Quantum Feild Theory Homework

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1. Consider $\psi^\top(x)C\Gamma\psi(x)$, which $C = i\gamma_0\gamma_2$. Analysis $\psi^\top(x)C\Gamma\psi(x)$ under the Lorentz transformation.

Answer : $\mathrm{d}x'^\mu = \dfrac{\partial x'^\mu}{\partial x^\nu} \equiv \Lambda^\mu_{\ \nu}\,\mathrm{d}x^\nu$, $\det(\Lambda^\mu_{\ \nu}) = 1$, $(\gamma^0)^\top = \gamma^0$, $(\gamma^1)^\top = -\gamma^1$, $(\gamma^2)^\top = \gamma^2$, $(\gamma^3)^\top = -\gamma^3$; $\psi'(x') = S\psi(x)$, $S^{-1}\gamma^\mu S = \Lambda^\mu_{\ \nu}\gamma^\nu$, $S^\dagger = \gamma^0 S^{-1}\gamma^0$, $S^* = \gamma^0\gamma^1\gamma^3 S\gamma^3\gamma^1\gamma^0$.

(1) $\Gamma = I_{4\times 4}$

We can write the Lorentz transformation below :

$$\psi'^\top(x')C\psi'(x') = \psi^\top(x)S^\top i\gamma_0\gamma_2 S\psi(x) = -\psi^\top(x)S^\top i\gamma^0\gamma^2 S\psi(x) \tag{1.1.1}$$

So we have :

$$\begin{aligned} S^\dagger = S^{*\dagger} &= (\gamma^0\gamma^1\gamma^3 S\gamma^3\gamma^1\gamma^0)^\top = (\gamma^0)^\top(\gamma^1)^\top(\gamma^3)^\top S(\gamma^3)^\top(\gamma^1)^\top(\gamma^0)^\top \\ &= \gamma^0(-\gamma^1)(-\gamma^3)S^\top(-\gamma^3)(-\gamma^1)\gamma^0 \\ &= \gamma^0 S^{-1}\gamma^0 \end{aligned} \tag{1.1.2}$$

So we can get $\gamma^1\gamma^3 S^\top\gamma^3\gamma^1 = S^{-1}$. Then we'll get $S^\top = \gamma^3\gamma^1 S^{-1}\gamma^1\gamma^3$.

Substitute Equ.1.1.2 into Equ.1.1.1:

$$\begin{aligned} \psi'^\top(x')C\psi'(x') &= -\psi^\top(x)\gamma^3\gamma^1 S^{-1}\gamma^1\gamma^3 i\gamma^0\gamma^2 S\psi(x) = \psi^\top(x)\gamma^3\gamma^1 S^{-1} i\gamma^0\gamma^1\gamma^2\gamma^3 S\psi(x) \\ &= \psi^\top(x)\gamma^3\gamma^1 S^{-1}\gamma^5 S\psi(x) \end{aligned} \tag{1.1.3}$$

For

$$S^{-1}\gamma^5 S = \det(\Lambda^\mu_{\ \nu})\gamma^5 = \gamma^5 \tag{1.1.4}$$

So we have $\psi'^{\top}(x')C\psi'(x') = \psi^{\top}(x)\gamma^3\gamma^1\gamma^5\psi(x)$, of which

$$\begin{aligned}\gamma^3\gamma^1\gamma^5 &= \gamma^3\gamma^1(i\gamma^0\gamma^1\gamma^2\gamma^3) = -i\gamma^0(\gamma^1)^2\gamma^2(\gamma^3)^2 \\ &= -i\gamma^0\gamma^2 = i\gamma_0\gamma_2 = C\end{aligned} \tag{1.1.5}$$

We can finally get $\psi'^{\top}(x')C\psi'(x') = \psi^{\top}(x)C\psi(x)$ when $\Gamma = I_{4\times4}$.

(2) $\Gamma = \gamma^\mu$

$$\begin{aligned}\psi'^{\top}(x')\gamma^\mu C\psi'(x') &= \psi^{\top}(x)S^{\top}\gamma^\mu CS\psi(x) = \psi^{\top}(x)\gamma^3\gamma^1S^{-1}\gamma^1\gamma^3\gamma^\mu i\gamma_0\gamma_2\psi(x) \\ &= -\psi^{\top}(x)\gamma^3\gamma^1S^{-1}\gamma^1\gamma^3\gamma^\mu i\gamma^0\gamma^2S\psi(x)\end{aligned} \tag{1.2.1}$$

When $\mu = 0, 2$

$$\begin{aligned}\psi'^{\top}(x')\gamma^\mu C\psi'(x') &= -\psi^{\top}(x)\gamma^3\gamma^1S^{-1}\gamma^\mu i\gamma^1\gamma^3\gamma^0\gamma^2S\psi(x) \\ &= \psi^{\top}(x)\gamma^3\gamma^1S^{-1}\gamma^\mu i\gamma^0\gamma^1\gamma^2\gamma^3S\psi(x) \\ &= \psi^{\top}(x)\gamma^3\gamma^1S^{-1}\gamma^\mu\gamma^5S\psi(x) \\ &= -\psi^{\top}(x)\gamma^3\gamma^1S^{-1}\gamma^5\gamma^\mu S\psi(x) \\ &= -\psi^{\top}(x)\gamma^3\gamma^1\gamma^5\Lambda^\mu_\nu\gamma^{nu}S\psi(x) \\ &= -\Lambda^\mu_\nu\psi^{\top}(x)C\gamma^\nu\psi(x)\end{aligned} \tag{1.2.2}$$

