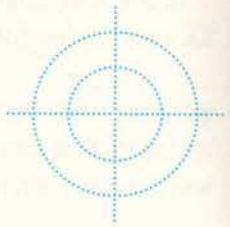
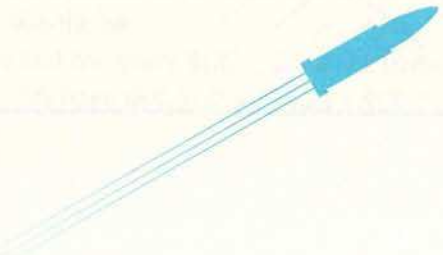



***Figure 4:*** *Expressing query T in terms of queries A, B.*

In conclusion, we can memorize the answers for queries $A$, $R$ in time complexity $O(n^2 \log n)$, since we sort all but one points from $S$ $n$ times in total. After that, we can answer the queries $T(X, Y, Z)$ in constant time. If we use this implementation of the oracle, then the presented algorithm runs in time $O(n^2 \log n + K)$, where $K$ is the total number of sides of polygons given in the input.







# Riddle

**Contest:** Algorithmic Engagements 2010
**Task authors:** Marek Cygan, Marcin Pilipczuk, Michał Pilipczuk, Jakub Wojtaszczyk
**Memory:** 128 MB
http://main.edu.pl/en/archive/pa/2010/zag

The evil sorcerer Voldebyte has imprisoned the brave knight Bytter the Bold in his tower. As is his custom, Voldebyte has promised to free Bytter if he can solve one of his (as yet unsolved) riddles. Bytter, unfortunately, has managed to kill Voldebyte's pet dragon, and came close to killing Voldebyte himself, so Voldebyte has decided that a really tough riddle is in order. This is the riddle posed by Voldebyte to Bytter:

*Byteland is divided into k counties, in which there are n towns altogether. Additionally, some pairs of towns are connected by bidirectional roads. I would like to choose a town in each county to be its capital in such a way that, for each road, at least one of its endpoints is a capital town. Is this possible?*

Help poor Bytter to save himself and solve the riddle for him.

## Input

In the first line of the input there are three integers: $n$ ($1 \le n \le 1{,}000{,}000$), specifying the number of towns in Voldebyte's riddle; $m$ ($0 \le m \le 1{,}000{,}000$), specifying the number of roads; and $k$ ($1 \le k \le n$), specifying the number of counties. The towns are numbered from 1 to $n$.

In the next $m$ lines there are $m$ pairs of integers $a_i$, $b_i$ ($1 \le a_i, b_i \le n$, $a_i \ne b_i$); the $i$-th pair specifies a road connecting towns $a_i$ and $b_i$. No pair of towns is connected by more than one road.

In the next $k$ lines there are descriptions of subsequent counties. The $j$-th line begins with an integer $w_j$ ($1 \le w_j \le n$), specifying the number of towns in the $j$-th county. Then $w_j$ integers are given, specifying the (distinct) numbers of the towns in the $j$-th county. The sum of all the numbers $w_j$ is equal to $n$.

**Output**

If the solution to the riddle is negative, your program should write a single line containing the word NIE (Polish for *no*) to the output.

Otherwise, your program should write two lines. The first should contain the word TAK (Polish for *yes*); the second should describe the solution. The second line should contain exactly $k$ integers. The $i$-th of these integers should be the number of the town that should be selected as the capital of the $i$-th county.

If there are multiple correct solutions, your program may output any of them.

**Example**

For the input data:

```
6 5 2
1 2
3 1
1 4
5 2
6 2
3 3 4 2
3 1 6 5
```

the correct result is:

```
TAK
2 1
```

whereas for the input:

```
3 3 1
1 2
2 3
3 1
3 1 2 3
```

the correct output is:

```
NIE
```



# / Solution

Evil Voldebyte gave Bytter a map of Byteland, on which the towns are divided between counties. Bytter's task is to choose exactly one town from each county as its capital in such a way that, for every road connecting two towns, at least one of these towns is chosen as a capital of its county.

Let us consider how to model this situation. For the $i$-th town of Byteland we introduce a Boolean variable $x_i$ that is true if this town is a capital of its county, and is false otherwise. In this language it is not hard to express the constraints given by the sorcerer:

- If the towns with indices $i$ and $j$ are connected by a road, then at least one of them must be the capital of its county; this constraint has the form $x_i \vee x_j$.
- In each county at most one town can be the capital. Equivalently, for every pair of towns within the same county, these towns cannot both be the capital at the same time. Therefore, we introduce a constraint $\neg x_i \vee \neg x_j$ for every pair of distinct indices $i$, $j$, such that the $i$-th town and the $j$-th town are in the same county.

Let us remark that such a representation of the problem allows for choosing none of the towns as the capital of some county. But if Bytter finds such a solution, then he will not spoil anything by simply making any town the capital in this county.

Hence, Voldebyte's riddle has a solution if and only if there exists such an evaluation of variables $x_i$ that satisfies every introduced constraint. Let us write a Boolean formula

$$\varphi = C_1 \wedge C_2 \wedge \cdots \wedge C_l$$

where $C_j$ are the introduced constraints which henceforth will be called *clauses*. Each clause is a disjunction of two *literals*, i.e. occurrences of a variable or of its negation. This means that $\varphi$ is a *conjunction of disjunctions of literals*, while every clause contains at most two literals. We say that the formula is in 2-CNF: it is in *conjunctive normal form* and 2 denotes the bound on the number of literals in a clause.

Checking satisfiability—i.e. whether there is an evaluation of variables that makes the formula true—for formulas in CNF is one of the oldest problems in theoretical computer science. For 3-CNF the problem is already NP-complete,

which means that we do not expect it to be solvable in polynomial time; however, for formulas in 2-CNF there exists a clever algorithm that checks satisfiability in linear time with respect to the number of variables and clauses. This algorithm is described in most of the classical books on graph algorithms, as well as on several algorithmic web pages.

We just need to check the size of the formula obtained by translating the description of the riddle. We have one variable per town. The number of clauses of the first type, i.e. those that encode roads between towns, is exactly the same as the number of roads. Unfortunately, in order to encode the constraints of the second type, we introduce one clause per pair of towns within the same county. The number of such pairs can be quadratic, so the algorithm will run in $O(n^2)$ time complexity. With one million towns in the input, this solution could not achieve a full score; we need something faster.

**Speedup via compactification**

The main problem in the presented approach is an overly expensive method of expressing the information that, in each county, at most one town can be a capital. Maybe we could encode it using only a linear number of clauses? In fact, it is possible; however, we will pay for a smaller number of clauses by increasing the number of variables.

The idea of the compactification is as follows: we will introduce new variables that encode whole groups of towns. Let us consider one county, containing towns with indices $i_1 < i_2 < \cdots < i_p$. For every prefix and every suffix of this sequence we introduce a variable, which intuitively says, "in this prefix (suffix) there is a capital." Formally, for every $j = 1,2,\ldots,p$ we introduce variables $y_{i_j}$ and $z_{i_j}$, which will satisfy the following properties:

- if among towns $i_1,i_2,\ldots,i_j$ there is a capital, then $y_{i_j}$ is true;
- if among towns $i_j,i_{j+1},\ldots,i_p$ there is capital, then $z_{i_j}$ is true.

Conditions imposed on variables $y_{i_j}$ and $z_{i_j}$ can easily be encoded via clauses containing two variables. The key observation is as follows: a capital is inside some prefix (suffix) if and only if it is on its end or in the prefix (suffix) shorter by one town. Therefore,

- for each $j = 1,2,\ldots,p$ we introduce a clause $x_{i_j} \Rightarrow y_{i_j}$;
- for each $j = 1,2,\ldots,p-1$ we introduce a clause $y_{i_j} \Rightarrow y_{i_{j+1}}$;
- for each $j = 1,2,\ldots,p$ we introduce a clause $x_{i_j} \Rightarrow z_{i_j}$;
- for each $j = 1,2,\ldots,p-1$ we introduce a clause $z_{i_{j+1}} \Rightarrow z_{i_j}$.

Let us note that in the formula we can use implication, as the clause $a \Rightarrow b$ is equivalent to $\neg a \vee b$.



Observe that in this way we have encoded only one direction of implication; it could happen that the variable $y_{i_j}$ is true even though in the corresponding prefix there is no capital. We only know that if the capital is actually there then $y_{i_j}$ is true. Fortunately, this turns out not to be a problem.

We are left with encoding the uniqueness of capitals using variables $y_{i_j}$ and $z_{i_j}$. For each $j = 1,2,\ldots,p-1$ we introduce a clause $\neg y_{i_j} \vee \neg z_{i_{j+1}}$. If there were two capitals in a county, say with indices $i_j$ and $i_{j'}$ for $j < j'$, then the introduced clause would not be satisfied for the index $j$, as both $y_{i_j}$ and $z_{i_{j+1}}$ would need to be true. On the other hand, if a county has at most one capital, say in the town with index $i_j$, then we can set $y_{i_{j'}}$ to true for all $j' \geq j$ and to false for $j' < j$, and symmetrically for variables $z_{i_{j'}}$. Thus we satisfy all the introduced clauses, as for every partition of the sequence $i_1,i_2,\ldots,i_p$ into a prefix and a suffix, in at least one of the parts the index $i_j$ does not occur.

We perform this construction for every county and add clauses encoding the roads just as in the previous section. Let us check the size of the resulting formula:

- we have three variables per town—in total $3n$ variables;
- for each road we introduced one clause—in total $m$ clauses;
- for each county we introduced $5p - 3$ clauses, where $p$ is the number of towns within the county—in total, we have constructed at most $5n$ clauses.

Hence, the constructed formula $\varphi$ has $3n$ variables and at most $5n + m$ clauses, so the linear-time algorithm for checking satisfiability of formulas in 2-CNF should be able to produce the answer within the time limit.

**And what is the problem really about?**

Let us view the map of Byteland as an undirected graph $(V,E)$, in which every town is represented by a vertex and two vertices are adjacent if and only if the corresponding towns are connected by a road. Moreover, the vertices of the first county will be colored with color 1, second with color 2, etc. Now Voldebyte asks for a subset of vertices $S$ such that each edge has at least one end in $S$. Such a set is called a *vertex cover* of the graph. In the sorcerer's riddle one has to find a vertex cover that contains exactly one vertex in each color. This is called the *colored* version of the problem.

In general, the problem of determining whether a given graph admits a vertex cover of size $k$ is NP-complete; this means that it seems implausible that it can be solved in polynomial time. The complexity of computing a small vertex cover is also considered in the description of the problem *Guilds* in this book; a simple branching algorithm with running time $O(2^k(n + m))$ is described there. We

now present how to obtain an algorithm with slightly worse time complexity, $O(e^k(n+m))$, which moreover will be randomized, but which will also be a lot cooler! The main idea is to use the colored version of the problem, solvable in linear time, as a subroutine.

The described algorithm will be *randomized with one-sided error*. This means that, during the computation, the program can make decisions based on random coin tosses. If the graph does not admit a vertex cover of size at most $k$, then the algorithm must always give a negative answer. However, if a vertex cover exists, then the algorithm can either find it or make an error and return a negative answer, but the probability of such a mistake must be at most $\frac{1}{2}$. Observe that by making 20 such tests consecutively, we can reduce the probability of a mistake to a minute chance.

Our algorithm will perform a simple test multiple times in hopes that, at some moment, it will be able to "pinpoint" the solution. In each test we draw at random a coloring of vertices with $k$ colors by independently assigning each vertex each color with the same probability; then, using the linear-time algorithm for the colored version, we check whether there exists a vertex cover that uses one vertex of each color. Therefore, in each test we count on the event that all the vertices from the solution will be assigned different colors, so the algorithm for the colored version will be able to extract the solution quickly.

---

**Algorithm** RandomizedVertexCover($(V,E),k$)
  **for** $t := 1$ **to** $\lceil e^k \rceil$ **do**
    choose a coloring of $V$ with $k$ colors at random
    invoke the algorithm for the colored version
    **if** the algorithm for the colored version has found a solution **then**
      **return** the solution
  **return** *no*

---

Clearly, if the algorithm has found a solution then it is a feasible vertex cover of size $k$. It remains to prove that if the graph admits some vertex cover of size $k$, then the algorithm finds an example solution with probability at least $\frac{1}{2}$. Let us fix some solution $S$ and consider the probability that all the vertices from $S$ are assigned different colors. It is easy to see that this probability is equal to $\frac{k!}{k^k}$, as $S$ can be colored using distinct colors in $k!$ ways, while there are $k^k$ colorings of $S$ in total. Now we use the inequality $\frac{k!}{k^k} \geq \frac{1}{e^k}$ to infer that the probability that, during a single test, the solution is colored distinctly is at least $\frac{1}{e^k}$. Hence, with

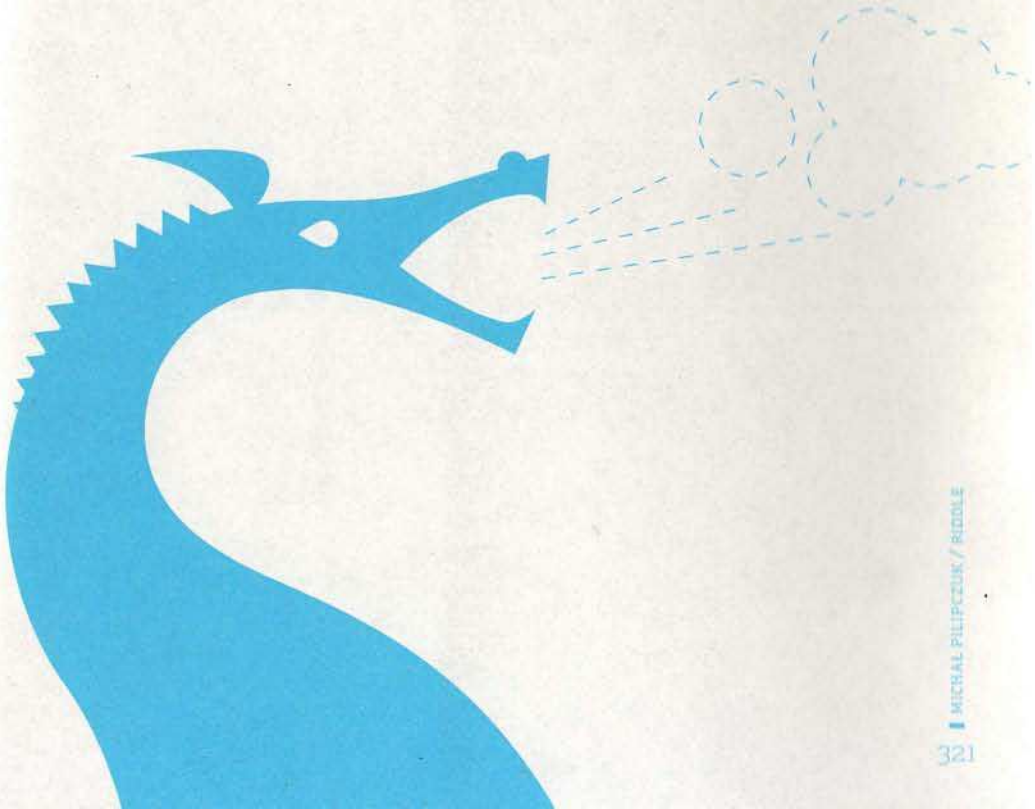


at least this probability the subroutine for the colored version will be able to find a solution (maybe a different one from $S$) in linear time.

Now let us bound the probability that in each of the $\lceil e^k \rceil$ tests the algorithm does not find a solution. The probability of a mistake in a single test is at most $1 - e^{-k}$, so, as the tests are independent, the computation will be erroneous with a probability of at most $(1-e^{-k})^{e^k}$. We can use the inequality $1 - x \leq e^{-x}$, which holds for all $x \in [0,1]$, to infer that the probability of an error is at most $e^{-e^{-k}e^k} = e^{-1} < \frac{1}{2}$.

Finally, let us remark that the algorithm invokes a subroutine running in $O(n+m)$ time at most $\lceil e^k \rceil$ times, so the total time complexity of the algorithm is $O(e^k(n+m))$.

The technique of randomly coloring a graph in order to make a solution distinguishable in polynomial time is called *color coding*, and was introduced by Alon, Yuster and Zwick in the early nineties. Nowadays, it is a classical method in the design of parameterized and exact exponential time algorithms. Let us also remark that the algorithm can be *derandomized* at the cost of worse running time. More precisely, based on the described method, one can construct a program that does not use randomness but that has a constant larger than $e$ in the complexity bound.



**JAKUB RADOSZEWSKI**

Jakub Radoszewski is a Ph.D. student of computer science at the Faculty of Mathematics, Informatics and Mechanics, University of Warsaw. His research interests focus on text algorithms and combinatorics. He is the chairman of the jury of the Polish Olympiad in Informatics and Algorithmic Engagements, an editor of booklets of the Polish Olympiad in Informatics, and an editor of a Polish popular science monthly, *Delta*. In 2011 he was the chairman of the jury of the Central European Olympiad in Informatics and the Polish Collegiate Programming Contest. As a student, together with his team, he reached second place in the Polish Collegiate Programming Contest and the ACM-ICPC Central European Regional Contest. He is a multiple semifinalist of TopCoder Open and finalist of Google Code Jam and Facebook Hacker Cup. He enjoys playing tennis, table tennis and badminton.

# / Fuel

**Contest:** Algorithmic Engagements 2011
**Task author:** Jacek Tomasiewicz
**Memory:** 128 MB
http://main.edu.pl/en/archive/pa/2011/pal

In the good old days, all $n$ towns in Byteland were connected by a dense network of two-way roads. The king of Byteland decided to reduce the number of roads, and asked his Chief Computer Scientist for advice. Now, as a result of this decision, Byteland has only $n - 1$ two-way roads connecting pairs of towns in such a way that there is exactly one route between any two towns in Byteland. All the roads are of the same length.

Equipped with a car whose tank holds exactly enough fuel to drive on $m$ roads in Byteland, Byteasar has decided to organize a trip that maximizes the number of visited towns. He can start his trip in any of the towns in Byteland, and, similarly, his trip can end anywhere—not necessarily back at the starting town. In maximizing the number of visited towns, Byteasar is allowed to drive multiple times on the same road, in the same or the reverse direction. Your task is to help Byteasar by finding the maximum number of towns that can be visited on one full tank of fuel.

## Input

The first line of the input contains two integers, $n$ and $m$ ($2 \le n \le 500{,}000$, $1 \le m \le 200{,}000{,}000$), where $n$ is the number of towns in Byteland (each numbered uniquely from $\{1,\ldots,n\}$), and $m$ is the number of roads that can be traveled on one tank of fuel.

The next $n - 1$ lines describe Byteland's road network. Each of these lines contains two integers, $a$ and $b$ ($1 \le a, b \le n$), indicating that towns $a$ and $b$ are connected with a two-way road.

## Output

The output consists of one line containing exactly one integer: the maximum number of towns that can be visited on one full tank of fuel.

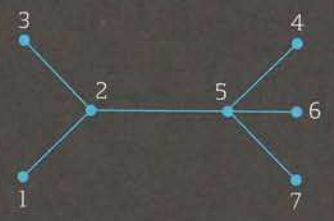
**Example**

For the input data:

7 6
1 2
2 3
2 5
5 6
5 7
4 5

the correct result is:

5




**Explanation of the example:** Byteasar can visit a maximum of five different towns. There are several different routes that visit five towns for this example input, including $4 \to 5 \to 7 \to 5 \to 6 \to 5 \to 2$ and $3 \to 2 \to 1 \to 2 \to 5 \to 6 \to 5$.



# Solution

I do not like hard problems. I am a person who likes to get his hands dirty at work, so instead of thinking about a solution for hours I much prefer to alternate phases of thinking and coding when tackling a problem. If I keep analyzing a problem for a long period of time, and I feel that I have not made any real progress towards the solution, I simply get bored with it and find something more productive to do.

I certainly do not like trivial problems, though; I learn nothing from solving such problems. Also, when solving such a problem, I feel like a monkey trained to hit a keyboard as fast as possible. This type of problem includes tasks that can be solved with a straightforward application of a well-known technique or a classical algorithm, even if this technique or algorithm requires writing many lines of very smart code.

I do not care for problems whose solutions require only abstract ideas: ideas that may or may not occur in one's mind arbitrarily. I maintain that, for contestants with similar knowledge and abilities, the time it takes to solve such a task depends mostly on luck. This introduces a considerable amount of randomness to a competition, especially if the idea seems intuitively correct, but is really hard to prove—though note that such a proof is not required during a competition.

It appears that I do like my own problems, however! A few years ago I used to devise lots of new problems. Unfortunately, nowadays I do not have enough time to look around and constantly search for interesting new ideas on which to base problems.

Regardless of what I have just said, the problems that I have chosen to describe in this book are *not* ones that I wrote myself.

What I consider a perfect problem is neither too easy nor too difficult. Solving it should require creativity—so that a contestant can enjoy the magic moment of a discovery—and also the ability to apply (in a smart way) important classical tools—known techniques and algorithmic bricks. Such a problem combines pure intellectual activity with a challenging task for an algorithms engineer. If, additionally, the formulation of the problem contains a neat story with a natural background, I will certainly remember such a problem for a long period of time.

For the purposes of this book, I have chosen two problems that, in my opinion, satisfy all the above requirements and, moreover, have made an especially good impression on me. Let us start with the easier one, the task *Fuel*.

**Solution**

The story from the task statement translates into the following graph-theory problem: we would like to visit nodes of a given tree in the most efficient way. More precisely, we are looking for a walk of a given length that visits as many different nodes as possible. By a *walk* we mean a path that may contain multiple occurrences of the same node. Sometimes such a sequence of nodes is also called just a path; however, we will stick to the term walk to avoid confusion with a simple path in which no node repetitions are allowed.

As in the task statement, by $n$ and $m$ we denote the number of nodes in the tree and the given length of the walk. As soon as we start thinking about the problem, we can immediately make several basic observations. First of all, from our point of view the best type of walk is a path without repeating nodes; that is, the aforementioned simple path. If $m$ is small enough, there is a chance that we will be able to find a simple path of length $m$. In order to describe all such clearly optimal walks, it is sufficient to find the longest simple path in the tree, the so-called *diameter* of the tree. Denote by $d$ the length of the diameter; if $m \le d$, the correct result is $m + 1$.

Thus we have identified the most optimistic case. Now let us consider the problem from a different perspective. What if we are interested in a really long walk? If $m$ is large enough with respect to $n$, we may expect to find a walk that visits all the nodes of the tree. This, surely, takes place if $m \ge 2 \cdot (n-1)$—in this case the desired walk can be constructed by a depth-first search (DFS) traversal of the tree in which each edge is visited exactly twice: once in each direction.

Let us try to state a precise condition for the existence of a walk that visits all the nodes of a tree. Consider the shortest such walk, $P$. Assume that it starts in the node $u$ and ends in the node $v$. If we added a path from $v$ to $u$ to $P$, we would obtain a walk that visits all the nodes of the tree and ends at its starting node. As we have already observed, the shortest such walk is the DFS traversal of the tree, of length $2 \cdot (n-1)$. Hence, the shortest possible walk $P$ can be obtained as a DFS traversal starting in such a node $u$ and ending in such a node $v$ that $u$ and $v$ are endpoints of a diameter of the tree. The length of this walk is $2 \cdot (n-1) - d$.

We have already shown how to solve the problem if either $m \le d$ or $m \ge 2 \cdot (n-1) - d$. What can we do in the remaining cases? Let us consider a walk of length $m \in (d, 2 \cdot (n-1) - d)$ that leads through as many different nodes of the tree as possible (see figure 1). Denote the number of nodes visited in this walk by $k$. It turns out that this value is the key to the construction of the final algorithm.



