Abstract

The aim of this thesis is to give an exposition to theory of triangulated categories, and give some constructions as well. The main goal is to show that the Verdier quotient, that the homotopy category and that the derived category is triangulated.

Sammendrag

Denne bacheloroppgaven har som mål i å gi en presentasjon av teorien til triangulerte kategorier, samt gi noen konstruksjoner i tillegg. Hovedmålet er å vise at Verdier kvotienten, homotopikategorien og den deriverte kategorien er triangulerte.

Introduction

Triangulated categories was defined by Puppe and Verdier independently. Puppes definition was motivated by the homotopy category of Spectra, but he missed the crucial Octahedron axiom. However, when Verdier introduced triangulated categories and derived categories in his PhD thesis published in 1967, he noticed the importance of the Octahedron axiom. As it stands, there are different ways of defining a triangulated category. For instance, Neeman showed that the octahedron axiom is equivalent to having a choice when applying the morphism axiom, such that the mapping cone becomes a triangle itself. Even though the Octahedron axiom is crucial for showing many of the important results, it is not known of a pre-triangulated category which is not triangulated.

In practice there are two different types of triangulated categories, topological and algebraic. A triangulated category is said to be topological if it is the stable category of a model category, and likewise algebraic if it is the stable category of a Frobenius category. In definition these types of categories are not similar, but in practice their differences are quite subtle. This thesis will solely focus on algebraic triangulated categories.

This thesis is split into three parts. The first parts aim to give an exposition to the classical theory of triangulated categories. Part two aims to introduce exact categories and show that the stable Frobenius category is triangulated, among with giving some examples of triangulated categories. The third part aims to introduce the derived category of the homotopy category, and give an exposition of some of the theory.

This thesis assumes that elementary category theory is known and the study of abelian categories and derived categories of abelian categories is known. It is not needed to know some representation theory of artin rings, but it is needed for the section on self-injective algebras.

Notation

Introduce notation which will be used in text. A list of notation and description would be nice, so that the reader might scroll back up if something is unclear.

Contents

Chapter 1

Triangulated Categories

1.1 Definition and First Properties

This section will present what a triangulated category is and show some properties of some functors from this category. The covariant functors which are of main interest are the ones which are called homological, while the contravariant are called cohomological. This family of functors will derive the elementary properties of the triangulation. In this section let $\mathcal{T}$ denote an additive category and $\Sigma_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}$ be an additive autoequivalence of $\mathcal{T}$, which is either called the translation or suspension functor.

Definition 1.1.1. A candidate triangle is a collection (A, B, C, a, b, c) of objects $A, B, C \in T$ and morphisms $a : A \to B$, $b : B \to C$, $c : C \to \Sigma_{\mathcal{T}}A$. These candidate triangles can be drawn as diagrams in the following way:

$$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} \Sigma_{\mathcal{T}}A$$

A morphism between candidate triangles is a triple of morphism (α, β, γ), where $\alpha : A \to A'$, $\beta : B \to B'$ and $\gamma : C \to C'$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\alpha"] & B \arrow[r, "b"] \arrow[d, "\beta"] & C \arrow[r, "c"] \arrow[d, "\gamma"] & \Sigma_{\mathcal{T}}A \arrow[d, "\Sigma_{\mathcal{T}}\alpha"] \\
A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C \arrow[r, "c'"] & \Sigma_{\mathcal{T}}A'
\end{tikzcd}$$

Why these objects are called triangles become apparent when an alternate description of the diagrams above are given. In order to remove confusion about the domain or codomain of the arrows to be presented, one arrow of the triangle will be decorated with "$\Sigma_{\mathcal{T}}|$". This decorator means that the functor $\Sigma_{\mathcal{T}}$ has to be applied to the corresponding edge of the arrow. With this notation the c arrow points to $\Sigma_{\mathcal{T}}A$, not A.

$$\begin{tikzcd}
A \arrow[dr, "a"] & \\
& B \arrow[dl, "b"] \\
C \arrow[uu, "c", "\Sigma_{\mathcal{T}}|"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rrrr, "\phi_a"] \arrow[dr, "a"] & & & & A' \arrow[dl, "a'"'] \\
& B \arrow[rr, "\phi_b"] \arrow[dl, "b"] & & B' \arrow[dr, "b'"] & \\
C \arrow[rrrr, "\phi_c"] \arrow[uu, dashed, "c", "\Sigma_{\mathcal{T}}|"'] & & & & C' \arrow[uu, dashed, "c'"', "\Sigma_{\mathcal{T}}|"]
\end{tikzcd}$$

These triangles will make up a triangulation on the category $\mathcal{T}$. Thus, a triangulated category is an additive category together with a translation functor $\Sigma_{\mathcal{T}}$ and a triangulation class $\Delta_{\mathcal{T}}$ consisting of candidate triangles. When a candidate triangle is an element of $\Delta_{\mathcal{T}}$ it is usually called a triangle, an exact triangle or a distinguished triangle. Note that if the candidate triangles are referred to as triangles it is common to either call the elements of $\Delta_{\mathcal{T}}$ for exact triangles or distinguished triangles. In this thesis the elements of $\Delta_{\mathcal{T}}$ will be called for triangles.

Definition 1.1.2. A triangulation of an additive category $\mathcal{T}$ with translation $\Sigma_{\mathcal{T}}$ is a collection $\Delta_{\mathcal{T}}$ of triangles consisting of candidate triangles in $\mathcal{T}$ satisfying the following axioms:

1. (TR1) Bookkeeping axiom
   a. A candidate triangle isomorphic to a triangle is a triangle.
   b. Every morphism $a : A \to B$ can be embedded into a triangle (A, B, C, a, b, c).
   $$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA$$
   c. For every object A there is a triangle $(A, A, 0, id_A, 0, 0)$.
   $$A \xrightarrow{id_A} A \xrightarrow{0} 0 \xrightarrow{0} \Sigma_{\mathcal{T}}A$$
2. (TR2) Rotation axiom
   For every triangle (A, B, C, a, b, c) there is a triangle $(B, C, TA, b, c, -\Sigma_{\mathcal{T}}a)$.
   $$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} \Sigma_{\mathcal{T}}A \implies B \xrightarrow{b} C \xrightarrow{c} TA \xrightarrow{-\Sigma_{\mathcal{T}}a} \Sigma_{\mathcal{T}}B$$
3. (TR3) Morphism axiom
   Given the two triangles (A, B, C, a, b, c) and (A', B', C', a', b', c'),
   $$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} \Sigma_{\mathcal{T}}A \qquad A' \xrightarrow{a'} B' \xrightarrow{b'} C' \xrightarrow{c'} \Sigma_{\mathcal{T}}A'$$
   and morphisms $\phi_A : A \to A'$ and $\phi_B : B \to B'$ such that square (1) commutes, then there is a morphism $\phi_C : C \to C'$ (not necessarily unique) such that (ϕ_A, ϕ_B, ϕ_C) is a morphism of triangles (2).
   $$(1)\quad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[d, "\phi_B"] \\ A' \arrow[r, "a'"] & B' \end{tikzcd} \qquad (2)\quad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[r, "b"] \arrow[d, "\phi_B"] & C \arrow[r, "c"] \arrow[d, dashed, "\phi_C"] & \Sigma_{\mathcal{T}}A \arrow[d, "\Sigma_{\mathcal{T}}\phi_A"] \\ A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & \Sigma_{\mathcal{T}}A' \end{tikzcd}$$
4. (TR4) Octahedron axiom
   Given the triangles (A, B, C', a, x, x') (1), (B, C, A', b, y, y') (2) and $(A, C, B', b \circ a, z, z')$ (3)
   $$(1)\quad A \xrightarrow{a} B \xrightarrow{x} C' \xrightarrow{x'} \Sigma_{\mathcal{T}}A$$
   $$(2)\quad B \xrightarrow{b} C \xrightarrow{y} A' \xrightarrow{y'} \Sigma_{\mathcal{T}}B$$
   $$(3)\quad A \xrightarrow{b\circ a} C \xrightarrow{z} B' \xrightarrow{z'} \Sigma_{\mathcal{T}}A$$

then there exist morphisms $f : C' \to B'$ and $g : B' \to A'$, the following diagram commutes and the third row is a triangle.

$$\begin{tikzcd}
\Sigma_{\mathcal{T}}^{-1}B' \arrow[r, "\Sigma_{\mathcal{T}}^{-1}z'"] \arrow[d, "\Sigma_{\mathcal{T}}^{-1}g"] & A \arrow[r, equal, "id_A"] \arrow[d, "a"] & A \arrow[d, "b\circ a"] & & \\
\Sigma_{\mathcal{T}}^{-1}A' \arrow[r, "\Sigma_{\mathcal{T}}^{-1}y'"] & B \arrow[r, "b"] \arrow[d, "x"] & C \arrow[r, "y"] \arrow[d, "z"] & A' \arrow[r, "y'"] \arrow[d, equal, "id_{A'}"] & \Sigma_{\mathcal{T}}B \arrow[d, "\Sigma_{\mathcal{T}}x'"] \\
 & C' \arrow[r, "f"] \arrow[d, "x'"] & B' \arrow[r, "g"] \arrow[d, "z'"] & A' \arrow[r, "\Sigma_{\mathcal{T}}i\circ y'"] & \Sigma_{\mathcal{T}}C' \\
 & \Sigma_{\mathcal{T}}A \arrow[r, equal, "id_{\Sigma_{\mathcal{T}}A}"] & \Sigma_{\mathcal{T}}A & &
\end{tikzcd}$$

A triangulated category is denoted as $(\mathcal{T}, \Sigma_{\mathcal{T}}, \Delta_{\mathcal{T}})$, where $\mathcal{T}$ is the additive category, $\Sigma_{\mathcal{T}}$ is the translation and $\Delta_{\mathcal{T}}$ is the triangulation. When $\mathcal{T}$ is called a triangulated category, it should be understood like a triple.

Remark. The third object in a triangle is usually called cone, fiber or cofiber. These names are in use due to historic reasons, rather than portraying their functionality. The names weak kernel or weak cokernel would be better in the sense that it tells what the function of this object is. In this thesis it will either be referred to as cone, weak kernel or weak cokernel.

Remark. The rotation axiom has a dual, and it can be thought of as a shift in the opposite direction. The dual rotation axiom goes as:

> Given a triangle $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} \Sigma_{\mathcal{T}}A$,
>
> there is a triangle $\Sigma_{\mathcal{T}}^{-1}C \xrightarrow{-\Sigma_{\mathcal{T}}^{-1}c} A \xrightarrow{a} B \xrightarrow{b} C$

To be able to prove this, some more lemmata are needed.

Remark. By the previous remark one may see that the definition of a triangulated category is self dual. That is a category $\mathcal{T}$ is triangulated if and only if $\mathcal{T}^{op}$ is triangulated.

Remark. The final axiom is referred to as the octahedron axiom. By using the alternative description of the triangle diagram, it is possible to rewrite the diagram as an octahedron. The axiom can be restated as the following.

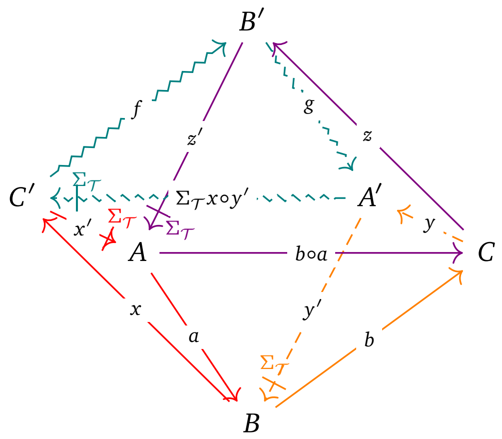
> Given the triangles (A, B, C', a, x, x') (1), (B, C, A', b, y, y') (2) and $(A, C, B', b\circ a, z, z')$ (3)

(1)
$$\begin{tikzcd}
A \arrow[dr, "a"] & \\
 & B \arrow[dl, "x"] \\
C' \arrow[uu, "\Sigma_{\mathcal{T}}", "x'"'] &
\end{tikzcd}$$

(2)
$$\begin{tikzcd}
B \arrow[dr, "b"] & \\
 & C \arrow[dl, "y"] \\
A' \arrow[uu, "\Sigma_{\mathcal{T}}", "y'"'] &
\end{tikzcd}$$

$$(3)\quad \begin{tikzcd} A \arrow[dr, "b\circ a"] & \\ & C \arrow[dl, "z"] \\ B' \arrow[uu, "z'", "\Sigma_{\mathcal{T}}"'] & \end{tikzcd}$$

then there exists morphisms $f : C' \to B'$ and $g : B' \to A'$, the following diagram commutes and the squiggly teal back face is a triangle.



Proposition 1.1.1. *The axiom TR3 can be proven from TR1 and TR4.*

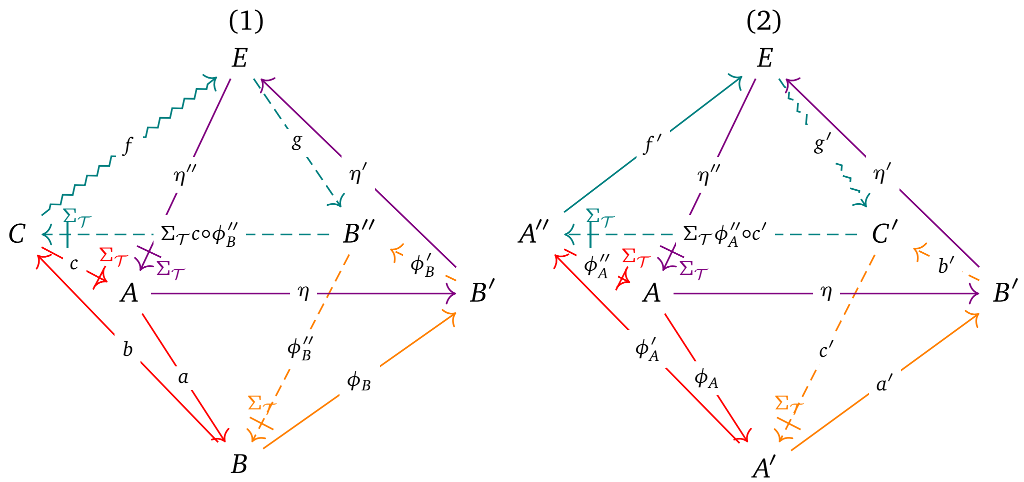
Proof. Suppose that there are two triangles and a commutative square as follows.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] \arrow[dr, "\eta"] & B \arrow[d, "\phi_B"] \\ A' \arrow[r, "a'"] & B' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[r, "b"] \arrow[d, "\phi_B"] & C \arrow[r, "c"] & \Sigma_{\mathcal{T}}A \arrow[d, "\Sigma_{\mathcal{T}}\phi_A"] \\ A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & \Sigma_{\mathcal{T}}A' \end{tikzcd}$$

The upper and lower simplex of the square may be completed to two sets of triangles satisfying the condition of TR4. Applying the Octahedron axiom twice yields the diagrams as below.

$$(1)\quad \begin{tikzcd} A \arrow[dr, "a"] & \\ & B \arrow[dl, "b"] \\ C \arrow[uu, "c", "\Sigma_{\mathcal{T}}"'] & \end{tikzcd} \quad \begin{tikzcd} B \arrow[dr, "\phi_B"] & \\ & B' \arrow[dl, "\phi'_B"] \\ B'' \arrow[uu, "\phi''_B", "\Sigma_{\mathcal{T}}"'] & \end{tikzcd} \quad \begin{tikzcd} A \arrow[dr, "\eta"] & \\ & B' \arrow[dl, "\eta'"] \\ E \arrow[uu, "\eta''", "\Sigma_{\mathcal{T}}"'] & \end{tikzcd}$$

$$(2)\quad \begin{tikzcd} A \arrow[dr, "\phi_A"] & \\ & A' \arrow[dl, "\phi'_A"] \\ A'' \arrow[uu, "\phi''_A", "\Sigma_{\mathcal{T}}"'] & \end{tikzcd} \quad \begin{tikzcd} A' \arrow[dr, "a'"] & \\ & B' \arrow[dl, "b'"] \\ C' \arrow[uu, "c'", "\Sigma_{\mathcal{T}}"'] & \end{tikzcd} \quad \begin{tikzcd} A \arrow[dr, "\eta"] & \\ & B' \arrow[dl, "\eta'"] \\ E \arrow[uu, "\eta''", "\Sigma_{\mathcal{T}}"'] & \end{tikzcd}$$



The teal squiggly lines at the back faces of each octahedra forms a morphism $g'f : C \to C'$. It remains to see that the morphism is a triangle morphism. Diagram chasing reveals that the following diagram is commutative, which is exactly the requirement for the collection $(\phi_A, \phi_B, g'f)$ to be a morphism of triangles.

$$
\begin{tikzcd}
B \arrow[r, "b"] \arrow[d, "\phi_B"] & C \arrow[d, "f"] \arrow[dr, "c"] & \\
B' \arrow[r, "\eta'"] \arrow[dr, "b'"] & E \arrow[r, "\eta''"] \arrow[d, "g'"] & \Sigma_{\mathcal{T}} A \arrow[d, "\Sigma_{\mathcal{T}}\phi_A"] \\
& C' \arrow[r, "c'"] & \Sigma_{\mathcal{T}} A'
\end{tikzcd}
$$

$\square$

Lemma 1.1.2. *Let (A, B, C, a, b, c) be a triangle, then $b \circ a = 0$*

Proof. By TR2 the triangle (A, B, C, a, b, c) can be rotated to $(B, C, \Sigma_{\mathcal{T}} A, b, c, -\Sigma_{\mathcal{T}} a)$.

$$
\begin{tikzcd}
A \arrow[dr, "a"] & & & B \arrow[dr, "b"] & \\
& B \arrow[dl, "b"] & \Longrightarrow & & C \arrow[dl, "c"] \\
C \arrow[uu, "c", "\Sigma_{\mathcal{T}}"'] & & & \Sigma_{\mathcal{T}} A \arrow[uu, "-\Sigma_{\mathcal{T}} a", "\Sigma_{\mathcal{T}}"'] &
\end{tikzcd}
$$

The triangle $(C, C, 0, id_C, 0, 0)$ exists by TR1 and TR3 states that there exists a morphism from $\Sigma_{\mathcal{T}} A$ to 0 making the diagram below commute.

$$
\begin{tikzcd}
B \arrow[r, "b"] \arrow[d, "b"] & C \arrow[r, "c"] \arrow[d, "id_C"] & \Sigma_{\mathcal{T}} A \arrow[r, "-\Sigma_{\mathcal{T}} a"] \arrow[d, dashed, "0"] & \Sigma_{\mathcal{T}} B \arrow[d, "\Sigma_{\mathcal{T}} b"] \\
C \arrow[r, "id_C"] & C \arrow[r, "0"] & 0 \arrow[r, "0"] & \Sigma_{\mathcal{T}} C
\end{tikzcd}
$$

Thus $0 = \Sigma_{\mathcal{T}} b \circ -\Sigma_{\mathcal{T}} a = \Sigma_{\mathcal{T}}(-ba) \implies b \circ a = 0$ as $\Sigma_{\mathcal{T}}$ is a translation. $\square$

One fundamental object to study when looking at categories are functors. In the case of triangulated categories there are two import type of functors, triangulated functors and homological functors. These are central to this discussion as one are able to relate triangulations to each other and derive information about triangulations through abelian categories. The result 5-lemma has an appearance in triangulated categories through the 2-out-of-3 property.