For $C\gamma^\nu = -i\gamma^0\gamma^2\gamma^\nu$ and $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I_{4\times4}$,
We have

$$\begin{aligned}C\gamma^\nu &= -i\gamma^0(2g^{2\nu}I_{4\times4} - \gamma^\nu\gamma^2) = -2ig^{2\nu}\gamma^0 + i(2g^{0\nu}I_{4\times4} - \gamma^\nu\gamma^0)\gamma^2 \\ &= i\cdot 2(g^{0\nu}\gamma^2 - g^{2\nu}\gamma^0) + \gamma^\nu C\end{aligned} \tag{1.2.3}$$

So we can say

$$\psi'^{\top}(x')\gamma^\mu C\psi'(x') = -\Lambda^\mu_\nu\left[i\cdot 2\psi^{\top}(x)(g^{0\nu}\gamma^2 - g^{2\nu}\gamma^0)\psi(x) + \psi^{\top}(x)\gamma^\mu C\psi(x)\right] \tag{1.2.4}$$

As the same goes, when $\mu = 1, 3$

$$\psi'^{\top}(x')\gamma^\mu C\psi'(x') = \Lambda^\mu_\nu[i\cdot 2\psi^{\top}(x)(g^{0\nu}\gamma^2 - g^{2\nu}\gamma^0)\psi(x) + \psi^{\top}(x)\gamma^\mu C\psi(x)] \tag{1.2.5}$$

We can get the flowing conclusion :

$$\psi'^{\top}(x')\gamma^{\mu}C\psi'(x') = \begin{cases} -\Lambda^{\mu}_{\ \nu}\left[i\cdot 2\psi^{\top}(x)(g^{0\nu}\gamma^{2}-g^{2\nu}\gamma^{0})\psi(x)+\psi^{\top}(x)\gamma^{\mu}C\psi(x)\right] & \mu=0,2 \\ \Lambda^{\mu}_{\ \nu}\left[i\cdot 2\psi^{\top}(x)(g^{0\nu}\gamma^{2}-g^{2\nu}\gamma^{0})\psi(x)+\psi^{\top}(x)\gamma^{\mu}C\psi(x)\right] & \mu=1,3 \end{cases} \tag{1.2.6}$$

Because $C = i\gamma_0\gamma_2$,

$$\begin{aligned} \psi'^{\top}(x')C\gamma^{\mu}\psi'(x') &= \psi^{\top}(x)S^{\top}C\gamma^{\mu}S\psi(x) = \psi^{\top}(x)\gamma^{3}\gamma^{1}S^{-1}\gamma^{1}\gamma^{3}i\gamma^{0}\gamma^{2}\gamma^{\mu}S\psi(x) \\ &= \psi^{\top}(x)\gamma^{3}\gamma^{1}S^{-1}\gamma^{5}\gamma^{\mu}\psi(x) \\ &= \psi^{\top}(x)\gamma^{3}\gamma^{1}(S^{-1}\gamma^{5}S)(S^{-1}\gamma^{\mu}S)\psi(x) \\ &= \Lambda^{\mu}_{\ \nu}\psi^{\top}(x)C\gamma^{\nu}\psi(x) \end{aligned} \tag{1.2.7}$$

So we can finally get $\psi'^{\top}(x')\gamma^{\mu}C\psi'(x') = \Lambda^{\mu}_{\ \nu}\psi^{\top}(x)C\gamma^{\nu}\psi(x)$ when $\Gamma = \gamma^{\mu}$.

(3) $\Gamma = \sigma^{\mu\nu} = \dfrac{i}{2}[\gamma^{\mu},\gamma^{\nu}]$

$$\begin{aligned} \psi'^{\top}(x')C\sigma^{\mu\nu}\psi'(x') &= \psi^{\top}(x)S^{\top}C\sigma^{\mu\nu}S\psi(x) = \frac{i}{2}\psi^{\top}(x)S^{\top}C(\gamma^{\mu}\gamma^{\nu}-\gamma^{\nu}\gamma^{\mu})S\psi(x) \\ &= \frac{i}{2}(\psi^{\top}(x)S^{\top}C\gamma^{\mu}\gamma^{\nu}S\psi(x) - \psi^{\top}(x)S^{\top}C\gamma^{\nu}\gamma^{\mu}S\psi(x)) \end{aligned} \tag{1.3.1}$$

