Review of Basics

Stuff you should (mostly) know

…already

EC516: Digital Signal Processing

Fall 2016

Prepared by

S. Hamid Nawab, Ph.D.

Professor

Boston University

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Review Readings

- O/S Chapter 2: Sections 2.0-2.9
- O/S Chapter 3: Sections 3.0-3.5
- O/S Chapter 4: Sections 4.0-4.6
- O/S Chapter 5: Sections 5.0-5.3

D-T LTI Systems: Response to Complex Exponentials

$$x[n] = e^{j\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow y[n] = H(e^{j\omega_0})e^{j\omega_0 n}$$

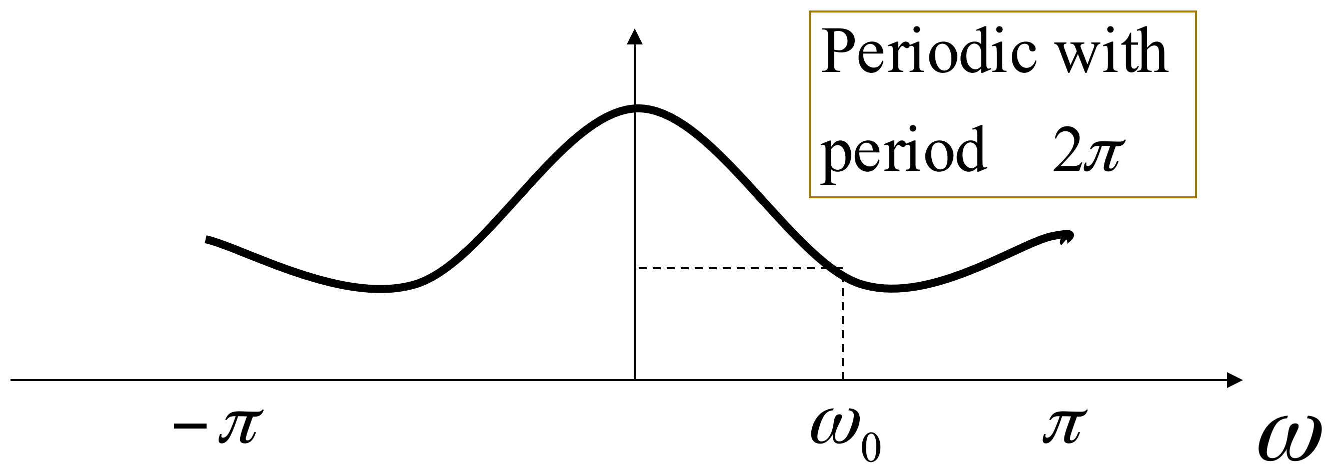
$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega_0(n-k)}$$

$$= e^{j\omega_0 n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega_0 k}$$

Frequency Response of D-T LTI systems

$$x[n] = e^{j\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow y[n] = H(e^{j\omega_0})e^{j\omega_0 n}$$

$$H(e^{j\omega_0}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega_0 n}$$



D-T Fourier Transform

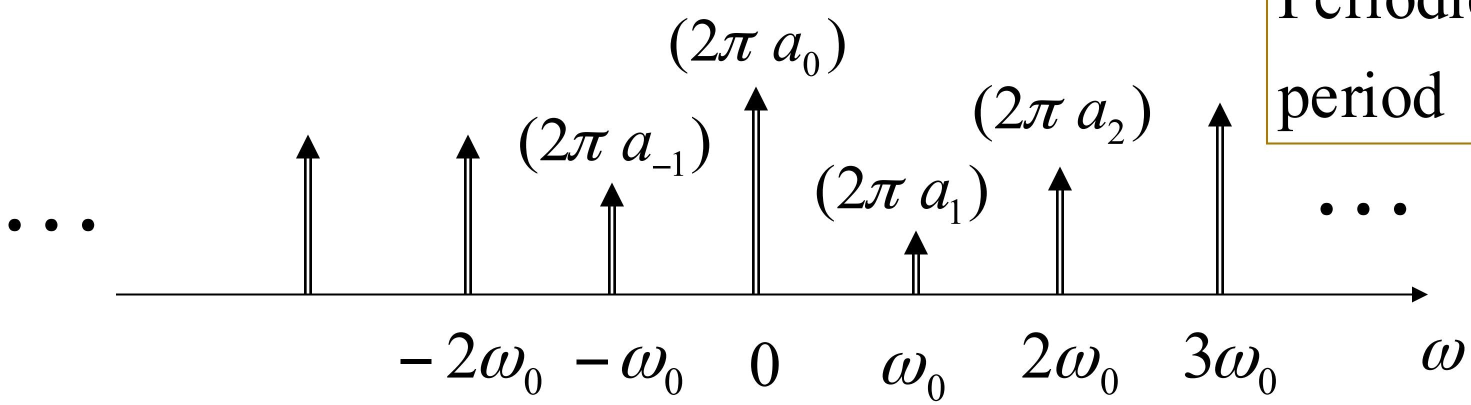
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}; \quad \textit{Analysis} \quad \textit{Equation}$$

$$x[n] = \frac{1}{2\pi} \int_{<2\pi>} X(e^{j\omega})e^{j\omega n} d\omega \qquad \textit{Synthesis} \quad \textit{Equation}$$

$$\boxed{x[n] \Leftrightarrow X(e^{j\omega}) \qquad \text{Fourier transform pair}}$$

D-T Fourier Transform of Periodic Signals

$$\boxed{x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \Leftrightarrow X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\, a_k \delta(\omega - k\omega_0)}$$



Periodic with period 2π

Basic D-T Fourier Transform Pairs

$$x[n] = \delta[n - n_0] \Leftrightarrow X(e^{j\omega}) = e^{-j\omega n_0}$$

$$x[n] = e^{j\omega_0 n} \Leftrightarrow X(e^{j\omega}) = \delta(\omega - \omega_0) \ over\ 1\ period$$

$$x[n] = a^n u[n] \Leftrightarrow X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}; \quad |a| < 1$$

