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Updated Analytical Partials for Covariance Transformations and Optimization

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Covariance estimates are becoming more widely available as flight dynamics operations work towards greater accuracy. Investigators have looked at how covariance matrices are propagated, to include orbital state formats and coordinate systems. Various equations to convert between orbital state formats and satellite coordinate systems are essential for proper use and analysis. The literature contains many examples. Vallado (2003) presented a complete set of equations, but advised that a few inconsistencies were found. We have corrected those errors and provide the results. Test results are given for several cases, and MatLab code is available.

INTRODUCTION[‡]

In Orbit Determination (OD), the covariance matrix is a bi-product of a Least Squares or Kalman filter process. Even with the advent of numerical operations for many space operations, widespread use remains limited – something we will not discuss in this paper. Making matters more difficult, organizations generally use different configurations (satellite state and coordinate systems) both in their formation of the covariance, and in any transmission to other organizations. Unfortunately, the covariance is often given in a format that's not consistent with another organization. To effectively process these data, transformations must be made between the two formats. Finally, we do not address how organizations choose to represent their uncertainty, simply how to transform the results they may have chosen.

The literature contains a lot of information relating to partial derivatives, as well as covariance matrices and their propagation characteristics. Broucke (1970), and Broucke and Cefola (1972) provided equations for some of the transformations. Later, Long (1989), Pon (1973), NORAD (1982), TRACE (1977), Wagner (1987) and Douglas (1987) conducted analyses using the covariance matrix and orbit determination methods to apply the results to gravity field determination, orbital selection criteria, and sea-surface determination. Some useful equations are given by ASTCM (1989), McClain (1992:79-91), and Cefola and Yurasov (1998) relating the partial derivatives for the direct transformation of equinoctial and Cartesian elements. Chobotov (1996) describes some of the earlier results, but uses a different notation. Danielson (1995) summarized the Cartesian to Equinoctial transformations. It turned out that notation was perhaps the single largest difficulty in using the earlier results – either through undefined variables, through type-setting errors, or from the orbital elements chosen.

With modern symbolic manipulation programs (MathCad ver 15), we have additional tools to verify and correct the lengthy mathematical derivations. In particular, MathCad enabled us to see the resulting equations in symbolic form, and to calculate numerical (finite differencing and chain rule) answers for testing purposes. We also created some MatLab code from these operations. Certain limitations exist with

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MathCad, especially in collecting trigonometric terms. It didn't recognize all the substitutions that could be made, but we were able to insert various substitutions to simplify the results. Thus, we've tried to consolidate the information for the specific equations, along with some advice for practical information concerning the implementation of these routines.

This paper details the transformations necessary to convert covariance matrices between Cartesian, Classical, Equinoctial, and Spherical (Flight) orbital state formats. In addition, the process to convert coordinate systems representations is given. Detailed equations are given in the Appendix including assumptions and limitations found during testing for the results section.

Figure 1 shows the various combinations of transformations. There are two types of transformations—those to different coordinate systems, and those to different orbital state formats. From the figure, an omission appears for some of the coordinate systems, such as Earth fixed (ECEF). This is because when the subsequent transformation is made to classical elements, the classical elements no longer represent the true orbit of the satellite due to the velocity vector changes in the transformation from "inertial" to "fixed" coordinates. Thus, a change to these alternate orbit state formats would not yield useful information.

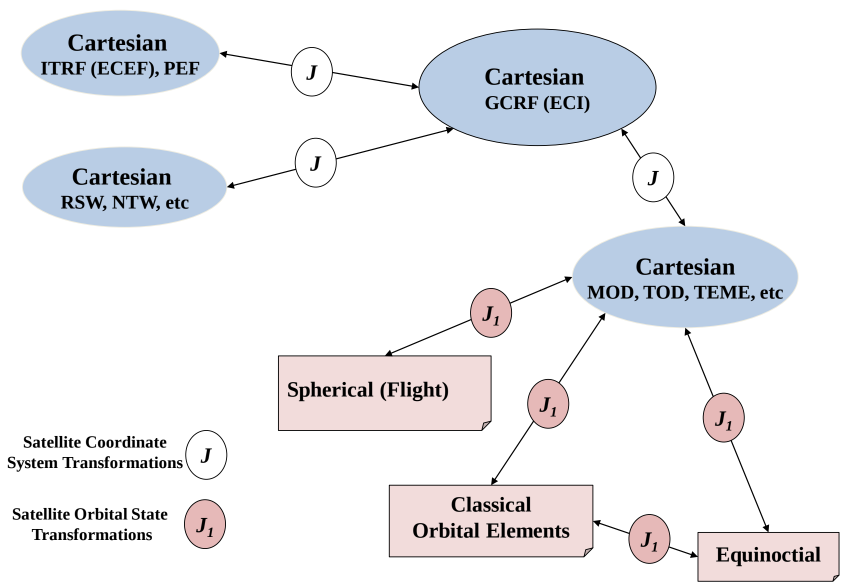

Figure 1: Covariance Transformations. This figure shows the types and data required to accomplish each transformation. The central point is the Cartesian Inertial covariance matrix. To transform to different coordinate systems, a simple transformation matrix (***J***) is required. To change the state format of the covariance matrix, the state is needed in the desired format (Cartesian, Classical, etc.), as well as the transformation matrix ($\boldsymbol{J_1}$).

A final comment. Because the routines are designed to transform a single position/velocity state, or orbital elements, at a single time, the routines need to be very efficient. The partials can be quite large, so we formulate both the state-element conversion and the partials in their native space, using the available elements or states. This also provides several checks to be sure each transformation is correct.

STATE AND COVARIANCE TRANSFORMATIONS

There are several transformations which we will use in this paper. State conversions are useful to find the required orbital elements to calculate other transformation. Some orbital elements are better for calculating the covariance transformations because the partial derivatives are simpler.

The similarity transformation is used for all covariance conversions. Eq. 1 shows a transformation from covariance "x" to covariance "y". The matrix $\boldsymbol{m}$ is a 6×6 matrix for position and velocity covariance matrices and because the equations are rather lengthy, they are included in the Appendix.

$$
\begin{aligned}
&P_y = mP_x m^T \\
&m_{\frac{y}{x}} = \frac{\partial y}{\partial x}
\end{aligned}
\tag{1}
$$

Chobotov (1996:346) shows the relationship of the partials of the elements with respect to the velocity with the Poisson brackets (α_α, α_β) and the partials of the elements with respect to the position vector (shown here with classical orbital elements and position and velocity vectors).

$$
\begin{aligned}
&\frac{\partial a_\alpha}{\partial \bar{v}} = -\sum_{\beta=1}^{6} (a_\alpha, a_\beta) \frac{\partial \bar{r}}{\partial a_\beta} \\
&\alpha \rightarrow a, e, i, \Omega, \omega, M \\
&\beta \rightarrow r_x, r_y, r_z, v_x, v_y, v_z
\end{aligned}
\tag{2}
$$

For terminology, we show a transformation matrix expanded with the partials. For example, if we look at the conversion of classical to equinoctial elements, the transformation matrix is:

$$
[tm]_{\frac{equinoctial}{classical}} = \begin{bmatrix}
\frac{\partial a}{\partial a} & \frac{\partial a}{\partial e} & \frac{\partial a}{\partial i} & \frac{\partial a}{\partial \Omega} & \frac{\partial a}{\partial \omega} & \frac{\partial a}{\partial M} \\
\frac{\partial a_f}{\partial a} & \frac{\partial a_f}{\partial e} & \frac{\partial a_f}{\partial i} & \frac{\partial a_f}{\partial \Omega} & \frac{\partial a_f}{\partial \omega} & \frac{\partial a_f}{\partial M} \\
\frac{\partial a_g}{\partial a} & \frac{\partial a_g}{\partial e} & \frac{\partial a_g}{\partial i} & \frac{\partial a_g}{\partial \Omega} & \frac{\partial a_g}{\partial \omega} & \frac{\partial a_g}{\partial M} \\
\frac{\partial \chi}{\partial a} & \frac{\partial \chi}{\partial e} & \frac{\partial \chi}{\partial i} & \frac{\partial \chi}{\partial \Omega} & \frac{\partial \chi}{\partial \omega} & \frac{\partial \chi}{\partial M} \\
\frac{\partial \psi}{\partial a} & \frac{\partial \psi}{\partial e} & \frac{\partial \psi}{\partial i} & \frac{\partial \psi}{\partial \Omega} & \frac{\partial \psi}{\partial \omega} & \frac{\partial \psi}{\partial M} \\
\frac{\partial \lambda_M}{\partial a} & \frac{\partial \lambda_M}{\partial e} & \frac{\partial \lambda_M}{\partial i} & \frac{\partial \lambda_M}{\partial \Omega} & \frac{\partial \lambda_M}{\partial \omega} & \frac{\partial \lambda_M}{\partial M}
\end{bmatrix}
\tag{3}
$$

Combining the transformation with the covariance matrix expressed in classical elements, and using x generically as the cross correlation terms, the transformed covariance matrix is

$$
\begin{aligned}
&\begin{pmatrix}
\sigma_a^2 & x\sigma_a\sigma_{a_f} & x\sigma_a\sigma_{a_g} & x\sigma_a\sigma_\chi & x\sigma_a\sigma_\psi & x\sigma_a\sigma_{\lambda_M} \\
x\sigma_{a_f}\sigma_a & \sigma_{a_f}^2 & x\sigma_{a_f}\sigma_{a_g} & x\sigma_{a_f}\sigma_\chi & x\sigma_{a_f}\sigma_\psi & x\sigma_{a_f}\sigma_{\lambda_M} \\
x\sigma_{a_g}\sigma_a & x\sigma_{a_g}\sigma_{a_f} & \sigma_{a_g}^2 & x\sigma_{a_g}\sigma_\chi & x\sigma_{a_g}\sigma_\psi & x\sigma_{a_g}\sigma_{\lambda_M} \\
x\sigma_\chi\sigma_a & x\sigma_\chi\sigma_{a_f} & x\sigma_\chi\sigma_i & \sigma_\chi^2 & x\sigma_\chi\sigma_\psi & x\sigma_\chi\sigma_{\lambda_M} \\
x\sigma_\psi\sigma_a & x\sigma_\psi\sigma_{a_f} & x\sigma_\psi\sigma_i & x\sigma_\psi\sigma_\chi & \sigma_\psi^2 & x\sigma_\psi\sigma_{\lambda_M} \\
x\sigma_{\lambda_M}\sigma_a & x\sigma_{\lambda_M}\sigma_{a_f} & x\sigma_{\lambda_M}\sigma_i & x\sigma_{\lambda_M}\sigma_\chi & x\sigma_{\lambda_M}\sigma_\psi & \sigma_{\lambda_M}^2
\end{pmatrix} = [tm]_{\frac{equinoctial}{classical}} P_{classical} [tm]^T_{\frac{equinoctial}{classical}} \\
&P_{classical} = \begin{pmatrix}
\sigma_a^2 & x\sigma_a\sigma_e & x\sigma_a\sigma_i & x\sigma_a\sigma_\Omega & x\sigma_a\sigma_\omega & x\sigma_a\sigma_M \\
x\sigma_e\sigma_a & \sigma_e^2 & x\sigma_e\sigma_i & x\sigma_e\sigma_\Omega & x\sigma_e\sigma_\omega & x\sigma_e\sigma_M \\
x\sigma_i\sigma_a & x\sigma_i\sigma_e & \sigma_i^2 & x\sigma_i\sigma_\Omega & x\sigma_i\sigma_\omega & x\sigma_i\sigma_M \\
x\sigma_\Omega\sigma_a & x\sigma_\Omega\sigma_e & x\sigma_\Omega\sigma_i & \sigma_\Omega^2 & x\sigma_\Omega\sigma_\omega & x\sigma_\Omega\sigma_M \\
x\sigma_\omega\sigma_a & x\sigma_\omega\sigma_e & x\sigma_\omega\sigma_i & x\sigma_\omega\sigma_\Omega & \sigma_\omega^2 & x\sigma_\omega\sigma_M \\
x\sigma_M\sigma_a & x\sigma_M\sigma_e & x\sigma_M\sigma_i & x\sigma_M\sigma_\Omega & x\sigma_M\sigma_\omega & \sigma_M^2
\end{pmatrix}
\end{aligned}
\tag{4}
$$

Note that although we give transformations using different equations for forward and reverse transformations, you can also invert a transformation to achieve the same result. Note that the transpose will not work for orbital state transformations, but will work for coordinate system transformations.

We do not discuss covariance propagation, but note that the state transition matrix (Φ_S) moves the state through time, and the error state transition matrix (Φ) moves the covariance through time (Vallado 2013:800-813).

STATE CONVERSIONS

Although the basic equations for the various orbital state representations are common, it's worth repeating them to be sure notation is precisely known. We provide equations for both directions of conversions for the various state representations. Some formulations proved easier to work with in the partial derivative process, and those are highlighted in the text.