Definition 1.1.3. A triangulated functor $F : \mathcal{S} \to \mathcal{T}$ between two triangulated categories $(\mathcal{S}, \Sigma_{\mathcal{S}}, \Delta_{\mathcal{S}})$ and $(\mathcal{T}, \Sigma_{\mathcal{T}}, \Delta_{\mathcal{T}})$, is an additive functor F along with a natural isomorphism $\phi_X : F(\Sigma_{\mathcal{S}}(X)) \to \Sigma_{\mathcal{T}}(F(X))$ such that $F(\Delta_{\mathcal{S}}) \subseteq \Delta_{\mathcal{T}}$. This means that for every triangle in $\mathcal{T}$ there is a triangle in $\mathcal{S}$.

$$\begin{tikzcd} A \arrow[dr, "a"] & \\ & B \arrow[dl, "b"] \\ C \arrow[uu, "c"', "\Sigma_{\mathcal{S}}"] & \end{tikzcd} \implies \begin{tikzcd} F(A) \arrow[dr, "F(a)"] & \\ & F(B) \arrow[dl, "F(b)"] \\ F(C) \arrow[uu, "F(c)"', "\Sigma_{\mathcal{T}}"] & \end{tikzcd}$$

Remark. Some literatures refer to triangulated functors as exact functors.

Definition 1.1.4. Let $\mathcal{T}$ be a triangulated category and $\mathcal{A}$ be an abelian category. A covariant functor $H : \mathcal{T} \to \mathcal{A}$ is called homological if $\forall (A, B, C, a, b, c) : \Delta_{\mathcal{T}}$ there is a long exact sequence in $\mathcal{A}$.

$$\begin{tikzcd} A \arrow[dr, "a"] & \\ & B \arrow[dl, "b"] \\ C \arrow[uu, "c"', "\Sigma_{\mathcal{T}}"] & \end{tikzcd} \implies \begin{array}{l} \cdots \longrightarrow H(\Sigma^i_{\mathcal{T}}A) \xrightarrow{H(\Sigma^i_{\mathcal{T}}a)} H(\Sigma^i_{\mathcal{T}}B) \xrightarrow{H(\Sigma^i_{\mathcal{T}}b)} H(\Sigma^i_{\mathcal{T}}C) \xrightarrow{H(\Sigma^i_{\mathcal{T}}c)} \\ H(\Sigma^{i+1}_{\mathcal{T}}A) \xrightarrow{H(\Sigma^{i+1}_{\mathcal{T}}a)} H(\Sigma^{i+1}_{\mathcal{T}}B) \xrightarrow{H(\Sigma^{i+1}_{\mathcal{T}}b)} H(\Sigma^{i+1}_{\mathcal{T}}C) \longrightarrow \cdots \end{array}$$

Dually, a contravariant functor $H : \mathcal{T} \to \mathcal{A}$ is called cohomological if $\forall (A, B, C, a, b, c) : \Delta$ there is a long exact sequence in $\mathcal{A}$.

$$\begin{tikzcd} A \arrow[dr, "a"] & \\ & B \arrow[dl, "b"] \\ C \arrow[uu, "c"', "\Sigma_{\mathcal{T}}"] & \end{tikzcd} \implies \begin{array}{l} \cdots \longleftarrow H(\Sigma^{i-1}_{\mathcal{T}}A) \xleftarrow[H(\Sigma^{i-1}_{\mathcal{T}}a)]{} H(\Sigma^{i-1}_{\mathcal{T}}B) \xleftarrow[H(\Sigma^{i-1}_{\mathcal{T}}b)]{} H(\Sigma^{i-1}_{\mathcal{T}}C) \xleftarrow{H(\Sigma^{i}_{\mathcal{T}}c)} \\ H(\Sigma^{i}_{\mathcal{T}}A) \xleftarrow[H(\Sigma^{i}_{\mathcal{T}}a)]{} H(\Sigma^{i}_{\mathcal{T}}B) \xleftarrow[H(\Sigma^{i}_{\mathcal{T}}b)]{} H(\Sigma^{i}_{\mathcal{T}}C) \longleftarrow \cdots \end{array}$$

Lemma 1.1.3. *Let $M : \mathcal{T}$ be any object of $\mathcal{T}$, then the represented functor $\mathcal{T}(M, \_)$ is homological and $\mathcal{T}(\_, M)$ is cohomological.*

Proof. Only the covariant case needs to be proved, as the contravariant case is dual. For $\mathcal{T}(M, \_)$ to be homological, it has to create long exact sequences for every triangle in Δ. Let $(A, B, C, a, b, c) : \Delta$ be a triangle, then sequences in Ab can be extracted for any $i : \mathbb{Z}$.

$$\begin{tikzcd} A \arrow[dr, "a"] & \\ & B \arrow[dl, "b"] \\ C \arrow[uu, "c"', "\Sigma_{\mathcal{T}}"] & \end{tikzcd} \implies \Sigma_{\mathcal{T}}(M, \Sigma^i_{\mathcal{T}}A) \xrightarrow{\Sigma^i_{\mathcal{T}}a_*} \Sigma_{\mathcal{T}}(M, \Sigma^i_{\mathcal{T}}B) \xrightarrow{\Sigma^i_{\mathcal{T}}b_*} \Sigma_{\mathcal{T}}(M, \Sigma^i_{\mathcal{T}}C)$$

It is enough to prove that these types of diagrams are exact at B, as the other diagrams are obtained by the rotation axiom. Thus it remains to prove that $Im(\Sigma^i_{\mathcal{T}} a_*) = Ker(\Sigma^i_{\mathcal{T}} b_*)$. Since $ba = 0$ it follows that $Im(\Sigma^i_{\mathcal{T}} a_*) \subseteq Ker(\Sigma^i_{\mathcal{T}} b_*)$. Assume that $f : Ker(\Sigma^i_{\mathcal{T}} b_*)$, that is $f : M \to \Sigma^i_{\mathcal{T}} B$ such that $b_*(f) = 0$. Showing that f factors through $\Sigma^i_{\mathcal{T}} A$ proves exactness, as this means that $Ker(\Sigma^i_{\mathcal{T}} b_*) \subseteq Im(\Sigma^i_{\mathcal{T}} a_*)$. Note that since T is a translation, it is necessarily a right adjoint to the inverse translation; $\mathcal{T}(M, \Sigma^i_{\mathcal{T}} B) \simeq \mathcal{T}(\Sigma^{-i}_{\mathcal{T}} M, B)$ and by this assertion it suffices to assume that $f : \Sigma^{-i}_{\mathcal{T}} M \to B$ such that $b \circ f = 0$. By TR1 and TR2 there exists triangles $(\Sigma^{-i}_{\mathcal{T}} M, 0, \Sigma^{-i+1}_{\mathcal{T}} M, 0, 0, -\Sigma^{-i+1}_{\mathcal{T}} id)$ and $(B, C, \Sigma_{\mathcal{T}} A, b, c, -\Sigma_{\mathcal{T}} a)$.

$$
\begin{tikzcd}
\Sigma^{-i}_{\mathcal{T}} M \arrow[r, "0"] \arrow[d, "f"] & 0 \arrow[r, "0"] \arrow[d, "0"] & \Sigma^{-i+1}_{\mathcal{T}} M \arrow[r, "-\Sigma^{-i+1}_{\mathcal{T}} id"] \arrow[d, dashed, "g"] & \Sigma^{-i+1}_{\mathcal{T}} M \arrow[d, "\Sigma_{\mathcal{T}} f"] \\
B \arrow[r, "b"] & C \arrow[r, "c"] & \Sigma_{\mathcal{T}} A \arrow[r, "-\Sigma_{\mathcal{T}} a"] & \Sigma_{\mathcal{T}} B
\end{tikzcd}
$$

The left square commutes by the assumption, thus the morphism g exist by TR3 such that $-\Sigma_{\mathcal{T}} a \circ h = -\Sigma_{\mathcal{T}} f \circ \Sigma^{-i+1}_{\mathcal{T}} id = -\Sigma_{\mathcal{T}} f \implies \Sigma_{\mathcal{T}} a \circ h = \Sigma_{\mathcal{T}} f$. This shows that $f = a \circ T^{-1} h$, asserting that f factors through A. $\square$

Lemma 1.1.4. ***2-out-of-3 property.*** *Let $(\phi_A, \phi_B, \phi_C) : (A, B, C, a, b, c) \to (A', B', C', a', b', c')$ be a morphism of triangles. If 2 of the maps are isomorphisms, then the last one is an isomorphism as well.*

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\phi_A", "\wr"'] & B \arrow[r, "b"] \arrow[d, "\phi_B", "\wr"'] & C \arrow[r, "c"] \arrow[d, dashed, "\phi_C", "\wr"'] & \Sigma_{\mathcal{T}} A \arrow[d, "\Sigma_{\mathcal{T}} \phi_A", "\wr"'] \\
A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & \Sigma_{\mathcal{T}} A'
\end{tikzcd}
$$

Proof. Without loss of generality, assume that ϕ_A and ϕ_B are the isomorphisms. This can be done as the rotation axiom reduce the other cases to this case. Then the diagram depicted below exists.

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\phi_A", "\wr"'] & B \arrow[r, "b"] \arrow[d, "\phi_B", "\wr"'] & C \arrow[r, "c"] \arrow[d, "\phi_C"] & \Sigma_{\mathcal{T}} A \arrow[d, "\Sigma_{\mathcal{T}} \phi_A", "\wr"'] \\
A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & \Sigma_{\mathcal{T}} A'
\end{tikzcd}
$$

Applying the functor $\mathcal{T}(C', \_)$ to the diagram yields the following diagram in Ab:

$$
\begin{tikzcd}
\mathcal{T}(C', A) \arrow[r, "a_*"] \arrow[d, "(\phi_A)_*", "\wr"'] & \mathcal{T}(C', B) \arrow[r, "b_*"] \arrow[d, "(\phi_B)_*", "\wr"'] & \mathcal{T}(C', C) \arrow[r, "c_*"] \arrow[d, "(\phi_C)_*"] & \mathcal{T}(C', \Sigma_{\mathcal{T}} A) \arrow[r, "\Sigma_{\mathcal{T}} a_*"] \arrow[d, "(\Sigma_{\mathcal{T}} \phi_A)_*", "\wr"'] & \mathcal{T}(C', TB) \arrow[d, "(\Sigma_{\mathcal{T}} \phi_B)_*", "\wr"'] \\
\mathcal{T}(C', A') \arrow[r, "a'_*"] & \mathcal{T}(C', B') \arrow[r, "b'_*"] & \mathcal{T}(C', C') \arrow[r, "c'_*"] & \mathcal{T}(C', \Sigma_{\mathcal{T}} A') \arrow[r, "\Sigma_{\mathcal{T}} a_*"] & \mathcal{T}(C', \Sigma_{\mathcal{T}} B)
\end{tikzcd}
$$

By the 5-lemma, $(\phi_C)_*$ is an isomorphisms, i.e. $(\phi_C)_*$ is both mono and epi. Thus for some unique s in $\mathcal{T}(C', C)$, $\phi_{C*}(s) = id_{C'}$.

Applying the functor $\mathcal{T}(\_, C)$ yields the diagram:

$$\begin{tikzcd}
\mathcal{T}(A,C) & \mathcal{T}(B,C) \arrow[l, "a^*"'] & \mathcal{T}(C,C) \arrow[l, "b^*"'] & \mathcal{T}(\Sigma_{\mathcal{T}}A,C) \arrow[l, "c^*"'] & \mathcal{T}(\Sigma_{\mathcal{T}}B,C) \arrow[l, "\Sigma_{\mathcal{T}}a^*"'] \\
\mathcal{T}(A',C) \arrow[u, "(\phi_A)^*", "\wr"'] & \mathcal{T}(B,C) \arrow[l, "a'^*"] \arrow[u, "(\phi_B)^*", "\wr"'] & \mathcal{T}(C',C) \arrow[l, "b'^*"] \arrow[u, "(\phi_C)^*"] & \mathcal{T}(\Sigma_{\mathcal{T}}A',C) \arrow[l, "c'^*"] \arrow[u, "(\Sigma_{\mathcal{T}}\phi_A)^*", "\wr"'] & \mathcal{T}(\Sigma_{\mathcal{T}}B',C) \arrow[l, "\Sigma_{\mathcal{T}}a'^*"] \arrow[u, "(\Sigma_{\mathcal{T}}\phi_B)^*", "\wr"']
\end{tikzcd}$$

Again, the 5-lemma asserts that $(\phi_C)^*$ is an isomorphisms, and by the same argument $id_C = s' \circ \phi_C$ for some unique s'. ϕ_C is both split mono and split epi, which means it is an isomorphism. □

Corollary 1.1.4.1. $(A, B, 0, a, 0, 0)$ *is a triangle if and only if a is an isomorphism.*

Proof. Assume that a is an isomorphism. Then it is seen that $(a, id_B, 0)$ is an isomorphism of triangles.

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "a", "\wr"'] & B \arrow[r, "0"] \arrow[d, "id_B", "\wr"'] & 0 \arrow[r, "0"] \arrow[d, "0", "\wr"'] & \Sigma_{\mathcal{T}}A \arrow[d, "\Sigma_{\mathcal{T}}a", "\wr"'] \\
B \arrow[r, "id_B"] & B \arrow[r, "0"] & 0 \arrow[r, "0"] & \Sigma_{\mathcal{T}}B
\end{tikzcd}$$

Conversely, assume that $(A, B, 0, a, 0, 0)$ is a triangle. Then the same diagram as above can be constructed, and by the 2 out of 3 property, a has to be an isomorphism. □

Lemma 1.1.5. *For a triangle (A, B, C, a, b, c) the following are equivalent:*

$$\begin{tikzcd}
A \arrow[dr, "a"] & \\
& B \arrow[dl, "b"] \\
C \arrow[uu, "\Sigma_{\mathcal{T}}", "c"'] &
\end{tikzcd}$$

- *a is split mono*
- *b is split epi*
- $c = 0$

Proof. The proof has two parts. First assume that a is split mono, then prove that b is split epi and $c = 0$. By duality, it is then known that b being split epi implies that a is split mono and $c = 0$. The final part is to assume that $c = 0$, and prove either a is split mono or b is split epi.

Assume that a is split mono, then there exist an a^{-1} such that $id_A = a^{-1}a$. Let $M : \mathcal{T}$ be any object, then there is a long exact sequence.

$$\begin{tikzcd}
\mathcal{T}(M,\Sigma_{\mathcal{T}}^{-1}C) \arrow[r, "\Sigma_{\mathcal{T}}^{-1}c_*"] & \mathcal{T}(M,A) \arrow[r, bend left, "a_*"] & \mathcal{T}(M,B) \arrow[l, bend left, dashed, "a_*^{-1}"] \arrow[r, "b_*"] & \mathcal{T}(M,C) \arrow[r, "c_*"] & \mathcal{T}(M,\Sigma_{\mathcal{T}}A)
\end{tikzcd}$$

By assumption a_* is split mono, thus $\Sigma_{\mathcal{T}}^{-1}c_* = 0$ and in particular $c = 0$. This implies that b_* is epi, making a split short exact sequence.

$$\begin{tikzcd}
0 \arrow[r, "0"] & \mathcal{T}(M,A) \arrow[r, bend left, "a_*"] & \mathcal{T}(M,B) \arrow[l, bend left, dashed, "a_*^{-1}"] \arrow[r, bend left, "b_*"] & \mathcal{T}(M,C) \arrow[l, bend left, dashed, "b_*^{-1}"] \arrow[r, "0"] & 0
\end{tikzcd}$$

This split short exact sequence shows that b is split epi, completing the first part of the proof.
For the final part, assume that $c = 0$; construct the following triangles.

$$(1)\quad \begin{tikzcd} A \arrow[dr, "a"] & \\ & B \arrow[dl, "b"] \\ C \arrow[uu, "\Sigma_{\mathcal{T}}", "0"'] & \end{tikzcd} \implies \begin{tikzcd} C \arrow[dr, "0"] & \\ & \Sigma_{\mathcal{T}} A \arrow[dl, "-\Sigma_{\mathcal{T}} a"] \\ \Sigma_{\mathcal{T}} B \arrow[uu, "\Sigma_{\mathcal{T}}", "-\Sigma_{\mathcal{T}} b"'] & \end{tikzcd}$$

$$(2)\quad \begin{tikzcd} A \arrow[dr, "id_A"] & \\ & A \arrow[dl, "0"] \\ 0 \arrow[uu, "\Sigma_{\mathcal{T}}", "0"'] & \end{tikzcd} \implies \begin{tikzcd} 0 \arrow[dr, "0"] & \\ & \Sigma_{\mathcal{T}} A \arrow[dl, "-id_{\Sigma_{\mathcal{T}} A}"] \\ \Sigma_{\mathcal{T}} A \arrow[uu, "\Sigma_{\mathcal{T}}", "0"'] & \end{tikzcd}$$

(1) is constructed by applying TR2 twice, while (2) is constructed with TR1 and the applying TR2 twice. Observe that there is a commutative square between the triangles, allowing for TR3 to make a morphism of triangles.

$$\begin{tikzcd} C \arrow[r, "0"] \arrow[d, "0"] & \Sigma_{\mathcal{T}} A \arrow[r, "-\Sigma_{\mathcal{T}} a"] \arrow[d, equal, "id_{\Sigma_{\mathcal{T}} A}"] & \Sigma_{\mathcal{T}} B \arrow[r, "-\Sigma_{\mathcal{T}} b"] \arrow[d, dashed, "\Sigma_{\mathcal{T}} a^{-1}"] & \Sigma_{\mathcal{T}} C \arrow[d, "0"] \\ 0 \arrow[r, "0"] & \Sigma_{\mathcal{T}} A \arrow[r, equal, "-id_{\Sigma_{\mathcal{T}} A}"] & \Sigma_{\mathcal{T}} A \arrow[r, "0"] & 0 \end{tikzcd}$$

Thus $T(a^{-1}a) = id_{\Sigma_{\mathcal{T}} A} = \Sigma_{\mathcal{T}}(id_A) \implies id_A = a^{-1}a$, making a split mono. $\square$

Lemma 1.1.6. *Given two triangles* (A, B, C, a, b, c) *and* (A', B', C', a', b', c') *the following are equivalent:*

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f"] & B \arrow[r, "b"] \arrow[d, "g"] & C \arrow[r, "c"] \arrow[d, "h"] & \Sigma_{\mathcal{T}} A \arrow[d, "\Sigma_{\mathcal{T}} f"] \\ A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & \Sigma_{\mathcal{T}} A' \end{tikzcd}$$

1. (f, g, h) *is a morphism of triangles*
2. $\exists g : B \to B'$ *such that* $b'ga = 0$

Moreover, if $\mathcal{T}(A, \Sigma_{\mathcal{T}}^{-1} C') \simeq 0$*, then f and h are unique.*

Proof. 1. $\implies$ 2. The composition $b'ga = ba = 0$ shows the claim.

