MIDTERM CONTENT START

Plasma flow.

$$mn\left(\underbrace{\frac{d\vec{v}}{dt}}_{1} + \underbrace{(\vec{v}\cdot\nabla)\vec{v}}_{2}\right) = -\nabla p + qn\underbrace{(\vec{E} + \vec{v}\times\vec{B})}_{\text{Lorentz}} \quad 3$$

$mn = \rho$

Consider motions $\perp \vec{B}$ and $f = \hat{f}e^{i\omega t}$

$$\frac{df}{dt} = i\omega f$$

$$\frac{1}{3} = \left|\frac{m\cancel{n} i\omega \cancel{v_\perp}}{q\cancel{n}\cancel{v_\perp}\times B}\right| = \frac{\omega}{\omega_c}; \qquad \omega_c = \frac{|q|B}{m}$$

If $\frac{\omega}{\omega_c} \ll 1$, $\frac{d}{dT}$ can be neglected

$\frac{2}{3} \ll 1$ by assumption

$$0 = qn(\vec{E} + \vec{V}_\perp\times\vec{B}) - \nabla p \;\Big|\; \times\vec{B}$$

$$qn(\vec{E}\times\vec{B} + (\vec{V}_\perp\times\vec{B})\times\vec{B}) - \nabla p\times\vec{B} = 0$$

$$\vec{a}\times(\vec{B}\cdot\vec{c}) = \vec{B}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{B})$$

$$qn\left(\vec{E}\times\vec{B} - \vec{V}_\perp B^2 + \underbrace{\vec{B}(\vec{V}_\perp\cdot\vec{B})}_{=0 \text{ by construction}}\right) - \nabla p\times\vec{B} = 0$$

$$\vec{V}_\perp = \underbrace{\frac{\vec{E}\times\vec{B}}{B^2}}_{\vec{V}_E} - \underbrace{\frac{\nabla p\times\vec{B}}{qnB^2}}_{V_D}$$

$\vec{V}_E \to$ E×B drift

$V_D \to$ magnetic drift

Recall $\nabla p = \gamma k_B T \nabla n$

$$\vec{V}_\perp = \frac{\vec{E}\times\vec{B}}{B^2} - \frac{\gamma K_B T \nabla n \times \vec{B}}{q n B^2}$$

assume charge $q = \pm e$

$$V_D = \pm \frac{\gamma K_B T}{eB} \frac{\vec{i}_z \times \nabla n}{n}$$

where $\vec{i}_z = \frac{\vec{B}}{|\vec{B}|}$

$B \parallel O_z$

Fluid motion $\parallel$ to $\vec{B}$

$$mn\left(\frac{dV_z}{dt} + \underbrace{V_z \frac{dV_z}{dz}}_{\text{assume to be small}}\right) = qnE_z - \frac{\partial p}{\partial z}$$

$$\frac{dV_z}{dt} = \frac{q}{m}E_z - \frac{\gamma K_B T}{mn}\frac{dn}{dz}$$

consider electrons only:

electrons quickly respond to the action of electric field so acceleration can be ignored

$$qE_z - \frac{\gamma K_B T}{n}\frac{dn}{dz} = 0$$

$$\boxed{e\frac{d\varphi}{dz} = \frac{\gamma K_B T}{n}\frac{dn}{dz}}$$

integrate

$$\boxed{n = n_0 \exp\left(\frac{e\varphi}{\gamma K_B T_e}\right)}$$ ← boltzmann distribution

if isothermal $T_e = \text{const}$ $\gamma = 1$

$$n = n_0 \exp\left(\frac{e\varphi}{K_B T}\right)$$

Plasma approximation: $n_i = n_e$

$\nabla \cdot E \sim n_i - n_e$

Waves in Plasma

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} \, dk$$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx$$

$\hat{f}(k)$ can be found in a table

k is called the wave #

also

$$f(\vec{r}, t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(\vec{k}, \omega) e^{\underbrace{i(\vec{k}\cdot\vec{r} - \omega t)}_{\text{the phase of a wave}}} d^3\vec{k} \, d\omega$$

$\vec{k}$ the wave vector

$\hat{f}(k_x, k_y, k_z, \omega)$: $d^3\vec{k} = dk_x \, dk_y \, dk_z$

$k \cdot r - \omega t = \text{const}$ defines a plane

$\Longleftrightarrow$

plane waves

if $k \cdot r - \omega t = \text{constant}$ we can calculate the derivative:

$$\vec{k} \cdot \frac{dr}{dt} = \omega \Longleftrightarrow \vec{k} \cdot \vec{v} = \omega$$

So $\vec{v} \equiv v_p = v_\varphi$ = phase speed = $\frac{\omega}{k} \hat{k}$; $\hat{k} = \frac{\vec{k}}{|\vec{k}|}$

assume density

$$n = \bar{n} \exp\left[i(\vec{k}\cdot\vec{r} - \omega t)\right]$$ and $\bar{n}$ is real

$$\mathrm{Re}\, n = \bar{n}\cos(\vec{k}\cdot\vec{r} - \omega t)$$

Then

real

$$\vec{E} = E\exp\left[i(\vec{k}\cdot\vec{r} - \omega t) + i\delta\right]$$

complex

$$\equiv E_c \exp\left[i(\vec{k}\cdot\vec{r} - \omega t)\right]$$

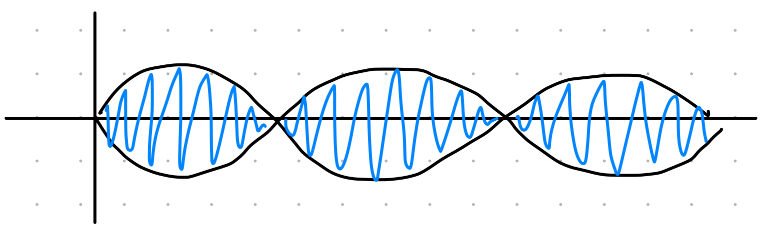
$$E_c = E\, e^{i\delta}$$

$$\tan\delta = \frac{\mathrm{Im}\, E_c}{\mathrm{Re}\, E_c}$$

Group Velocity

$V_\varphi = \frac{\omega}{k}$ if $k \to 0$ $V_\varphi \to \infty$

If $\Delta\omega \to 0$, $\Delta k \to 0$



result

$$V_g = \frac{d\omega}{dk}$$

group velocity

$$E_1 = E_0\cos\left[(k+\Delta k)x - (\omega + \Delta\omega)t\right]$$

$$E_2 = E_0\cos\left[(k-\Delta k)x - (\omega - \Delta\omega)t\right]$$

$E_1 + E_2 = ?$

$a = kx - \omega t$

$b = \Delta k x - \omega t$

$$E_1 + E_2 = E_0\left[\cos(a+b) + \cos(a-b)\right]$$

$$= E_0\left[\cos a\cos b - \cancel{\sin a \sin b} + \cos a\cos b + \cancel{\sin a\sin b}\right]$$

