Norwegian University of Science and Technology

TRIANGULATED CATEGORIES

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Outline

Introduction

Triangulated Categories
- The axioms
- Homological functors
- Subcategories and Verdier Quotient

Frobenius Categories
- Exact categories
- Stable Frobenius categories

Constructions
- Homotopy categories
- Derived categories

Introduction

- Why study triangulated categories?
- Stable Frobenius Categories vs. Stable Homotopy Categories

Candidate Triangles

Assume that:

- $\mathcal{T}$ an additive category
- $\Sigma_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ an additive autoequivalence

Definition (Candidate triangle)

Candidate triangle: $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} \Sigma_{\mathcal{T}} A$

Morphism:
$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[r, "b"] \arrow[d, "\phi_B"] & C \arrow[r, "c"] \arrow[d, "\phi_C"] & \Sigma_{\mathcal{T}} A \arrow[d, "\Sigma_{\mathcal{T}} \phi_A"] \\
A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & \Sigma_{\mathcal{T}} A'
\end{tikzcd}$$

Triangulation axioms; I

A triangulated category is a triple $(\mathcal{T}, \Sigma_{\mathcal{T}}, \Delta_{\mathcal{T}})$ where $\Delta_{\mathcal{T}}$ is a triangulation.

Definition (Triangulation)

- $\Delta_{\mathcal{T}}$ class of candidate triangles
- Element of $\Delta_{\mathcal{T}}$ is called triangle
- $\Delta_{\mathcal{T}}$ is a triangulation if it satisfies the following axioms:

TR1 Bookkeeping axiom

1. A candidate triangle isomorphic to a triangle is a triangle
2. For every morphism $a : A \to B$ there is a triangle

$$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} \Sigma_{\mathcal{T}} A$$

3. For every object $A : \mathcal{T}$ there is a triangle

$$A \xrightarrow{id_A} A \xrightarrow{0} 0 \xrightarrow{0} \Sigma_{\mathcal{T}} A$$

Triangulation axioms; II

TR2 Rotation axiom

Given a triangle

$$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} \Sigma_{\mathcal{T}} A$$

there are triangles

$$B \xrightarrow{b} C \xrightarrow{c} \Sigma_{\mathcal{T}} A \xrightarrow{-\Sigma_{\mathcal{T}} a} \Sigma_{\mathcal{T}} B$$

$$\Sigma_{\mathcal{T}}^{-1} C \xrightarrow{-\Sigma_{\mathcal{T}}^{-1} c} A \xrightarrow{a} B \xrightarrow{b} C$$

TR3 Morphism axiom

Two triangles and a square of morphisms between the triangles may be completed to a triangle morphism.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[d, "\phi_B"] \\ A' \arrow[r, "a'"] & B' \end{tikzcd} \Longrightarrow \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[r, "b"] \arrow[d, "\phi_B"] & C \arrow[r, "c"] \arrow[d, dashed, "\phi_C"] & \Sigma_{\mathcal{T}} A \arrow[d, "\Sigma_{\mathcal{T}} \phi_A"] \\ A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C \arrow[r, "c'"] & \Sigma_{\mathcal{T}} A' \end{tikzcd}$$

Triangulation axioms; III

TR4 <u>Octahedron axiom</u>

Given three triangles

(1) $A \xrightarrow{a} B \xrightarrow{x} C' \xrightarrow{x'} \Sigma_{\mathcal{T}} A$

(2) $B \xrightarrow{b} C \xrightarrow{y} A' \xrightarrow{y'} \Sigma_{\mathcal{T}} B$

(3) $A \xrightarrow{b \circ a} C \xrightarrow{z} B' \xrightarrow{z'} \Sigma_{\mathcal{T}} A$

such that there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[dr, "b \circ a"'] & B \arrow[d, "b"] \\ & C \end{tikzcd}$$

Triangulation axioms; III

TR4 <u>Octahedron axiom</u>

then there exists morphisms f and g making the third row a triangle.

