Thesis

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1 Introduction

This is an introduction, welcome!

Introduce notation which will be used in text. A list of notation and description would be nice, so that the reader might scroll back up if something is unclear.

2 Triangulated Categories

Probably introduce this section, what is happening and what will be done etc. I can maybe say something about algebraic triangulated categories and topological triangulated categories, and explaining the name cone, fiber and cofiber.

2.1 Definition and First Properties

In this section $\mathcal{T}$ denotes an additive category and $T : \mathcal{T} \to \mathcal{T}$ is an additive autoequivalence of $\mathcal{T}$, which is often called translation or suspension functor.

Definition 2.1. A sextuple is a collection (A, B, C, a, b, c) of objects $A, B, C \in T$ and morphisms $a : A \to B$, $b : B \to C$, $c : C \to TA$. These sextuples can be drawn as diagrams in the following way:

$$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA$$

A morphism between sextuples is a triple of morphism (α, β, γ), where $\alpha : A \to A'$, $\beta : B \to B'$ and $\gamma : C \to C'$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\alpha"] & B \arrow[r, "b"] \arrow[d, "\beta"] & C \arrow[r, "c"] \arrow[d, "\gamma"] & TA \arrow[d, "T\alpha"] \\
A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C \arrow[r, "c'"] & TA'
\end{tikzcd}$$

The naming convention of the sextuples isn't standarized, some literatures calls the sextuples for triangles instead [literature here, learn bibtex you lazy fuck]. This name arises from an alternate description of the diagrams given above. To remove confusion about the domain or codomain of the arrows, one arrow of the triangle is decorated with ”$_T$—”. This decorator means that the functor T has to be applied to the corresponding edge of the arrow. Thus the c arrow points to TA, not A.

$$\begin{tikzcd}
A \arrow[dr, "a"] & \\
& B \arrow[dl, "b"] \\
C \arrow[uu, "c", "T"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rrr, "\phi_a"] \arrow[dr, "a"] & & & A' \arrow[dl, "a'"'] \\
& B \arrow[r, "\phi_b"] \arrow[dl, "b"] & B' \arrow[dr, "b'"] & \\
C \arrow[uu, dashed, "c", "T"'] \arrow[rrr, "\phi_c"] & & & C' \arrow[uu, dashed, "c'"', "T"]
\end{tikzcd}$$

A triangulated category is an additive category together with a translation functor T and a triangulation Δ consisting of sextuples. When a sextuple is an element of Δ it is usually called a distinguished triangle, an exact triangles or just a triangle. Note that if sextuples are referred to as triangles it is common to either call the elements of Δ for distinguished triangles or exact triangles. As this is not the case for this thesis these objects will be referred to as triangles.

Definition 2.2. A triangulation of an additive category $\mathcal{T}$ with translation T is a collection Δ of triangles consisting of sextuples in $\mathcal{T}$ satisfying the following axioms:

1. (TR1) Formation axiom

   (a) A sextuple isomorphic to a triangle is a triangle.

   (b) Every morphism $a : A \to B$ can be embedded into a triangle (A, B, C, a, b, c):

   $$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA$$

   (c) For every object A there is a triangle $(A, A, 0, id_A, 0, 0)$:

   $$A \xrightarrow{id_A} A \xrightarrow{0} 0 \xrightarrow{0} TA$$

2. (TR2) Rotation axiom

   For every triangle (A, B, C, a, b, c) there is a triangle (B, C, TA, b, c, Ta)

   $$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA \quad \Longrightarrow \quad B \xrightarrow{b} C \xrightarrow{c} TA \xrightarrow{-Ta} TB$$

3. (TR3) Morphism axiom

   Given the two triangles (A, B, C, a, b, c) (1) and (A', B', C', a', b', c') (2)

   $$(1) \quad A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA \quad (2) \quad A' \xrightarrow{a'} B' \xrightarrow{b'} C' \xrightarrow{c'} TA'$$

   and morphisms $\phi_A : A \to A'$ and $\phi_B : B \to B'$ such that the square (1) commutes, then there is a morphism $\phi_C : C \to C'$ (not necessarily unique) such that (ϕ_A, ϕ_B, ϕ_C) is a morphism of triangles (2).

   $$(1) \quad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[d, "\phi_B"] \\ A' \arrow[r, "a'"] & B' \end{tikzcd} \qquad (2) \quad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[r, "b"] \arrow[d, "\phi_B"] & C \arrow[r, "c"] \arrow[d, dashed, "\phi_C"] & TA \arrow[d, "T\phi_A"] \\ A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & TA' \end{tikzcd}$$

4. (TR4) Octahedron axiom

   Given the triangles (A, B, C', a, x, x') (1), (B, C, A', b, y, y') (2)
   and $(A, C, B', b \circ a, z, z')$ (3)

$$(1)\quad A \xrightarrow{a} B \xrightarrow{x} C' \xrightarrow{x'} TA$$

$$(2)\quad B \xrightarrow{b} C \xrightarrow{y} A' \xrightarrow{y'} TB$$

$$(3)\quad A \xrightarrow{b\circ a} C \xrightarrow{z} B' \xrightarrow{z'} TA$$

then there exist morphisms $f : C' \to B'$ and $g : B' \to A'$, the following diagram commutes and the third row is a triangle:

$$\begin{tikzcd}
T^{-1}B' \arrow[r, "T^{-1}z'"] \arrow[d, "T^{-1}g"] & A \arrow[r, equal, "id_A"] \arrow[d, "a"] & A \arrow[d, "b\circ a"] & & \\
T^{-1}A' \arrow[r, "T^{-1}y'"] & B \arrow[r, "b"] \arrow[d, "x"] & C \arrow[r, "y"] \arrow[d, "z"] & A' \arrow[r, "y'"] \arrow[d, equal, "id_{A'}"] & TB \arrow[d, "Tx'"] \\
 & C' \arrow[r, "f"] \arrow[d, "x'"] & B' \arrow[r, "g"] \arrow[d, "z'"] & A' \arrow[r, "Ti\circ y'"] & TC' \\
 & TA \arrow[r, equal, "id_{TA}"] & TA & &
\end{tikzcd}$$

A triangulated category is denoted as $(\mathcal{T}, T, \Delta)$, where $\mathcal{T}$ is the additive category, T is the translation and Δ is the triangulation. When $\mathcal{T}$ is called a triangulated category, it should be understanded as the triple given above.