Basic D-T Fourier Transform Pairs (cont'd)

$$x[n] = \text{box} \Leftrightarrow X(e^{j\omega}) = \text{periodic sinc}$$

$$x[n] = \text{sinc} \Leftrightarrow X(e^{j\omega}) = \text{box over 1 period}$$

Basic D-T Fourier Transform Properties

$$x_1[n] \Leftrightarrow X_1(e^{j\omega}) \qquad x_2[n] \Leftrightarrow X_2(e^{j\omega})$$

$$\boxed{\alpha\, x_1[n] + \beta\, x_2[n] \Leftrightarrow \alpha\, X_1(e^{j\omega}) + \beta\, X_2(e^{j\omega})}$$

$$\boxed{x_1[n-n_0] \Leftrightarrow e^{-j\omega n_0} X_1(e^{j\omega})}$$

$$\boxed{e^{j\omega_0 n} x_1[n] \Leftrightarrow X_1(e^{j(\omega-\omega_0)})}$$

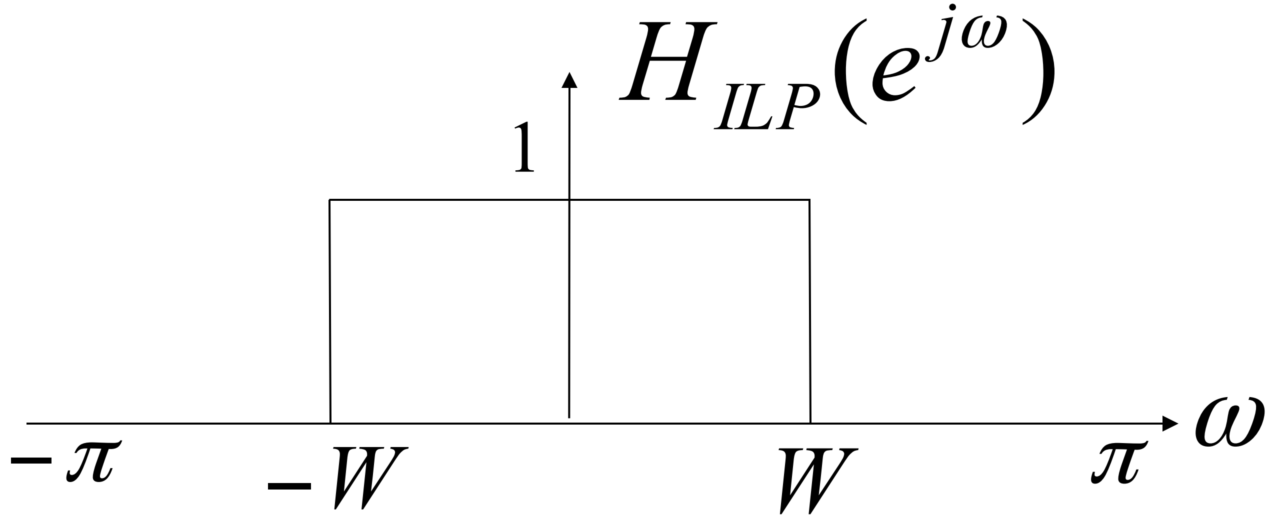
Basic D-T Fourier Transform Properties (cont'd)

$$\boxed{x_1[-n] \Leftrightarrow X_1(e^{-j\omega})}$$

$$\boxed{x_1^*[n] \Leftrightarrow X_1^*(e^{-j\omega})}$$

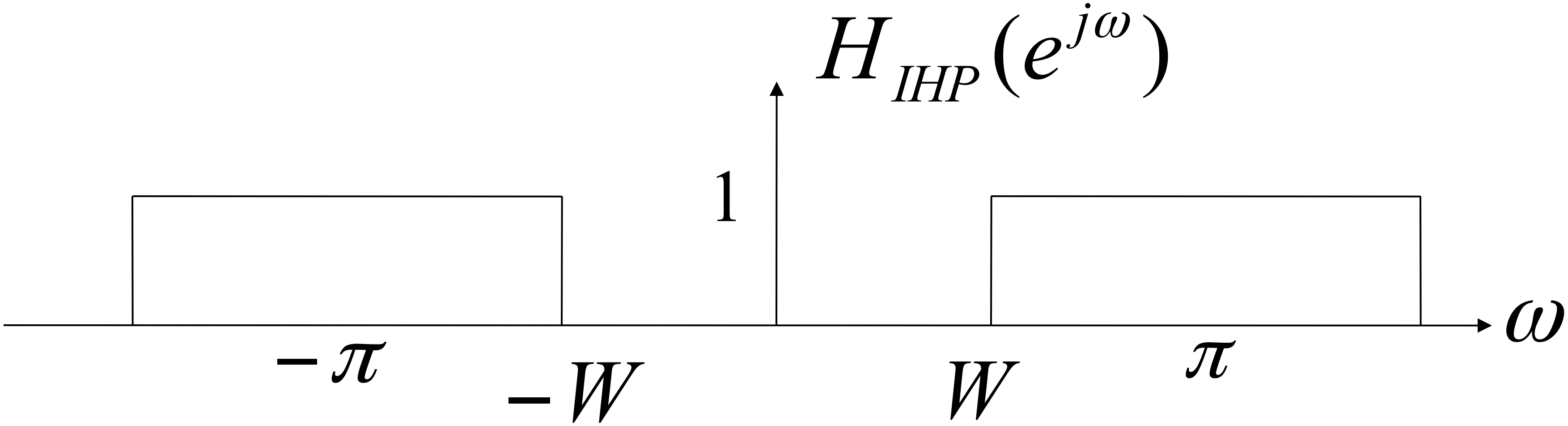
$$\boxed{\begin{aligned} &x_1[n]\, real \Leftrightarrow \\ &\qquad X_1(e^{j\omega}) = X_1^*(e^{-j\omega}) \end{aligned}}$$