2. $\implies$ 1. The existence of f and h are will be seen to be a consequence of the long exact sequence of the bottom triangle at the covariant functor represented by A.

$$\mathcal{T}(A, \Sigma_{\mathcal{T}}^{-1} C') \xrightarrow{\Sigma_{\mathcal{T}}^{-1} c'_*} \mathcal{T}(A, A') \xrightarrow{a'_*} \mathcal{T}(A, B') \xrightarrow{b'_*} \mathcal{T}(A, C')$$

The morphism $ga : \mathcal{T}(A,B')$ has the property that $b'ga = b'_*(ga) = 0$, thus $ga : Ker(b'_*)$. By exactness, $\exists f : \mathcal{T}(A,A')$ such that $a'f = ga$, and by TR3 $\exists h : C \to C'$ such that (f,g,h) is a morphism of triangles. This have shown that f and g exists, it remains to check uniqueness if the assumption is true. Now assume that $\mathcal{T}(A, \Sigma_{\mathcal{T}}^{-1}C') \simeq 0$. Exactness determines that a'_* is a monomorphism, and f is therefore unique. Since $\Sigma_{\mathcal{T}}$ is a translation, one gets that $\mathcal{T}(A, \Sigma_{\mathcal{T}}^{-1}C') \simeq \mathcal{T}(\Sigma_{\mathcal{T}}A, C')$. By using the functor $\mathcal{T}(\_, C')$ at the top triangle, it is seen that b^* is a monomorphism, thus h is also unique. □

Lemma 1.1.7. ***Opposite Rotation Axiom;*** *$TR2^{op}$. If (A,B,C,a,b,c) is a triangle, then $(\Sigma_{\mathcal{T}}^{-1}C, A, B, -\Sigma_{\mathcal{T}}^{-1}c, a, b)$ is a triangle.*

Remark. It is known a priori that the direct sum of triangles is a candidate triangle, thus it remains to check if it is isomorphic to a triangle.

Proof. Apply TR2 twice to construct the triangle below.

$$\begin{tikzcd} A \arrow[dr, "a"] & \\ & B \arrow[dl, "b"] \\ C \arrow[uu, "\Sigma_{\mathcal{T}}", "c"'] & \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} C \arrow[dr, "c"] & \\ & \Sigma_{\mathcal{T}}A \arrow[dl, "-\Sigma_{\mathcal{T}}a"] \\ \Sigma_{\mathcal{T}}B \arrow[uu, "\Sigma_{\mathcal{T}}", "-\Sigma_{\mathcal{T}}b"'] & \end{tikzcd}$$

The morphism $\Sigma_{\mathcal{T}}^{-1}c$ has a triangle $(\Sigma_{\mathcal{T}}^{-1}C, A, B', \Sigma_{\mathcal{T}}^{-1}c, a', b')$ by TR1. Use TR3 to find a morphism between these associated candidate triangles.

$$\begin{tikzcd} C \arrow[r, "c"] \arrow[d, equal, "id_C"] & \Sigma_{\mathcal{T}}A \arrow[r, "\Sigma_{\mathcal{T}}a'"] \arrow[d, equal, "id_{\Sigma_{\mathcal{T}}A}"] & \Sigma_{\mathcal{T}}B' \arrow[r, "\Sigma_{\mathcal{T}}b'"] \arrow[d, dashed, "h"] & \Sigma_{\mathcal{T}}C \arrow[d, equal, "id_{\Sigma_{\mathcal{T}}C}"] \\ C \arrow[r, "c"] & \Sigma_{\mathcal{T}}A \arrow[r, "-\Sigma_{\mathcal{T}}a"] & \Sigma_{\mathcal{T}}B \arrow[r, "-\Sigma_{\mathcal{T}}b"] & \Sigma_{\mathcal{T}}C \end{tikzcd}$$

By the 2 out of 3 property it is seen that h is an isomorphism, so the triple $(id_{\Sigma_{\mathcal{T}}^{-1}C}, id_A, \Sigma_{\mathcal{T}}^{-1}h)$ is an isomorphism of candidate triangles, and by TR1, is an isomorphism of triangles, asserting that $(\Sigma_{\mathcal{T}}^{-1}C, A, B, -\Sigma_{\mathcal{T}}^{-1}c, a, b)$ is in fact a triangle. □

Lemma 1.1.8. *Let (A,B,C,a,b,c) and (A',B',C',a',b',c') be two triangles, then the direct sum of these triangles is a triangle.*

Proof. Observe that direct sums of triangles admits long exact sequences of hom-functor, as $\mathcal{T}(K, A \oplus A') \simeq \mathcal{T}(K,A) \oplus \mathcal{T}(K,A')$. Thus the direct sum of the triangles has the following exact sequence.

$$A \oplus A' \xrightarrow{\begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix}} B \oplus B' \xrightarrow{\begin{pmatrix} b & 0 \\ 0 & b' \end{pmatrix}} C \oplus C' \xrightarrow{\begin{pmatrix} c & 0 \\ 0 & c' \end{pmatrix}} TA \oplus TC$$

$$\Downarrow$$

$$\ldots \longrightarrow \mathcal{T}(K,A) \oplus \mathcal{T}(K,A') \longrightarrow \mathcal{T}(K,B) \oplus \mathcal{T}(K,B') \longrightarrow$$

$$\longrightarrow \mathcal{T}(K,C) \oplus \mathcal{T}(K,C') \to \mathcal{T}(K,\Sigma_{\mathcal{T}}A) \oplus \mathcal{T}(K,\Sigma_{\mathcal{T}}A') \longrightarrow \ldots$$

The 2-out-of-3 property holds for the direct sum, via 5-lemma. By TR1 there is a triangle

$$A \oplus A' \longrightarrow B \oplus B' \longrightarrow D \longrightarrow \Sigma_{\mathcal{T}} A \oplus \Sigma_{\mathcal{T}} A'$$

By TR3 there are morphisms from this triangle to to each of the direct summands. Adding these maps together, there is a map from this triangle to direct sum. Using the 2-out-of-3 property this is an isomorphism of between a candidate triangle and a triangle, showing that the direct sum is a triangle.

$$\begin{tikzcd}
A \oplus A' \arrow[r] \arrow[d] & B \oplus B' \arrow[r] \arrow[d] & D \arrow[r] \arrow[d, dashed] & \Sigma_{\mathcal{T}} A \oplus \Sigma_{\mathcal{T}} A' \arrow[d] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & \Sigma_{\mathcal{T}} A
\end{tikzcd}$$

$$\&$$

$$\begin{tikzcd}
A \oplus A' \arrow[r] \arrow[d] & B \oplus B' \arrow[r] \arrow[d] & D \arrow[r] \arrow[d, dashed] & \Sigma_{\mathcal{T}} A \oplus \Sigma_{\mathcal{T}} A' \arrow[d] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & \Sigma_{\mathcal{T}} A'
\end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd}
A \oplus A' \arrow[r] \arrow[d, equal] & B \oplus B' \arrow[r] \arrow[d, equal] & D \arrow[r] \arrow[d, dashed, "\wr"] & \Sigma_{\mathcal{T}} A \oplus \Sigma_{\mathcal{T}} A' \arrow[d, equal] \\
A \oplus A' \arrow[r] & B \oplus B' \arrow[r] & A'' \oplus B'' \arrow[r] & \Sigma_{\mathcal{T}} A \oplus \Sigma_{\mathcal{T}} A'
\end{tikzcd}$$

□

Lemma 1.1.9. *The direct summands of a triangle is a triangle.*

Proof. The proof can be found in [1] □

1.2 Mapping Cones, Homotopies and Contractibility

Up until now the Octahedron axiom have not yet been used once, other than for proving TR3. Only by assuming TR1, TR2 and TR3 all of the results from the previous section follows. This is what will motivate the next definition.

Definition 1.2.1. A pre-triangulation of an additive category $\mathcal{T}$ with translation $\Sigma_{\mathcal{T}}$ is a collection $\Delta'_{\mathcal{T}}$ of triangles consisting of candidate triangles in $\mathcal{T}$ satisfying TR1, TR2 and TR3.
The category $\mathcal{T}$ with the pre-triangulation $\Delta'_{\mathcal{T}}$ is called a pre-triangulated category, and the candidate triangles in $\Delta'_{\mathcal{T}}$ are called triangles.

Remark. This notion of triangles will only be used in this section.

This section aims see how candidate triangles are constructed and formed. More importantly it will be discussed when these objects are triangles. These results are essential to motivate the definition of good morphisms between triangles. Lastly, other equivalent versions of TR4 will be presented, and the construction of weak kernels and cokernels will be shown. For this section it is assumed that $\mathcal{T}$ pre-triangulated.

Definition 1.2.2. Let $\phi : (A, B, C, a, b, c) \to (A', B', C', a', b', c')$ be a morphism of candidate triangles.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[r, "b"] \arrow[d, "\phi_B"] & C \arrow[r, "c"] \arrow[d, "\phi_C"] & \Sigma_{\mathcal{T}} A \arrow[d, "\Sigma_{\mathcal{T}}\phi_A"] \\ A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C \arrow[r, "c'"] & \Sigma_{\mathcal{T}} A' \end{tikzcd}$$

The mapping cone of ϕ is defined to be the candidate triangle below.

$$A' \oplus B \xrightarrow{\begin{pmatrix} b & \phi_B \\ 0 & -a' \end{pmatrix}} B' \oplus C \xrightarrow{\begin{pmatrix} c & \phi_C \\ 0 & -b' \end{pmatrix}} C' \oplus \Sigma_{\mathcal{T}} A \xrightarrow{\begin{pmatrix} \Sigma_{\mathcal{T}} a & \Sigma_{\mathcal{T}}\phi_A \\ 0 & -c' \end{pmatrix}} \Sigma_{\mathcal{T}} A' \oplus \Sigma_{\mathcal{T}} B$$

Definition 1.2.3. A morphism $\alpha : (A, B, C, a, b, c) \to (A', B', C', a', b', c')$ between candidate triangles is called null-homotopic if it factors through a homotopy. A homotopy is defined to be a triple of maps Θ, Φ, Ψ in the following sense.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\alpha_A"] & B \arrow[r, "b"] \arrow[d, "\alpha_B"] \arrow[dl, "\Theta"] & C \arrow[r, "c"] \arrow[d, "\alpha_C"] \arrow[dl, "\Phi"] & \Sigma_{\mathcal{T}} A \arrow[d, "\Sigma_{\mathcal{T}}\alpha_A"] \arrow[dl, "\Psi"] \\ A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C \arrow[r, "c'"] & \Sigma_{\mathcal{T}} A' \end{tikzcd}$$

It is required that $\alpha_A = \Theta a + \Sigma_{\mathcal{T}}^{-1}(c'\Psi)$, $\alpha_B = \Phi b + a'\Theta$ and $\alpha_C = \Psi c + b'\Phi$ for the triple to be a homotopy. Two maps are called homotopic if their difference is null-homotopic

Lemma 1.2.1. *The mapping cone only depends on morphisms up to homotopy. I.e. if two maps are homotopic, their mapping cones are isomorphic.*

Proof. Suppose that (f, g, h) and (f', g', h') are two homotopic morphisms of triangles:

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d] & B \arrow[r, "b"] \arrow[d] & C \arrow[r, "c"] \arrow[d] & \Sigma_{\mathcal{T}} A \arrow[d] \\ A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & \Sigma_{\mathcal{T}} A' \end{tikzcd}$$

Let (Θ, Φ, Ψ) be the homotopy between the triangle morphisms. Then there is an isomorphism of triangles.

$$\begin{tikzcd}[column sep=large] A' \oplus B \arrow[r, "\begin{pmatrix} b & g \\ 0 & -a' \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & \Theta \\ 0 & 1 \end{pmatrix}"] & B' \oplus C \arrow[r, "\begin{pmatrix} c & h \\ 0 & -b' \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & \Phi \\ 0 & 1 \end{pmatrix}"] & C' \oplus \Sigma_{\mathcal{T}} A \arrow[r, "\begin{pmatrix} \Sigma_{\mathcal{T}} a & \Sigma_{\mathcal{T}} f \\ 0 & -c' \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & \Psi \\ 0 & 1 \end{pmatrix}"] & \Sigma_{\mathcal{T}} A' \oplus \Sigma_{\mathcal{T}} B \arrow[d, "\begin{pmatrix} 1 & \Sigma_{\mathcal{T}}\Theta \\ 0 & 1 \end{pmatrix}"] \\ A' \oplus B \arrow[r, "\begin{pmatrix} b & g' \\ 0 & -a' \end{pmatrix}"'] & B' \oplus C \arrow[r, "\begin{pmatrix} c & h' \\ 0 & -b' \end{pmatrix}"'] & C' \oplus \Sigma_{\mathcal{T}} A \arrow[r, "\begin{pmatrix} \Sigma_{\mathcal{T}} a & \Sigma_{\mathcal{T}} f' \\ 0 & -c' \end{pmatrix}"'] & \Sigma_{\mathcal{T}} A' \oplus \Sigma_{\mathcal{T}} B \end{tikzcd}$$

□

Lemma 1.2.2. *Let A denote the candidate triangle (A, A', A'') and B denote (B, B', B''). Suppose $\alpha, \beta : A \to B$ are two homotopic morphisms of candidate triangles. Then for any map $\gamma : \widetilde{A} \to A$ and any map $\delta : B \to \widetilde{B}$ the maps $\delta\alpha\gamma$ and $\delta\beta\gamma$ are homotopic as well.*

Proof. To prove this statement it is enough to prove that $\alpha\gamma$ is homotopic to $\beta\gamma$ du to the symmetry of the statement. The goal is then to show that $(\Theta\gamma', \Phi\gamma'', \Psi\Sigma_{\mathcal{T}}\gamma)$ is the homotopy between these maps. This can be seen as

$$\alpha'\gamma' - \beta'\gamma' \;=\; (\alpha' - \beta')\gamma' \;=\; (b\Theta + \Phi a')\gamma' \;=\; b\Theta\gamma' + \Phi a'\gamma' \;=\; b(\Theta\gamma') + (\Phi\gamma'')\widetilde{a}'$$

. □

Definition 1.2.4. A candidate triangle A is called a contractible triangle if id_A is null-homotopic.

Remark. If A is a contractible triangle and $F : \mathcal{T} \to \mathcal{A}$ is an additive functor to an abelian category, then the identity of the cochain is null-homotopic as well.

$$\ldots \longrightarrow F(A) \longrightarrow F(A') \longrightarrow F(A'') \longrightarrow F(\Sigma_{\mathcal{T}}A) \longrightarrow \ldots$$

The homology of this sequence is therefore 0 everywhere, asserting that it is an exact sequence. The exactness of such sequences allow us to use the 2 out of 3 property on morphisms between contractible triangles.

Corollary 1.2.2.1. *If A is a contractible triangle, then any map in $\mathcal{T}(A, \_)$ or $\mathcal{T}(\_, A)$ is null-homotopic.*

Proof. By definition, being contractible is the same as the existence of a homotopy between the map and the zero map. If $id_A \sim 0 \implies f \circ id_A = f \sim f \circ 0 = 0$. So any map f is null-homotopic. □

Lemma 1.2.3. *A contractible triangle is a triangle.*

Proof. Let A be the contractible triangle (A, A', A''). Writing everything out, there is a homotopy between candidate triangles.

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "id_A"] & A' \arrow[r, "a'"] \arrow[d, "id_{A'}"] \arrow[dl, "\Theta"] & A'' \arrow[r, "a''"] \arrow[d, "id_{A''}"] \arrow[dl, "\Phi"] & \Sigma_{\mathcal{T}}A \arrow[d, "id_{\Sigma_{\mathcal{T}}A}"] \arrow[dl, "\Psi"] \\
A \arrow[r, "a"'] & A' \arrow[r, "a'"'] & A'' \arrow[r, "a''"'] & \Sigma_{\mathcal{T}}A
\end{tikzcd}$$

By using TR1 there is a triangle, and consequently, a long exact sequence.

$$A \xrightarrow{a} A' \xrightarrow{e} E \xrightarrow{e'} \Sigma_{\mathcal{T}}A$$
$$\Downarrow$$
$$\ldots \longrightarrow \mathcal{T}(\Sigma_{\mathcal{T}}A, A) \longrightarrow \mathcal{T}(\Sigma_{\mathcal{T}}A, A') \longrightarrow \mathcal{T}(\Sigma_{\mathcal{T}}A, E) \xrightarrow{e'_*} \mathcal{T}(\Sigma_{\mathcal{T}}A, \Sigma_{\mathcal{T}}A) \xrightarrow{\Sigma_{\mathcal{T}}a_*} \ldots$$

Since the map $\Sigma_{\mathcal{T}} a \circ a''\Psi = 0$ and by exactness at $\mathcal{T}(\Sigma_{\mathcal{T}}A, \Sigma_{\mathcal{T}}A)$, the kernel $Ker\Sigma_{\mathcal{T}}a_* = Ime'_* \neq 0$. This shows that there is a map $\Psi' : \mathcal{T}(\Sigma_{\mathcal{T}}A, E)$ such that $e'\Psi' = a''\Psi$, and the map $(id_A, id_{A'}, e\Theta + \Psi' a'')$ is a well defined map of candidate triangles. By the remark, one may use the 2 out of 3 properties to assert that the map found is an isomorphism, giving an isomorphism of triangles, showing that the contractible triangle is a triangle by the Bookkeeping axiom. □

Corollary 1.2.3.1. *The mapping cone of the zero map between triangles is a triangle.*

Proof. The mapping cone of the zero map can be seen to be the direct sum of two triangles. Thus it is a triangle. □

Corollary 1.2.3.2. *The mapping cone of a null-homotopic map between triangles is a triangle.*

A natural question to ask is when does the map between triangles admit a mapping cone which is a triangle. It has been shown that this is true whenever the map is null-homotopic. Then it can be seen that if either of the triangles the map is between is contractible, the mapping cone is a triangle. This sections main result shows the connection between triangulations and the realization of mapping cones as triangles.

Definition 1.2.5. A map between triangles will be called good if the mapping cone is a triangle.

Theorem 1.2.4. *A pre-triangulated category $\mathcal{T}$ is triangulated if given two triangles (A, B, C, a, b, c) and (A', B', C', a', b', c') and diagram (1) commutes, then diagram (1) can be completed to diagram (2) such that ϕ is good.*

$$\text{(1)}\quad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[d, "\phi_B"] \\ A' \arrow[r, "a'"] & B' \end{tikzcd} \qquad \text{(2)}\quad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[r, "b"] \arrow[d, "\phi_B"] & C \arrow[r, "c"] \arrow[d, dashed, "\phi_C"] & \Sigma_{\mathcal{T}}A \arrow[d, "\Sigma_{\mathcal{T}}\phi_A"] \\ A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & \Sigma_{\mathcal{T}}A' \end{tikzcd}$$

This result was shown by [1], however it also holds in the opposite direction. That is, if $\mathcal{T}$ is a triangulated category, then every pair of morphisms as in (1) may be completed to a good map between triangles. This highlights an interesting connection between mapping cones and the Octahedron axiom.