Remark. The rotation axiom has a dual, and it can be thought of as a rotation in the opposite direction. This dual can be proved by the other axioms, so it is here omitted as an axiom. The dual roation axiom goes as:

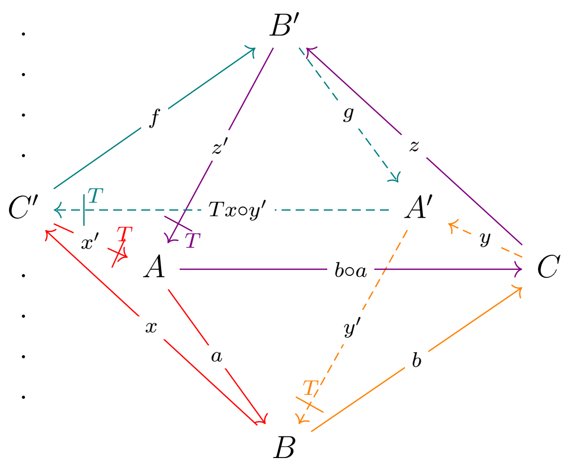
Given a triangle $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA$, there is a triangle $T^{-1}C \xrightarrow{-T^{-1}c} A \xrightarrow{a} B \xrightarrow{b} C$
To be able to prove this, some more lemmata are needed.

Remark. The final axiom is referred to as the octahedron axiom. By using the alternative description of the triangle diagram, it is possible to rewrite the diagram as an octahedron. The axiom can be restated as the following:

Given the triangles (A, B, C', a, x, x') (1), (B, C, A', b, y, y') (2)
and $(A, C, B', b\circ a, z, z')$ (3)

$$(1)\quad \begin{tikzcd}
A \arrow[dr, "a"] & \\
 & B \arrow[dl, "x"] \\
C' \arrow[uu, "T", "x'"'] &
\end{tikzcd}
\qquad
(2)\quad \begin{tikzcd}
B \arrow[dr, "b"] & \\
 & C \arrow[dl, "y"] \\
A' \arrow[uu, "T", "y'"'] &
\end{tikzcd}
\qquad
(3)\quad \begin{tikzcd}
A \arrow[dr, "b\circ a"] & \\
 & C \arrow[dl, "z"] \\
B' \arrow[uu, "T", "z'"'] &
\end{tikzcd}$$

then there exists morphisms $f : C' \to B'$ and $g : B' \to A'$, the following diagram commutes and the teal back face is a triangle.



Lemma 2.1. *Let (A, B, C, a, b, c) be a triangle, then $b \circ a = 0$*

Proof. By TR2 the triangle (A, B, C, a, b, c) can be rotated to (B, C, TA, b, c, Ta).

$$\begin{tikzcd}
A \arrow[dr, "a"] & & & B \arrow[dr, "b"] & \\
& B \arrow[dl, "b"] & \Longrightarrow & & C \arrow[dl, "c"] \\
C \arrow[uu, "T", "c"'] & & & TA \arrow[uu, "T", "-Ta"'] &
\end{tikzcd}$$

The triangle exists $(C, C, 0, id_C, 0, 0)$ by TR1 and TR3 says there exists a morphism from TA to 0 making the diagram below commute.

$$\begin{tikzcd}
B \arrow[r, "b"] \arrow[d, "b"] & C \arrow[r, "c"] \arrow[d, "id_C"] & TA \arrow[r, "-Ta"] \arrow[d, dashed, "0"] & TB \arrow[d, "Tb"] \\
C \arrow[r, "id_C"] & C \arrow[r, "0"] & 0 \arrow[r, "0"] & TC
\end{tikzcd}$$

Thus $0 = Tb \circ -Ta = T(-ba) \implies b \circ a = 0$ as T is a translation. $\square$

Definition 2.3. An additive functor between triangulated categories $F : (\mathcal{T}, T, \Delta) \to (\mathcal{R}, R, \Gamma)$ is called exact or triangulated if there exist a natural isomorphisms $\alpha : FT \to RF$ such that $F(\Delta) \subseteq \Gamma$.

A functor $F : \mathcal{T} \to \mathcal{R}$ is called a triangle-equivalence if it is triangulated and an equivalence of categories. In this case $\mathcal{T}$ and $\mathcal{R}$ are called triangle-equivalent.

Definition 2.4. Let $\mathcal{T}$ be a triangulated category and $\mathcal{A}$ be an abelian category. A covariant functor $H : \mathcal{T} \to \mathcal{A}$ is called a homological functor if $\forall (A, B, C, a, b, c) : \Delta$ there is a long exact sequence in $\mathcal{A}$.

$$
\begin{tikzcd}
A \arrow[dr, "a"] & & \\
 & B \arrow[dl, "b"] & \\
C \arrow[uu, "c", "T"'] & &
\end{tikzcd}
\implies
\begin{tikzcd}
\dots \arrow[r] & H(T^iA) \arrow[r, "H(T^ia)"] & H(T^iB) \arrow[r, "H(T^ib)"] & H(T^iC) \arrow[dll, "H(T^ic)"] & \\
 & H(T^{i+1}A) \arrow[r, "H(T^{i+1}a)"] & H(T^{i+1}B) \arrow[r, "H(T^{i+1}b)"] & H(T^{i+1}C) \arrow[r] & \dots
\end{tikzcd}
$$

Dually, a contravariant functor $H : \mathcal{T} \to \mathcal{A}$ is called cohomological if $\forall (A, B, C, a, b, c) : \Delta$ there is a long exact sequence in $\mathcal{A}$.

$$
\begin{tikzcd}
A \arrow[dr, "a"] & & \\
 & B \arrow[dl, "b"] & \\
C \arrow[uu, "c", "T"'] & &
\end{tikzcd}
\implies
\begin{tikzcd}
\dots & H(T^{i-1}A) \arrow[l] & H(T^{i-1}B) \arrow[l, "H(T^{i-1}a)"'] & H(T^{i-1}C) \arrow[l, "H(T^{i-1}b)"'] & \\
 & H(T^iA) \arrow[urr, "H(T^ic)"] & H(T^iB) \arrow[l, "H(T^ia)"] & H(T^iC) \arrow[l, "H(T^ib)"] & \dots \arrow[l]
\end{tikzcd}
$$