Classical Elements to Cartesian State

For Cartesian transformations, it's useful to relate the classical orbital elements to the position and velocity state vector in the perifocal coordinate system, defined in Vallado (2013:155).

$$\vec{r}_{PQW} = \begin{bmatrix} \dfrac{p\cos(\nu)}{1+e\cos(\nu)} \\ \dfrac{p\sin(\nu)}{1+e\cos(\nu)} \\ 0 \end{bmatrix} \qquad \vec{v}_{PQW} = \begin{bmatrix} -\sqrt{\dfrac{\mu}{p}}\sin(\nu) \\ \sqrt{\dfrac{\mu}{p}}\left(e+\cos(\nu)\right) \\ 0 \end{bmatrix} \tag{5}$$

When using the PQW system, you must also perform a coordinate transformation from inertial (GCRF) to PQW to complete the processing (Eq. 14 later).

Cartesian State to Classical Elements

There are several choices for the fast variable with common choices of true anomaly (ν), mean anomaly (M), Eccentric anomaly (E), or time since perigee passage (Δt). We chose to use the mean anomaly (because of its suspected use in AFSPC operations) and true anomaly. The following relations are useful.

$$\begin{aligned}
&E = \tan^{-1}\left[\frac{\vec{r}\cdot\vec{v}}{\sqrt{\mu a}\,(1-\frac{r}{a})}\right] && a = \frac{\mu r}{-v^2 r + 2\mu} \qquad \vec{e} = \frac{\left(v^2 - \frac{\mu}{r}\right)\vec{r} - \left(\vec{r}\cdot\vec{v}\right)\vec{v}}{\mu} \\
&e\sin(E) = \frac{\vec{r}\cdot\vec{v}}{\sqrt{\mu a}} && \vec{h} = \vec{r}\times\vec{v} \qquad \vec{n} = \vec{K}\times\vec{h} \\
&\nu = 2\tan^{-1}\left[\sqrt{\frac{1-e}{1-e}}\tan(\frac{E}{2})\right] && \cos(i) = \frac{h_K}{|\vec{h}|} \qquad \sin(i) = \frac{r_K}{|\vec{r}|} \\
&\sin(E) = \frac{\sin(\nu)\sqrt{1-e^2}}{1+e\cos(\nu)} && \cos(\Omega) = \frac{n_I}{|\vec{n}|} \qquad \cos(\Omega) = \frac{-h_J}{\sqrt{h_I^2+h_J^2}} \qquad \tan(\Omega) = \frac{h_I}{h_J} \\
&\cos(E) = \frac{e+\cos(\nu)}{1+e\cos(\nu)} && \cos(\omega) = \frac{\vec{n}\cdot\vec{e}}{|\vec{n}||\vec{e}|} \\
& && \cos(\nu) = \frac{\vec{e}\cdot\vec{r}}{|\vec{e}||\vec{r}|} \qquad \sin(\nu) = \sqrt{\frac{p}{\mu}}\frac{\vec{r}\cdot\vec{v}}{re} \\
& && M = E - e\sin(E)
\end{aligned} \tag{6}$$

Classical Elements to Equinoctial Elements

Equinoctial elements (Broucke and Cefola, 1972) are popular because they do not suffer from the singularity problems that classical and other elements do. However, there are a few "standard" parameters. Wright (2009) discusses the various forms over the last 50 years. Two issues stand out. First, the use of the mean motion (n) or the semi-major axis (a) is not always known, or documented. Next is the use of true anomaly (ν) or mean anomaly (M) in the longitude term. AFSPC presumably uses mean motion (n) and mean anomaly (M) (TP-008 1982:69-70). Equinoctial elements have a singularity for exact 180º inclinations. Most sources suggest a multiplier (f_r), but the application is not standard. Some centers use the multiplier only for exact retrograde orbits (switching the singularity for that case to an inclination of 0º), while others use the multiple for all retrograde orbits (f_r = 1 for direct orbits, -1 for retrograde orbits).

We have chosen to go with the original definition for which all but exact retrograde orbits are unique because we believe that AFSPC has used this approach for many decades. The f_r multiplier is included in the equations for completeness.

$$
\begin{aligned}
& & n &= \sqrt{\frac{\mu}{a^3}} \qquad a = \left(\frac{\mu}{n^2}\right)^{\frac{1}{3}} \\
M &= E - e\sin(E) & a_f &= k_e = e\cos(\omega + f_r\Omega) \\
\sin(\nu) &= \frac{\sin(E)\sqrt{1-e^2}}{1-e\cos(E)} & a_g &= h_e = e\sin(\omega + f_r\Omega) \\
\cos(\nu) &= \frac{\cos(E)-e}{1-e\cos(E)} & \chi &= p_e = \left(\tan(\frac{i}{2})\right)^{f_r}\sin(\Omega) \\
& & \psi &= q_e = \left(\tan(\frac{i}{2})\right)^{f_r}\cos(\Omega) \\
& & \lambda_M &= f_r\Omega + \omega + M \\
& & \lambda_\nu &= L = f_r\Omega + \omega + \nu
\end{aligned}
\tag{7}
$$

Equinoctial Elements to Classical Elements

Another set of equations relating the classical orbital elements to the equinoctial elements are needed to form an alternate formulation of the equinoctial to classical elements partials.

$$
\begin{aligned}
a &= \left(\frac{\mu}{n^2}\right)^{\frac{1}{3}} \\
e &= \sqrt{a_f^2 + a_g^2} \\
i &= \cos^{-1}\left(\frac{\left(1-\chi^2-\psi^2\right)f_r}{1+\chi^2+\psi^2}\right) = \pi\frac{1-f_r}{2} + 2f_r\tan^{-1}(\sqrt{\chi^2+\psi^2}) \\
\Omega &= \tan^{-1}(\frac{\chi}{\psi}) \\
\omega &= \zeta - f_r\Omega = \tan^{-1}(\frac{a_g}{a_f}) - f_r\tan^{-1}(\frac{\chi}{\psi}) \\
M &= \lambda_M - f_r\Omega - \omega = \lambda_M - \zeta = \lambda_M - \tan^{-1}(\frac{a_g}{a_f}) \\
\nu &= \lambda_\nu - f_r\Omega - \omega = \lambda_\nu - \zeta = L - \zeta = \lambda_\nu - \tan^{-1}(\frac{a_g}{a_f})
\end{aligned}
\tag{8}
$$

$$
\sin(\zeta) = \frac{a_g}{\sqrt{a_f^2 + a_g^2}} \qquad \cos(\zeta) = \frac{a_f}{\sqrt{a_f^2 + a_g^2}}
$$

Cartesian State to Equinoctial Elements

First find the axis vectors for the equinoctial coordinate system (fgw). From McClain (1992:87-88) and Danielson (1995:9), Long et al. (1989:3-66 to 3-68), and CCT (6-30).

$$\chi = \frac{w_e}{1+f_r w_w}$$

$$\psi = -\frac{w_q}{1+f_r w_w}$$

$$a = \frac{\mu r}{-v^2 r + 2\mu}$$

$$\bar{e} = \frac{\left(v^2 - \frac{\mu}{r}\right)\bar{r} - \left(\bar{r}\cdot\bar{v}\right)\bar{v}}{\mu}$$

$$\hat{w} = \frac{\bar{r}\times\bar{v}}{\left|\bar{r}\times\bar{v}\right|}$$

$$\hat{f} = \frac{1}{1+\chi^2+\psi^2}\begin{bmatrix} 1-\chi^2+\psi^2 \\ 2\chi\psi \\ -2f_r\chi \end{bmatrix} = \begin{bmatrix} 1-\dfrac{w_e^2}{1+w_w} \\ \dfrac{-w_e w_q}{1+w_w} \\ -w_e \end{bmatrix}$$

$$X = r\cos(L) = r\hat{f}$$

$$Y = r\sin(L) = r\hat{g}$$

$$b = \frac{1}{1+\sqrt{1-a_g^2-a_f^2}}$$

$$\hat{g} = \frac{1}{1+\chi^2+\psi^2}\begin{bmatrix} 2f_r\chi\psi \\ \left(1+\chi^2-\psi^2\right)f_r \\ 2\psi \end{bmatrix} = \begin{bmatrix} \dfrac{-w_e w_q}{1+w_w} \\ 1-\dfrac{w_q^2}{1+w_w} \\ -w_w \end{bmatrix} \qquad (9)$$

$$\sin(F) = a_g + \frac{\left(1-a_g^2 b\right)Y - a_g a_f bX}{a\sqrt{1-a_g^2-a_f^2}}$$

$$\cos(F) = a_f + \frac{\left(1-a_f^2 b\right)X - a_g a_f bY}{a\sqrt{1-a_g^2-a_f^2}}$$

$$a_f = \bar{e}\cdot\hat{f}$$

$$a_g = \bar{e}\cdot\hat{g}$$

$$\lambda_M = F + a_g\cos(F) - a_f\sin(F)$$

$$\lambda_\nu = \nu + \omega + \Omega$$

Equinoctial Elements to Cartesian State

Begin by finding the axes representations in the equinoctial frame (fgw) shown in Eq 10. Newton's equation must be solved for F – here shown in equinoctial form. From Danielson (1995:7-8), Long et al. (1989:3-65 to 3-66).

$$F_0 \approx \lambda_M$$

$$F_{i+1} = F_i - \frac{F_i + a_g\cos(F_i) - a_f\sin(F_i) - \lambda_M}{1 - a_g\sin(F_i) - a_f\cos(F_i)} \qquad i = 0,1,2\ldots \qquad (10)$$

Then proceed to find the following quantities.

$$\hat{f} = \frac{1}{1+\chi^2+\psi^2}\begin{bmatrix} 1-\chi^2+\psi^2 \\ 2\chi\psi \\ -2f_r\chi \end{bmatrix}$$

$$\hat{g} = \frac{1}{1+\chi^2+\psi^2}\begin{bmatrix} 2f_r\chi\psi \\ \left(1+\chi^2-\psi^2\right)f_r \\ 2\psi \end{bmatrix}$$

$$\vec{w} = \frac{1}{1+\chi^2+\psi^2}\begin{bmatrix} 2\chi \\ -2\psi \\ \left(1-\chi^2-\psi^2\right)f_r \end{bmatrix}$$

$$B = \sqrt{1-a_g^2-a_f^2}$$

$$b = \frac{1}{1+B}$$

$$\sin(L) = \frac{(1-a_g^2 b)\sin(F)+a_g a_f b\cos(F)-a_f}{1-a_f\sin(F)-a_g\cos(F)} \qquad (11)$$

$$\cos(L) = \frac{(1-a_f^2 b)\cos(F)+a_g a_f b\sin(F)-a_g}{1-a_f\sin(F)-a_g\cos(F)}$$

$$X = r\cos(L) = a\left[(1-a_g^2 b)\cos(F)+a_g a_f b\sin(F)-a_f\right]$$

$$Y = r\sin(L) = a\left[(1-a_f^2 b)\sin(F)+a_g a_f b\cos(F)-a_g\right]$$

$$\dot{X} = -\frac{na(a_g+\sin(L))}{B} = \frac{na^2}{r}\left[a_g a_f b\cos(F)-(1-a_g^2 b)\sin(F)\right]$$

$$\dot{Y} = \frac{na(a_f+\cos(L))}{B} = \frac{na^2}{r}\left[\left(1-a_f^2 b\right)\cos(F)-a_g\sin(F)\right]$$

$$\vec{r} = X\hat{f}+Y\hat{g}$$

$$\vec{v} = \dot{X}\hat{f}+\dot{Y}\hat{g}$$

Cartesian State to Spherical Elements

Spherical elements are useful for some applications. They are sometimes called flight elements as well. There are several variants, the main being the use of right ascension (α) and declination (δ) instead of geocentric latitude (ϕ_{gc}) and longitude (λ) values. Some centers find longitude and declination to be useful. Note that right-ascension and declination use Earth inertial vectors while geographic locations use Earth fixed coordinates. The remaining elements are flight path angle from the local horizontal (ϕ_{fpa}), azimuth, and position and velocity magnitudes (r, v). Fortunately, the partial derivatives assume the same form for each set of parameters. From Long et al. (1989:3-42 to 3-44).

$$\phi_{gc} = \sin^{-1}\left(\frac{r_z}{\sqrt{r_x^2+r_y^2+r_z^2}}\right)$$

$$\lambda = \tan^{-1}\left(\frac{r_y}{r_x}\right)$$

$$\sin(\delta) = \frac{r_z}{r} \qquad \cos(\delta) = \frac{\sqrt{r_x^2+r_y^2}}{r}$$

$$\sin(\alpha) = \frac{r_y}{\sqrt{r_x^2+r_y^2}} \qquad \cos(\alpha) = \frac{r_x}{\sqrt{r_x^2+r_y^2}}$$

$$\phi_{fpa} = 90° - \tan^{-1}\left(\frac{h}{\vec{r}\cdot\vec{v}}\right)$$

$$\vec{A} = \vec{h}\times\vec{r} \qquad (12)$$

$$k_1 = r(r_x v_y - r_y v_x)$$

$$k_2 = r_y(r_y v_z - r_z v_y) - r_x(r_z v_x - r_x v_z)$$

$$\sin(\beta) = \frac{k_1}{\sqrt{k_1^2+k_2^2}}$$

$$\cos(\beta) = \frac{k_2}{\sqrt{k_1^2+k_2^2}}$$

$$\beta = \tan^{-1}\left(\frac{r_I A_J - r_J A_I}{A_K r}\right)$$

$$r = \sqrt{r_x^2+r_y^2+r_z^2}$$

$$v = \sqrt{v_x^2+v_y^2+v_z^2}$$

Spherical Elements to Cartesian State

From Escobal (1985: 397) we find the relations directly. Note that some references (Trace 1970:64) had different starting equations. The vectors here are all inertial (ECI).

$$
\begin{aligned}
r_x &= r\cos(\delta)\cos(\alpha) \\
r_y &= r\cos(\delta)\sin(\alpha) \\
r_z &= r\sin(\delta) \\
v_x &= v\left[-\cos(\alpha)\sin(\delta)\cos(\beta)\cos(\phi_{fpa}) - \sin(\alpha)\sin(\beta)\cos(\phi_{fpa}) + \cos(\alpha)\cos(\delta)\sin(\phi_{fpa})\right] \\
v_y &= v\left[-\sin(\alpha)\sin(\delta)\cos(\beta)\cos(\phi_{fpa}) + \cos(\alpha)\sin(\beta)\cos(\phi_{fpa}) + \sin(\alpha)\cos(\delta)\sin(\phi_{fpa})\right] \\
v_z &= v\left[\sin(\delta)\sin(\phi_{fpa}) + \cos(\delta)\cos(\beta)\cos(\phi_{fpa})\right]
\end{aligned}
\tag{13}
$$

COVARIANCE TRANSFORMATIONS

There are two broad categories of transforming covariance matrices: between coordinate systems, and between orbital element types. The similarity transformation from Eq. 1 is used in both cases.