D-T Ideal Lowpass Filters



$$h_{ILP}[n] = \frac{\sin Wn}{\pi n}$$

D-T Ideal Highpass Filters



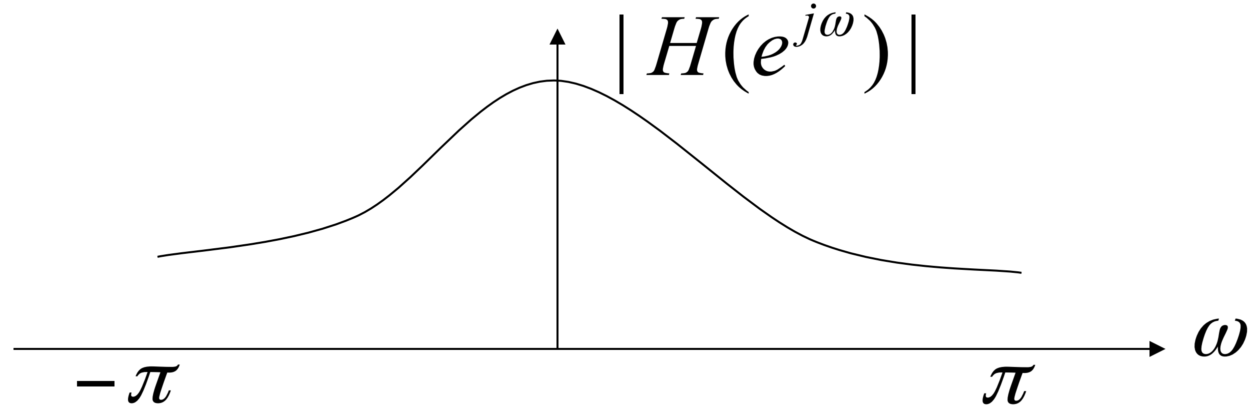
$$h_{IHP}[n] = \delta[n] - \frac{\sin Wn}{\pi n} = (-1)^n \frac{\sin((\pi - W)n)}{\pi n}$$

D-T LTI Systems Based on Difference Equations

$$y[n] = 0.5y[n-1] + x[n]$$

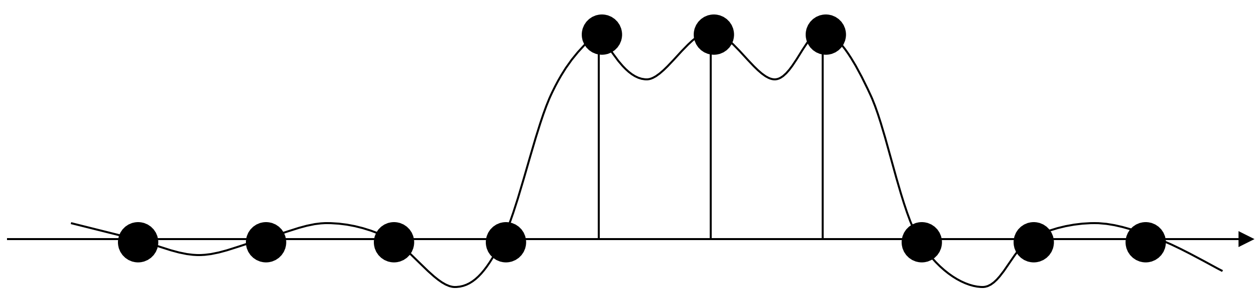
$$Y(j\omega) = 0.5e^{-j\omega}Y(e^{j\omega}) + X(e^{j\omega})$$

$$\therefore H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1-0.5e^{-j\omega}}$$



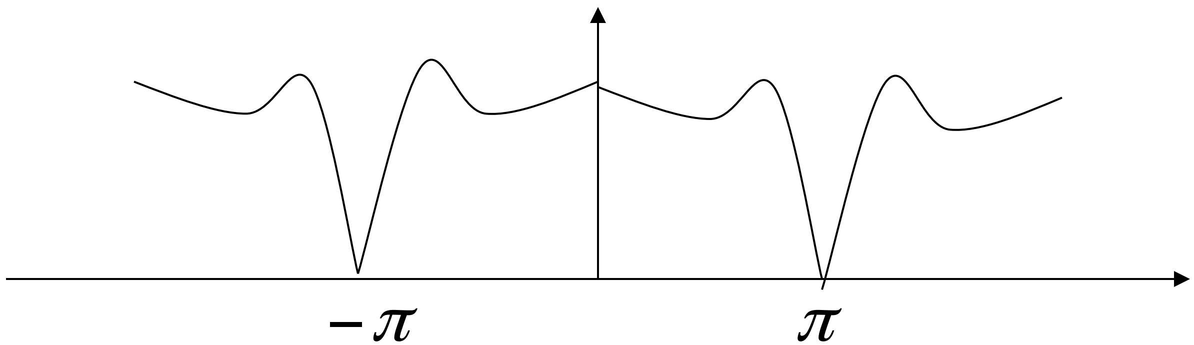
Analog Envelope of a Digital Signal: Time Domain

There is a unique analog envelope for every digital signal!!!!!!