Lemma 2.2. *Let $M : \mathcal{T}$ be any object of $\mathcal{T}$, then the represented functors $\mathcal{T}(M, \_)$ is homological and $\mathcal{T}(\_, M)$ is cohomological.*

Proof. Only the covariant case needs to be proved, as the contravariant case is dual. For $\mathcal{T}(M, \_)$ to be homological, it has to create long exact sequences for every triangle in Δ. Let $(A, B, C, a, b, c) : \Delta$ be a triangle, then there can be extracted sequences in Ab for any $i : \mathbb{N}$.

$$
\begin{tikzcd}
A \arrow[dr, "a"] & & \\
 & B \arrow[dl, "b"] & \\
C \arrow[uu, "c", "T"'] & &
\end{tikzcd}
\implies
\mathcal{T}(M, T^iA) \xrightarrow{T^ia_*} \mathcal{T}(M, T^iB) \xrightarrow{T^ib_*} \mathcal{T}(M, T^iC)
$$

Observe that it is enough to prove that these types of diagrams are exact, as the other diagrams can be obtained by the rotation axiom, thus reducing it to same case.

The goal is then to prove that $Im(T^ia_*) = Ker(T^ib_*)$. Since $ba = 0$ it follows that $Im(T^ia_*) \subseteq Ker(T^ib_*)$. Assume that $f : Ker(T^ib_*)$, that is $f : M \to T^iB$ such that $b_*(f) = 0$. The current goal is to show that f factors through T^iA, as this means that $Ker(T^ib_*) \subseteq Im(T^ia_*)$. Note that since T is a translation, it is necessarily a right adjoint to the inverse translation; thus $\mathcal{T}(M, T^iB) \simeq \mathcal{T}(T^{-i}M, B)$, and by this assertion it suffices to assume that $f : T^{-i}M \to B$ such that $b \circ f = 0$. By TR1 and TR2 there exists triangles $(T^{-i}M, 0, T^{-i+1}M, 0, 0, -T^{-i+1}id)$ and $(B, C, TA, b, c, -Ta)$.

$$
\begin{tikzcd}
T^{-i}M \arrow[r, "0"] \arrow[d, "f"] & 0 \arrow[r, "0"] \arrow[d, "0"] & T^{-i+1}M \arrow[r, "-T^{-i+1}id"] \arrow[d, dashed, "g"] & T^{-i+1}M \arrow[d, "Tf"] \\
B \arrow[r, "b"] & C \arrow[r, "c"] & TA \arrow[r, "-Ta"] & TB
\end{tikzcd}
$$

The left square commutes by the assumption, thus the morphism g exist by TR3, such that $-Ta \circ h = -Tf \circ T^{-i+1}id = -Tf \implies Ta \circ h = Tf$, thus $f = a \circ T^{-1}h$ asserting that f factors through A. $\square$

Lemma 2.3. *Let $(\phi_A, \phi_B, \phi_C) : (A, B, C, a, b, c) \to (A', B', C', a', b', c')$ be a morphism of triangles. If 2 of the maps are isomorphisms, then the last one is an isomorphism as well.*

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\phi_A", "\wr"'] & B \arrow[r, "b"] \arrow[d, "\phi_B", "\wr"'] & C \arrow[r, "c"] \arrow[d, dashed, "\phi_C", "\wr"'] & TA \arrow[d, "T\phi_A", "\wr"'] \\
A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & TA'
\end{tikzcd}$$

Proof. Without loss of generality, assume that ϕ_A and ϕ_B are the isomorphisms. This can be done as the rotation axiom reduce the other cases to this case. Then we have the following diagram:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\phi_A", "\wr"'] & B \arrow[r, "b"] \arrow[d, "\phi_B", "\wr"'] & C \arrow[r, "c"] \arrow[d, "\phi_C"] & TA \arrow[d, "T\phi_A", "\wr"'] \\
A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & TA'
\end{tikzcd}$$

By applying the functor $\mathcal{T}(C', \_)$ we get the following diagram in Ab:

$$\begin{tikzcd}
\mathcal{T}(C', A) \arrow[r, "a_*"] \arrow[d, "(\phi_A)_*", "\wr"'] & \mathcal{T}(C', B) \arrow[r, "b_*"] \arrow[d, "(phi_B)_*", "\wr"'] & \mathcal{T}(C', C) \arrow[r, "c_*"] \arrow[d, "(\phi_C)_*"] & \mathcal{T}(C', TA) \arrow[r, "Ta_*"] \arrow[d, "(\phi_T A)_*", "\wr"'] & \mathcal{T}(C', TB) \arrow[d, "(T\phi_B)_*", "\wr"'] \\
\mathcal{T}(C', A') \arrow[r, "a'_*"] & \mathcal{T}(C', B') \arrow[r, "b'_*"] & \mathcal{T}(C', C') \arrow[r, "c'_*"] & \mathcal{T}(C', TA') \arrow[r, "Ta_*"] & \mathcal{T}(C', TB)
\end{tikzcd}$$

By the 5-lemma, we get that $(\phi_C)_*$ is an isomorphisms, i.e. $(\phi_C)_*$ is both mono and epi. Thus for some unique s in $\mathcal{T}(C', C)$, $\phi_{C*}(s) = id_{C'}$.

By applying the functor $\mathcal{T}(\_, C)$ we get the diagram:

$$\begin{tikzcd}
\mathcal{T}(A, C) & \mathcal{T}(B, C) \arrow[l, "a^*"'] & \mathcal{T}(C, C) \arrow[l, "b^*"'] & \mathcal{T}(TA, C) \arrow[l, "c^*"'] & \mathcal{T}(TB, C) \arrow[l, "Ta^*"'] \\
\mathcal{T}(A', C) \arrow[u, "(\phi_A)^*", "\wr"'] & \mathcal{T}(B, C) \arrow[l, "a'^*"] \arrow[u, "(\phi_B)^*", "\wr"'] & \mathcal{T}(C', C) \arrow[l, "b'^*"] \arrow[u, "(\phi_C)^*"] & \mathcal{T}(TA', C) \arrow[l, "c'^*"] \arrow[u, "(\phi_T A)^*", "\wr"'] & \mathcal{T}(TB', C) \arrow[l, "Ta'^*"] \arrow[u, "(\phi_T B)^*", "\wr"']
\end{tikzcd}$$

By the 5-lemma, we get that $(\phi_C)^*$ is an isomorphisms. By the same argument $id_C = s' \circ \phi_C$ for some unique s'. ϕ_C is both split mono and split epi, which means it is an isomorphism. $\square$

Corollary 2.3.1. $(A, B, 0, a, 0, 0)$ *is a triangle if and only if a is an isomorphism.*

Proof. Assume that a is an isomorphism. Then it is seen that $(a, id_B, 0)$ is an isomorphism of triangles.