$$= 2E_0\cos a\cos b \rightarrow 2E_0\left[\cos(kx-\omega t)\cos(\Delta k x - \Delta\omega t)\right]$$

$$V_g = \frac{d\omega}{dk}$$

V_g can never be infinite

The phase speed is the property of the median

$$\frac{\omega}{K} = \frac{\omega+\Delta\omega}{K+\Delta K} = \frac{\omega-\Delta\omega}{K-\Delta K} \quad *$$

For $e^{i(\vec{K}\cdot\vec{r}-\omega t)} = e^{i(kx+ky+kz-\omega t)}$

$$\nabla\cdot(\exp(\ldots)) = (ik_x+ik_y+ik_z)\exp(\ldots)$$

$$* \quad \nabla\cdot\vec{E} = i\vec{K}\cdot\vec{E}$$

$$\frac{\partial E}{\partial t} = -i\omega\vec{E}$$

$$\nabla\times\vec{E} = i\vec{K}\times\vec{E}$$

Plasma oscillations

- $\vec{B}=0$
- $T=0$
- ions are motionless
- Infinite volume
- electrons are moving $\parallel Ox$

Equations:

(1) $$mn_e\left(\frac{\partial\vec{V}_e}{\partial t}+\left((\vec{V}_e\cdot\nabla)\vec{V}_e\right)\right) = -en_e\vec{E}$$

$\rightarrow$ normalization

$m(-i\omega v_1) = -eE_1$

$v_1 = \frac{eE_1}{i\omega m}$ (4)

(2) $$\frac{\partial n_e}{\partial t}+\nabla\cdot(n_e\vec{V}_e) = 0$$

(3) $$\varepsilon_0\nabla\cdot\vec{E} = e(n_i-n_e)$$

$\parallel$

$\varepsilon_0\frac{dE}{dx}$ for 1D $\qquad E = E_x$

Linearization

$n_e = n_0 + n_1$ $n_0 =$ const $n_1 \rightarrow$ small perturbation

$V_e = V_{e_0} + V_1$ for this approx w/ no movement $V_{e_0} = 0$

$V_e = V_1$

$E = E_0 + E_1$ no initial E b/c no shift

$E = E_1$

from (2)

$$\frac{\partial}{\partial t}(n_0 + n_1) + \frac{\partial}{\partial x}\left((n_0+n_1)(v_0+v_1)\right) = 0$$

0 small

$$\frac{\partial n_1}{\partial t} + \frac{\partial}{\partial x}\left[\cancel{n_0 V_0}^{0} + n_0 V_1 + \cancel{n_1 V_0}^{0} + \cancel{n_1 V_1}^{0}\right] = 0$$

from $V_0 = 0$

$$\frac{\partial n_1}{\partial t} + \frac{\partial}{\partial x}(n_0 v_1) = 0$$

$$-i\omega n_1 + ik n_0 V_1 = 0$$

$$n_1 = \frac{n_0 k}{\omega} V_1 \quad (5)$$

$$E_1 = \frac{-en_1}{ik\epsilon_0} \quad (6)$$

put others into (6)

$$E_1 = \frac{n_0 e^2}{\epsilon_0 m \omega^2} \rightarrow \frac{\omega_p^2}{\omega^2} E_1$$

$$\left(1 - \frac{\omega_p^2}{\omega^2}\right) E_1 = 0$$

so $\omega_p^2 = \omega^2$

dispersion relation for plasma

1. $$mn_e\left(\frac{\partial \vec{v}_e}{\partial t} + ((\vec{v}_e \cdot \nabla)\vec{v}_e\right) = -en_e\vec{E}$$ If $T \neq 0$

$$mn_e\left(\frac{\partial \vec{v}_e}{\partial t} + ((\vec{v}_e \cdot \nabla)\vec{v}_e\right) = -en_e\vec{E} - \nabla P_e$$

$$mn_e\left(\frac{\partial \vec{v}_e}{\partial t} + ((\vec{v}_e \cdot \nabla)\vec{v}_e\right) = -en_e\vec{E} - \gamma K_B T \nabla n$$

D.R $T \neq 0$ $$w^2 = w_p^2 + \frac{3K_B T_e}{2?\, m} K^2$$

Bohm-Gross wave

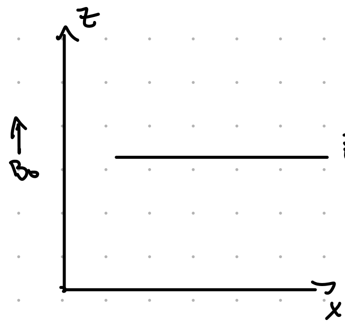
Lecture Notes from

10/8

general oscillation in cold plasma

whereas transverse are

1: lecture 14 perpendicular, electrostatic (longitudinal) waves in electron fluid



$\vec{E}$, $\vec{K} \parallel \vec{E}$, $\vec{E} = \vec{E}_1$ $(\vec{E}_0 = 0)$

$\Updownarrow$

$\vec{K} \parallel \vec{E}_1$

recall $\nabla \times \vec{E} = -\dot{\vec{B}}$

so $\vec{K} \times \vec{E}_1 = \omega \vec{B}_1$

If $\vec{B}_1 = 0$, then $\vec{K} \parallel \vec{E}_1$

$\vec{V}_0 = 0$

$\vec{E}_1 \parallel Ox$

$$m \frac{\partial \vec{V}_{e_1}}{\partial t} = -e\left(\vec{E}_1 + \vec{V}_{e_1} \times B_0\right) \quad (1)$$

$T_e = 0$.