Indeed, let us remove all the nodes of the tree that are not present in the walk. What remains is a subtree of the initial tree. Our walk visits all the nodes of this subtree, so, due to the previous considerations, it satisfies $m \ge 2 \cdot (k-1) - d'$, where $d'$ is the diameter of this subtree. Hence, $k \le (m + d' + 2)/2$. In order to maximize the value of $k$ we choose a subtree of maximal diameter, that is, a subtree containing a diameter of the initial tree. Then, $d' = d$. We obtain a walk that visits exactly $\lfloor (m + d + 2)/2 \rfloor$ tree nodes.

Using the parameters $n$, $m$ and $d$ one can state the final algorithm as follows:

- If $m \le d$, return $m + 1$.
- If $m \ge 2 \cdot (n-1) - d$, return $n$.
- If none of these cases holds, return $\lfloor (m + d + 2)/2 \rfloor$.

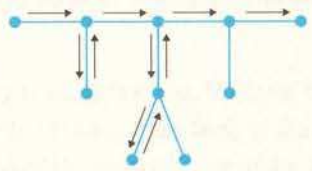


*Figure 1: An example of an optimal walk for $n = 10$, $d = 4$, $m = 10$.*

What is still missing is an efficient algorithm for finding the diameter of a tree. Here I could actually conclude the whole description, since a linear time algorithm solving this problem is folklore knowledge, or I could leave it as an exercise for the reader. However, since the most straightforward approach may often not be the simplest one, in the remaining part of the description I will show my favorite algorithm for this problem, which I learned only after several years of dealing with algorithms.

**Diameter of a tree**

What is the simplest algorithm for finding the diameter of a tree? I am quite sure that it is as follows:

- Choose an arbitrary node of the tree; denote it by $u$.
- Walk from $u$ to the furthest node of the tree; denote that node as $v$.
- Walk from $v$ to the furthest node of the tree; denote that node as $w$.
- The path from $v$ to $w$ is the diameter of the tree.

The only nontrivial operation here is walking to the furthest node of a tree. This operation is very easy to implement if we know the distances from $u$ ($v$ respectively) to all the nodes of the tree. These distances can be computed using any tree search algorithm; e.g. a BFS or DFS algorithm.

It is much more interesting to know why this algorithm actually works. To answer this question it is convenient to present the tree as "hanging by its diameter". Let $x$ and $y$ denote the endpoints of a diameter (initially unknown to us). Imagine that the tree edges are strings of unit length and the tree nodes are knobs. Let us hold the node $x$ in one hand and the node $y$ in the other hand, and pull the strings in the diameter connecting $x$ and $y$. How long are the strings that hang from the respective nodes of the diameter? Obviously, no strings can hang from $x$ and $y$. From a direct neighbor of $x$ or $y$ only a string of length 1 can hang, since otherwise the tree would contain a path longer than its diameter. Similarly, from neighbors of neighbors of $x$ and $y$, only strings of length 2 can hang, and so on (see figure 2).



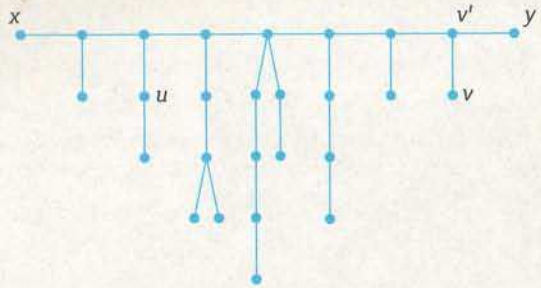



***Figure 2:*** *A tree hanging by its diameter. A sample location of the nodes u and v from the algorithm is presented.*

Now, $u$ is an arbitrary node of the tree. Without loss of generality, assume that $u$ is located not further from $x$ than from y. Clearly, the node $v$ from the algorithm must be located on the opposite side of the diameter, either at the end of one of the hanging strings or in the node $y$. In the former case, the distance from $v$ to the corresponding node on the diameter, $v'$, must be equal to the distance from $v'$ to $y$ (e.g. in figure 2, $v$ could not be the node located one unit left from the node denoted as $v$ in this figure). In both cases, the path from $v$ to $x$ is a diameter of the tree, hence the path from $v$ to the furthest node $w$ is also a diameter of the tree. This proves the correctness of the algorithm.

# / Ploughing

**Contest:** 13th Polish Olympiad in Informatics
**Task author:** Marcin Kubica
**Memory:** 64 MB
http://main.edu.pl/en/archive/oi/13/ork

Byteasar, the farmer, wants to plough his rectangular field. He can begin by ploughing a slice from any of the field's edges, then he can plough a slice from any unploughed area's edges, and so on, until the whole field is ploughed. After the ploughing of every successive slice, the still-unploughed field has a rectangular shape. Each slice has a span of 1, and the length and width of the field are the integers $n$ and $m$.

Unfortunately, Byteasar has only one puny and frail nag (horse) at his disposal for the ploughing. Once the nag starts to plough a slice, it will not stop until the slice is completely ploughed. However, if the slice is too much for the nag to bear, it will die of exhaustion, so Byteasar has to be careful. After every ploughed slice, the nag can rest and regain its strength. The difficulty of certain parts of the field varies, but Byteasar is a good farmer and knows his field well; hence, he knows the difficulty of ploughing every part.

Let us divide the field into $m \times n$ unitary squares called *tiles*. We identify them by their coordinates $(i, j)$, for $1 \le i \le m$ and $1 \le j \le n$. Each tile has its own ploughing-difficulty—a nonnegative integer. Let $t_{i,j}$ denote the difficulty of ploughing the tile whose coordinates are $(i, j)$. For every slice, the sum of ploughing-difficulties of the tiles constituting the slice may not exceed a certain constant $k$, lest the nag dies.

A difficult task awaits Byteasar: before ploughing each successive slice, he has to decide which edge of the field he will plough, so that the nag will not die. On the other hand, he would like to plough as few slices as possible.

## Task

Write a program that:

- reads the numbers $k$, $m$ and $n$ from the input file, as well as the ploughing-difficulty coefficients,
- determines the best way to plough Byteasar's field,
- writes the result to the output file.



## Input

There are three positive integers in the first line of the input file, separated by single spaces: $k$, $m$ and $n$ ($1 \le k \le 200{,}000{,}000$, $1 \le m, n \le 2000$). In the following $n$ lines there are the ploughing-difficulty coefficients. The $(j+1)$-th line contains the coefficients $t_{1,j}, t_{2,j}, \ldots, t_{m,j}$ ($0 \le t_{i,j} \le 100{,}000$), separated by single spaces.

## Output

Your program should write one integer to the output file: the minimum number of slices required to plough the field while satisfying the given conditions. Since we care for animals, we guarantee that the field can be ploughed according to the above rules. But remember, saving the nag is up to you!

## Example

For the input data:

```
12 6 4
6 0 4 8 0 5
0 4 5 4 6 0
0 5 6 5 6 0
5 4 0 0 5 4
```

the correct result is:

```
8
```

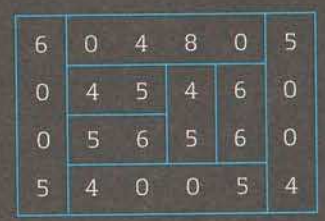



***Figure:*** *The illustration shows the optimal ploughing pattern for the field in the example.*

# / Solution

This problem is much harder than the task *Fuel*. I consider it the most elegant problem from the Polish Olympiad in Informatics. It satisfies all the requirements from the introduction to the previous description, but that is not the only reason why I have chosen this task.

For several years now I have been looking at competition problems not only from a contestant's point of view but also, even more often, from a jury member's perspective. Problems that allow the contestants to be differentiated are invaluable in all "olympiad in informatics"-style contests. An important category of such problems is those that admit several natural solutions with different time complexities. When solving the task *Ploughing*, one arrives at solutions with time complexity $O(N^5)$, $O(N^4)$, $O(N^3)$ and finally $O(N^2)$. The respective solutions require an increasing amount of conceptual effort: applying dynamic programming, a greedy approach and a very simple amortized cost analysis—a so-called sliding window technique (or, alternatively, a smarter greedy algorithm). What an awesome combination!

I remember the day when Marcin Kubica gave me this task. He came up with it while mowing his lawn. As I was walking in a park and thinking about the possible solutions, each time I was able to reduce the time complexity by a factor of $N$ I was more and more amazed with this task. At that time I was a rookie member of the Jury of the Polish Olympiad in Informatics. It is thanks to problems like this one that I have decided to participate in the organization of the Olympiad for a longer period of time.

### Dynamic programming

Let us proceed with the problem analysis. We are given a rectangular board divided into $m \times n$ cells, each cell filled with a nonnegative integer. We would like to remove all the cells of the board. The only operation we may perform is to remove a slice of the board from any of its edges—that is, a peripheral row or column. Such an operation is allowed only if the sum of numbers written in the cells to be removed does not exceed a specified parameter $k$. Our task is to remove all the cells of the board with the minimum number of operations. We assume that the columns are numbered from 1 to $m$ from left to right (the first coordinate) and the rows from 1 to $n$ from top to bottom (the second coordinate). We state that $N = n + m$.



To obtain the first solution, let us make the following basic observation: after an arbitrary sequence of operations the board still has the shape of a rectangle. More precisely, it is a subrectangle of the initial board. This allows us to apply a standard approach based on dynamic programming.

A state in our dynamic programming is a quartet of numbers $(i_1,j_1,i_2,j_2)$ representing a rectangle whose upper-left corner is $(i_1,j_1)$ and lower-right corner is $(i_2,j_2)$. In a single operation we can transition from such a state to each of the states: $(i_1+1,j_1,i_2,j_2)$, $(i_1,j_1+1,i_2,j_2)$, $(i_1,j_1,i_2-1,j_2)$ and $(i_1,j_1,i_2,j_2-1)$, provided that the corresponding operation can be performed; that is, that the sum of elements in the $i_1$-th column, the $j_1$-th row, the $i_2$-th column and the $j_2$-th row of the rectangle, respectively, does not exceed $k$. The result associated with a state is the minimum number of operations required to remove all the cells of the rectangle. The final state of the dynamic programming is $(1,1,m,n)$, and the initial states are all empty rectangles, e.g. rectangles for which $i_1 = i_2 + 1$ and $j_1 \le j_2$ and symmetric cases. In this classical approach we have $O(N^4)$ states and at most four transitions from a single state.

A straightforward implementation of this approach yields $O(N^5)$ time complexity, since $O(N)$ time is required to check which transitions are valid. To obtain $O(N^4)$ time complexity, we need to be able to compute in $O(1)$ time the sum of numbers written in the part of the $i$-th column located between the $j_1$-th and the $j_2$-th row, and, similarly, the sum of the part of the $j$-th row located between the $i_1$-th and the $i_2$-th column. Thus we are left with a one-dimensional problem of finding the sum of given contiguous subsequences in a number sequence $a_1,\ldots,a_p$ (that represents the numbers in a given row or column of the board). A simple solution to this problem uses the so-called sequence of partial sums, i.e. sums of the form $s_i = a_1 + \cdots + a_i$. Once we compute the partial sums, each $a_i + \cdots + a_j$ can be computed in constant time with a single subtraction $s_j - s_{i-1}$. This enables us to answer the requested queries in $O(1)$ time after $O(N^2)$ time preprocessing of partial sums in all rows and columns of the board. Thus we obtain a solution with $O(N^4)$ time complexity.

### Greedy approach

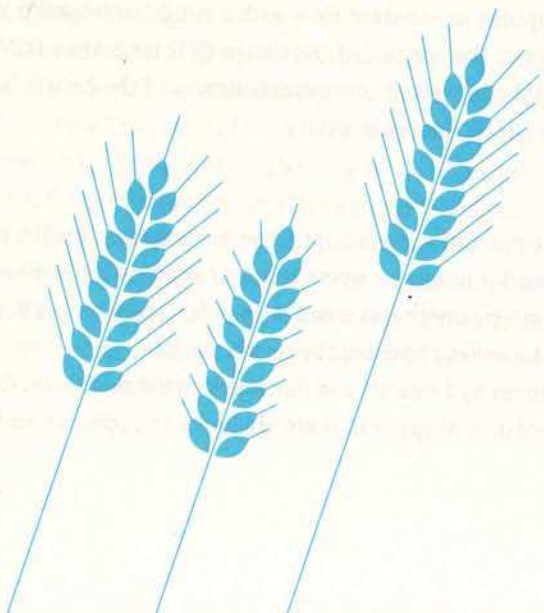
Clearly our first solutions did not attempt to gain a serious insight into the problem. When we hold a hammer, everything around us looks like a nail. In order to make an improvement, we need to look for some properties of the solution. Let us start by asking how big the result can be.

Each operation reduces by 1 exactly one dimension of the rectangle. We finish when one of the dimensions drops to 0. If we are lucky enough, we will end up

performing $\min(n, m)$ operations. On the other hand, after at most $n + m - 1$ operations the board will already be empty. Thus we can conclude that the range of possible results is not so wide.

As a byproduct of the above analysis we obtain the useful corollary that one of the dimensions must eventually be reduced to 0, and this can be obtained only using slices that reduce this dimension by 1. Hence, in the whole solution we remove either $m$ vertical slices or $n$ horizontal slices. From now on, we assume that in the solution we remove $m$ vertical slices (as the alternative case is symmetric). Under this assumption, the goal is to remove as few horizontal slices as possible (see figure 1). Let us just note that, having made this assumption, we are no longer sure that a valid solution can still be obtained.

So far our analysis has been only of a structural nature. The time has come to introduce a bit of greediness to the solution. We know that sooner or later we will remove $m$ vertical slices. Consequently, if it is possible to remove a vertical slice at some point, it pays off to perform this action instantly. Unfortunately, there are cases when no vertical slice can be removed from the board. Then, a horizontal slice must be removed. But which of the two possibilities should we choose? Apparently no obvious greedy criterion can be stated here. Therefore we will apply a "less greedy" approach—we will check both possible options.





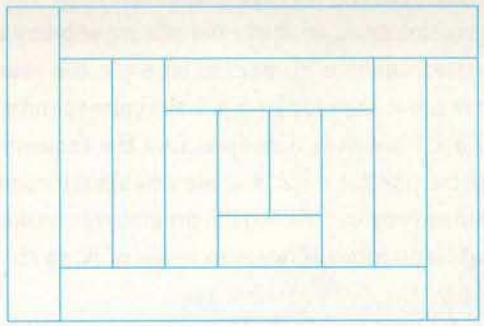


*Figure 1: We remove m vertical slices.*

Assume that, apart from the $m$ vertical slices, in our solution we remove $a$ upper and $b$ lower horizontal slices ($a + b < n$). With this assumption in mind, we can afford a completely greedy approach. Each time we are able to remove a vertical slice, one of the $a$ upper horizontal slices or one of the $b$ lower horizontal slices, we can simply perform this action, since we would have done it sooner or later anyway. Note that this greedy routine exits with a success if and only if there exists a solution that removes $m$ vertical slices, at most $a$ upper horizontal slices and at most $b$ lower horizontal slices.

Let us analyze the complexity of this solution. First of all, after the parameters $a$ and $b$ have been selected, the slice removal can be simulated really efficiently. Indeed, in each step we choose just one of four possible slices and remove it. Checking whether a slice is valid is performed in constant time using the partial-sums trick described in the previous section. In conclusion, the cost of a single simulation is $O(N)$, so the time complexity of the whole solution is $O(N^3)$.

**Final tweaks**

It is easy to see where the previous solution performs in a suboptimal way. It is doubtful whether the whole simulation can be performed in time sublinear in $N$; however, the number of simulations performed seems rather excessive. Indeed, there are a few ways of improving this process.

The first idea is to apply a binary search. We set the value of the parameter $a$ and look for the smallest such $b$ that the problem can be solved by removing at most $a$ upper horizontal slices, at most $b$ lower horizontal slices and, obviously,

$m$ vertical slices. Finding the minimal value of $b$ requires performing $O(\log N)$ simulations, each of which works in $O(N)$ time. We obtain an $O(N^2 \log N)$ time algorithm, which is not too far from being optimal.

To get an optimal solution, we apply the sliding window technique. Let us note that if, for a given value of the parameter $a = x$, the smallest value of the second parameter is $b = y$, then for $a = x + 1$ the corresponding value of $b$ does not exceed $y$. For $a = 1$ we can, therefore, find the requested $b$ by applying an exhaustive search, and for $a = 2, 3, \ldots$ we can simply consider consecutive values of $b$ obtained only by decreasing the previously considered values. Here the total number of simulations is linear in terms of $N$, so the whole algorithm runs in $O(N^2)$ time.

Same time complexity can be obtained with a smarter implementation of the initial greedy approach. It suffices to introduce just one of the auxiliary parameters, e.g. the parameter $a$. Then, according to the greedy approach, in the simulation we can freely remove any vertical slices and at most $a$ upper horizontal slices. Only if we cannot remove any slice of the aforementioned form do we choose to remove a lower horizontal slice. This modified simulation procedure still runs in $O(N)$ time, so the whole algorithm is quadratic in terms of $N$. And this solution to the task *Ploughing* is the one I admire the most.

**Exercise**

We conclude with an exercise for the reader. Note that the majority of the solutions described here do not work correctly if the board is allowed to contain negative numbers. Which solutions fall into this category? What are the corresponding counterexamples? How should this phenomenon be explained?

**WOJCIECH RYTTER**

Wojciech Rytter is a professor of computer science and the initiator of the algorithms section at the Faculty of Mathematics, Informatics and Mechanics, University of Warsaw. He is an author and co-author of over 100 publications and several textbooks on algorithmics.

His scientific interests are centered on the design and analysis of algorithms and data structures for parallel computing, discrete mathematics, graph theory and text algorithms as well as formal languages and automata theory. Wojciech Rytter is also interested in algorithms for heavily compressed objects that do not require decompression.

Wojciech Rytter is one of the co-founders of the Polish Olympiad in Informatics launched in 1993, and has been involved in this event since its inception. He is the author of a few dozen olympic programming tasks.

# / Canoes

**Contest:** 4th Polish Olympiad in Informatics
**Task author:** Wojciech Rytter
**Memory:** 32 MB
http://main.edu.pl/en/archive/oi/4/kaj

We are organizing a canoe tour. Canoes can be hired at the harbor. Canoes are all alike. A canoe can take at most two persons. The sum of weights of these persons may not exceed the fixed maximum weight. We want to pay as little as possible, so we should try to place all the participants of our tour in the minimum number of canoes.

## Task

Write a program that:

- reads from the input the maximum weight of a canoe crew, the number of participants of the tour and the weight of each participant,
- computes the minimum number of canoes that should be hired to accommodate all the participants of the tour according to the rules,
- writes the result to the output.

## Input

In the first line of the input there is one integer $w$ ($80 \le w \le 200$), specifying the maximum weight of a canoe crew. In the second line there is one integer $n$ ($1 \le n \le 30{,}000$), specifying the number of participants of the tour.

Each of the following $n$ lines contains one integer within the range $[5,w]$, representing the weights of the participants.

## Output

In the first line of the output one integer should be written: the minimum number of canoes that should be hired.



## Example

For the input data:

```
100
9
90
20
20
30
50
60
70
80
90
```

the correct result is:

```
6
```

# / Solution

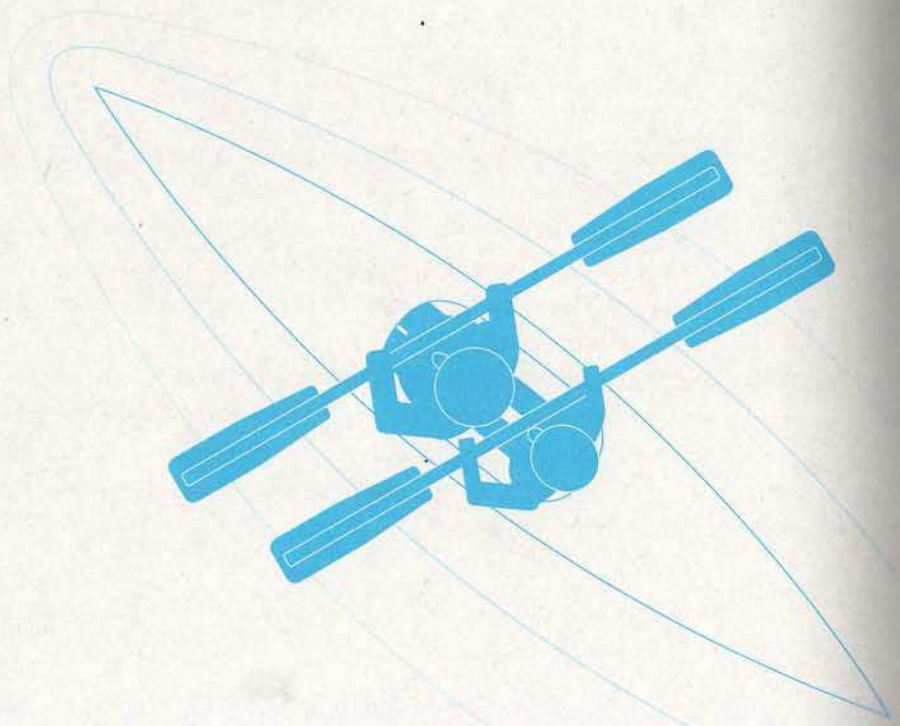
This is a simple problem that appeared in the Polish Olympiad in Informatics. A fast algorithmic solution is almost trivial, but not uninteresting. The algorithm is designed using a straightforward greedy approach:

---

**Algorithm** Greedy (informally)
  **while** there is still any person on the bank of the river **do**
    the heaviest of them is loaded into the next canoe with the person
    of the smallest weight, if they fit together;
    otherwise, only the heaviest one is loaded
  **return** the number of canoes used

---

In fact, we can place the heaviest person with anybody with whom he fits in a single canoe.





Why is such a strategy correct? Consider the following fact, which implies correctness immediately.

***Fact.*** There exists an optimal solution (configuration) in which the heaviest person, say Mr. H, is in the same canoe with the lightest person, say Mr. L, or alone.

*Proof.* There are two possibilities in a fixed optimal configuration:

1. Mr. H sails by himself: obviously we can take this configuration and our fact is self-evidently true.
2. Mr. H sails with some person $X$. Then, $X$ can exchange places with Mr. L, and we get a configuration that satisfies our requirements. □

The greedy algorithm can be written in the pseudocode as follows:

---

**Algorithm** Greedy
  {Input: array $weight[0..n-1]$, capacity of the canoe $MAX$}
  {Assumptions: $weight[i] \le MAX$ for $i = 0, 1, \ldots, n-1$}
  sort the array of weights in non-increasing order
  $i := 0$; $j := n-1$
  **while** $i \le j$ **do**
    **if** $weight[i] + weight[j] \le MAX$ **then**
      $j := j - 1$
    $i := i + 1$
  **return** $i$

---

The efficiency depends on the sorting method; we have chosen bucket sort. Thus, the whole algorithm has linear time complexity.

The simplicity of the solution disappears if the problem is slightly changed:

1. How should we count the number of optimal configurations?
2. What happens if we allow three people into the same canoe? How, then, do we efficiently compute the minimum number of canoes?

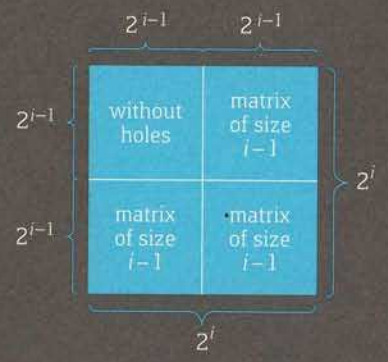
We suggest finding a polynomial time algorithm for the case where the weight array is a permutation of the numbers $1, \ldots, n$ ($MAX$ can be arbitrary). Observe that, in this case and for (at most) two-person canoes, there is exactly one solution if $n \le MAX \le n+1$, but for $MAX = 2n$ there are exponentially many solutions. What about $MAX = n+2$?

