General Relativity Homework

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January 15, 2019

1. Prove the torsion $\Gamma^{\lambda}_{[\mu\nu]}$ (the anti-symmetric part of an affine connection) is a tensor.

Proof:

$$\begin{aligned}\widetilde{\Gamma}^{\lambda}_{[\mu\nu]} &= \frac{1}{2}(\widetilde{\Gamma}^{\lambda}_{\mu\nu} - \widetilde{\Gamma}^{\lambda}_{\nu\mu}) \\ &= \frac{1}{2}(\Gamma^{\rho}_{\alpha\sigma} - \Gamma^{\rho}_{\sigma\alpha})\frac{\partial x^{\alpha}}{\partial \widetilde{x}^{\mu}}\frac{\partial x^{\sigma}}{\partial \widetilde{x}^{\nu}}\frac{\partial \widetilde{x}^{\lambda}}{\partial x^{\rho}} + \frac{1}{2}(\frac{\partial^2 x^{\rho}}{\partial \widetilde{x}^{\mu}\partial \widetilde{x}^{\nu}}\frac{\partial \widetilde{x}^{\lambda}}{\partial x^{\rho}} - \frac{\partial^2 x^{\rho}}{\partial \widetilde{x}^{\nu}\partial \widetilde{x}^{\mu}}\frac{\partial \widetilde{x}^{\lambda}}{\partial x^{\rho}}) \\ &= \Gamma^{\rho}_{[\alpha\sigma]}\frac{\partial x^{\alpha}}{\partial \widetilde{x}^{\mu}}\frac{\partial x^{\sigma}}{\partial \widetilde{x}^{\nu}}\frac{\partial \widetilde{x}^{\lambda}}{\partial x^{\rho}}\end{aligned} \tag{1.1}$$

From Equ.1.1 we can prove that the torsion $\Gamma^{\lambda}_{[\mu\nu]}$ is a tensor.

Q.E.D.

2. We know that Φ is a scalar. Please verify that $A_\mu = \dfrac{\partial \Phi}{\partial x^\mu}$ is a covariant vector.

Proof:

$$\widetilde{A}_\mu = \frac{\partial \Phi}{\partial \widetilde{x}^\mu}, A_\nu = \frac{\partial \Phi}{\partial x^\nu} \tag{2.1}$$

then we have

$$\widetilde{A}_\mu = \frac{\partial \Phi}{\partial x^\nu}\frac{\partial x^\nu}{\partial \widetilde{x}^\mu} = A_\nu \frac{\partial x^\nu}{\partial \widetilde{x}^\mu} \tag{2.2}$$

So $A_\mu = \dfrac{\partial \Phi}{\partial x^\mu}$ is a covariant vector.

Q.E.D.

3. Prove $T^{\mu\nu}A_{\mu\nu} = 0$ when $T^{\mu\nu}$ is a symmetric tensor and $A_{\mu\nu}$ is an antisymmetric tensor.

Proof:

$$T^{\mu\nu}A_{\mu\nu} = T^{\nu\mu}A_{\nu\mu} = T^{\mu\nu}(-A_{\mu\nu}) = -T^{\mu\nu}A_{\mu\nu} \tag{3.1}$$

then we can get

$$T^{\mu\nu}A_{\mu\nu} = -T^{\mu\nu}A_{\mu\nu} \tag{3.2}$$

So we can say $T^{\mu\nu}A_{\mu\nu}=0$.

Q.E.D.

4. Known that g is the metric of $g_{\mu\nu}$. Try to verify that $\Gamma^{\mu}_{\alpha\mu}=\frac{1}{2}g^{\mu\nu}g_{\mu\nu,a}=\frac{\partial}{\partial x^{\alpha}}(\ln\sqrt{-g})$.

Proof:

$$\begin{aligned}\Gamma^{\mu}_{\alpha\beta}&=\frac{1}{2}g^{\mu\nu}(g_{\alpha\nu,\mu}+g_{\nu\mu,\alpha}-g_{\alpha\mu,\nu})\\&=\frac{1}{2}g^{\mu\nu}g_{\mu\nu,\alpha}\end{aligned}\tag{4.1}$$

when

$$\frac{\partial}{\partial x^{\alpha}}(\ln\sqrt{-g})=-\frac{1}{\sqrt{-g}}\cdot\frac{1}{2}(-g)^{-1/2}\frac{\partial g}{\partial x^{\alpha}}=\frac{1}{2g}\frac{\partial g}{\partial x^{\alpha}}\tag{4.2}$$

And because of the equation $\mathrm{d}g=g\cdot g^{\mu\nu}\,\mathrm{d}g^{\mu\nu}$, we can get $\frac{\partial g}{\partial x^{\alpha}}=g\cdot g^{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}$.
So we have $\Gamma^{\mu}_{\alpha\mu}=\frac{1}{2}g^{\mu\nu}g_{\mu\nu,a}=\frac{\partial}{\partial x^{\alpha}}(\ln\sqrt{-g})$.