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "a", "\wr"'] & B \arrow[r, "0"] \arrow[d, "id_B", "\wr"'] & 0 \arrow[r, "0"] \arrow[d, "0", "\wr"'] & TA \arrow[d, "Ta", "\wr"'] \\
B \arrow[r, "id_B"] & B \arrow[r, "0"] & 0 \arrow[r, "0"] & TB
\end{tikzcd}$$

Converesly, assume that $(A, B, 0, a, 0, 0)$ is a triangle. Then the same diagram as above can be constructed, and by the 2 out of 3 property, a has to be an isomorphism. $\square$

Lemma 2.4. *For a triangle (A, B, C, a, b, c) the following are equivalent:*

$$\begin{tikzcd}
A \arrow[dr, "a"] & \\
& B \arrow[dl, "b"] \\
C \arrow[uu, "c", "T"'] &
\end{tikzcd}$$

- *a is split mono*
- *b is split epi*
- *$c = 0$*

Proof. The proof has two parts. First assume that a is split mono, and prove that b is split epi, and $c = 0$. By duality, it is then known that b being split epi implies that a is split mono and $c = 0$. The final part is to assume that $c = 0$, and prove either a is split mono or b is split epi.

Assume that a is split mono, then there exist an a^{-1} such that $id_A = a^{-1}a$. Let $M : \mathcal{T}$ be any object, then we can make a long exact sequence:

$$\begin{tikzcd}
\mathcal{T}(M, T^{-1}C) \arrow[r, "T^{-1}c_*"] & \mathcal{T}(M, A) \arrow[r, bend left, "a_*"] & \mathcal{T}(M, B) \arrow[l, dashed, bend left, "a_*^{-1}"] \arrow[r, "b_*"] & \mathcal{T}(M, C) \arrow[r, "c_*"] & \mathcal{T}(M, TA)
\end{tikzcd}$$

By assumption a_* is split mono, thus $T^{-1}c_* = 0$ and in particular $c = 0$. This implies that b_* is epi, making a split short exact sequence.

$$\begin{tikzcd}
0 \arrow[r, "0"] & \mathcal{T}(M, A) \arrow[r, bend left, "a_*"] & \mathcal{T}(M, B) \arrow[l, dashed, bend left, "a_*^{-1}"] \arrow[r, bend left, "b_*"] & \mathcal{T}(M, C) \arrow[l, dashed, bend left, "b_*^{-1}"] \arrow[r, "0"] & 0
\end{tikzcd}$$

This gives that b is split epi, completing the first part of the proof.

For the next part, assume that $c = 0$; then we can construct the following triangles.

$$
(1)\quad
\begin{tikzcd}
A \arrow[dr, "a"] & \\
 & B \arrow[dl, "b"] \\
C \arrow[uu, "T", "0"'] &
\end{tikzcd}
\implies
\begin{tikzcd}
C \arrow[dr, "0"] & \\
 & TA \arrow[dl, "-Ta"] \\
TB \arrow[uu, "T", "-Tb"'] &
\end{tikzcd}
$$

$$
(2)\quad
\begin{tikzcd}
A \arrow[dr, "id_A"] & \\
 & A \arrow[dl, "0"] \\
0 \arrow[uu, "T", "0"'] &
\end{tikzcd}
\implies
\begin{tikzcd}
0 \arrow[dr, "0"] & \\
 & TA \arrow[dl, "-id_{TA}"] \\
TA \arrow[uu, "T", "0"'] &
\end{tikzcd}
$$

(1) is constructed by applying TR2 twice, while (2) is constructed with TR1 and TR2 twice. Observe that there is a commutative square between the triangles, allowing for TR3 to make a morphism of triangles.

$$
\begin{tikzcd}
C \arrow[r, "0"] \arrow[d, "0"] & TA \arrow[r, "-Ta"] \arrow[d, equal, "id_{TA}"] & TB \arrow[r, "-Tb"] \arrow[d, dashed, "Ta^{-1}"] & TC \arrow[d, "0"] \\
0 \arrow[r, "0"] & TA \arrow[r, equal, "-id_{TA}"] & TA \arrow[r, "0"] & 0
\end{tikzcd}
$$

Thus $T(a^{-1}a) = id_{TA} = T(id_A) \implies id_A = a^{-1}a$, making a split mono. $\square$

Lemma 2.5. *Given two triangles (A, B, C, a, b, c) and (A', B', C', a', b', c') the following are equivalent:*

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "f"] & B \arrow[r, "b"] \arrow[d, "g"] & C \arrow[r, "c"] \arrow[d, "h"] & TA \arrow[d, "Tf"] \\
A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & TA'
\end{tikzcd}
$$

1. *(f, g, h) is a morphism of triangles*
2. *$\exists g : B \to B'$ such that $b'ga = 0$*

Moreover, if $\mathcal{T}(A, T^{-1}C') \simeq 0$, then f and h are unique.

Proof. 1. $\implies$ 2. as the composition $ba = 0 = b'a'$, so assume 2. The existence of f and h is evident from the long exact sequence of the bottom triangle at the functor represented by A.

$$
\mathcal{T}(A, T^{-1}C') \xrightarrow{T^{-1}c'_*} \mathcal{T}(A, A') \xrightarrow{a'_*} \mathcal{T}(A, B') \xrightarrow{b'_*} \mathcal{T}(A, C')
$$

The morphism $ga : \mathcal{T}(A, B')$ such that $b'ga = b'_*(ga) = 0$, thus $ga : Ker(b'_*)$. By exactness $\exists f : \mathcal{T}(A, A')$ such that $a'f = ga$, and by TR3 $\exists h : C \to C'$ such that (f, g, h) is a morphism of triangles. Now assume that $\mathcal{T}(A, T^{-1}C') \simeq 0$. Exactness determines that a'_* is a monomorphism, and f is then unique. Since T is a translation, we have that

$\mathcal{T}(A, T^{-1}C') \simeq \mathcal{T}(TA, C')$. By using the functor $\mathcal{T}(\_, C')$ at the top triangle, we get that b^* is a monomorphism, and thus h is chosen uniquely. □

