A 3d curvilinear skeletonization algorithm with application to bidirectional path tracing

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Abstract. What we propose is really great...

1 Introduction

Blah blah blah...

2 The cubical complex framework

2.1 Basic definitions

In the 3d cubical complex framework, objects are no more made of voxel, but of cubes, squares, lines and vertices $\mathbb{R}$. Let $\mathbb{Z}$ be the set of integers, we consider the family of sets $\mathbb{F}_0^1$ and $\mathbb{F}_1^1$, such that $\mathbb{F}_0^1 = \{\{a\} \mid a \in \mathbb{Z}\}$ and $\mathbb{F}_1^1 = \{\{a, a+1\} \mid a \in \mathbb{Z}\}$. Any subset f of $\mathbb{Z}^n$ such that f is the cartesian product of m elements of $\mathbb{F}_1^1$ and $(n-m)$ elements of $\mathbb{F}_0^1$ is called a face or an *m-face* of $\mathbb{Z}^n$, m is the dimension of f, we write $dim(f) = m$. A 0-face is called a *vertex*, a 1-face is an *edge*, a 2-face is a *square*, and a 3-face is a *cube*.

We denote by $\mathbb{F}^n$ the set composed of all faces in $\mathbb{Z}^n$. Given $m \in \{0, \ldots, n\}$, we denote by $\mathbb{F}_m^n$ the set composed of all m-faces in $\mathbb{Z}^n$.

Let $f \in \mathbb{F}^n$. We set $\hat{f} = \{g \in \mathbb{F}^n | g \subseteq f\}$, and $\hat{f}^* = \hat{f} \setminus \{f\}$. Any element of $\hat{f}$ is *a face of* f, and any element of $\hat{f}^*$ is *a proper face of* f. We call *star of* f the set $\check{f} = \{g \in \mathbb{F}^n | f \subseteq g\}$, and we write $\check{f}^* = \check{f} \setminus \{f\}$: any element of $\check{f}$ is a *coface of* f. It is plain that $g \in \hat{f}$ iff $f \in \check{g}$.

A set X of faces in $\mathbb{F}^n$ is a *cell*, or *m-cell*, if there exists an m-face $f \in X$ such that $X = \hat{f}$. The *closure* of a set of faces X is the set $X^- = \cup\{\hat{f} | f \in X\}$. The set $\overline{X}$ is $\mathbb{F}^n \setminus X$.

Definition 1. *A finite set X of faces in $\mathbb{F}^n$ is a* cubical complex *if $X = X^-$, and we write $X \preceq \mathbb{F}^n$.*

Any subset Y of X which is also a complex is a subcomplex of X, *and we write $Y \preceq X$.*

Informally, in 3d, a set of face is a cubical complex if, for each cube of the complex, the six squares (sides) of the cube also belong to the complex, for each square of the complex, the four edges (sides) of the square also belong to the complex, and for each edge of the complex, the two vertices (extremities) of the edge also belong to the complex.

A face $f \in X$ is *a facet of* X if f is not a proper face of any face of X. We denote by X^+ the set composed of all facets of X . A complex X is *pure* if all its facets have the same dimension. The *dimension of* X is $\dim(X) = \max\{\dim(f) \mid f \in X\}$. If $\dim(X) = d$, then we say that X is a d-complex. The notions of purity and dimensions can be trivially extended to sets of faces.

In $\mathbb{F}^n$, a complex X is *thin* if $\dim(X) < n$. In $\mathbb{F}^3$, a complex is thin if it contains no cube.

2.2 From binary images to cubical complex

Traditionally, a binary image is defined as a finite subset of $\mathbb{Z}^n$ (with $n = 2$ or $n = 3$). Given $S \subseteq \mathbb{Z}^n$, the object voxels are the elements of S. This kind of image is the most common one in the field of image processing so, in order to work in cubical complex framework, we need to find a way to transpose a binary image to cubical complex framework.