Q.E.D.

5. Known that $A_{\mu;\nu}=A_{\mu,\nu}-\Gamma^{\lambda}_{\mu\nu}A_{\lambda}$. Use coordinate differential relation $U_{;\mu}=U_{,\mu}$ and Leibnitzs law to prove $B^{\mu}_{;\nu}=B^{\mu}_{,\nu}+\Gamma^{\mu}_{\lambda\nu}B^{\lambda}$.

Proof: From Leibnitzs law, we have

$$(A_{\mu}B^{\mu})_{;\nu}=A_{\mu;\nu}B^{\mu}+A_{\mu}B^{\mu}_{;\nu},(A_{\mu}B^{\mu})_{,\nu}=A_{\mu,\nu}B^{\mu}+A_{\mu}B^{\mu}_{,\nu}\tag{5.1}$$

while $U_{;\mu}=U_{,\mu}$, we have

$$A_{\mu;\nu}B^{\mu}+A_{\mu}B^{\mu}_{;\nu}=A_{\mu,\nu}B^{\mu}+A_{\mu}B^{\mu}_{,\nu}\tag{5.2}$$

Then substitute $A_{\mu;\nu}=A_{\mu,\nu}-\Gamma^{\lambda}_{\mu\nu}A_{\lambda}$ into Equ.5.2, we'll have

$$(A_{\mu;\nu}-\Gamma^{\lambda}_{\mu\nu}A_{\lambda})B^{\mu}+A_{\mu}B^{\mu}_{;\nu}=A_{\mu,\nu}B^{\mu}+A_{\mu}B^{\mu}_{,\nu}\tag{5.3}$$

The formula can be obtained as

$$A_{\mu}B^{\mu}_{;\nu}=A_{\mu}B^{\mu}+A_{\mu}\Gamma^{\mu}_{\sigma\nu}B^{\sigma}\tag{5.4}$$

it equals to $B^{\mu}_{;\nu}=B^{\mu}_{,\nu}+\Gamma^{\mu}_{\lambda\nu}B^{\lambda}$.

Q.E.D.

6. Known that $\mathrm{d}s^2=g_{\mu\nu}\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu}=-\mathrm{d}\tau^2$. Derive the geodesic equation from variational principle $\delta\int_A^B\mathrm{d}s=0$ or $\delta\int_A^B(\frac{\mathrm{d}\tau}{\mathrm{d}\lambda})^2\,\mathrm{d}\lambda=0$.

Answer : We can get $\mathrm{d}s = (g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta)^{1/2}$, and introduce scalar parameter λ.
Then we have

$$\mathrm{d}s = (g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta)^{1/2}\,\mathrm{d}\lambda \tag{6.1}$$

and $\dot{x}^\alpha = \dfrac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}$, $\dot{x}^\beta = \dfrac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}$.
So we have

$$\delta\int_A^B L\,\mathrm{d}\lambda = 0 \tag{6.2}$$

and $L = (g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta)^{1/2}$ is the Lagrangian. From Lagrange equation $\dfrac{\partial L}{\partial x^\nu} - \dfrac{\mathrm{d}}{\mathrm{d}\lambda}\dfrac{\partial L}{\partial \dot{x}^\nu} = 0$, we have :

$$\frac{1}{(g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta)^{1/2}}\frac{\partial g_{\alpha\beta}}{\partial x^\nu}\dot{x}^\alpha\dot{x}^\beta - \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{g_{\alpha\nu}\dot{x}^\alpha + g_{\beta\nu}\dot{x}^\beta}{(g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta)^{1/2}} = 0 \tag{6.3}$$

when we select λ as s, we have

$$(g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta)^{1/2} = g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s} = 1 \tag{6.4}$$

Now we can rewrite the Lagrange equation as

$$\begin{aligned} g_{\alpha\nu,\beta}\dot{x}^\alpha\dot{x}^\beta - \frac{\mathrm{d}}{\mathrm{d}s}(g_{\alpha\nu}\dot{x}^\alpha) &= 0 \\ g_{\alpha\nu}\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}s^2} + (g_{\alpha\nu,\beta} - \frac{1}{2}g_{\alpha\beta,\nu})\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s} &= 0 \end{aligned} \tag{6.5}$$

and

$$g_{\alpha\nu,\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s} = g_{\beta\nu,\alpha}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s} = g_{\alpha\beta,\nu}\frac{\mathrm{d}x^\beta}{\mathrm{d}s}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \tag{6.6}$$

so

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}s^2} + \frac{1}{2}g^{\mu\nu}(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu})\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s} = 0 \tag{6.7}$$