Definition 1.2.6. A commutative square (1) is called homotopy cartesian if and only if (2) is a triangle. One would say that homotopy cartesian squares arises from triangles.

$$\text{(1)}\quad \begin{tikzcd} D \arrow[r] \arrow[d] \arrow[dr, phantom, "HO"] & A \arrow[d] \\ B \arrow[r] & C \end{tikzcd} \quad \Longrightarrow \quad \text{(2)}\quad \begin{tikzcd} D \arrow[dr] & \\ & A \oplus B \arrow[dl] \\ C \arrow[uu, "\Sigma_{\mathcal{T}}"] & \end{tikzcd}$$

Remark. One method to construct homotopy cartesian squares is with homotopy pullbacks. A homotopy pullback is created with the application of TR1 and TR2, the procedure is drawn out below.

$$
\begin{tikzcd}
 & A \arrow[d, "a"] \\
B \arrow[r, "b"] & C
\end{tikzcd}
\overset{Collapse}{\Longrightarrow}
A \oplus B \xrightarrow{\begin{pmatrix} a & b \end{pmatrix}} C
$$

$$
\overset{TR1}{\Longrightarrow} \quad A \oplus B \xrightarrow{\begin{pmatrix} a & b \end{pmatrix}} C \longrightarrow \Sigma_{\mathcal{T}} D \longrightarrow \Sigma_{\mathcal{T}} A \oplus \Sigma_{\mathcal{T}} B \quad \overset{TR2}{\Longrightarrow}
\begin{tikzcd}
D \arrow[r] \arrow[d] \arrow[dr, phantom, "HO"] & A \arrow[d] \\
B \arrow[r] & C
\end{tikzcd}
$$

Dually, one may use homotopy push-outs to construct homotopy cartesian squares.

Remark. A remark about good maps and homotopy cartesian squares???

Lemma 1.2.5. *Suppose that there is a homotopy cartesian square (1). Then there are triangles and a morphism of triangles as in (2).*

$$
(1) \quad
\begin{tikzcd}
D \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "HO"] & A \arrow[d, "f"] \\
B \arrow[r, "g"] & C
\end{tikzcd}
\qquad (2) \quad
\begin{tikzcd}
D \arrow[r] \arrow[d, "f'"] & A \arrow[r] \arrow[d, "f"] & E \arrow[r] \arrow[d, equal] & \Sigma_{\mathcal{T}} D \arrow[d, "\Sigma_{\mathcal{T}} f'"] \\
B \arrow[r] & C \arrow[r] & E \arrow[r] & \Sigma_{\mathcal{T}} B
\end{tikzcd}
$$

Proof. There is a commutative square (1) which satisfies the requirements of the octahedron axiom (2), yielding a triangle (3).

$$
(1) \quad
\begin{tikzcd}
D \arrow[r, "\begin{pmatrix} g' \\ f' \end{pmatrix}"] \arrow[d, equal] & A \oplus B \arrow[d, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] \\
D \arrow[r, "g'"] & A
\end{tikzcd}
$$

$$
(2) \quad
\begin{tikzcd}
D \arrow[dr, "\begin{pmatrix} g' \\ f' \end{pmatrix}"] & \\
 & A \oplus B \arrow[dl, "\begin{pmatrix} f & g \end{pmatrix}"] \\
C \arrow[uu, "\Sigma_{\mathcal{T}} 0"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \oplus B \arrow[dr, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] & \\
 & A \arrow[dl, "0"] \\
\Sigma_{\mathcal{T}} B \arrow[uu, "\Sigma_{\mathcal{T}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
D \arrow[dr, "f'"] & \\
 & A \arrow[dl] \\
E \arrow[uu, "\Sigma_{\mathcal{T}}"] &
\end{tikzcd}
$$

$$
(3) \quad C \longrightarrow E \longrightarrow \Sigma_{\mathcal{T}} B \xrightarrow{\Sigma_{\mathcal{T}} g} \Sigma_{\mathcal{T}} C
$$

By TR4 (3) is a triangle, and two commutative squares as below. Every arrow should be understood to be the arrow from its corresponding triangle.

$$
\begin{tikzcd}
A \oplus B \arrow[r] \arrow[d] & A \arrow[d] \arrow[dl, dotted, "f"] \\
C \arrow[r] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r] \arrow[d] & \Sigma_{\mathcal{T}} D \arrow[d] \arrow[dl, dotted, "\Sigma_{\mathcal{T}} g"'] \\
\Sigma_{\mathcal{T}} B \arrow[r] & \Sigma_{\mathcal{T}} A \oplus \Sigma_{\mathcal{T}} B
\end{tikzcd}
$$

Since the composition $B \xrightarrow{g} C \longrightarrow E$ is 0, it is seen that the lower simplex in the first diagram commute. Dually, the upper simplex in the second triangle also commutes. This is exactly the condition that the triple of morphisms is a morphism of triangles, as illustrated below.

$$\begin{tikzcd} D \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, equal] & \Sigma_{\mathcal{T}} D \arrow[d] \\ B \arrow[r] & C \arrow[r] & E \arrow[r] & \Sigma_{\mathcal{T}} B \end{tikzcd}$$

□

1.3 Calculus of Fractions and the Verdier Quotient

One important construction of triangulated categories is the Verdier Quotient. This construction is a localization of a triangulated category at some set related to a triangulated subcategory, this gives the construction some resemblance of classical quotients. This section aims to introduce the concept localization of categories, as well as show how triangulated categories fit within this theory. Localization is most notably known in commutative algebra where elements are given formal inverses. The idea for categories is to attach formal inverses of morphisms onto the category.

Definition 1.3.1. Let S be a collection of morphisms in the category $\mathcal{C}$. The Localization of $\mathcal{C}$ on $\mathcal{S}$ is the category $\mathcal{C}[S^{-1}]$ together with a functor $q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ such that:

- $\forall s : S | q(s)$ is an isomorphism
- Any functor $F : \mathcal{C} \to \mathcal{D}$ such that $\forall s : S\ F(s)$ is an isomorphism, then F factors through q. That is to say that there is a natural isomorphism $\eta : F \to F' \circ q$ so that $\mathcal{C}[S^{-1}]$ is the universal category where morphisms in S are isomorphisms.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "q"'] & & \mathcal{D} \\ & S^{-1}\mathcal{C} \arrow[ur, dashed, "F'"'] & \end{tikzcd}$$

Remark. Even though it is known that $\mathcal{C}$ is locally small, it is not clear a priori that the category $\mathcal{C}[S^{-1}]$ is again locally small. Thus it is not evident that these localizations exists.

Remark. Suppose $X : \mathcal{C}$, then one may always assume that $id_X : S$. To see this, let $T = \{id_X | \forall X : \mathcal{C}\}$, then it is evident that $Id_{\mathcal{C}}$ is the universal functor in which morphisms in T are inverted. Thus adding identities to a set, does not alter its localization. Thus it will be assumed that the sets used in localization have identity morphisms.

In general, it is difficult to describe a method to construct the localization of a category at a set. This discussion will however be much easier if one are to put assumptions on the set S of morphisms. In order to mimic the construction of localization of rings, one wants to assume that S is a multiplicative system.

Definition 1.3.2. A set S of morphisms in a category $\mathcal{C}$ is called right multiplicative if it satisfies the following conditions:

- S is closed under composition, i.e. if $f,g : S$ are composable then $gf : S$. Every identity morphism in $\mathcal{C}$ is in S.
- (Right Ore condition) If $t : X \to Y$ is a morphism in S, then $\forall g : Z \to Y$ there is a commutative square (1) such that $f : W \to X$ and $s : W \to Z$ exists, where $s : S$.

$$(1) \quad \begin{tikzcd} W \arrow[r, dashed, "f"] \arrow[d, dashed, "s"] & X \arrow[d, "t"] \\ Z \arrow[r, "g"] & Y \end{tikzcd}$$

- (Left cancellation) Suppose $f,g : X \to Y$ are parallel morphisms in $\mathcal{C}$, then 1. $\Longrightarrow$ 2.:
  1. $sf = sg$ for som $s : S$ starting at Y
  2. $ft = gt$ for som $t : S$ ending at X

Remark. The previous definition has a dual statement. A set S of morphisms is left multiplicative if it satisfies:

- S is closed under composition, i.e. if $f,g : S$ are composable then $gf : S$. Every identity morphism in $\mathcal{C}$ is in S.
- (Left Ore condition) If $s : Y \to Z$ is a morphism in S, then $\forall f : Y \to X$ there is a commutative square (1) such that $g : Z \to W$ and $t : X \to W$ exists and $t : S$ as well.

$$(1) \quad \begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "s"] & X \arrow[d, dashed, "t"] \\ Z \arrow[r, dashed, "g"] & W \end{tikzcd}$$

- (Right cancellation) Suppose $f,g : X \to Y$ are parallel morphisms in $\mathcal{C}$, then 1. $\Longrightarrow$ 2.:
  1. $ft = gt$ for som $t : S$ ending at X
  2. $sf = sg$ for som $s : S$ starting at Y

If S is both right multiplicative and left multiplicative then it is called multiplicative.

As with the definition of localization of rings, localization of a category $\mathcal{C}$ at a multiplicative system will be defined with fractions. That is the morphisms will be "fractions" of morphisms. These morphisms will be described as diagrams over spans for right multiplicative systems (or dually cospans for left multiplicative systems), together with an equivalence relation.

Definition 1.3.3. A span is a diagram of the form:

$$\cdot \longleftarrow \cdot \longrightarrow \cdot$$

Definition 1.3.4. Let S be a right multiplicative system of morphisms in a category $\mathcal{C}$. Given a morphism $s : Y \to X$ in S and a morphism $t : Y \to Z$, define the right fraction of s and t to be the span of the morphisms. That is s and t fit in the diagram below.

$$\begin{tikzcd} X & Y \arrow[l, "s"'] \arrow[r, "t"] & Z \end{tikzcd}$$

Right fractions are denoted as ts^{-1}. Let $\sim$ be the equivalence relation of right fractions given by the diagram (1) such that $ts^{-1} \sim t's'^{-1}$ if and only if $\exists w, w' : \mathcal{C}$ making the diagram commute and that the middle row is a right fraction.

$$\begin{tikzcd} & Y \arrow[dl, "s"'] \arrow[dr, "t"] & \\ X & W \arrow[l, dashed] \arrow[r, dashed] \arrow[u, "w"] \arrow[d, "w'"] & Z \\ & Y' \arrow[ul, "s'"] \arrow[ur, "t'"'] & \end{tikzcd}$$

Dually, define left fractions as diagrams over cospans such that if $t : S$, then there is a left fraction $t^{-1}s$ as the diagram below.

$$\begin{tikzcd} X \arrow[r, "s"] & Y & Z \arrow[l, "t"'] \end{tikzcd}$$

The equivalence relation $\sim$ is given by the diagram in the same manner as above.

$$\begin{tikzcd} & Y \arrow[d, "w"] & \\ X \arrow[ur, "s"] \arrow[r, dashed] \arrow[dr, "s'"'] & W & Z \arrow[ul, "t"'] \arrow[l, dashed] \arrow[dl, "t'"] \\ & Y' \arrow[u, "w'"'] & \end{tikzcd}$$

Proposition 1.3.1. *Suppose that S is a right multiplicative system, then the relation stated above is in fact an equivalence relation.*

Proof. An equivalence relation is proven by showing that $\sim$ is reflexive, symmetric and transitive.

- (Reflexive) Let fs^{-1} be a right fraction. Then diagram (1) shows that $fs^{-1} \sim fs^{-1}$.

$$(1) \quad \begin{tikzcd} & W \arrow[dl, "s"'] \arrow[dr, "f"] \arrow[d, equal] & \\ X & W \arrow[l, dashed, "s"] \arrow[r, dashed, "f"] \arrow[d, equal] & Y \\ & W \arrow[ul, "s"] \arrow[ur, "f"'] & \end{tikzcd}$$

- (Symmetric) Let fs^{-1} and gt^{-1} be two right fractions such that $fs^{-1} \sim gt^{-1}$, that is diagram (2) commute. Due to inherent symmetric nature of the diagram it follows that $gt^{-1} \sim fs^{-1}$.

$$(2)\quad \begin{tikzcd} & W \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & \widetilde{W} \arrow[l, dashed] \arrow[r, dashed] \arrow[u, "w"] \arrow[d, "w'"] & Y \\ & W' \arrow[ul, "t"] \arrow[ur, "g"'] & \end{tikzcd} \implies \begin{tikzcd} & W' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & \widetilde{W} \arrow[l, dashed] \arrow[r, dashed] \arrow[u, "w'"] \arrow[d, "w"] & Y \\ & W \arrow[ul, "s"] \arrow[ur, "f"'] & \end{tikzcd}$$

- (Transitive) Suppose that there are three right fractions fs^{-1}, gt^{-1} and hu^{-1} such that $fs^{-1} \sim gt^{-1}$ and $gt^{-1} \sim hu^{-1}$. This may be written as diagram (3) and (4).

$$(3)\quad \begin{tikzcd} & W' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & \widetilde{W} \arrow[l, dashed] \arrow[r, dashed] \arrow[u, "w'"] \arrow[d, "\widetilde{w'}"] & Y \\ & W \arrow[ul, dashed, "t"] \arrow[ur, "g"'] & \end{tikzcd} \quad (4)\quad \begin{tikzcd} & W \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & \widetilde{\widetilde{W}} \arrow[l, dashed] \arrow[r, dashed] \arrow[u, "\widetilde{w''}"] \arrow[d, "w''"] & Y \\ & W'' \arrow[ul, "u"] \arrow[ur, "h"'] & \end{tikzcd}$$

Diagram (5) may be created by using the Ore condition on the maps $\widetilde{w'}$ and $\widetilde{w''}$. Since both morphisms are assumed to be in S, it follows that both $\widetilde{w'}$ and $\widetilde{w''}$ are in S as well. Diagram (6) then shows that $fs^{-1} \sim hu^{-1}$.

$$(5)\quad \begin{tikzcd} \widetilde{\widetilde{\widetilde{W}}} \arrow[r, "\widetilde{\widetilde{w''}}"] \arrow[d, "\widetilde{\widetilde{w'}}"] & \widetilde{\widetilde{W}} \arrow[d, "\widetilde{w''}"] \\ \widetilde{W} \arrow[r, "\widetilde{w'}"] & W \end{tikzcd} \quad (6)\quad \begin{tikzcd} & W' \arrow[ddl, "s"'] \arrow[ddr, "f"] & \\ & \widetilde{W} \arrow[u, "\widetilde{w'}"] \arrow[dl] \arrow[dr] & \\ X & \widetilde{\widetilde{\widetilde{W}}} \arrow[l, dashed] \arrow[r, dashed] \arrow[u, "\widetilde{\widetilde{w'}}"] \arrow[d, "\widetilde{\widetilde{w''}}"] & Y \\ & \widetilde{\widetilde{W}} \arrow[ul] \arrow[ur, "h"'] \arrow[d, "\widetilde{w''}"] & \\ & W'' \arrow[uul, "u"] \arrow[uur] & \end{tikzcd}$$

$\square$

Definition 1.3.5. Let S be a multiplicate system in a category $\mathcal{C}$. Given two right fractions fs^{-1} and gt^{-1}

$$X \xleftarrow[s]{} W \xrightarrow{f} Y \quad \& \quad Y \xleftarrow[t]{} W' \xrightarrow{g} Z$$

the composition of the fractions are defined to be $gt^{-1} \circ fs^{-1}$. The Ore condition describes how this composition should be defined,

$$\begin{tikzcd}
& \widetilde{W} \arrow[r, dashed, "h"] \arrow[d, dashed, "u"] & W' \arrow[r, "g"] \arrow[d, "t"] & Z \\
X & W \arrow[l, "s"] \arrow[r, "f"] & Y
\end{tikzcd}$$

the composite is the right fraction $gt^{-1} \circ fs^{-1} = gh(su)^{-1}$.

Proposition 1.3.2. *The composition of right fractions is well-defined up to equivalence.*

Proof. In order to prove that the composite is well-defined one must prove that the composite is independent from the different options of morphisms provided by the right Ore condition, and that it is therefore independent from the choice of right fraction. There will only be presented a proof for that the choice of Ore maps is independent, as the other other case is analogous.

Suppose there are two right fractions fs^{-1} and gt^{-1} as indicated by the diagrams.

$$X \xleftarrow[s]{} W_1 \xrightarrow{f} Y \quad \& \quad Y \xleftarrow[t]{} W_2 \xrightarrow{g} Z$$

Further suppose that there are at least two different choices for the morphisms provided by the right Ore condition, for example $\widetilde{W}$ and $\widehat{W}$. The two compositions may be drawn as the diagrams below.

$$\begin{tikzcd}
& \widetilde{W} \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{s}"] & W_2 \arrow[r, "g"] \arrow[d, "t"] & Z \\
X & W_1 \arrow[l, "s"] \arrow[r, "f"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
& \widehat{W} \arrow[r, "\hat{f}"] \arrow[d, "\hat{s}"] & W_2 \arrow[r, "g"] \arrow[d, "t"] & Z \\
X & W_1 \arrow[l, "s"] \arrow[r, "f"] & Y
\end{tikzcd}$$

Combining the diagrams at W_1 by using the right Ore condition, the object W exists as in the diagram below together with its corresponding maps.

$$\begin{tikzcd}
\bar{W} \arrow[dr, dashed, "\xi"] & & & & \\
& W \arrow[r, "\hat{w}"] \arrow[d, "\tilde{w}"] & \widehat{W} \arrow[r, "\hat{f}"] \arrow[d, "\hat{s}"] & W_2 \arrow[r, "g"] \arrow[d, "t"] & Z \\
& \widetilde{W} \arrow[r, "\tilde{s}"] \arrow[urr, "\tilde{g}"] & W_1 \arrow[r, "f"] \arrow[d, "s"] & Y & \\
& & X & &
\end{tikzcd}$$

Observe that the three squares commute, as by the definition of right Ore condition. Thus it follows that $s\tilde{s}\tilde{w} = s\hat{s}\hat{w}$, and that $t\hat{f}\hat{w} = t\tilde{g}\tilde{w}$. As $t : S$ one may use right cancellation to find a $\xi : \bar{W} \to W$ such that $\hat{f}\hat{w}\xi = \tilde{g}\tilde{w}\xi \implies g\hat{f}\hat{w}\xi = g\tilde{g}\tilde{w}\xi$. Thus the equivalence relation diagram commutes.

$$\begin{tikzcd}
 & \widehat{W} \arrow[dl, "s\widehat{s}"'] \arrow[dr, "g\widehat{f}"] & \\
X & \bar{W} \arrow[l, dashed] \arrow[r, dashed] \arrow[u, "\widehat{w}\xi"'] \arrow[d, "\widetilde{w}\xi"] & Z \\
 & \widetilde{W} \arrow[ul, "s\widetilde{s}"] \arrow[ur, "g\widetilde{g}"'] &
\end{tikzcd}$$

□

Proposition 1.3.3. *The composition of right fractions is associative.*

Proof. Sketch. Let fs^{-1}, gt^{-1} and hu^{-1} be right fractions as in the diagrams below.

$$A \xleftarrow[s]{} X \xrightarrow{f} B \quad , \quad B \xleftarrow[t]{} Y \xrightarrow{g} C \quad \& \quad C \xleftarrow[u]{} Z \xrightarrow{h} D$$

There are two different ways of calculating the composition. Every morphism in S will be marked blue.

$\underline{hu^{-1}\circ(gt^{-1}\circ fs^{-1})}$

$$\begin{tikzcd}
W \arrow[rr] \arrow[d] & & Z \arrow[r] \arrow[d] & D \\
V \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & C & \\
X \arrow[r] \arrow[d] & B & & \\
A & & &
\end{tikzcd}$$

$\underline{(hu^{-1}\circ gt^{-1})\circ fs^{-1}}$

$$\begin{tikzcd}
V' \arrow[r] \arrow[dd] & W' \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & D \\
 & Y \arrow[r] \arrow[d] & C & \\
X \arrow[r] \arrow[d] & B & & \\
A & & &
\end{tikzcd}$$

To be able to find a relation between these diagrams create another diagram with the right Ore condition.

$$\begin{tikzcd}
T \arrow[r, dashed] \arrow[d, dashed] & V' \arrow[d] \\
W \arrow[r] & X
\end{tikzcd}$$

To finish the proof, one would need to show that the maps to A and D commute. The maps to A commute right out of the bat, by the right Ore condition. To prove that the maps to D commute, first apply right cancellation on the maps to B, then on the maps to C.