Informally, to do so, we associate to each element of $S \subseteq \mathbb{Z}^n$ an n-face of $\mathbb{F}^n$ (to a pixel we associate a square, to a voxel we associate a cube). More precisely, let $x = (x_1, ..., x_n) \in S$, we define the n-face $\Phi(x) = \{x_1, x_1+1\} \times \ldots \times \{x_n, x_n+1\}$. We can extend the map Φ to sets: $\Phi(S) = \{\Phi(x) | x \in S\}$. Given a binary image S, we associate to it the cubical complex $\Phi(S)^-$ (see Fig. **??**).

In the following, most of the objects we consider were indeed binary images which were then transposed into cubical complex framework: this is why most of the two-dimensional complexes we show are pure 2-complexes, and most of the three-dimensional complexes we show are pure 3-complexes.

2.3 Thinning: the collapse operation

The collapse operation is the basic operation for performing homotopic thinning of a complex. It consists of removing two distinct elements (f, g) from a complex X under the condition that g is contained in f and is not contained in any other element of X. This operation may be repeated several times.

Definition 2. *Let $X \preceq \mathbb{F}^n$, and let f, g be two faces of X. The face g is* free for X*, and the pair (f, g) is* a free pair for X *if f is the only face of X such that g is a proper face of f.*

In other terms, (f, g) is a free pair for X whenever $\breve{g}^* \cap X = \{f\}$ (g is included only in f). It can be easily seen that if (f, g) is a free pair for a complex X and $\dim(f) = m$, then f is a facet and $\dim(g) = m - 1$.

Definition 3. *Let $X \preceq \mathbb{F}^n$, and let (f,g) be a free pair for X. The complex $X \setminus \{f,g\}$ is an* elementary collapse *of X.*

Let $Y \preceq \mathbb{F}^n$, the complex X collapses onto *Y if there exists a sequence of complexes $(X_0, ..., X_\ell)$ of $\mathbb{F}^n$ such that $X = X_0$, $Y = X_\ell$ and for all $i \in \{1, \ldots, \ell\}$, X_i is an elementary collapse of X_{i-1}. We also say, in this case, that Y* is a collapse *of X.*

We insist here on the fact that, if Y is a collapse of a complex X, then Y is a complex.

Let us now introduce some elements that will serve later for proving the thinness of our skeletons. Let f_0, f_ℓ be two n-faces of $\mathbb{F}^n$ (with ℓ being even). An *$(n-1)$-path from f_0 to f_ℓ* is a sequence $\pi = (f_0, ..., f_\ell)$ of faces of $\mathbb{F}^n$ such that for all $i \in \{0, \ldots, \ell\}$, either i is even and f_i is an n-face, or i is odd and f_i is an $(n-1)$-face with $\check{f_i}^* = \{f_{i-1}, f_{i+1}\}$ (such path always exists).

Proposition 4. *Let $X \preceq \mathbb{F}^n$ be an n-complex, with $n > 0$. Then X has at least one free $(n-1)$-face.*

Proof. Since X is an n-complex (hence X is finite) there exists an n-face a in X and an n-face b in $\overline{X}$. Obviously, there exists an $(n-1)$-path from a to b. Let h be the first n-face of π that is not in X, let k be the last n-face of π before h (thus k is in X), and let $e = k \cap h$ be the $(n-1)$-face of π between k and h. Since k and h are the only two n-faces of $\mathbb{F}^n$ that contain e, we see that the pair (k, e) is free for X. □

In conclusion, in $\mathbb{F}^3$, as long as a complex still contains 3-faces, it has a free 2-face and more collapse operations can be performed. Therefore, it is possible to perform collapse on a complex until no more 3-faces (volumes) can be found (until it is thin).