$$
\begin{tikzcd}
\Sigma_{\mathcal{T}}^{-1}B' \arrow[r, "\Sigma_{\mathcal{T}}^{-1}z'"] \arrow[d, "\Sigma_{\mathcal{T}}^{-1}g"] & A \arrow[r, equal, "id_A"] \arrow[d, "a"] & A \arrow[d, "b\circ a"] & & \\
\Sigma_{\mathcal{T}}^{-1}A' \arrow[r, "\Sigma_{\mathcal{T}}^{-1}y'"] & B \arrow[r, "b"] \arrow[d, "x"] & C \arrow[r, "y"] \arrow[d, "z"] & A' \arrow[r, "y'"] \arrow[d, equal, "id_{A'}"] & \Sigma_{\mathcal{T}}B \arrow[d, "\Sigma_{\mathcal{T}}x"] \\
& C' \arrow[r, dashed, "f"] \arrow[d, "x'"] & B' \arrow[r, dashed, "g"] \arrow[d, "z'"] & A' \arrow[r, "\Sigma_{\mathcal{T}}x\circ y'"] & \Sigma_{\mathcal{T}}C' \\
& \Sigma_{\mathcal{T}}A \arrow[r, equal, "id_{\Sigma_{\mathcal{T}}A}"] & \Sigma_{\mathcal{T}}A & &
\end{tikzcd}
$$

Triangulation axioms; III

TR4 Octahedron axiom

$$(1)\qquad \begin{tikzcd} A \arrow[dr, "a"] & \\ & B \arrow[dl, "x"] \\ C' \arrow[uu, "x'", "\Sigma_{\mathcal{T}}"' pos=0.9] & \end{tikzcd}$$

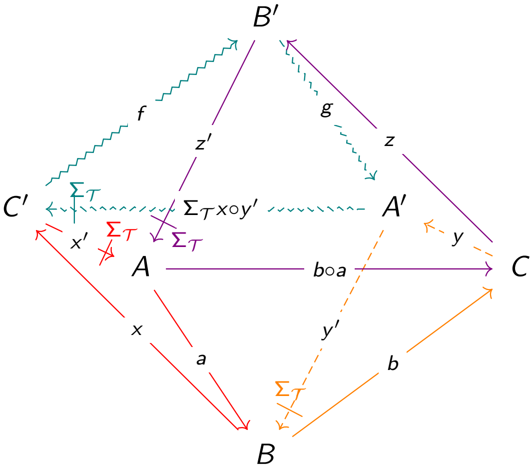
$$(2)\qquad \begin{tikzcd} B \arrow[dr, "b"] & \\ & C \arrow[dl, "y"] \\ A' \arrow[uu, "y'", "\Sigma_{\mathcal{T}}"' pos=0.9] & \end{tikzcd}$$

$$(3)\qquad \begin{tikzcd} A \arrow[dr, "b\circ a"] & \\ & C \arrow[dl, "z"] \\ B' \arrow[uu, "z'", "\Sigma_{\mathcal{T}}"' pos=0.9] & \end{tikzcd}$$

Triangulation axioms; III

TR4 <u>Octahedron axiom</u>

There exist morphisms $f : C' \to B'$ and $g : B' \to A'$, and the squiggly teal back face is a triangle.



Functors

Definition (Triangulated functor)

A functor $F : \mathcal{T} \to \mathcal{S}$ between triangulated categories is called triangulated if:

- $\phi : F \circ \Sigma_{\mathcal{T}} \implies \Sigma_{\mathcal{S}} \circ F$ is a natural isomorphism
- $F(\Delta_{\mathcal{T}}) \subseteq \Delta_{\mathcal{S}}$

Functors

Definition (Triangulated functor)

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- $\phi : F \circ \Sigma_{\mathcal{T}} \implies \Sigma_{\mathcal{S}} \circ F$ is a natural isomorphism
- $F(\Delta_{\mathcal{T}}) \subseteq \Delta_{\mathcal{S}}$

Definition (Homological functor)

A covariant functor $H : \mathcal{T} \to \mathcal{A}$ from a triangulated category and an abelian category is called homological if it sends triangles to long exact sequences.