# / Painter's *Studio*

**Contest:** 5th Polish Olympiad in Informatics
**Task author:** Wojciech Rytter
**Memory:** 32 MB
http://main.edu.pl/en/archive/oi/5/mal

The Painter's Studio is preparing the mass production of paintings. Paintings are going to be made with the aid of square matrices of various sizes. A matrix of size $i$ consists of $2^i$ rows and $2^i$ columns. There are holes at the intersections of some rows and columns. A matrix of size 0 has one hole.

For $i > 0$, a matrix of size $i$ is composed of four squares of size $2^{i-1} \times 2^{i-1}$. Look at the following figure:




Both squares on the right side and the bottom-left square are matrices of size $i-1$. The top-left square has no holes. Pictures are constructed in the following way. First, we decide on the values of three nonnegative integers $n$, $x$, $y$. Next, we take two matrices of size $n$, place one of them onto the other and shift the upper one by $x$ columns right and $y$ rows up. We place such a pattern on a white canvas and cover the common part of the matrices with yellow paint. In this way, we get yellow stains on the canvas in the places where the holes in both matrices align.



**Example**
Consider two matrices of size 2.

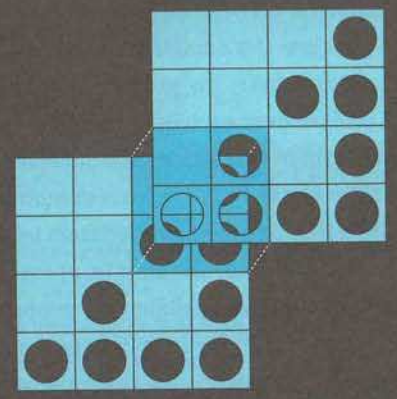


*Figure: The upper matrix was shifted by 2 columns right and 2 rows up. There are three places where holes align.*

**Task**
Write a program that:
- reads from the input the sizes of two matrices and the numbers of columns and rows by which the upper matrix should be shifted,
- computes the number of yellow stains on the canvas,
- writes the result to the output.

**Input**
There is one integer $n$ ($0 \le n \le 100$) in the first line of the input, specifying the size of the matrices used for the production of paintings. In the second line there is one integer $x$, and in the third line one integer $y$, where $0 \le x, y \le 2^n$. The integers $x$ and $y$ are the numbers of columns and rows by which the upper matrix should be shifted.

**Output**
The first and only line of the output should contain the number of stains on the canvas.

**Example**

| For the input data: | the correct result is: |
|---|---|
| 2<br>2<br>2 | 3 |

# / Solution

For this task we use the general conceptual technique of *dynamic programming*. The solution of a given problem is a combination (in our case, sum of values) of several subproblems. In this case, the problem is to compute the size of the overlap of two squares shifted over each other: let $N((x,y),n)$ be the *number* of common holes in the overlap of both matrices, the second matrix shifted by $(x,y)$ over the first one. Denote by $M_n$ the $2^n \times 2^n$ matrix containing holes as described in the task statement. The columns are numbered from left to right and the rows from bottom to top, starting from zero. The shift $(x,y)$ corresponds in the matrix to the position $(x,y)$ called the *active position*. The subproblems correspond to overlaps of matrices of shape $2^i \times 2^i$, for $i \le n$, where the number $i$ is called the *rank* of the problem (and of the matrix). The main point is the dependence of overlaps of rank $i$ on a constant number of overlaps of rank $i-1$.

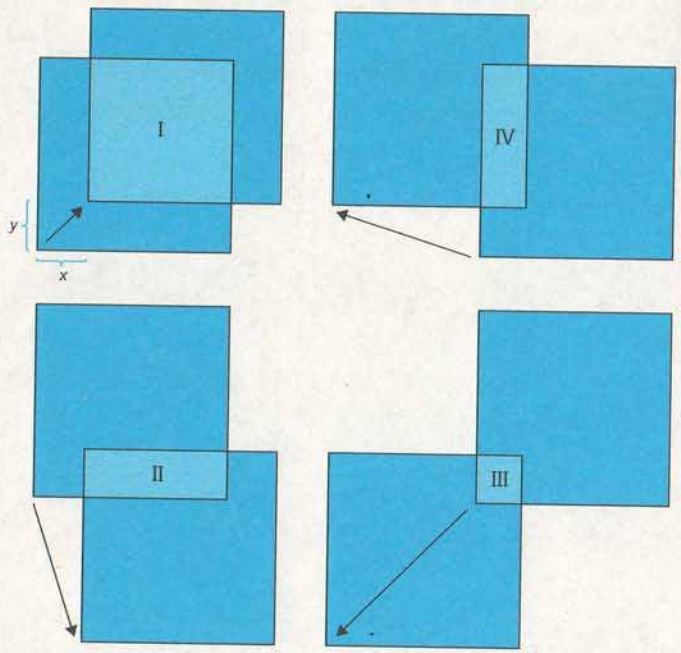



***Figure 1:*** *The common parts of type I, II, III and IV of two matrices shifted one over the other. The arrows correspond to shifts $(x,y)+\beta_k(i)$, for $k=1,2,3,4$. $W_k(i)$ is the number of common holes in the corresponding common part.*



Define:

$$\beta_1(i) = (0,0), \quad \beta_2(i) = (0,-2^i), \quad \beta_3(i) = (-2^i,-2^i), \quad \beta_4(i) = (-2^i,0);$$

$$W_k(i) = N((x,y)+\beta_k(i),i), \quad \text{for } k = 1, 2, 3, 4.$$

Instead of computing a single value we want to compute the 4-tuple

$$W(i) = [W_1(i), W_2(i), W_3(i), W_4(i)].$$

In other words, we compute the number of common holes in the overlapping parts I, II, III, IV shown in figure 1.

The main idea is to express *in a simple way* the 4-tuple $W(i)$ in terms of a 4-tuple $W(i-1)$, corresponding to the matrix of the smaller rank. The 4-tuple $W(i)$ is located with respect to the active position $(x,y)$, while the new 4-tuple is considered with respect to a new active position, $(x',y') = reduce(x,y)$. The coordinates of the position $(x',y')$ are the coordinates of $(x,y)$ with respect to the quarter of the matrix $M_i$ in which $(x,y)$ is located. The operation *reduce* is computable in constant time.

The value of the new 4-tuple corresponding to the matrix of rank $i-1$ is similarly determined by the location of the active position $(x',y')$ in this matrix, as before.

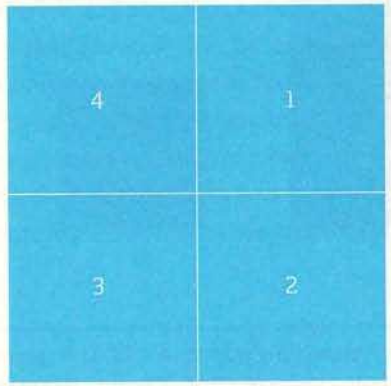



***Figure 2:*** *Numbering of quarters.*

Hence, knowing four numbers related to objects of rank $i-1$, we want to compute in constant time four numbers related to objects of rank $i$, and can write

$$W(i) = \Phi(W(i-1), x, y).$$

The crucial point is to find the dependency $\Phi$ of $W(i)$ with respect to $W(i-1)$. The form of $\Phi$ depends crucially on the number of the quarter of the matrix in which the position $(x,y)$ is located. Hence we switch to a quartile representation of positions in matrices. Let us number the quarters of the matrix as in figure 2.

In this representation, the operation of reducing $(x,y)$ to $(x',y')$ has an especially simple form. We describe how to compute the quartile representation $\lambda = address(x,y)$.

Initially, $\lambda$ is an empty sequence. The initial matrix is of rank $n$, so its shape is $2^n \times 2^n$. Its active position belongs to one of four quarters. Let us write the number $p$ of this quarter at the beginning of the address: in other words, $\lambda := p \cdot \lambda$. Let us remove the other quarters of the matrix. We get a new matrix of a smaller rank, and the active position is now located in a possibly different way with respect to the beginning of this new matrix; its new coordinates are $(x',y')$. We repeat this process until we arrive at a $1 \times 1$ matrix (see figure 3).

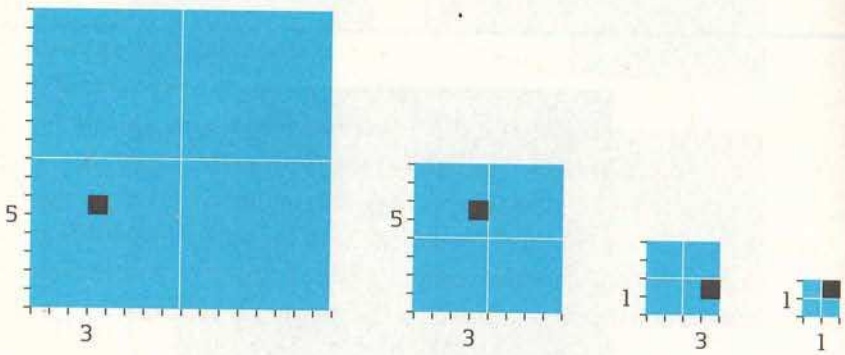



***Figure 3:*** *The sequence of consecutive positions of* $(3,5)$ *in quarters of decreasing ranks. This position is successively in quarters* 3, 4, 2 *and* 1. *Hence* $address(3,5) = (1,2,4,3)$ *in the matrix of rank* 4. *The addresses (quartile representations) of successive positions result from cutting off the last digit.*



The sequence of consecutive numbers of quarters written in reverse order is the quartile address of $(x,y)$ denoted by $address(x,y)$.

We now show the dependence of $W(i)$ on $W(i-1)$ for the case when $(x,y)$ is in the first quarter of the matrix of rank $i$. In this case, the position $(x',y') = reduce(x,y)$ is of the form:

$$(x-2^{i-1}, y-2^{i-1}).$$

Let us express the numbers $W_1(i)$, $W_2(i)$, $W_3(i)$, $W_4(i)$ in terms of $W_1(i-1)$, $W_2(i-1)$, $W_3(i-1)$, $W_4(i-1)$. It is convenient to represent this dependence by $4 \times 4$ tables, called dependency tables. We introduce our first dependency table:

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix},$$

$$\begin{aligned}
W_1(i) &= a_{11} \cdot W_1(i-1) + a_{12} \cdot W_2(i-1) + a_{13} \cdot W_3(i-1) + a_{14} \cdot W_4(i-1), \\
W_2(i) &= a_{21} \cdot W_1(i-1) + a_{22} \cdot W_2(i-1) + a_{23} \cdot W_3(i-1) + a_{24} \cdot W_4(i-1), \\
W_3(i) &= a_{31} \cdot W_1(i-1) + a_{32} \cdot W_2(i-1) + a_{33} \cdot W_3(i-1) + a_{34} \cdot W_4(i-1), \\
W_4(i) &= a_{41} \cdot W_1(i-1) + a_{42} \cdot W_2(i-1) + a_{43} \cdot W_3(i-1) + a_{44} \cdot W_4(i-1).
\end{aligned}$$

In other words, the rows of the table are the multiplicities of components of $W(i-1)$. The $k$-th row of $A_j$ is the sequence of dependency coefficients of $W_k(i)$ from $(W_1(i-1), W_2(i-1), W_3(i-1), W_4(i-1))$, assuming that $(x,y)$ is in the $j$-th quarter of the current matrix. The index of the table (in this case it is 1, since we are considering the first quarter) corresponds to the number of the quarter containing the active position. We can see here the useful properties of the quartile system. When the active position is in the first quarter, it is easy to see, by considering the mutual placements of the quarters of both matrices, that:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Example 1.** We show how the first row of $A_3$ is computed (by hand). In this case, $(x, y)$ is in the third quarter. The common part consists of three smaller parts of type I, and one of each of the types II, III and IV (in the sense of overlaps of matrices of rank $i-1$); see figure 4. Hence the first row of $A_3$ is $[3,1,1,1]$. Other rows of $A_3$ are computed by shifting the overlapping matrix of rank $i$ in three remaining directions over the first matrix of rank $i$, as in figure 1.

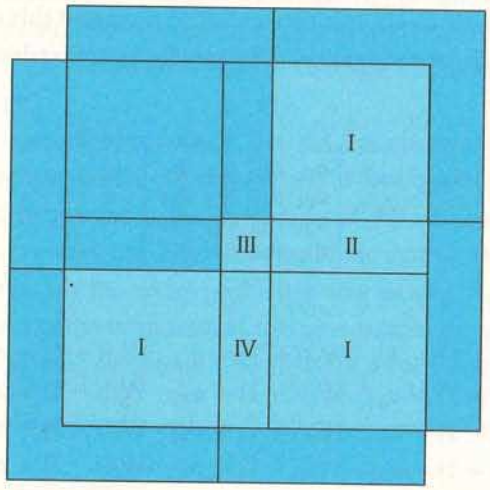



***Figure 4:*** *The first row of $A_3$ is $[3,1,1,1]$; this means that if the left-bottom corner of the upper matrix is in the first quarter, then we have three smaller overlaps of type I, and one of each of the types II, III and IV, corresponding to matrices of smaller rank. (The parts which are not labeled contain no holes, and can be ignored.)*

In a similar way, we construct *by hand* the tables $A_2$, $A_3$, $A_4$, thus obtaining

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 3 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



**Example 2.** Assume $(x, y)$ is in the fourth quarter. According to $A_4$, the computation of $(W_1(i), W_2(i), W_3(i), W_4(i))$ consists in executing:

$$W_1(i) := W_1(i-1) + W_4(i-1); \quad W_2(i) := W_1(i-1) + 3 \cdot W_2(i-1) + W_3(i-1);$$

$$W_3(i) := W_3(i-1) + W_4(i-1); \quad W_4(i) := 0.$$

To simplify the notation, we introduce the operation $\otimes$, which transforms one 4-tuple into another using a table:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \otimes \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix},$$

$$\begin{aligned} r_1 &= v_1 \cdot a_{11} + v_2 \cdot a_{12} + v_3 \cdot a_{13} + v_4 \cdot a_{14}, \\ r_2 &= v_1 \cdot a_{21} + v_2 \cdot a_{22} + v_3 \cdot a_{23} + v_4 \cdot a_{24}, \\ r_3 &= v_1 \cdot a_{31} + v_2 \cdot a_{32} + v_3 \cdot a_{33} + v_4 \cdot a_{34}, \\ r_4 &= v_1 \cdot a_{41} + v_2 \cdot a_{42} + v_3 \cdot a_{43} + v_4 \cdot a_{44}. \end{aligned}$$

Using all these notations, the pseudocode of our algorithm is as follows.

```
Algorithm Compute-N((x,y),n)
    compute the quartile address of (x,y)
    W(0) := [1,0,0,0]
    for i := 1 to n do
        k := the i-th digit of the quartile address
        W(i) := A_k ⊗ W(i-1)
    return W_1(n)
```

We set the initial values:

$$W_1(0) = 1 \quad \text{and} \quad W_2(0) = W_3(0) = W_4(0) = 0.$$

It is easy to check that they generate correct values of $W(1)$ for all four possible shifts $(x,y)$ in the $2 \times 2$ matrix. We have:

$$address(x,y) = 1 \Rightarrow W_1 = [1,0,1,0], \quad address(x,y) = 2 \Rightarrow W_1 = [1,0,0,1].$$

$$address(x,y) = 3 \Rightarrow W_1 = [3,0,0,0], \quad address(x,y) = 4 \Rightarrow W_1 = [1,1,0,0].$$

***Example 3.*** For the $16 \times 16$ matrix and the shift $(3,5)$ (compare with figure 3), the correct answer is 14. The algorithm computes it as follows:

| | |
|---|---|
| initialization: | $address(3,5) = (1,2,4,3)$ |
| iteration 1: | $A_1 \otimes [1,0,0,0] = [1,0,1,0]$ |
| iteration 2: | $A_2 \otimes [1,0,1,0] = [1,0,1,2]$ |
| iteration 3: | $A_4 \otimes [1,0,1,2] = [3,2,3,0]$ |
| iteration 4: | $A_3 \otimes [3,2,3,0] = [14,2,3,0]$ |
| termination: | $output = 14$ |

**KRZYSZTOF STENCEL**

Krzysztof Stencel is an associate professor at the Institute of Informatics, University of Warsaw. He is interested in databases and software engineering. He is an author of numerous publications, textbooks and translations.

As a contestant, he has taken part in algorithmic competitions only once. In 1989 he represented Poland during the first International Olympiad in Informatics. However, he has been much more involved in such competitions as a judge and organizer. Between 1993 and 2008 he was the chairman of the jury of the Polish Olympiad in Informatics. He also chaired the jury of several international contests: Central European Olympiad in Informatics (1997, 2004), Baltic Olympiad in Informatics (2000, 2001, 2008) and International Olympiad in Informatics (2005). In 2011 he was the President of the Central European Olympiad in Informatics.

In 2008 Krzysztof Stencel began solving programming tasks on the famous contest server of University of Valladolid for fun. Between September 19, 2011, and the time of writing this text, he has been the leader in the rankings, with over 3000 algorithmic problems solved.

# / Triangles

**Contest:** 1st Polish Olympiad in Informatics
**Task author:** Piotr Chrząstowski
**Memory:** 16 MB
http://main.edu.pl/en/archive/oi/1/tro

A finite, at least three-element set $A$ of rational-length line segments is given. We want to examine whether a triangle may be created from every three segments in $A$.

### Input

The input contains a sequence of lengths of all line segments in the set $A$. Every segment length is a rational number written in the form $a/b$, where numerator and denominator are positive integers not greater than 9999. Consecutive lengths in this sequence are separated by single spaces.

### Output

Your program should give one of the following answers: TAK (Polish for *yes*) if a triangle may be formed from every three line segments of $A$, or NIE (Polish for *no*) if some three line segments of $A$ cannot be used to create a triangle.

### Examples

For the input data:

13/10 1/2 6/5 11/6 9/7 3/5 9/7 13/10 9/5 8/5

the correct result is:

NIE

whereas for the input:

1/2 3/5 2/3 4/7 1/1 4/6

the correct output is:

TAK

**Explanation of the first example:** One cannot create a triangle from line segments of lengths 6/5, 3/5 and 9/5.



# / Triangles 2

**Contest:** 2nd Polish Olympiad in Informatics
**Task author:** Piotr Chrząstowski
**Memory:** 32 MB
http://main.edu.pl/en/archive/oi/2/tro

In a finite sequence of positive integers representing the lengths of line segments, we want to find three numbers such that a triangle can be created from segments of such lengths.

### Task

Write a program that examines whether there exist three line segments—whose lengths are supplied in the input—from which one can create a triangle. If so, the program should write the lengths of those three segments in the output. If no such set of three segments exists, the program should write one word, NIE (Polish for *no*), in the output file.

If there exist multiple such sets of three line segments from which triangles may be created, then your program should find and write only one of them (the choice of set is arbitrary).

### Input

In the input there is a finite sequence of at least three positive integers not greater than 1,000,000,000, terminated by the number 0. Each number is written on a separate line. These integers are lengths of line segments, and the number 0 denotes the end of the data.

### Output

In the output there should be either one word, NIE, or three line segment lengths chosen from the input, from which one can create a triangle. The lengths are separated by single spaces.

**Examples**

For the input data:

105
325
55
12555
1700
0

the correct result is:

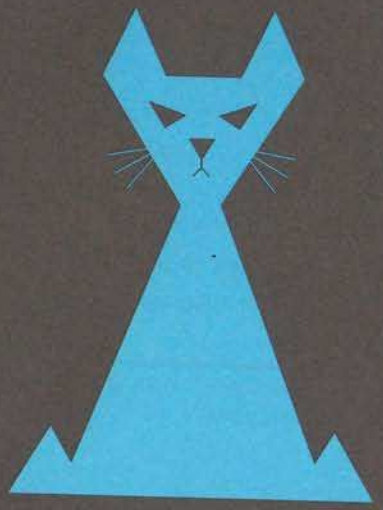
NIE

whereas for the input:

250
1
105
150
325
99999
73
0

the correct output is:

250 105 150





# / Solution

**Triangles**

The lengths of the sides of each proper triangle on the Euclidean plane satisfy so-called *triangle inequality*. If the sides of a triangle have lengths $a$, $b$, $c$ and $c$ is not smaller than the others, then it is true that

$$a + b > c. \tag{1}$$

This inequality is so common that it is hard to imagine basing upon it an interesting problem for programming contests. Surprise! Piotr Chrząstowski managed to prepare two interesting problems based solely on triangle inequality for the 1st and 2nd Polish Olympiads in Informatics. Personally, I am not a problem setter at all; I have set problems neither for the national Olympiad nor for Algorithmic Engagements. For years I have been a judge of programming contests, however. Nowadays, my hobby is to solve problems on the famous server of the University of Valladolid. I have solved a significant number of them; yet, my ability to set new problems is nil. This makes my esteem for problem setters very high: especially for those like Piotr Chrząstowski, who can make a nice contest problem out of almost nothing. They are like brilliant poets who can find subject matter under every stone, while we ordinary people can only trample the stones underfoot.

The input to the task from the 1st Olympiad is a sequence of segments of rational lengths. The problem consists in checking whether, from each three of those segments, one can form a proper triangle in Euclidean geometry. Note that—fortunately—only in the 1st Olympiad were contestants requested to check the correctness of the input data. If they had been faulty, the program would have printed the message NONSENSE. It was a nonsense indeed, and in subsequent Olympiads a contestant could assume that input data were correct. This rule is currently commonly adopted in all programming contests all over the world*.

Let us return to our problem. We do not have to check each trio of segments; it would be too expensive. Instead, we initially compute the length of the longest segment $c$ and the lengths of the two shortest segments $a$ and $b$ ($a \le b$). Then we check whether they satisfy the inequality (1). If not, we have a trio of segments

* That is why the task statement presented in this book has been modified to remove the requirement to check the validity of the input data.

that do not form a proper triangle. However, if this inequality holds, all other trios also form correct triangles. In order to prove it, we pick three arbitrary segments $a_1, b_1, c_1$ from the input sequence. We will assume that $a_1 \le b_1 \le c_1$. Since $a$ and $b$ are the shortest and $c$ is the longest, then $a \le a_1$, $b \le b_1$ and also $c \ge c_1$. Thus, using (1) we know that:

$$a_1 + b_1 \ge a + b > c \ge c_1.$$

Therefore, the trio $a_1, b_1, c_1$ satisfies the triangle inequality. Since the choice of $a_1, b_1, c_1$ was arbitrary, we know that all trios of segments form proper triangles.

The algorithm to solve this problem is now obvious: read the data; find the two shortest segments and the longest segment; then check if they satisfy (1) and output the answer. This means linear time and constant space consumption.

Can you believe that no contestant presented a program that scored the maximum number of points? Why not? Because we have forgotten the mouse traps in the input data. They are rational numbers with numerators and denominators up to 9999. "So what? Only $10^4$?" you ask. When comparing two rational numbers with such denominators, they can reach only $10^8$ and comfortably fit in a 32-bit signed integer. This is what our diabolic problem setter counted on! Eventually, we have to compute the answer. Assume that the smallest numbers are $\frac{n_1}{d_1}$ and $\frac{n_2}{d_2}$, while the largest is $\frac{n_3}{d_3}$. We have to check whether

$$\frac{n_1}{d_1} + \frac{n_2}{d_2} > \frac{n_3}{d_3}.$$

After we multiply both sides by $d_1 d_2 d_3$, we get

$$n_1 d_2 d_3 + n_2 d_1 d_3 > n_3 d_1 d_2.$$

Input data are from the range 1 to 9999. Thus we can obtain numbers of magnitude $10^{12}$, which is too large for 32 bits!

Twenty years ago, tricking the contestants in this way was even easier! A machine word usually contained only two bytes, and the same was also true of the standard integer type. Thus, it was very easy to overlook the magnitude of the values to be processed. Tests were chosen in such a way that precision errors in the conversion of rational to floating point (*real* or *double*) caused wrong answers. No student had yet heard about epsilons needed to compare floating point values. This is no surprise. Even today, many programmers still believe in the perfect precision of such data types.