cont. eq

$$\frac{\partial n_{e1}}{\partial t} + n_0 \nabla \cdot \vec{V}_{e_1} = 0 \quad (2)$$

$$\nabla \cdot (n \vec{V}_e) = \nabla \cdot \left[(n_0 + n_1)(\underbrace{\vec{V}_0}_{=0} + V_{e_1})\right]$$

$$\varepsilon_0 \nabla \cdot (\underbrace{\vec{E}_0}_{=0} + \vec{E}_1) = e(n_i - n_e)$$

but $n_{i0} = n_{e0}$ (without perterbation there is no charge separation)

$$\varepsilon_0 \nabla \cdot (\vec{E}_1) = -e n_{e1} \quad (3)$$

$n_e = n_{e0} + n_{e_1}$

$$\vec{K} \parallel O_x \quad \text{so} \quad \vec{V}_1 \times \vec{B}_0 = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & \vec{B}_0 \end{vmatrix} = \hat{i}\, V_y B_0 - \hat{j}\, V_x B_0$$

now substitute into equation of motion

$$\begin{cases} -i\omega m V_{ex} = -e E_1 - e V_y B_0 \\ -i\omega m V_{ey} = e V_x B_0 \\ -i\omega m V_{ez} = 0 \Rightarrow V_{ez} = 0 \end{cases}$$

continuity equation:

$$-i\omega n_1 + i K n_0 V_x = 0 \longrightarrow n_1 = n_0 \frac{K}{\omega} V_x$$

maxwell:

$$ik\varepsilon_0 E_1 = -en_1$$

(A) find v_y

$$v_y = i\frac{eB_0}{m\omega}v_x = i\frac{\omega_c}{\omega}v_x \quad \text{and } i = e^{i\pi/2}$$

(B) substitute v_y into equation for v_x

$$i\omega m v_x = eE_1 + ev_yB_0$$

$$v_x = \frac{eE_1}{\cancel{i}\omega m} + \cancel{i}\frac{eB_0}{\omega m}\frac{\omega_c}{\omega}v_x \quad \text{so} \quad v_x = \frac{eE_1/i\omega m}{1-\omega_c^2/\omega^2} \qquad \omega_c = \frac{eB_0}{m}$$

(C) $n_1 = n_0\frac{k}{\omega}v_x$

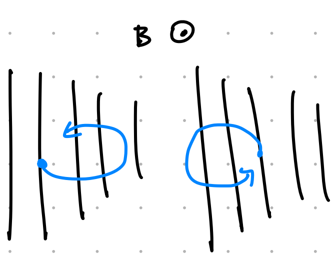
(D) $ik\varepsilon_0 \cancel{E_1} = -en_0\frac{k}{\omega}\left(\frac{e\cancel{E_1}/i\omega m}{1-\omega_c^2/\omega^2}\right)$

$$\varepsilon_0 = \frac{e^2 n_0}{m}\frac{1}{\omega^2-\omega_c^2}$$

$$1 = \underbrace{\frac{e^2 n_0}{\varepsilon_0 m}}_{\omega_p^2 \text{ electron plasma frequency}}\frac{1}{\omega^2-\omega_c^2}$$

$$\omega^2 = \omega_p^2 + \omega_c^2 \rightarrow \text{the upper hybrid frequency (squared)}$$

Planes of constant density

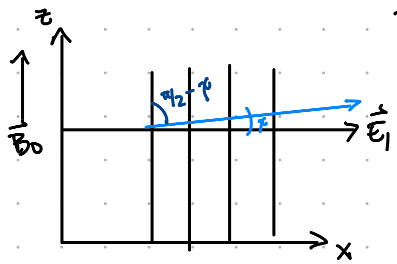


paths are ellipses

2: electrostatic (longitudinal) ion waves $\perp \vec{B}_0$

$\Rightarrow \vec{B}_1 = 0$ $\quad T_i = 0$ (cold); $\vec{K} \parallel \vec{E}_1$

$T_e \neq 0$



χ is small if it is $< \chi_0$

$$\frac{mv_{\parallel}^2}{2} = \frac{Mv_{\perp}^2}{2}; \quad \frac{v_{\parallel}}{v_{\perp}} \sim \tan\chi \approx \tan\chi_0 \quad \text{but } \chi_0 = \left(\frac{m}{M}\right)^{1/2}$$

Now, $\vec{E}_1 = -\nabla\varphi_1 = -iK\varphi_1$

Equation of motion for ions

$$M\frac{d\vec{v}_{i1}}{\partial t} = e\vec{E}_1 + e(\vec{v}_{i1} \times B_0)$$

χ is small from the ion perspective

Electric field $\vec{E}_1 = E_1 \vec{i}_x$ $\quad$ with $\vec{K} \parallel \vec{i}_x$

$$\begin{cases} -i\omega M v_x = -ike\varphi_1 + ev_y B_0 \\ -i\omega M v_y = \qquad\qquad -ev_x B_0 \end{cases}$$

$v_x = v_{ix} \qquad v_y = v_{iy}$

find: $\quad v_y = \dfrac{eB_0}{i\omega M} v_x$

substitute:

ion cyclotron freq.

$$-i\omega M v_x = -ike\varphi_1 + \frac{e^2B_0^2 M}{i\omega M^2} v_x \qquad \text{so } v_x = \frac{ke\varphi_1/\omega M}{1 - \frac{\Omega_c^2}{\omega^2}}$$

$$\omega M v_x = ke\varphi_1 + \frac{\Omega_c^2 M}{\omega} v_x$$

continuity:

$$n_{i1} = n_0 \frac{k}{\omega} v_x$$

Electrons are in the Boltzmann equilibrium

$$n_e = n_0 \exp\left(\frac{e\varphi_1}{k_B T_e}\right) \approx n_0 \frac{e\varphi_1}{k_B T_e}$$

Quasineutrality: $n_{e1} = n_{i1}$

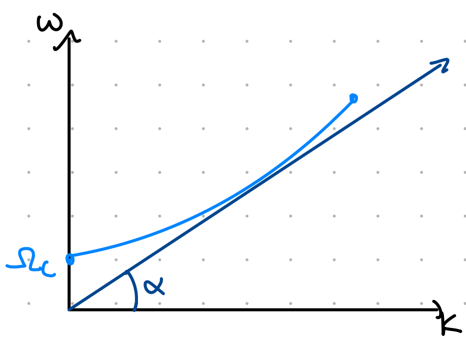
$$\Rightarrow v_x = \frac{\omega}{k} \frac{e\varphi_1}{k_B T_e}$$

$$\frac{ek}{\omega M} \cancel{\varphi_1} \left(1 - \frac{\Omega_c^2}{\omega^2}\right)^{-1} = \frac{\omega}{k} \frac{e\cancel{\varphi_1}}{k_B T_e}$$ (only ω, k → disp rel.)