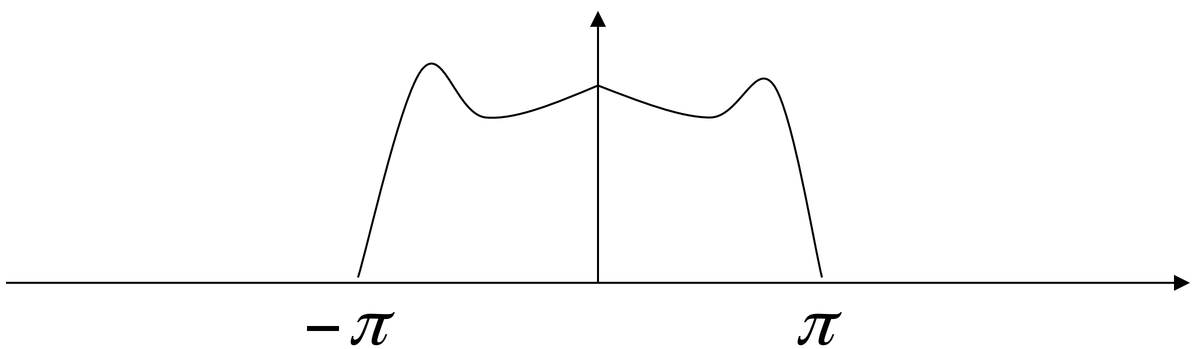


Analog Envelope of a Digital Signal: Frequency Domain

Discrete-Time Fourier Transform



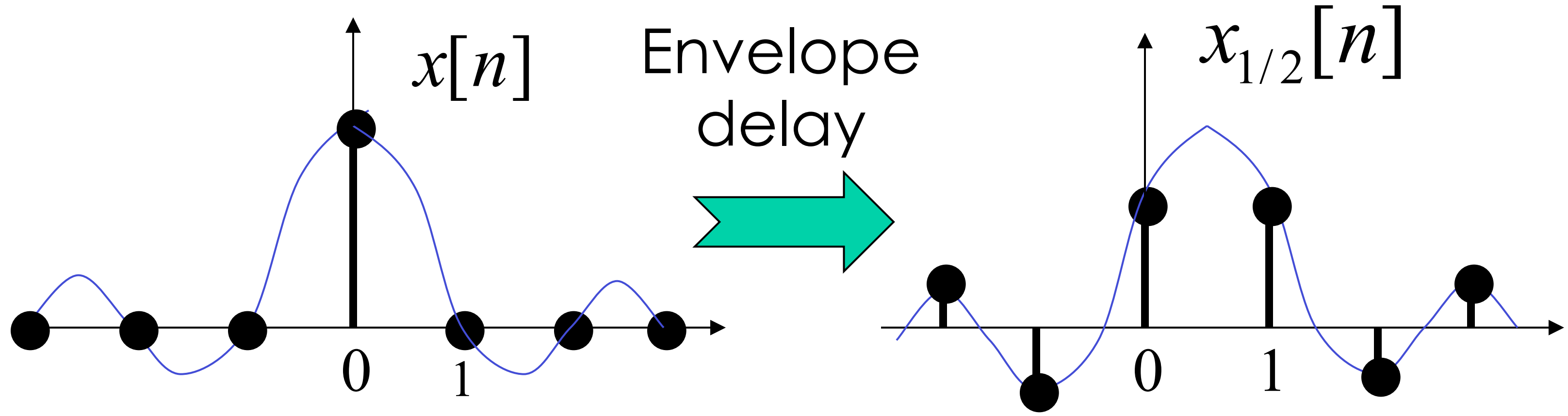
Analog Fourier Transform of Envelope



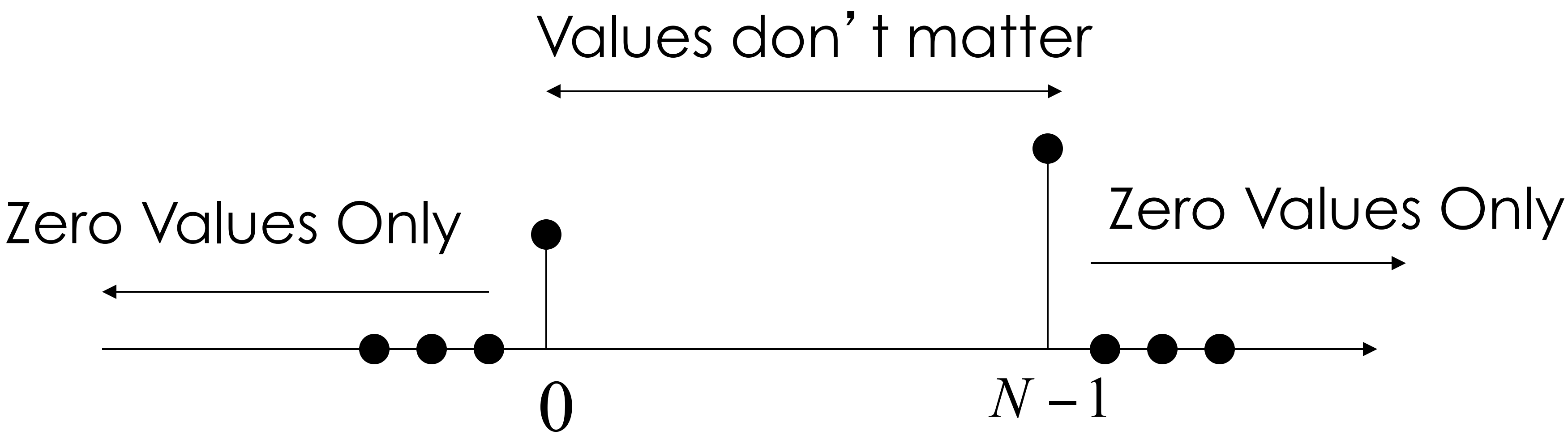
One can go back and forth between the two without information loss.

Non-Integer Delay of a Digital Signal

- **Non-integer Delay** in a digital signal $x[n]$ produces the digital signal $x_\alpha[n]$ corresponding to the non-integer shifted (by α) analog envelope of $x[n]$.
- **Example**: Delay of ½ in Unit Impulse.

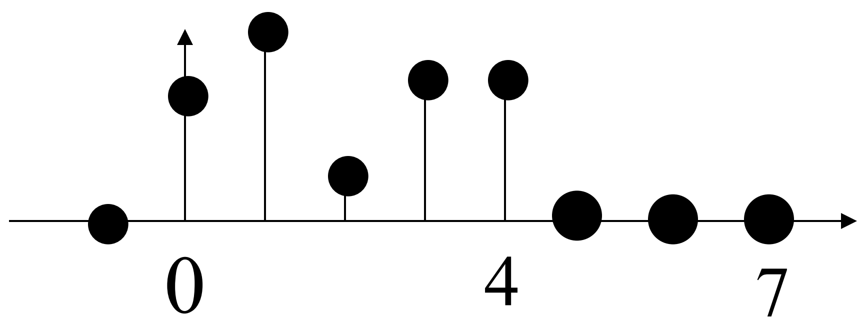


An N-point signal



Zero Padding

When a signal may be considered an N-point signal as well as an M-point signal ($M<N$), it is said to be a **zero-padded** N-point signal.



This is zero-padded as a 8-point signal.

This is not zero-padded as a 5-point signal.