As in the DT framework, it is sometimes necessary to perform collapse in the cubical complex framework while preserving some faces safe from deletion: these faces are the so-called inhibitor set. When using an inhibitor set during collapse, the guarantee of having a thin result does no more hold. As illustrated on figure ffjkz, the inhibitor set can be thin and a result of the constrained thinning can still not be thin. However, under some conditions, it is possible to use an inhibitor set while collapsing, and still have the guarantee of getting a thin result. In order to prove this, let us state the following:

Lemma 5. *Let $X \preceq \mathbb{F}^n$, let (f_1, g_1) be a free pair of X, and (f_2, g_2) be a free pair of $(X \backslash \{f_1, g_1\})$. If $\dim(f_2) > \dim(f_1)$, then (f_2, g_2) is free for X.*

Proof. If (f_2, g_2) is a free pair of $(X \backslash \{f_1, g_1\})$, then g_2 is included in only one face of $(X \backslash \{f_1, g_1\})$, which is f_2. As $(\dim(g_2) = \dim(f_2) - 1)$ and $(\dim(f_1) \leq (\dim(f_2) - 1))$, then $g_2 \nsubseteq f_1$ and $g_2 \nsubseteq g_1$. Therefore, g_2 is included in only one face of X, which is f_2: therefore, (f_2, g_2) is free for X.□

This lemma implies that, when one has a sequence of removal of free pairs of faces from a complex, one can only keep the free pairs of highest dimension and still have a sequence of removal of free pairs from the complex.

The following proposition explains which conditions on the inhibitor set (denoted W) need to be matched in order to guarantee the thinness of the result of a thinning constrained by W.

Proposition 6. *Let X be an n-complex with $n > 0$, let S be a collapse of X such that $\dim(S) \leq (n-1)$ and let $W \preceq S$. Let Y be a collapse of X such that $W \subseteq Y$ and such that there are no free pairs in Y included in $Y \backslash W$. Then, the dimension of Y is inferior or equal to $(n-1)$.*

Proof. In the following, we show that if there exists an n-face in Y, then it belongs to a free pair for Y that is included in $Y \backslash W$, a contradiction with the hypothesis of the proposition.

Let $C = ((a_1, b_1), ..., (a_k, b_k))$ be a sequence of removal of free faces which allows to obtain S from X: for all $i \in [1;k]$, (a_i, b_i) is free for $X \backslash \{a_1, b_1, ..., a_{i-1}, b_{i-1}\}$ and $S = X \backslash \{a_1, b_1, ..., a_k, b_k\}$. Let C' be the sequence C restrained only to free pairs containing an n-face: $C' = ((f_1, g_1), ..., (f_h, g_h))$. A consequence of lemma 5 is that, for all $i \in [1;h]$, (f_i, g_i) is free for $X \backslash \{f_1, g_1, ..., f_{i-1}, g_{i-1}\}$ and $(X \backslash \{f_1, g_1, ..., f_k, g_k\})$ is a collapse of X.

Any n-face $c \in Y$ is such that $c \in X$ and $c \notin S$, and therefore there exists $j \in [1;h]$ such that $c = f_j$. Without loss of generality, let j be the smallest integer such that $f_j \in Y$: for all $k \in [1; j-1]$, $f_k \notin Y$ and $g_k \notin Y$. As previously said, (f_j, g_j) is free for $(X \backslash \{f_1, g_1, ..., f_{j-1}, g_{j-1}\})$: g_j is included in only one face of $(X \backslash \{f_1, g_1, ..., f_{j-1}, g_{j-1}\})$, and this face is f_j.

As $\{f_1, g_1, ..., f_{j-1}, g_{j-1}\} \cap Y = \emptyset$ and that $f_j \in Y$, then g_j is included in only one face of Y, and that face is f_j. Consequently, the pair (f_j, g_j) is free for Y. Moreover, the pair (f_j, g_j) belongs to the sequence C, therefore $f_j \notin S$ and $g_j \notin S$. As $W \subseteq S$, the pair (f_j, g_j) is included in $Y \backslash W$.$\square$

3 A parallel directional thinning based on cubical complex

3.1 Removing free pairs in parallel

In the cubical complex framework, parallel removal of simple pairs can be easily achieved when following simple rules that we will give now. First, we need to define the *direction* and the *orientation* of a free face.