COORDINATE SYSTEM TRANSFORMATIONS

For converting covariance matrices between coordinate systems the transformation itself is all that's required. Because the coordinate systems are described in Vallado (2013:Ch. 3), we present only the transformations themselves to identify direction and nomenclature.

For example, the transformation from the Geocentric Equatorial Coordinate System (GCRF, or ECI) to the Perifocal system (PQW) (Vallado, 2013:173), we find

$$
\begin{aligned}
\vec{r}_{GCRF} &= \left[rot3(-\Omega)\right]\left[rot1(-i)\right]\left[rot3(-\omega)\right]\vec{r}_{PQW} = [\tfrac{GCRF}{PQW}]\vec{r}_{PQW} \\
\vec{r}_{PQW} &= \left[rot3(\omega)\right]\left[rot1(i)\right]\left[rot3(\Omega)\right]\vec{r}_{GCRF} = [\tfrac{GCRF}{PQW}]^T\vec{r}_{GCRF} \\
\left[\frac{GCRF}{PQW}\right] &= \begin{bmatrix} \cos(\Omega)\cos(\omega) - \sin(\Omega)\sin(\omega)\cos(i) & -\cos(\Omega)\sin(\omega) - \sin(\Omega)\cos(\omega)\cos(i) & \sin(\Omega)\sin(i) \\ \sin(\Omega)\cos(\omega) + \cos(\Omega)\sin(\omega)\cos(i) & -\sin(\Omega)\sin(\omega) + \cos(\Omega)\cos(\omega)\cos(i) & -\cos(\Omega)\sin(i) \\ \sin(\omega)\sin(i) & \cos(\omega)\sin(i) & \cos(i) \end{bmatrix}
\end{aligned}
\tag{14}
$$

The transformation is actually a 6×6 transformation when applied to position and velocity vectors.

$$
\vec{X}_{GCRF} = \begin{pmatrix} [\frac{GCRF}{PQW}]_{3x3} & [0]_{3x3} \\ [0]_{3x3} & [\frac{GCRF}{PQW}]_{3x3} \end{pmatrix} \vec{X}_{PQW}
\tag{15}
$$

Another example would be coordinate systems that are fixed within the satellite orbital plane (RSW aligned to radial direction, NTW aligned to velocity direction). Here the 3×3 transformation component is shown formed using vectors instead of angle rotations.

$$
\begin{gathered}
\hat{R} = \frac{\vec{r}}{|\vec{r}|} \qquad \hat{W} = \frac{\vec{r}\times\vec{v}}{|\vec{r}\times\vec{v}|} \qquad \hat{S} = \hat{W}\times\hat{R} \qquad \hat{T} = \frac{\vec{v}}{|\vec{v}|} \qquad \hat{W} = \frac{\vec{r}\times\vec{v}}{|\vec{r}\times\vec{v}|} \qquad \hat{N} = \hat{T}\times\hat{W} \\
\vec{r}_{IJK} = \left[\hat{R}\right]\left[\hat{S}\right]\left[\hat{W}\right]\vec{r}_{RSW} \qquad\qquad \vec{r}_{IJK} = \left[\hat{N}\right]\left[\hat{T}\right]\left[\hat{W}\right]\vec{r}_{NTW}
\end{gathered}
\tag{16}
$$

ORBITAL ELEMENT TYPE TRANSFORMATIONS

When converting between orbital element types, partial derivatives are required. The length of these expressions required us to present them in the appendix. As discussed earlier, there are several parameters that may be considered for the variables. The classical and equinoctial elements have the following options.

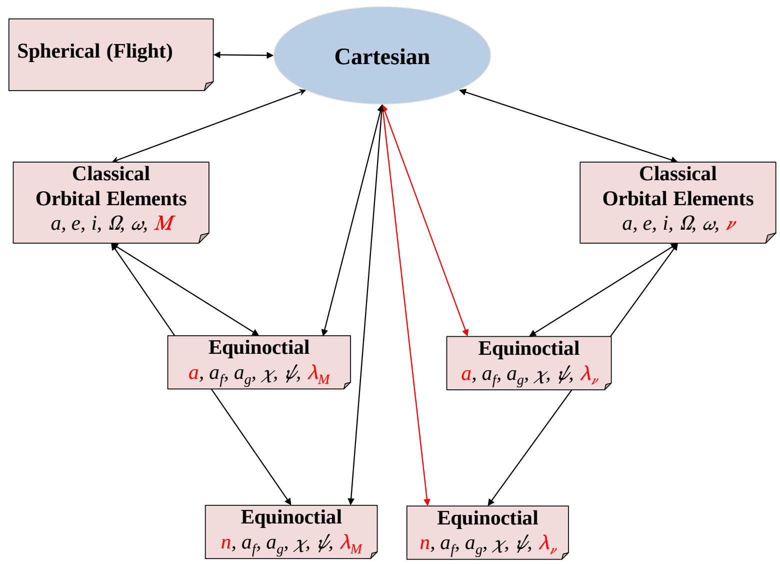

Figure 2: Parameter Options. This figure shows the various options when including different orbital element types. The red lines represent the options that remain unaddressed in this paper. We address the options in black. Note that we do not "mix" mean and true anomalies in the classical and equinoctial formulations – thus, if the classical element uses mean anomaly, the equinoctial uses mean longitude.

The transformation matrices are not simple transposes of the reverse operation, but rather are inverses of the reverse operation. The partial derivatives do not mix the classical and equinoctial element options. If the mean anomaly is used, the mean longitude is used. We did this because it kept the partial derivatives cleaner. If you were to mix, say having true anomaly and mean longitude, you need the following relations.

$$\frac{\partial M}{\partial e} = -\frac{\sin(\nu)\left[\dfrac{(1+e\cos(\nu))(e+\cos(\nu))}{\sqrt{(e+\cos(\nu))^2}} + 1 - 2e^2 - e^3\cos(\nu)\right]}{(1+e\cos(\nu))^2\sqrt{1-e^2}} \qquad \frac{\partial M}{\partial \nu} = \frac{(1-e^2)^{3/2}}{1+e\cos(\nu)} \tag{17}$$

USEFUL UTILITIES

We ran across a useful associated item relating to covariance and we describe it here.

In launch and early orbit stages of flight, significant events in powered flight segments (notably, stage ignition, cutoff and burnout) have uncertainties in time, in addition to our more "normal" uncertainties in position and velocity. As a result, launch simulations (typically Monte Carlo, sampling across launch performance, guidance, winds aloft and environment) often produce 7×7 covariance matrices linked to such key events (as opposed to covariances linked to specified times).

For further analysis and use, this time uncertainty can be eliminated by effectively mapping these 7×7 matrices into their equivalent 6×6 position and velocity. Alan Jenkin derived a transformation of this in the mid-1990s and we present the summary here for formal documentation with his permission. The 7×7 covariance is P_A, shown below.

$$P_A = \begin{bmatrix} P_{x_0} & P_{x_0 t_0} \\ P_{t_0 x_0} & P_{t_0} \end{bmatrix} \tag{18}$$

The upper left hand term is the 6×6 covariance, but it is not the 6×6 covariance we seek since it will not include the timing uncertainties. Upon conversion, the resulting 6×6 covariance will incorporate the timing uncertainties present in the 7×7. The following equation performs the conversion to a 6×6 ($P_x(t_o)$).

$$P_x(t_0) = ff^T\Big|_{x_0} P_{t_0} - f(\vec{x}_0)P_{t_0 x_0} - P_{x_0 t_0} f^T(\vec{x}_0) + P_{\vec{x}_0} \tag{19}$$

For implementation, we show the terms for lower triangular representations of P_A and $P_x(t_o)$, with v as the velocity vector and a as the acceleration vector. Dan Oltrogge provided the algorithm in this section.

$$\begin{array}{ll}
P_1 = v_x(v_x P_{A1} - 2P_{A2}) + P_{A3} & P_{12} = v_z(v_z P_{A1} - 2P_{A7}) + P_{A10} \\
P_2 = v_x(v_y P_{A1} - P_2) - v_y P_{A2} + P_{A5} & P_{13} = v_z(a_x P_{A1} - P_4) - a_x P_{A7} + P_{A14} \\
P_3 = v_x(v_z P_{A1} - P_3) - v_z P_{A2} + P_{A8} & P_{14} = v_z(a_y P_{A1} - P_5) - a_y P_{A7} + P_{A19} \\
P_4 = v_x(a_x P_{A1} - P_4) - a_x P_{A2} + P_{A12} & P_{15} = v_z(a_z P_{A1} - P_6) - a_z P_{A7} + P_{A25} \\
P_5 = v_x(a_y P_{A1} - P_5) - a_y P_{A2} + P_{A17} & \\
P_6 = v_x(a_z P_{A1} - P_6) - a_z P_{A2} + P_{A23} & P_{16} = a_x(a_x P_{A1} - 2P_{A11}) + P_{A15} \\
 & P_{17} = a_x(a_y P_{A1} - P_5) - a_y P_{A11} + P_{A20} \\
P_7 = v_y(v_y P_{A1} - 2P_{A4}) + P_{A6} & P_{18} = a_x(a_z P_{A1} - P_6) - a_z P_{A11} + P_{A26} \\
P_8 = v_y(v_z P_{A1} - P_3) - v_z P_{A4} + P_{A9} & \\
P_9 = v_y(a_x P_{A1} - P_4) - a_x P_{A4} + P_{A13} & P_{19} = a_y(a_y P_{A1} - 2P_{A16}) + P_{A21} \\
P_{10} = v_y(a_y P_{A1} - P_5) - a_y P_{A4} + P_{A18} & P_{20} = a_y(a_z P_{A1} - P_{A22}) - a_z P_{A16} + P_{A27} \\
P_{11} = v_y(a_z P_{A1} - P_6) - a_z P_{A4} + P_{A24} & P_{21} = a_z(a_z P_{A1} - 2P_{A22}) + P_{A28}
\end{array} \tag{20}$$

We can then propagate the resulting 6×6 covariance ($P_x(t_o)$) using 6 × 6 covariance matrix integration or 6×6 state transition matrix methods. There are differences between propagating the state (state transition matrix, Φ_S) and the covariance (error state transition matrix, Φ) as detailed in Vallado (2013:809-813). For single instances of time, either approach will work, but finding the full variational equations to propagate the state error matrix may be complex, especially if thrusting is present. In this case, the state propagation may be simpler to implement.

RESULTS

There are a variety of input and outputs used by organizations. Because the covariance matrix is [usually] symmetric, a common shorthand is to only provide the upper or lower triangular covariance matrix elements. It's quite important to understand which is in use for a particular problem as the elements of each do not line up, and the most common resulting error will be a "singular element on the diagonal" message. The number of each term is listed in Eq. 21 (left is Lower triangular and right is Upper triangular).

$$\begin{bmatrix} 1 & & & & & \\ 2 & 3 & & & & \\ 4 & 5 & 6 & & & \\ 7 & 8 & 9 & 10 & & \\ 11 & 12 & 13 & 14 & 15 & \\ 16 & 17 & 18 & 19 & 20 & 21 \end{bmatrix} or \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ & 7 & 8 & 9 & 10 & 11 \\ & & 12 & 13 & 14 & 15 \\ & & & 16 & 17 & 18 \\ & & & & 19 & 20 \\ & & & & & 21 \end{bmatrix} \tag{21}$$

The order of the states is also very important. This paper lists each orbital element in a particular order, and the partial derivatives are in that order too. Changing the order of the state implies a commensurate change in the order of the covariance partial derivatives, as well as any orbital states that are required for a transformation.