□

Definition 1.3.6. Let S be a right multiplicative system in a category $\mathcal{C}$. Define a category $\mathfrak{r}S^{-1}\mathcal{C}$ to have objects $\mathfrak{Obr}S^{-1}\mathcal{C} = \mathfrak{Ob}\mathcal{C}$ and morphisms $\mathfrak{Arr}S^{-1}\mathcal{C} = \{$right fractions of $S\}/\sim$. This means that the morphisms $\mathfrak{r}S^{-1}\mathcal{C}(X,Y)$ are spans in $\mathcal{C}$ where one of the maps are in S up to equivalence.

$$X \longleftarrow A \longrightarrow Y$$

This is well-defined by the previous results and the identity morphisms are the right fractions of the form:

$$X =\!=\!= X =\!=\!= X$$

Remark. Dually there is a category $\mathfrak{l}S^{-1}\mathcal{C}$ for a left multiplicative system S in a category $\mathcal{C}$. It is defined in the same manner as $\mathfrak{r}S^{-1}\mathcal{C}$, but with left fractions instead.

A priori it does still not make sense to ask for these kind of categories to always exist. The class of morphisms $S^{-1}\mathcal{C}(A,B)$ consists of a large collection of morphism from $\mathcal{C}$. This collection has been described as a disjoint union of sets on the form $(f,g) : \mathcal{C}(A,X) \times \mathcal{C}(X,B)$ modulo an equivalence relation. This disjoint union spans over the whole category $\mathcal{C}$ with the X index. Since $\mathcal{C}$ is not assumed to be small, there is no reason for this set to be small as well. To see how this problem might be untangled, this collection will be further studied. This discussion is based on the results and definitions from [2] and [3].

Definition 1.3.7. Let $\mathcal{C}$ be a category and S a collection of morphisms from $\mathcal{C}$, such that every identity morphism is in S and that it is closed under composition. Define the subcategory $\mathcal{C}|_S \subseteq \mathcal{C}$ to have $\mathfrak{Ob}\mathcal{C}|_S = \mathfrak{Ob}\mathcal{C}$ and $\mathfrak{Ar}\mathcal{C} = S$.

This definition is seen to be well-defined, as it is closed under composition of morphisms and every identity morphism is in $\mathcal{C}|_S$ by assumption. Suppose now that S is a right multiplicative system, then the category $\mathcal{C}|_S$ is defined. For any $A : \mathcal{C}$, look at the category $\mathcal{C}|_S \downarrow A$ which have objects $\mathfrak{Ob}\mathcal{C}|_S \downarrow A = \{s : X \to A \mid s : S\}$ and morphisms as indicated by the squiggly arrow in the commutative diagram below.

$$\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[d, squiggly, "\sigma"] & & X \arrow[d, squiggly, "\sigma"] \\ A & Y \arrow[l, "t"] & \overset{\delta_A}{\Longrightarrow} & Y \end{tikzcd}$$

This category has a forgetful functor which associate each morphism to its domain $\delta_A : \mathcal{C}|_S \downarrow A \to \mathcal{C}$, and forgets the commutativity of the arrows. Choose a morphism $s : X \to A$ from $\mathcal{C}|_S \downarrow A$, then a morphism $g : \mathcal{C}(\delta_A(s), B)$ may be regarded as a right fraction gs^{-1}. In order to describe every possible right fraction from A to B, consider the following colimit $\varinjlim \mathcal{C}(\delta_A(\_), B)$ over the category $\mathcal{C}|_S \downarrow A$. Since S is a right multiplicative system it follows that $\mathcal{C}|_S$ is a cofiltered category, and the colimit is therefore filtered. As the colimit is filtered, it is created from the coproduct modulo an equivalence relation $\sim$.

The relation $\sim$ can be described with maps from S. Suppose that there are two morphisms $f : \mathcal{C}(\delta_A(s), B)$ and $g : \mathcal{C}(\delta_A(t), B)$, and that there exists some morphism $\sigma : \mathcal{C}|_S(s,t)$. The induced morphism $\sigma^* : \mathcal{C}(\delta_A(t), B) \to \mathcal{C}(\delta_A(s), B)$ defines the relation, where $f \sim t$ if $f = g\sigma$. Observe that this relation is not an equivalence relation, so $\sim$ has to be the smallest equivalence relation generated by such relations. The smallest such equivalence relation may be seen to consist of zig-zags between morphisms in S, connecting two morphisms. Luckily, the right Ore condition simplifies this picture, reducing to at most 1 zig-zag. To illustrate with 2 zig-zags, consider two maps $f : \mathcal{C}(\delta_A(s), B)$ and $g : \mathcal{C}(\delta_A(t), B)$, where there are zig-zag morphisms $\sigma : s \to v$, $\tau : u \to v$ and $\upsilon : u \to t$ such that the diagram below commute, relating f and g, i.e. $fs^{-1} = gt^{-1}$. It is evident that this equivalence relation is exactly the same as stated earlier in this section.

$$\begin{tikzcd}
 & \delta_A(s) \arrow[dl, "s"'] \arrow[d, "\sigma"] \arrow[dr, "f"] & \\
A \arrow[d, equal] & \delta_A(u) \arrow[l, dotted, "u"] \arrow[r, dotted] & B \arrow[d, equal] \\
A & \delta_A(v) \arrow[l, dotted, "v"] \arrow[r, dotted] \arrow[u, "\tau"] \arrow[d, "\upsilon"] & B \\
 & \delta_A(t) \arrow[ul, "t"] \arrow[ur, "g"'] &
\end{tikzcd}$$

If S is instead a left multiplicative system one would have to consider the colimit $\varinjlim \mathcal{C}(A, \gamma_B(\_))$ over the category $B \downarrow \mathcal{C}|_S$. Here γ_B is the codomain functor, and $B \downarrow \mathcal{C}|_S$ may be seen to be filtered as S is left multiplicative. The discussion would be dual to the right multiplicative case.

The localization of a category exists whenever each hom-set is in fact a set. That is to say that $\mathfrak{r}S^{-1}\mathcal{C}(A, B) \simeq \varinjlim \mathcal{C}(\delta_A(\_), B)$ is a Set, which is to ask for the colimit to exist for every A and B. One assumption which does this is to assume that the colimit is equivalent to a smaller colimit.

Definition 1.3.8. A right multiplicative system S in a locally small category $\mathcal{C}$ is called locally small on the right if for every object $X : \mathcal{C}$ there is a set $\widehat{X}$, with a small category $\mathcal{C}|_{\widehat{X}} \downarrow X$ and a forgetful functor $\widehat{\delta_X} : \mathcal{C}|_{\widehat{X}} \downarrow X \to \mathcal{C}$, such that the colimit functor $\varinjlim \mathcal{C}(\widehat{\delta_X}(\_), \_) : \mathcal{C} \to Set$ actually evaluates in Set. Moreover there is an isomorphism, natural in both arguments X and Y, $\varinjlim \mathcal{C}(\widehat{\delta_X}(\_), Y) \simeq \varinjlim \mathcal{C}(\delta_X(\_), Y)$.
Dually, a locally left multiplicative system would require the contravariant colimit functor to evaluate in Set.

Theorem 1.3.4. ***Gabriel-Zisman***. *Let S be a locally small right multiplicative system of morphisms in a category $\mathcal{C}$. Then the category $\mathfrak{r}S^{-1}\mathcal{C}$ exists and it is the localization of $\mathcal{C}$ on S. This mean that there is an equivalence of categories $\mathcal{C}[S^{-1}] \simeq \mathfrak{r}S^{-1}\mathcal{C}$ together with a functor $q : \mathcal{C} \to \mathfrak{r}S^{-1}\mathcal{C}$ sending a morphism $f : X \to Y$ to the right fraction $f id_X^{-1}$.*

Proof. To prove the theorem one must show that q is a functor, and that it is universal. Suppose that $f : X \to Y$ and $g : Y \to Z$ are morphisms in $\mathcal{C}$. Then $q(gf) = (gf)id_X^{-1}$ and $q(g)q(f) = (gid_Y^{-1}) \circ (f id_X^{-1})$. Choose the composition to be defined by the diagram below.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \\ X \arrow[r, "f"] \arrow[d, equal] & Y & \\ X & & \end{tikzcd}$$

Observe that $(gid_Y^{-1}) \circ (fid_X^{-1}) = (gf)id_X^{-1}$, asserting the functoriality of q.
To see that q is universal let $\mathcal{D}$ be a category where every morphism of S is an isomorphism, and suppose there is a functor $F : \mathcal{C} \to \mathcal{D}$. Define a functor $\mathfrak{r}S^{-1}F : \mathfrak{r}S^{-1}\mathcal{C} \to \mathcal{D}$ by $\mathfrak{r}S^{-1}F(fs^{-1}) = F(f)F(s)^{-1}$. One may see that $F = \mathfrak{r}S^{-1}F \circ q$, it remains to show that it is well-defined. Suppose $fs^{-1} = gt^{-1}$, that means there is a diagram in $\mathcal{C}$ with the blue arrows in S.

$$\begin{tikzcd} & W' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & W \arrow[u, "w'"'] \arrow[d, "w''"] & Y \\ & W'' \arrow[ul, "t"] \arrow[ur, "g"'] & \end{tikzcd}$$

Thus there is a relationship in $\mathcal{D}$ such that $F(t) = F(sw')F(w'')^{-1}$ and $F(g) = F(fw')F(w'')^{-1}$. This again shows that

$$\begin{aligned} \mathfrak{r}S^{-1}F(gt^{-1}) &= F(g)F(t)^{-1} \\ &= F(fw')F(w'')^{-1}(F(fw')F(w'')^{-1})^{-1} = F(fw')F(w'')^{-1}F(w'')F(sw')^{-1} \\ &= F(f)F(w')F(w')^{-1}F(s)^{-1} = F(f)F(s)^{-1} = \mathfrak{r}S^{-1}F(fs^{-1}) \end{aligned}$$

It follows that $\mathfrak{r}S^{-1}F$ is well-defined and is unique by construction. □

Corollary 1.3.4.1. *If S is a locally small left multiplicative system instead then $\mathfrak{l}S^{-1}\mathcal{C}$ is the localization of $\mathcal{C}$ on S.*
If moreover S is a locally small multiplicative system, then there is an equivalence of categories $\mathfrak{r}S^{-1}\mathcal{C} \simeq \mathfrak{l}S^{-1}\mathcal{C}$.

Proof. The first statement is dual to the theorem.
To see the other statement, note that both $\mathfrak{r}S^{-1}\mathcal{C}$ and $\mathfrak{l}S^{-1}\mathcal{C}$ are the universal categories where the morphisms of S are isomorphisms. Thus it follows that these categories have to be equivalent. □

Remark. Since righthandedness of lefthandedness of the multiplicative system S doesn't affect the localization, one simply call the localization of a (left/right) multiplicative system for $S^{-1}\mathcal{C}$.

Remark. A morphisms $f : \mathcal{C}(X, Y)$ will be invertible in the localized category if it is in the same equivalence class as the identity, both id_X and id_Y. This forces a morphism f to be invertible in $S^{-1}\mathcal{C}$ if and only if there is $g, h : S$ such that $fg, hf : S$.

Proposition 1.3.5. *Let $\mathcal{C}$ be a category, and S a right multiplicative set of morphisms. The canonical functor $q : \mathcal{C} \to S^{-1}\mathcal{C}$ commutes with finite limits.*

Proof. Let $T : \mathcal{D} \to \mathcal{C}$ be a diagram over a finite category $\mathcal{D}$. Then for any object $A : S^{-1}\mathcal{C}$ one may find the following equation.

$$\begin{aligned} S^{-1}\mathcal{C}(qA, q(\varprojlim T\_)) &\simeq \varinjlim \mathcal{C}(\delta\_, \varprojlim T\_) \\ &\simeq \varinjlim \varprojlim \mathcal{C}(\delta\_, T\_) \simeq \varprojlim \varinjlim \mathcal{C}(\delta\_, T\_) \simeq \varprojlim S^{-1}\mathcal{C}(qA, q(T\_)) \end{aligned}$$

The first isomorphism is given by the remark and the second is given by the representative nature of finite limits. The third isomorphism is given by that filtered colimits commute with finite limits in the category Set, this is shown as theorem 3.8.9 in [4]. The colimits have been shown to be filtered by the discussion of $\mathcal{C}|_S \downarrow A$, as the category is cofiltered, but considered as a contravariant diagram. $\square$

Remark. For the purpose of this thesis, it is also needed that the proposition above also holds for categories enriched over abelian groups. Luckily, there are other arguments which allow for the interchange of the filtered colimit and the finite limit.

Proposition 1.3.6. *Let $\mathcal{C}$ be a category with a zero. That is an object which is both initial and terminal. Suppose that S is a right multiplicative system, then $q0$ is a zero object in $S^{-1}\mathcal{C}$.*

Proof. The claim that $q0$ is initial follows from that initial is a limit of a diagram over the empty category. To see that $q0$ is terminal one have to prove that every right fraction of the form $0f^{-1}$ is equivalent to $0id_A^{-1}$, where A is the codomain of f. This fact can be seen with the diagram below.

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, "f"] \arrow[dr, "0"] & \\ A \arrow[r, equal] & A \arrow[r, "0"] & 0 \end{tikzcd}$$

$\square$

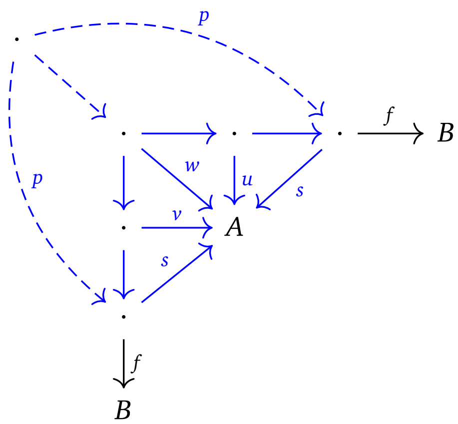
Proposition 1.3.7. *If $\mathcal{A}$ is an additive category and S is a right multiplicative system, then $S^{-1}\mathcal{A}$ is additive as well.*

Proof. From the previous propositions it is known that $q0$ is the zero object and that $q(A \times B) \simeq qA \times qB$. By proving that there is an addition induced by $\mathcal{A}$ and that q preserves this addition one obtains that the product is the biproduct induced by the maps in $\mathcal{A}$.

Suppose that there are fractions $fs^{-1}, gt^{-1} : S^{-1}\mathcal{C}(A, B)$. Define their addition by using the right Ore condition to find new morphisms f', g' and u such that $fs^{-1} = f'u^{-1}$ and $gt^{-1} = g'u^{-1}$.

$$fs^{-1} + gt^{-1} = (f' + g')u^{-1}$$

To prove that this is an addition one must prove that it is well defined; associativity, inverses and commutativity will be inherited from $\mathcal{A}$. Let $\bar{f}$, $\bar{g}$ and v be another choice provided by the right Ore condition. To summarize, the equations $\bar{f}v^{-1} = fs^{-1} = f'u^{-1}$ and $\bar{g}v^{-1} = gt^{-1} = g'u^{-1}$ have been established. In order to prove well-definedness one must show that $(\bar{f}+\bar{g})v^{-1} - (f'+g')u^{-1} = 0$. By definition $(\bar{f}+\bar{g})v^{-1} - (f'+g')u^{-1} = \bar{f}v^{-1} - f'u^{-1} + \bar{g}v^{-1} - g'u^{-1}$. Proving that the whole sum is 0, is the same as proving that $\bar{f}v^{-1} + (-f')u^{-1} = (\bar{\bar{f}} - f'')w^{-1} = 0$. This can be done by writing out the diagrams after repeatedly applying the right Ore condition.



The line to the bottom represents $\bar{\bar{f}}$ and the line to the right represents f''. Using left cancellation on the common morphism s into A one obtains the morphism p, which relates the two fractions and make the sum go to zero.

It remains to show that $q : \mathcal{C} \to S^{-1}\mathcal{C}$ respects addition. Assume that $f, g : \mathcal{C}(X, Y)$, then

$$q(f+g) = (f+g)id_X^{-1} = f\,id_X^{-1} + g\,id_X^{-1} = qf + qg.$$

$\square$

Corollary 1.3.7.1. *If $\mathcal{A}$ is abelian and S is a multiplicative system, then $S^{-1}\mathcal{A}$ is abelian as well.*

General descriptions on how to localize categories have been discussed. The next natural step is to look at localization of triangulated categories. The goal is to define the Verdier quotient for triangulated categories. The idea of this localization is to mimic quotient modules from algebra in a categorical setting. Thus triangulated subcategories will be in the center of this discussion. This method has been described by [1] in a broader term than what was originally proposed by Verdier.

Definition 1.3.9. A triangulated subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$ is a full additive subcategory such that the inclusion functor is triangulated.

Definition 1.3.10. Let $F : \mathcal{S} \to \mathcal{T}$ be a triangulated functor. The kernel of F is defined to be the full subcategory $Ker(F)$ of $\mathcal{S}$ such that every object in $Ker(F)$ gets mapped to 0 by F. That is, $Ker(F)$ is the class of objects $\{K : \mathcal{S} | F(K) \simeq 0\}$.

Lemma 1.3.8. *The kernel of a triangulated functor $F : \mathcal{C} \to D$ is a triangulated subcategory.*

Proof. Let $X : KerF$, since F is a triangulated functor $\Sigma_{\mathcal{C}} X : KerF$ as $F(\Sigma_{\mathcal{C}} X) = \Sigma_{\mathcal{D}}(FX) = \Sigma_{\mathcal{D}} 0 = 0$. As F is triangulated, one has that every triangle maps to a triangle. Let $X, Y : KerF$, then:

$$\begin{tikzcd} X \arrow[dr] & & & & 0 \arrow[dr] & \\ & Y \arrow[dl] & \Longrightarrow & & & 0 \arrow[dl] \\ Z \arrow[uu, "\Sigma_{\mathcal{C}}"] & & & & F(Z) \arrow[uu, "\Sigma_{\mathcal{D}}"] & \end{tikzcd}$$

By TR3 and the 2 out of 3 property $F(Z) \simeq 0 \implies Z : KerF$. Thus $KerF$ is a triangulated subcategory of $\mathcal{C}$. $\square$

Definition 1.3.11. A subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$ is called thick if it contains all the direct summands of its objects.

Lemma 1.3.9. *The kernel of a triangulated functor $F : \mathcal{C} \to \mathcal{D}$ is thick.*

Proof. Let $X \oplus Y : KerF$, since F is additive one may see that $0 \simeq F(X \oplus Y) \simeq F(X) \oplus F(Y)$, but then there is a splitmono $F(X) \to 0 \implies F(X) \simeq 0 \simeq F(Y)$. $\square$

Lemma 1.3.10. *Let $F : \mathcal{C} \to \mathcal{D}$ be a triangulated functor. Suppose that $f : X \to Y$ is a morphism such that $F(f)$ is an isomorphism. Then the cone of f is in $KerF$.*

Proof. There is an isomorphism of triangles in $\mathcal{D}$, showing that the cone of f is in $KerF$.