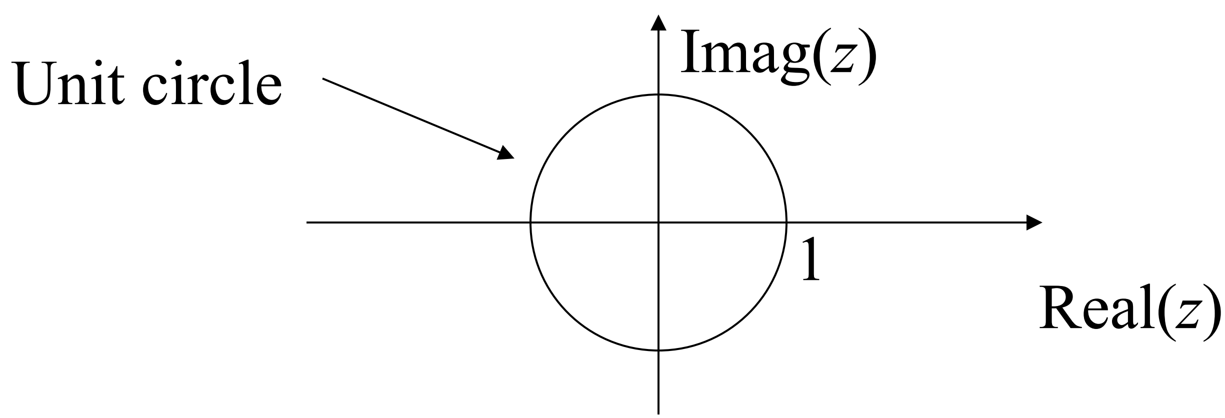
The Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \quad with \quad L < |z| < U$$

Region of Convergence

The Z-transform and the DTFT

- When $|z|=1$, we can write $z = e^{j\omega}$
- These values of z constitute the **unit circle** in the z plane.
- Unit circle is in the ROC of $X(z)$ if and only if $X(e^{j\omega})$ converges.



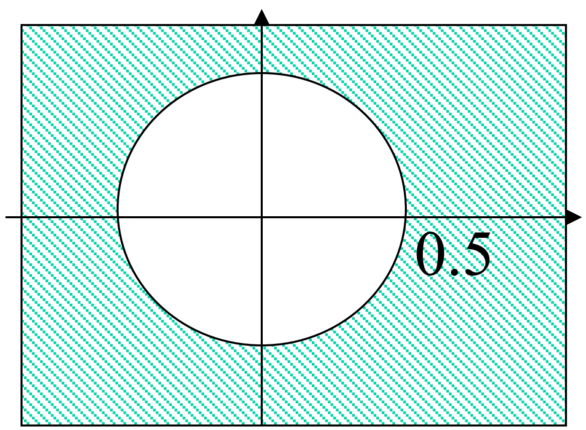
Z-transform Example I

- **Decaying Exponential:**

$$h[n] = (0.5)^n u[n]$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} = \sum_{n=0}^{\infty} (0.5)^n z^{-n} = \sum_{n=0}^{\infty} (0.5 z^{-1})^n$$

$$= \frac{1}{1 - 0.5 z^{-1}} \quad \textit{provided} \quad |z| > 0.5$$



Clearly, this ROC includes the unit circle. So the frequency response converges, and the filter is **stable**.

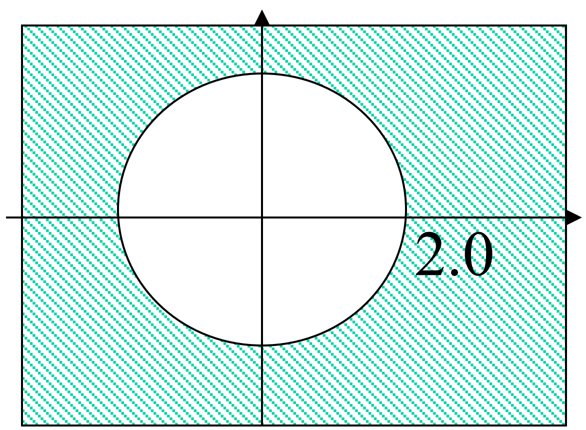
Z-transform Example II

- Growing Exponential:

$$h[n] = (2.0)^n u[n]$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} = \sum_{n=0}^{\infty} (2.0)^n z^{-n} = \sum_{n=0}^{\infty} (2.0 z^{-1})^n$$

$$= \frac{1}{1 - 2.0 z^{-1}} \quad provided \quad |z| > 2.0$$



Clearly, this ROC does not include the unit circle. So the frequency response diverges, and the filter is unstable.

Z-transform Examples: General

- Right-Sided Exponential:

$$\alpha^n u[n] \Leftrightarrow \frac{1}{1-\alpha z^{-1}} \quad ROC: |z| > |\alpha|$$

- Left-Sided Exponential:

$$-\alpha^n u[-n-1] \Leftrightarrow \frac{1}{1-\alpha z^{-1}} \quad ROC: |z| < |\alpha|$$

Z-transform Properties

- Similar to DTFT Properties—

$$1.\ \alpha\, x[n] + \beta\, y[n] \Leftrightarrow \alpha\, X(z) + \beta\, Y(z)$$

$$2.\ x[n - n_0] \Leftrightarrow z^{-n_0} X(z)$$

$$3.\ x[n] * h[n] \Leftrightarrow X(z) H(z)$$

- What happens to the ROC?

Z-transform Properties (cont’d)

- **Property 1 (Linearity):** ROC is at least the intersection of the two original ROCs.
- **Property 2 (Shift):** ROC is the original ROC except possibly at z=0 or infinity.
- **Property 3 (Convolution):** ROC is at least the intersection of the two original ROCs.