$$\begin{tikzcd} A \arrow[dr, "a"] & \\ & B \arrow[dl, "b"] \\ C \arrow[uu, "c", "\Sigma_{\mathcal{T}}"'] & \end{tikzcd} \implies \begin{array}{l} \cdots \longrightarrow H(\Sigma^i_{\mathcal{T}} A) \xrightarrow{H(\Sigma^i_{\mathcal{T}} a)} H(\Sigma^i_{\mathcal{T}} B) \xrightarrow{H(\Sigma^i_{\mathcal{T}} b)} H(\Sigma^i_{\mathcal{T}} C) \xrightarrow{H(\Sigma^i_{\mathcal{T}} c)} \\ H(\Sigma^{i+1}_{\mathcal{T}} A) \xrightarrow{H(\Sigma^{i+1}_{\mathcal{T}} a)} H(\Sigma^{i+1}_{\mathcal{T}} B) \xrightarrow{H(\Sigma^{i+1}_{\mathcal{T}} b)} H(\Sigma^{i+1}_{\mathcal{T}} C) \longrightarrow \cdots \end{array}$$

Functors

Definition (Triangulated functor)

A functor $F : \mathcal{T} \to \mathcal{S}$ between triangulated categories is called triangulated if:

- $\phi : F \circ \Sigma_{\mathcal{T}} \implies \Sigma_{\mathcal{S}} \circ F$ is a natural isomorphism
- $F(\Delta_{\mathcal{T}}) \subseteq \Delta_{\mathcal{S}}$

Definition (Cohomological functor)

A contravariant functor $H : \mathcal{T} \to \mathcal{A}$ from a triangulated category and an abelian category is called cohomological if it sends triangles to long exact sequences.

$$
\begin{tikzcd}
A \arrow[dr, "a"] & \\
& B \arrow[dl, "b"] \\
C \arrow[uu, "c", "\Sigma_{\mathcal{T}}"'] &
\end{tikzcd}
\implies
\begin{tikzcd}
\dots & H(\Sigma_{\mathcal{T}}^{i-1}A) \arrow[l] & H(\Sigma_{\mathcal{T}}^{i-1}B) \arrow[l, "H(\Sigma_{\mathcal{T}}^{i-1}a)"'] & H(\Sigma_{\mathcal{T}}^{i-1}C) \arrow[l, "H(\Sigma_{\mathcal{T}}^{i-1}b)"'] \\
& H(\Sigma_{\mathcal{T}}^{i}A) \arrow[urr, "H(\Sigma_{\mathcal{T}}^{i}c)"] & H(\Sigma_{\mathcal{T}}^{i}B) \arrow[l, "H(\Sigma_{\mathcal{T}}^{i}a)"'] & H(\Sigma_{\mathcal{T}}^{i}C) \arrow[l, "H(\Sigma_{\mathcal{T}}^{i}b)"'] & \dots \arrow[l]
\end{tikzcd}
$$

Hom-functor

Lemma (Hom is (co)homological)

For any $M : \mathcal{T}$

- $\mathcal{T}(M, \_) : \mathcal{T} \to \mathcal{A}$ *is a homological functor.*
- $\mathcal{T}(\_, M) : \mathcal{T} \to \mathcal{A}$ *is a cohomological functor*

Hom-functor

Lemma (Hom is (co)homological)

For any $M : \mathcal{T}$

- $\mathcal{T}(M, \_) : \mathcal{T} \to \mathcal{A}$ *is a homological functor.*
- $\mathcal{T}(\_, M) : \mathcal{T} \to \mathcal{A}$ *is a cohomological functor*

Lemma (2-out-of-3 property)

If 2-out-of-3 of the triangle morphism are isomorphism, the final one is as well.