As you already know, there was no happy ending to this story. One student identified the problem with data types and implemented his own arithmetic to cope with it. Unfortunately, instead of computing maximum and minimum values on the fly, he initially loaded all the data into main memory. This could not succeed, since the input data and memory limits were set up to prevent such solutions from getting full points. Thus, this contestant also had to settle for less than the maximum score. So who's laughing at triangle inequality now?

**Triangles Reloaded**

The triangular problem from the 2nd Olympiad in Informatics is dual. Given a sequence of lengths of segments, you are to find a trio of them that forms a proper triangle; otherwise, your program has to declare the nonexistence of such a trio. Again, we do not have to check all possible trios. Let us sort our segments in increasing order. A triangle can be formed if, in the sorted sequence $a_1, \ldots, a_n$, there exist three neighboring items $a_k, a_{k+1}, a_{k+2}$ such that:

$$a_k + a_{k+1} > a_{k+2}. \tag{2}$$

Indeed, assume there are three elements of our ordered sequence $a_r, a_s, a_t$ such that $a_r + a_s > a_t$ and $r < s < t$. Note that $a_{s+1} \le a_t$ and $a_r \le a_{s-1}$. Thus:

$$a_{s-1} + a_s \ge a_r + a_s > a_t \ge a_{s+1}.$$

Therefore it holds that $a_{s-1} + a_s > a_{s+1}$, so $a_{s-1}, a_s, a_{s+1}$ are three neighboring items of the sequence that satisfy the triangle inequality. We have proven that if the input sequence contains three segments that form a triangle, then in the sorted input sequence there will be three appropriate neighboring segments. In order to solve the problem, we sort the input data and check whether the inequality (2) holds for any k. The algorithm consumes $O(n \log n)$ time and linear space.

It remains only to collect our pay check, but to our surprise they say that they will pay only for constant time complexity (of course, in this case the space complexity must also be constant). Have we missed something? Yes! The range of input data. The lengths of the segments are positive integers up to $10^9$. If there are no trios forming proper triangles, in the sorted input sequence each item (except the first two) must be no smaller than the sum of its two predecessors. (Who called up Fibonacci's spirit again? Anyone still not convinced about *The Da Vinci*

*Code*?) Among all possible input sequences that have no trio forming a triangle, the least ascending is the following:

1, 1, 2, 3, 5, 8, 13, 21, 34,...

This sequence increases very quickly, and its 45th element is already greater than $10^9$: its value is 1,134,903,170. Thus it is enough to read only 45 first input numbers, sort them and find a trio that satisfies (2) or announce its nonexistence. The time needed by this algorithm is thus constant; we do not have to read all the input data.

The contestants of the 2nd Olympiad solved this problem much better than the previous one. Almost half of them got full marks. Nearly everyone noted that the length of the sequence without any triangle was limited; the estimations varied from 43 to 16,000. That lower limit was in fact too small, and was caused by neglecting initial repeating values. Of course, the input data contained a test case where the 44th item closed a triangle and made such a solution lose points. That upper limit was too big, since the time limit disallowed the reading of so much data. Remember that it was almost twenty years ago, and computers were significantly slower than today. Nowadays, in many problems you can still find a warning like: the input data is huge, use faster input/output methods.

**Triangles Revolutions**

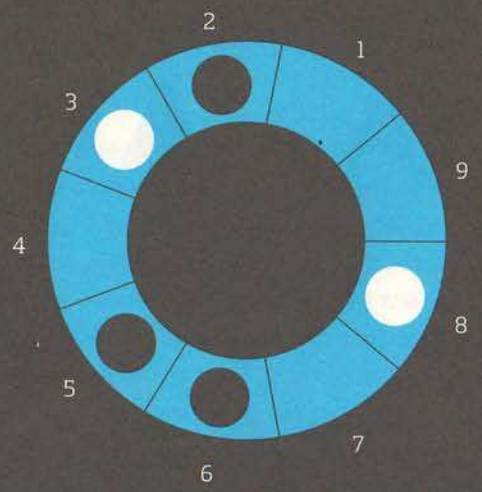
In 2009 three former contestants of the Polish Olympiad in Informatics founded Codility. This company offers automatic interviewing facilities for businesses that hire programmers. One of the Codility tests is almost the same as the second triangle problem discussed above; the only difference is that the answer is only YES or NO, depending on whether one can form a triangle from input items or not. The Codility servers now have 3,500 solutions to this problem, submitted by IT professionals from 85 countries. Only 4.5% of them have noticed that it is sufficient to check only trios of neighbouring items in the sorted sequence. About 54.5% of candidates have been unable to submit a fully correct solution, even if we consider very slow programs. Unfortunately, so far none of the programmers tested by Codility have noticed that if the number of segments were big enough (here: 45), the answer to the problem would always be positive.

On the other hand, programmers can be lucky with this problem. They can present an optimal solution without even noticing the aforementioned limit. It is enough to read numbers one by one and register them in a sorted sequence. If, at each number, the program checks whether the number forms a triangle together with its neighbours, it will lead to the optimal solution. We already know that such a procedure cannot last for too long, and either answers that a triangle cannot be formed or finds such a triangle in the 45th step at the latest. Unfortunately, programming contests require only correct and fast programs; no-one cares for proofs of correctness or complexity. It is only effectiveness that counts.

# / Circular *Game*

**Contest:** Algorithmic Engagements 2009
**Task author:** Tomasz Idziaszek
**Memory:** 64 MB
http://main.edu.pl/en/archive/pa/2009/gra

In the *Circular Game*, the board consists of $m$ fields arranged on a circle and numbered from 1 to $m$. On the board there are $b$ white and $c$ black pieces, at most one on each field. Two players are playing the game: the white player and the black player. Starting with the white player, the players perform their moves on the board alternately. A move consists of moving a piece of the player's color any number of free fields forward or backward. For instance, for the board depicted below, the white player can move the piece from field 3 to field 4 or the piece from field 8 to any of the fields 7, 9 and 1.




If a player can perform no move in his turn, he loses. Knowing that both players play optimally, check who will win the game. It is possible for neither player to win (the game never ends).



**Input**

The first line of the input contains one integer $t$ representing the number of boards to be considered. The following lines contain descriptions of the respective boards, each of which consists of three lines. In the first line there are three integers, $m$, $b$ and $c$ ($1 \le m \le 10^9$, $1 \le b, c$), separated by single spaces and specifying the length of the board, the number of white pieces and the number of black pieces. In the second line there is an increasing sequence of $b$ integers (in the range 1 to $m$) representing the positions of the white pieces. In the third line there is an increasing sequence of $c$ integers (in the range 1 to $m$) representing the positions of the black pieces. The total number of pieces in all boards does not exceed $10^6$.

**Output**

Exactly $t$ lines, containing answers for the consecutive boards, should be written to the output. The answer is always a single character: B, C, or R, depending on whether the white player wins (B), the black player wins (C) or the game never ends (R).

**Example**

For the input data:

```
3
9 2 3
3 8
2 5 6
6 2 2
5 6
2 4
7 1 1
3
4
```

the correct result is:

```
C
B
R
```

# / Solution

### Friends and relations of Nim

Consider the classic game of *Nim*. Pieces in this game are divided into $k$ heaps of size $n_1, n_2, \ldots, n_k$. Two players take turns, one after the other, and in each turn the player has to remove some positive number of pieces from one of the heaps. The player who collects the last piece wins. If we express the sizes of heaps as binary numbers, and at each position we count ones and all these counts are even, then the game is lost for the player whose turn it is. Namely, if the opponent plays perfectly, our poor player will be doomed. More formally, if the bitwise *xor* of numbers $n_1, n_2, \ldots, n_k$ is zero, our opponent has a winning strategy. Hence, everything depends on the value of the following expression (the symbol $\oplus$ denotes the bitwise xor):

$$n_1 \oplus n_2 \oplus \cdots \oplus n_k.$$

We can also permit the addition of new pieces to heaps, and under a certain condition we will still have a game equivalent to Nim. This condition is a limit for the number of reverse moves (addition of pieces) performed by players. Note that if a player has a losing position, he or she can try adding some pieces somewhere in order to avoid the loss. Unfortunately, the opponent will easily annihilate these efforts simply by removing the added pieces. Since the number of additions is limited, the losing player cannot prolong this procedure forever and will eventually lose. We have just described *Northcott's Game*. It can also easily be explained using a chessboard. In each of its columns we place two checkers: white and black. Checkers can move arbitrarily within their initial column, with a condition that they cannot jump over one another. The distance between two checkers corresponds to the size of a heap. Moving a checker towards the enemy position is like removing pieces from Nim heaps. Moving backward is equivalent to the addition of pieces. However, the board is finite; thus it is impossible to retreat endlessly. Northcott's Game is therefore a variant of Nim.



### Analysis of the Circular Game

Wait a moment! Black and white checkers? Moving, not jumping? This is almost our *Circular Game*! Easy! There are some differences; notably, the problem statement allows for the game to be tied. Of course, there are also some contest problems, whereby some answers are just baits for incautious contestants—e.g. one is either to find a sequence or answer NO if no such sequence exists. After the contest, during the discussion of the problem, it emerged that such a sequence always exists—resulting in much wailing and gnashing of teeth! However, that's not a part of this solution. Note that if a player has a row of at least three pieces without any opponent piece between them, and there is at least one free field among them, the player cannot lose as it is always possible to move the pieces inside such a structure. If both players have such a strong formation, the game will be tied.

**Partition into blocks.** Let us formalize the problem a little. A maximal sequence of a player's pieces, arranged such that there is no opponent piece between any of them, shall be called a *block* and denoted as follows:

$$\langle d, n, l \rangle,$$

where $d$ is the length of the block (the number of fields between its extreme pieces inclusive), $n$ is the number of pieces and $l$ is the length of the free space after the block (see figure 1). The state of a game can be depicted as the sequence of blocks on the board. The rotational symmetry of the board allows any block to be taken as the first one. The similarity to Northcott's Game is now apparent. A player moving their pieces can reduce or enlarge the space around their blocks. As in Northcott's Game, a move shrinking our blocks never leads to victory, since we cannot perform it forever. If all blocks consist of two pieces, they can be used to conquer territory (i.e. to collect pieces from heaps); our game simply reduces to Northcott's Game and eventually Nim. If the board contains $k$ such blocks with gaps $l_1, l_2, \ldots, l_k$, the result of the game will depend on the value of the following expression:

$$l_1 \oplus l_2 \oplus \cdots \oplus l_k. \qquad (1)$$

If it is zero and our opponent plays faultlessly, we will lose. Otherwise, we have a winning strategy.

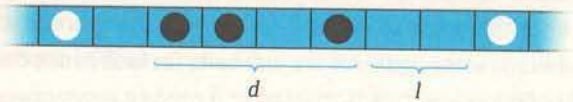



*Figure 1: Sample block for $d = 4$, $n = 3$, $l = 2$.*

**Strong blocks.** However, there may be blocks with some other number of pieces than two. Let us start with more abundant blocks, and suppose that every block on the board has at least two pieces. The key role is played by blocks with more than two pieces whose length is bigger than their number of pieces. We will call them *strong blocks*. A player who has a strong block cannot lose, since pieces can always be moved inside such a block. Moreover, that player will be the winner if the opponent has no strong block and cannot form one. A strong block is, in fact, like a tank: we can use it to crush any weaker squad of the enemy. In order to foretell the result of the game, we count the number of strong blocks possessed by each player and the number of strong blocks that can be formed in one move. We can form a strong block if a non-strong block with a least three pieces has free space before or after itself.

1. If Black has a strong block, or at least two moves forming such a block, it cannot lose. Thus if White has a strong block, or a move to form one, the game will be tied. Otherwise Black loses.
2. Assume Black has no strong block and no move to form one. If White has a strong block or a move to form one, White wins. Otherwise, the result depends on the Nim-value (1).
3. Now assume that Black has no strong block, but can make exactly one move to form such a block. White moves first, thus has the option to block this move. If White has a strong block, it will prevent Black from forming a strong block and thus win. If White is able to form a strong block, it can form one to assure a tie, or prevent the creation of the Black strong block in order to pursue victory. Therefore, we check the result of the game after this prevention move (technically we reverse the colors of the pieces). Note that each player can have only one such move (other possibilities are considered in the previous two points), so the recursive descent will be performed at most twice.



**Singular blocks.** Now it remains to discuss blocks containing single pieces. We will call them *singular*. Their fate depends on the surrounding blocks. If the board contains only singular blocks, and each player can move, the game will be tied. Singular blocks are weak, so they are not able to conquer the territory. It can also happen that, in the initial board, White cannot move or can block the only possible move of Black. Such a game of course is not tied.

If singular blocks lie between two non-singular blocks of the opponent's color, they are surrounded in a deadly formation. The owner of these two opposite blocks can easily squeeze what is inside them. We can thus remove such singular blocks. Remember also to remove the space they occupy, because, although deceased, they will always engage the field in which they appear. However, we can achieve the same goal simply by reversing the color of late singular blocks. Consider the following group of blocks:

$$\langle d_i, n_i, l_i \rangle \langle d_{i+1}, n_{i+1}, l_{i+1} \rangle \cdots \langle d_{i+k}, n_{i+k}, l_{i+k} \rangle, \tag{2}$$

where $n_i > 1$, $n_{i+k} > 1$ and $n_{i+j} = 1$ for each $j = 1, 2, \ldots, k-1$.

If $k$ is even, extreme blocks belong to the same player. We replace this group of blocks by the following block:

$$\left\langle \sum_{j=i}^{i+k} d_j,\ n_{i+k} + \sum_{j=i}^{i+k-1} (n_j + l_j),\ l_{i+k} \right\rangle.$$

This operation is associative. Therefore, if we can perform a number of such reductions, their order will not matter.

If $k$ is odd, the singular blocks are inside non-singular blocks of different players. The elimination of these single blocks requires an additional observation. For the player who controls the left end of the group, it only pays to move pieces rightward. For the other player, it pays to move pieces in the opposite direction. The pairs of neighboring pieces will change into fighting pairs of Northcott's checkers. The group of blocks (2) can be thus transformed into the following sequence:

$$\langle d'_i, n'_i, l'_i \rangle \langle d'_{i+1}, n'_{i+1}, l'_{i+1} \rangle \cdots \langle d'_{i+k}, n'_{i+k}, l'_{i+k} \rangle,$$

where for each $j = 1, 2, \ldots, k-1$ it holds that

$$d'_{i+j} = n'_{i+j} = 2$$

and

$$l'_{i+j} = \begin{cases} 0 & \text{if } j \text{ is odd,} \\ l_{i+j} & \text{if } j \text{ is even.} \end{cases}$$

**The solution**

At last we know how to deal with blocks containing any number of pieces. Therefore we can formulate the solution to the problem. It consists of the following steps:

1. the computation of the initial partition into blocks,
2. the check whether White has at least one move or can block the single possible move of Black,
3. the elimination of singular blocks according to the two last prescriptions,
4. the analysis of existing and potential strong blocks,
5. if strong blocks do not resolve the game, the computation of the value of the expression (1) for remaining gaps between blocks.

All these steps can be performed in linear time and linear space with respect to the number of pieces on the board.

**Epilogue**

A similar task, but on a smaller (up to 60 fields) and linear (non-circular) board, was used in 1998 at ACM-ICPC South Pacific Regional in New Zealand. I could not find the exact results of that contest on the Internet; however, it appears that no-one solved the problem. It is indeed not easy; nonetheless, it can easily be transformed into the problem discussed above. It is enough to add a field at each end of the board and put there a piece of the same color as the extreme piece from this side. For example, if the leftmost piece is white, we will add a field on the left with a white piece. After this operation we glue together the ends of the board to make it circular. The resulting circular game is equivalent to the original linear game.

It could be interesting to ask for the result of the game on other shapes of board: A planar graph? An arbitrary graph? A tree? A cactus? A Euro 2012 ball? To be continued at the next ACM-ICPC in Poland...

**WOJCIECH ŚMIETANKA**

Wojciech Śmietanka is a fifth-year student of computer science at the Faculty of Mathematics, Informatics and Mechanics, University of Warsaw. His scientific interests include algorithmics, number theory and combinatorics. As far as programming competitions are concerned, Wojciech Śmietanka was a finalist of TopCoder Open in 2010 and in the same year qualified for the Google Code Jam finals. However, he did not participate in the latter after all, as he was doing an internship at Google at the time. He has been taking part in team programming competitions since 2007. Together with his team, he has won the Polish Collegiate Programming Contest three times and the ACM-ICPC Central European Regional Contest twice. He also represented the University of Warsaw at the ACM-ICPC World Finals in 2011 and 2012. His non-scientific hobbies include history, economics, dance, volleyball and basketball.

# / Byteland

**Contest:** 1st Junior Polish Olympiad in Informatics
**Task author:** Wojciech Śmietanka
**Memory:** 32 MB
http://main.edu.pl/en/archive/oig/1/baj

King Byteasar the Great is the ruler of the rich and mighty country of Byteland. There are $n$ towns in the country. King Byteasar would like to improve the infrastructure of Byteland, and has ordered the royal architects to prepare plans for building highways across the country. He has received $m$ proposals, each of which is described by three numbers, $p$, $k$, $w$, where $p$ and $k$ are the towns at the endpoints of the highway, and $w$ denotes the cost of building this highway. Each highway is bidirectional and does not visit any town except for its endpoints.

The king would like to choose the highways so as to create at least one way of traveling between any pair of towns, possibly by visiting several towns along the way. Byteasar would also like to build the highway network as cheaply as possible.

## Task

Write a program that:

- reads from the input the number of towns, the number of proposed highways and the descriptions of those highways,
- finds out, for each highway, whether a highway network exists that contains this highway and satisfies Byteasar's requirements,
- writes the results to the output.

## Input

The first line of the input contains two integers: the number of towns, $n$, and the number of proposed highways, $m$, separated by a single space and satisfying the conditions: $2 \le n \le 7000$, $1 \le m \le 300{,}000$. Each of the following $m$ lines contains three space-separated integers, $p$, $k$, $w$, describing the proposed highway. Here $p$ and $k$ denote the numbers of the towns at the endpoints of the highway, and $w$ is the price of the highway ($1 \le p, k \le n$, $1 \le w \le 100{,}000$).



## Output

The output should consist of $m$ lines. The $i$-th of those lines should contain a word, TAK (Polish for *yes*) or NIE (Polish for *no*), depending on whether a highway network exists that contains the $i$-th highway from the input and satisfies Byteasar's requirements. You can safely assume that at least one highway network satisfying Byteasar's requirements exists.

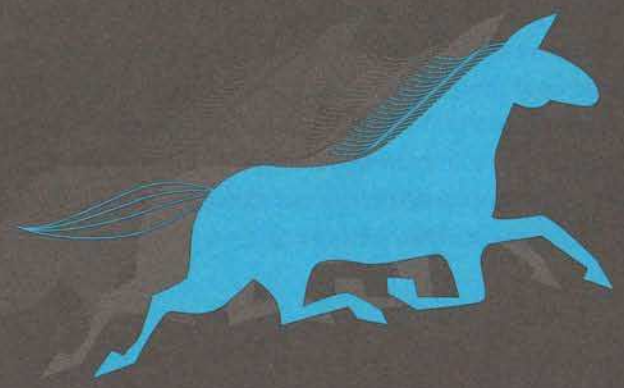
## Example

For the input data:

```
6 10
1 2 2
1 6 1
1 5 3
4 1 5
2 6 2
2 3 5
4 3 4
3 5 4
4 5 4
5 6 3
```

the correct result is:

```
TAK
TAK
TAK
NIE
TAK
NIE
TAK
TAK
TAK
TAK
```



# / Solution

In the description of the solution, it is convenient to use graph theory language. An undirected graph with weighted edges is given, and we would like to tell, for each edge, whether there exists a minimum spanning tree that contains this edge. In the following, we show two different approaches to this task. We use the same symbols as the task statement: $n$ denotes the number of cities and $m$ the number of highways.

**Author's solution**

At the start we build the minimum spanning tree using Kruskal's algorithm. It follows trivially that, for every edge of this tree, there is a minimum spanning tree that contains this edge. Hence, we get the result for $n-1$ edges. But what about the remaining $m-(n-1)$ edges?

For each of these edges, we would like to know whether it can belong to a minimum spanning tree. We first present the easiest solution. Let us focus on an edge $ab$. We build the minimum spanning tree by taking this edge and constructing the rest of the tree with Kruskal's algorithm. Then we can check whether the weight of the resulting tree is the same as the weight of the minimum spanning tree obtained previously. Since all edges can be sorted only once at the start, this algorithm can be implemented to run in $O(m^2 \log^* n)$ time. However, this is too slow.

We can do better. Observe that edges that were rejected by Kruskal's algorithm would be also rejected by the modified algorithm. If we compare the run of Kruskal's algorithm—in which we inserted edge $ab$ at the beginning—with the ordinary run, then in corresponding steps, the current spanning tree (or, rather, spanning forest) is "more connected" in the former. What does "more connected" mean? It means that if two vertices are connected in the second of these runs, then they are also connected in the first. Thus we can limit our attention only to edges from the minimum spanning tree.

As a result, when we take an edge $ab$ and construct the rest of the tree, we add only edges that belong to the (optimal) minimum spanning tree. This observation allows us to give an alternate description of our problem. We are given the minimum spanning tree and insert an edge $ab$, thus creating a cycle. We need to check whether we can remove an edge from this cycle, so that the remaining tree still has minimal cost.



Of course, we can only remove one edge and it has to belong to the cycle; any other deletion would disconnect the tree. Which edge should we delete? Obviously, it is best to remove the most expensive one. Thus, for every edge $ab$ that does not belong to the minimum spanning tree, we check whether its weight $w$ is equal to the weight of the heaviest edge on the path that connects $a$ and $b$ in the tree.

If every edge on the path connecting $a$ and $b$ has weight lower than $w$, then $ab$ does not belong to any minimum spanning tree. Otherwise, if the heaviest edge on the path had weight $w$, it could be exchanged with $ab$ without affecting the total cost. The above condition can be checked in $O(n)$ time. Hence, the algorithm runs in $O(mn)$ time, which is still too slow.

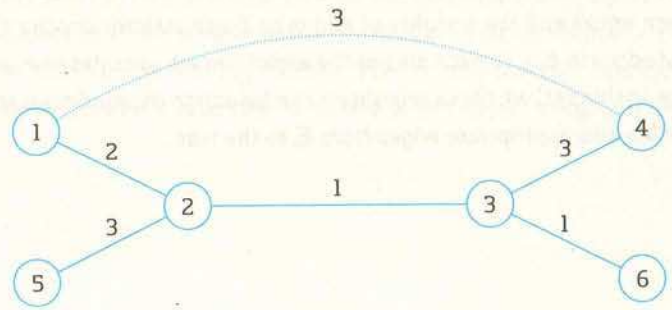



***Figure 1:*** *The dotted edge from 1 to 4 has the same weight as the most expensive edge on the tree path from 1 to 4. This means that this edge can replace the edge connecting 3 and 4 in the minimum spanning tree without changing the total cost.*

We can improve this further. Observe that the costliest operation, performed $m-n+1$ times, is traversing the tree in order to find the maximum weight of an edge on a particular path. We can do this more efficiently. Start from a vertex $v$ and compute the heaviest edges on paths from $v$ to all other vertices. Having done that, we can easily test whether an edge whose endpoint is $v$ can belong to the minimum spanning tree. This would be the case if the maximum weight of the path connecting its two endpoints were equal to the weight of the edge itself.

We need $O(n^2)$ total time for computing the heaviest edges on all paths in the tree. In addition, we use constant time to perform the check for each edge and $O(m \log m)$ time to run Kruskal's algorithm. The total time is thus $O(n^2 + m \log m)$.

**The best solution from the contestants**

The contestants were more creative than the author of this task, and came up with an interesting idea: an edge is in some minimum spanning tree if there exists a run of Kruskal's algorithm that includes this edge in a minimum spanning tree.

Kruskal's algorithm may have many different runs. This is caused by the fact that edges of equal weight can be put in any order by the sorting algorithm. It can easily be seen that an edge is in some minimum spanning tree if it is selected by a run of Kruskal's algorithm, in which the edge is in the first position among edges of equal weight.