$$\frac{1}{1 - \Omega_c^2/\omega^2} = \frac{\omega^2}{k^2} \frac{M}{k_B T_e}$$

$$1 - \frac{\Omega_c^2}{\omega^2} = \frac{k^2}{\omega^2} \frac{k_B T_e}{M}$$ → ion acoustic wave for $T_i = 0$ v_s^2

$$\Omega^2 = \Omega_c^2 + k^2 v_s^2$$ dispersion relation electrostatic ion cyclotron waves



$\tan\alpha = v_s$ $k \to \infty$ $\omega \to k v_s$

$v_\varphi = \frac{\omega}{k} \to v_s$ at $k \to \infty$,

$k = \frac{2\pi}{\lambda}$ as wavelength goes to 0

group velocity $d\omega/dk$

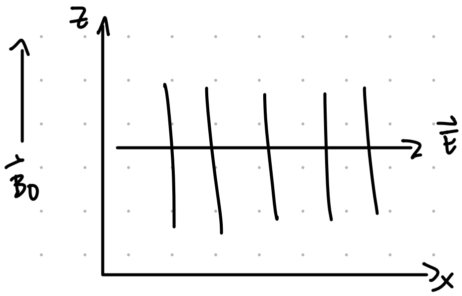
$$2\omega \underbrace{\frac{d\omega}{dK}}_{\text{group velocity}} = 2KV_s^2$$

group velocity $= \dfrac{KV_s^2}{\omega}$ but $\omega = \sqrt{\Omega_c^2 + K^2 V_s^2}$ for $K \to \infty$ ($\lambda \to 0$)

$V_g \to V_s$

even if $T_i = 0$ $V_s \neq 0$

3: electrons not allowed to move $\parallel$ to magnetic field



$\vec{E}$ $\vec{K} \parallel \vec{E}_1 \parallel \partial_x$

options: a) $n_{i1} = n_{e1}$

b) $\varepsilon_0 \nabla \cdot \vec{E}_1 = e(n_{i1} - n_{e1})$

DO THESE DERIVATIONS

a)

Solutions:

$$V_{ix} = \frac{eK}{\omega M}\varphi_1 \left(1 - \frac{\Omega_c^2}{\omega^2}\right)^{-1}$$

$$V_{ex} = \frac{-eK}{\omega M}\varphi_1 \left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1}$$

of course we remember $\omega_c \gg \Omega_c$

Continuity:

$$\left.\begin{aligned} n_{i1} &= n_0 \frac{K}{\omega} V_{ix} \\ n_{e1} &= n_0 \frac{K}{\omega} V_{ex} \end{aligned}\right\}$$

if $n_{i1} = n_{e1} \Rightarrow V_{ix} = V_{ex}$

$$\frac{eK}{\omega M}\cancel{\varphi_1}\left(1 - \frac{\Omega_c^2}{\omega^2}\right)^{-1} = \frac{-eK}{\omega m}\cancel{\varphi_1}\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1}$$ (only ω & K → dispersion)

$$\frac{1}{M}\left(1 - \frac{\omega_c^2}{\omega^2}\right) = -\frac{1}{m}\left(1 - \frac{\Omega_c^2}{\omega^2}\right) \to m(\omega^2 - \omega_c^2) = -M(\omega^2 - \Omega_c^2)$$

$$\to (m+M)\omega^2 = M\Omega_c^2 + m\,\omega_c^2 \to e^2 B_0^2\left(\frac{1}{M} + \frac{1}{m}\right)$$

$$\omega^2 = \omega_c \Omega_c \;; \quad \omega = \sqrt{\omega_c \Omega_c} \equiv \omega_\ell$$ lower hybrid frequency

we can consider the wave breaks quasineutrality in chen:

if $n_{e1} \neq n_{i1}$ then $\frac{1}{\omega^2} = \frac{1}{\omega_c \Omega_c} + \frac{1}{\Omega_p^2}$ ← not true!

but this <u>IS</u> true only for $\omega_c^2 \gg \omega^2 \gg \Omega_c$

this will be shown in HW

Electromagnetic waves with $0 = 0$ in vacuum → $(\vec{J} = 0)$ no current

$\nabla \cdot B$

E/m waves in plasma

1) E/m waves with $\vec{B}_0 = 0$
↳ transverse → $\vec{k} \perp \vec{E}_1$

Vacuum	Plasma
$\nabla \times \vec{E}_1 = -\dot{\vec{B}}_1$ $(i\vec{k} \times \vec{E}_1 = i\omega \vec{B}_1)$	$\nabla \times \vec{E}_1 = -\dot{\vec{B}}_1$
$c^2 \nabla \times \vec{B}_1 = \dot{\vec{E}}_1$ $(ic^2 \vec{k} \times \vec{B}_1 = -i\omega \vec{E}_1)$	$c^2 \nabla \times \vec{B}_1 = \frac{\vec{J}_1}{\epsilon_0} + \dot{\vec{E}}_1$

$$\begin{cases} c^2 \nabla \times \dot{\vec{B}}_1 = \frac{1}{\epsilon_0} \frac{\partial j_1}{\partial t} + \ddot{\vec{E}}_1 \\ \nabla \times (\nabla \times \vec{E}_1) = \nabla(\nabla \cdot \vec{E}_1) - \nabla^2 \vec{E}_1 \\ \qquad = -\dot{\vec{B}}_1 \end{cases}$$

$$\nabla^2 \vec{E}_1 - \nabla(\nabla \cdot \vec{E}_1) = \frac{1}{\epsilon_0 c^2} \frac{\partial j_1}{\partial t} + \frac{1}{c^2} \ddot{\vec{E}}_1$$