Z-transform and Difference Equations: System Function

$$y[n] = \sum_{k=1}^{N} \alpha_k y[n-k] + \sum_{m=0}^{M} \beta_m x[n-m]$$

Taking the Z-transform of both sides of this equation,

$$Y(z) = \sum_{k=1}^{N} \alpha_k z^{-k} Y(z) + \sum_{m=0}^{M} \beta_m z^{-m} X(z)$$

$$\therefore H(z) = \left(\sum_{m=0}^{M} \beta_m z^{-m}\right) \Big/ \left(1 - \sum_{k=1}^{N} \alpha_k z^{-k}\right)$$

System Function

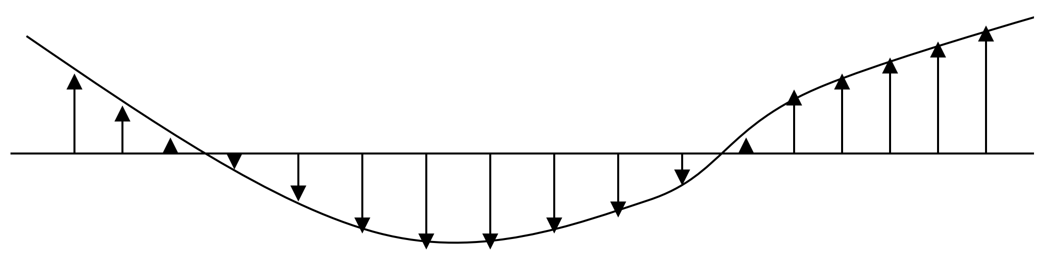
Poles and Zeroes

1. Write $H(z)$ as a ratio of a numerator polynomial in non-negative powers of z and a denominator polynomial in non-negative powers of z. That is, $H(z) = N(z)/D(z)$
2. Cancel out any common factors between N(z) and D(z).
3. Solve for the poles with $N(z) = 0$. and for the zeroes with $D(z) = 0$

Basics: Sampling
(Time Domain View)

$$x(t) \longrightarrow \otimes \longrightarrow g(t) = x(t)m(t)$$

$$m(t) = \sum_{k=-\infty}^{\infty} \delta\,(t - kT)$$



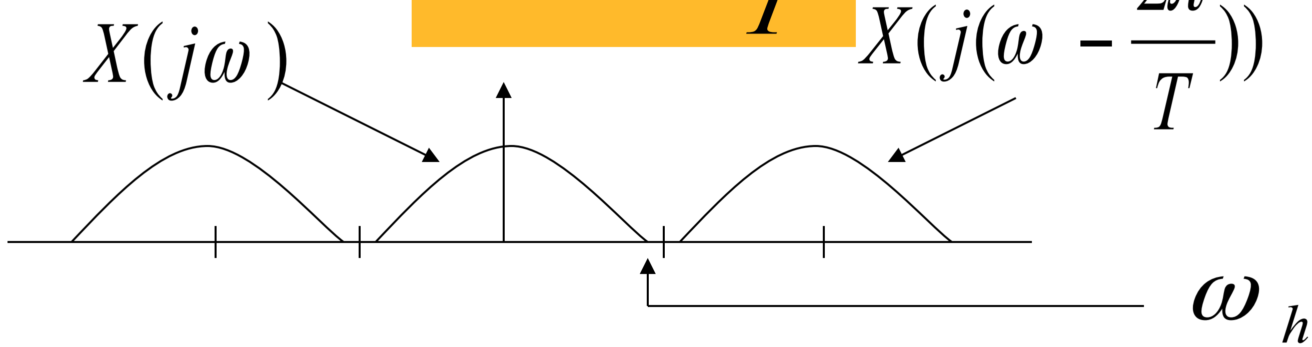
Basics: Sampling
(Frequency Domain View)

$$g(t) = x(t)m(t) = \sum_{k=-\infty}^{\infty} x(kT)\delta(t - kT)$$

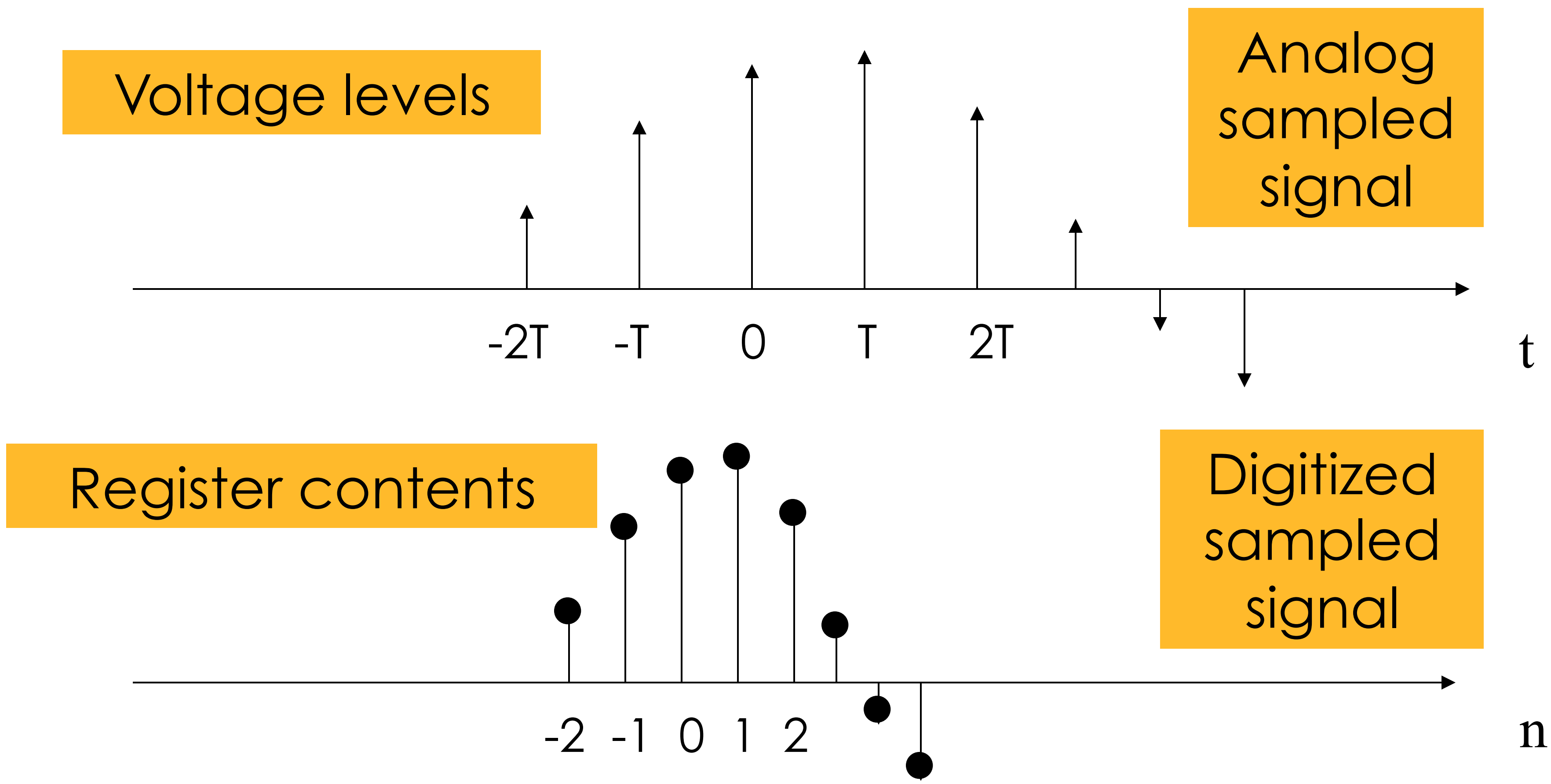
$$G(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - \frac{2\pi k}{T}))$$