Let $f \in \mathbb{F}^n$, the *center of* f is the center of mass of the points in f, that is, $c_f = \frac{1}{|f|} \sum_{a \in f} a$. The center of f is an element of $[\frac{\mathbb{Z}}{2}]^n$, where $\frac{\mathbb{Z}}{2}$ denotes the set of half integers. Let $X \preceq \mathbb{F}^n$, let (f, g) be a free pair for X, and let c_f and c_g be the respective centers of the faces f and g. We denote by $V(f, g)$ the vector $(c_f - c_g)$ of $[\frac{\mathbb{Z}}{2}]^n$.

We define a surjective function $Dir() : \mathbb{F}^n \times \mathbb{F}^n \to \{0, \dots, n-1\}$ such that, for all free pairs (f, g) and (i, j) for X, $Dir(f, g) = Dir(i, j)$ if and only if $V(f, g)$ and $V(i, j)$ are collinear (we don't bother defining $Dir()$ for non free pairs as it won't be useful in this case). The number $Dir(f, g)$ is called the *direction* of the free pair (f, g). Let (f, g) be a free pair, the vector $V(f, g)$ has only one non-null coordinate: the pair (f, g) has a *positive orientation*, and we write $Orient(f, g) = 1$, if the non-null coordinate of $V(f, g)$ is positive; otherwise (f, g) has a *negative orientation*, and we write $Orient(f, g) = 0$. On Fig. **??**, the free pair (a, c) and the free pair (d, e) have different directions; the free pairs (a, b) and (d, e) have the same direction, but opposite orientations.

Now, we give a property of collapse which brings a necessary and sufficient condition for removing two free pairs of faces in parallel from a complex, while preserving topology (see Fig. **??**c).

Proposition 7. *Let $X \preceq \mathbb{F}^n$, and let (f, g) and (k, ℓ) be two distinct free pairs for X. The complex X collapses onto $X \setminus \{f, g, k, \ell\}$ if and only if $f \neq k$.*

Proof. If $f = k$, then it is plain that (k, ℓ) is not a free pair for $Y = X \setminus \{f, g\}$ as $k = f \notin Y$. Also, (f, g) is not free for $X \setminus \{k, \ell\}$. If $f \neq k$, then we have $g \neq \ell$, $\breve{g}^* \cap X = \{f\}$ (g is free for X) and $\breve{\ell}^* \cap X = \{k\}$ (ℓ is free for X). Thus, we have $\breve{\ell}^* \cap Y = \{k\}$ as $\ell \neq g$ and $k \neq f$. Therefore, (k, ℓ) is a free pair for Y. $\square$

From Prop. 7, the following corollary is immediate.

Corollary 8. *Let $X \preceq \mathbb{F}^n$, and let $(f_1, g_1) \ldots (f_m, g_m)$ be m distinct free pairs for X such that, for all $a, b \in \{1, \ldots, m\}$ (with $a \neq b$), $f_a \neq f_b$. The complex X collapses onto $X \setminus \{f_1, g_1 \ldots f_m, g_m\}$.*

Considering two distinct free pairs (f, g) and (i, j) for $X \preceq \mathbb{F}^n$ such that $Dir(f, g) = Dir(i, j)$ and $Orient(f, g) = Orient(i, j)$, we have $f \neq i$. From this observation and Cor. 8, we deduce the following property.

Corollary 9. *Let $X \preceq \mathbb{F}^n$, and let $(f_1, g_1) \ldots (f_m, g_m)$ be m distinct free pairs for X having all the same direction and the same orientation. The complex X collapses onto $X \setminus \{f_1, g_1 \ldots f_m, g_m\}$.*