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\phi_A", "\wr"'] & B \arrow[r, "b"] \arrow[d, "\phi_B", "\wr"'] & C \arrow[r, "c"] \arrow[d, dashed, "\phi_C", "\wr"'] & \Sigma_{\mathcal{T}} A \arrow[d, "\Sigma_{\mathcal{T}} \phi_A", "\wr"'] \\
A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & \Sigma_{\mathcal{T}} A'
\end{tikzcd}
$$

Localization; I

Definition (Localization)

Let S be a collection of morphisms in the category $\mathcal{C}$. The Localization of $\mathcal{C}$ on $\mathcal{S}$ is the category $\mathcal{C}[S^{-1}]$ together with a functor $q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ such that:

- $\forall s : S$ such that $q(s)$ is an isomorphism
- Any functor $F : \mathcal{C} \to \mathcal{D}$ such that for any $s : S$ such that $F(s)$ is an isomorphism, then F factors through q. That is to say that there is a natural isomorphism $\eta : F \to F' \circ q$ so that $\mathcal{C}[S^{-1}]$ is the universal category where morphisms in S are isomorphisms.

$$
\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "q"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{D} \\
& \mathcal{C}[S^{-1}] \arrow[ur, dashed, "F'"'] &
\end{tikzcd}
$$

Calculus of Fractions

Definition (Right multiplicative system)

A set S of morphisms in a category $\mathcal{C}$ is called right multiplicative if it satisfies the following conditions:

- S is closed under composition, and has every identity morphism.
- (Right Ore condition)

$$(1) \quad \begin{tikzcd} W \arrow[r, dashed, "f"] \arrow[d, dashed, "s"] & X \arrow[d, "t"] \\ Z \arrow[r, "g"] & Y \end{tikzcd}$$

- (Left cancellation) Suppose $f, g : X \to Y$ are parallel morphisms in $\mathcal{C}$, then 1. $\implies$ 2.:
  1. $sf = sg$ for som $s : S$ starting at Y
  2. $ft = gt$ for som $t : S$ ending at X

Calculus of Fractions

Definition (Right fractions)

S is a right multiplicative system

$$X \xleftarrow[s]{} Y \xrightarrow{t} Z$$

Right fractions are denoted as ts^{-1}.

- Let $\sim$ be the equivalence relation of right fractions such that $ts^{-1} \sim t's'^{-1}$ if and only if $\exists w, w' : \mathcal{C}$ making the diagram below commute and the middle row a right fraction.

$$\begin{tikzcd}
& Y \arrow[dl, "s"'] \arrow[dr, "t"] & \\
X & W \arrow[l, dashed] \arrow[r, dashed] \arrow[u, "w"] \arrow[d, "w'"] & Z \\
& Y' \arrow[ul, "s'"] \arrow[ur, "t'"'] &
\end{tikzcd}$$

Calculus of Fractions

Definition (Right fractions)

S is a right multiplicative system

$$X \xleftarrow[s]{} Y \xrightarrow{t} Z$$

Right fractions are denoted as ts^{-1}.

- Let $S^{-1}\mathcal{C}$ denote the category with objects from $\mathcal{C}$ and arrows are right fractions modulo $\sim$.

Set theory issues

There is no reason for this category to have small homsets between objects.

Localization; II

Theorem (Gabriel-Zisman)

Let S be a locally small right multiplicative system of morphisms in a category $\mathcal{C}$. Then the category $\mathfrak{r}S^{-1}\mathcal{C}$ exists and it is the localization of $\mathcal{C}$ on S. This mean that there is an equivalence of categories $\mathcal{C}[S^{-1}] \simeq \mathfrak{r}S^{-1}\mathcal{C}$ together with a functor $q : \mathcal{C} \to \mathfrak{r}S^{-1}\mathcal{C}$ sending a morphism $f : X \to Y$ to the right fraction $f \circ id_X^{-1}$.

Subcategories

Definition (Triangulated subcategory)

A triangulated subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$ is a full additive subcategory such that the inclusion functor is triangulated.

Definition ($Mor_{\mathcal{S}}$)

Let $\mathcal{C}$ be a triangulated category and $\mathcal{S} \subseteq \mathcal{C}$ be a triangulated subcategory. Define the collection $Mor_{\mathcal{S}}$ to be a collection of morphisms in $\mathcal{C}$ such that for any $f : Mor_{\mathcal{S}}$ there is a triangle with $C : \mathcal{S}$.