Let us partition all edges into sets $E_1, E_2, \ldots, E_k$, where inside each $E_i$ the edges' weights are equal and the weights of edges in $E_i$ are strictly smaller than the weights of edges in $E_{i+1}$. In each step of the algorithm we consider one set $E_i$. For every edge in this set, we check whether it can belong to the minimum spanning tree. Then we add appropriate edges from $E_i$ to the tree.



```
Algorithm Byteland
  T := ∅
  partition all edges into sets E_i
  for i := 1 to k do
    for ab ∈ E_i do
      if vertices a and b are not connected in T then
        ab belongs to some minimum spanning tree
      else
        ab does not belong to any minimum spanning tree
    for ab ∈ E_i do
      if a and b are not connected in T then
        T := T ∪ {ab}
```

In this way we can get to know, for each edge, whether it belongs to some minimum spanning tree. If we use a union-find algorithm to implement a disjoint-set data structure that checks whether vertices are connected in $T$, we obtain an algorithm that runs in $O(m \log m)$ time. We can also achieve $O(m \log^* m + W)$ time complexity where $W$ is the maximum of the edges' weights in the graph.

# / Cakes

**Contest:** Algorithmic Engagements 2009
**Task author:** Jakub Łącki
**Memory:** 256 MB
http://main.edu.pl/en/archive/pa/2009/cia

Masters of baking from all over the country have come to this year's Bytean confectioners' convention. During the convention there will be a competition for the best cake baked by a team of three confectioners. Each team of three confectioners may submit at most one cake. However, each contestant can be a member of multiple teams. Unfortunately, some confectioners do not like each other and would not wish to bake cakes together.

It is known how many decagrams of flour each individual contestant requires to bake a cake. A three-person team needs as much flour as is required by the team member with the highest requirements. How much flour will be used during the competition, provided that each team of three confectioners who all like each other will bake exactly one cake?

## Input

The first line of the input contains two integers, $n$ and $m$ ($1 \le n \le 100{,}000$, $1 \le m \le 250{,}000$), separated by a single space and specifying the number of confectioners at the convention and the number of pairs of them that like each other. The participants at the convention are numbered from 1 to $n$. The second line contains $n$ integers $p_i$ ($1 \le p_i \le 1{,}000{,}000$) separated by single spaces and specifying the requirements of the respective confectioners for flour (in decagrams). The following $m$ lines contain data about the pairs of contestants that like each other. Each of these lines contains two integers, $a_i$ and $b_i$ ($1 \le a_i, b_i \le n$, $a_i \ne b_i$), separated by a single space, and represent pairs of confectioners $a_i$ and $b_i$ who like each other. We assume that all pairs of participants of the convention, other than the ones listed in the input, would not wish to bake cakes together. Each pair of confectioners appears at most once in the input.

## Output

The first and only line of the output should contain a single integer: the quantity of flour that will be used by all teams in total, in decagrams.



## Example

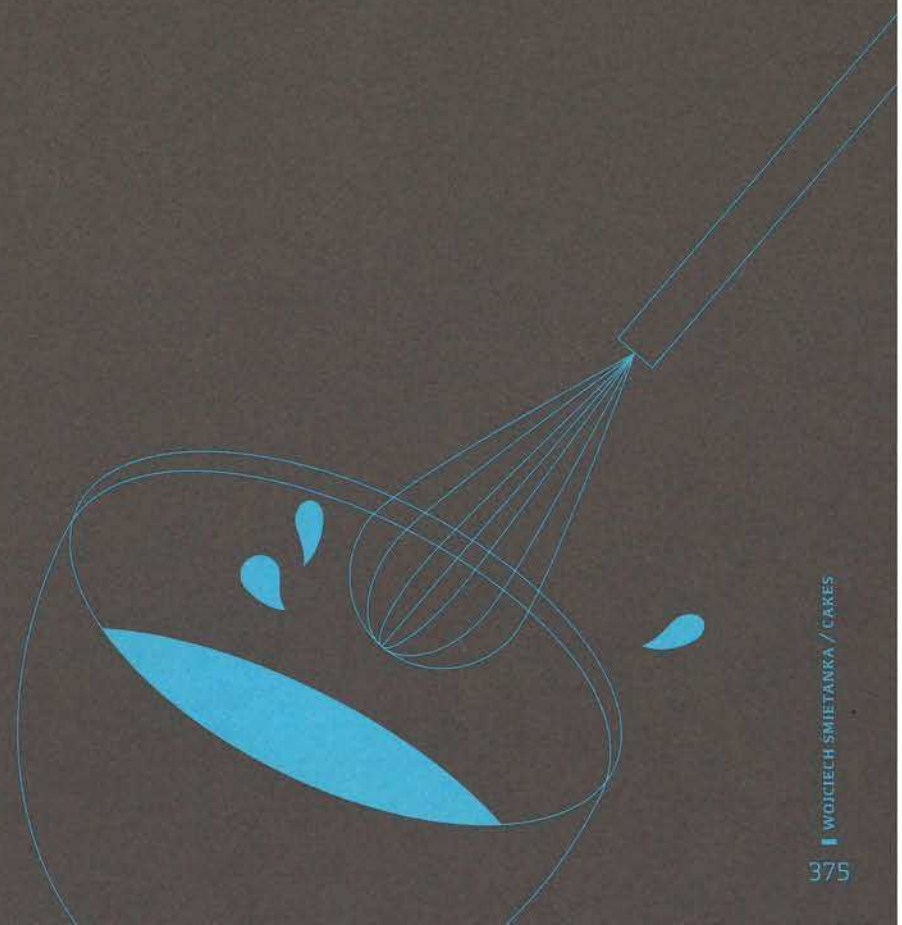
For the input data:

```
5 7
1 5 3 4 2
1 2
2 3
5 2
4 3
3 1
1 4
5 1
```

the correct result is:

```
14
```

**Explanation of the example:** The following three-person teams: (1,2,3), (1,2,5) and (1,3,4) require 5, 5 and 4 decagrams of flour to bake the cakes. In total, $5 + 5 + 4 = 14$ decagrams of flour are required.



# / Solution

In this task it is convenient to think in terms of graph theory. Each confectioner is a vertex in a graph, and for each pair of confectioners who like each other there is an undirected edge connecting them. We would like to find all trios of vertices in which each pair of vertices is connected with an edge—we will call such trios *triangles*.

Let us think of the simplest solution. For each trio of vertices we can check whether they form a triangle. Such a solution works in $O(n^3)$ time, which is too long for the time limit. However, by making some cunning observations, this can be greatly improved.

**Solution**

In the simplest solution we perform many wasteful operations, so let us try to prune it. If we choose vertex $i$ as one of the vertices of a triangle, then the remaining vertices $j$ and $k$ have to be neighbours of $i$. If the degree of a vertex $i$ is equal to $d$ then there are $\frac{d(d-1)}{2}$ pairs of neighbours to consider. We would like $d$ to be as small as possible. Observe that after considering vertex $i$, we can erase it as well as all its incident edges. How should we choose $i$? It seems reasonable to consider the vertex with the lowest degree. It turns out that the degree of such a vertex is always quite low.

***Lemma.*** In any simple graph with $n$ vertices and $m$ edges, there exists a vertex of degree lower than or equal to $\sqrt{2m}$.

*Proof.* Suppose that the degree of every vertex in the graph is greater than $\sqrt{2m}$. Then $n < \sqrt{2m}$, as otherwise the sum of degrees would be greater than $\sqrt{2m} \cdot \sqrt{2m} = 2m$, which is impossible. Since there are fewer than $\sqrt{2m}$ vertices, none of them can have a degree greater than $\sqrt{2m}$: a contradiction. □

Thus, the solution can be summarized as follows. We choose the vertex with the lowest degree, and for each pair of its neighbours we check if they are connected. Then we erase the chosen vertex. We continue this process until we have deleted all vertices.



Observe that we go through all pairs of edges incident to the chosen vertex. From the lemma we infer that each edge would belong to at most $\sqrt{2m} - 1$ such pairs. Hence, we can bound the total number of pairs considered by $m \cdot \sqrt{2m}$. Each of these pairs can be used to construct a triangle, but for this we need the third edge. How can we check whether such an edge exists? At the start, we sort adjacency lists of all vertices and then use a binary search to perform every check in $O(\log n)$ time. This gives a solution that works in $O(m\sqrt{m} \log n)$ time.

**A different and faster approach**

Let us consider what happens if we direct the edges. If we do this in such a way that there are no directed cycles of length 3, then each triangle has exactly one vertex with two out-edges. In such a case the triangle can be considered from the point of view of this vertex. Moreover, we would like the out-degrees of vertices to be not too big, as checking all pairs of out-edges of a vertex requires time that is quadratic in the number of its out-edges.

Observe that in any undirected graph there are at most $\sqrt{2m}$ vertices of degree greater than $\sqrt{2m}$. Let us order all vertices from left to right in ascending order of their degrees and direct all edges from left to right. Note that the first $n - \sqrt{2m}$ vertices (from the left) have out-degrees lower than $\sqrt{2m}$, as their degrees had been lower than $\sqrt{2m}$ even before the edges were directed. The last $\sqrt{2m}$ vertices also have out-degrees lower than $\sqrt{2m}$, since there are at most $\sqrt{2m} - 1$ vertices to the right of them. In addition, there are no cycles of length 3 in such a graph, as there are no cycles at all.

The out-degree of every vertex of the graph is now at most $\sqrt{2m}$. We iterate through all edges. For an edge $a \to b$ we compute the intersection of adjacency lists of $a$ and $b$. This intersection is exactly the list of vertices that can complement a triangle with $a$ and $b$. If we first sort the adjacency lists of all vertices, computing the intersection of two lists takes time that is linear with regard to their total length—in this case it is bounded by $O(\sqrt{m})$. As every edge is processed in $O(\sqrt{m})$ time, the time complexity of the entire algorithm is $O(m\sqrt{m})$.

**TOMASZ WALEŃ**

Tomasz Waleń works at the Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, where in 2010 he earned his Ph.D. degree in algorithms in computational biology. His scientific research is centered on various aspects of algorithmics, with special focus on text algorithms.

In 1997 Tomasz Waleń won the 4th Polish Olympiad in Informatics and a bronze medal at the International Olympiad in Informatics. He is the first author of SIO—an electronic web-based system that has been supporting the online facilitation of programming competitions in Poland for the past decade. The system makes it possible for solutions to competition tasks to be evaluated in real time. SIO has facilitated hundreds of competitions, including the 2005 and 2008 International Olympiad in Informatics as well as Central European Regional Contests in Budapest, Warsaw and Wrocław. Tomasz is one of the founders of the software company Codility.com.

In his spare time, Tomasz enjoys mountain trekking and photography.

# / Building *Blocks*

**Contest:** 15th Polish Olympiad in Informatics
**Task author:** Tomasz Waleń
**Memory:** 32 MB
http://main.edu.pl/en/archive/oi/15/klo

Byteasar loved to play with building blocks as a child. He used to arrange the blocks into $n$ columns of random height and then organize them: Byteasar would choose a number $k$ and try to arrange the blocks in such a way that some $k$ consecutive columns would be of equal height. Furthermore, he always tried to achieve this goal in the minimum number of moves possible, where a single move consisted of:

- laying one block on top of any column (Byteasar had a huge box of spare blocks, ensuring that this move could always be performed), or
- removing the uppermost block from the top of any column.

However, Byteasar was never quite sure whether his sequence of moves was indeed optimal; therefore he has asked you to write a program that will help him solve the problem.

## Task

Write a program that:

- reads the number $k$ of consecutive columns, along with the initial setup of the blocks, from the input,
- determines the optimal solution (the shortest possible sequence of moves),
- writes out the solution to the output.

## Input

In the first line of the input there are two integers, $n$ and $k$ ($1 \le k \le n \le 100{,}000$), separated by a single space. Each of the following $n$ lines contains the height of some column; the $(i+1)$-th line contains the integer $0 \le h_i \le 1{,}000{,}000$—the height of the $i$-th column, i.e. the number of blocks it consists of.

**Output**

An optimal solution should be written to the output—i.e. such an arrangement of blocks that:

- contains $k$ consecutive columns of equal height,
- can be obtained from the initial setup in the minimum possible number of moves.

The output should consist of $n + 1$ lines, each one containing a single integer. The number in the first line should be the minimum number of moves needed to get the desired arrangement. The $(i + 1)$-th line (for $1 \le i \le n$) should contain the number $h'_i$—the final height of the $i$-th column. If more than one optimal solution exists, output just one (chosen arbitrarily).

**Example**

For the input data:

5 3
3
9
2
3
1

the correct result is:

2
3
9
2
2
2



# / Solution

**A simplified version of the task**

Before we proceed with the solution to the original problem, let us consider its simplified version in which $k = n$. In this version we do not have to worry about choosing the part of the sequence of columns that we have to equalize. Nevertheless, the computation of the minimum cost is still not very obvious.

Of course, if we knew the height to which the columns should be equalized, the computation would be very simple. If this height was equal to $h$, we would just add blocks to columns for which $h_i < h$ to obtain the height $h$, and for columns with $h_i > h$ we would just remove the additional blocks.

Checking all possible values of $h$ would, obviously, be too time-consuming, so we have to come up with a better method for finding the optimal height.

Let us assume that we have $k$ columns of blocks of heights $x_1, \ldots, x_k$. Let $c_x(h)$ be the cost of equalizing all columns to obtain the height $h$. This function is described using the following formula:

$$c_x(h) = \sum_{i=1}^{k} |h - x_i|.$$

***Example.*** Let us consider the configuration of blocks from the example in the task statement; that is, $x = (3, 9, 2, 3, 1)$. A plot of the function $c_x(h)$ for this sequence is shown in figure 1.

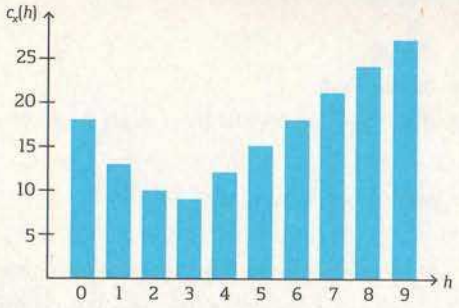



***Figure 1:*** *A plot of the function $c_x(h)$ for the sequence $x = (3,9,2,3,1)$.*

Looking at the plot we can observe that the function $c_x(h)$ decreases, then attains its minimum, and then increases. For us it is important to determine the value of $h$ for which the function $c_x(h)$ attains its minimum.

To find the optimal height we will take a closer look at the function $c_x(h)$. First of all, for $h = 0$ the value of $c_x(0)$ is:

$$c_x(0) = \sum_{i=1}^{k} x_i.$$

Let us now check how the function $c_x(h)$ behaves when its argument increases by 1. To obtain $c_x(h)$ from $c_x(h-1)$, we have to:

- add a single block to each column whose height is less than or equal to $h-1$,
- for each column of height greater than or equal to $h$, we avoid removing a single block (compared with the solution for $h-1$).

These observations can be transformed into the following formula:

$$\begin{aligned} c_x(h) &= c_x(h-1) + |\{x_i : x_i \le h-1\}| - |\{x_i : x_i \ge h\}| = \\ &= c_x(h-1) + |\{x_i : x_i \le h-1\}| - (k - |\{x_i : x_i \le h-1\}|) = \\ &= c_x(h-1) + 2 \cdot |\{x_i : x_i \le h-1\}| - k. \end{aligned}$$

Let $p_x(h)$ be the number of elements of the sequence $x$ with value less than or equal to $x$; that is, $p_x(h) = |\{x_i : x_i \le h\}|$. Using this notation we can write $c_x(h)$ as:



$$c_x(h) = c_x(h-1) + 2p_x(h-1) - k = c_x(0) + \sum_{i=0}^{h-1} (2p_x(i) - k).$$

The function $p_x(h)$ is non-decreasing, therefore $c_x(h)$ decreases as long as the elements of the above sum are negative. This means that $c_x(h)$ attains its minimum for the height $h = H$ such that:

$$2p_x(H-1) - k < 0 \quad \text{and} \quad 2p_x(H) - k \ge 0. \tag{1}$$

The value of $H$ that satisfies the inequalities (1) is always well-defined, since the range of the function $p_x(i)$ is $0,\ldots,k$ (for $i = -1,\ldots, \max\{x_j : 1 \le j \le k\}$). Hence, the function $2p_x(i) - k$, for the same domain, may attain values from $-k$ (for $i = -1$) to $k$ (for the maximal value of $i$).

Once more, let us take a quick look at the inequalities (1). For the requested height $H$ we would like to have in the sequence $x$:

- fewer than $\lceil k/2 \rceil$ columns of heights less than or equal to $H-1$, and
- at least $\lceil k/2 \rceil$ columns of heights less than or equal to $H$.

By assigning to $H$ the height of the $\lceil k/2 \rceil$-th (in the order of increasing heights) column of the sequence $x$, we fulfill both conditions. This element is called the median of the sequence $x$.

***Corollary.*** The function $c_x(h)$ attains its minimum for $h$ equal to the $\lceil k/2 \rceil$-th (in the order of increasing values) element of the sequence $x$, where $k$ is the length of the sequence $x$.

**Searching for the optimal segment**

From the corollary we can infer a draft algorithm that solves the general version of the task, in which we consider the whole sequence $h_1,\ldots,h_n$:

```
Algorithm ArrangeBlocks((h_1, ..., h_n), k)
    result := +∞
    for i := 1 to n + 1 − k do
        find the median H of the sequence x = (h_i, ..., h_{i+k−1})
        compute the optimal cost for the sequence x, that is, c_x(H)
        result := min(result, c_x(H))
    return result
```

The median of a $k$-element sequence can be computed in (expected) $O(k)$ time using the Hoare algorithm (unfortunately, the worst-case time complexity of this algorithm is $O(k^2)$). We could also use a more complicated (and not really practical) algorithm with worst-case $O(k)$ time complexity. The value of $c_x(H)$ can easily be computed in $O(k)$ time. This gives us a solution with total time complexity $O(nk)$. If $k$ were a small constant, such a solution would be satisfactory. Unfortunately, in our problem the parameter $k$ can be of magnitude $\Theta(n)$, and in such a case the time complexity of our solution is $O(n^2)$. This is clearly not sufficient, so we need to obtain a more efficient solution.

First, let us observe that the consecutive sequences $x$, for which we compute the median, are very similar. In the $i$-th iteration we process the sequence $(h_i, h_{i+1}, \ldots, h_{i+k-1})$, and in the $(i+1)$-th iteration, the sequence $(h_{i+1}, \ldots, h_{i+k-1}, h_{i+k})$. These two sequences differ only by two elements. Using this observation we can obtain a more efficient algorithm, as long as we can implement a data structure $S$ that provides the following methods:

- add($y$)—add an element $y$ to the structure,
- remove($y$)—remove the element $y$ from the structure,
- median()—return the median of the sequence stored in the structure,
- minCost()—return the value of the function $c_x(median())$ for the sequence stored in the structure.

Using the structure $S$ we can transform the draft algorithm into the following solution:

```
Algorithm ArrangeBlocks2((h_1, ..., h_n), k)
    S := ∅
    for i := 1 to k do
        S.add(h_i)
    result := S.minCost()
    for i := k + 1 to n do
        S.remove(h_{i-k})
        S.add(h_i)
        result := min(result, S.minCost())
    return result
```

Now it all reduces to an efficient implementation of this data structure. This is, however, a more sophisticated problem.

The presence of add and remove operations suggests that we can use some kind of a dictionary data structure; for example, red-black trees. And, in fact, if we add some extra information to the nodes of such trees, each of the requested operations on the structure $S$ can be performed in $O(\log n)$ time. The complexity of the whole algorithm is then $O(n \log n)$. The main drawback of this solution is that it is quite inconvenient to implement. The implementation of the dictionary structures available in the common libraries (like set or multiset in the STL in C++) does not allow for the storing of additional information in the nodes of the tree. So, in order to use this approach, we have to implement the red-black trees by ourselves.

Luckily, we can implement the data structure using less complicated tools. We start the new construction with a more detailed analysis of the information that is necessary to compute the values of the function *minCost*. Assume that all the elements of the multiset $S$ have been divided into: the median (*median*) and two multisets, *le* and *ge*, such that *le* contains elements that do not exceed the median, whereas *ge* contains elements that are at least equal to the median. Denote by *sumLE* and *sumGE* the sum of elements in the multiset *le* and *ge* respectively. Using this additional data, the value $c_x(median)$ can be obtained with the following formula:

$$c_x(median) = (|le| \cdot median - sumLE) + (sumGE - |ge| \cdot median). \qquad (2)$$

We add one more assumption: the multisets *le* and *ge* should be of equal or almost equal cardinality (there can be at most one extra element in the multiset *ge*). This means that the following condition should be an invariant of the algorithm:

$$|le| = \left\lceil \frac{|le| + |ge| + 1}{2} \right\rceil - 1. \qquad (3)$$

Let us notice that the location of elements equal to the median is not strictly determined. Such elements can be stored both in *le* and in *ge*; as long as the invariant is satisfied, we can choose the location arbitrarily. This time we can implement the multisets *le* and *ge* using a standard data structure from the STL; that is, the multiset container.

We have to show how to use the pair of multisets to implement the operations on the data structure $S$:

- *median*()—returns the stored median;
- *minCost*()—can be computed in constant time using the formula (2);
- *add*($y$)—first we add the new element to one of the multisets *le* or *ge*: if $y \le median$, then we add $y$ to *le*; in the opposite case we add $y$ to *ge*.

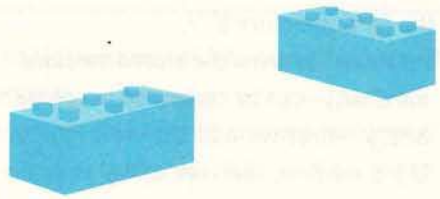
Unfortunately, after this insertion, the invariant (3) might not hold. To restore the invariant, we compare the cardinality of *le* and *ge*, and:

- if one element is missing in *le*, then we add an element equal to *median* to *le*, extract the minimal element from *ge* and save it as the new value of *median*;
- similarly, if one element is missing in *ge*, then we add an element equal to *median* to *ge*, extract the maximal element from *le* and save it as the new *median*.

In both cases we update *sumLE* and *sumGE* accordingly. If we choose the multiset container from STL, then adding, removing and extracting the minimal/maximal elements can be performed in $O(\log k)$ time.

→ *remove*($y$)—if $y < median$, we remove the element $y$ from *le*, and if $y > median$, we remove $y$ from *ge*; it is also possible that $y = median$, in which case we remove $y$ from the field *median*, and as the new *median* we set the minimal element from *ge*. In all cases we may have to restore the invariant, but this can be done using the same procedure as in *add*($y$).

The whole solution works in $O(n \log k)$ time. From the theoretical point of view, the multisets *le* and *ge* could be implemented with simpler data structures; note that all the necessary operations are provided by the ordinary heap data structure. However, once again we encounter implementation obstacles caused by the limitations of standard programming libraries. Indeed, the algorithm requires the operation of removing an element (function *remove*($y$)), and such an operation is not available in the heap implementation of priority_queue in STL.





# / Matching

**Contest:** Central European Olympiad in Informatics 2011
**Task authors:** Tomasz Kulczyński, Tomasz Waleń
**Memory:** 64 MB
http://main.edu.pl/en/archive/ceoi/2011/mat

As a part of a new advertising campaign, a big company in Gdynia wants to put its logo somewhere in the city. The company is going to spend its whole advertising budget for this year on the logo, so it has to be really huge. One of the managers has decided to use entire buildings as parts of the logo.

The logo consists of $n$ vertical stripes of different heights. The stripes are numbered from 1 to $n$ from left to right. The logo is described by a permutation $(s_1, s_2, \ldots, s_n)$ of numbers $1, 2, \ldots, n$. Stripe number $s_1$ is the shortest, stripe number $s_2$ is the second shortest etc., and finally stripe $s_n$ is the tallest. The actual heights of the stripes do not really matter.