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, equal] & FY \arrow[r] \arrow[d, equal] & F(cone(f)) \arrow[r] \arrow[d, dashed, "\wr"] & F\Sigma_{\mathcal{C}} X \arrow[d, equal] \\ FX \arrow[r, "Ff"] & FY \arrow[r] & 0 \arrow[r] & F\Sigma_{\mathcal{C}} X \end{tikzcd}$$

$\square$

The goal for the rest of this section is to prove that there is a localization at any triangulated subcategory $\mathcal{S} \subseteq \mathcal{C}$. This localization will yield a functor $q : \mathcal{C} \to \mathcal{C}/\mathcal{S}$ such that $\mathcal{S} \subseteq Kerq$. There is a set of morphism $Mor_{\mathcal{S}}$ related to $\mathcal{S}$ such that this set is multiplicative.

Definition 1.3.12. Let $\mathcal{C}$ be a triangulated category and $\mathcal{S} \subseteq \mathcal{C}$ be a triangulated subcategory. Define the collection $Mor_{\mathcal{S}}$ to be a collection of morphisms in $\mathcal{C}$ such that for any $f : Mor_{\mathcal{S}}$ there is a triangle with $C : \mathcal{S}$.

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow \Sigma_{\mathcal{C}} A$$

Remark. Every isomorphism is in $Mor_{\mathcal{S}}$. This is because isomorphisms are found in triangles $(A, B, 0, f, 0, 0)$ and $0 : \mathcal{S}$ for any triangulated subcategory.

Lemma 1.3.11. *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms. If any two of the morphisms f, g and gf are in $Mor_{\mathcal{S}}$ then so is the third.*

Proof. We are able to find three triangles in $\mathcal{C}$.

(1)
$$\begin{tikzcd} X \arrow[dr, "f"] & \\ & Y \arrow[dl] \\ Z' \arrow[uu, "\Sigma_{\mathcal{C}}"] & \end{tikzcd}$$

(2)
$$\begin{tikzcd} Y \arrow[dr, "g"] & \\ & Z \arrow[dl] \\ X' \arrow[uu, "\Sigma_{\mathcal{C}}"] & \end{tikzcd}$$

(3)
$$\begin{tikzcd} A \arrow[dr, "g\circ f"] & \\ & C \arrow[dl] \\ Y' \arrow[uu, "\Sigma_{\mathcal{C}}"] & \end{tikzcd}$$

By the Octahedron axiom there exist another triangle in $\mathcal{C}$:

$$Z' \longrightarrow X' \longrightarrow Y' \longrightarrow \Sigma_{\mathcal{C}} Z'$$

Note that f is in $Mor_{\mathcal{S}}$ if and only if $Z' : S$. WLOG assume that f and g is in $Mor_{\mathcal{S}}$, this can be done by the rotation axiom. Thus one may find the triangle in $\mathcal{S}$ by TR1 (Z', X', Y'') proving that $Y' \simeq Y''$.

$$\begin{tikzcd} Z' \arrow[r] \arrow[d, equal] & X' \arrow[r] \arrow[d, equal] & Y' \arrow[r] \arrow[d, "\wr"] & \Sigma_{\mathcal{C}} Z' \arrow[d, equal] \\ Z' \arrow[r] & X' \arrow[r] & Y'' \arrow[r] & \Sigma_{\mathcal{C}} Z' \end{tikzcd}$$

To see that gf is in $Mor_{\mathcal{S}}$ one can construct the triangle below with the isomorphism given above.

$$A \xrightarrow{g\circ f} C \longrightarrow Y'' \longrightarrow \Sigma_{\mathcal{C}} A$$

□

Proposition 1.3.12. *Let $\mathcal{S} \subseteq \mathcal{C}$ be a triangulated subcategory, then $Mor_{\mathcal{S}}$ satisfies the Ore condition.*

Proof. To prove that a system satisfies the Ore condition there has to be a proof for both right and left condition. Luckily, the arguments presented here can be dualized to give a proof for the other condition. Thus there will only be presented a proof for the right Ore condition. Let $f : A \to C$ be in $Mor_{\mathcal{S}}$ and $g : B \to C$ in $\mathcal{C}$. Then one may form a homotopy pullback creating a homotopy cartesian square as below.

$$\begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd} \Longrightarrow \begin{tikzcd} D \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "HO"] & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

By Lemma 1.2.5 there are triangles along this homotopy cartesian square identifying the cones. Since the cone of f is assumed to be in $\mathcal{S}$, the cone of f' is also in $\mathcal{S}$. This proves that $f' : Mor_{\mathcal{S}}$. □

Proposition 1.3.13. *For any parallel morphism $f, g : X \to Y$ in $\mathcal{C}$ the following are equivalent:*

1. $sf = sg$ *for some* $s : Mor_{\mathcal{S}}$ *starting at* Y.
2. $ft = gt$ *for some* $t : Mor_{\mathcal{S}}$ *ending at* X.
3. $f - g$ *factors through an object* $C : \mathcal{S}$.

Proof. (1. $\iff$ 3.): Suppose that there exists an $s : Y \to Z$ such that $s(f - g) = 0$. By TR1 there is a triangle $Y \xrightarrow{s} Z \xrightarrow{\Sigma_{\mathcal{C}} s'} \Sigma_{\mathcal{C}} C \longrightarrow \Sigma_{\mathcal{C}} Y$ and a long exact sequence.

$$\mathcal{C}(X, C) \xrightarrow{s'_*} \mathcal{C}(X, Y) \xrightarrow{s_*} \mathcal{T}(X, Z)$$

$$p \longmapsto^{s'_*} f - g \longmapsto^{s_*} 0$$

Since $s(f - g) = 0$ there exists a $p : \mathcal{C}(X, C)$ such that $f - g = s'_* p$. By definition, $s : Mor_{\mathcal{S}} \iff C : \mathcal{S}$, but $s : Mor_{\mathcal{S}} \implies f - g$ factors through $\Sigma_{\mathcal{C}}$, and vice versa. (2. $\iff$ 3.): This argument is dual. □

This has shown that $Mor_{\mathcal{S}}$ is a multiplicative system, and Theorem 1.3.4 say that the localization exists given that $Mor_{\mathcal{S}}$ is locally small. The category $Mor_{\mathcal{S}}^{-1}$ will be denoted as $\mathcal{C}/\mathcal{S}$ and it is called the Verdier quotient. As $\mathcal{C}$ is additive, it is known that $\mathcal{C}/\mathcal{S}$ is additive as well by Proposition 2.20. The remaining part is to show that $\mathcal{C}/\mathcal{S}$ is triangulated and that localization functor $q : \mathcal{C} \to \mathcal{C}/\mathcal{S}$ is a triangulated functor.

Theorem 1.3.14. *Let $\mathcal{S} \subseteq \mathcal{C}$ be triangulated categories. Then the Verdier quotient $\mathcal{C}/\mathcal{S}$ together with the functor $q : \mathcal{C} \to \mathcal{C}/\mathcal{S}$ is the universal triangulated category where morphisms in $Mor_{\mathcal{S}}$ are isomorphisms.*

Proof. The triangulation on $\mathcal{C}/\mathcal{S}$ is defined as the following. Let $\Sigma_{\mathcal{C}/\mathcal{S}} : \mathcal{C}/\mathcal{S} \to \mathcal{C}/\mathcal{S}$ be the additive autoequivalence defined by its action on objects $\Sigma_{\mathcal{C}/\mathcal{S}}(A) = \Sigma_{\mathcal{C}}(A)$ and maps $\Sigma_{\mathcal{C}/\mathcal{S}}(f) = f \circ id^{-1}$. Since $q : \mathcal{C} \to \mathcal{C}/\mathcal{S}$ maps every object to itself it follows that $q(\Sigma_{\mathcal{C}}(A)) \simeq \Sigma_{\mathcal{C}/\mathcal{S}}(A) = \Sigma_{\mathcal{C}/\mathcal{S}}(q(A))$, and define $\Delta_{\mathcal{C}/\mathcal{S}} \supseteq q(\Delta_{\mathcal{C}})$ such that $\Delta_{\mathcal{C}/\mathcal{S}}$ has every isomorphism class of $q(\Delta_{\mathcal{C}})$.

$$\begin{tikzcd} qX \arrow[dr] & & & X \arrow[dr] & \\ & qY \arrow[dl] & \Longleftarrow & & Y \arrow[dl] \\ qZ \arrow[uu, "\Sigma_{\mathcal{C}/\mathcal{S}}"] & & & Z \arrow[uu, "\Sigma_{\mathcal{C}}"] & \end{tikzcd}$$

Then by definition q is triangulated if the category $\mathcal{C}/\mathcal{S}$ is triangulated. By definition, the triangles are closed under isomorphisms, $(X, X, 0, id_X, 0, 0)$ is a triangle, and TR2 holds. Thus it remains to show TR1 and TR4 (TR3 is implied by the other axioms). To prove TR1, let $fs^{-1} : \mathcal{C}/\mathcal{S}(qW, qY)$. Expand $f : \mathcal{C}(X, Y)$ to a triangle in $\mathcal{C}$ with TR1, it will induce a triangle in $\mathcal{C}/\mathcal{S}$.

$$qX \xrightarrow{fid_X^{-1}} qY \xrightarrow{gid_Y^{-1}} qZ \xrightarrow{hid_Z^{-1}} q\Sigma_{\mathcal{C}}X$$

There is an isomorphism to the following candidate triangle from the induced triangle, proving TR1.

$$\begin{tikzcd}
qX \arrow[r, "fid_X^{-1}"] \arrow[d, "sid_X^{-1}", "\wr"'] & qY \arrow[r, "gid_Y^{-1}"] \arrow[d, equal] & qZ \arrow[r, "hid_Z^{-1}"] \arrow[d, equal] & q\Sigma_{\mathcal{C}}X \arrow[d, "(\Sigma_{\mathcal{C}}s)id_{\Sigma_{\mathcal{C}}X}^{-1}", "\wr"'] \\
qW \arrow[r, "fs^{-1}"] & qY \arrow[r, "gid_Y^{-1}"] & qZ \arrow[r, "(\Sigma_{\mathcal{C}}s)hid_Z^{-1}"] & q\Sigma_{\mathcal{C}}W
\end{tikzcd}$$

To show the Octahedron axiom, suppose that there are three triangles in $\mathcal{C}/\mathcal{S}$. By construction, these triangles can be chosen such that only the first map is a fraction up to isomorphism of triangles.

$$(1)\quad \begin{tikzcd}
Z \arrow[r, "t'"] & X \arrow[dl, "s"'] \arrow[d, "a"] \\
A \arrow[r, "as^{-1}"] & B \arrow[dl, "x"] \\
C' \arrow[u, "x'", "\Sigma_{\mathcal{C}/\mathcal{S}}"'] &
\end{tikzcd}
\quad (2)\quad \begin{tikzcd}
& Y \arrow[dl, "t"'] \arrow[d, "b"] \\
B \arrow[r, "bt^{-1}"] & C \arrow[dl, "y"] \\
A' \arrow[u, "y'", "\Sigma_{\mathcal{C}/\mathcal{S}}"'] &
\end{tikzcd}
\quad (3)\quad \begin{tikzcd}
& Z \arrow[dl, "st'"'] \arrow[d, "ba'"] \\
A \arrow[r, "b\circ a"] & C \arrow[dl, "z"] \\
B' \arrow[u, "z'", "\Sigma_{\mathcal{C}/\mathcal{S}}"'] &
\end{tikzcd}$$

This is possible, as when composing the fractions from A to B and B to C one may find an object Z as in the diagram by using the Ore condition. To illustrate with triangle (1), there is a correspondence of triangles in $\mathcal{C}/\mathcal{S}$ and $\mathcal{C}$ by the following isomorphism.

$$\begin{tikzcd}
Z \arrow[r, "at'"] \arrow[d, "t", "\wr"'] & B \arrow[r] \arrow[d, equal] & Z' \arrow[r] \arrow[d, dashed, "\wr"] & \Sigma_{\mathcal{C}}Z \arrow[d, "\Sigma_{\mathcal{C}}t", "\wr"'] \\
X \arrow[r, "a"] \arrow[d, "s"] & B \arrow[r] & C' \arrow[r] & \Sigma_{\mathcal{C}}X \arrow[d, "\Sigma_{\mathcal{C}}s"] \\
A & & & \Sigma_{\mathcal{C}}A
\end{tikzcd}$$

The result of the octahedron axiom follows as one instead consider the triangles found by the composition of morphisms as below.

$$\begin{tikzcd}
Z \arrow[d, "f'"] \arrow[dr, "bf'"] & \\
Y \arrow[r, "b"] & C
\end{tikzcd}$$

$\square$

Proposition 1.3.15. *Let $\mathcal{S} \subseteq \mathcal{C}$ be triangulated categories. If $0 : X \to 0$ is an isomorphism in $\mathcal{C}/\mathcal{S}$, then there is an object Y such that $X \oplus Y : \mathcal{S}$.*

Proof. If $0 : X \to 0$ is invertible, then there exist a map $0 : 0 \to Y$, such that $0 : X \to Y$ is in $Mor_{\mathcal{S}}$. By definition $X \oplus Y$ is in $\mathcal{S}$. $\square$

This proposition shows that the kernel of $q : \mathcal{C} \to \mathcal{C}/\mathcal{S}$ is the smallest thick subcategory of $\mathcal{C}$ such that $\mathcal{C}/Kerq$ is the universal category where every morphism in $Mor_{\mathcal{S}}$ is an isomorphism. For this reason $\widehat{\mathcal{S}} = Kerq$ is called the thick closure of $\mathcal{S}$.

Chapter 2

Exact Categories

2.1 Definitions and First Properties

This section will focus on defining what an exact category is and its elementary properties. The main result from this section is proposition 2.1.3. This proposition is similar to the characterization theorem of push-outs and pullbacks in abelian categories. Another important result is the obscure axiom, this will be proved and hopefully seen to be not as obscure as its name suggests. Lastly, variants of some homological diagram lemmata, like 5-lemma, will also be proved for exact categories.

To start with exact categories one should first take a look towards the abelian categories first. Short exact sequences are of great interest, and they can be characterized with two morphisms $p : A \to B$ and $q : B \to C$ such that p is the kernel of q and q is the cokernel of p. This leads to the first definition.

Definition 2.1.1. Let $\mathcal{A}$ be an additive category. A kernel-cokernel pair is a pair of maps (p, q) such that p is the kernel of q and q is the cokernel of p. A morphism of kernel-cokernel pairs (p, q) and (p', q') is a triple (f, g, h) such that the following diagram commutes. An isomorphism of a kernel-cokernel pair is a triple in which each morphism is an isomorphism.

$$\begin{tikzcd}
A \arrow[r, tail, "p"] \arrow[d, two heads, "f"] & B \arrow[r, "q"] \arrow[d, "g"] & C \arrow[d, "h"] \\
A' \arrow[r, tail, "p'"] & B' \arrow[r, two heads, "q'"] & C'
\end{tikzcd}$$

Lemma 2.1.1. *Let (p, q) be a kernel-cokernel pair, then the image and coimage of p exists and are isomorphic. I.e. this diagram exists, such that the left square is a push-out and the right square is a pull-back:*

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[dr, "0"'] & A \arrow[rr, tail, "p"] \arrow[d, two heads] & & B \arrow[r, two heads, "q"] & C \\
& Coim(p) \arrow[rr, tail, two heads, "iso"] & & Im(p) \arrow[u, tail] \arrow[ur, "0"'] &
\end{tikzcd}$$

Proof. Since (p,q) is a kernel-cokernel pair one may see that the first simplex is bicartesian and the second simplex is a push-out.

$$\begin{tikzcd} A \arrow[r, tail, "p"] \arrow[dr, "0"'] & B \arrow[d, two heads, "q"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, "0"] \arrow[dr, "0"'] & A \arrow[d, equal] \\ & A \end{tikzcd}$$

Thus $Im(p) = Coim(p) = A$, asserting the isomorphism as the identity in the diagram.

$$\begin{tikzcd} 0 \arrow[r, "0"] \arrow[dr, "0"'] & A \arrow[r, tail, "p"] \arrow[d, equal] & B \arrow[r, two heads, "q"] & C \\ & A \arrow[r, equal] & A \arrow[u, tail, "p"] \arrow[ur, "0"] & \end{tikzcd}$$

□

Corollary 2.1.1.1. *Suppose that (p,q) is a kernel-cokernel pair. If p is an epimorphism, then p is an isomorphism.*

Definition 2.1.2. An exact structure for an additive category $\mathcal{A}$ is a class $\mathcal{E}$ of kernel-cokernel pairs which are closed under isomorphisms. A pair $(p,q) : \mathcal{E}$ is called a conflation, here p is called an inflation and q is called a deflation. $(\mathcal{A},\mathcal{E})$ is called exact when the following axioms holds:

- (QE0) $\forall A : \mathcal{A}$, id_A is both an inflation and a deflation.
- (QE1) Both inflations and deflations are closed under composition.
- (QE2) The push-out of an inflation is an inflation.
- (QE2^{op}) The pull-back of a deflation is a deflation.

An exact category is the additive category $\mathcal{A}$ together with an exact structure $\mathcal{E}$.

Remark. Decorated arrows will be used when writing diagrams to indicate that a morphism is either an inflation or a deflation. A tail with a circle means inflation: $A \overset{\circ}{\rightarrowtail} B$. Double heads with a circle means deflation: $A \overset{\circ}{\twoheadrightarrow} B$. $(QE2^*)$ axioms can now be written as the diagrams below.

$$\begin{tikzcd} A \arrow[r, tail, "\circ"] \arrow[d] & B \arrow[d] \\ C \arrow[r, tail, "\circ"] & D \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, two heads, "\circ"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d] \\ C \arrow[r, two heads, "\circ"] & D \end{tikzcd}$$

Remark. In literature, inflations are also referred to as admissable monomorphisms, and deflations are referred to as admissable epimorphisms while conflations are also called short exact sequences.

Remark. Observe that the axioms for an exact structure is self dual. This allows for reasoning with duality, as a category has an exact structure $(\mathcal{A},\mathcal{E})$ if and only if $(\mathcal{A}^{op},\mathcal{E}^{op})$ is an exact structure.

Remark. For any category $\mathcal{C}$, there is a category $\mathcal{C}^{\rightarrow} = \mathcal{C} \downarrow \mathcal{C}$ consisting of arrows and $\mathcal{C}^{\rightarrow\rightarrow} = \mathcal{C} \downarrow \mathcal{C} \downarrow \mathcal{C}$ consisting of pairs of composable arrows. If $\mathcal{A}$ is additive, then $\mathcal{A}^{\rightarrow}$ and $\mathcal{A}^{\rightarrow\rightarrow}$ are additive as well. It can be seen that $\mathcal{E}$ may be considered as an extension closed additive subcategory of $\mathcal{A}^{\rightarrow\rightarrow}$.

The aim of exact categories is to characterize the fundamental properties of abelian categories. In nature, exact categories are quite common and one additive category usually admits more than one exact structure. Thus there may exist a chain of exact structures in the sense of subsets $\mathcal{E}_{min} \subseteq \mathcal{E}_1 \subseteq ... \subseteq \mathcal{E}_{max}$.

Example. Any abelian category is exact with every short exact sequence as the exact structure. This exact structure us $\mathcal{E}_{max}$.