Finally we can get

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}s^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s} = 0 \tag{6.8}$$

7. If an ordinary spherical space is embedded in three - dimensional Euclidean space and spherical polar coordinate system is selected, it has line elements with the form $\mathrm{d}s^2 = a^2\,\mathrm{d}\theta^2 + a^2\sin^2\theta\,\mathrm{d}\phi^2$ **:**

(1) derive $g^{\mu\nu}$**;**
(2) derive all Christoffel connection $\Gamma^\mu_{\alpha\beta}$**;**
(3) derive all $R^\alpha_{\mu\nu\lambda}$**;**
(4) derive all $R_{\mu\nu}$**;**
(5) derive R**;**

(6) derive the geodesic equation of spherical space represented by the metric.

Answer :

(1) From the definition we can get $g^{11} = \dfrac{1}{a^2}$, $g^{22} = \dfrac{1}{a^2 \sin^2\theta}$, and $g^{12} = g^{21} = 0$.

(2) From (1), we can get

$$\Gamma^1_{11} = \frac{1}{2} g^{1\mu}(g_{\mu 1,1} + g_{1\mu,1} - g_{11,\mu}) = \frac{1}{2} g^{11} g_{11,1} = 0 \tag{7.1}$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2} g^{11}(g_{11,2} + g_{21,1} - g_{12,1}) = \frac{1}{2} g^{11} g_{11,2} = 0 \tag{7.2}$$

$$\begin{aligned} \Gamma^1_{22} &= \frac{1}{2} g^{11}(g_{12,2} + g_{21,2} - g_{22,1}) \\ &= \frac{1}{2} g^{11}(-g_{22,1}) = \frac{1}{2}\frac{1}{a^2}(-2a^2 \sin\theta\cos\theta) = -\sin\theta\cos\theta \end{aligned} \tag{7.2}$$

$$\Gamma^2_{11} = \frac{1}{2} g^{22}(g_{21,1} + g_{12,1} - g_{11,2}) = 0 \tag{7.3}$$

$$\begin{aligned} \Gamma^2_{12} = \Gamma^2_{21} &= \frac{1}{2} g^{22}(g_{22,1} + g_{12,2} - g_{21,2}) \\ &= \frac{1}{2} g^{22} g_{22,1} = \frac{1}{2}\frac{1}{a^2 \sin^2\theta}(2a^2 \sin\theta\cos\theta) = \cot\theta \end{aligned} \tag{7.4}$$

$$\Gamma^2_{22} = \frac{1}{2} g^{22} g_{22,2} = 0 \tag{7.5}$$

So we can finally get $\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{11} = \Gamma^2_{22} = 0$, $\Gamma^1_{22} = -\sin\theta\cos\theta$, and $\Gamma^2_{12} = \cot\theta$.

(3) From (2), we can get

$$\begin{aligned} R^1_{212} &= \Gamma^1_{22,1} - \Gamma^1_{21,2} + \Gamma^1_{\sigma 1}\Gamma^\sigma_{22} - \Gamma^1_{\sigma 2}\Gamma^\sigma_{21} \\ &= \Gamma^1_{22,1} - \Gamma^1_{22}\Gamma^2_{21} = \sin^2\theta \end{aligned} \tag{7.6}$$

and

$$R^2_{121} = g^{22} R_{2121} = g^{22} R_{1212} \tag{7.7}$$

for $R^2_{212} = g^{11} R^1_{1212}$, then we can get $R_{2121} = \dfrac{R^1_{212}}{g^{11}} = \dfrac{\sin^2\theta}{\frac{1}{a^2}} = a^2 \sin^2\theta$.

So we have $R^2_{121} = \dfrac{1}{a^2 \sin^2\theta} \cdot a^2 \sin^2\theta = 1$. We can finally get $R^1_{212} = \sin^2\theta$, $R^2_{121} = 1$.

(4) Because $R_{\mu\nu} = R^\lambda_{\mu\nu\lambda} = R^1_{\mu\nu 1} + R^2_{\mu\nu 2}$, we can get Equ.7.8 below :

$$\begin{aligned} R_{11} &= R^\lambda_{11\lambda} = R^1_{111} + R^2_{112} = -1 \\ R_{12} &= R^\lambda_{12\lambda} = R^1_{121} + R^2_{122} = 0 \\ R_{22} &= R^\lambda_{22\lambda} = R^1_{221} + R^2_{222} = -\sin^2\theta \end{aligned} \tag{7.8}$$

(5) $R = g^{\mu\nu} R_{\mu\nu} = g^{11} R_{11} + g^{22} R_{22} = -\dfrac{1}{a^2} - \dfrac{1}{a^2} = -\dfrac{2}{a^2}$.