There are $m$ buildings along the main street in Gdynia. To your surprise, the heights of the buildings are distinct. The problem is to find all positions where the logo matches the buildings.

Help the company and find all the contiguous parts of the sequence of buildings that match the logo. A contiguous sequence of buildings matches the logo if building number $s_1$ within this sequence is the shortest, building number $s_2$ is the second shortest, etc. For example, a sequence of buildings of heights 5, 10, 4 matches a logo described by a permutation (3,1,2), since building number 3 (of height 4) is the shortest, building number 1 is the second shortest and building number 2 is the tallest.

## Input

The first line of the input contains two integers, $n$ and $m$ ($2 \le n \le m \le 1{,}000{,}000$). The second line contains $n$ integers $s_i$, forming a permutation of the numbers $1, 2, \ldots, n$. That is, $1 \le s_i \le n$ and $s_i \ne s_j$ for $i \ne j$. The third line contains $m$ integers $h_i$—the heights of the buildings ($1 \le h_i \le 10^9$ for $1 \le i \le m$). All the numbers $h_i$ are different. In each line the integers are separated by single spaces.

## Output

The first line of the output should contain an integer $k$: the number of matches. The second line should contain $k$ integers—1-based indices of buildings that

correspond to the stripe number 1 from the logo in a proper match. The numbers should be listed in increasing order and separated by single spaces. If $k = 0$, your program should output an empty second line.

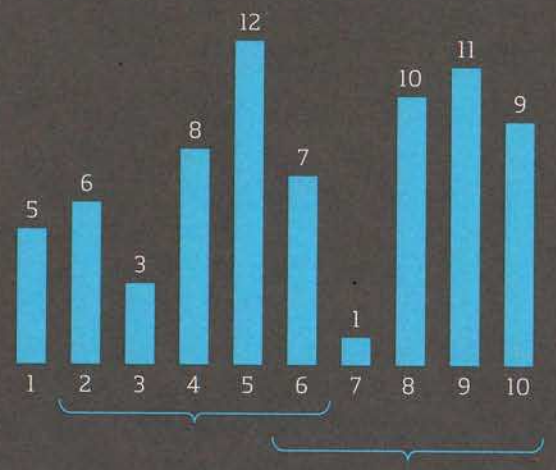
**Example**

For the input data:

5 10
2 1 5 3 4
5 6 3 8 12 7 1 10 11 9

the correct result is:

2
2 6




**Explanation of the example:** Both the sequences 6, 3, 8, 12, 7 and 7, 1, 10, 11, 9 match the logo described by the permutation (2, 1, 5, 3, 4). In particular, in the first sequence, building number 2 (of height 3) is the shortest, building number 1 (of height 6) is the second shortest, building number 5 (of height 7) is the third shortest, and so on.



# Solution

This problem is a variant of the pattern-matching problem, which is a fundamental task in text processing.

In the pattern-matching problem we are given two strings, $P$ and $T$, and we need to find all occurrences of the pattern $P$ in the text $T$; i.e. find the positions of all contiguous parts of $T$ that are equal to $P$. Several efficient algorithms for this problem are known: for example, the Knuth-Morris-Pratt (KMP) algorithm that runs in $O(|P| + |T|)$ time.

In this task we also have to find all occurrences of a pattern in a text; however, the definition of an *occurrence* is not so obvious as in the classical version of the problem.

**How to compare sequences**

Let us first observe that in this task the pattern is not given in the same format as the text. The input sequence $s_1,\ldots,s_n$ can be used to restore the pattern. This reconstruction is not unique; for one sequence $s$ there could be several possible patterns. To make it unique, we can restrict the elements of the pattern to the range $1,\ldots,n$. For example, for the sequence $s = (2, 1, 5, 3, 4)$ the pattern $p$ has the following form: $(2, 1, 4, 5, 3)$. In general, we have $p[s_i] = i$ for $i = 1,\ldots,n$.

Once we have settled on the canonical form of the pattern, we need to define when a part of the text (denote it by $w$) matches the pattern (denoted as $p$). The first and obvious condition is the equal lengths of the two sequences; that is, $|w| = |p|$. The second, more important condition is the identical ordering of elements in $w$ and $p$, which can be expressed as:

$$w[i] < w[j] \Leftrightarrow p[i] < p[j] \quad \text{for } 1 \le i, j \le |w| = |p|.$$

If $p$ matches $w$, we will denote it as $p \approx w$. For example, $(2,1,4,5,3) \approx (6,3,8,12,7)$.

Unfortunately this definition has a few disadvantages. Verification of a match is quite time-consuming. In the classical pattern-matching problem it requires linear time, and the verification of the relation $\approx$ directly (from the definition) requires quadratic time.

Luckily, based on a deeper analysis of the relation $\approx$, we can propose a more efficient method of verification.

For the pattern $p$ let us define the following arrays:

$$LPred_p[i] = i - \text{maxarg}_j\{p[j] : j < i \text{ and } p[j] < p[i]\},$$
$$LSucc_p[i] = i - \text{minarg}_j\{p[j] : j < i \text{ and } p[j] > p[i]\}.$$

In the above formulas, $\text{maxarg}_j$ and $\text{minarg}_j$ denote the value of $j$ for which the maximum and minimum, respectively, of the argument set is attained. For some indexes $i$ the value of $LPred_p$ or $LSucc_p$ may be undefined: this happens when the argument set of the function maxarg/minarg is empty.

| $i$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $p[i]$ | 2 | 1 | 4 | 5 | 3 |
| $LPred_p[i]$ | - | - | 2 | 1 | 4 |
| $LSucc_p[i]$ | - | 1 | - | - | 2 |

***Figure 1:*** *The arrays $LPred_p$ and $LSucc_p$ for the pattern $p = (2, 1, 4, 5, 3)$.*

The arrays *LPred/LSucc* can be used to locate the positions of the predecessor/successor (as to the value) to the left of an element of the pattern.

Using those two arrays, we can significantly reduce the verification time. Now only two comparisons are required for each character of the text. The pattern $p$ matches the text fragment $w$ ($p \approx w$) if:

$$w[i - LPred_p[i]] < w[i] \text{ and } w[i] < w[i - LSucc_p[i]] \text{ for } 1 \le i \le |p| = |w|. \quad (1)$$

Indeed, if $p[j] = p[i] + 1$ (these are two elements of the pattern of adjacent values), then among the above comparisons a comparison between $w[i]$ and $w[j]$ will take place; more precisely, it will take place for the greater of the indices $i$, $j$. If for some index $i$ the value $LPred_p[i]$ (or $LSucc_p[i]$) is undefined, the corresponding comparison is not performed. In this case we may assume that $LPred_p[i] = 0$ (or $LSucc_p[i] = 0$) and use a subacute inequality when checking the condition.



The above test can be performed in linear time. Now we have to show that the arrays *LPred/LSucc* can be computed efficiently. Both arrays can be computed in linear time by inspecting the pattern $p$ from the right to the left:

```
Algorithm ComputeLPredLSucc(p)
    let L be a bidirected list of pairs (p[i],i) ordered by the values of p[i]
    for i := |p| downto 1 do
        let x be the element of the list L corresponding to the pair (p[i],i)
        let (a,l) and (b,r) be the predecessor and successor of x in the list L
        LPred_p[i] := i − l (or NULL, if x is the first element of L)
        LSucc_p[i] := i − r (or NULL, if x is the last element of L)
        remove x from the list L
    return (LPred_p, LSucc_p)
```

Now we have all the pieces needed to prepare the first solution. We can adapt the naïve pattern-matching algorithm to obtain a solution with $O(mn)$ running time (where $n$ is the length of the pattern $p$ and $m$ is the length of the text $h$).

```
Algorithm FindMatchesSimple(h,p)
    (LPred_p, LSucc_p) := ComputeLPredLSucc(p)
    for i := 1 to |h|−|p| + 1 do
        check if p ≈ h[i, ..., (i + |p| − 1)] (using the condition (1))
```

According to the constraints from the problem statement, this solution is not efficient enough. We need to look for a faster algorithm.

**How to find all occurrences efficiently**

Fortunately, we can modify the Knuth-Morris-Pratt algorithm to fit our definition of a match. The KMP algorithm is a tricky improvement of the naïve pattern-matching algorithm—by precomputing the pattern to find a partial matches function, the algorithm can find all the matches efficiently, spending (on average) constant time on each character of text.

In the original KMP algorithm, the partial matches function is defined as follows:

$$\pi[i] = \max\{j : j < i \text{ and } p[1,\ldots,j] = p[(i-j+1),\ldots,i]\}.$$

A natural application of this definition to our model requires the relation = to be changed to ≈.

$$\pi'[i] = \max\{j : j < i \text{ and } p[1,\ldots,j] \approx p[(i-j+1),\ldots,i]\}.$$

| $i$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $p[i]$ | 2 | 1 | 4 | 5 | 3 |
| $\pi'[i]$ | 0 | 1 | 1 | 1 | 2 |

***Figure 2:*** *The values of $\pi'$ for the pattern $p = (2,1,4,5,3)$.*

The modified function $\pi'$ can be computed using the same algorithm as the original function $\pi$. The only change that has to be made is the modification of the test "if a given part of the pattern can be extended by a single character". In the classical KMP algorithm, this test is trivial (we just compare two characters). Here, as in the verification of ≈, we can use information from the arrays *LPred* and *LSucc*. We deal with the undefined elements of *LPred/LSucc* just as before.

```
Algorithm ComputePartialMatchFunction(p, LPred_p, LSucc_p)
  π'[1] := j := 0
  for i := 2 to |p| do
    while j > 0 and not (p[i − LPred_p[j + 1]] < p[i] < p[i − LSucc_p[j + 1]]) do
      j := π'[j]
    if p[i − LPred_p[j + 1]] < p[i] < p[i − LSucc_p[j + 1]] then
      j := j + 1
    π'[i] := j
  return π'
```



Once we have computed the function $\pi'$, we can find all the occurrences of the pattern efficiently. As in the previous case, we can modify the original KMP algorithm slightly, by changing those parts of the code that include character comparison.

```
Algorithm FindMatches(h, p)
  (LPred_p, LSucc_p) := ComputeLPredLSucc(p)
  π' := ComputePartialMatchFunction(p, LPred_p, LSucc_p)
  j := 0
  for i := 1 to |h| do
    while j > 0 and not (h[i − LPred_p[j + 1]] < h[i] < h[i − LSucc_p[j + 1]]) do
      j := π'[j]
    if h[i − LPred_p[j + 1]] < h[i] < h[i − LSucc_p[j + 1]] then
      j := j + 1
    if j = |p| then
      there is a match ending at the element i
      j := π'[j]
```

This solution runs in linear time. Here, only the two inner while-loops in the functions ComputePartialMatchFunction and FindMatches may look confusing. However, we can observe that each iteration of each of the loops causes a decrease of the counter $j$, and this variable is increased by $|p| - 1$ (or $|h|$ respectively) during the whole execution of the algorithm.

**JAKUB WOJTASZCZYK**

Jakub Wojtaszczyk graduated from the University of Warsaw with a Ph.D. degree in mathematics and a M.S. degree in computer science. He is currently a software engineer at Google. He is known for authoring programming tasks, some of which were used at the Polish olympiads both in informatics and mathematics, Algorithmic Engagements, ACM-ICPC World Finals, International Mathematical Competition and Google Code Jam. However, Jakub Wojtaszczyk also has some notable achievements as a contestant: he was a semifinalist of TopCoder Open in 2010, placed second in Algorithmic Engagements in 2011 and came third in the Internet Problem Solving Contest in 2011. Besides his technical interests, he is an active scout leader and an organizer of over twenty summer camps, more than ten of them abroad.

# / Party

**Contest:** 18th Polish Olympiad in Informatics
**Task author:** Jakub Wojtaszczyk
**Memory:** 64 MB
http://main.edu.pl/en/archive/oi/18/imp

Byteasar intends to throw a party. Naturally, he would like it to be a success. Furthermore, Byteasar is quite certain that, for it to be a success, it is sufficient that all the invited guests know each other. He is currently trying to come up with a list of friends he would like to invite.

Byteasar has $n$ friends, where $n$ is divisible by 3. Fortunately, most of Byteasar's friends know one another. Furthermore, Byteasar recalls that he once attended a party where there were $\frac{2}{3}n$ of his friends, and where everyone knew everyone else. Unfortunately, Byteasar does not quite remember anything else from that party... In particular, he has no idea which of his friends attended it.

Byteasar does not feel obliged to throw a huge party, but he would like to invite at least $\frac{n}{3}$ of his friends. He has no idea how to choose them, so he asks you for help.

## Input

In the first line of the input, two integers, $n$ and $m$,

$$(3 \le n \le 3000,\ \frac{\frac{2}{3}n(\frac{2}{3}n-1)}{2} \le m \le \frac{n(n-1)}{2}),$$

are given, separated by a single space. These specify, respectively, the number of Byteasar's friends and the number of pairs of his friends who know each other. Byteasar's friends are numbered from 1 to $n$. Each of the following $m$ lines holds two integers separated by a single space. The numbers in the $(i+1)$-th line (for $i = 1, 2, \ldots, m$) are $a_i$ and $b_i$ ($1 \le a_i < b_i \le n$), separated by a single space, which specify that persons $a_i$ and $b_i$ know each other. Every pair of numbers appears at most once in the input.

## Output

In the first and only line of the output, your program should print $\frac{n}{3}$ numbers, separated by single spaces, in increasing order. These integers should specify

the numbers of Byteasar's friends whom he should invite to the party. As there are multiple solutions, pick one arbitrarily.

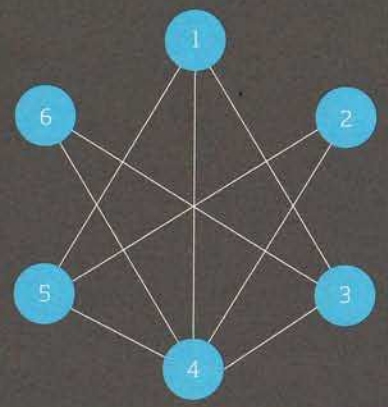
**Example**

For the input data:

6 10
2 5
1 4
1 5
2 4
1 3
4 5
4 6
3 5
3 4
3 6

the correct result is:

2 4




**Explanation of the example:** Byteasar's friends numbered 1, 3, 4, 5 know one another. However, any pair of Byteasar's friends who know each other—like 2 and 4, for instance—constitutes a correct solution; i.e. such a pair needs not be part of the aforementioned group of four.



# / Solution

As a problem author I like my problems to be appreciated. The situation that gives me the greatest satisfaction is when someone who does not know that I devised a particular problem starts telling me about it: "Man, what a cool problem I saw recently...!" I would like to present two problems for which exactly this situation occurred.

In both cases, the reason why the problem stuck in my interlocutor's memory was *surprise*—a hard-to-define moment in which some previously unsuspected fact became obvious, and a simple path appeared in what had previously been a tangled combination of questions. To give oneself the opportunity of experiencing this moment—regardless of whether it comes from one's own insight or from reading the solutions I provide—one has to try and tackle the problem oneself: to try to grasp, on one's own, where the real difficulty lies in the problem and spend some time trying to solve it. If you read the solutions without giving the problems an honest try, you will miss out on the opportunity of experiencing this magic moment.

On the other hand, if you have expended the effort on facing the problems yourself, welcome!

**About the problem**

This problem was created for the finals of the Polish Olympiad in Informatics in 2011. It can be expressed in the language of graph theory. For a given graph $G = (V, E)$ a set $K$ of vertices is called a *clique* if any two vertices of $K$ are connected by an edge. If $n$ denotes the number of vertices of the input graph, we know that there is a clique of cardinality at least $\frac{2}{3}n$ in the graph, while our job is to find a clique of cardinality at least $\frac{1}{3}n$.

It is natural to wonder why the problem statement is so suboptimal: if we know that a clique containing two thirds of the vertices exists, why doesn't the problem ask us to find it? The short answer is that finding a clique of size $\frac{2}{3}n$ is very difficult; I will expand on this later. As a matter of fact, it turned out that even finding the "small" clique proved pretty tricky for quite a few of the participants of the finals of the Olympiad...

One more piece of trivia about the problem statement... When creating a new programming task, it sometimes happens that the problem begins with a story, and an algorithm is discovered later on—this was the case with the second presented task, *Dice*. In this task the algorithmic problem, formulated in the language of graph theory, came first. I spent a bit of time trying to figure

out a story that would justify this atypical situation—we know a clique can exist, but we cannot actually put a finger on it. The moment the idea of a party from which you don't remember very much dawned on me, I knew I had the story I was looking for.

**Solution**

Consider any two vertices of the graph that are not connected by an edge. Regardless of the clique we are aiming to find, at least one of the two has to be rejected. Which one? It's hard to guess. So, in order not to make a mistake, we will just remove both of them from the graph.

What next? Well, we take some other two vertices not connected by an edge, and throw those two out as well. And so on, as the pseudocode below illustrates:

```
Algorithm Party(V,E)
  for all v ∈ V do
    take[v] := true {Initially all vertices are in the solution}
  for all v, w ∈ V do
    if vw ∉ E and take[v] and take[w] then
      take[v] := false
      take[w] := false
  return take
```

Of course, the real question is why this simple algorithm works (as opposed to a number of other, more or less simple, algorithms that do not). Let $K$ denote the mysterious clique with $\frac{2}{3}n$ elements that we know to be somewhere out there. Each time we remove a pair of vertices from the graph, at most one of them belongs to $K$, and at least one of them does not. As there are $\frac{1}{3}n$ vertices outside $K$ initially, we can remove a pair of vertices at most $\frac{1}{3}n$ times, so at the end we will have at least $n - 2 \cdot \frac{1}{3}n = \frac{1}{3}n$ vertices left, and each pair will be connected by an edge (otherwise we would have removed it). So the leftover vertices are indeed the clique we were searching for.

**Testing**

Luckily for me, the Polish Olympiad in Informatics is one of these contests in which the author of the problem is not responsible for creating the test data—"luckily" because this is one of these problems for which a plethora of creative, incorrect solutions exists, and the author of the test data should attempt to eliminate all of them (and, in order to do so, should think quite a few of them up). Creating tests is actually quite an art, as can be seen in the case of this problem. Consider the following pseudocode:

```
Algorithm IncorrectParty(V,E)
  for all v ∈ V do
    reject[v] := false   {Initially we treat all vertices as candidates.}
  clique := ∅
  while |clique| < n/3 do
    for all v ∈ V do
      if not reject[v] then
        deg[v] := 0
        for all w ∈ V do
          if (not reject[w]) and vw ∈ E then
            deg[v] := deg[v] + 1   {We calculate the degree of all
                                    vertices}
    candidate := v0
    for all v ∈ V do
      if ((not reject[v]) and deg[v] > deg[candidate] and v ∉ clique)
        or reject[candidate] or candidate ∈ clique then
        candidate := v   {We search for the candidate with the
                          highest degree}
    clique := clique ∪ {candidate}
    for all v ∈ V do
      if (v, candidate) ∉ E then
        reject[v] := true
  return clique
```

The idea here is to have a greedy algorithm—one by one we add the vertices with the highest degree to our clique. At first sight this does not look any better or worse than the solution that turned out to be the correct one—so maybe this one is also correct?

To verify that it is *not*, we should find a graph for which this solution will not work. We begin by creating the clique $K$ of size $\frac{2}{3}n$ that our graph has to contain. We add a vertex $v_0$—this is the vertex we want our algorithm to add to the clique in the first step. As the clique vertices have degree at least $\frac{2}{3}n - 1$, because they are all adjacent, we have to give $v_0$ a larger degree; say $\frac{3}{4}n - 1$. Suppose it neighbours all the vertices from outside $K$ and as many vertices from $K$ as are needed to sum up to $\frac{3}{4}n - 1$ (it will give $\frac{5}{12}n$ vertices from $K$).

We have quite a few vertices left, so we will prepare $v_1$ for our algorithm. At this point we have not rejected around $\frac{1}{3}n$ vertices from outside $K$ from the graph, and around $\frac{5}{12}n$ vertices from $K$—so let us set the degree of $v_1$ to be equal to $\frac{1}{2}n$: a bit more than $\frac{5}{12}n$. We achieve this, again, by connecting it to $\frac{1}{6}n+1$ vertices from $K$. This guarantees all the vertices from outside of $K$ and that the above algorithm chooses $v_1$ in the second step.

Now, if all the vertices outside $K$ have only the two edges we already created (to $v_0$ and $v_1$), the next step of the algorithm will take something from $K$ into its clique, and eventually construct a clique with $\frac{1}{6}n+3$ vertices—too few. So this algorithm is indeed incorrect.

Unfortunately, the job of the test setter is far from done at this point. What if we calculated the degree taking into account all vertices, and not only the ones left in the graph? Or—a nastier case—what if we improved the algorithm by removing all vertices of degree below $\frac{1}{3}n$ (or $\frac{2}{3}n$) from the graph at the beginning (they are of no use to us anyway)? This will pass the test created above. It is, however, still incorrect—although I will leave the creation of a graph to show this as an exercise for the reader.

### A bit of background

**NP-hardness.** The majority of readers will probably have heard about NP-hard problems; this is a class of algorithmic problems for which a polynomial-time algorithm is very unlikely to exist (and also connected to a number of prizes, if somebody does actually devise one). One of the best-known NP-hard problems is to find the largest possible clique in an arbitrary graph. This serves as a partial explanation of why we ask for a "suboptimal" clique in the problem.

Some of you might doubt this; does the fact that we *know* there exists a large (in our case, of size $\frac{2}{3}n$) clique in the graph not help us? After all, we know from experience that searching for something becomes easier if we know that it is actually there to be found... This is not the case here, however, and some easy reasoning proves it. Say we have an algorithm $A$ that allows us to find a clique of cardinality $k$ in a graph, assuming we know such a clique exists. Moreover, assume $A$ is polynomial, i.e. for some polynomial $p$ we have the guarantee that $A$ will find the clique in at most $p(n)$ operations. We will show how to convert this algorithm into one that answers the question whether there is a clique of size $k$ in it. Assume we have a graph $G$ that we know nothing about. Let us run $A$ on $G$ and see what happens. Of course, anything can happen—$A$ can crash, loop, work for a very long time or return some random junk: we have almost no guarantee as to what will occur. Why



"almost"? Well, we do know that if no clique of size $k$ exists, $A$ will not find it. It might *claim* to find it, but we will easily be able to verify that what $A$ found is not, in fact, a clique of size $k$.

So, if $A$ does end and return something that is not a clique of size $k$, we know that no such clique was there to be found (that is the assumption about $A$—if there were anything to find, it would find it; so if it returned junk, there wasn't anything to find). If $A$ works for longer than $p(n)$ steps or crashes, we can also safely claim that there is no clique of size $k$. So indeed, whatever happens when we run $A$ on an arbitrary graph does tell us whether the graph contains a requested clique. Formally, we turned an algorithm for finding a clique, knowing that it exists, into an algorithm that checks whether a clique exists. Let us add, for clarity, that if $A$ is known to run in polynomial time, but $p$ is unknown, we will not be able to pinpoint the requested polynomial-time algorithm checking for a clique, but we will be able to prove that one exists. Reasonings like the one presented above are called *reductions*, and are standard tools for proving that various problems are NP-hard.

I encourage the reader to find a reduction in the other direction—to show that, given an algorithm that can only answer *whether* there is a clique of size $k$ in a graph, we can also create an algorithm that will *find* it if it exists.

**Approximation.** The NP-hardness of the clique problem is quite widely known. A somewhat less well known fact is that the clique problem is also *hard to approximate*. For many NP-hard problems, seeing no chance of finding an effective algorithm to give the optimal solution, computer scientists settle on some approximation of the optimum—a solution that we can guarantee is not too far away from the optimal one.

It turns out that the approximation of the clique problem is actually also pretty hard. What does that mean, exactly? For example, the following problem is NP-hard (meaning that it's unlikely to have a polynomial time algorithm):

***Problem 1.*** Given a graph $G=(V,E)$, find a clique in this graph that is at most two times smaller than the largest one.