The following tests were run to ensure we had the process correct.

1. Equation consistency and accuracy. Use a default covariance matrix and convert to and from an orbital state format. The answers should be identical.
    a. Cartesian to Classical and Classical to Cartesian
    b. Classical to Equinoctial and Equinoctial to Classical
    c. Cartesian to Equinoctial and Equinoctial to Cartesian
    d. Cartesian to Spherical and Spherical to Cartesian
2. Examine each transformation matrix to ensure it is not singular. Take the transformation and multiply it by its inverse to obtain the identity matrix.
3. Examine the matrix inverse of the forward and reverse transformations and ensure both formulations result in the same conversion matrix.
4. Test the combinations of conversions to ensure the proper state is arrived at.
    a. These tests combined transformations and compared to a direct transformation. We combined the Cartesian to Classical with the Classical to Equinoctial and compared to Cartesian to the Equinoctial.
5. Test special cases of i = 0 º, 90 º, and e = 0.0. Each of these cases have problems with the classical formulation, but equinoctial works for each.

The following state vector and timing data (Earth Orientation Parameters, EOP, ΔUT1, ΔAT, x_p, y_p) were used. From this information, the following orbital state formats were obtained using standard relations.

```
year  2000 mon   12 day  15 hr  16:58:50.208000
dut1 0.105970 s dat  32 s xp 0.000000 " yp 0.000000 " lod 0.000000 s
reci   -605.7922166 -5870.2295111  3493.0531990 veci  -1.568254290  -3.702348910  -6.479483950
recef  1230.4987461  4839.1009687 -4700.7418589 vecef -4.346646037   4.671315446   3.660258994
          p km       a km      ecc      incl deg     raan deg     argp deg      nu deg      m deg
coes   6860.7554   6860.7631   0.0010640    97.65184    79.54701   83.86041    65.21303    65.10238
           a           af           ag           chi        psi           meanlonM     meanLonNu   fr
eq     6860.7631490      0.0010610      0.0000800       0.8601197      0.1586839     69.4157838     69.5264380 -1
       lon deg       latgc deg     rtasc deg      decl deg      fpa deg        az deg       magr km      magv km/s
flt        75.7330540     -43.2725437     -95.8919168      30.6213739       0.0553210    -171.0988678   6857.6963605
7.6256489
```

From the original Cartesian vector, and using the results of Fig. 2, seven matrices are required to cover all the options.

```
0.  Cartesian Covariance
       x   m           y m            z   m          xdot   m/s       ydot   m/s       zdot   m/s
    1.000000e+02    1.000000e-02    1.000000e-02    1.000000e-04    1.000000e-04    1.000000e-04
    1.000000e-02    1.000000e+02    1.000000e-02    1.000000e-04    1.000000e-04    1.000000e-04
    1.000000e-02    1.000000e-02    1.000000e+02    1.000000e-04    1.000000e-04    1.000000e-04
    1.000000e-04    1.000000e-04    1.000000e-04    1.000000e-04    1.000000e-06    1.000000e-06
    1.000000e-04    1.000000e-04    1.000000e-04    1.000000e-06    1.000000e-04    1.000000e-06
    1.000000e-04    1.000000e-04    1.000000e-04    1.000000e-06    1.000000e-06    1.000000e-04
1.  Classical Covariance from Cartesian #1 above
         a   m           ecc           incl rad      raan rad         argp rad         m rad
    7.299847e+02    3.262250e-05    1.988270e-07   -1.526735e-07    6.571931e-02   -6.571976e-02
    3.262250e-05    4.791316e-12    1.698441e-14   -1.304184e-14    1.909371e-09   -1.914428e-09
    1.988270e-07    1.698441e-14    1.821030e-12    1.857460e-13    2.217579e-11   -2.216053e-11
   -1.526735e-07   -1.304184e-14    1.857460e-13    2.051627e-12   -1.673598e-11    1.701644e-11
    6.571931e-02    1.909371e-09    2.217579e-11   -1.673598e-11    7.206135e-06   -7.203997e-06
   -6.571976e-02   -1.914428e-09   -2.216053e-11    1.701644e-11   -7.203997e-06    7.201868e-06

         a   m           ecc           incl rad      raan rad         argp rad         nu rad
```

7.299847e+02	3.262250e-05	1.988270e-07	-1.526735e-07	6.571931e-02	-6.571927e-02
3.262250e-05	4.791316e-12	1.698441e-14	-1.304184e-14	1.909371e-09	-1.907438e-09
1.988270e-07	1.698441e-14	1.821030e-12	1.857460e-13	2.217579e-11	-2.214950e-11
-1.526735e-07	-1.304184e-14	1.857460e-13	2.051627e-12	-1.673598e-11	1.700796e-11
6.571931e-02	1.909371e-09	2.217579e-11	-1.673598e-11	7.206135e-06	-7.206970e-06
-6.571927e-02	-1.907438e-09	-2.214950e-11	1.700796e-11	-7.206970e-06	7.207807e-06

2. Equinoctial Covariance from Classical #2 above

a m	af	ag	chi	psi	meanlonM rad
7.299847e+02	2.727093e-05	7.218012e-05	-1.967784e-07	9.948340e-08	-2.946262e-07
2.727093e-05	4.505630e-12	1.756086e-12	-1.499603e-14	7.462691e-15	-5.202264e-12
7.218012e-05	1.756086e-12	8.443544e-12	-2.467493e-14	1.404855e-14	1.909008e-12
-1.967784e-07	-1.499603e-14	-2.467493e-14	1.372028e-12	1.069441e-13	-1.331001e-13
9.948340e-08	7.462691e-15	1.404855e-14	1.069441e-13	1.615648e-12	1.550717e-12
-2.946262e-07	-5.202264e-12	1.909008e-12	-1.331001e-13	1.550717e-12	1.048619e-11

a m	af	ag	chi	psi	meanlonNu rad
7.299847e+02	2.727093e-05	7.218012e-05	-1.967784e-07	9.948340e-08	1.966375e-07
2.727093e-05	4.505630e-12	1.756086e-12	-1.499603e-14	7.462691e-15	2.006030e-12
7.218012e-05	1.756086e-12	8.443544e-12	-2.467493e-14	1.404855e-14	-7.194654e-13
-1.967784e-07	-1.499603e-14	-2.467493e-14	1.372028e-12	1.069441e-13	-1.440202e-13
9.948340e-08	7.462691e-15	1.404855e-14	1.069441e-13	1.615648e-12	1.556237e-12
1.966375e-07	2.006030e-12	-7.194654e-13	-1.440202e-13	1.556237e-12	3.670403e-12

n rad	af	ag	chi	psi	meanlonM rad
4.306971e-17	-6.624132e-15	-1.753261e-14	4.779764e-17	-2.416461e-17	7.156497e-17
-6.624132e-15	4.505630e-12	1.756086e-12	-1.499603e-14	7.462691e-15	-5.202264e-12
-1.753261e-14	1.756086e-12	8.443544e-12	-2.467493e-14	1.404855e-14	1.909008e-12
4.779764e-17	-1.499603e-14	-2.467493e-14	1.372028e-12	1.069441e-13	-1.331001e-13
-2.416461e-17	7.462691e-15	1.404855e-14	1.069441e-13	1.615648e-12	1.550717e-12
7.156497e-17	-5.202264e-12	1.909008e-12	-1.331001e-13	1.550717e-12	1.048619e-11

n rad	af	ag	chi	psi	meanlonNu rad
4.306971e-17	-6.624132e-15	-1.753261e-14	4.779764e-17	-2.416461e-17	-4.776343e-17
-6.624132e-15	4.505630e-12	1.756086e-12	-1.499603e-14	7.462691e-15	2.006030e-12
-1.753261e-14	1.756086e-12	8.443544e-12	-2.467493e-14	1.404855e-14	-7.194654e-13
4.779764e-17	-1.499603e-14	-2.467493e-14	1.372028e-12	1.069441e-13	-1.440202e-13
-2.416461e-17	7.462691e-15	1.404855e-14	1.069441e-13	1.615648e-12	1.556237e-12
-4.776343e-17	2.006030e-12	-7.194654e-13	-1.440202e-13	1.556237e-12	3.670403e-12

3. Flight Covariance from Cartesian #1 above

rtasc rad	decl rad	fpa rad	az rad	r m	v m/s
2.871312e-12	3.128972e-16	-3.834926e-13	1.463584e-12	-6.575239e-10	-2.329193e-11
3.128972e-16	2.126609e-12	-2.102216e-12	-1.954182e-15	-9.003905e-10	-3.189516e-11
-3.834926e-13	-2.102216e-12	3.834949e-12	-1.884557e-13	3.449826e-09	1.222056e-10
1.463584e-12	-1.954182e-15	-1.884557e-13	2.456036e-12	3.439029e-09	1.218231e-10
-6.575239e-10	-9.003905e-10	3.449826e-09	3.439029e-09	9.999189e+01	6.702467e-05
-2.329193e-11	-3.189516e-11	1.222056e-10	1.218231e-10	6.702467e-05	1.013743e-04

lon rad	latgc rad	fpa rad	az rad	r m	v m/s
4.010875e-12	-3.295511e-16	2.341301e-13	-1.711666e-12	-2.509405e-06	2.230299e-11
-3.295511e-16	2.126699e-12	-2.076406e-12	1.237412e-13	-3.102657e-06	-3.508231e-11
2.341301e-13	-2.076406e-12	3.834949e-12	-1.884557e-13	3.449826e-09	1.222056e-10
-1.711666e-12	1.237412e-13	-1.884557e-13	2.456036e-12	3.439029e-09	1.218231e-10
-2.509405e-06	-3.102657e-06	3.449826e-09	3.439029e-09	9.999189e+01	6.702467e-05
2.230299e-11	-3.508231e-11	1.222056e-10	1.218231e-10	6.702467e-05	1.013743e-04

2. Check that the inverses of each approach are consistent. This section lists the actual transformation matrices and then the inverse(tm) * tm to see if the identity matrix is returned. There is some noise in each

transformation, but it is very small, generally less than 1×10^{-13}. Examine the Cartesian to Classical transformation.

```
-------- tm ct2cl ---------
   -1.768332e-01   -1.713544e+00    1.019636e+00   -3.703851e+02   -8.744086e+02   -1.530303e+03
   -3.261695e-08   -1.166025e-07   -8.128501e-08   -3.315927e-05   -1.553315e-04   -3.295410e-05
    7.316107e-08   -1.349752e-08   -9.995027e-09   -1.096404e-04    2.022761e-05    1.497871e-05
    1.229425e-07   -2.268173e-08   -1.679601e-08    6.627761e-05   -1.222759e-05   -9.054632e-06
    8.613283e-07   -7.860480e-05    1.123284e-04   -4.143953e-02   -6.424291e-02   -2.166397e-01
   -8.749347e-07    7.853110e-05   -1.124546e-04    4.144835e-02    6.424128e-02    2.166385e-01

    1.000000e+00    1.421085e-14    0.000000e+00    0.000000e+00    0.000000e+00    0.000000e+00
    0.000000e+00    1.000000e+00   -5.684342e-14    0.000000e+00    0.000000e+00    1.164153e-10
    8.881784e-16    0.000000e+00    1.000000e+00   -5.820766e-11   -5.820766e-11    0.000000e+00
   -1.084202e-19    0.000000e+00    0.000000e+00    1.000000e+00    0.000000e+00    0.000000e+00
   -8.673617e-19   -1.110223e-16    0.000000e+00    0.000000e+00    1.000000e+00   -2.273737e-13
    8.673617e-19    5.551115e-17    0.000000e+00   -5.684342e-14    0.000000e+00    1.000000e+00
```

3. Next, check the pair of transformations. Because we programmed each direction in different variables, the operations should close. That is, ct2eq should match the inverse of eq2ct.

Comparing to the original covariance, the following percentage differences were noted. In this analysis, individual differences, or values below 1×10^{-18} were considered to be negligible. This limit was determined by examining the performance of the Cartesian to Equinoctial covariance transformation. The goal was to determine how 1 cm and 1 mm/s errors would translate to an equinoctial (or other) covariance. When these initial variances were input in the Cartesian covariance, terms on the order of 1×10^{-16} to 1×10^{-21} were obtained in the equinoctial result, and hence the tolerance selection of 1×10^{-18}. This worked well with Matlab's double precision.