Example. Any additive category is exact with every split short exact sequence as the exact structure. This structure will always be $\mathcal{E}_{min}$, and moreover, it is always contained inside another exact structure.

Lemma 2.1.2. *The map* $0 : 0 \to A$ *is an inflation. Dually, the map* $0 : A \to 0$ *is a deflation.*

Proof. Consider the diagram $0 \overset{0}{\rightarrowtail} A \overset{id_A}{\twoheadrightarrow} A$. The left morphism is the kernel of the right morphism making a kernel-cokernel pair $(0, id_A)$. The identity id_A is assumed to be a deflation, implying that the pair is a conflation. □

Remark. It can be seen that isomorphisms are deflations. Let $f : A \to B$ be an isomorphism, then there are two kernel-cokernel pairs: $(0, id_A)$ and $(0, f)$. Between these there is an isomorphism which is the triple $(0, id_A, f^{-1})$. As the conflations are closed under isomorphism, $(0, f)$ is a conflation, making f into a deflation. By dualizing this argument, f is also an inflation.

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[d, "0"] & A \arrow[r, tail, two heads, "f"] \arrow[d, "id_A"] & B \arrow[d, tail, two heads, "f^{-1}"] \\
0 \arrow[r, tail, "0"] & A \arrow[r, two heads, "id_A"] & A
\end{tikzcd}$$

Corollary 2.1.2.1. *A kernel-cokernel pair* (i, p) *found as a split short-exact sequence (1) is a conflation.*

$$\text{(1)} \quad A \overset{i}{\rightarrowtail} A \oplus B \overset{p}{\twoheadrightarrow} B$$

Proof. In a category with an initial object the coproduct can be thought of as the push-out with the initial in the upper right corner. This can be assembled into push-out (1). By the lemma the zero morphisms are inflations, asserting that i and i' are inflations by (QE2). Thus there are conflations (i, p) and (i', p').

$$\text{(1)} \quad \begin{tikzcd}
0 \arrow[r, tail, "0"] \arrow[d, tail, "0"] & A \arrow[d, tail, "i"] \\
B \arrow[r, tail, "i'"] & A \oplus B
\end{tikzcd}$$

□

Corollary 2.1.2.2. *The direct sum of conflations is a conflation. I.e. there is a diagram:*

$$\begin{gathered} A \overset{i}{\rightarrowtail} B \overset{p}{\twoheadrightarrow} C \;,\; A' \overset{i'}{\rightarrowtail} B' \overset{p'}{\twoheadrightarrow} C' \\ \Downarrow \\ A \oplus A' \overset{i \oplus i'}{\rightarrowtail} B \oplus B' \overset{p \oplus p'}{\twoheadrightarrow} C \oplus C' \end{gathered}$$

Proof. Start by only considering the conflation (i, p). For any $D : \mathcal{A}$ there is a conflation $(i \oplus id_D, p \oplus 0)$, drawn as the diagram.

$$A \oplus D \overset{\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}}{\rightarrowtail} B \oplus D \overset{\begin{pmatrix} p & 0 \end{pmatrix}}{\twoheadrightarrow} C$$

As kernels and cokernels are preserved by direct sums, this pair is in fact a kernel-cokernel pair. The epimorphism is a deflation as it can be factored by the deflations:

$$B \oplus D \overset{\begin{pmatrix} 1 & 0 \end{pmatrix}}{\twoheadrightarrow} B \overset{p}{\twoheadrightarrow} C$$

Thus it is seen that $(i \oplus id_D, p \oplus 0)$ is a conflation, and dually $(i \oplus 0, p \oplus id_D)$ is also a conflation. To finish off the proof it is seen that the morphism $i \oplus i'$ factors as $i \oplus id_{A'} \circ id_A \oplus i'$, asserting that it is an inflation by (QE1). By dualizing the argument, one get that the direct sum of conflations is a conflation. □

Definition 2.1.3. A square is bicartesian if it is both a pull-back and a push-out.
$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d] \\ C \arrow[r] & D \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

Proposition 2.1.3. *The following statements are equivalent:*

1. *The square (1) is a push-out.*
2. *The sequence (2) is a conflation.*
3. *The square (1) is bicartesian.*
4. *The square (1) is a part of the commutative diagram (3)*

$$(1)\; \begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[d, "g"] \\ C \arrow[r, tail, "j"] & D \end{tikzcd} \quad (2)\; A \overset{\begin{pmatrix} i \\ -f \end{pmatrix}}{\rightarrowtail} B \oplus C \overset{\begin{pmatrix} g & j \end{pmatrix}}{\twoheadrightarrow} D \quad (3)\; \begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[r, two heads, "p"] \arrow[d, "g"] & E \arrow[d, equal] \\ C \arrow[r, tail, "j"] & D \arrow[r, two heads, "q"] & E \end{tikzcd}$$

Before the proof for this proposition there will be presented a useful lemma, which will be proved first.

Lemma 2.1.4. *Assume that there is a commutative square (1) and an associated sequence (2). (1) is a push-out square if and only if* $\begin{pmatrix} p & q \end{pmatrix}$ *is the cokernel of the morphism* $\begin{pmatrix} i \\ -j \end{pmatrix}$

$$
(1)\quad \begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, "p"] \\ C \arrow[r, "q"] & D \end{tikzcd}
\qquad
(2)\quad A \xrightarrow{\begin{pmatrix} i \\ -j \end{pmatrix}} B \oplus C \xrightarrow{\begin{pmatrix} p & q \end{pmatrix}} D
$$

Proof. For any test object T and two maps $t_1 : B \to T$ and $t_2 : C \to T$, one may construct the diagrams for the universal properties of both the cokernel and the push-out. It is seen that these diagrams are equivalent, proving the lemma.

$$
\begin{tikzcd} A \arrow[r, "\begin{pmatrix} i \\ -f \end{pmatrix}"] \arrow[dr, "0"'] & B \oplus C \arrow[d, two heads, "\begin{pmatrix} g & j \end{pmatrix}"] \arrow[dr, bend left, "\begin{pmatrix} t_1 & t_2 \end{pmatrix}"] & \\ & D \arrow[r, dashed, "t'"] & T \end{tikzcd}
\quad \Leftrightarrow \quad
\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[ddr, bend left, "t_1"] & \\ C \arrow[r, "j"] \arrow[drr, bend right, "t_2"'] & D \arrow[dr, dashed, "t'"] & \\ & & T \end{tikzcd}
$$

$\square$

Corollary 2.1.4.1. *For the same diagrams (1) and (2) as above the dual statement is also true. (1) is a pull-back square if and only if* $\begin{pmatrix} i \\ -j \end{pmatrix}$ *is kernel of the morphism* $\begin{pmatrix} p & q \end{pmatrix}$*. Thus it follows that (1) is bicartesian (i.e. both a pull-back and a push-out) if and only if the morphisms make a kernel-cokernel pair.*

Proof. of Proposition 2.1.3. 1. $\Rightarrow$ 2.: By the previous lemma it is known that $\begin{pmatrix} g & j \end{pmatrix}$ is the cokernel of $\begin{pmatrix} i \\ -j \end{pmatrix}$. Thus proving that $\begin{pmatrix} i \\ -j \end{pmatrix}$ is an inflation, will prove that the pair is a conflation.
Observe that the morphism $\begin{pmatrix} i \\ -f \end{pmatrix}$ can be factored through the sequence.

$$
A \overset{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\rightarrowtail} A \oplus C \underset{\cong}{\overset{\begin{pmatrix} 1 & 0 \\ -f & 1 \end{pmatrix}}{\rightarrowtail\!\!\!\twoheadrightarrow}} A \oplus C \overset{\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}}{\rightarrowtail} B \oplus C
$$

By corollary 3.2.1 the first map is an inflation, as the second map is an isomorphism it is also an inflation and the last map is the direct sum of two inflations. Thus the composite of all these maps is an inflation by (QE1), proving the first implication.

2. $\Rightarrow$ 3.: This follows from corollary 3.4.1.

3 $\Rightarrow$ 1.: This is by definition.

1. $\Rightarrow$ 4.: Let p be the cokernel of i, then form the diagram below using the push-out property.

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[ddr, bend left, two heads, "p"] & \\ C \arrow[r, tail, "j"] \arrow[drr, bend right, "0"] & D \arrow[dr, dashed, two heads, "p'"] & \\ & & T \end{tikzcd}$$

p' is an epimorphism as $p = p'g$ is epi. To prove that p' is the cokernel of j let T' be another test object with a map $t' : D \to T'$ such that $0 = t'j$. By doing some diagram chases one may see that $0 = t'jf = t'gi$, thus by the universal property of p the morphism $t'g$ factors through T such that $t'g = tp$ for some unique t. This shows that $t'g = tp'g = tp$, and $t'j = tp'j = 0$. Since t' is the unique morphism satisfying this equation we demand that $t' = tp'$. t is also unique, for if there exist another map h such that $tp' = hp'$, then $t = h$ as p' is epic. The unique existence proves the universal property, and p' is the cokernel of j.

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"'] & B \arrow[r, two heads, "p"] \arrow[d, "g"] & T \arrow[d, equal] \\ C \arrow[r, tail, "j"] \arrow[drr, bend right, "0"] & D \arrow[r, dashed, two heads, "p'"] \arrow[dr, "t'"] & T \arrow[d, dashed, "t"] \\ & & T' \end{tikzcd}$$

4. $\Rightarrow$ 2.: Start by taking the pullback of p and q using $(QE2^{op})$. The diagrams below are determined by using the dual statement of the last implication.

$$\begin{tikzcd} & A \arrow[r, equal] \arrow[d, tail, "i'"] & A \arrow[d, tail, "i"] \\ C \arrow[r, tail, "j'"] \arrow[d, equal] & T \arrow[r, two heads, "q'"] \arrow[d, two heads, "p'"] & B \arrow[d, two heads, "p"] \arrow[dl, dashed, "g"'] \\ C \arrow[r, tail, "j"] & D \arrow[r, two heads, "q"] & E \end{tikzcd} \Rightarrow \begin{tikzcd} B \arrow[drr, bend left, equal] \arrow[dr, dashed, "k"] \arrow[ddr, bend right, "g"'] & & \\ & T \arrow[r, two heads, "q'"] \arrow[d, two heads, "p'"] & B \arrow[d, two heads, "p"] \\ & D \arrow[r, two heads, "q"] & E \end{tikzcd}$$

From these diagrams one can see that q' is a split epimorphism. The composite $q'(id_T - kq') = q' - q'kq' = q' - q' = 0$ as q' is split epi, so $(id_T - kq')$ factors over j' as in the following diagram.

$$\begin{tikzcd} T \arrow[dr, dashed, "l"] \arrow[drr, bend left, "id_T - kq'"] & & \\ & C \arrow[r, tail, "j'"] \arrow[dr, "0"] & T \arrow[d, two heads, "q'"] \\ & & B \end{tikzcd}$$

From these diagrams one may find three different equations:

- $0 = k - k = k - kq'k = (id_T - kq')k = j'lk \implies lk = 0$ as j' is monic

- $j'lj' = (id_T - kq')j' = j' \implies lj' = id_C$ as j' is monic
- $jli' = (p'j')li' = p'(id_T - kq')i' = -(p'k)(q'i') = -gi = -jf \implies li' = -f$ as j is monic

The morphisms $\begin{pmatrix} k & j' \end{pmatrix}$ and $\begin{pmatrix} q' \\ l \end{pmatrix}$ are inverses:

- $\begin{pmatrix} k & j' \end{pmatrix}\begin{pmatrix} q' \\ l \end{pmatrix} = kq' + j'l = kq' + id_T - kq' = id_T$
- $\begin{pmatrix} q' \\ l \end{pmatrix}\begin{pmatrix} k & j' \end{pmatrix} = \begin{pmatrix} q'k & q'j' \\ lk & lj' \end{pmatrix} = \begin{pmatrix} id_B & 0 \\ 0 & id_C \end{pmatrix}$

Thus there is an isomorphism of kernel-cokernel pairs $(id_A, , \begin{pmatrix} q' \\ l \end{pmatrix}\begin{pmatrix} k & j' \end{pmatrix})$, from $(\begin{pmatrix} i \\ -f \end{pmatrix}, \begin{pmatrix} f' & i' \end{pmatrix})$ to (i', p'). This proves 2. □

Corollary 2.1.4.2. *The pull-back of an inflation along a deflation is an inflation.*

$$\begin{tikzcd}
A \arrow[r, tail, "i'"] \arrow[d, two heads, "e'"] \arrow[dr, phantom, very near start, "\ulcorner"] & B \arrow[d, two heads, "e"] \\
C \arrow[r, tail, "i"] & D
\end{tikzcd}$$

Proof. By (QE2) this pullback exists, as there is a deflation in the pullback. Extend the diagram by adding the deflation of the inflation in the following manner.

$$\begin{tikzcd}
 & T \arrow[dl, dashed, "t'"'] \arrow[d, "t"] \arrow[dr, bend left, "0"] & \\
A \arrow[r, tail, "i'"] \arrow[d, two heads, "e'"] \arrow[dr, phantom, very near start, "\ulcorner"] & B \arrow[r, two heads, "pe"] \arrow[d, two heads, "e"] \arrow[dl, dashed, "t''"'] & C \arrow[d, equal] \\
C \arrow[r, tail, "i"] & D \arrow[r, two heads, "p"] & C
\end{tikzcd}$$

pe is a deflation by (QE1), and i' is a mono as a limit of a mono is a mono. The goal is to prove that i' is the kernel of pe. Let T be a test object such that $pet = 0$, then if follows that te factorizes over i, such that we can apply the universal property of the pullback to factorize te over i'. Uniqueness of t' is achieved with i' being monic. This proves that (i', pe) is a conflation. □

Theorem 2.1.5. ***The Obscure Axiom.*** *Assume that $i : A \to B$ is a morphism with a cokernel. If there is a morphism $j : B \to C$ such that ji is an inflation, then i is an inflation.*

Proof. Let $k : B \to D$ be the cokernel of i. Start by forming the push-out of i and ji.

$$\begin{tikzcd}
A \arrow[r, tail, "ji"] \arrow[d, "i"] & C \arrow[d] \\
B \arrow[r, tail] \arrow[ur, phantom, very near end, "\lrcorner"] & E
\end{tikzcd}$$

By proposition 2.1.3 $\begin{pmatrix} i \\ ji \end{pmatrix}$ is an inflation. $\begin{pmatrix} i \\ 0 \end{pmatrix} = \begin{pmatrix} id_B & 0 \\ -j & id_C \end{pmatrix}\begin{pmatrix} i \\ ji \end{pmatrix}$, this is an inflation by (QE1) as the 2x2 matrix is an isomorphism. Observe that the cokernel of $\begin{pmatrix} i \\ 0 \end{pmatrix}$ is $\begin{pmatrix} k & 0 \\ 0 & id_C \end{pmatrix}$. The final trick will be to show that there is a pullback square, and then use (QE2) to say that k is a deflation.

$$\begin{tikzcd}
T \arrow[drr, bend left, "t_1"] \arrow[dr, dashed, "t"] \arrow[ddr, bend right, "\begin{pmatrix} t_2 \\ 0 \end{pmatrix}"'] & & \\
& B \arrow[r, "k"] \arrow[d, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & D \arrow[d, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] \\
& B \oplus C \arrow[r, two heads, "\begin{pmatrix} k & 0 \\ 0 & id_C \end{pmatrix}"'] & D \oplus C
\end{tikzcd}$$

Note that setting $t = t_2$ one get the universal property. This is well defined as $kt_2 = t_1$ by assumption, thus $kt = t_1$. This is what is needed to prove that the square is a pullback, proving the obscure axiom. □

Contrary to its name, The Obscure Axiom is a very natural result. To motivate this, let $\mathcal{A}$ be an abelian category. Here every map has a cokernel, and moreover if $i : A \to B$ is a map and the composition $ji : A \to C$ is a monomorphism, this result follow. Since ji is mono, it follows that i has to be mono, thus the kernel of i is 0. This gives a short exact sequence with the same implications as the obscure axiom.

$$0 \longrightarrow A \overset{i}{\rightarrowtail} B \overset{\pi_i}{\twoheadrightarrow} Cok(i) \longrightarrow 0$$

The classical diagram lemmata which will be proven are the 5-lemma and the 3x3-lemma. In this context they wil be dubbed the short five lemma and Noethers isomorphism lemma respectively. In order to prove the short 5 lemma, a lemma is needed.

Lemma 2.1.6. *Let (p,q) and (p',q') be the conflations:*

- (p,q)*:* $A \overset{p}{\rightarrowtail} B \overset{q}{\twoheadrightarrow} C$
- (p',q')*:* $A' \overset{p'}{\rightarrowtail} B' \overset{q'}{\twoheadrightarrow} C'$

A morphism of the conflations $(f,g,h) : (p,q) \to (p',q')$ factors through the conflation $A \rightarrowtail D \twoheadrightarrow C'$ *such that the following diagram exists, where $g = g_2 g_1$.*

$$\begin{tikzcd}
A \arrow[r, tail, "p"] \arrow[d, "f"] \arrow[dr, phantom, "\ulcorner" near start, "\lrcorner" near end] & B \arrow[r, two heads, "q"] \arrow[d, "g_1"] & C \arrow[d, equal] \\
A' \arrow[r, tail] \arrow[d, equal] & D \arrow[r, two heads] \arrow[d, "g_2"] \arrow[dr, phantom, "\ulcorner" near start, "\lrcorner" near end] & C \arrow[d, "h"] \\
A' \arrow[r, tail, "p'"] & B' \arrow[r, two heads, "q'"] & C'
\end{tikzcd}$$

Proof. Observe that the upper part of the diagram is made by taking a push-out of p and f, where the right part is gained from proposition 2.1.3. Combine the upper part with the lower part using the push-out property.

$$
\begin{tikzcd}
A \arrow[r, tail, "p"] \arrow[d, "f"] & B \arrow[d, "g_1"] \arrow[ddr, bend left, "g"] & \\
A' \arrow[r, tail] \arrow[ul, phantom, "\lrcorner", very near end] \arrow[drr, tail, bend right, "p'"] & D \arrow[dr, dashed, "g_2"] & \\
& & B'
\end{tikzcd}
\implies
\begin{tikzcd}
A \arrow[r, tail, "p"] \arrow[d, "f"] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[r, two heads, "q"] \arrow[d, "g_1"] & C \arrow[d, equal] \\
A' \arrow[r, tail, "a"] \arrow[d, equal] & D \arrow[r, two heads, "c"] \arrow[d, "g_2"] & C \arrow[d, "h"] \\
A' \arrow[r, tail, "p'"] & B' \arrow[r, two heads, "q'"] & C'
\end{tikzcd}
$$

It remains to show that the lower right square is commutative, then use the dual of proposition 2.1.3 to see that the square is bicartesian. Note that $q = cg_1$ by prop 2.1.3 thus $q'g_2g_1 = q'g = hq = hcg_1$. Uniqueness of the push-out property asserts that $hc = q'g_2$. $\square$

Corollary 2.1.6.1. ***The short five lemma***. *Suppose that there is a morphism of conflations (f, g, h) as above. If f and h are isomorphisms, then g is an isomorphism.*

Proof. Since f is an isomorphism it is at least an inflation, thus g_1 is an inflation by (QE2). As colimits preserve epis, g_1 is also an epimorphism. Corollary 2.1.1.1 states that g_1 is an iso, and dually that g_2 is an iso. Since isomorphisms are closed under composition it follows that g is an isomorphism.

$$
\begin{tikzcd}
A \arrow[r, tail, "p"] \arrow[d, tail, "f", "\wr"'] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[r, two heads, "q"] \arrow[d, tail, "g_1", "\wr"'] & C \arrow[d, equal] \\
A' \arrow[r, tail] \arrow[d, equal] & D \arrow[r, two heads] \arrow[d, two heads, "g_2", "\wr"'] \arrow[dr, phantom, "\ulcorner", very near start] & C \arrow[d, two heads, "h", "\wr"'] \\
A' \arrow[r, tail, "p'"] & B' \arrow[r, two heads, "q'"] & C'
\end{tikzcd}
$$

$\square$

Lemma 2.1.7. ***Noethers isomorphism lemma***. *Suppose there is a diagram with rows as conflations and the first column as a conflation. Then the final column is also a conflation.*

$$
\begin{tikzcd}
A \arrow[r, tail] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, tail, dashed] \\
A \arrow[r, tail] & C \arrow[r, two heads] \arrow[d, two heads] & Y \arrow[d, two heads, dashed] \\
& Z \arrow[r, equal] & Z
\end{tikzcd}
$$

Proof. Assume that only the solid part of the diagram above exists. By the universal property of cokernels, the upper dashed map exists, and by the dual of proposition 2.1.3 the square is bicartesian. This infers that the upper dashed map is an inflation, and since the square is a push-out it follows that the lower dashed map exists such that the final column is a conflation by proposition 2.1.3. $\square$

2.2 The Stable Frobenius Category

This section aims to introduce Frobenius categories and show that their stabilization are triangulated. Categories which can be realized as stable Frobenius categories are called the algebraic triangulated categories. In order to define this construction, one must define projective and injective objects in an exact category. It will then be shown that the stable Frobenius category is a quotient category by the injective objects. One of the important ideas from this section is that conflations from the Frobenius category will be the class generating the triangles in the stable Frobenius category.