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow \Sigma_{\mathcal{C}} A$$

Verdier quotient

Lemma

Let $\mathcal{S} \subseteq \mathcal{C}$ be triangulated categories, then $Mor_{\mathcal{S}}$ is a multiplicative system.

Theorem (Verdier Quotient)

The Verdier quotient $\mathcal{C}/\mathcal{S}$, defined as $Mor_{\mathcal{S}}^{-1}\mathcal{C}$, together with the functor $q : \mathcal{C} \to \mathcal{C}/\mathcal{S}$ is the universal triangulated category where morphisms in $Mor_{\mathcal{S}}$ are isomorphisms.

Exact categories

Definition (Kernel-cokernel pair)

- $\mathcal{A}$ is an additive category
- (p, q) is a kernel-cokernel pair if p is the kernel of q and q is the cokernel of p
- A morphism of kernel-cokernel pairs are diagrams

$$\begin{tikzcd} A \arrow[r, tail, "p"] \arrow[d, "f"] & B \arrow[r, two heads, "q"] \arrow[d, "g"] & C \arrow[d, "h"] \\ A' \arrow[r, tail, "p'"] & B' \arrow[r, two heads, "q'"] & C' \end{tikzcd}$$

Exact categories

An exact structure for an additive category $\mathcal{A}$ is a class $\mathcal{E}$ of kernel-cokernel pairs which are closed under isomorphisms. A pair $(p, q) : \mathcal{E}$ is called a conflation, here p is called an inflation and q is called a deflation. $(\mathcal{A}, \mathcal{E})$ is called exact when the following axioms holds:

- (QE0) $\forall A : \mathcal{A}$, id_A is both an inflation and a deflation.
- (QE1) Both inflations and deflations are closed under composition.
- (QE2) The push-out of an inflation is an inflation.
- (QE2^{op}) The pull-back of a deflation is a deflation.

An exact category is the additive category $\mathcal{A}$ together with an exact structure $\mathcal{E}$.

Examples of exact categories

Example

Any abelian category is exact with every short exact sequence as the exact structure. This exact structure is $\mathcal{E}_{max}$.

Example

Any additive category is exact with every split short exact sequence as the exact structure. This structure will always be $\mathcal{E}_{min}$, and it is always contained inside another exact structure.

Projective and injective objects

Definition (Exact functors)

A functor $F : (\mathcal{A}, \mathcal{E}) \to (\mathcal{A}', \mathcal{E}')$ between exact categories is called exact if it is additive and $F(\mathcal{E}) \subseteq \mathcal{E}'$.

Definition (Projective object)

$P : \mathcal{A}$ is called projective if $\mathcal{A}(P, \_) : (\mathcal{A}, \mathcal{E}) \to \mathbf{Ab}$ is an exact functor.

Lemma

$P : \mathcal{A}$ is projective if and only if for every deflation $q : A \to B$ and morphism $f : P \to B$, there is a morphism $f' : P \to A$ rendering the diagram below commutative.

$$\begin{tikzcd}
& P \arrow[dl, dashed, "f'"] \arrow[d, "f"] \\
A \arrow[r, two heads, "q"'] & B
\end{tikzcd}$$

Projective and injective objects

Definition (Exact functors)

A functor $F : (\mathcal{A}, \mathcal{E}) \to (\mathcal{A}', \mathcal{E}')$ between exact categories is called exact if it is additive and $F(\mathcal{E}) \subseteq \mathcal{E}'$.

Definition (Injective object)

$I : \mathcal{A}$ is called injective if $\mathcal{A}(\_, I) : (\mathcal{A}, \mathcal{E})^{op} \to \mathbf{Ab}$ is an exact functor.