We should calculate $\psi^{\top}(x)S^{\top}C\gamma^{\mu}\gamma^{\nu}S\psi(x)$ and $\psi^{\top}(x)S^{\top}C\gamma^{\nu}\gamma^{\mu}S\psi(x)$ first :

$$\begin{aligned} \psi^{\top}(x)S^{\top}C\gamma^{\mu}\gamma^{\nu}S\psi(x) &= \psi^{\top}(x)\gamma^{3}\gamma^{1}S^{-1}i\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3}\gamma^{\mu}\gamma^{\nu}S\psi(x) \\ &= \Lambda^{\mu}_{\ \alpha}\Lambda^{\nu}_{\ \beta}\psi^{\top}(x)C\gamma^{\alpha}\gamma^{\beta}\psi(x) \end{aligned} \tag{1.3.2}$$

$$\begin{aligned} \psi^{\top}(x)S^{\top}C\gamma^{\nu}\gamma^{\mu}S\psi(x) &= -\psi^{\top}(x)\gamma^{3}\gamma^{1}S^{-1}\gamma^{1}\gamma^{3}i\gamma^{0}\gamma^{2}\gamma^{\mu}\gamma^{\nu}S\psi(x) \\ &= \Lambda^{\mu}_{\ \alpha}\Lambda^{\nu}_{\ \beta}\psi^{\top}(x)C\gamma^{\beta}\gamma^{\alpha}\psi(x) \end{aligned} \tag{1.3.3}$$

Then substitute Equ.1.3.2 and Equ.1.3.3 into Equ.1.3.1, we'll get :

$$\begin{aligned}\psi'^{\top}(x')C\sigma^{\mu\nu}\psi'(x') &= \frac{i}{2}(\Lambda^{\mu}_{\ \alpha}\Lambda^{\nu}_{\ \beta}\psi^{\top}(x)C\gamma^{\alpha}\gamma^{\beta}\psi(x) - \Lambda^{\mu}_{\ \alpha}\Lambda^{\nu}_{\ \beta}\psi^{\top}(x)C\gamma^{\beta}\gamma^{\alpha}\psi(x)) \\ &= \frac{i}{2}\Lambda^{\mu}_{\ \alpha}\Lambda^{\nu}_{\ \beta}\psi^{\top}(x)C(\gamma^{\alpha}\gamma^{\beta} - \gamma^{\beta}\gamma^{\alpha})\psi(x) \\ &= \Lambda^{\mu}_{\ \alpha}\Lambda^{\nu}_{\ \beta}\psi^{\top}(x)C\sigma^{\alpha\beta}\psi(x)\end{aligned} \tag{1.3.4}$$

We can finally get $\psi'^{\top}(x')C\sigma^{\mu\nu}\psi'(x') = \Lambda^{\mu}_{\ \alpha}\Lambda^{\nu}_{\ \beta}\psi^{\top}(x)C\sigma^{\alpha\beta}\psi(x)$ when $\Gamma = \sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu},\gamma^{\nu}]$

(4) $\Gamma = \gamma^{\mu}\gamma^{5}$

$$\begin{aligned}\psi'^{\top}(x')C\gamma^{\mu}\gamma^{5}\psi'(x') &= \psi^{\top}(x)S^{\top}C\gamma^{\mu}\gamma^{5}S\psi(x) = -\psi^{\top}(x)\gamma^{3}\gamma^{1}S^{-1}\gamma^{1}\gamma^{3}i\gamma^{0}\gamma^{2}\gamma^{\mu}\gamma^{5}S\psi(x) \\ &= \psi^{\top}(x)\gamma^{3}\gamma^{1}S^{-1}\gamma^{s}\gamma^{\mu}\gamma^{5}S\psi(x) = \psi^{\top}(x)\gamma^{3}\gamma^{1}\gamma^{5}\Lambda^{\mu}_{\ \nu}\gamma^{\nu}\gamma^{5}S\psi(x)\end{aligned} \tag{1.4.1}$$

Then we have

$$\psi'^{\top}(x')C\gamma^{\mu}\gamma^{5}\psi'(x') = \Lambda^{\mu}_{\ \nu}\psi^{\top}(x)C\gamma^{\nu}\gamma^{5}\psi(x) \tag{1.4.2}$$

We can finally get $\psi'^{\top}(x')C\gamma^{\mu}\gamma^{5}\psi'(x') = \Lambda^{\mu}_{\ \nu}\psi^{\top}(x)C\gamma^{\nu}\gamma^{5}\psi(x)$ when $\Gamma = \gamma^{\mu}\gamma^{5}$.

(5) $\Gamma = \gamma^{5}$

$$\begin{aligned}\psi'^{\top}(x')C\gamma^{5}\psi'(x') &= \psi^{\top}(x)S^{\top}C\gamma^{5}S\psi(x) = -\psi^{\top}(x)\gamma^{3}\gamma^{1}S^{-1}\gamma^{1}\gamma^{3}i\gamma^{0}\gamma^{2}\gamma^{5}S\psi(x) \\ &= \psi^{\top}(x)\gamma^{3}\gamma^{1}S^{-1}\gamma^{5}\gamma^{5}S\psi(x) = \psi^{\top}(x)\gamma^{3}\gamma^{1}\gamma^{5}\gamma^{5}S\psi(x) \\ &= \psi^{\top}(x)C\gamma^{5}\psi(x)\end{aligned} \tag{1.5.1}$$

We can finally get $\psi'^{\top}(x')C\gamma^{5}\psi'(x') = \psi^{\top}(x)C\gamma^{5}\psi(x)$ when $\Gamma = \gamma^{5}$.