4. We also checked the combination of transformations. Thus, we could convert from Cartesian to classical to equinoctial, or directly from Cartesian to equinoctial, and reverse. This proved the toughest check to get right but after many trials, it worked. An example comparison follows.

```
-------- tm combined ct2cl, cl2eq ---------
   -1.768332e-01   -1.713544e+00    1.019636e+00   -3.703851e+02   -8.744086e+02   -1.530303e+03
   -3.258365e-08   -1.099837e-07   -9.004515e-08   -2.974387e-05   -1.497515e-04   -1.552500e-05
   -1.669780e-09   -9.214334e-08    1.130815e-07   -4.653048e-05   -7.982976e-05   -2.323176e-04
   -4.398365e-08    8.114567e-09    6.008903e-09    1.056683e-04   -1.949480e-05   -1.443606e-05
   -1.174591e-07    2.167009e-08    1.604688e-08   -3.945220e-05    7.278557e-06    5.389831e-06
   -7.658289e-08    9.041530e-08    1.385271e-07   -3.430747e-05    2.350701e-04   -1.258655e-04

-------- tm ct2eq ---------
   -1.768332e-01   -1.713544e+00    1.019636e+00   -3.703851e+02   -8.744086e+02   -1.530303e+03
   -3.258365e-08   -1.099837e-07   -9.004515e-08   -2.974387e-05   -1.497515e-04   -1.552500e-05
   -1.669780e-09   -9.214334e-08    1.130815e-07   -4.653048e-05   -7.982976e-05   -2.323176e-04
   -4.398365e-08    8.114567e-09    6.008903e-09    1.056683e-04   -1.949480e-05   -1.443606e-05
   -1.174591e-07    2.167009e-08    1.604688e-08   -3.945220e-05    7.278557e-06    5.389831e-06
   -7.658289e-08    9.041530e-08    1.385271e-07   -3.430747e-05    2.350701e-04   -1.258655e-04

-------- tm combined eq2cl, cl2ct ---------
   -8.829808e-02   -2.430153e+06    1.556428e+06   -2.902619e+06   -5.721404e+06   -1.411582e+06
   -8.556234e-01   -4.175372e+06    7.832891e+06   -1.089294e+05    4.548598e+06   -3.332477e+06
    5.091348e-01   -1.214958e+07    8.149317e+05   -6.864554e+05    6.651869e+06   -5.832170e+06
    1.142915e-04    8.143420e+01   -1.704747e+03    8.007979e+03   -3.608485e+03    6.739326e+02
    2.698205e-04    4.819935e+03   -5.754917e+03   -2.819928e+02   -5.813753e+03    6.530521e+03
    4.722131e-04   -5.912878e+03   -4.706736e+03   -1.777072e+03    4.195328e+03   -3.885957e+03

-------- tm ct2eq ---------
   -8.829808e-02   -2.430153e+06    1.556428e+06   -2.902619e+06   -5.721404e+06   -1.411582e+06
   -8.556234e-01   -4.175372e+06    7.832891e+06   -1.089294e+05    4.548598e+06   -3.332477e+06
    5.091348e-01   -1.214958e+07    8.149317e+05   -6.864554e+05    6.651869e+06   -5.832170e+06
```

1.142915e-04	8.143420e+01	-1.704747e+03	8.007979e+03	-3.608485e+03	6.739326e+02
2.698205e-04	4.819935e+03	-5.754917e+03	-2.819928e+02	-5.813753e+03	6.530521e+03
4.722131e-04	-5.912878e+03	-4.706736e+03	-1.777072e+03	4.195328e+03	-3.885957e+03

This test proved difficult to match and we ended up having to re-derive partials for the fast variable element. Some texts mention an assumption about the eccentric longitude being dependent on just 3 variables. We confirmed this assumption, but found that if you had "mixed" elements, such as Cartesian to Classical using true anomaly, but wanted to convert from that Classical to Equinoctial with mean longitude, there were additional cross dependencies. We did not include these in the formulations as it made for too many additional options. We did not complete transformations with true anomaly for the Cartesian and equinoctial cases as we had the relations with classical elements (see Fig. 2) and we felt the preponderance of existing data would use mean anomaly if indeed AFSPC had chosen this approach.

Finally, we tested limits in the orbital elements for cases of i = 0.0º, 90.0º, and 180.0º, and e = 0.0. As expected, the classical formulation did not work for these cases, but the equinoctial form did. For eccentricity, the results are good to about e = 0.00001º. For inclination, the limit is about i = 0.00001º.

CONCLUSIONS

The use of covariance continues to increase in space surveillance. With accurate transformations, one can easily convert between coordinate systems and orbital state formats and choose which ever is best suited for their needs. This paper has presented an essentially mechanical approach for performing covariance transformations. Classical, Cartesian, Equinoctial, and Spherical orbital state formats were considered, and a variety of inertial and rotating coordinate systems were presented. Sample data was given to allow reconstruction of the results. Accuracies were calculated for a variety of conditions to ensure that the transformations were correct. Full partial derivative equations are presented in the appendix to let the reader enjoy the lengthy calculations!

The equations (and code) should provide a solid baseline to work from. Complete treatment of mean motion and semimajor axis, true and mean anomalies, and the retrograde factor were important omissions in many of the past literature articles.

FUTURE WORK

During work on the first version of this paper, the best ideas seem to come a few days before the deadline for the paper. We accomplished several of the previous future work items, but a couple remain.

1. Obtain additional actual data for a wide variety of satellite orbits to better establish the envelope of applicability for the transformations.
2. Investigate transformations when mean elements are used for covariance formation (e.g. DSST, SGP4, etc) and how transformations could be made to Cartesian and other forms identified in this paper.
3. Investigate how the covariance is formed and propagated in mean elements. While often cited, proof of superior behavior of the mean element propagation would be a useful result.
4. Understand if there are any covariance issues that come up when estimating the mean elements and dynamic parameters from osculating data.

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Appendix

Equations are given for most of the transformations. In a few cases, the length of equations prevented accurate typesetting and spacing for a reasonable presentation. The computer code is structured the same way to facilitate comparisons. A source is given for each case, but beware that each source had some notational or parameter problem that is corrected in these relations. The notation conforms to that presented earlier in the paper.

Cartesian to Classical:

From Alfano 2015, TRACE A4-A5, Broucke (1970) (uses *a*, *M*, notation not defined):

$$
\begin{array}{c}
 \\
\frac{\partial a}{\partial} \\
\frac{\partial i}{\partial} \\
\frac{\partial \Omega}{\partial}
\end{array}
\begin{array}{c}
\begin{array}{cccccc} \quad r_x \quad & \quad r_y \quad & \quad r_z \quad & \quad v_x \quad & \quad v_y \quad & \quad v_z \quad \end{array} \\
\begin{bmatrix}
\frac{2r_x a^2}{r^3} & \frac{2r_y a^2}{r^3} & \frac{2r_z a^2}{r^3} & -\frac{v_x}{2a} & -\frac{v_y}{2a} & -\frac{v_z}{2a} \\
\frac{-1}{|\vec{n}|}\left(v_y - \frac{h_z(v_y h_z - v_z h_y)}{h^2}\right) & \frac{1}{|\vec{n}|}\left(v_x - \frac{h_z(v_x h_z - v_z h_x)}{h^2}\right) & -\frac{h_z(v_y h_x - v_x h_y)}{|\vec{n}|h^2} & \frac{1}{|\vec{n}|}\left(r_y - \frac{h_z(r_y h_z - r_z h_y)}{h^2}\right) & \frac{-1}{|\vec{n}|}\left(r_x - \frac{h_z(r_x h_z - r_z h_x)}{h^2}\right) & \frac{v_z n_x}{|\vec{n}|h^2} \\
\frac{-v_z n_y}{|\vec{n}|^2} & \frac{v_z n_x}{|\vec{n}|^2} & \frac{v_x n_y - v_y n_x}{|\vec{n}|^2} & \frac{r_z n_y}{|\vec{n}|^2} & \frac{-r_z n_x}{|\vec{n}|^2} & \frac{r_y n_x - r_x n_y}{|\vec{n}|^2}
\end{bmatrix}
\end{array}
$$

$$
\frac{\partial e^T}{\partial} = \begin{bmatrix}
-\frac{1}{\mu e}\left[\left(v_x v_y - \frac{\mu r_x r_y}{r^3}\right)e_y + \left(v_x v_z - \frac{\mu r_x r_z}{r^3}\right)e_z - \left(v_y^2 + v_z^2 - \frac{\mu}{r} + \frac{\mu r_x^2}{r^3}\right)e_x\right] \\
-\frac{1}{\mu e}\left[\left(v_x v_y - \frac{\mu r_x r_y}{r^3}\right)e_x + \left(v_y v_z - \frac{\mu r_y r_z}{r^3}\right)e_z - \left(v_x^2 + v_z^2 - \frac{\mu}{r} + \frac{\mu r_y^2}{r^3}\right)e_y\right] \\
-\frac{1}{\mu e}\left[\left(v_x v_z - \frac{\mu r_x r_y}{r^3}\right)e_x + \left(v_y v_z - \frac{\mu r_y r_z}{r^3}\right)e_y - \left(v_y^2 + v_x^2 - \frac{\mu}{r} + \frac{\mu r_z^2}{r^3}\right)e_z\right] \\
-\frac{1}{\mu e}\left[\left(r_x v_y - 2r_y v_x\right)e_y + \left(r_y v_y - r_z v_z\right)e_x + \left(r_x v_z - 2r_z v_x\right)e_z\right] \\
-\frac{1}{\mu e}\left[\left(r_y v_x - 2r_x v_y\right)e_x + \left(r_x v_x - r_z v_z\right)e_y + \left(r_y v_z - 2r_z v_y\right)e_z\right] \\
-\frac{1}{\mu e}\left[\left(r_x v_x + r_y v_y\right)e_z + \left(r_z v_x - 2r_x v_z\right)e_x + \left(r_z v_y - 2r_y v_z\right)e_y\right]
\end{bmatrix}
$$

$$
\frac{\partial \omega^{T}}{\partial} = \begin{bmatrix}
-\frac{sign\left(\left(v^2-\frac{\mu}{r}\right)r_z-\bar{r}\cdot\bar{v}v_z\right)}{\left(1-\cos^2(\omega)\right)\mu|\bar{n}|e}\left[-h_y\left(v_y^2+v_z^2-\frac{\mu}{r}+\frac{\mu r_x^2}{r^3}\right)-h_x\left(v_xv_y-\frac{\mu r_xr_y}{r^3}\right)+v_z\mu e_x+\frac{\bar{n}\cdot\bar{e}}{e}\left(\frac{v_zh_y}{|\bar{n}|^3}-\frac{\frac{\partial e}{\partial r_x}}{|\bar{n}|e}\right)\right] \\
-\frac{sign\left(\left(v^2-\frac{\mu}{r}\right)r_z-\bar{r}\cdot\bar{v}v_z\right)}{\left(1-\cos^2(\omega)\right)\mu|\bar{n}|e}\left[h_x\left(v_x^2+v_z^2-\frac{\mu}{r}+\frac{\mu r_y^2}{r^3}\right)+h_y\left(v_xv_y-\frac{\mu r_xr_y}{r^3}\right)+v_z\mu e_y+\frac{\bar{n}\cdot\bar{e}}{e}\left(\frac{v_zh_x}{|\bar{n}|^3}-\frac{\frac{\partial e}{\partial r_y}}{|\bar{n}|e}\right)\right] \\
\frac{sign\left(\left(v^2-\frac{\mu}{r}\right)r_z-\bar{r}\cdot\bar{v}v_z\right)}{\left(1-\cos^2(\omega)\right)\mu|\bar{n}|e}\left[-h_y\left(v_xv_z-\frac{\mu r_xr_z}{r^3}\right)+h_x\left(v_zv_y-\frac{\mu r_yr_z}{r^3}\right)+v_x\mu e_x+v_y\mu e_y+\frac{\bar{n}\cdot\bar{e}}{e}\left(\frac{v_yh_x-v_xh_y}{|\bar{n}|^3}-\frac{\frac{\partial e}{\partial r_z}}{|\bar{n}|e}\right)\right] \\
\frac{sign\left(\left(v^2-\frac{\mu}{r}\right)r_z-\bar{r}\cdot\bar{v}v_z\right)}{\left(1-\cos^2(\omega)\right)\mu|\bar{n}|e}\left[h_x\left(r_xv_y-2r_yv_x\right)-h_y\left(r_yv_y+r_zv_z\right)+r_z\mu e_x+\frac{\bar{n}\cdot\bar{e}}{e}\left(\frac{r_zh_y}{|\bar{n}|^3}-\frac{\frac{\partial e}{\partial v_x}}{|\bar{n}|e}\right)\right] \\
\frac{sign\left(\left(v^2-\frac{\mu}{r}\right)r_z-\bar{r}\cdot\bar{v}v_z\right)}{\left(1-\cos^2(\omega)\right)\mu|\bar{n}|e}\left[-h_y\left(r_yv_x-2r_xv_y\right)+h_x\left(r_xv_x+r_zv_z\right)+r_z\mu e_y+\frac{\bar{n}\cdot\bar{e}}{e}\left(\frac{r_zh_x}{|\bar{n}|^3}-\frac{\frac{\partial e}{\partial v_y}}{|\bar{n}|e}\right)\right] \\
\frac{sign\left(\left(v^2-\frac{\mu}{r}\right)r_z-\bar{r}\cdot\bar{v}v_z\right)}{\left(1-\cos^2(\omega)\right)\mu|\bar{n}|e}\left[-h_y\left(r_zv_x-2r_xv_z\right)+h_x\left(r_zv_y-2r_yv_z\right)-r_x\mu e_x-r_y\mu e_y+\frac{\bar{n}\cdot\bar{e}}{e}\left(\frac{r_xh_y-r_yh_x}{|\bar{n}|^3}-\frac{\frac{\partial e}{\partial v_z}}{|\bar{n}|e}\right)\right]
\end{bmatrix}
$$