A wary reader is ready to protest at this point: isn't that exactly the problem we just solved a page or two ago? Well, not exactly. The difference is that in the *Party* problem we know that the optimal size is at least $\frac{2}{3}n$. On the other hand, it turns out that the difficult cases for the general approximation problem begin when the largest clique is rather small, e.g. $\frac{1}{20}n$. This small difference (and my initially futile attempts to grasp the precise meaning of the statement "it is NP-hard to approximate a clique") is the source of the *Party* problem.

An interested reader might like to know that the problem of approximating the clique is far harder than what has been stated above. One can start by replacing the constant 2 with any other constant, and the problem will remain NP-hard. But that's not all! Even the following problem is NP-hard:

***Problem 2.*** Given a graph $G = (V, E)$ with $n$ vertices, find a clique in this graph that is at most $\sqrt{n}$ times smaller than the largest clique in $G$.

Here, again, the square root can be replaced by any power of $n$ of exponent smaller than 1. The proof of this fact is complex and requires quite a lot of theory (although it is also quite beautiful, and rather well explained in Irit Dinur's paper, "The PCP theorem by gap amplification").

# / Dice

**Contest:** Algorithmic Engagements 2009
**Task author:** Jakub Wojtaszczyk
**Memory:** 32 MB
http://main.edu.pl/en/archive/pa/2009/kos

Jack claims to have telekinetic powers. He has made a bet with Mike that he will throw a die $n$ times and get exactly $k$ pips in total. Jack's die has six faces, numbered from 1 to 6, and is perfectly symmetrical. Mike does not believe in Jack's telekinetic powers (in general, being a tenacious rationalist, he does not believe in telekinesis), but he is concerned that Jack could win the bet simply by luck. Therefore, he would like to know the chance (as a percentage) of such an unlucky event occurring, and has asked you to write a program that will help him solve this problem.

**Input**

The first line of the input contains a single integer $t$ ($1 \le t \le 20$) specifying the number of test cases. Each of the following $t$ lines contains a description of one test case in the form of two integers, $n_i$ and $k_i$ ($1 \le n_i, k_i \le 10^6$), separated by a single space.

**Output**

To the output $t$ lines should be written, containing the answers to respective test cases. The answer to each test case is the probability (as a percentage) that Jack wins the bet, rounded *down* to the nearest integer.

**Example**

For the input data:

```
1
1 6
```

the correct result is:

```
16
```

# Solution

This problem appeared in the second round of Algorithmic Engagements and went rather well—not surprisingly, as it is not mindblowingly hard. I came up with it while teaching a *Probability* class about the Central Limit Theorem—which will be described in more detail later on.

**Solution**

Suppose for a moment that $n$ is bounded from above by 100 or 1000. This would turn out to be a very standard problem in dynamic programming. Let us see—what is the probability of getting exactly $k$ pips on $n$ dice? If $n = 0$, then either $k = 0$ and we have a probability of 100%, or $k \neq 0$ and the probability is zero.

Now we move on to the slightly more interesting case of $n > 0$. Let us throw the first $n - 1$ dice and see what happens. If the total number of pips is smaller than $k - 6$ or larger than $k - 1$, we do not have a chance to get exactly $k$ pips after the last throw. Otherwise, if the number of pips in the first $n - 1$ throws is between $k - 6$ and $k - 1$, we have exactly one chance in six to succeed. Thus if $P(n, k)$ denotes the probability of getting $k$ pips in $n$ throws, we see that

$$P(n,k) = \frac{\sum_{i=1}^{6} P(n-1, k-i)}{6}.$$

Moreover, if $k < n$ or $k > 6n$ then $P(n, k) = 0$.

Thus if we take, say, $1 \leq n \leq 1000$, we can calculate all the 6,000,000 values $P(n, k)$ for $1 \leq n \leq 1000$ and $1 \leq k \leq 6000$ as follows:

```
Algorithm DiceUpTo1000(N,K)
    SetToZeros(P)
    P[0][0] := 100   {The array P holds exact values, not rounded to the
                      nearest percent}
    for n := 1 to 1000 do
        for k := 1 to 6000 do
            for i := 1 to 6 do
                if k − i ≥ n − 1 then
                    P[n][k] := P[n][k] + P[n − 1][k − i] / 6
    return ⌊P[N][K]⌋   {Here we round downwards to the nearest percent}
```



For each feasible pair $n, k$ we add at most six numbers, which gives 36,000,000 additions in total (and a similar number of other operations)—well in time in most contests.

Attempts to use this approach for the full range of $n$ and $k$ cannot succeed. We need to be somewhat smarter.

The key observation is that for large $n$ there are so many different possible results that the probability of hitting a particular one is pretty small—likely to be smaller than 1%. So in this problem we will not need to calculate the exact value; we can just output zero. Having had this idea, we can do one of two things:

- We can take a piece of paper and try to calculate how big $n$ has to be to guarantee that $P[n][k]$ is below 1% for all $k$; or
- We can simply run the program above and check whether 1000 is enough.

A programmer will naturally opt for the second solution. In just a few minutes we write a program that calculates $P[n][k]$ up to 1000 and learn that the furthest value of $P[n][k]$ that is at least 1% is $P[545][1908] \simeq 1.0002\%$. So we could have written the above program for $n \leq 545$, and if $n > 545$, we can output zero without a single calculation. The complete solution is as follows:

```
Algorithm Dice(N,K)
    if N > 545 then
        return 0
    else
        SetToZeros(P)
        P[0][0] := 100   {Exact values, again}
        for n := 1 to 545 do
            for k := 1 to 545 · 6 do
                for i := 1 to 6 do
                    if k − i ≥ n − 1 then
                        P[n][k] := P[n][k] + P[n − 1][k − i] / 6
    return ⌊P[N][K]⌋   {Rounding to the nearest percent}
```

**A bit of background**

The background here, as already mentioned, is intuition tied to the Central Limit Theorem (often abbreviated to CLT); it is also a pretty useful tidbit of knowledge. The Theorem itself states that when we add together a large number of independent variables (for instance, dice throws), the sums will approximate a *normal* (or *Gaussian*) *distribution*. Here we do not care exactly what this means (although

it does pay to know sometimes); we care how the CLT translates to knowledge about the highest $n$ for which $P(n,k)$ is not smaller than 1%.

The key fact is that the probability of hitting the most probable outcome which is roughly the average—in this case $3.5n$, after rounding to an integer—is proportional to $1/\sqrt{n}$. The constant of proportionality in this case is roughly $1/\sqrt{6\pi} \approx 0.23$. Thus we would expect that $P(n, 3.5n) \approx 0.23/\sqrt{n}$. If we want this to drop below 1%, we solve $0.01 \approx 0.23/\sqrt{n}$, obtaining $n \approx 529$—a pretty good approximation of the real value of 545.

For applications in other problems, it is worth knowing that, while the exact constant of proportion depends on the type of variables we add (it decreases with the variety and spread of the variable; for instance, if we add pips on twenty-sided dice, it is about 0.07, while for a coin toss, which corresponds to a "two-sided die", it is about 0.56), the probability of hitting the most probable solution is always proportional to $1/\sqrt{n}$. This rough estimate often serves to tell us whether a given approach has any chance of success at all.

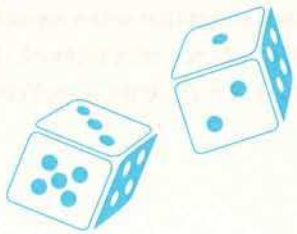
**What if we had to count exactly?**

In this problem we avoided the issue of calculating $P[n][k]$ for large $n$ thanks to the low precision of the output that was required. It is worth considering whether we could solve this problem if the requested precision were the standard $10^{-6}$ required in similar problems.

We will not follow the solution to the end, but we will create a rough sketch that can be extended to a full solution (with enough work). We will use two observations to lead us to the solution.

First, notice that the recursive formula we used for $P(n,k)$ was not the only possible one. For example, instead of looking at the last die throw, we could consider the last three throws (assuming $n > 3$), and write

$$P(n,k) = \sum_{i=3}^{18} P(n-3, k-i) \cdot P(3,i).$$





In this formula and all the following formulas we assume that, for negative arguments, $P$ equals 0. We will use yet another formula, obtained by splitting $n$ into two equal halves. If $n$ is even, then

$$P(n,k) = \sum_{i=n/2}^{3n} P(n/2, k-i) \cdot P(n/2, i).$$

Let us assume for a moment that $N = 2^l$ is a power of two. Then, instead of calculating the whole $P[n][k]$ array for $n \le 2^l$, we can just calculate $P[2^m][k]$ for $m \le l$—so instead of $O(N^2)$ values we calculate $O(N \log N)$.

If $n$ is not even, we first calculate $P[n-1][k]$ for all $k$ (using the equality above), and then we go to $P[n][k]$ by considering the last die.

Does this help? We need to calculate $O(N \log N)$ values of $P$—but unfortunately, calculating one of those values now takes $O(n)$ additions instead of six. So we could say we gained nothing (a more detailed analysis would also show that we did not really lose a log factor, as one might expect, but that is not really satisfactory). Time to pull a second trick out of our sleeves.

With enough algorithmic experience one can notice that the expression

$$P(n,k) = \sum_{i} P(n/2, k-i) \cdot P(n/2, i)$$

looks pretty familiar: it is the so-called *convolution*, and it can also be written as polynomial multiplication. The *Fast Fourier Transform* (FFT) algorithm allows one to calculate all the values of $P(n,k)$ (if the values of $P(n/2,i)$ are given for all $i$) in $O(n \log n)$ time, as opposed to the $O(n^2)$ one gets by applying the recursive formula directly. Thus, connecting the observation about half-splitting with the FFT, we obtain a solution that works in $O(N \log^2 N)$ time—which will run in time for $N \le 10^5$ (and could even work for $N \le 10^6$ if the time limits were lenient enough).

**FILIP WOLSKI**

Filip Wolski started participating in programming competitions in Seventh Grade, when Ryszard Szubartowski—a teacher at the Third High School in Gdynia, and now the president of the "Talent" Society—became his mentor. He has won four gold medals at the International Olympiad in Informatics (including first place in 2006) as well as a number of medals and a few victories in regional computer science and mathematics competitions (Baltic Olympiad in Informatics, Central European Olympiad in Informatics, Baltic Way). He has represented the University of Warsaw at the ACM-ICPC World Finals twice, winning the competition with his teammates Marek Cygan and Marcin Pilipczuk in Tokyo in 2007.

Being a water sports enthusiast, and especially passionate about scuba diving, Filip Wolski spent a year traveling and diving around the globe as an OWUSS European Rolex Scholar. He is also a scuba instructor.

His interests include innovative uses of informatics in society and economics. Recently Filip Wolski graduated from his joint program at the Faculty of Mathematics, Informatics and Mechanics, University of Warsaw with an M.S. degree in computer science and a B.S. degree in mathematics.

# / Two *Parties*

**Contest:** 12th Polish Olympiad in Informatics
**Task author:** Wojciech Guzicki
**Memory:** 32 MB
http://main.edu.pl/en/archive/oi/12/dwa

King Byteasar has decided to throw two great parties, and wants to invite to them every single inhabitant of Byteland. Of course, he wants to invite each inhabitant to exactly one of the parties. The king knows from his own vast experience that a person has a good time when there is an even number of their own friends at the party. Thus he has asked you to divide the country's inhabitants between the two parties in such a way that as many people as possible have an even number of friends at their party. A trivial division—i.e. one that leaves one party with no guests invited—is allowed. An acquaintanceship is a symmetric relation; i.e. whenever person $A$ knows person $B$, person $B$ also knows person $A$.

## Task

Write a program that:

- reads from the input the number of Byteland's inhabitants and the descriptions of their acquaintances,
- splits the inhabitants into two parties in such a way that the number of people who have an even number of friends at their party is as large as possible,
- writes to the output a list of people who should be invited to the first party.

## Input

In the first line of the input there is one integer $N$ ($1 \le N \le 200$), specifying the number of inhabitants of Byteland. The inhabitants are numbered from 1 to $N$. In the following $N$ lines there are the descriptions of the inhabitants' acquaintances. At the beginning of the $(i+1)$-th line there is an integer $l_i$ ($0 \le l_i \le N-1$), specifying the number of friends of the $i$-th inhabitant. It is followed by $l_i$ pairwise distinct numbers of the $i$-th inhabitant's friends. We assume that no inhabitant is his/her own friend. It follows that each acquaintance is written twice: if $A$ and $B$ know each other, then $B$ appears on $A$'s list of friends and $A$ appears on $B$'s list.

**Output**

In the first line of the output your program should write one integer $M$: the number of people who are to come to the first party. In the second line the $M$ numbers of these people should be written. The rest shall come to the second party.

There are many correct results for this task; your program should write any one of them.

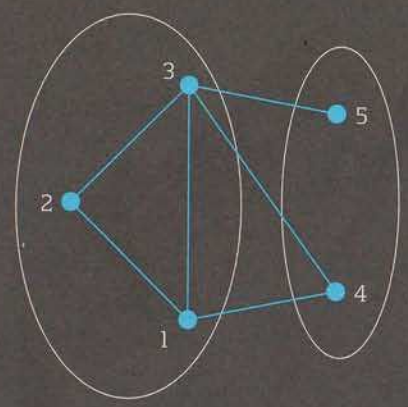
**Example**

For the input data:

```
5
3 2 3 4
2 1 3
4 2 1 4 5
2 1 3
1 3
```

the correct result is:

```
3
1 2 3
```




**Explanation of the example:** In the above example, everyone will have an even number of friends at the party.

# / Solution

Reading the problem statement immediately creates associations. Vertices having even degrees—that brings to mind the problem of determining whether the graph has an Euler cycle. On the other hand, splitting the graph into two parts reminds us of bipartite graphs—and, in fact, if the friendship graph in this task is bipartite, it is easy to split it into two parts so that no vertex has any friend whatsoever in its part, which solves our problem. Going further, determining whether a graph is bipartite involves searching for odd-length cycles in the graph. However, we will actually use a completely different technique to solve this task.

As it turns out, it is always possible to split Bytelanders so that they will all be happy! In other words, the set of vertices of any undirected graph can be split into two subsets, so that each vertex will have an even number of neighbors in its set. We will call such a division *good*. A subgraph in which each vertex has an even number of neighbors will be called a *party*. Given a division of a graph, we will also be talking about the *inner degree* of a vertex—that is, the number of its neighbors that belong to the same subset.

Our proof will use mathematical induction. Additionally, it will be constructive: it will show directly how to build an algorithm that solves this problem. For very small graphs it is easy to see that good division is always possible. It is obvious for a graph with only one or two vertices, and we could also manually check that this is also the case for each graph with three vertices. Let us assume, then, that our hypothesis is true for each graph of $N$ vertices.

Now let us take a graph $G$ with $N+1$ vertices. $G=(V,E)$, where $V$ is the set of its vertices and $E$ is the set of its edges. We will attempt to split the graph into two sets, $V_1$ and $V_2$, so that for each $v \in V_i$, set $\{w \in V_i : (v,w) \in E\}$ is of an even size.

If each vertex of $G$ has an even number of neighbors, then it is easy: $V_1 = V$ and $V_2 = \varnothing$. Let us assume, then, that there is at least one vertex $v$ with an odd number of neighbors. Denote by $S$ the set of its neighbors (see figure 1).

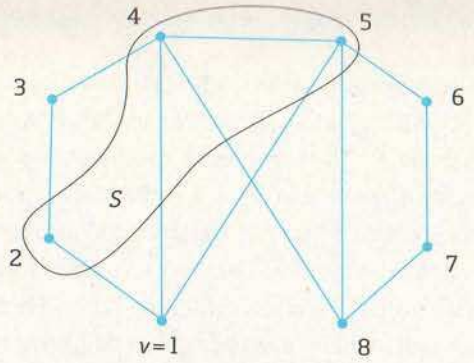



*Figure 1: Graph G, in which we have chosen vertex v = 1 of odd degree. S is the set of its neighbors.*

We now try to build a good division recursively. First we find a good division of a smaller graph, and on that basis we try to find a good division of the whole graph $G$. If the smaller graph is obtained by removing vertex $v$ then, after finding a good division of the smaller graph, it would be nice if we could just add $v$ to one of the parties to get a good division of $G$. If the whole division is to be good, $v$ has to be added to a party containing an even number of elements of $S$. But if we did that, we would reverse the parity of the inner degrees of elements of $S$ in that subset, and it would no longer be a party (see figure 2)!

Let us try to deal with this problem by correcting $G$ a bit after removing $v$ and before searching for a good division of the smaller graph. Let $G'$ be that corrected graph. After we find a good division of $G'$, in one of its two parties there is an even number of elements of $S$. We would like these elements to have an odd (instead of even) inner degree. Can we achieve that just by manipulating the edges? Most definitely! What we need to do is to "invert" edges between vertices of $S$ in graph $G'$. This means that, for each pair of vertices in $S$, if there is an edge between them in $G$ there is none in $G'$, and if there is no edge in $G$, there is one in $G'$. In graph theory language, $G'$ contains an edge supplement of an $S$-induced subgraph of $G$.

A vertex with degree $s$ in a graph with $n$ vertices will have degree $n-s-1$ in this graph's supplement, so the difference between these degrees is $|n-2s-1|$. Therefore, if $n$ is even, then the parity of the degree changes, and if it is odd, it stays the same.



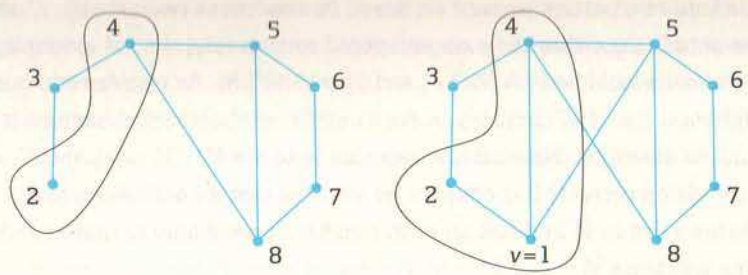



*Figure 2: Graph G after removing vertex v and its good division (on the left). If we add v back again, then the marked subgraph is no longer a party.*

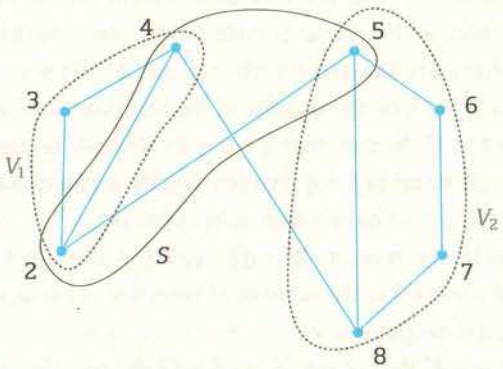



*Figure 3: Graph G' created based on v's selection in G.*

What happens to the parity of inner degrees of vertices in $S$? Exactly what we expect. The good division of $G'$ splits $S$ into two subsets $S_1$ and $S_2$, such that one of them has even size and the other has odd size. Assume that $S_1$ is the subset of odd size. Each element of $S$ has an even inner degree in $G'$. Vertices that belong to $S_1$ will also have an even inner degree in $G$. Vertices that belong to $S_2$ will have an odd inner degree in $G$—but they become even again after adding $v$.

Let us make it more formal (see also figure 3). Let $G' = (V', E')$. Vertex set $V'$ is created by removing $v$ from $G$: $V' = V \setminus \{v\}$. Edge set $E'$ is similar to $E$, except for edges whose endpoint is $v$ (they are removed) and edges between vertices in $S$:

- If either $u$ or $w$ is not in $S$, then $(u, w) \in E' \Leftrightarrow (u, w) \in E$.
- If both $u$ and $w$ are in $S$, then $(u, w) \in E' \Leftrightarrow (u, w) \notin E$.

From our assumption we know that graph $G'$, which has $N$ vertices, can be split into two parties (sets of vertices). Denote these two sets by $V_1'$ and $V_2'$ (in the actual algorithm they are computed recursively). In our example, one such division would be $V_1' = \{2,3,4\}$ and $V_2' = \{5,6,7,8\}$. As $v$ has an odd number of neighbors, one of $V_i'$ contains an even number of them—let us assume it is $V_1'$ (like in the example). Now all that remains is to set $V_1 = V_1' \cup \{v\}$ and $V_2 = V_2'$. Is this division correct? It is obvious for vertices that do not belong to $S$:

- Vertex $v$ lies in $V_1$ and has an even number of neighbors in that set—this is how we chose $V_1$.
- Vertex $w \neq v$, which does not lie in $S$, has the same set of neighbors in this division as in the division of $G'$, and that is—as follows from the induction—an even number. In the example, that would be vertices 3 or 6 in figures 1 and 3.

Vertices from $S$ are split into two groups based on the division, $S_1 = S \cap V_1$ and $S_2 = S \cap V_2$. Vertex $w \in S_i$ has an even number of neighbors in $V_i'$, but with regard to the edge set $E'$. So exactly what neighbors does $w$ have within $V_i$? It might have neighbors within $V_i$ but outside $S \cup \{v\}$—let $p$ be their number. Note that these neighbors are the same both in $E$ and $E'$. Now $w$ might also have neighbors within $S$. Let $q$ be the number of its neighbors in $S_i$ with regard to $E$, and $q'$ with regard to $E'$. Notice, that $q + q' + 1 = |S_i|$. Moreover, the number of neighbors of $w$ in $G'$, equal to $p + q'$, is even. Lastly, it is possible that, if $w \in S_1$, $v$ is a neighbor of $w$. Let us have a look at both cases:

- If $w \in V_1$ then $|S_i|$ is even, so $(p + q') - (q + q' + 1) + 2(q + 1) = p + q + 1$ is also even and $p + q + 1$ is the number of neighbors of $w$ in $V_1$ with regard to $E$ (one additional neighbor is $v$).

  Example: Let $w = 4$. Vertex $w \in V_1$, so $S_i = \{2,4\}$, $p = |\{3\}| = 1$, $q = |\varnothing|$ (that is because, in $G$, vertex 4 is a neighbor of 1, 3, 5 and 8, of which only 5 is from $S$, but it is not in $V_1$) and $q' = |\{2\}| = 1$. Then $(p + q') - (q + q' + 1) + 2(q + 1) = 2$, and this is equal to $|\{1,3\}|$—the number of $w$'s neighbors within $V_1'$ with regard to $E$.
- If $w \in V_2$ then $|S_i|$ is odd, while $q + q'$ is even. Hence, $(p + q') - (q + q') + 2q = p + q$ is even as well, and $p + q$ is number of $w$'s neighbors in $V_2$ with regard to $E$.

  Example: Let $w = 5$. Then $p = 2$, $q = q' = 0$ and $(p+q')-(q+q')+2q = 2$, and this is equal to $|\{6,8\}|$—the number of $w$'s neighbors within $V_2$ with regard to $E$.

This ends the proof for graphs with $N + 1$ vertices—and, by induction, for any undirected graph.



A sample solution is shown in the pseudocode below. At the beginning, all entries in the *Active* array have to be set to **true**, whereas in array *Even* we assign **true** or **false** to each vertex, depending on whether it has an even number of neighbors. After running the algorithm, the set *Division* contains one of the two subsets of a good division of the graph.

```
Algorithm DivideGraph()
    v := 1
    while v ≤ N and (Even[v] or not Active[v]) do
        v := v + 1
    if v > N then
        Division := {w ∈ V : Active[w]}
        return
    S := ∅
    for w ∈ V ∧ Active[w] ∧ Neighbors[v, w] do
        S := S ∪ {w}
    if |S| mod 2 ≠ 0 then
        for w ∈ S do
            Even[w] := ¬Even[w]
    for (w, u) ∈ S × S ∧ w ≠ u do
        Neighbors[w, u] := ¬Neighbors[w, u]
    Active[v] := false
    DivideGraph()
    if |Division ∩ S| mod 2 = 0 then
        Division := Division ∪ {v}
```

Let $N$ be the number of vertices of graph $G$. The function *DivideGraph* will be executed no more than $N$ times (each time, one of the vertices gets removed). Its execution time is dictated by the loop that reverses the parities of vertices, which requires $O(N^2)$ steps. Therefore, the running time of the entire algorithm is bounded by $O(N^3)$.