For plane waves:

$$\vec{k}^2\vec{E}_1 - \vec{k}(\underbrace{\vec{k}\cdot\vec{E}_1}_{=0\text{ b/c the wave is transverse}}) = \frac{i\omega}{\epsilon_0 c^2}\vec{J}_1 + \frac{\omega^2}{c^2}\vec{E}_1$$

$$(\omega^2 - c^2k^2)\vec{E}_1 = -\frac{i\omega}{\epsilon_0}\vec{J}_1$$

need to express J in terms of E_1

$\vec{J}_1 = -en\vec{v}_{e1}$ (waves in the electron fluid <=> ions are motionless)

equation of motion for electrons

$$m\frac{\partial \vec{v}_{e1}}{\partial t} = -e\vec{E}_1$$

$$-i\omega v_{e1} = -e\vec{E}_1$$

$$v_{e1} = \frac{e\vec{E}_1}{i\omega}$$

substitute in:

$$\Rightarrow (\omega^2 - c^2k^2)\cancel{E_1} = \frac{\cancel{i\omega}}{\epsilon_0} n_0 e \frac{e\cancel{E_1}}{\cancel{i\omega} m}$$

$$\Rightarrow \omega^2 - c^2k^2 = \underbrace{\frac{n_0 e^2}{\epsilon_0 m}}_{\omega_p^2}\vec{E}_1$$

$$\omega^2 = \omega_p^2 + c^2k^2 \quad \text{dispersion relation}$$

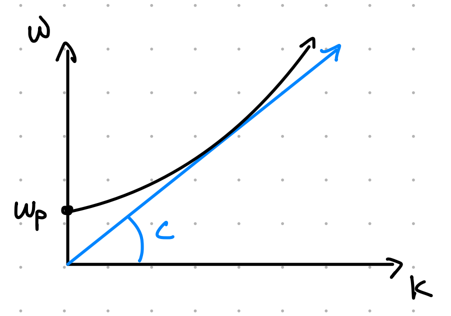
phase velocity

$$\frac{\omega^2}{k^2} = v_\varphi^2 = c^2 + \frac{\omega_p^2}{k^2} \quad \text{so } v_p > c$$

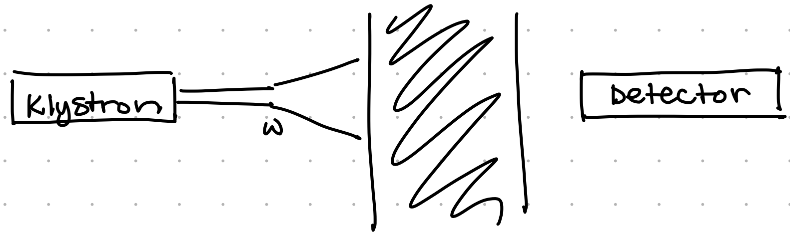
HW

group velocity $\quad 2\omega\frac{d\omega}{\partial k} = 2c^2k$

$$\frac{d\omega}{dk} = \frac{c^2}{v_\varphi} \quad \text{as } v_\varphi > c;\ v_g = \frac{\partial\omega}{\partial k} < c$$



Wave Cutoff



Plasma of density n_0

Inside plasma $k = \frac{2\pi}{\lambda}$ where k is taken from dispersion relation and λ = wavelength

$n_0 \uparrow = \omega_p \uparrow = k \downarrow$

k decreases until it is equal to 0!

$$\Rightarrow \omega = \omega_p \Rightarrow \frac{n_0 e^2}{\epsilon_0 m} = \omega_p{}^2 = \omega^2$$

so $n_0 = \frac{e^2 \omega^2}{\epsilon_0 m}$ → waves will stop propagating and be cut off

Suppose this happened:

$$ck = (\omega^2 - \omega_p^2)^{1/2}$$

$\omega^2 - \omega_p^2$ will be negative so k becomes imag.

$$= i|\omega_p^2 - \omega^2|^{1/2}$$

$$f \propto e^{ikx} = e^{-|k|x} = e^{-x/\delta}$$

where $\delta = |k|^{-1} = \frac{c}{(\omega_p^2 - \omega^2)^{1/2}}$

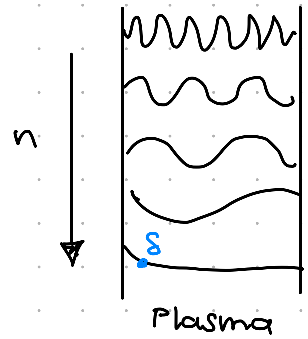
so δ is called the skin depth

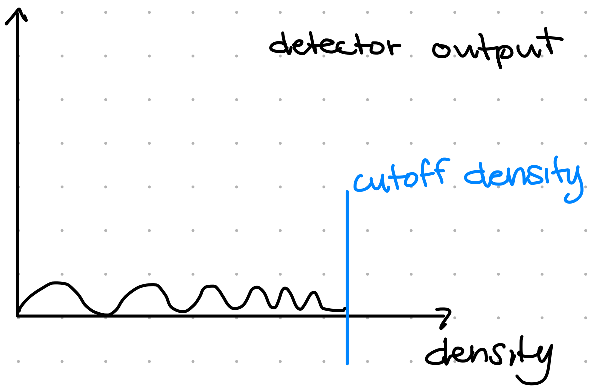
We frequently use the refraction index

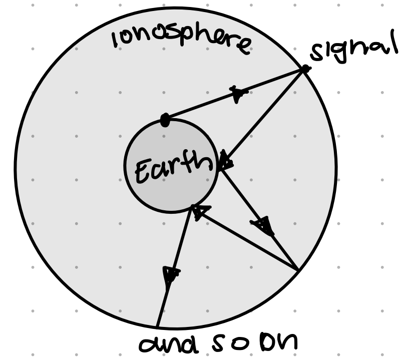
HW

$$\tilde{n} = \frac{c}{v_\varphi} = \frac{ck}{\omega}$$

let density(n) decrease







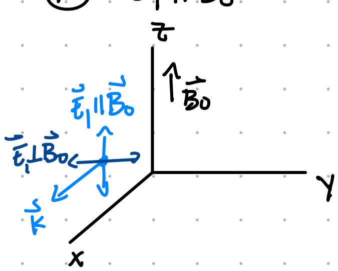
2) E/m waves $\perp \vec{B}_0$

↳ transverse $\Longleftrightarrow$ $\vec{k} \perp \vec{E}_1$

Two options 1) $\vec{E}_1 \parallel \vec{B}_0$ (ordinary waves)

2) $\vec{E}_1 \perp \vec{B}_0$ (extraordinary waves)