Lemma

$I : \mathcal{A}$ is injective if and only if for every inflation $p : B \to A$ and morphism $f : B \to I$, there is a morphism $f' : A \to I$ rendering the diagram below commutative.

$$
\begin{tikzcd}
 & I \\
A \arrow[ur, dashed, "f'"] & B \arrow[l, tail, "p"] \arrow[u, "f"']
\end{tikzcd}
$$

Cosyzygies and stabilization

Definition (Zysygy)

A syzygy of an object X, if it exists, is denoted ΩX. It is defined to be the kernel object of any deflation $p : P \to X$, where P is projective.

Syzygy is not a functor

Cosyzygies and stabilization

Definition (Cosyzygy)

A cosyzygy of an object X, if it exists, is denoted $\mho X$. It is defined to be the cokernel object of any inflation $i : X \to I$, where I is injective.

Cosyzygy is not a functor

Cosyzygies and stabilization

Definition (Cosyzygy)

A cosyzygy of an object X, if it exists, is denoted $\mho X$. It is defined to be the cokernel object of any inflation $i : X \to I$, where I is injective.

Cosyzygy is not a functor

Definition (Frobenius category)

$\mathcal{A}$ is a Frobenius category if

- it has enough injectives and projectives
- injectives and projectives coincide

The quotient category $\underline{\mathcal{A}} = \mathcal{A}/\sim$ is the stable Frobenius category. $f \sim g \iff f - g$ factors over an injective/projective.

Triangulation

- $\underline{\mathcal{A}}$ is an additive category
- $\mho\colon \underline{\mathcal{A}} \to \underline{\mathcal{A}}$ is an additive autoequivalence
- What is $\Delta_{\underline{\mathcal{A}}}$?

Triangulation

- $\mathcal{A}$ is an additive category
- $\mho\colon \mathcal{A} \to \mathcal{A}$ is an additive autoequivalence

<u>Triangles from morphisms</u>

$$X \xrightarrow{x} Y \quad \Longrightarrow \quad \begin{tikzcd} X \arrow[r, "x"] \arrow[d, tail, "i"] & Y \arrow[d, tail, "y"] \arrow[ddr, bend left, "0"] & \\ I(X) \arrow[r, "p'"] \arrow[drr, bend right, two heads, "p"] & Z \arrow[dr, dashed, two heads, "z"] & \\ & & \mho X \end{tikzcd}$$

Triangulation

- $\underline{\mathcal{A}}$ is an additive category
- $\mho\colon \underline{\mathcal{A}} \to \underline{\mathcal{A}}$ is an additive autoequivalence

Triangles from conflations

$$
\begin{tikzcd}
\Omega Z \arrow[r, "-\Omega z"] \arrow[d, tail, "i"] & X \arrow[d, tail, "x"] \arrow[ddr, bend left, "0"] & & & X \arrow[d, tail, "x"] \\
P(Z) \arrow[r, "p'"] \arrow[drr, bend right, two heads, "p"'] & Y \arrow[dr, dashed, two heads, "y"] & & \Longleftarrow & Y \arrow[d, two heads, "y"] \\
& & Z & & Z
\end{tikzcd}
$$

Homotopy categories are triangulated

Definition (Chain complexes)

Let $\mathcal{A}$ be an additive category. Define $Ch(\mathcal{A})$ to be the category of diagrams in $\mathcal{A}$ on the form

$$\dots \xrightarrow{d_{A^\bullet}^{-2}} A^{-1} \xrightarrow{d_{A^\bullet}^{-1}} A^0 \xrightarrow{d_{A^\bullet}^{0}} A^1 \xrightarrow{d_{A^\bullet}^{1}} \dots$$

such that $d_{A^\bullet}^{i} \circ d_{A^\bullet}^{i-1} = 0$ for every $i : \{-\infty, \dots, \infty\}$. A morphism $\phi^\bullet : A^\bullet \to B^\bullet$ between chain complexes is a collection of morphisms from $\mathcal{A}$, such that the diagram commutes.