$$
\frac{\partial \nu^T}{\partial} = \begin{bmatrix}
-\frac{sign(\vec{r}\cdot\vec{v})}{\left(1-\cos^2(\nu)\right)\mu|\vec{r}|e}\left[r_y\left(v_xv_y-\frac{\mu r_xr_y}{r^3}\right)-r_x\left(v^2-\frac{\mu}{r}\right)+r_z\left(v_xv_z-\frac{\mu r_xr_z}{r^3}\right)-r_x\left(v_y^2+v_z^2-\frac{\mu}{r}+\frac{\mu r_x^2}{r^3}\right)+v_x\vec{r}\cdot\vec{v}-\frac{\vec{r}\cdot\vec{e}}{re}\left(\frac{r_x}{r^2}+\frac{\frac{\partial e}{\partial r_x}}{e}\right)\right] \\
-\frac{sign(\vec{r}\cdot\vec{v})}{\left(1-\cos^2(\nu)\right)\mu|\vec{r}|e}\left[r_x\left(v_xv_y-\frac{\mu r_xr_y}{r^3}\right)-r_y\left(v^2-\frac{\mu}{r}\right)+r_z\left(v_yv_z-\frac{\mu r_yr_z}{r^3}\right)-r_y\left(v_x^2+v_z^2-\frac{\mu}{r}+\frac{\mu r_y^2}{r^3}\right)+v_y\vec{r}\cdot\vec{v}-\frac{\vec{r}\cdot\vec{e}}{re}\left(\frac{r_y}{r^2}+\frac{\frac{\partial e}{\partial r_y}}{e}\right)\right] \\
-\frac{sign(\vec{r}\cdot\vec{v})}{\left(1-\cos^2(\nu)\right)\mu|\vec{r}|e}\left[r_x\left(v_xv_z-\frac{\mu r_xr_z}{r^3}\right)-r_z\left(v^2-\frac{\mu}{r}\right)+r_y\left(v_yv_z-\frac{\mu r_yr_z}{r^3}\right)-r_z\left(v_x^2+v_y^2-\frac{\mu}{r}+\frac{\mu r_z^2}{r^3}\right)+v_z\vec{r}\cdot\vec{v}-\frac{\vec{r}\cdot\vec{e}}{re}\left(\frac{r_z}{r^2}+\frac{\frac{\partial e}{\partial r_z}}{e}\right)\right] \\
-\frac{sign(\vec{r}\cdot\vec{v})}{\left(1-\cos^2(\nu)\right)\mu|\vec{r}|e}\left[r_y\left(r_xv_y-2r_yv_x\right)+r_x\left(r_yv_y-r_zv_z\right)+r_z\left(v_xv_z-2r_zv_x\right)-\frac{\vec{r}\cdot\vec{e}\,\frac{\partial e}{\partial v_x}}{re^2}\right] \\
-\frac{sign(\vec{r}\cdot\vec{v})}{\left(1-\cos^2(\nu)\right)\mu|\vec{r}|e}\left[r_x\left(r_yv_x-2r_xv_y\right)+r_y\left(r_xv_x-r_zv_z\right)+r_z\left(v_yv_z-2r_zv_y\right)-\frac{\vec{r}\cdot\vec{e}\,\frac{\partial e}{\partial v_y}}{re^2}\right] \\
-\frac{sign(\vec{r}\cdot\vec{v})}{\left(1-\cos^2(\nu)\right)\mu|\vec{r}|e}\left[r_z\left(r_xv_x-r_yv_y\right)+r_x\left(r_zv_x-r_xv_z\right)+r_y\left(v_zv_y-2r_yv_z\right)-\frac{\vec{r}\cdot\vec{e}\,\frac{\partial e}{\partial v_z}}{re^2}\right]
\end{bmatrix}
$$

The partials for mean anomaly are the same as those for true anomaly except for the eccentricity and mean anomaly terms which are as follows:

$$
\frac{\partial M}{\partial \vec{r}\vec{v}} = \frac{\frac{\partial \nu}{\partial \vec{r}\vec{v}}}{\frac{\partial M}{\partial \nu}} \qquad \frac{\partial e}{\partial \vec{r}\vec{v}} = \frac{\partial e}{\partial \vec{r}\vec{v}} - \frac{\partial M}{\partial \vec{r}\vec{v}}\frac{\partial M}{\partial e}
$$

Classical to Cartesian

The reverse process is also somewhat tedious due to the lengthy derivatives created by the transformation from IJK to PQW (See also Long, 1989, 3-58 to 3-59). From the TRACE Vol III, B-1 to B-4, Broucke (1970) (notation problems). From Alfano 2015 and using p_{ij} as the components of the PQW to ECI transformation in Eq. 14,

$$
\begin{array}{cccc}
 & a & e & i \\
\frac{\partial r_x}{\partial} & \frac{1-e^2}{1+e\cos(\nu)}\left(\cos(\nu)p_{11}+\sin(\nu)p_{12}\right) & -\frac{2ae+a\cos(\nu)+a\cos(\nu)e^2}{\left(1+e\cos(\nu)\right)^2}\left(\cos(\nu)p_{11}+\sin(\nu)p_{12}\right) & \frac{a(1-e^2)}{1+e\cos(\nu)}p_{13}\left(\cos(\nu)\sin(\omega)+\sin(\nu)\cos(\omega)\right) \\
\frac{\partial r_y}{\partial} & \frac{1-e^2}{1+e\cos(\nu)}\left(\cos(\nu)p_{21}+\sin(\nu)p_{22}\right) & -\frac{2ae+a\cos(\nu)+a\cos(\nu)e^2}{\left(1+e\cos(\nu)\right)^2}\left(\cos(\nu)p_{21}+\sin(\nu)p_{22}\right) & \frac{a(1-e^2)}{1+e\cos(\nu)}p_{23}\left(\cos(\nu)\sin(\omega)+\sin(\nu)\cos(\omega)\right) \\
\frac{\partial r_z}{\partial} & \frac{1-e^2}{1+e\cos(\nu)}\left(\cos(\nu)p_{31}+\sin(\nu)p_{32}\right) & -\frac{2ae+a\cos(\nu)+a\cos(\nu)e^2}{\left(1+e\cos(\nu)\right)^2}\left(\cos(\nu)p_{31}+\sin(\nu)p_{32}\right) & \frac{a(1-e^2)}{1+e\cos(\nu)}\cos(i)\left(\cos(\nu)\sin(\omega)+\sin(\nu)\cos(\omega)\right) \\
\frac{\partial v_x}{\partial} & \frac{1}{2a}\sqrt{\frac{\mu}{a(1-e^2)}}\left(\sin(\nu)p_{11}-(e+\cos(\nu))p_{12}\right) & \frac{1}{1-e^2}\sqrt{\frac{\mu}{a(1-e^2)}}\left(e\sin(\nu)p_{11}+(e+\cos(\nu))p_{12}\right) & \sqrt{\frac{\mu}{a(1-e^2)}}\sin(\Omega)\left(\sin(\nu)p_{31}+\left(e+\cos(\nu)\right)p_{32}\right) \\
\frac{\partial v_y}{\partial} & \frac{1}{2a}\sqrt{\frac{\mu}{a(1-e^2)}}\left(\sin(\nu)p_{21}-(e+\cos(\nu))p_{22}\right) & \frac{1}{1-e^2}\sqrt{\frac{\mu}{a(1-e^2)}}\left(e\sin(\nu)p_{21}+(e+\cos(\nu))p_{22}\right) & \sqrt{\frac{\mu}{a(1-e^2)}}\cos(\Omega)\left(\sin(\nu)p_{31}-\left(e+\cos(\nu)\right)p_{32}\right) \\
\frac{\partial v_z}{\partial} & \frac{1}{2a}\sqrt{\frac{\mu}{a(1-e^2)}}\left(\sin(\nu)p_{31}-(e+\cos(\nu))p_{32}\right) & \frac{1}{1-e^2}\sqrt{\frac{\mu}{a(1-e^2)}}\left(e\sin(\nu)p_{31}+(e+\cos(\nu))p_{32}\right) & -\sqrt{\frac{\mu}{a(1-e^2)}}\left(\sin(\nu)\sin(\omega)\cos(i)+\left(e+\cos(\nu)\right)\cos(\omega)\cos(i)\right)
\end{array}
$$

$$
\begin{array}{c}
 \\
\frac{\partial r_x}{\partial} \\[2ex]
\frac{\partial r_y}{\partial} \\[2ex]
\frac{\partial r_z}{\partial} \\[2ex]
\frac{\partial v_z}{\partial} \\[2ex]
\frac{\partial v_z}{\partial} \\[2ex]
\frac{\partial v_z}{\partial}
\end{array}
\left[
\begin{array}{cccc}
\Omega & \omega & M & \nu \\
\frac{a\left(1-e^2\right)}{1+e\cos(\nu)}\left(-\cos(\nu)p_{21}-\sin(\nu)p_{22}\right) & \frac{a\left(1-e^2\right)}{1+e\cos(\nu)}\left(\cos(\nu)p_{12}-\sin(\nu)p_{11}\right) & \frac{a\left(1-e^2\right)}{\left(1+e\cos(\nu)\right)^2}\left(-\sin(\nu)p_{11}+\left(e+\cos(\nu)\right)p_{12}\right) & * \\
\frac{a\left(1-e^2\right)}{1+e\cos(\nu)}\left(\cos(\nu)p_{21}+\sin(\nu)p_{22}\right) & \frac{a\left(1-e^2\right)}{1+e\cos(\nu)}\left(\cos(\nu)p_{22}-\sin(\nu)p_{21}\right) & \frac{a\left(1-e^2\right)}{\left(1+e\cos(\nu)\right)^2}\left(-\sin(\nu)p_{21}+\left(e+\cos(\nu)\right)p_{22}\right) & \\
0 & \frac{a\left(1-e^2\right)}{1+e\cos(\nu)}\sin(i)\left(\cos(\nu)\cos(\omega)-\sin(\nu)\sin(\omega)\right) & \frac{a\left(1-e^2\right)}{\left(1+e\cos(\nu)\right)^2}\left(-\sin(\nu)p_{31}+\left(e+\cos(\nu)\right)p_{32}\right) & \\
\sqrt{\frac{\mu}{a(1-e^2)}}\left(\sin(\nu)p_{21}-\left(e+\cos(\nu)\right)p_{22}\right) & \sqrt{\frac{\mu}{a(1-e^2)}}\left(\sin(\nu)p_{12}+\left(e+\cos(\nu)\right)p_{21}\right) & \sqrt{\frac{\mu}{a(1-e^2)}}\left(-\cos(\nu)p_{11}-\sin(\nu)p_{12}\right) & \\
\sqrt{\frac{\mu}{a(1-e^2)}}\left(-\sin(\nu)p_{11}+\left(e+\cos(\nu)\right)p_{12}\right) & \sqrt{\frac{\mu}{a(1-e^2)}}\left(-\sin(\nu)p_{22}-\left(e+\cos(\nu)\right)p_{21}\right) & \sqrt{\frac{\mu}{a(1-e^2)}}\left(-\cos(\nu)p_{21}-\sin(\nu)p_{22}\right) & \\
0 & \sqrt{\frac{\mu}{a(1-e^2)}}\left(-\sin(\nu)p_{32}-\left(e+\cos(\nu)\right)p_{31}\right) & \sqrt{\frac{\mu}{a(1-e^2)}}\left(-\cos(\nu)p_{31}-\sin(\nu)p_{32}\right) &
\end{array}
\right]
$$

and

$$
\frac{\partial v_z}{\partial}^T =
\left[
\begin{array}{c}
\frac{a(e^2-1)\left(e\cos(\Omega)\sin(\omega)+\cos(\Omega)\cos(\omega)\sin(\nu)+\cos(\Omega)\sin(\omega)\cos(\nu)+e\cos(i)\sin(\Omega)\cos(\omega)+\cos(i)\sin(\Omega)\cos(\omega)\cos(\nu)-\cos(i)\sin(\Omega)\sin(\omega)\sin(\nu)\right)}{\left(e\cos(\nu)+1\right)^2} \\
\frac{a(e^2-1)\left(e\sin(\Omega)\sin(\omega)+\sin(\Omega)\cos(\omega)\sin(\nu)+\sin(\Omega)\sin(\omega)\cos(\nu)-e\cos(i)\cos(\Omega)\cos(\omega)-\cos(i)\cos(\Omega)\cos(\omega)\cos(\nu)+\cos(i)\cos(\Omega)\sin(\omega)\sin(\nu)\right)}{\left(e\cos(\nu)+1\right)^2} \\
\frac{-a(e^2-1)\sin(i)\left(\cos(\omega+\nu)+e\cos(\omega)\right)}{\left(e\cos(\nu)+1\right)^2} \\
\sqrt{\frac{-\mu}{a(e^2-1)}}\left(\cos(\Omega)\sin(\omega)\sin(\nu)-\cos(\Omega)\cos(\omega)\cos(\nu)+\cos(i)\sin(\Omega)\cos(\omega)\sin(\nu)+\cos(i)\sin(\Omega)\sin(\omega)\cos(\nu)\right) \\
-\sqrt{\frac{-\mu}{a(e^2-1)}}\left(\sin(\Omega)\cos(\omega)\cos(\nu)-\sin(\Omega)\sin(\omega)\sin(\nu)+\cos(i)\cos(\Omega)\cos(\omega)\sin(\nu)+\cos(i)\cos(\Omega)\sin(\omega)\cos(\nu)\right) \\
-\sqrt{\frac{-\mu}{a(e^2-1)}}\left(\sin(\omega+\nu)\sin(i)\right)
\end{array}
\right]
$$