Replication

To Avoid Aliasing $\boxed{2\omega_h \le \frac{2\pi}{T}}$



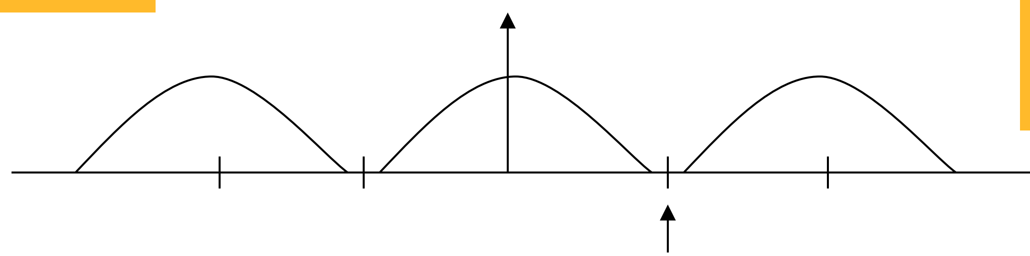
Digitization (Time Domain View)



Digitization
(Frequency Domain View)

$G(j\omega)$

For
Analog
sampled
signal

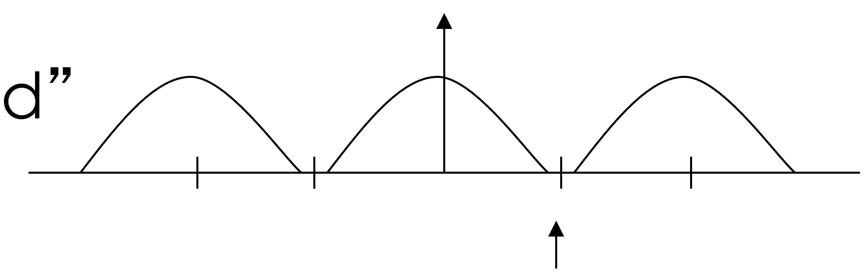


π / T

$G(j\omega / T)$

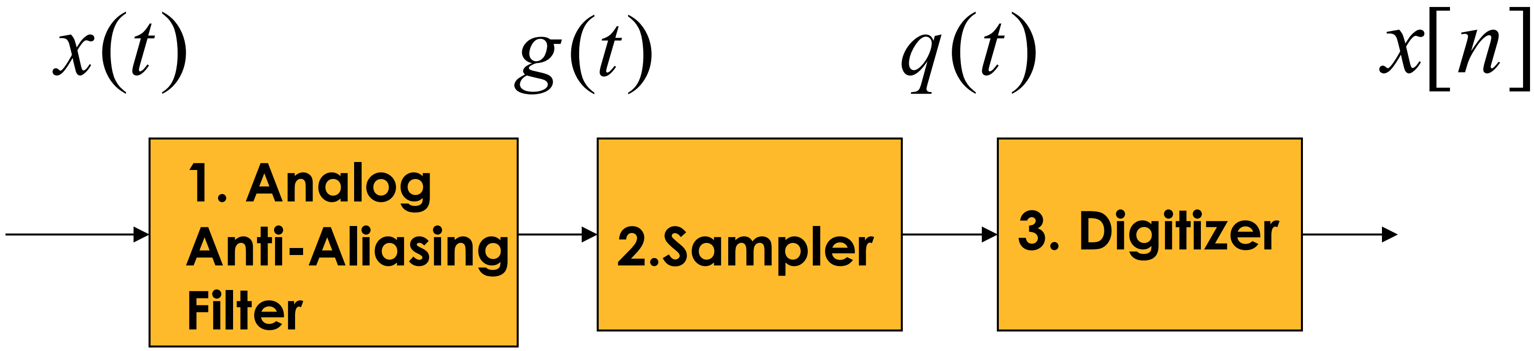
For
Digitized
sampled
signal

“compressed”



π

A/D Conversion – Frequency View



$$G(j\omega) = X(j\omega)H_{LP}(j\omega)$$

1. Tail Clipping

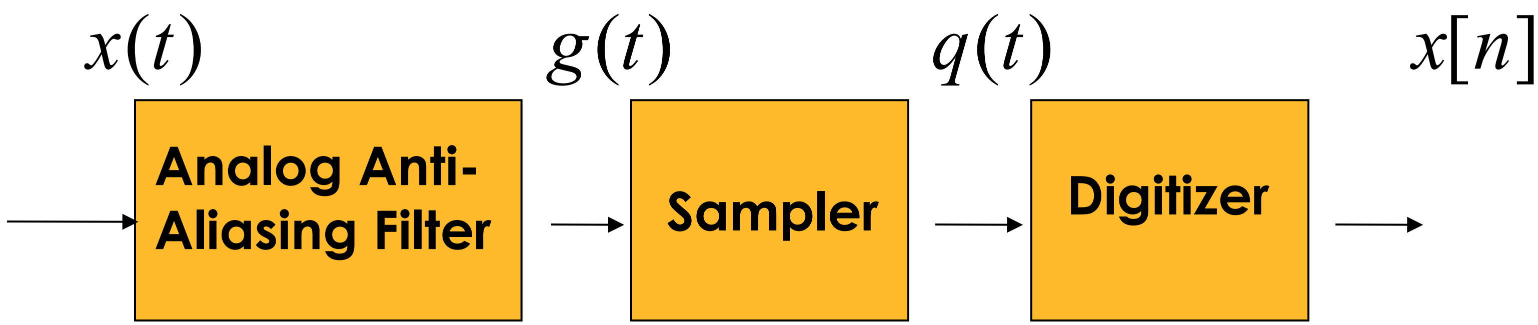
$$Q(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G\left(j\left(\omega - \frac{2\pi k}{T}\right)\right)$$