2. Analyze the spin of Dirac field .

Answer : We make S_i to represent the component of spin angular momentum on the $\hat{\boldsymbol{e}}_i$-axis, so we have

$$s_3 = \pi_a S^{12}_{ab}\psi_b = \pi_a(\frac{1}{2}\gamma^1\gamma^2)\psi_b = -\frac{i}{2}\pi_a(\Sigma_3)_{ab}\psi_b \tag{2.1.1}$$

Equ.2.1.1 describes the the spin density of fermions.

(1) Fermion

For $\pi_a = i\psi^*_a$, we have $s_3 = \dfrac{1}{2}\psi^*_a(\Sigma_3)_{ab}\psi_b$, then we'll get S_3 :

$$S_3 = \int \mathrm{d}^3x s_3 = \int \mathrm{d}^3x \frac{1}{2}\psi^*_a(\Sigma_3)_{ab}\psi_b \tag{2.1.2}$$

For quantized Dirac field, we have

$$\hat{S}_3 = \int \mathrm{d}^3x \frac{1}{2}\hat{\psi}^\dagger_a(\Sigma_3)_{ab}\hat{\psi}_b \tag{2.1.3}$$

which

$$\hat{\psi}_a(\boldsymbol{x},0) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\sum_{s=1}^{2}\left[\hat{a}^s_{\boldsymbol{p}}u^s_a(\boldsymbol{p}) + \hat{b}^{s\dagger}_{-\boldsymbol{p}}v^s_a(-\boldsymbol{p})\right] \tag{2.1.4}$$

and

$$\hat{\psi}^\dagger_a(\boldsymbol{x},0) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\sum_{s=1}^{2}\left[\hat{a}^{s\dagger}_{\boldsymbol{p}}u^{s*}_a(\boldsymbol{p}) + \hat{b}^{s}_{-\boldsymbol{p}}v^{s*}_a(-\boldsymbol{p})\right] \tag{2.1.5}$$

for $t = 0$, $\boldsymbol{p}$ is 3-momentum, then we can factor $e^{\pm i\boldsymbol{p}\cdot\boldsymbol{x}}$ out before the sum.

Substitute Equ.2.4 and Equ.2.5 into Equ.2.3, we'll get the expansion of S_3 :

$$\begin{aligned}
\hat{S}_3 &= \int \mathrm{d}^3x \int \frac{\mathrm{d}^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}'}}}e^{i\boldsymbol{p}'\cdot\boldsymbol{x}}\sum_{s'=1}^{2}\left[\hat{a}^{s'}_{\boldsymbol{p}}u^{s'}_a(\boldsymbol{p}') + \hat{b}^{s'\dagger}_{-\boldsymbol{p}'}v^{s'}_a(-\boldsymbol{p}')\right] \\
&\times \frac{1}{2}(\Sigma_3)_{ab}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\sum_{s=1}^{2}\left[\hat{a}^{s\dagger}_{\boldsymbol{p}}u^{s*}_b(\boldsymbol{p}) + \hat{b}^{s}_{-\boldsymbol{p}}v^{s*}_b(-\boldsymbol{p})\right] \\
&= \int \mathrm{d}^3x \int \frac{\mathrm{d}^3p'\,\mathrm{d}^3p}{(2\pi)^6}\frac{1}{\sqrt{2E_{\boldsymbol{p}'}2E_{\boldsymbol{p}}}}e^{i\boldsymbol{p}'\cdot\boldsymbol{x}}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\sum_{s,s'}\left[\hat{a}^{s'}_{\boldsymbol{p}}u^{s'}_a(\boldsymbol{p}') + \hat{b}^{s'\dagger}_{-\boldsymbol{p}'}v^{s'}_a(-\boldsymbol{p}')\right] \\
&\times \frac{1}{2}(\Sigma_3)_{ab}\left[\hat{a}^{s\dagger}_{\boldsymbol{p}}u^{s*}_b(\boldsymbol{p}) + \hat{b}^{s}_{-\boldsymbol{p}}v^{s*}_b(-\boldsymbol{p})\right]
\end{aligned} \tag{2.1.6}$$

Suppose $\textcircled{1} = \sum_{s,s'} \left[\hat{a}_{\boldsymbol{p}}^{s'} u_a^{s'}(\boldsymbol{p'}) + \hat{b}_{-\boldsymbol{p'}}^{s'\dagger} v_a^{s'}(-\boldsymbol{p'})\right] \times \frac{1}{2}(\Sigma_3)_{ab} \left[\hat{a}_{\boldsymbol{p}}^{s\dagger} u_b^{s*}(\boldsymbol{p}) + \hat{b}_{-\boldsymbol{p}}^{s} v_b^{s*}(-\boldsymbol{p})\right]$,