A $\vec{E}_1 \parallel B_0$



$\vec{B}_0 = B_0 \hat{z}$

$\vec{E}_1 = \vec{E}_1 \hat{z}$

$\vec{k} = k \hat{x}$

The wave equation remains unchanged for ordinary

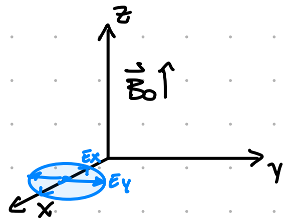
$$(\omega^2 - c^2k^2)\vec{E}_1 = -\frac{i\omega}{\epsilon_0}\vec{J}_1 = i n_0 e \omega v_{e1}/\epsilon_0$$

we project onto the z-axis since all others remain the same
(only proj on z is meaningful)

$$m\frac{dv_{e1}}{dt} = -eE_1 \qquad (E_1 \parallel Oz\ ;\ \vec{v}_{e1} \parallel Oz$$

B_0 does not contribute, $v_{e1} = \frac{eE_1}{i\omega m}$ $\Rightarrow$ $\underline{\omega^2 = \omega_p^2 + c^2k^2}$

(B) $\vec{E}_1 \perp \vec{B}_0$ (X-waves)



assume $T_e = 0$; $v_{e0} = 0$

$$\vec{E}_1 = E_x \hat{x} + E_y \hat{y}$$

$$m\dot{\vec{v}}_{e1} = -e\left[\vec{E}_1 + \vec{v}_{e1} \times \vec{B}_0\right]$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & 0 \\ 0 & 0 & B_0 \end{vmatrix}$$

$$\vec{E} = \cancel{\vec{E}_0}^{\,0} + \vec{E}_1 \qquad \vec{B} = \vec{B}_0 + \vec{B}_1; \qquad \vec{v}_e = \vec{v}_{e1} + \cancel{\vec{v}_{e0}}^{\,0}$$

$$\begin{cases} -i\omega m v_x = -eE_x - e v_y B_0 \\ -i\omega m v_y = -eE_y + e v_x B_0 \end{cases}$$

$$\begin{cases} v_x = \dfrac{e}{m\omega}\left(-iE_x - \dfrac{\omega_c}{\omega}E_y\right)\left(1 - \dfrac{\omega_c^2}{\omega^2}\right)^{-1} \\ v_y = \dfrac{e}{m\omega}\left(-iE_y + \dfrac{\omega_c}{\omega}E_x\right)\left(1 - \dfrac{\omega_c^2}{\omega^2}\right)^{-1} \end{cases}$$

use maxwell's eqn's

$$\begin{cases} \nabla \times \vec{E}_1 = -\dot{\vec{B}}_1 & (1) \\ c^2 \nabla \times \vec{B}_1 = \dfrac{1}{\epsilon_0}\vec{j}_1 + \dot{\vec{E}}_1 & (2) \end{cases} \qquad \nabla \times \vec{E}_1 = i(\vec{k} \times \vec{E}_1)$$

take time derivative of (2) to find $\nabla \times \dot{\vec{B}}_1$

$$c^2 \nabla \times \dot{\vec{B}}_1 = \frac{1}{\epsilon_0}\frac{\partial j_1}{\partial t} + \ddot{\vec{E}}_1$$

substitute in (1)

$$k^2 E_1 - \vec{k}(\vec{k} \cdot \vec{E}_1) = -\frac{i\omega}{\epsilon_0 c^2}\vec{J}_1 + \frac{\omega^2}{c^2}\vec{E}_1$$

$$\vec{k} \parallel Ox; \quad \nabla \cdot \vec{E}_1 = \vec{k} \cdot \vec{E}_1 = ikE_x \rightarrow \vec{k} \cdot \vec{E}_1 = kE_x$$

$$(\omega^2 - c^2k^2)\vec{E}_1 + c^2 k E_x \vec{k} = -\frac{i\omega}{\epsilon_0}\vec{J}_1$$

$$= i n_0 \omega e \vec{v}_{e1} / \epsilon_0$$

This can be projected onto x & y, z is useless

$$\begin{cases} \omega^2 E_x = \dfrac{i\omega n_0 e}{\epsilon_0} \dfrac{e}{m\omega}\left(-iE_x - \dfrac{\omega_c}{\omega} E_y\right)\left(1 - \dfrac{\omega_c^2}{\omega^2}\right)^{-1} \\ (\omega^2 - c^2k^2) E_y = \dfrac{-i\omega n_0 e}{\epsilon_0} \dfrac{e}{m\omega} = x\left(iE_y - \dfrac{\omega_c}{\omega} E_x\right)\left(1 - \dfrac{\omega_c^2}{\omega^2}\right)^{-1} \end{cases}$$

two equations for two unknowns

for a nontrivial solution the determinant of coeff $= 0$

$$\det \left\| \begin{matrix} A & B \\ C & D \end{matrix} \right\| : \quad A = \omega^2 - \omega_k^2 = \omega^2 = \omega_c^2 - \omega_p^2$$

$$\underbrace{\left[\omega^2\left(1 - \frac{\omega_c^2}{\omega^2}\right) - \omega_p^2\right]}_{A} E_x + \underbrace{i\,\frac{\omega_p^2 \omega_c}{\omega}}_{B} E_y = 0 \qquad \text{x-axis}$$

$$\underbrace{-i\,\frac{\omega_p^2 \omega_c}{\omega}}_{C} E_x + \underbrace{\left[(\omega^2 - c^2k^2)\left(1 - \frac{\omega_c^2}{\omega^2}\right) - \omega_p^2\right]}_{D} E_y = 0 \qquad \text{y-axis}$$

so $AD = BC$ for non trivial sol$^{\underline{n}}$

$$(\omega^2 - \omega_h^2)\left[\omega^2 - \omega_h^2 - c^2k^2\left(1 - \frac{\omega_c^2}{\omega^2}\right)\right] = \left(\frac{\omega_p^2 \omega_c}{\omega}\right)^2 \qquad \text{disp. relation}$$

simplify

$$\frac{c^2k^2}{\omega^2} = \frac{\omega^2 - \omega_h^2 - \left[\left(\frac{\omega_p^2 \omega_c}{\omega}\right)^2 \big/ (\omega^2 - \omega_h^2)\right]}{\omega^2 - \omega_c^2}$$