Lemma 2.6. *If (A, B, C, a, b, c) is a triangle, then $(T^{-1}C, A, B, T^{-1}c, a, b)$ is a triangle.*

Proof. By TR2 we can construct a triangle.

$$
\begin{tikzcd}
A \arrow[dr, "a"] & & & C \arrow[dr, "c"] & \\
& B \arrow[dl, "b"] & \Longrightarrow & & TA \arrow[dl, "Ta"] \\
C \arrow[uu, "c", "T"' near end] & & & TB \arrow[uu, "Tb", "T"' near end] &
\end{tikzcd}
$$

By TR1 we can create a triangle $(T^{-1}C, A, B', T^{-1}c, a', b')$, and then use TR3 to find a morphism.

$$
\begin{tikzcd}
C \arrow[r, "c"] \arrow[d, equal, "id_C"] & TA \arrow[r, "Ta'"] \arrow[d, equal, "id_{TA}"] & TB' \arrow[r, "Tb'"] \arrow[d, dashed, "h"] & TC \arrow[d, equal, "id_{TC}"] \\
C \arrow[r, "c"] & TA \arrow[r, "Ta"] & TB \arrow[r, "Tb"] & TC
\end{tikzcd}
$$

By the 2 out of 3 property it is seen that h is an isomorphism, so the triple $(id_{T^{-1}C}, id_A, T^{-1}h)$ is an isomorphism of sextuples, and by TR1, is an isomorphism of triangles, asserting that $(T^{-1}C, A, B, T^{-1}c, a, b)$ is in fact a triangle. □

Remark. The Octahedron axiom have not been used once in this section. This motivates the definition of a pre-triangulated category, that is a triangulation satisfying all axioms except TR4. Note that these properties also holds for pre-triangulated categories. However, the octahedron axiom is still important, since TR3 can be proved from the other 3 axioms. [Need sources here]

2.2 Triangulation of $\underline{\textbf{mod}}$-$K[x]/(x^n)$

In this section a quotient of the finitely genereated module categories of the K-algebras $K[x]/(x^n)$ will be studied and proved to be triangulated. As a first example it would be more instructive with a more canonical triangulated category, such as the homotopy category of chain complexes. For such type of categories the underlying triangulated structure would be more intuitive, and it is easier to see how it interplays with long exact sequences in an abelian category. Never the less, this class of categories will be more instructive in showing how flexible the notion of a triangulated category may be.

The more skilled reader may notice that the algebra $K[x]/(x^n)$ is self-injective and is of finite length. Therefore the category of finitely generated modules over $K[x]/(x^n)$ is a Frobenius category, and the stabilization admits a triangulation. What all of this means will be clarified in the next section, for now we will focus on the structure of this very specific category, and prove that it's stabilized module category has a triangulation.

When it comes to stabilization of module categories, there is a projective stabilization and an injective stabilization. They use the same method to construct a new category, but uses different classes of objects to form the quotient. We are however interested in the case where these two notions coincide, so we can talk about a stable category.

I don't really like how this section is written, it feels very clumsy, should I postpone the example? I think that would be a lot cleaner as the Frobenius category is the way we actually construct these categories. I AM POSTPONING THIS CHAPTER

Definition 2.5. The injective stable mod-$K[x]/(x^n)$ category is the quotient category of morphisms factoring over injective objects. This means that if I_1, I_2 is injective and $f, g : A \to B$ are morphisms factoring over I_1 and I_2 respectively, then f and g are related in the quotient. We will denote the category as $\underline{\text{mod}}$-$K[x]/(x^n)$.

The projective stable categories are defined in the same way, but with projective objects instead.

Remark. Cosyzygies are in fact a functor for injective stable categories, and syzygies are functors for the projective stable categories.

Write about the indecomposable objects in this category. Find the projective and injective modules, i.e. show that $K[x]/(x^n)$ is injective.

2.3 Localizations of Triangulated Categories

Verdier quotient goes here. And more stuff about these localizations, probably state Gabriel-Zisman and how this is often a good trick too prove that the localization actually exists.

2.4 Discussion of Triangulations

Maybe do some Yoneda-embedding into sheaves and use that to deduce how the triangulation are in fact a shadow of some other abelian category

3 Exact Categories

I can maybe write some of the history of the development of the idea of exact categories.

3.1 Definitions and First Properties

In this section we will focus on defining what an exact category is and the first elementary properties. We will prove the axiom dubbed as "the obscure axiom" and motivate that it is not as obscure as its name suggest. Some "short" variants of some homological diagram lemmas will also be proved.

To start with the exact categories we will first take a look towards the abelian ones first. Short exact sequences are of great interest, and they can be characterized with two morphisms $p : A \to B$ and $q : B \to C$ such that p is the kernel of q and q is the cokernel of p. This leads to the first definition.

Definition 3.1. A kernel-cokernel pair is a pair of maps (p, q) such that p is the kernel of q and q is the cokernel of p. A morphism of kernel-cokernel pairs (p, q) and (p', q') is a triple (f, g, h) such that the following diagram commutes. An isomorphisms is a triple in which each morphism is an isomorphism.

$$\begin{tikzcd}
A \arrow[r, "p"] \arrow[d, "f"] & B \arrow[r, "q"] \arrow[d, "g"] & C \arrow[d, "h"] \\
A' \arrow[r, "p'"] & B' \arrow[r, "q'"] & C'
\end{tikzcd}$$

Lemma 3.1. *Let (p, q) be a kernel-cokernel pair, then the image and coimage of p exists and are isomorphic. I.e. this diagram exists, such that the left square is a push-out and the right square is a pull-back:*

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[dr, "0"] & A \arrow[rr, "p"] \arrow[d] & & B \arrow[r, "q"] & C \\
& Coim(p) \arrow[rr, "iso"] & & Im(p) \arrow[u] \arrow[ur, "0"] &
\end{tikzcd}$$

Proof. Since (p, q) is a kernel-cokernel pair we have that the first square is bicartesian and the second square is a push-out.

$$\begin{tikzcd}
A \arrow[r, "p"] \arrow[dr, "0"] & B \arrow[d, "q"] \\
& C
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r, "0"] \arrow[dr, "0"] & A \arrow[d, equal] \\
& A
\end{tikzcd}$$

Thus $Im(p) = Coim(p) = A$, asserting the isomorphism as the identity in the diagram.