$$\begin{tikzcd}
\dots \arrow[r, "d_{A^\bullet}^{-2}"] & A^{-1} \arrow[r, "d_{A^\bullet}^{-1}"] \arrow[d, "\phi^{-1}"] & A^0 \arrow[r, "d_{A^\bullet}^{0}"] \arrow[d, "\phi^{0}"] & A^1 \arrow[r, "d_{A^\bullet}^{1}"] \arrow[d, "\phi^{1}"] & \dots \\
\dots \arrow[r, "d_{B^\bullet}^{-2}"] & B^{-1} \arrow[r, "d_{B^\bullet}^{-1}"] & B^0 \arrow[r, "d_{B^\bullet}^{0}"] & B^1 \arrow[r, "d_{B^\bullet}^{1}"] & \dots
\end{tikzcd}$$

Homotopy categories are triangulated

Definition (Homotopies)

A chain map $f^\bullet : A^\bullet \to B^\bullet$ is called null-homotopic if there is a map $\varepsilon^\bullet : A^\bullet \to B^\bullet[-1]$ such that $f^\bullet = d_{B^\bullet}^{\bullet-1}\varepsilon^\bullet + \varepsilon^{\bullet+1}d_{A^\bullet}^\bullet$.

$$\begin{tikzcd}
\dots \arrow[r, "d_{A^\bullet}^{-2}"] & A^{-1} \arrow[r, "d_{A^\bullet}^{-1}"] \arrow[d, "f^{-1}"] & A^0 \arrow[r, "d_{A^\bullet}^{0}"] \arrow[d, "f^{0}"] \arrow[dl, "\varepsilon^0"'] & A^1 \arrow[r, "d_{A^\bullet}^{1}"] \arrow[d, "f^{1}"] \arrow[dl, "\varepsilon^1"'] & \dots \\
\dots \arrow[r, "d_{B^\bullet}^{-2}"] & B^{-1} \arrow[r, "d_{B^\bullet}^{-1}"] & B^0 \arrow[r, "d_{B^\bullet}^{0}"] & B^1 \arrow[r, "d_{B^\bullet}^{1}"] & \dots
\end{tikzcd}$$

$\varepsilon^\bullet$ is called the homotopy. Two chain maps $f^\bullet$ and $g^\bullet$ are said to be homotopic $f^\bullet \sim g^\bullet$ if their difference $f^\bullet - g^\bullet$ is null-homotopic.

Homotopy categories are triangulated

Definition (Homotopy category)

$K(\mathcal{A}) = Ch(\mathcal{A})/\sim$

Lemma (Exact structure on $Ch(\mathcal{A})$)

For every chain map $r^\bullet : A^\bullet \to B^\bullet$ there is a conflation as the diagram below. This makes $Ch(\mathcal{A})$ an exact category with structure $\mathcal{E}$.

$$B^\bullet \overset{\begin{pmatrix}1\\0\end{pmatrix}}{\rightarrowtail} cone(r^\bullet) \overset{\begin{pmatrix}0 & -1^\bullet\end{pmatrix}}{\twoheadrightarrow} A^\bullet[1]$$

Homotopy categories are triangulated

Definition (Homotopy category)

$K(\mathcal{A}) = Ch(\mathcal{A})/\sim$

Lemma (Triangulation on $K(\mathcal{A})$)

For every chain map $r^\bullet : A^\bullet \to B^\bullet$ there is a triangle as the diagram below.

$$A^\bullet \xrightarrow{r^\bullet} B^\bullet \overset{\begin{pmatrix}1\\0\end{pmatrix}}{\rightarrowtail\!\!\circ\!\!\rightarrow} cone(r^\bullet) \overset{\begin{pmatrix}0 & -1^\bullet\end{pmatrix}}{\longrightarrow\!\!\circ\!\!\twoheadrightarrow} A^\bullet[1]$$

Derived categories are triangulated

Definition

Let $\mathcal{A}$ be an abelian category. Define the category $Ac(\mathcal{A}) \subset K(\mathcal{A})$ to be the full category whose objects are exact sequences.

Lemma

Let $f^\bullet : A^\bullet \to B^\bullet$ be a chain map between acyclic chain complexes, then $\mathrm{cone}(f^\bullet)$ is acyclic as well.

Theorem

The derived category is the Verdier quotient $K(\mathcal{A})/Ac(\mathcal{A})$