so

$$
\begin{aligned}
\textcircled{1} &= \frac{1}{2}(\Sigma_3)_{ab} \sum_{s,s'} \left[\hat{a}_{\boldsymbol{p}}^{s'} u_a^{s'}(\boldsymbol{p'}) + \hat{b}_{-\boldsymbol{p'}}^{s'\dagger} v_a^{s'}(-\boldsymbol{p'})\right] \left[\hat{a}_{\boldsymbol{p}}^{s\dagger} u_b^{s*}(\boldsymbol{p}) + \hat{b}_{-\boldsymbol{p}}^{s} v_b^{s*}(-\boldsymbol{p})\right] \\
&= \frac{1}{2}(\Sigma_3)_{ab} \sum_{s,s'} \left[\hat{a}_{\boldsymbol{p}}^{s'} u_a^{s'}(\boldsymbol{p'}) \hat{a}_{\boldsymbol{p}}^{s\dagger} u_b^{s*}(\boldsymbol{p}) + \hat{b}_{-\boldsymbol{p'}}^{s'\dagger} v_a^{s'}(-\boldsymbol{p'}) \hat{a}_{\boldsymbol{p}}^{s\dagger} u_b^{s*}(\boldsymbol{p}) + \right. \\
&\left. \hat{a}_{\boldsymbol{p}}^{s'} u_a^{s'}(\boldsymbol{p'}) \hat{b}_{-\boldsymbol{p}}^{s} v_b^{s*}(-\boldsymbol{p}) + \hat{b}_{-\boldsymbol{p'}}^{s'\dagger} v_a^{s'}(-\boldsymbol{p'}) \hat{b}_{-\boldsymbol{p}}^{s} v_b^{s*}(-\boldsymbol{p})\right]
\end{aligned}
\tag{2.1.7}
$$

When $\boldsymbol{p} = \boldsymbol{0}$

$$
\hat{S}_3 \hat{a}_{\boldsymbol{0}}^{s\dagger} |0\rangle = \int \mathrm{d}^3 x \int \frac{\mathrm{d}^3 p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p'}} 2E_{\boldsymbol{p}}}} e^{i\boldsymbol{p'}\cdot\boldsymbol{x}} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \textcircled{1} \hat{a}_{\boldsymbol{0}}^{s\dagger} |0\rangle \tag{2.1.8}
$$

and

$$
\begin{aligned}
\textcircled{1} \hat{a}_{\boldsymbol{0}}^{s\dagger} |0\rangle = \frac{1}{2}(\Sigma_3)_{ab} \sum_{s,s'} &\left[\hat{a}_{\boldsymbol{p}}^{s'} u_a^{s'}(\boldsymbol{p'}) \hat{a}_{\boldsymbol{p}}^{s\dagger} u_b^{s*}(\boldsymbol{p}) \hat{a}_{\boldsymbol{0}}^{s\dagger} |0\rangle + \hat{b}_{-\boldsymbol{p'}}^{s'\dagger} v_a^{s'}(-\boldsymbol{p'}) \hat{a}_{\boldsymbol{p}}^{s\dagger} u_b^{s*}(\boldsymbol{p}) \hat{a}_{\boldsymbol{0}}^{s\dagger} |0\rangle \right. \\
&\left. + \hat{a}_{\boldsymbol{p}}^{s'} u_a^{s'}(\boldsymbol{p'}) \hat{b}_{-\boldsymbol{p}}^{s} v_b^{s*}(-\boldsymbol{p}) \hat{a}_{\boldsymbol{0}}^{s\dagger} |0\rangle + \hat{b}_{-\boldsymbol{p'}}^{s'\dagger} v_a^{s'}(-\boldsymbol{p'}) \hat{b}_{-\boldsymbol{p}}^{s} v_b^{s*}(-\boldsymbol{p}) \hat{a}_{\boldsymbol{0}}^{s\dagger} |0\rangle\right]
\end{aligned}
\tag{2.1.9}
$$

of which

$$
\{\hat{a}_{\boldsymbol{p}}^{r}, \hat{a}_{\boldsymbol{q}}^{s\dagger}\} = \{\hat{b}_{\boldsymbol{p}}^{r}, \hat{b}_{\boldsymbol{q}}^{s\dagger}\} = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q}) \delta_{rs} \tag{2.1.10}
$$

So

$$
\hat{S}_3 \hat{a}_{\boldsymbol{0}}^{s\dagger} |0\rangle = \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} (-1)^{s+1} 2E_{\boldsymbol{p}} (2\pi)^3 \delta^3(\boldsymbol{p}) \hat{a}_{\boldsymbol{p}}^{s\dagger} |0\rangle = \frac{(-1)^{s+1}}{2} \hat{a}_{\boldsymbol{0}}^{s\dagger} |0\rangle \tag{2.1.11}
$$

For positive fermion field, we have $\hat{S}_3 \hat{a}_{\boldsymbol{0}}^{1\dagger} |0\rangle = \frac{1}{2} \hat{a}_{\boldsymbol{0}}^{1\dagger} |0\rangle$, and $\hat{S}_3 \hat{a}_{\boldsymbol{0}}^{2\dagger} |0\rangle = -\frac{1}{2} \hat{a}_{\boldsymbol{0}}^{2\dagger} |0\rangle$.