Classical to Equinoctial:
From Trace (77:A-6 to A-8), Chobotov (359),

$$
\begin{array}{cc}
 & \begin{matrix} \overbrace{a \quad a} & e & i & \Omega & \omega & \overbrace{M \quad \nu} \end{matrix} \\
\begin{matrix} \left\{ \begin{matrix} \frac{\partial a}{\partial} \\ \frac{\partial n}{\partial} \end{matrix} \right. \\ \frac{\partial a_f}{\partial} \\ \frac{\partial a_g}{\partial} \\ \frac{\partial \chi}{\partial} \\ \frac{\partial \psi}{\partial} \\ \left\{ \begin{matrix} \frac{\partial \lambda_M}{\partial} \\ \frac{\partial \lambda_\nu}{\partial} \end{matrix} \right. \end{matrix} &
\begin{bmatrix}
1 & N/A & 0 & 0 & 0 & 0 & 0 & 0 \\
N/A & -\frac{3}{2a}\sqrt{\frac{\mu}{a^3}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cos(\omega + f_r\Omega) & 0 & -ef_r\sin(\omega + f_r\Omega) & e\sin(\omega + f_r\Omega) & 0 & 0 \\
0 & 0 & \sin(\omega + f_r\Omega) & 0 & -ef_r\cos(\omega + f_r\Omega) & e\cos(\omega + f_r\Omega) & 0 & 0 \\
0 & 0 & 0 & f_r\tan(\frac{i}{2})^{f_r-1}\sin(\Omega)\left(\frac{\tan(\frac{i}{2})^2}{2}+\frac{1}{2}\right) & \tan(\frac{i}{2})^{f_r}\cos(\Omega) & 0 & 0 & 0 \\
0 & 0 & 0 & f_r\tan(\frac{i}{2})^{f_r-1}\cos(\Omega)\left(\frac{\tan(\frac{i}{2})^2}{2}+\frac{1}{2}\right) & -\tan(\frac{i}{2})^{f_r}\sin(\Omega) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & f_r & 1 & 1 & N/A \\
0 & 0 & 0 & 0 & f_r & 1 & N/A & 1
\end{bmatrix}
\end{array}
$$

Equinoctial to Classical:

Mostly from Chobotov (1989:359).

$$
\left[\begin{array}{ccccccccc}
 & a & n & a_f & a_g & \chi & \psi & \lambda_M & \lambda_\nu \\[6pt]
\frac{\partial a}{\partial} & 1 & N/A & 0 & 0 & 0 & 0 & 0 & 0 \\[6pt]
\frac{\partial a}{\partial} & N/A & -\frac{2}{3n}\left(\frac{\mu}{n^2}\right)^{1/3} & 0 & 0 & 0 & 0 & 0 & 0 \\[6pt]
\frac{\partial e}{\partial} & 0 & 0 & \frac{a_f}{\sqrt{a_f^2+a_g^2}} & \frac{a_g}{\sqrt{a_f^2+a_g^2}} & 0 & 0 & 0 & 0 \\[6pt]
\frac{\partial i}{\partial} & 0 & 0 & 0 & 0 & \frac{2 f_r \chi}{(1+\chi^2+\psi^2)\sqrt{\chi^2+\psi^2}} & \frac{2 f_r \psi}{(1+\chi^2+\psi^2)\sqrt{\chi^2+\psi^2}} & 0 & 0 \\[6pt]
\frac{\partial \Omega}{\partial} & 0 & 0 & 0 & 0 & \frac{\psi}{\chi^2+\psi^2} & \frac{-\chi}{\chi^2+\psi^2} & 0 & 0 \\[6pt]
\frac{\partial \omega}{\partial} & 0 & 0 & \frac{-a_g}{a_f^2+a_g^2} & \frac{a_f}{a_f^2+a_g^2} & \frac{-f_r \psi}{\chi^2+\psi^2} & \frac{f_r \chi}{\chi^2+\psi^2} & 0 & 0 \\[6pt]
\frac{\partial M}{\partial} & 0 & 0 & \frac{-a_g}{a_f^2+a_g^2} & \frac{a_f}{a_f^2+a_g^2} & 0 & 0 & 1 & N/A \\[6pt]
\frac{\partial \nu}{\partial} & 0 & 0 & \frac{a_g}{a_f^2+a_g^2} & \frac{-a_f}{a_f^2+a_g^2} & 0 & 0 & N/A & 1
\end{array}\right]
$$

Equinoctial to Cartesian:

The direct formulation of Cartesian and Equinoctial elements may be found in Broucke & Cefola (1972:306) (uses *a*, some notation, uses t?), ASTCM (1989, 20-90 to 20-102), McClain (1992:89-90, position only), TRACE (1977:B10 to B-11), Cefola and Yurasov (1998), and Danielson (1995:10-11).

$$A = na^2$$
$$B = \sqrt{1 - a_g^2 - a_f^2}$$
$$b = \frac{1}{1+B}$$
$$C = 1 + \chi^2 + \psi^2$$

$$\frac{\partial X}{\partial a_f} = \frac{a_g \dot{X}}{n(1+B)} + \frac{aY\dot{X}}{AB} - a \qquad \frac{\partial \dot{X}}{\partial a_f} = \frac{a\dot{X}\dot{Y}}{AB} - \frac{A}{r^3}\left(\frac{aa_g X}{1+B} + \frac{XY}{B}\right)$$

$$\frac{\partial Y}{\partial a_f} = \frac{a_g \dot{Y}}{n(1+B)} - \frac{aX\dot{X}}{AB} \qquad \frac{\partial \dot{Y}}{\partial a_f} = -\frac{a\dot{X}^2}{AB} - \frac{A}{r^3}\left(\frac{aa_g Y}{1+B} - \frac{X^2}{B}\right)$$

$$\frac{\partial X}{\partial a_g} = -\frac{a_f \dot{X}}{n(1+B)} + \frac{aY\dot{Y}}{AB} \qquad \frac{\partial \dot{X}}{\partial a_g} = \frac{a\dot{Y}^2}{AB} + \frac{A}{r^3}\left(\frac{aa_f X}{1+B} - \frac{Y^2}{B}\right)$$

$$\frac{\partial Y}{\partial a_g} = -\frac{a_f \dot{Y}}{n(1+B)} - \frac{aX\dot{Y}}{AB} - a \qquad \frac{\partial \dot{Y}}{\partial a_g} = -\frac{a\dot{X}\dot{Y}}{AB} + \frac{A}{r^3}\left(\frac{aa_f Y}{1+B} + \frac{XY}{B}\right)$$

$$
\begin{array}{c|cccccccc}
 & a & n & a_f & a_g & \chi & \psi & \lambda_M & \lambda_\nu \\
\hline
\frac{\partial r_x}{\partial} & \frac{r_x}{a} & -\frac{2r_x}{3n} & \frac{\partial X}{\partial a_f} f_e + \frac{\partial Y}{\partial a_f} g_e & \frac{\partial X}{\partial a_g} f_e + \frac{\partial Y}{\partial a_g} g_e & \frac{2(f_r\psi(Yf_e - Xg_e) - Xw_e)}{C} & \frac{2(f_r\chi(Xg_e - Yf_e) + Yw_e)}{C} & \frac{v_x}{n} & * \\
\frac{\partial r_y}{\partial} & \frac{r_y}{a} & -\frac{2r_y}{3n} & \frac{\partial X}{\partial a_f} f_q + \frac{\partial Y}{\partial a_f} g_q & \frac{\partial X}{\partial a_g} f_q + \frac{\partial Y}{\partial a_g} g_q & \frac{2(f_r\psi(Yf_q - Xg_q) - Xw_q)}{C} & \frac{2(f_r\chi(Xg_q - Yf_q) + Yw_q)}{C} & \frac{v_y}{n} & * \\
\frac{\partial r_z}{\partial} & \frac{r_z}{a} & -\frac{2r_z}{3n} & \frac{\partial X}{\partial a_f} f_w + \frac{\partial Y}{\partial a_f} g_w & \frac{\partial X}{\partial a_g} f_w + \frac{\partial Y}{\partial a_g} g_w & \frac{2(f_r\psi(Yf_w - Xg_w) - Xw_w)}{C} & \frac{2(f_r\chi(Xg_w - Yf_w) + Yw_w)}{C} & \frac{v_z}{n} & * \\
\frac{\partial v_x}{\partial} & -\frac{v_x}{2a} & \frac{v_x}{3n} & \frac{\partial \dot{X}}{\partial a_f} f_e + \frac{\partial \dot{Y}}{\partial a_f} g_e & \frac{\partial \dot{X}}{\partial a_g} f_e + \frac{\partial \dot{Y}}{\partial a_g} g_e & \frac{2(f_r\psi(\dot{Y}f_e - \dot{X}g_e) - \dot{X}w_e)}{C} & \frac{2(f_r\chi(\dot{X}g_e - \dot{Y}f_e) + \dot{Y}w_e)}{C} & -\frac{na^3 r_x}{r^3} & * \\
\frac{\partial v_y}{\partial} & -\frac{v_y}{2a} & \frac{v_y}{3n} & \frac{\partial \dot{X}}{\partial a_f} f_q + \frac{\partial \dot{Y}}{\partial a_f} g_q & \frac{\partial \dot{X}}{\partial a_g} f_q + \frac{\partial \dot{Y}}{\partial a_g} g_q & \frac{2(f_r\psi(\dot{Y}f_q - \dot{X}g_q) - \dot{X}w_q)}{C} & \frac{2(f_r\chi(\dot{X}g_q - \dot{Y}f_q) + \dot{Y}w_q)}{C} & -\frac{na^3 r_y}{r^3} & * \\
\frac{\partial v_z}{\partial} & -\frac{v_z}{2a} & \frac{v_z}{3n} & \frac{\partial \dot{X}}{\partial a_f} f_w + \frac{\partial \dot{Y}}{\partial a_f} g_w & \frac{\partial \dot{X}}{\partial a_g} f_w + \frac{\partial \dot{Y}}{\partial a_g} g_w & \frac{2(f_r\psi(\dot{Y}f_w - \dot{X}g_w) - \dot{X}w_w)}{C} & \frac{2(f_r\chi(\dot{X}g_w - \dot{Y}f_w) + \dot{Y}w_w)}{C} & -\frac{na^3 r_z}{r^3} & *
\end{array}
$$

The * terms were not formulated in this paper.