(6) The geodesic equation in triangular rectangular coordinates system is $\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2}+\Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}=0$. Then we can transform it into spherical polar coordinate system :

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2}+(-\sin\theta\cos\theta)\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau}=0 \tag{7.9}$$

and we will finally get

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\tau^2}+2\cot\theta\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau}=0 \tag{7.10}$$

Equ.7.10 is the geodesic equation we want.

8. Prove Einstein field equation $R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\kappa T_{\mu\nu}$ can be rewritten as $R_{\mu\nu}=\kappa(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T)$.

Proof: Transform $R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\kappa T_{\mu\nu}$ into

$$g^{\mu\nu}R_{\mu\nu}-\frac{1}{2}g^{\mu\nu}g_{\mu\nu}R=\kappa g^{\mu\nu}T_{\mu\nu} \tag{8.1}$$

then we'll get

$$R-\frac{1}{2}\cdot 4R=\kappa T \quad\Rightarrow\quad R=-\kappa T \tag{8.2}$$

So we have

$$\begin{aligned} R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\kappa T=\kappa T_{\mu\nu} \\ \Rightarrow R_{\mu\nu}=-\kappa(\frac{1}{2}g_{\mu\nu}T-T_{\mu\nu})=\kappa(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T) \end{aligned} \tag{8.3}$$

Q.E.D.

9. Under the linear approximation of a weak gravitational field, the metric can be written as $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$. Find the form of a linearized Einstein field equation.

Answer: Under weak gravitational field we can represent metric as $g_{\mu\nu}=h_{\mu\nu}+\eta_{\mu\nu}$, and we make $|h_{\mu\nu}|\ll 1$. In linear approximation theory, we just keep the linear terms for $h_{\mu\nu}$, so we have

$$\begin{aligned} \Gamma^\mu_{\alpha\beta} &= \frac{1}{2}\eta_{\mu\nu}(h_{\alpha\nu,\beta}+h_{\beta\nu,\alpha}-h_{\alpha\beta,\nu}) \\ &= \frac{1}{2}(h^\mu_{\alpha,\beta}+h^\mu_{\beta,\alpha}-h^{;\mu}_{\alpha\beta}) \end{aligned} \tag{9.1}$$

And we have linearized Ricci tensor

$$\begin{aligned} R_{\mu\nu} &= \Gamma^\lambda_{\mu\lambda,\,\nu}-\Gamma^\lambda_{\mu\nu,\,\lambda} \\ &\equiv \frac{1}{2}(h^{;\alpha}_{\mu\nu}+h_{,\mu,\nu}-h^\alpha_{\mu,\nu,\alpha}-h^\alpha_{\nu,\mu,\alpha}) \end{aligned} \tag{9.2}$$

for $h \equiv h_\alpha^\alpha = \eta_{\alpha\beta} h_{\alpha\beta}$.

We define

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \tag{9.3}$$

and its inverse transformation is

$$\bar{\bar{h}}_{\mu\nu} \equiv \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} = h_{\mu\nu} \tag{9.4}$$

which can be easily proved.
With the help of Equ.9.3 and Equ.9.4, we'll get linearized field equation

$$\bar{R}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R = -8\pi G T_{\mu\nu} \tag{9.5(a)}$$

The specific form is

$$\bar{h}_{\mu\nu,\alpha}^{;\alpha} + \eta_{\mu\nu}\bar{h}_{\alpha\beta}^{;\alpha,\beta} - \bar{h}_{\mu\alpha,\nu}^{;\alpha} - \bar{h}_{\nu\alpha,\mu}^{;\alpha} = -16\pi G T_{\mu\nu} \tag{9.5(b)}$$

Consider the harmonic condition, then we have

$$\bar{h}_{\mu\alpha}^{;\alpha} = 0 \tag{9.6}$$

So we can finally get the simplified field equation

$$\bar{h}_{\mu\nu,\alpha}^{;\alpha} = -16\pi G T_{\mu\nu} \tag{9.7}$$

Equ.9.7 is the answer we want to get.

10. Suppose $\mathrm{d}s^2 = -(x^0)^4(\mathrm{d}x^0)^2 + 2e^{x^1}(\mathrm{d}x^1)^2 + e^{-x^2}(\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2$**, and prove the space-time is flat.**

Proof: Under the coordinate transformation in Equ.10.1

$$\begin{cases} t = \frac{1}{3}(x_0)^3 \\ x = 2\sqrt{2}e^{x^1/2} \\ y = -2e^{-x^2/2} \\ z = x^3 \end{cases} \tag{10.1}$$

We can get $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, so the space-time is flat.

Q.E.D.