# / Guesswork

**Contest:** 11th Polish Olympiad in Informatics
**Task author:** Wojciech Guzicki
**Memory:** 32 MB
http://main.edu.pl/en/archive/oi/11/zga

Byteasar has become a chronic gambler. He throws away all his money in the casino playing *Guesswork*. He claims that it is possible to devise a game system that allows him to beat the casino.

A single game of *Guesswork* consists of drawing nine consecutive real numbers belonging to (0, 1) using uniform distribution. Upon drawing each number, the player must immediately guess the position of the number (with respect to the standard order of numbers) among the nine to be chosen. Obviously, it is not possible to foresee the position—one has to take a guess. If nine positions are guessed correctly, the player wins; otherwise, the player loses. If, for instance, we guess that the first drawn number is the second biggest, and then draw two numbers that are smaller, we lose.

## Task

Help Byteasar! Program the best guessing strategy you can. Write a program that plays multiple games of *Guesswork*, winning as many as possible. The greater the number of games won, the better the evaluation of your program will be. More precisely, for each game won the casino pays the player 423.99 bytean dollars, and for each game lost it collects 13.53 bytean dollars from the player. Your program shall play $10^6$ games. The number of points you shall receive will equal the total profit divided by $10^4$ and rounded to the nearest integer within [0, 100].

You are to program a module containing the following three functions:

- void inicjalizuj() (Polish for *initialize*)—this procedure shall be called only once, at the outset, before any game begins. You can use it to initialize your data structures.
- void nowa_rozgrywka() (Polish for *new game*)—this procedure will be called at the beginning of each game. You can use it to initialize the variables concerning a single game.
- int kolejna_liczba(double x) (Polish for *next number*)—this function receives a drawn number $x$ as a parameter ($0 < x < 1$), with a precision of 12 decimal places. The function should calculate the position of $x$ among the 9 numbers drawn in this round and return the outcome. If the game is lost, the function may return any integer from 1 to 9.

## Example

A hypothetical interaction with the program could be as follows (the numbers returned by the function kolejna_liczba are given in italics; two games have been played.

| GAME 1 | | GAME 2 | |
|---|---|---|---|
| 0.043319026120 | | 0.164020610610 | |
| *1* | | *2* | |
| 0.933801434041 | | 0.205594263575 | |
| *8* | | *3* | |
| 0.992050359372 | | 0.042637391231 | |
| *9* | | *1* | |
| 0.145189967093 | | 0.147521628974 | |
| *4* | | *1* | Defeat |
| 0.518803250649 | | 0.946549875333 | |
| *6* | | *1* | |
| 0.093583048537 | | 0.772946216573 | |
| *2* | | *1* | |
| 0.764309529654 | | 0.152956276544 | |
| *7* | | *1* | |
| 0.338653748790 | | 0.539653928563 | |
| *5* | | *1* | |
| 0.119437652934 | | 0.552047936535 | |
| *3* | Victory | *1* | |

# / Solution

*Guesswork* is a very interesting problem—and not only for the gamblers among us. Much like the movie *21*, in which some American students were winning big in blackjack, Byteasar—as we learn from the problem description—intends to outplay casinos in *Guesswork*, using a special strategy. In this game, Byteasar's opponent—the dealer or the casino—does not have any strategy, so Byteasar's strategy is enough to determine the probability of his winning.

If we assume that, before each game, Byteasar picks a permutation of numbers 1 through 9 and uses its consecutive elements during the game, then the probability of his winning a single game would be $\frac{1}{9!} = 0.00000275...$; this means that, with each game, he would be losing 13.52879431... bytean dollars on average.

We can try to determine how good a strategy we need to come up with. If its probability of winning is $p$, then Byteasar's expected income in each round would be $423.99p - 13.53(1 - p)$. Byteasar will be happy if this is positive, because that would make him a millionaire (someday). We, on the other hand, would like Byteasar to earn at least $10^6$ bytean dollars after $10^6$ games—that is, when we can expect the number of points scored by our solution to be close to 100. We are therefore expecting probability $p$ to be at least

$$423.99p - 13.53(1 - p) \geq 1 \Rightarrow p \geq 0.03320991...$$

**Brief discussion about the strategy**

Let us ponder first how we might represent our strategy. One thing to consider is which events are independent of each other and which are not. For example, in a game where we are to guess the score of the next dice roll, we could incorporate previous results into our strategy; but this would not affect our results. This is because the score of every roll is independent of the previous ones.

In our case, we get a number in the first move. We are to decide how big this number is in comparison with the other eight numbers that we do not know yet. Let us say that we pick an answer and claim that the first number is the $i$-th smallest.

We can start by computing the probability that we are right. Let $(x_i)_{i=1}^{9}$ be a sequence of numbers given to us by the casino. So far, we know only $x_1$. We would like to compute the probability that $i - 1$ among $x_2, \ldots, x_9$ will be smaller than $x_1$, and that the other numbers will be greater. First, let us pick those $i - 1$ numbers. There are $\binom{8}{i-1}$ ways to pick $i - 1$ numbers from a sequence of eight numbers. Once we pick them, for each of the numbers $x_i$ (with $i = 2, 3, \ldots, 9$) we can tell that we would like it to be either smaller or greater than $x_1$. The probability that a specific number is smaller than $x_1$ is $x_1$, whereas the probability of its being greater amounts to $1 - x_1$. Picking each of them is independent, so the probability that all of them will meet our requirements is $x_1^{i-1}(1 - x_1)^{9-i}$. Therefore, the cumulative probability that our guess about $x_1$ is correct is $\binom{8}{i-1}x_1^{i-1}(1 - x_1)^{8-i}$, or

$$\binom{n-1}{i-1}x_1^{i-1}(1 - x_1)^{n-i} \quad \text{for } n = 9.$$

Let us generalize the *Guesswork* game now: we will say that $n$-guesswork is a game of *Guesswork*, but played on a sequences of *n elements*, instead of 9. We will also say that it is played on a segment $(a, b)$, if *Guesswork*'s elements are picked with uniform distribution from segment $(a, b)$ (in original *Guesswork*, $a = 0$ and $b = 1$). Note that $n$-guesswork played on $(a, b)$ is equivalent to $n$-guesswork played on $(0, 1)$: it just boils down to scaling the elements you receive from the casino ($x$ in the former corresponds to $\frac{x-a}{b-a}$ in the latter).

We now observe that the full description of a strategy is not necessary. All we need is an algorithm for figuring out the first move. Why is that? Assume that in the first move we get 0.4 and respond that it is the fourth element. Therefore we limit the space of games we can still win to those games in which the first element, equal to 0.4, is the fourth smallest one. Hence, all probabilities should now be computed within this game space. If we get 0.3 in the next move, then in general there is a positive probability that it is the fifth element of the whole sequence—but we have already guessed that 0.4 is the fourth one, and the probability of both these events happening together is 0. What we need here is conditional probability: the probability, that 0.3 in the second position is the $i$-th smallest number *under the condition* that 0.4 in the first position is the fourth smallest number.

Thus, the first move splits the rest of the game into two independent games:

- 3-guesswork played on (0,0.4),
- 5-guesswork played on (0.4,1).

Further on, if our guess was correct, then each of the following moves will take one of the independent games still being played and split it again into two (down to 0-guesswork).

Before we describe it more formally, let us introduce some notation. Recall that $(x_i)_{i=1}^{9}$ denotes the sequence of values given by the casino—at the $i$-th step we

know only $x_j$, for all $j \le i$. Let $(t_i)_{i=1}^{9}$ be the sequence of our responses—at the $i$-th step we are supposed to give element $t_i$. To make further considerations simpler, let us assume that we also have these sentinels: $x_0 = 0$, $t_0 = 0$, $x_{-1} = 1$, $t_{-1} = 10$.

Each time we receive a new element $x_i$, we should pick from $x_{-1},\dots,x_{i-1}$ the largest element smaller than $x_i$ (call it $x_s$) and the smallest element greater than $x_i$ (call it $x_g$). Thus, $x_s < x_i < x_g$. We have already decided on $t_s$ and $t_g$ values, so we have already assumed that, among elements $x_{\max(s,g)+1},\dots,x_9$, there are exactly $t_g - t_s - 1$ falling into segment $(x_s, x_g)$. The first one of them is $x_i$. Now we should "ask" our strategy for the best first move in the $(t_g - t_s - 1)$-guesswork played on $(x_s, x_g)$, with the first element equal to $x_i$; or, equivalently, for the best first move in the $(t_g - t_s - 1)$-guesswork with the first element equal $\frac{x_i - x_s}{x_g - x_s}$.

**Simple strategy**

We have already figured out that all we need from our strategy is the ability to tell us which number, in sorted order, is the first element $x$ in $n$-guesswork, for a given $x \in (0,1)$. But we do not yet know how to compute that.

Let us start with the simplest strategy: divide $(0,1)$ into $n$ equal segments. If a given $x$ falls into the $i$-th segment, then our response is $i$. If we represent this strategy with $S^0$, it could be written as:

$$S^0(n,x) = \lceil xn \rceil.$$

This means that if we play $n$-guesswork and the casino gives us a number $x$, we say that it is the $\lceil xn \rceil$-th smallest number. We can also represent any strategy in a different way (this will prove useful later on). Let us fix $n$. Assume that, for $x < y$, we have $S^*(n,x) \le S^*(n,y)$. Strategies that do not fulfill this requirement seem to be less reasonable. Furthermore, the values of $S^*(n,x)$ are $1, 2, \dots, n$. Hence, we can fully specify $S^*(n,x)$ (for this $n$) by giving values $s^*_{n,0} = 0 \le s^*_{n,1} \le \dots \le s^*_{n,n-1} \le s^*_{n,n} = 1$ and saying that $S^*(n,x) = i \iff s^*_{n,i-1} < x \wedge x < s^*_{n,i}$. In case of $S^0$ we would have $s^0_{n,i} = \frac{i}{n}$. Observe that this description does not say what to do when the casino gives us one of the numbers $s^*_{n,i}$. This is not a problem—we can give any answer in such a case—as this happens with probability 0.

Before we find out whether this simple strategy is of any use, let us try to describe a slightly more complicated one. If we are to answer the question of which number in sorted order is a given number, why do we not respond with the one with the maximum likelihood? This is analogous to so-called maximum-likelihood estimation.



We already know how to compute the probability that the first element $x$ is the $i$-th number in sorted order in $n$-guesswork:

$$p(n,x,i) = \binom{n-1}{i-1} x^{i-1}(1-x)^{n-i}. \tag{1}$$

In the example given before, i.e. $n = 9$ and $x = 0.4$, the right side is maximal for $i = 4$ (for $i = 3$ the right side equals 0.20901888, and for $i = 4$ it is 0.27869184).

Let us call this strategy $S^1$ and try to compute $s^1_{n,i}$. We already know that $s^1_{n,0} = 0$ and $s^1_{n,n} = 1$ for any $n$. How do we find the value of $s^1_{n,i}$ for $0 < i < n$? Notice that in $s^1_{n,i}$'s neighborhood strategy, $S^1$ always responds with either $i$ or $i+1$. More precisely, for $x < s^1_{n,i}$ we have $S^1(n,x) = i$ and for $x > s^1_{n,i}$ we have $S^1(n,x) = i+1$. Additionally, we know that $p(n,x,i)$ is continuous. Therefore, $x = s^1_{n,i}$ must be a value for which $p(n,x,i) = p(n,x,i+1)$. Let us try to find such an $x$:

$$p(n,x,i) = p(n,x,i+1),$$

$$\binom{n-1}{i-1} x^{i-1}(1-x)^{n-i} = \binom{n-1}{i} x^{i}(1-x)^{n-i-1},$$

$$\frac{1-x}{n-i} = \frac{x}{i},$$

$$x = \frac{i}{n}.$$

Wow! It turns out that this "more complicated" strategy is in fact the same as 0-strategy. Or, in other words, the simple 0-strategy is equivalent to picking the answer with the greatest likelihood. Is this enough to beat the casino?

**Strategy analysis**

Let us try to compute probability $P^*$—the probability of winning in a game of *Guesswork*, assuming we are playing with some strategy $S^*$. Let $P^*(n)$ be the probability of winning $n$-guesswork in general, while $P^*(n,x)$ is the probability of winning $n$-guesswork if the first element given by the casino is equal to $x$. These probabilities depend directly on our strategy:

$$P^*(n,x) = p(n,x,S^*(n,x))P^*(S^*(n,x)-1)P^*(n-S^*(n,x)).$$

Below we will discuss two ways of computing these probabilities. One of them uses the Monte Carlo method to approximate them. The other is more precise, but requires a bit of integral calculus.

**Monte Carlo.** The Monte Carlo method is often used when we want to compute some values that might be difficult to obtain analytically. Instead of the exact result, we get an approximation by taking enough samples. In our case, we have already decided on a strategy, so we have all we need to write a program to play the game of *Guesswork* with strategy $S^1$.

What about the casino program? If we were at the competition, we could just use the program provided by the organizers. It would run our solution $10^6$ times and tell us how many points we got for this execution. That should be sufficient to decide whether our solution is good enough to score many points.

On the other hand, outside the competition, we could write a simple program to perform as a dealer. This program would also query our solution. We could have it play $10^7$-$10^8$ games and report the fraction of the games won. That fraction would be a good approximation of the value of $P^1(9)$.

Programming all this would be a good exercise for the reader. If you are impatient, though, have a look at the end of the next subsection to find out whether strategy $S^1$ would score any points...

**Precise analysis.** Independent of the choice of strategy, the probability of winning the game of guesswork can be described recursively:

$$P^*(n) = \int_0^1 P^*(n,x)dx,$$

$$P^*(n,x) = p(n,x,S^*(n,x))P^*(S^*(n,x)-1)P^*(n-S^*(n,x)).$$

We can make it a bit simpler. For that, we will utilize values $s_{n,i}$ that we have defined before:

$$\begin{aligned} P^*(n) &= \int_0^1 p(n,x,S^*(n,x))P^*(S^*(n,x)-1)P^*(n-S^*(n,x))dx \\ &= \sum_{i=1}^{n} \int_{s^*_{n,i-1}}^{s^*_{n,i}} p(n,x,i)P^*(i-1)P^*(n-i)dx \\ &= \sum_{i=1}^{n} P^*(i-1)P^*(n-i)\int_{s^*_{n,i-1}}^{s^*_{n,i}} p(n,x,i)dx. \end{aligned}$$



Now let us see if we can deal with this kind of integral:

$$\begin{aligned} \int_a^b p(n,x,i)dx &= \int_a^b \binom{n-1}{i-1} x^{i-1}(1-x)^{n-i}dx \\ &= \binom{n-1}{i-1}\int_a^b x^{i-1}\sum_{j=0}^{n-1}\binom{n-i}{j}(-x)^j dx \\ &= \binom{n-1}{i-1}\int_a^b \sum_{j=0}^{n-1}\binom{n-i}{j}(-1)^j x^{i+j-1}dx \\ &= \binom{n-1}{i-1}\sum_{j=0}^{n-1}\binom{n-i}{j}(-1)^j\int_a^b x^{i+j-1}dx \\ &= \binom{n-1}{i-1}\sum_{j=0}^{n-1}\binom{n-i}{j}(-1)^j\frac{b^{i+j}-a^{i+j}}{i+j}. \end{aligned}$$

We can compute that numerically with any precision we need. It is easy to see that $P^1(0) = P^1(1) = 1$. Now for further values we can use the following formula:

$$P^1(n) = \sum_{i=1}^{n} P^1(i-1)P^1(n-i)\int_{\frac{i-1}{n}}^{\frac{i}{n}} p(n,x,i)dx.$$

After we implement this, we find out that $P^1(9) = 0.03075546...$; so strategy $S^1$ is not sufficient in this problem. But what can we improve?

**Better strategy**

So far, we have had a very simple strategy: each time we pick the response with the greatest likelihood. This seems reasonable—the very same value we are maximizing is the one we use later when computing the probability of winning. Recall that it can be calculated from the formula:

$$P^*(n,x) = p(n,x,S^*(n,x))P^*(S^*(n,x)-1)P^*(n-S^*(n,x)).$$

We choose $S^1(n,x)$ so that $p(n,x,\ S^1(n,x))$ is maximal. But, as we look at it now... why not pick $S^*(n,x)$ to maximize the entire right-hand side?

Our mistake led us to pick the response with the greatest likelihood in the first step. Instead, we should be picking the response that will maximize our chance to win the entire game—and that includes further steps as well.

Here is an example: 3-guesswork with first element $x = 0.3$. Should we say it is the first or the second number in sorted order? It is more likely that it is the first one—the probability that the next two elements are larger is 0.49, while the probability that exactly one is larger is 0.42. In first case, though, we have one more decision to make later on—whether the second element is the second or the third smallest one. We can win this 2-guesswork with probability 0.75. So if we respond that 0.3 is smallest element of the sequence, then we win the whole game with probability $0.49 \cdot 0.75 = 0.3675$. On the other hand, if we respond that it is the second smallest number, then although that is true with a probability of only 0.42, there are no more choices to take later.

Let us call this strategy $S^2$. What are the values $s^2$? We compute them in a similar way to $s^1$. In strategy $S^2$ we are maximizing the value of $q(n,x,i) = p(n,x,i)P^2(i-1)P^2(n-i)$ by picking response $i$. Therefore we find the border value of $s^2_{n,i}$ by equating:

$$q(n,x,i) = q(n,x,i+1),$$

$$p(n,x,i)P^2(i-1)P^2(n-i) = p(n,x,i+1)P^2(i)P^2(n-i-1),$$

$$\frac{1-x}{n-i}P^2(i-1)P^2(n-i) = \frac{x}{i}P^2(i)P^2(n-i-1),$$

This eventually gives us

$$s^2_{n,i} = x = \frac{iP^2(i-1)P^2(n-i)}{iP^2(i-1)P^2(n-i) + (n-i)P^2(i)P^2(n-i-1)}.$$

In the $S^1$ strategy we computed $P^1$ values to discover whether it pays off to use it. Here, in turn, we need to compute the values in order to define the strategy. As with $S^1$, we will have $P^2(0) = P^2(1) = 1$, and for $n \geq 2$:

$$P^2(n) = \sum_{i=1}^{n} P^2(i-1)P^2(n-i)\int_{s^2_{n,i-1}}^{s^2_{n,i}} p(n,x,i)dx.$$



As in the previous analysis, we do not need integral formulas here. If, for a fixed $n$, we know $P^2(n')$ for each $n' < n$, then based on that we can compute $s^2_{n,i}$ (for all $0 \leq i \leq n$). This is enough to play $n$-guesswork with strategy $S^2$.

However, here we do not have to use integrals either. Instead, we can use the Monte Carlo method to approximate $P^2(n)$. This way we can precompute border values $s^2_{n,i}$ in advance, using the Monte Carlo method for consecutive values of $n$. If we incorporate them directly into our solution's code, we will not need to recompute them when our solution is playing the game.

Now we have everything we need to compute not only the strategy but also the probability of winning *Guesswork* when playing with strategy $S^2$: $P^2(9) = 0.03337533...$; this should already be enough to get 100 points.

How do we figure out the probability of getting the maximum score using strategy $S^2$? For that we need a bit of knowledge from probability theory. Say that after $10^6$ games we win $X$ of them ($X$ is a random variable). We would like $423.99X - 13.53(10^6 - X) \geq 10^6$, which is equivalent to $X \geq 33210$. The probability of winning in a single game of guesswork is $p = 0.03337533...$. From the Central Limit Theorem we know that $X$ has (approximately) normal distribution, with expected value $\mathbb{E}X = 10^6 p > 33375$ and standard deviation $\sigma_X = \sqrt{10^6 p(1-p)} < 180$. Therefore the probability of $\{X < 33210\}$ is less than:

$$\mathbb{P}(X < 33210) < \mathbb{P}(X - \mathbb{E}X < -165) < \mathbb{P}\left(\frac{X - \mathbb{E}X}{\sigma_X} < -\frac{165}{180}\right) =$$

$$= \mathbb{P}\left(\mathcal{N}(0,1) < -\frac{165}{180}\right) < \Phi\left(-\frac{165}{180}\right) < 0.18.$$

So by playing strategy $S^2$ we will get 100 points with a probability of at least 0.82.

**Closing remarks**

The problem *Guesswork* is pretty original. It first appeared on a trial day of the final stage of the Polish Olympiad in Informatics, so it was probably treated as an experimental problem by the Olympiad's Committee. Some of the competitors solved it correctly. Others finished without a full solution—e.g. they played *Guesswork* using the simple strategy described above. One of the competitors managed to totally beat the casino by figuring out future game progressions based on pseudo-random number generator variables.

I remembered this problem, years after the competition, while on a lengthy trip. After a long SMS exchange with an unknown number, the time came for me finally to guess which of several girls I had recently met was on the other end. Well trained by the Olympiad, I guessed it was the one I liked most.

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OFEK
OGÓLNOPOLSKA FUNDACJA
EDUKACJI KOMPUTEROWEJ

**Ogólnopolska Fundacja Edukacji Komputerowej** (OFEK) was established in Wrocław by Founding Members included in the notary deed of February 24, 1986. The Foundation was brought to life at the initiative of directors of Wrocław-based enterprises and institutions, especially Mr. Bogusław Lisowski, the first President of OFEK, and Mr. Andrzej Myszkier, the current President of OFEK. During the 25 years of its existence, the Foundation has been cooperating closely first with the Ministry of Education and Upbringing and its later equivalent—Ministry of National Education, thanks to which it was able to supply more than 8700 computer labs in schools across the country with over 91,000 computers and 15,000 printers.

OFEK has been fulfilling its statutory goal by coordinating and participating in training events for over 80,000 teachers. These activities have not only brought financial benefits but, as Mr. Lisowski predicted, they enabled Polish schools to catch up with the most modern schools in the world—turn-of-the-century schools that emerged in the midst of the information revolution.

Since the very beginning the main goal of the Foundation has been to provide computer education to people of all ages, especially the youth, by popularizing the use of computers for teaching and learning purposes.

**The Foundation's activities largely contribute to:**

- wider application of computers in didactic and development processes, thanks to which former students are able to make use of computers in everyday life once they have already graduated,
- society's engagement in activities supporting computer education,
- popularization of achievements in the field of science and technology that facilitate practical use of computers,
- continuous development of computer culture within the society.

**The Foundation fulfills its goals by:**

- raising financial means, computer equipment and other means as well as increasing their volume in the course of its own business activities,
- creating favorable conditions supporting the learning process, creative and productive efforts aiming at popularization of computer education in the society, especially among pupils and students,
- granting scholarships, organizing contests, exhibitions and olympiads in informatics,
- cooperating with local and national institutions and other parties interested in activities supported by the Foundation.

Contest tasks say a lot about the quality of a programming competition. They should be original, engaging and of different levels of difficulty. Finding a solution should cause the contestant to feel great satisfaction, whereas being unable to solve a given task should encourage an individual to broaden their knowledge and develop new skills. This book contains the best tasks from algorithmic and programming competitions organized or co-organized by the University of Warsaw, together with their exemplary solutions.

The selection of the tasks was undertaken by people who have played significant roles in the history of Polish algorithmic and programming competitions as their participants or organizers. All of the authors of texts presented in this book are closely affiliated with the Faculty of Mathematics, Informatics and Mechanics at the University of Warsaw, whether as former or current students or as academic staff.

Each of the tasks presented and discussed in this book was used during one of the following events: Polish Olympiad in Informatics, Junior Polish Olympiad in Informatics, Polish Olympiad in Informatics Training Camp, Central European Olympiad in Informatics, Polish Collegiate Programming Contest and Algorithmic Engagements.

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