Thesis

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1 Introduction

Write about reasons for writing this text, who is it meant for etc?

Maybe write some history of triangulated and derived categories and where they find their uses, etc?

Introduce notation which will be used in text.

2 Triangulated Categories

Probably introduce this section, what is happening and what will be done etc. I can maybe say something about algebraic triangulated categories and topological triangulated categories, and explaining the name cone, fiber and cofiber.

2.1 Definition and first properties

In this section $\mathcal{T}$ denotes an additive category and $T : \mathcal{T} \to \mathcal{T}$ is an additive autoequivalence of $\mathcal{T}$.

Definition 2.1. A sextuple is a collection (A, B, C, a, b, c) of objects $A, B, C \in T$ and morphisms $a : A \to B$, $b : B \to C$, $c : C \to TA$. These sextuples can be drawn as diagrams in the following way:

$$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA$$

A morphism between sextuples is a triple of morphism (α, β, γ), where $\alpha : A \to A'$, $\beta : B \to B'$ and $\gamma : C \to C'$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\alpha"] & B \arrow[r, "b"] \arrow[d, "\beta"] & C \arrow[r, "c"] \arrow[d, "\gamma"] & TA \arrow[d, "T\alpha"] \\
A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C \arrow[r, "c'"] & TA'
\end{tikzcd}$$

The naming convention of the sextuples isn't standarized, some literatures calls the sextuples for triangles instead [literature here, learn bibtex you lazy fuck]. This name arises from an alternate description of the diagrams given above. To remove confusion about the domain or codomain of the arrows, one arrow of the triangle is decorated with "$_T$—". This decorator means that the functor T has to be applied to the corresponding edge of the arrow. Thus the c arrow points to TA, not A.

$$\begin{tikzcd}
A \arrow[dr, "a"] & \\
& B \arrow[dl, "b"] \\
C \arrow[uu, "c", "T"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rrr, "\phi_a"] \arrow[dr, "a"] & & & A' \arrow[dl, "a'"'] \\
& B \arrow[r, "\phi_b"] \arrow[dl, "b"] & B' \arrow[dr, "b'"] & \\
C \arrow[uu, dashed, "c", "T"'] \arrow[rrr, "\phi_c"] & & & C' \arrow[uu, dashed, "c'"', "T"]
\end{tikzcd}$$

A triangulated category is an additive category together with an autoequivalence T and a triangulation Δ consisting of sextuples. When a sextuple is an element of Δ it is usually called a distinguished triangle, an exact triangles or just a triangle. Note that if sextuples are referred to as triangles it is common to either call the elements of Δ for distinguished triangles or exact triangles. As this is not the case for this thesis these objects will be referred to as triangles.

Definition 2.2. A triangulation of an additive category $\mathcal{T}$ with autoequivalence T is a collection Δ of sextuples in $\mathcal{T}$ satisfying the following axioms:

1. (TR1) Formation axiom

   (a) A sextuple isomorphic to a triangle is a triangle.

   (b) Every morphism $a : A \to B$ can be embedded into a triangle:

   $$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA$$

   (c) For every object A there is a triangle:

   $$A \xrightarrow{id_A} A \xrightarrow{0} 0 \xrightarrow{0} TA$$

2. (TR2) Rotation axiom

   For every triangle $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA$ in Δ,

   there is a triangle $B \xrightarrow{b} C \xrightarrow{c} TA \xrightarrow{-Ta} TB$ in Δ

3. (TR3) Morphism axiom

   Given the two triangles $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA$ and $A' \xrightarrow{a'} B' \xrightarrow{b'} C' \xrightarrow{c'} TA'$, and morphism $\phi_A : A \to A'$ and $\phi_B : B \to B'$ such that the square (1) commutes, then there is a morphism $\phi_C : C \to C'$ (not necessarily unique) such that (ϕ_A, ϕ_B, ϕ_C) is a morphism of triangles (2).

$$(1) \quad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[d, "\phi_B"] \\ A' \arrow[r, "a'"] & B' \end{tikzcd} \qquad (2) \quad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[r, "b"] \arrow[d, "\phi_B"] & C \arrow[r, "c"] \arrow[d, dashed, "\phi_C"] & TA \arrow[d, "T\phi_A"] \\ A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & TA' \end{tikzcd}$$

4. (TR4) Octahedron axiom

   Given the triangles $A \xrightarrow{a} B \xrightarrow{x} C' \xrightarrow{x'} TA$, $B \xrightarrow{b} C \xrightarrow{y} A' \xrightarrow{y'} TB$ and

   $A \xrightarrow{b \circ a} C \xrightarrow{z} B' \xrightarrow{z'} TA$ then there exist morphisms $f : C' \to B'$ and $g : B' \to A'$ following diagram commutes and the third row is a triangle:

$$\begin{tikzcd} T^{-1}B' \arrow[r, "T^{-1}z'"] \arrow[d, "T^{-1}g"] & A \arrow[r, equal, "id_A"] \arrow[d, "a"] & A \arrow[d, "b \circ a"] & & \\ T^{-1}A' \arrow[r, "T^{-1}y'"] & B \arrow[r, "b"] \arrow[d, "x"] & C \arrow[r, "y"] \arrow[d, "z"] & A' \arrow[r, "y'"] \arrow[d, equal, "id_{A'}"] & TB \arrow[d, "Tx'"] \\ & C' \arrow[r, "f"] \arrow[d, "x'"] & B' \arrow[r, "g"] \arrow[d, "z'"] & A' \arrow[r, "Ti \circ y'"] & TC' \\ & TA \arrow[r, equal, "id_{TA}"] & TA & & \end{tikzcd}$$

Remark. The rotation axiom have a dual, and that is a rotation in the opposite direction. This dual can be proved by the other axioms, so it is omitted as an axiom.

The dual to the roation axiom goes as: Given a triangle $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA$, there is a triangle $T^{-1}C \xrightarrow{-T^{-1}c} A \xrightarrow{a} B \xrightarrow{b} C$

Proof. □

Remark. Why is it called the Octahedron axiom. Draw that Octahedron!

Lemma 2.1. *composition of morphisms is zero*

Corollary 2.1.1. *isomorphisms in triangles*

Definition 2.3. An additive functor between triangulated categories $F : (\mathcal{T}, T, \Delta) \to (\mathcal{R}, R, \Gamma)$ is called exact or triangulated if there exist a natural transformation $\alpha : FT \to RF$ such that $F(\Delta) \subseteq \Gamma$

Definition 2.4. Define homological functors

Lemma 2.2. *long exact sequence of hom(ology)*

Lemma 2.3. *2 out of 3 property*

Lemma 2.4. *splitmonos, splitepis and zeros*

Corollary 2.4.1. *existence of maps*

3 Exact Categories (and the Frobenius category)

4 The Derived Categories (of Exact Categories)

5 Auslander-Reiten Triangles (in the Derived category)