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2\left(\omega^2 - \omega_h^2 + (\omega_p^4 \omega_c^2/\omega^2)\right)}{(\omega^2 - \omega_c^2)(\omega^2 - \omega_h^2)}$$

$$\boxed{\frac{c^2k^2}{\omega^2} = \frac{c^2}{V_p^2} = 1 - \frac{\omega_p^2}{\omega^2}\left(\frac{\omega^2 - \omega_p^2}{\omega^2 - \omega_h^2}\right)} \qquad \text{dispersion relation for X-waves}$$

when $\omega = \omega_h$ the expression on the right $\to \infty$

in principle we can find ω where the right $\to 0$ (next lecture) transverse ($\vec{k} \perp \vec{E}_1$)

Electromagnetic waves $\perp$ to $\vec{B}_0$

① O-waves

$$\omega^2 = \omega_p^2 + c^2 k^2$$

$\vec{E}_1 \parallel \vec{B}_0$

② X-waves

$$\frac{c^2 k^2}{\omega^2} = \frac{c^2}{V_\varphi^2} = 1 - \frac{\omega_p^2}{\omega^2} \frac{\omega^2 - \omega_p^2}{\omega^2 - \omega_h^2}$$

upper hybrid freq.

$\omega_h^2 = \omega_p^2 + \omega_c^2$

(refraction coeff)^2

$\vec{E}_1 \perp \vec{B}_0$

Some definitions:

① $\tilde{n} = 0 \Rightarrow k = 0 \Rightarrow \lambda = \infty$ (cutoff) wave reflection

② $\tilde{n} = \infty \Rightarrow k = \infty \Rightarrow \lambda = 0$ resonance $\Leftrightarrow$ the wave is absorbed

For E/m waves w/ $\vec{B}_0 = 0$ and O-waves where $\vec{k} \perp \vec{B}_0 \neq 0$

then $\tilde{n} = \sqrt{1 - \omega_p^2/\omega^2} \Rightarrow \omega = \omega_p$ is a cutoff

For X-waves

$$\tilde{n} = \frac{c^2}{V_\varphi^2} = 1 - \frac{\omega_p^2}{\omega^2} \frac{\omega^2 - \omega_p^2}{\omega^2 - \omega_h^2}$$

if $\omega = \omega_h \rightarrow \tilde{n} \rightarrow \infty$ resonance

$$\tilde{n} = 0 \Rightarrow 1 = \frac{\omega_p^2}{\omega^2} \frac{\omega^2 - \omega_p^2}{\omega^2 - \omega_p^2 - \omega_c^2}$$

$$1 = \frac{\omega_p^2}{\omega^2} \frac{1}{1 - \omega^2/(\omega^2 + \omega_p^2)}$$

$$1 - \frac{\omega_c^2}{\omega^2 - \omega_p^2} = \frac{\omega_p^2}{\omega^2}$$

$$1 - \frac{\omega_p^2}{\omega^2} = \frac{\omega_c^2}{\omega^2} \left(\frac{1}{1 - \omega_p^2/\omega^2} \right)$$

$$\left(1 - \frac{\omega_p^2}{\omega^2}\right)^2 = \frac{\omega_c^2}{\omega^2}$$

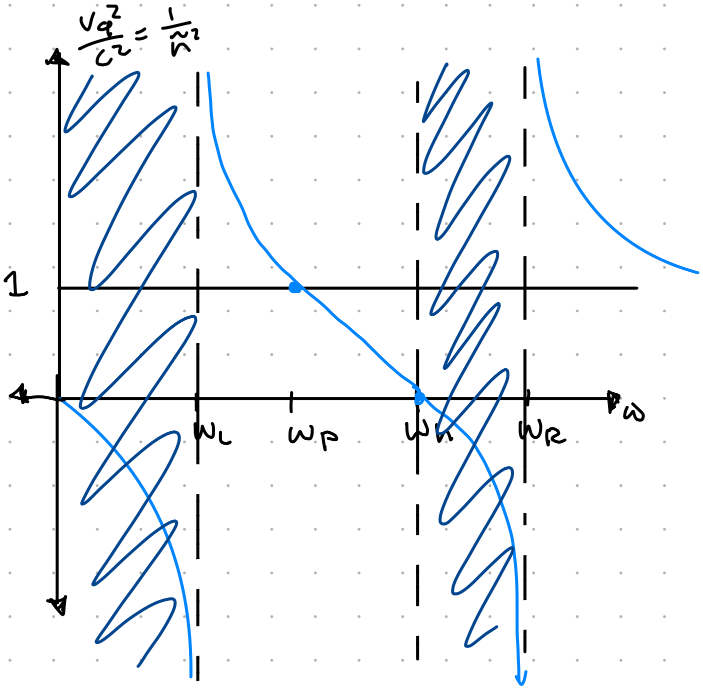
cont.

$$1 - \frac{\omega_p^2}{\omega^2} = \pm \frac{\omega_c}{\omega}$$

$$\omega^2 \mp \omega_c\omega - \omega_p^2 = 0$$

$$\omega_R = \frac{1}{2}\left[\omega_c + \sqrt{\omega_c^2 + 4\omega_p^2}\right]$$ right cutoff

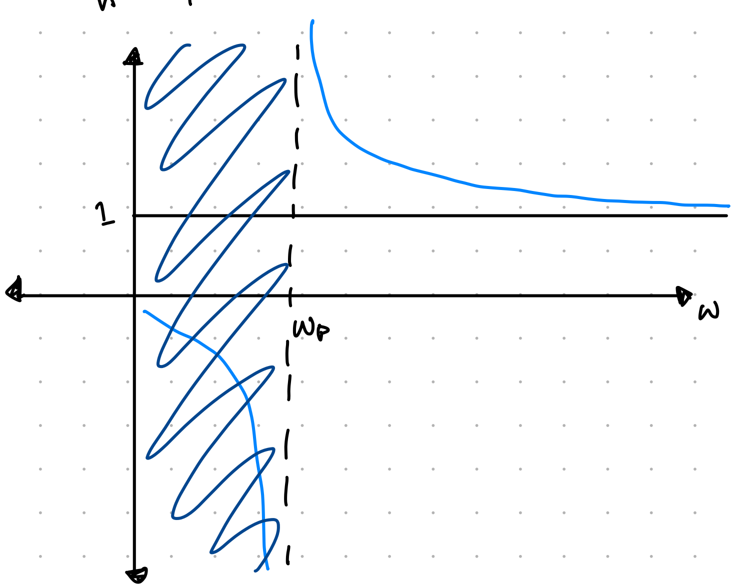
$$\omega_L = \frac{1}{2}\left[-\omega_c + \sqrt{\omega_c^2 + 4\omega_p^2}\right]$$ left cutoff