Cartesian to Equinoctial:

See Danielson (1995:11-12), McClain (1992:91 for velocity only), Broucke (1970:180), Broucke and Cefola use *a* (1972:308).

$$
\begin{array}{c}
 \\
\frac{\partial a}{\partial} \\[6pt]
\frac{\partial n}{\partial} \\[6pt]
\frac{\partial a_f}{\partial} \\[6pt]
\frac{\partial a_g}{\partial} \\[6pt]
\frac{\partial \chi}{\partial} \\[6pt]
\frac{\partial \psi}{\partial} \\[6pt]
\frac{\partial \lambda_M}{\partial} \\[6pt]
\frac{\partial \lambda_\nu}{\partial}
\end{array}
\begin{array}{ccc}
r_x & r_y & r_z
\end{array}
$$

$$
\begin{bmatrix}
\frac{2a^2 r_x}{r^3} & \frac{2a^2 r_y}{r^3} & \frac{2a^2 r_z}{r^3} \\[6pt]
\frac{-3nar_x}{r^3} & \frac{-3nar_y}{r^3} & \frac{-3nar_z}{r^3} \\[6pt]
-\frac{aba_f Br_x}{r^3} - \frac{a_g(\chi\dot{X} - f_r\psi\dot{Y})w_e}{AB} + \frac{B}{A}\frac{\partial v_x}{\partial a_g} & -\frac{aba_f Br_y}{r^3} - \frac{a_g(\chi\dot{X} - f_r\psi\dot{Y})w_q}{AB} + \frac{B}{A}\frac{\partial v_y}{\partial a_g} & -\frac{aba_f Br_z}{r^3} - \frac{a_g(\chi\dot{X} - f_r\psi\dot{Y})w_w}{AB} + \frac{B}{A}\frac{\partial v_z}{\partial a_g} \\[6pt]
-\frac{aba_g Br_x}{r^3} + \frac{a_f(\chi\dot{X} - f_r\psi\dot{Y})w_e}{AB} - \frac{B}{A}\frac{\partial v_x}{\partial a_f} & -\frac{aba_g Br_y}{r^3} + \frac{a_f(\chi\dot{X} - f_r\psi\dot{Y})w_q}{AB} - \frac{B}{A}\frac{\partial v_y}{\partial a_f} & -\frac{aba_g Br_z}{r^3} + \frac{a_f(\chi\dot{X} - f_r\psi\dot{Y})w_w}{AB} - \frac{B}{A}\frac{\partial v_z}{\partial a_f} \\[6pt]
-\frac{C\dot{Y}w_e}{2AB} & -\frac{C\dot{Y}w_q}{2AB} & \frac{CYw_w}{2AB} \\[6pt]
-\frac{Cf_r\dot{X}w_e}{2AB} & -\frac{Cf_r\dot{X}w_q}{2AB} & -\frac{Cf_r\dot{X}w_w}{2AB} \\[6pt]
-\frac{v_x}{A} + \frac{(\chi\dot{X} - f_r\psi\dot{Y})w_e}{AB} - \frac{bB}{A}\left(a_g\frac{\partial v_x}{\partial a_g} + a_f\frac{\partial v_x}{\partial a_f}\right) & -\frac{v_y}{A} + \frac{(\chi\dot{X} - f_r\psi\dot{Y})w_q}{AB} - \frac{bB}{A}\left(a_g\frac{\partial v_y}{\partial a_g} + a_f\frac{\partial v_y}{\partial a_f}\right) & -\frac{v_z}{A} + \frac{(\chi\dot{X} - f_r\psi\dot{Y})w_w}{AB} - \frac{bB}{A}\left(a_g\frac{\partial v_z}{\partial a_g} + a_f\frac{\partial v_z}{\partial a_f}\right) \\[6pt]
* & * & *
\end{bmatrix}
$$

Where the * terms are found on the following page:

$$
\begin{array}{c}
\\
\dfrac{\partial a}{\partial} \\[2ex]
\dfrac{\partial n}{\partial} \\[2ex]
\dfrac{\partial a_f}{\partial} \\[2ex]
\dfrac{\partial a_g}{\partial} \\[2ex]
\dfrac{\partial \chi}{\partial} \\[2ex]
\dfrac{\partial \psi}{\partial} \\[2ex]
\dfrac{\partial \lambda_M}{\partial} \\[2ex]
\dfrac{\partial \lambda_\nu}{\partial}
\end{array}
\begin{bmatrix}
v_x & v_y & v_z \\[2ex]
\dfrac{2v_x}{n^2 a} & \dfrac{2v_y}{n^2 a} & \dfrac{2v_z}{n^2 a} \\[2ex]
\dfrac{-3v_x}{na^2} & \dfrac{-3v_y}{na^2} & \dfrac{-3v_z}{na^2} \\[2ex]
\dfrac{(2X\dot{Y} - \dot{X}Y)g_e - Y\dot{Y}f_e}{\mu} - \dfrac{a_g(f_r\psi Y - \chi X)w_e}{AB} & \dfrac{(2X\dot{Y} - \dot{X}Y)g_q - Y\dot{Y}f_q}{\mu} - \dfrac{a_g(f_r\psi Y - \chi X)w_q}{AB} & \dfrac{(2X\dot{Y} - \dot{X}Y)g_w - Y\dot{Y}f_w}{\mu} - \dfrac{a_g(f_r\psi Y - \chi X)w_w}{AB} \\[2ex]
\dfrac{(2\dot{X}Y - X\dot{Y})f_e - X\dot{X}g_e}{\mu} + \dfrac{a_f(f_r\psi Y - \chi X)w_e}{AB} & \dfrac{(2\dot{X}Y - X\dot{Y})f_q - X\dot{X}g_q}{\mu} + \dfrac{a_f(f_r\psi Y - \chi X)w_q}{AB} & \dfrac{(2\dot{X}Y - X\dot{Y})f_w - X\dot{X}g_w}{\mu} + \dfrac{a_f(f_r\psi Y - \chi X)w_w}{AB} \\[2ex]
\dfrac{CYw_e}{2AB} & \dfrac{CYw_q}{2AB} & \dfrac{CYw_w}{2AB} \\[2ex]
\dfrac{f_r CXw_e}{2AB} & \dfrac{f_r CXw_q}{2AB} & \dfrac{f_r CXw_w}{2AB} \\[2ex]
-\dfrac{2r_x}{A} + \dfrac{a_f \frac{\partial a_g}{\partial v_x} - a_g \frac{\partial a_f}{\partial v_x}}{1+B} + \dfrac{(f_r\psi Y - \chi X)w_e}{A} & -\dfrac{2r_y}{A} + \dfrac{a_f \frac{\partial a_g}{\partial v_y} - a_g \frac{\partial a_f}{\partial v_y}}{1+B} + \dfrac{(f_r\psi Y - \chi X)w_q}{A} & -\dfrac{2r_z}{A} + \dfrac{a_f \frac{\partial a_g}{\partial v_z} - a_g \frac{\partial a_f}{\partial v_z}}{1+B} + \dfrac{(f_r\psi Y - \chi X)w_w}{A} \\[2ex]
* & * & *
\end{bmatrix}
$$

Where the * terms are the same as ct2cl.

Cartesian to Spherical

From Trace (1977:A-1 to A-3). The fpa and azimuth partials were re-calculated. The fpa partials turned out to be the negative of the published ones in Trace. From Long et al. (1989, 3-44 to 3-46),

$$\dot{r} = \frac{\vec{r} \cdot \vec{v}}{r}$$

$$
\begin{array}{cc}
 & \begin{array}{ccc} r_x & r_y & r_z \end{array} \\
\begin{array}{c}
\dfrac{\partial \alpha}{\partial} \\[3ex]
\dfrac{\partial \delta}{\partial} \\[3ex]
\dfrac{\partial \phi_{fpa}}{\partial} \\[3ex]
\dfrac{\partial \beta}{\partial} \\[3ex]
\dfrac{\partial r}{\partial} \\[3ex]
\dfrac{\partial v}{\partial v_z}
\end{array}
&
\begin{bmatrix}
\dfrac{-r_y}{r_x^2 + r_y^2} & \dfrac{r_x}{r_x^2 + r_y^2} & 0 \\[3ex]
\dfrac{-r_x r_z}{r^2 \sqrt{r_x^2 + r_y^2}} & \dfrac{-r_y r_z}{r^2 \sqrt{r_x^2 + r_y^2}} & \dfrac{\sqrt{r_x^2 + r_y^2}}{r^2} \\[3ex]
\dfrac{1}{r^2 h}\left(v_x (r_y^2 + r_z^2) - r_x (r_y v_y + r_z v_z) \right) & \dfrac{1}{r^2 h}\left(v_y (r_x^2 + r_z^2) - r_y (r_x v_x + r_z v_z) \right) & \dfrac{1}{r^2 h}\left(v_z (r_x^2 + r_y^2) - r_z (r_x v_x + r_y v_y) \right) \\[3ex]
\dfrac{v_y \left(r v_z - r_z \dot{r} \right) - \dfrac{r_x v_y - r_y v_x}{r} \left(r_x v_z - r_z v_x + \dfrac{r_x r_y \dot{r}}{r} \right)}{\left(v^2 - \dot{r}^2 \right)\left(r_x^2 + r_y^2 \right)} & \dfrac{-v_x \left(r v_z - r_z \dot{r} \right) + \dfrac{r_x v_y - r_y v_x}{r} \left(r_y v_z - r_z v_y + \dfrac{r_y r_z \dot{r}}{r} \right)}{\left(v^2 - \dot{r}^2 \right)\left(r_x^2 + r_y^2 \right)} & \dfrac{\dot{r}\left(r_x v_y - r_y v_x \right)}{r^2 \left(v^2 - \dot{r}^2 \right)} \\[3ex]
\dfrac{r_x}{r} & \dfrac{r_y}{r} & \dfrac{r_z}{r} \\[3ex]
0 & 0 & 0
\end{bmatrix}
\end{array}
$$

$$
\begin{array}{cc}
 & \begin{matrix} \qquad\qquad v_x \qquad\qquad & \qquad\qquad v_y \qquad\qquad & \qquad\qquad v_z \qquad\qquad \end{matrix} \\
\begin{matrix}
\dfrac{\partial \alpha}{\partial} \\[2ex]
\dfrac{\partial \delta}{\partial} \\[2ex]
\dfrac{\partial \phi_{fpa}}{\partial} \\[2ex]
\dfrac{\partial \beta}{\partial} \\[2ex]
\dfrac{\partial r}{\partial} \\[2ex]
\dfrac{\partial v}{\partial}
\end{matrix}
&
\begin{bmatrix}
0 & 0 & 0 \\[2ex]
0 & 0 & 0 \\[2ex]
\dfrac{1}{v^2 h}\left(r_x(v_y^2+v_z^2)-v_x(r_y v_y+r_z v_z)\right) & \dfrac{1}{v^2 h}\left(r_y(v_x^2+v_z^2)-v_y(r_x v_x+r_z v_z)\right) & \dfrac{1}{v^2 h}\left(r_z(v_x^2+v_y^2)-v_z(r_x v_x+r_y v_y)\right) \\[2ex]
\dfrac{-\left(r_y v_z - r_z v_y\right)}{r\left(v^2-\dot{r}^2\right)} & \dfrac{\left(r_x v_z - r_z v_x\right)}{r\left(v^2-\dot{r}^2\right)} & \dfrac{-\left(r_x v_y - r_y v_x\right)}{r\left(v^2-\dot{r}^2\right)} \\[2ex]
0 & 0 & 0 \\[2ex]
\dfrac{v_x}{v} & \dfrac{v_y}{v} & \dfrac{v_z}{v}
\end{bmatrix}
\end{array}
$$

<u>Spherical to Cartesian</u>

From Long et al. 1989:3-41 to e-42).

$$
\begin{bmatrix}
 & \alpha & \delta & \phi_{fpa} & \beta & r & v \\
\frac{\partial r_x}{\partial} & -r\sin(\alpha)\cos(\delta) & -r\cos(\alpha)\sin(\delta) & 0 & 0 & \cos(\alpha)\cos(\delta) & 0 \\
\frac{\partial r_y}{\partial} & r\cos(\alpha)\cos(\delta) & -r\sin(\alpha)\sin(\delta) & 0 & 0 & \sin(\alpha)\cos(\delta) & 0 \\
\frac{\partial r_z}{\partial} & 0 & r\cos(\delta) & 0 & 0 & \sin(\delta) & 0 \\
\frac{\partial v_x}{\partial} & -v_y & -v_z\cos(\alpha) & * & v\left[\cos(\alpha)\sin(\delta)\sin(\beta)\cos(\phi_{fpa})-\sin(\alpha)\cos(\beta)\cos(\phi_{fpa})\right] & 0 & \frac{v_x}{v} \\
\frac{\partial v_y}{\partial} & v_x & -v_z\sin(\alpha) & * & v\left[\sin(\alpha)\sin(\delta)\sin(\beta)\cos(\phi_{fpa})+\cos(\alpha)\cos(\beta)\cos(\phi_{fpa})\right] & 0 & \frac{v_y}{v} \\
\frac{\partial v_z}{\partial} & * & * & * & * & * & *
\end{bmatrix}
$$

Where the * terms are:

$$
\frac{\partial v_z}{\partial}^{T} = \begin{bmatrix}
0 \\
v\left[\cos(\delta)\sin(\phi_{fpa})-\sin(\delta)\cos(\beta)\cos(\phi_{fpa})\right] \\
v\left[\sin(\delta)\cos(\phi_{fpa})-\cos(\delta)\cos(\beta)\sin(\phi_{fpa})\right] \\
-v\left[\cos(\delta)\sin(\beta)\cos(\phi_{fpa})\right] \\
0 \\
\frac{v_z}{v}
\end{bmatrix}
$$

$$
\frac{\partial v_x}{\partial\phi_{fpa}} = v\left[\cos(\alpha)\sin(\delta)\cos(\beta)\sin(\phi_{fpa})+\sin(\alpha)\sin(\beta)\sin(\phi_{fpa})+\cos(\alpha)\cos(\delta)\cos(\phi_{fpa})\right]
$$

$$
\frac{\partial v_y}{\partial\phi_{fpa}} = v\left[\sin(\alpha)\sin(\delta)\cos(\beta)\sin(\phi_{fpa})-\cos(\alpha)\sin(\beta)\sin(\phi_{fpa})+\sin(\alpha)\cos(\delta)\cos(\phi_{fpa})\right]
$$