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[dr, "0"'] & A \arrow[r, "p"] \arrow[d, equal] & B \arrow[r, "q"] & C \\
& A \arrow[r, equal] & A \arrow[u, "p"] \arrow[ur, "0"'] &
\end{tikzcd}$$

$\square$

Definition 3.2. An exact structure for an additive category $\mathcal{A}$ is a class $\mathcal{E}$ of kernel-cokernel pairs which are closed under isomorphisms. A pair $(p, q) : \mathcal{E}$ is called a conflation, here p is called an inflation and q is called a deflation. $(\mathcal{A}, \mathcal{E})$ is called exact when the following axioms holds:

- (QE0) $\forall A : \mathcal{A}\ id_A$ is both an inflation and a deflation.
- (QE1) Both inflations and deflations are closed under composition.
- (QE2) The push-out of an inflation is an inflation.
- (QE2^{op}) The pull-back of a deflation is a deflation.

An exact category is the additive category $\mathcal{A}$ together with an exact structure $\mathcal{E}$.

Remark. When writing diagrams we use decorated arrows to indicate that a morphism is either an inflation or a deflation. A tail with a circle means inflation: $A \overset{\circ}{\rightarrowtail} B$. Double heads with a circle means deflation: $A \overset{\circ}{\twoheadrightarrow} B$. We can now rewrite the (QE2) axioms as:

$$\begin{tikzcd}
A \arrow[r, tail, "\circ" marking] \arrow[d] & B \arrow[d] & A \arrow[r, two heads, "\circ" marking] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & B \arrow[d] \\
C \arrow[r, tail, "\circ" marking] \arrow[ur, phantom, very near end, "\lrcorner"] & D & C \arrow[r, two heads, "\circ" marking] & D
\end{tikzcd}$$

Remark. Inflations are also called admissable monomorphisms, deflations are also called admissable epimorphisms and conflations are also called short exact sequences.

Remark. The axioms for an exact category is made in such a way that $\mathcal{E}$ is an exact structure for $\mathcal{A}$ if and only if $\mathcal{E}^{op}$ is an exact structure for $\mathcal{A}^{op}$.

Lemma 3.2. *The map $0 : 0 \to A$ is an inflation. Dually, the map $0 : A \to 0$ is a deflation.*

Proof. Consider the diagram $0 \overset{0}{\rightarrowtail} A \overset{id_A}{\twoheadrightarrow} A$. The left morphism is the kernel of the right morphism making a kernel-cokernel pair $(0, id_A)$. The identity id_A is assumed to be a deflation, implying that the pair is a conflation. $\square$

Remark. It can be seen that isomorphisms are deflations. Let $f : A \to B$ be an isomorphism, then there are two kernel-cokernel pairs: $(0, id_A)$ and $(0, f)$. Between these there is an isomorphism which is the triple $(0, id_A, f^{-1})$. As the conflations are closed under isomorphism, $(0, f)$ is a conflation, making f into a deflation. By dualizing this argument f is also an inflation.

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[d, "0"] & A \arrow[r, "f"] \arrow[d, "id_A"] & B \arrow[d, "f^{-1}"] \\
0 \arrow[r, tail, "0"] & A \arrow[r, two heads, "id_A"] & A
\end{tikzcd}$$

Corollary 3.2.1. *A kernel-cokernel pair (i, p) found as a split short-exact sequence (1) is a conflation.*

$$\text{(1)} \quad A \overset{i}{\rightarrowtail} A \oplus B \overset{p}{\twoheadrightarrow} B$$

Proof. In a category with an initial object the coproduct can be thought of as the push-out with the initial in the upper right corner. This can be assembled into push-out (1). By the lemma the zero morphisms are inflations, asserting that i and i' are inflations by (QE2). Thus there are conflations (i, p) and (i', p').

$$\text{(1)} \quad \begin{tikzcd}
0 \arrow[r, tail, "0"] \arrow[d, tail, "0"] & A \arrow[d, tail, "i"] \\
B \arrow[r, tail, "i'"] & A \oplus B
\end{tikzcd}$$

□

Corollary 3.2.2. *The direct sum of conflations is a conflation. I.e. there is a diagram:*

$$A \overset{i}{\rightarrowtail} B \overset{p}{\twoheadrightarrow} C \quad \& \quad A' \overset{i'}{\rightarrowtail} B' \overset{p'}{\twoheadrightarrow} C'$$
$$\Downarrow$$
$$A \oplus A' \overset{i \oplus i'}{\rightarrowtail} B \oplus B' \overset{p \oplus p'}{\twoheadrightarrow} C \oplus C'$$

Proof. We start with only considering the conflation (i, p). $\forall D$ there is a conflation $(i \oplus id_D, p \oplus 0)$, drawn as the diagram.

$$A \oplus D \overset{\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}}{\rightarrowtail} B \oplus D \overset{\begin{pmatrix} p & 0 \end{pmatrix}}{\twoheadrightarrow} C$$

As kernels and cokernels are preserved by direct sums, this pair is in fact a kernel-cokernel pair. The epimorphism is a deflation as it can be factored by the deflations:

$$B \oplus D \xrightarrow{\begin{pmatrix}1 & 0\end{pmatrix}} B \xrightarrow{p} C$$

Thus it is seen that $(i \oplus id_D, p \oplus 0)$ is a conflation, and dually $(i \oplus 0, p \oplus id_D)$ is also a conflation. To finish off the proof it is seen that the morphism $i \oplus i'$ factors as $i \oplus id_{A'} \circ id_A \oplus i'$ asserting that it is an inflation by (QE1). By the dual argument we then get that the direct sum of conflations is a conflation. ☐

Definition 3.3. A square is bicartesian if it is both a pull-back and a push-out.
$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner" near start, "\lrcorner" near end] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

Proposition 3.3. *The following statements are equivalent:*

1. *The square (1) is a push-out.*
2. *The sequence (2) is a conflation.*
3. *The square (1) is bicartesian.*
4. *The square (1) is a part of the commutative diagram (3)*

$$(1)\ \begin{tikzcd} A \arrow[r, rightarrowtail, "i"] \arrow[d, "f"] & B \arrow[d, "g"] \\ C \arrow[r, rightarrowtail, "j"] & D \end{tikzcd} \quad (2)\ A \xrightarrow{\begin{pmatrix} i \\ -f \end{pmatrix}} B \oplus C \xrightarrow{\begin{pmatrix} g & j \end{pmatrix}} D \quad (3)\ \begin{tikzcd} A \arrow[r, rightarrowtail, "i"] \arrow[d, "f"] & B \arrow[r, twoheadrightarrow, "p"] \arrow[d, "g"] & E \arrow[d, equal] \\ C \arrow[r, rightarrowtail, "j"] & D \arrow[r, twoheadrightarrow, "q"] & E \end{tikzcd}$$

Before the proof for this proposition will be presented a useful lemma will be proved first.