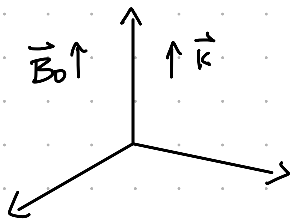
= no waves possible

for o-waves;

$$\frac{1}{\tilde{n}^2} = \frac{1}{1 - \omega_p^2/\omega^2}$$



E/m waves parallel to $\vec{B_0}$



$$\vec{k} = k\hat{z} \qquad \vec{B_0} = B_0\hat{z}$$

$$\vec{E_1} = E_x\hat{x} + E_y\hat{y}$$

$$\vec{k} \perp \vec{E_1}$$

recall:

$$(\omega^2 - c^2k^2)\vec{E_1} + c^2\underbrace{(\vec{E_1}\cdot\vec{k})}_{=0 \;\; E_1 \perp k} = i\omega n_0 e\vec{v}_{e1}/\epsilon_0$$

① $$(\omega^2 - c^2k^2)E_x = \frac{\omega_p^2}{1 - \frac{\omega_c^2}{\omega^2}}\left(E_x - \frac{i\omega_c}{\omega}E_y\right)$$

② $$(\omega^2 - c^2k^2)E_x = \frac{\omega_p^2}{1 - \frac{\omega_c^2}{\omega^2}}\left(E_y + \frac{i\omega_c}{\omega}E_x\right)$$

let $$\alpha = \frac{\omega_p^2}{1 - \frac{\omega_p^2}{\omega^2}}$$

$$(\omega^2 - c^2k^2 - \alpha)E_x + i\alpha\frac{\omega_c}{\omega}E_y = 0$$

$$-i\alpha\frac{\omega_c}{\omega}E_x + (\omega^2 - c^2k^2 - \alpha)E_y = 0$$

$$(\omega^2 - c^2k^2 - \alpha)^2 = \alpha^2\frac{\omega_c^2}{\omega^2}$$

$$\omega^2 - c^2k^2 - \alpha = \pm\alpha\frac{\omega_c}{\omega}$$

$$\omega^2 - c^2k^2 = \alpha\left(1 \pm \frac{\omega_c}{\omega}\right)$$

$$= \frac{\omega_p^2}{1 - \omega_c^2/\omega^2}\left(1 \pm \omega_c/\omega\right)$$

$$= \omega_p^2\frac{1 \pm \omega_c/\omega}{(1 + \omega_c/\omega_p)(1 - \omega_c/\omega_p)}$$

$$= \omega_p^2\frac{1}{1 \mp \omega_c/\omega}$$

$$n^2 = \frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}\left(\frac{1}{1 - \omega_c/\omega}\right) \rightarrow \text{R wave}$$

$$n^2 = \frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}\left(\frac{1}{1-\omega_c/\omega}\right) \rightarrow \text{R wave} \quad \text{CCW}$$

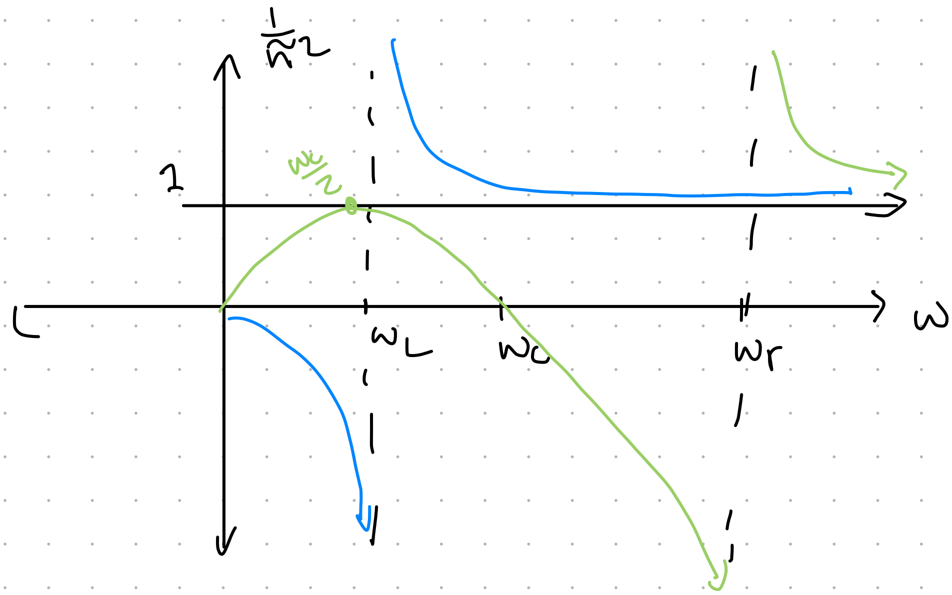
$$n^2 = \frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}\left(\frac{1}{1+\omega_c/\omega}\right) \rightarrow \text{L wave} \quad \text{CW}$$

R-waves: for $K=0$

$$\omega^2 - \omega_c\omega - \omega_p^2 = 0 \iff \omega = \omega_R$$

L-waves:

$$\omega^2 + \omega_c\omega - \omega_p^2 = 0 \iff \omega = \omega_L$$



L wave $\omega_L \le \omega < \infty$

R wave $\omega \le \omega_c$
$\omega > \omega_r$

R waves with $0 < \omega < \omega_c$ are called the whistler waves

⋮

$E_y = f(z)\, e^{-i\omega t + i\pi/2}$

$E_y = f(z) = e^{-i(\omega t - \pi/2)} \longrightarrow$ $E_x = \mathrm{Re}\left[f(z)\left(\cos(-\omega t) - i\sin(-\omega t)\right)\right]$

$= f(z)\cos\omega t$

$E_y = \mathrm{Re}\left[f(z)\left(\cos(\omega t - \pi/2) - i\sin(\omega t - \pi/2)\right)\right]$

$= f(z)\sin(\omega t)$