2a. Replication
2b. Amplitude Scaling

$$X(e^{j\omega}) = Q\left(j\frac{\omega}{T}\right)$$

3. Frequency Scaling

A/D Conversion – Time View

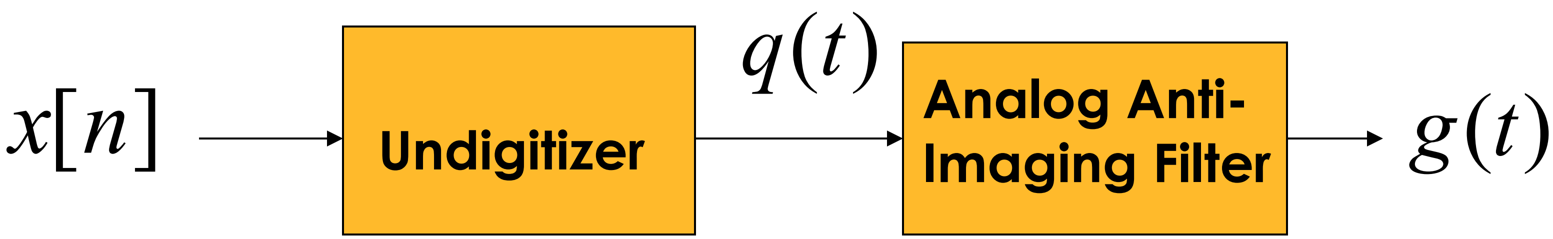


$$g(t) = x(t) * h_{LPF}(t)$$

$$q(t) = \sum_{k=-\infty}^{\infty} g(kT)\delta(t - kT)$$

$$x[n] = g(nT)$$

D/A Conversion – Time View



$$q(t) = \sum_{k=-\infty}^{\infty} x[k]\delta(t - kT)$$

$$g(t) = q(t) * h_{LPF}(t);$$

$$ideally, \quad h_{LPF}(t) = T\frac{\sin(\frac{\pi t}{T})}{\pi t}$$

D/A Conversion – Frequency View

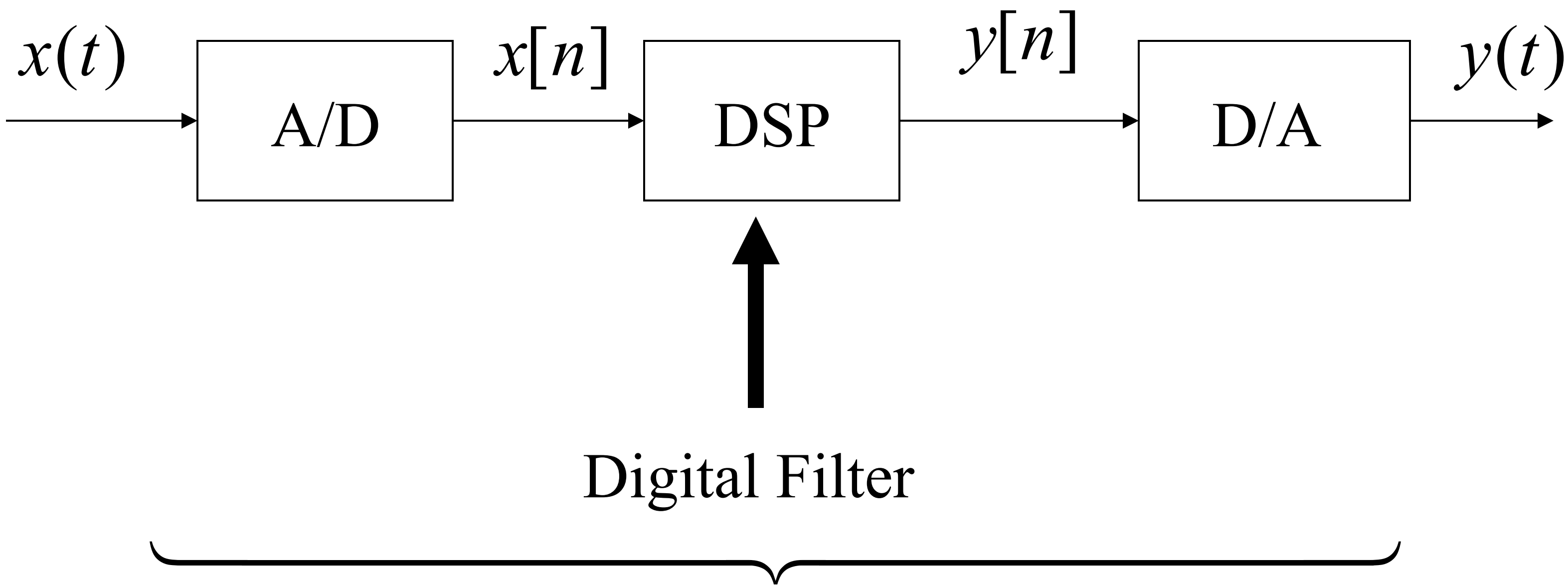
$$x[n] \longrightarrow \boxed{\text{Undigitizer}} \xrightarrow{q(t)} \boxed{\text{Analog Anti-Imaging Filter}} \longrightarrow g(t)$$

$$Q(j\omega) = X(e^{j\omega T})$$ Frequency scaling

$$G(j\omega) = Q(j\omega) H_{LPF}(j\omega);$$

$$ideally, \quad H_{LPF}(j\omega) = \begin{cases} T & |\omega| \le \pi / T \\ 0 & elsewhere \end{cases}$$

DSP Framework for Analog Filtering



Overall System is equivalent to an analog filter