Lemma 3.4. *Assume that there is a commutative square (1) and an associatied sequence (2). (1) is a push-out square if and only if* $\begin{pmatrix} p & q \end{pmatrix}$ *is the cokernel of the morphism* $\begin{pmatrix} i \\ -j \end{pmatrix}$

$$(1)\ \begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"] & B \arrow[d, "p"] \\ C \arrow[r, "q"] & D \end{tikzcd} \quad (2)\ A \xrightarrow{\begin{pmatrix} i \\ -j \end{pmatrix}} B \oplus C \xrightarrow{\begin{pmatrix} p & q \end{pmatrix}} D$$

Proof. For any test object T and two maps $t_1 : B \to T$ and $t_2 : C \to T$, we can construct the diagrams for the universal properties of both the cokernel and the push-out. It is seen that these diagrams are equivalent, proving the lemma.

$$
\begin{tikzcd}
A \arrow[r, "\begin{pmatrix} i \\ -f \end{pmatrix}"] \arrow[dr, "0"'] & B \oplus C \arrow[d, two heads, "\begin{pmatrix} g & j \end{pmatrix}"] \arrow[dr, bend left, "\begin{pmatrix} t_1 & t_2 \end{pmatrix}"] & \\
& D \arrow[r, dashed, "t'"] & T
\end{tikzcd}
\Leftrightarrow
\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[d, "g"] \arrow[ddr, bend left, "t_1"] & \\
C \arrow[r, "j"] \arrow[drr, bend right, "t_2"'] & D \arrow[dr, dashed, "t'"] & \\
& & T
\end{tikzcd}
$$

$\square$

Corollary 3.4.1. *For the same diagrams (1) and (2) as above the dual statement is also true. (1) is a pull-back square if and only if* $\begin{pmatrix} i \\ -j \end{pmatrix}$ *is kernel of the morphism* $\begin{pmatrix} p & q \end{pmatrix}$. *Thus we have that (1) is bicartesian (i.e. both a pull-back and a push-out) if and only if the morphisms make a kernel-cokernel pair.*

Proof. **of Proposition 3.3** 1. $\Rightarrow$ 2.: By the previous lemma we know that $\begin{pmatrix} g & j \end{pmatrix}$ is the cokernel of $\begin{pmatrix} i \\ -j \end{pmatrix}$. Thus proving that $\begin{pmatrix} i \\ -j \end{pmatrix}$ is an inflation, will prove that the pair is a conflation.

Observe that the morphism $\begin{pmatrix} i \\ -f \end{pmatrix}$ can be factored through the sequence.

$$
\begin{tikzcd}
A \arrow[r, tail, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & A \oplus C \arrow[r, tail, two heads, "\begin{pmatrix} 1 & 0 \\ -f & 1 \end{pmatrix}", "\cong"'] & A \oplus C \arrow[r, tail, "\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}"] & B \oplus C
\end{tikzcd}
$$

By corollary 3.2.1 the first map is an inflation, as the second map is an isomorphism it is also an inflation and the last map is the direct sum of two inflations. Thus the composite of all these maps surely is an inflation by (QE1), proving the first implication.

2. $\Rightarrow$ 3.: This follows from corollary 3.4.1.

3 $\Rightarrow$ 1.: This is by definition.

1. $\Rightarrow$ 4.: Let p be the cokernel of i, then we can form the diagram below.

$$
\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[d, "g"] \arrow[ddr, bend left, two heads, "p"] & \\
C \arrow[r, tail, "j"] \arrow[drr, bend right, "0"'] & D \arrow[dr, dashed, two heads, "p'"] & \\
& & T
\end{tikzcd}
$$

p' is an epimorphism as $p = p'g$ is epi. To prove that p' is the cokernel of j we let T' be another test object with a map $t' : D \to T'$ such that $0 = t'j$. By doing some diagram chases we have that $0 = t'jf = t'gi$, thus by the universal property of p the morphism $t'g$ factors through T such that $t'g = tp$ for some unique t. By rearranging we have that $t'g = tp'g$, and with some push-out magic $t' = tp'$**I MUST FIGURE THIS OUT**. t is also unique, for if there exist another map h such that $tp' = hp'$, then $t = h$ as p' is epic. The unique existence proves the universal property, and p' is the cokernel of j.

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[r, two heads, "p"] \arrow[d, "g"] & T \arrow[d, equal] \\
C \arrow[r, tail, "j"] \arrow[rrd, bend right, "0"] & D \arrow[r, dashed, two heads, "p'"] \arrow[rd, "t'"] \arrow[ul, phantom, "\lrcorner"] & T \arrow[d, dashed, "t"] \\
& & T'
\end{tikzcd}$$

4. $\Rightarrow$ 2.: We start by taking the pullback of p and p' using $(QE2^{op})$, and determines the diagram with the dual statement of the last implication.

$$\begin{tikzcd}
& A \arrow[r, equal] \arrow[d, tail, "i'"] & A \arrow[d, tail, "i"] \\
C \arrow[r, tail, "j'"] \arrow[d, equal] & E \arrow[r, two heads, "q'"] \arrow[d, two heads, "q"] \arrow[dr, phantom, "\ulcorner", near start] & B \arrow[d, two heads, "p"] \arrow[dl, dashed, "g"'] \\
C \arrow[r, tail, "j"] & D \arrow[r, two heads, "p'"] & T
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
B \arrow[dr, dashed, "k"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, "g"'] & & \\
& E \arrow[r, two heads, "q'"] \arrow[d, two heads, "q"] & B \arrow[d, two heads, "p"] \\
& D \arrow[r, two heads, "p'"] & T
\end{tikzcd}$$

From these diagrams we can deduce that q' is a split-epimorphism. The composite $q'(id_E - kq') = q' - q'kq' = q' - q' = 0$ as q' is split-epi, so $(id_E - kq')$ factors over j' as in the following diagram.