(2) Antifermion

As the same goes, we can get

$$
\hat{S}_3 \hat{b}_{\boldsymbol{0}}^{s\dagger} |0\rangle = \int \mathrm{d}^3 x \int \frac{\mathrm{d}^3 p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p'}} 2E_{\boldsymbol{p}}}} e^{i\boldsymbol{p'}\cdot\boldsymbol{x}} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \textcircled{1} \hat{b}_{\boldsymbol{0}}^{s\dagger} |0\rangle \tag{2.2.1}
$$

and

$$
\begin{aligned}
①\hat{b}_{\mathbf{0}}^{s\dagger}|0\rangle = \frac{1}{2}(\Sigma_3)_{ab}\sum_{s,s'}\Big[&\hat{a}_{\boldsymbol{p}}^{s'}u_a^{s'}(\boldsymbol{p}')\hat{a}_{\boldsymbol{p}}^{s\dagger}u_b^{s*}(\boldsymbol{p})\hat{b}_{\mathbf{0}}^{s\dagger}|0\rangle + \hat{b}_{-\boldsymbol{p}'}^{s'\dagger}v_a^{s'}(-\boldsymbol{p}')\hat{a}_{\boldsymbol{p}}^{s\dagger}u_b^{s*}(\boldsymbol{p})\hat{b}_{\mathbf{0}}^{s\dagger}|0\rangle \\
&+\hat{a}_{\boldsymbol{p}}^{s'}u_a^{s'}(\boldsymbol{p}')\hat{b}_{-\boldsymbol{p}}^{s}v_b^{s*}(-\boldsymbol{p})\hat{b}_{\mathbf{0}}^{s\dagger}|0\rangle + \hat{b}_{-\boldsymbol{p}'}^{s'\dagger}v_a^{s'}(-\boldsymbol{p}')\hat{b}_{-\boldsymbol{p}}^{s}v_b^{s*}(-\boldsymbol{p})\hat{b}_{\mathbf{0}}^{s\dagger}|0\rangle\Big]
\end{aligned}
\tag{2.2.2}
$$

So we have

$$
\begin{aligned}
\hat{S}_3\hat{b}_{\mathbf{0}}^{s\dagger}|0\rangle &= -\frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}(-1)^{s+1}2E_{\boldsymbol{p}}(2\pi)^3\delta^3(\boldsymbol{p})\hat{b}_{-\boldsymbol{p}}^{s\dagger}|0\rangle = -\frac{(-1)^{s+1}}{2}\hat{b}_{\mathbf{0}}^{s\dagger}|0\rangle \\
&= \frac{(-1)^s}{2}\hat{b}_{\mathbf{0}}^{s\dagger}|0\rangle
\end{aligned}
\tag{2.2.3}
$$

For negative fermion field, we have $\hat{S}_3\hat{b}_{\mathbf{0}}^{1\dagger}|0\rangle = -\frac{1}{2}\hat{b}_{\mathbf{0}}^{1\dagger}|0\rangle$, and $\hat{S}_3\hat{b}_{\mathbf{0}}^{2\dagger}|0\rangle = \frac{1}{2}\hat{b}_{\mathbf{0}}^{2\dagger}|0\rangle$.

3. Prove : (1) $C\bar{\psi}\gamma^\mu\psi C = -\bar{\psi}\gamma^\mu\psi$**; (2)** $C\bar{\psi}\gamma^\mu\gamma^5\psi C = \bar{\psi}\gamma^\mu\gamma^5\psi$**; (3)** $C\bar{\psi}\sigma^{\mu\nu}\psi C = -\bar{\psi}\sigma^{\mu\nu}\psi$**, which** C **is unitary linear operator to represent charge conjugation.**

Proof: We have $\bar{\psi} = \psi^\dagger\gamma^0$, and $(\gamma^5)^\top = \gamma^5$

For bilinear Dirac field under charge conjugation transformation, we have $C\psi C = (-i\bar{\psi}\gamma^0\gamma^2)^\top$, and $C\bar{\psi}C = (-i\gamma^0\gamma^2\psi)^\top$

$$
\begin{aligned}
C\bar{\psi}\psi C = C\bar{\psi}CC\psi C &= (-i\bar{\psi}\gamma^0\gamma^2)^\top(-i\gamma^0\gamma^2\psi)^\top \\
&= -(\gamma^0\gamma^2\psi)_a^\top(\bar{\psi}\gamma^0\gamma^2)_a^\top = -\gamma^0_{ab}\gamma^2_{bc}\psi_c\bar{\psi}_d\gamma^0_{de}\gamma^2_{ea} \\
&= -\gamma^0_{ab}\gamma^2_{bc}\psi_c\psi^\dagger_e\gamma^2_{ea} \\
&= -\gamma^0_{ab}\gamma^2_{bc}\gamma^2_{ea}\left[\{\psi_c(x),\psi^\dagger_e(x)\} - \psi^\dagger_e(x)\psi_c(x)\right] \\
&= -\gamma^0_{ab}\gamma^2_{bc}\gamma^2_{ca}\delta^{(3)}(\mathbf{0}) + \gamma^2_{ea}\psi^\dagger_e(x)\gamma^0_{ab}\gamma^2_{bc}\psi_c(x) \\
&= \gamma^0\delta^{(3)}(\mathbf{0}) + \bar{\psi}(x)\gamma^0\gamma^2\gamma^0\gamma^2\psi(x)
\end{aligned}
$$

Therefore $C\bar{\psi}\psi C = \bar{\psi}\psi$. As the same goes, we have

(1)

$$\begin{aligned} C\bar{\psi}\gamma^\mu\psi C &= (-i\gamma^0\gamma^2\psi)^\top\gamma^\mu(-i\bar{\psi}\gamma^0\gamma^2)^\top \\ &= -\psi_b(\gamma^2)^\top_{bc}(\gamma^0)^\top_{ca}\gamma^\mu_{ab}(\gamma^2)^\top_{bd}\psi^\dagger_d \\ &= \psi^\dagger_d\gamma^2_{db}(\gamma^\mu)^\top_{ba}\gamma^0_{ac}\gamma^2_{cb}\psi_b \\ &= \psi^\dagger_d\gamma^0_{de}\gamma^0_{ef}\gamma^2_{fb}(\gamma^\mu)^\top_{ba}\gamma^0_{ac}\gamma^2_{cb}\psi_b \\ &= \bar{\psi}_e\gamma^0_{ef}\gamma^2_{fb}(\gamma^\mu)^\top_{ba}\gamma^0_{ac}\gamma^2_{cb}\psi_b \\ &= \bar{\psi}\gamma^0\gamma^2(\gamma^\mu)^\top\gamma^0\gamma^2\psi \end{aligned} \tag{3.1.1}$$

When $\mu = 0, 2$, $(\gamma^\mu)^\top = \gamma^\mu$

$$\bar{\psi}\gamma^0\gamma^2(\gamma^\mu)^\top\gamma^0\gamma^2 = \bar{\psi}(-1)\gamma^\mu\psi = -\bar{\psi}\gamma^\mu\psi \tag{3.1.2}$$

and if $\mu = 1, 3$, $(\gamma^\mu)^\top = -\gamma^\mu$

$$\bar{\psi}\gamma^0\gamma^2(\gamma^\mu)^\top\gamma^0\gamma^2 = \bar{\psi}(-\gamma^\mu)\psi = -\bar{\psi}\gamma^\mu\psi \tag{3.1.3}$$

So we can say $C\bar{\psi}\gamma^\mu\psi C = -\bar{\psi}\gamma^\mu\psi$.

Q.E.D.

(2)

$$\begin{aligned} C\bar{\psi}\gamma^\mu\psi C &= (-i\gamma^0\gamma^2\psi)^\top\gamma^\mu\gamma^5(-i\bar{\psi}\gamma^0\gamma^2)^\top \\ &= -\psi_f(\gamma^2)^\top_{fe}(\gamma^0)^\top_{ea}\gamma^\mu_{ab}\gamma^5_{bc}(\gamma^2)^\top_{cd}\psi^\dagger_d \\ &= -\psi^\dagger_d\gamma^2_{dc}(\gamma^5)^\top_{cb}(\gamma^\mu)^\top_{ba}\gamma^0_{ae}\gamma^2_{ef}\psi_f \\ &= -\bar{\psi}_d\gamma^0_{de}\gamma^2_{ec}\gamma^5_{cb}(\gamma^\mu)^\top_{ba}\gamma^0_{ae}\gamma^2_{ef}\psi_f \\ &= -\bar{\psi}\gamma^0\gamma^2\gamma^5(\gamma^\mu)^\top\gamma^0\gamma^2\psi \end{aligned} \tag{3.2.1}$$

When $\mu = 0, 2$, $(\gamma^\mu)^\top = \gamma^\mu$

$$-\bar{\psi}\gamma^0\gamma^2\gamma^5(\gamma^\mu)^\top\gamma^0\gamma^2\psi = -\bar{\psi}\gamma^5\gamma^\mu\psi = \bar{\psi}\gamma^\mu\gamma^5\psi \tag{3.2.2}$$

and if $\mu = 1, 3$, $(\gamma^\mu)^\top = -\gamma^\mu$

$$-\bar{\psi}\gamma^0\gamma^2\gamma^5(\gamma^\mu)^\top\gamma^0\gamma^2\psi = -\bar{\psi}(-\gamma^\mu)\gamma^5\psi = \bar{\psi}\gamma^\mu\gamma^5\psi \tag{3.2.3}$$

So we can say $C\bar{\psi}\gamma^\mu\gamma^5\psi C = \bar{\psi}\gamma^\mu\gamma^5\psi$.

Q.E.D.

(3) For $\sigma^{\mu\nu} = \frac{i}{2}[\,\gamma^\mu, \gamma^\nu]\; = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$,

we have

$$\begin{aligned} C\bar{\psi}\gamma^\mu\gamma^\nu\psi C &= (-i\gamma^0\gamma^2\psi)^\top\gamma^\mu\gamma^\nu(-i\bar{\psi}\gamma^0\gamma^2)^\top \\ &= -\psi_f(\gamma^2)^\top_{fe}(\gamma^0)^\top_{ea}\gamma^\mu_{ab}\gamma^\nu_{bc}(\gamma^2)^\top_{cd}\psi^\dagger_d \\ &= -\psi^\dagger_d\gamma^2_{dc}(\gamma^\nu)^\top_{cb}(\gamma^\mu)^\top_{ba}\gamma^0_{ae}\gamma^2_{ef}\psi_f \\ &= -\bar{\psi}\gamma^0\gamma^2(\gamma^\nu)^\top(\gamma^\mu)^\top\gamma^0\gamma^2\psi \end{aligned} \tag{3.3.1}$$