$$\begin{tikzcd}
E \arrow[dr, dashed, "l"] \arrow[drr, bend left, "id_E - kq'"] & & \\
& C \arrow[r, tail, "j'"] \arrow[dr, "0"] & E \arrow[d, two heads, "q'"] \\
& & B
\end{tikzcd}$$

From these diagrams we can extract three different equations:

- $0 = k - k = k - kq'k = (id_E - kq')k = j'lk \implies lk = 0$ as j' is monic
- $j'lj' = (id_E - kq')j' = j' \implies lj' = id_C$ as j' is monic

- $jli' = (qj')li' = q(id_E - kq')i' = -(qk)(q'i') = -gi = -jf \implies li' = -f$ as j is monic

The morphisms $\begin{pmatrix} k & j' \end{pmatrix}$ and $\begin{pmatrix} q' \\ l \end{pmatrix}$ are inverses:

- $\begin{pmatrix} k & j' \end{pmatrix} \begin{pmatrix} q' \\ l \end{pmatrix} = kq' + j'l = kq' + id_E - kq' = id_E$
- $\begin{pmatrix} q' \\ l \end{pmatrix} \begin{pmatrix} k & j' \end{pmatrix} = \begin{pmatrix} q'k & q'j' \\ lk & lj' \end{pmatrix} = \begin{pmatrix} id_B & 0 \\ 0 & id_C \end{pmatrix}$

Thus we have an isomorphism of kernel-cokernel pairs $(id_A, , \begin{pmatrix} q' \\ l \end{pmatrix} \begin{pmatrix} k & j' \end{pmatrix})$,
from $(\begin{pmatrix} i \\ -f \end{pmatrix}, \begin{pmatrix} f' & i' \end{pmatrix})$ to (i', q). This proves 2. ☐

Corollary 3.4.2. *The pull-back of an inflation along a deflation is an inflation.*

$$\begin{tikzcd}
A \arrow[r, tail, "i'"] \arrow[d, two heads, "e'"'] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d, two heads, "e"] \\
C \arrow[r, tail, "i"] & D
\end{tikzcd}$$

Proof. By (QE2) this pullback exists, as there is a deflation in the pullback. Extend the diagram by adding the deflation of the inflation in the following manner.

$$\begin{tikzcd}
 & T \arrow[dl, dashed, bend right, "t'"'] \arrow[d, "t"] \arrow[dr, bend left, "0"] \arrow[ddl, dashed, bend left, "t''"] & \\
A \arrow[r, tail, "i'"] \arrow[d, two heads, "e'"'] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[r, two heads, "pe"] \arrow[d, two heads, "e"] & C \arrow[d, equal] \\
C \arrow[r, tail, "i"] & D \arrow[r, two heads, "p"] & C
\end{tikzcd}$$

pe is a deflation by (QE1), and i' is a mono as a limit of a mono is a mono. Our goal is to prove that i' is the kernel of pe. Let T be a test object such that $pet = 0$. Then we have that te factorizes over i, such that we can apply the universal property of the pullback to factorize te over i'. Uniqueness of t' is achieved with i' being monic. This proves that (i', pe) is a conflation. ☐

Theorem 3.5. *Assume that $i : A \to B$ is a morphism with a cokernel. If there is a morphism $j : B \to C$ such that ji is an inflation, then i is an inflation.*

Proof. ☐

Lemma 3.6. *Let (p,q) be the conflation $A \overset{p}{\rightarrowtail} B \overset{q}{\twoheadrightarrow} C$, (p',q') be the conflation $A' \overset{p'}{\rightarrowtail} B' \overset{q'}{\twoheadrightarrow} C'$. A morphism of the conflations $(f,g,h) : (p,q) \to (p',q')$ factors through the conflation $A \rightarrowtail D \twoheadrightarrow C'$*

Proof. Let $p : B \to D$ be the cokernel of i. We start the proof with forming the push-out of i and ji.

$$\begin{tikzcd}
A \arrow[r, tail, "ji"] \arrow[d, "i"] & C \arrow[d] \\
B \arrow[r, tail] \arrow[ur, phantom, "\lrcorner" very near end] & E
\end{tikzcd}$$

By proposition 3.3 we get that $\begin{pmatrix} i \\ ji \end{pmatrix}$ is an inflation. $\begin{pmatrix} i \\ 0 \end{pmatrix} = \begin{pmatrix} id_B & 0 \\ -j & id_C \end{pmatrix} \begin{pmatrix} i \\ ji \end{pmatrix}$, this is an inflation by (QE1) as the 2x2 matrix is an isomorphism. Observer that the cokernel of this map is $\begin{pmatrix} k & 0 \\ 0 & id_C \end{pmatrix}$. Our final trich will be to show that there is a pullback square, and then use (QE2) to say that k is a deflation.

$$\begin{tikzcd}
T \arrow[dr, dashed, "t"] \arrow[drr, bend left, "t_1"] \arrow[ddr, bend right, "\begin{pmatrix} t_2 \\ 0 \end{pmatrix}"'] & & \\
& B \arrow[r, "k"] \arrow[d, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & D \arrow[d, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] \\
& B \oplus C \arrow[r, two heads, "\begin{pmatrix} k & 0 \\ 0 & id_C \end{pmatrix}"'] & D \oplus C
\end{tikzcd}$$

Note that setting $t = t_2$ we get the universal property. This is well defined as $kt_2 = t_1$ by assumption, thus $kt = t_1$. This is what we need to prove that the square is a pullback, proving the obscure axiom. $\square$

Remark. Write a bit about the dual of the obscure axiom.

Corollary 3.6.1. *The short five lemma*

Proof. $\square$

Lemma 3.7. *Noethers isomorphism lemma*

Proof. $\square$

3.2 The Frobenis Category

3.3 Self-injective Algebras

3.4 The Homotopy Category

4 The Derived Category

4.1 Admissable Morphisms

4.2 Homology and Long Exact Sequences

4.3 The Derived Category

4.4 If time, derived functors as well

5 Auslander-Reiten Triangles

5.1 Krull-Schmidt Categories

5.2 Definition and First Properties

5.3 Description of Derived Categories

Appendix A: Category theory

5.4 Quotient Categories

5.5 Additive and Abelian Categories

5.6 Freyd-Mitchell Embedding???