• When $\mu, \nu = 0, 2$, or $\mu, \nu = 1, 3$. We have

$$-\bar{\psi}\gamma^0\gamma^2(\gamma^\nu)^\top(\gamma^\mu)^\top\gamma^0\gamma^2\psi = \bar{\psi}\gamma^\nu\gamma^\mu\psi = -\bar{\psi}\gamma^\mu\gamma^\nu\psi \tag{3.3.2}$$

• Or $\mu, \nu = 0, 2$, or $\mu, \nu = 1, 3$. We have

$$-\bar{\psi}\gamma^0\gamma^2(\gamma^\nu)^\top(\gamma^\mu)^\top\gamma^0\gamma^2\psi = \bar{\psi}(-\gamma^\nu)(-\gamma^\mu)\psi = -\bar{\psi}\gamma^\mu\gamma^\nu\psi \tag{3.3.3}$$

From Equ.3.3.2 and Equ.3.3.3, we can get $C\bar{\psi}\gamma^\mu\gamma^\nu\psi C = -\bar{\psi}\gamma^\mu\gamma^\nu\psi$.

And as the same goes, we have $C\bar{\psi}\gamma^\nu\gamma^\mu\psi C = -\bar{\psi}\gamma^\nu\gamma^\mu\psi$.

Then we obtain

$$
\begin{aligned}
C\bar{\psi}\sigma^{\mu\nu}\psi C &= \frac{i}{2}C\bar{\psi}[\gamma^\mu, \gamma^\nu]\psi C \\
&= \frac{i}{2}C\bar{\psi}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\psi C \\
&= \frac{i}{2}\bar{\psi}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\psi \\
&= \bar{\psi}\sigma^{\nu\mu}\psi = -\bar{\psi}\sigma^{\mu\nu}\psi
\end{aligned}
\tag{3.3.3}
$$

So we can say $C\bar{\psi}\sigma^{\mu\nu}\psi C = -\bar{\psi}\sigma^{\mu\nu}\psi$.

Q.E.D.

4. Prove the commutation relation of creation-annihilation operators of photon field.

Proof: With the expressions of vector field operator in Equ.4.1.1 and conjugate momentum in Equ.4.1.2 below

$$
\boldsymbol{A}(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{k}}}}\sum_{\lambda=1,2}\boldsymbol{\epsilon}^{(\lambda)}(\boldsymbol{k})\left(a_\lambda(\boldsymbol{k})e^{ik\cdot x} + a_\lambda^\dagger(\boldsymbol{k})e^{-ik\cdot x}\right) \tag{4.1.1}
$$

$$
\boldsymbol{\pi}(x) = i\int \frac{\mathrm{d}^3k}{(2\pi)^3}\sqrt{\frac{E_{\boldsymbol{k}}}{2}}\sum_{\lambda=1,2}\boldsymbol{\epsilon}^{(\lambda)}(\boldsymbol{k})\left(a_\lambda(\boldsymbol{k})e^{ik\cdot x} - a_\lambda^\dagger(\boldsymbol{k})e^{-ik\cdot x}\right) \tag{4.1.2}
$$

which polarization vector $\boldsymbol{\epsilon}$ satisfies $\boldsymbol{k}\cdot\boldsymbol{\epsilon}^{(\lambda)}(\boldsymbol{k}) = 0$, and three space-like polarization vectors $\boldsymbol{\epsilon}^{(\lambda)}(\boldsymbol{k})$ are orthogonal to $\boldsymbol{k}$

$$
\boldsymbol{\epsilon}^{(\lambda)}(\boldsymbol{k})\cdot\boldsymbol{\epsilon}^{(\lambda')}(\boldsymbol{k}) = -\delta_{\lambda\lambda'} \qquad \sum_{\lambda=1}^{3}\epsilon_\rho^{(\lambda)}(k)\epsilon_\nu^{(\lambda)}(k) = -\left(g_{\rho\nu} - \frac{k_\rho k_\nu}{\mu^2}\right) \tag{4.1.3}
$$

Now we can work out the expression of annihilation operator

$$
a_{\lambda=1,2}(\boldsymbol{k}) = \int \mathrm{d}^3x e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{4.1.4}
$$

And creation operator $a_\lambda^\dagger(\boldsymbol{k})$ is

$$a_{\lambda=1,2}^\dagger(\boldsymbol{k}) = \int \mathrm{d}^3x e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{4.1.5}$$

Then we have

$$\begin{aligned}\left[a_\lambda(\boldsymbol{k}), a_{\lambda'}^\dagger(\boldsymbol{k}')\right] &= -\int \mathrm{d}^3x\,\mathrm{d}^3x' e^{-ik\cdot x} e^{ik'\cdot x'} \\ &= \\ &= \delta_{\lambda\lambda'}\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')\end{aligned} \tag{4.1.6}$$

So we verify that creation-annihilation operators of vector field have commutation relation which is $\left[a_\lambda(\boldsymbol{k}), a_{\lambda'}^\dagger(\boldsymbol{k}')\right] = \delta_{\lambda\lambda'}\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')$