Definition 2.2.1. Let $(\mathcal{A},\mathcal{E})$ and $(\mathcal{A}',\mathcal{E}')$ be two exact categories. A functor $F:(\mathcal{A},\mathcal{E})\to(\mathcal{A}',\mathcal{E}')$ is called exact if it is additive and $F(\mathcal{E})\subseteq\mathcal{E}'$. That is to say that conflations gets mapped to conflations.

Definition 2.2.2. Let $(\mathcal{A},\mathcal{E})$ be an exact category. An object $P:\mathcal{A}$ is called projective if $\mathcal{A}(P,\_):(\mathcal{A},\mathcal{E})\to\mathbf{Ab}$ is an exact functor. Objects $I:\mathcal{A}$ are called injective whenever $\mathcal{A}(\_,I):\mathcal{A}^{op}\to\mathbf{Ab}$ is an exact functor.

Remark. In the case of exact functors $F:(\mathcal{A},\mathcal{E})\to\mathbf{Ab}$, one generally speaks of a functor which maps conflations to short exact sequences.

Remark. The hom-functor is called left-exact. This means that conflations gets mapped to sequences which is only exact in the first two terms.

Proposition 2.2.1. *Let $(\mathcal{A},\mathcal{E})$ be an exact category. $P:\mathcal{A}$ is projective if and only if for every deflation $q:A\twoheadrightarrow B$ and morphism $f:P\to B$ there is a morphism $f':P\to A$ rendering the diagram below commutative.*

$$\begin{tikzcd} & P \arrow[dl, dashed, "f'"'] \arrow[d, "f"] \\ A \arrow[r, two heads, "q"'] & B \end{tikzcd}$$

Proof. Suppose that P is projective, then $\mathcal{A}(P,\_)$ is an exact functor. Let $(p:A\to B, q:B\to C)$ be a conflation, then there is a short exact sequence.

$$0 \xrightarrow{0} \mathcal{A}(P,A) \xrightarrow{p_*} \mathcal{A}(P,B) \xrightarrow{q_*} \mathcal{A}(P,C) \xrightarrow{0} 0$$

Pick $f:\mathcal{A}(P,C)$, since q_* is a surjection there exists an $f':\mathcal{A}(P,B)$ such that $pf'=f$. Now, suppose that P has the property described by the diagram in the proposition and that $(p:A\to B, q:B\to C)$ is a conflation, then there is an exact sequence in $\mathbf{Ab}$ by $\mathcal{A}(P,\_)$.

$$0 \xrightarrow{0} \mathcal{A}(P,A) \xrightarrow{p_*} \mathcal{A}(P,B) \xrightarrow{q_*} \mathcal{A}(P,C)$$

To see that q_* is a surjection, let $f:P\to C$. As q is a deflation there exists an $f':P\to B$ such that $q_*(f')=f$. Thus the sequence above is short exact and P is projective. □

Corollary 2.2.1.1. *Let P be projective, then if $q:A\twoheadrightarrow P$ is a deflation, it is split-epi.*

Corollary 2.2.1.2. *Two objects P and Q are projective if and only if $P \oplus Q$ is projective.*

Corollary 2.2.1.3. *$I : \mathcal{A}$ is injective if and only if for every inflation $p : B \to A$ and morphism $g : B \to I$ there is a morphism $g' : A \to I$ rendering the diagram below commutative.*

$$\begin{tikzcd} & I \\ A \arrow[ur, dashed, "g'"] & B \arrow[l, tail, "p"] \arrow[u, "f"'] \end{tikzcd}$$

Definition 2.2.3. A category $(\mathcal{A}, \mathcal{E})$ has enough projective objects if for any object $A : \mathcal{A}$ there is a projective object P along with a deflation $q : P \to A$. Dually, it has enough injective objects if for any object $A : \mathcal{A}$ there is an injective object I along with an inflation $p : A \to I$.

Definition 2.2.4. An exact category is called a Frobenius category if it has enough projective and injective objects and the class of projective objects coincide with the injective objects.

The stable Frobenius category will be defined as the quotient of every morphism factoring through an injective object. This will be made precise, following the construction of the quotient category provided by [5].

Definition 2.2.5. A congruence relation $\sim$ on a category $\mathcal{C}$ is a relation on the hom-sets, such that:

1. $\forall A, B : \mathcal{C}$ the relation $\sim_{A,B}$ is an equivalence relation.
2. Given that $f, f' : A \to B$ is related ($f \sim f'$) and morphisms $g : A' \to A$ and $h : B \to B'$, then $hfg \sim hf'g$.

Proposition 2.2.2. *Let $\mathcal{C}$ be a category and $\sim$ be a congruence relation. Then there is a universal category $\mathcal{C}/\sim$ together with a functor $q : \mathcal{C} \to \mathcal{C}/\sim$ such that morphisms $f, g : A \to B$ are identified if $f \sim g$. Universality means that if there is a functor $H : \mathcal{C} \to \mathcal{D}$ such that $Hf = Hg$ for any f, g iff $\sim g$, then H factors uniquely through $\mathcal{C}/\sim$.*

Proof. Define the category $\mathcal{C}/\sim$ to have the same objects as $\mathcal{C}$, and define $\mathcal{C}/\sim(A, B) = \mathcal{C}(A, B)/\sim_{a,b}$. This definition is well defined as $\sim$ is a congruence relation. A sketch of this proof can be found in [5]. □

Remark. Any functor gives rise to a congruence relation. That is, if $F : \mathcal{C} \to \mathcal{D}$ is a functor, then there is a congruence relation $\sim$ defined as follows: $\forall A, B : \mathcal{C}$ and $f, g : A \to B$, we define $f \sim_{A,B} g \iff Ff = Fg$. This is a congruence as equality within $\mathcal{D}$ gives rise to an equivalence relation, and functoriality gives the congruence.

Remark. For any relation $\sim$ the universal category $\mathcal{C}/\sim$ exists. As in the case for the Verdier quotient, $\mathcal{C}/\sim$ is the same as the quotient category of the smallest congruence relation having the same relations as $\sim$.

If $\mathcal{A}$ is an additive category, the quotient categories which respects the additive structures are of interest. That is to say that the functor $q : \mathcal{A} \to \mathcal{C}/\sim$ is additive and the equivalence relation $\sim$ should respect the additive structure. Then a quotient category is additive if $f \sim f'$ and $g \sim g'$, then $f + g \sim f' + g'$. This leads to the following definition.

Definition 2.2.6. Let $\mathcal{A}$ be an additive category. $\mathcal{I}$ is an ideal of $\mathcal{A}$ if:

1. (subgroup) for every abelian group $\mathcal{A}(A,B)$ there is a subgroup $\mathcal{I}(A,B) \subseteq \mathcal{A}(A,B)$.
2. (absorption) For every $g : A' \to A$, $h : B \to B'$ and $f : \mathcal{I}(A,B)$ it follows that $hfg : \mathcal{I}(A',B')$

This is equivalent of saying that the equivalence relation $f \sim g \iff f - g : \mathcal{I}(A,B)$ is a congruence relation.

Corollary 2.2.2.1. *Let $\mathcal{A}$ be an additive category and $\mathcal{I}$ be an ideal of $\mathcal{A}$, then $\mathcal{A}/\mathcal{I}$ is an additive category.*

Let $\mathcal{A}$ be a Frobenius category. Define the ideal $\mathcal{I}$ as the subgroups of every morphism factoring through injective objects.

Proposition 2.2.3. *For any Frobenius category $\mathcal{A}$ the ideal $\mathcal{I}$ is well defined and $\underline{\mathcal{A}} = \mathcal{A}/\mathcal{I}$ is the stable Frobenius category.*

Proof. To prove this one must show that $\mathcal{I}(A,B)$ is a subgroup for any $A, B : \mathcal{A}$, and that it is absorptive. First observe that $0 : \mathcal{I}(A,B)$. Let $f, g : \mathcal{I}(A,B)$. Since $\mathcal{A}$ has enough injectives, there exists an injective object with an inflation from A.

$$\begin{tikzcd}
& J_1 \arrow[dr, "f_2"] & \\
A \arrow[ur, "f_1"] \arrow[r, tail, "i"] \arrow[dr, "g_1"'] & I \arrow[u, dashed, "f_1'"] \arrow[d, dashed, "g_1'"] & B \\
& J_2 \arrow[ur, "g_2"'] &
\end{tikzcd}$$

$f - g = f_2 \circ f_1 - g_2 \circ g_1 = (f_2 \circ f_1' - g_2 \circ g_1') \circ i$. Thus $f - g$ factors through an injective, and $\mathcal{I}(A,B)$ is a subgroup. To see that it is absorptive is to see that if f factors through an injective, then gf factors through an injective as well. □

Objects in the stable Frobenius category is denoted as $\underline{X}$ and morphisms are denoted as $\underline{f}$. That is the functor $q : \mathcal{A} \to \underline{\mathcal{A}}$ is defined as $q(X) = \underline{X}$ and $q(f) = \underline{f}$. One important property of the stable Frobenius category is that taking syzygies or cosyzygies is a functor.

Definition 2.2.7. A syzygy of an object X, if it exists, is denoted ΩX. The syzygy is defined to be the kernel object of a deflation $p : P \to X$, where P is projective. A cosyzygy, denoted as $\mho X$ is defined to be the cokernel of an inflation $i : X \to I$, where I is injective.

Remark. Note that this choice is not necessarily unique up to isomorphism. Thus syzygies and cosyzygies are not in general functors.

Lemma 2.2.4. *Let $\mathcal{A}$ be a Frobenius category and suppose that there are two conflations with injectives as below. Then $\underline{X'} \simeq \underline{X''}$.*

$$\begin{tikzcd} X \arrow[r, tail, "i"] & I \arrow[r, tail, "i'"] & X' \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, tail, "j"] & J \arrow[r, tail, "j'"] & X'' \end{tikzcd}$$

Proof. Observe that there are morphisms in the diagram as I and J are injective.

$$\begin{tikzcd} X \arrow[r, tail, "i"] \arrow[d, equal] \arrow[dr, "j"] & I \arrow[d, dashed, "f"] \\ X \arrow[r, tail, "j"] & J \end{tikzcd}$$

The commutative diagram below is created by the cokernel property.

$$\begin{tikzcd} X \arrow[r, tail, "i"] \arrow[d, equal] & I \arrow[r, two heads, "i'"] \arrow[d, "f"] & X' \arrow[d, dashed, "g"] \\ X \arrow[r, tail, "j"] \arrow[d, equal] & J \arrow[r, two heads, "j'"] \arrow[d, "f'"] & X'' \arrow[d, dashed, "g'"] \\ X \arrow[r, tail, "i"] & I \arrow[r, two heads, "i'"] & X' \end{tikzcd}$$

A diagram chase shows that $i - f'fi = (id_I - f'f) \circ i = 0$. This means that $(f'f - id_I)$ factors through X', i.e. there exists $h : X' \to I$ and $f'f = hi' + id_I$. Diagram chasing also reveals that $g'gi' = i'f'f = i'(hi' + id_I) = i'hi' + i' = (i'h + id_{X'})i'$. As i' is an epi one obtains that $g'g = i'h + id_{X'} \implies \underline{g'g} = id_{\underline{X'}}$ as $i'h$ factors through I. $\underline{gg'} = id_{\underline{X''}}$ is dual. $\square$

Corollary 2.2.4.1. *Cosyzygy is a well defined functor* $\mho$: $\underline{\mathcal{A}} \to \underline{\mathcal{A}}$

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{A}$. Then the following diagrams representing the different choices of syzygies.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & I \arrow[r, "p"] \arrow[d, dashed] & \mho X \arrow[d, dashed, "\mho f"] \\ Y \arrow[r, "j"] & J \arrow[r, "q"] & \mho Y \end{tikzcd} \quad \begin{tikzcd} X \arrow[r, "i'"] \arrow[d, "f"] & I' \arrow[r, "p'"] \arrow[d, dashed] & \mho' X \arrow[d, dashed, "\mho' f"] \\ Y \arrow[r, "j'"] & J' \arrow[r, "q'"] & \mho' Y \end{tikzcd}$$

By the previous proof there are maps between the diagrams making an almost commutative diagram where all the 8 outer squares commute.

$$
\begin{tikzcd}
X \arrow[rr, tail, "i"] \arrow[dr, equal, "\alpha(X)"'] \arrow[dd, "f"'] & & I \arrow[rr, two heads, "p"] \arrow[dr, "\beta(X)"] \arrow[dd, dashed, "If" near start] & & \mho X \arrow[dr, "\gamma(X)"] \arrow[dd, dashed, "\mho f" near start] \arrow[ddl, squiggly, dashed, "\chi"] & \\
& X \arrow[rr, tail, "i'" near start, crossing over] & & I' \arrow[rr, two heads, "p'" near start, crossing over] \arrow[dd, dashed, "I'f" near end] & & \mho' X \arrow[dd, dashed, "\mho' f"] \\
Y \arrow[rr, tail, "j" near start] \arrow[dr, equal, "\alpha(Y)"'] & & J \arrow[rr, two heads, "q" near start] \arrow[dr, "\beta(Y)"'] & & \mho Y \arrow[dr, "\gamma(Y)"] & \\
& Y \arrow[rr, tail, "j'"] \arrow[from=uu, crossing over, "f" near start] & & J' \arrow[rr, two heads, "q'"] & & \mho' Y
\end{tikzcd}
$$

To see that the definition of the cosyzygy is well defined is to show that the 3 inner squares is commutative in the quotient category, i.e. that the diagram commutes in the quotient.

Observe that the left inner square commutes by definition, and that the central inner square commutes in the quotient as every morphism gets related to 0. Thus it remains to show that $\underline{\gamma(Y)}\circ\underline{\mho f}=\underline{\mho' f}\circ\underline{\gamma(X)}$, which is the same as to say that $\gamma(Y)\circ\mho f - \mho' f\circ\gamma(X)$ factors over an injective.

By doing a diagram chase in the left cube one may find the following equation $(I'f\circ\beta(X)-\beta(Y)\circ If)i=0$. This means that the map $I'f\circ\beta(X)-\beta(Y)\circ If$ factors through the cokernel of i as χp. By chasing the right cube one may assert the equation $q'\chi p = (\gamma(Y)\circ\mho f - \mho' f\circ\gamma(X))p$, thus $q'\chi = \gamma(Y)\circ\mho f - \mho' f\circ\gamma(X)$. □

Corollary 2.2.4.2. *Cosyzygy $\mho$ is an autoequivalence with syzygy Ω as quasi-inverse.*

Proof. The goal is to show that there is a natural isomorphisms $\Omega\mho\simeq Id_{\underline{\mathcal{A}}}$ and $\mho\Omega\simeq Id_{\underline{\mathcal{A}}}$. As these are inverse operations one have that taking syzygy then cosyzygy is the same as taking cosyzygy then syzygy in $\mathcal{A}^{op}$. Let $X:\mathcal{A}$, the goal is to show that the following diagram gives a natural isomorphism at the rightmost arrow in $\underline{\mathcal{A}}$.

$$
\begin{tikzcd}
\Omega X \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d, dashed] & X \arrow[d, dashed] \\
\Omega X \arrow[r] & I \arrow[r] & \mho\Omega X
\end{tikzcd}
$$

Observe that this case is identical as the one previous proved. This shows that there is a natural isomorphism from X to $\mho\Omega$X. □

Remark. A subtle, but important point is that the category $\mathcal{A}$ has enough projectives and injectives. This enables one to find the syzygies and cosyzygies. It is also important that the projectives are the same as the injectives for this construction to give the isomorphisms as well.

With the category $\underline{\mathcal{A}}$ and the functor $\mho$ it remains to find the triangulation $\Delta_{\underline{\mathcal{A}}}$. The triangulation of $\underline{\mathcal{A}}$ will be defined as the set of candidate triangles in $\underline{\mathcal{A}}$ called standard triangles. Let $\underline{x}:\underline{X}\to\underline{Y}$ be a morphism, then by (QE2) there is a push-out in $\mathcal{A}$. Moreover, by Proposition 2.1.3 (y,z) is a conflation.

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, tail, "i"] & Y \arrow[d, tail, "y"] \arrow[ddr, bend left, "0"] & \\
I(X) \arrow[r, "x'"] \arrow[drr, bend right, two heads, "p"] & Z \arrow[dr, dashed, two heads, "z"] & \\
 & & \mho X
\end{tikzcd}$$

The set of standard triangles will be of the form $(\underline{X}, \underline{Y}, \underline{Z}, \underline{x}, \underline{y}, \underline{z})$. Thus a triangle $(A, B, C, a, b, c) : \Delta_{\underline{A}}$ if and only if (A, B, C, a, b, c) is isomorphic to a standard triangle.

Proposition 2.2.5. $\Delta_{\underline{A}}$ *is a triangulation of* $\underline{A}$.

Proof. Sketch. Most of the details of this proof will be omitted, and they can be found in [6] or [7]. The proof is structured into 3 different parts, namely showing TR1, TR2 and TR4. Note that TR1 is satisfied by definition of $\Delta_{\underline{A}}$. To see this observe that the following diagram is a push-out.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, tail, "i"] & X \arrow[d, tail, "i"] \\
I(X) \arrow[r, equal] & I(X)
\end{tikzcd}$$

Thus $(\underline{X}, \underline{X}, \underline{0}, \underline{id_X}, \underline{0}, \underline{0})$ is a standard triangle.
(TR2) Consider the standard triangle $(\underline{X}, \underline{Y}, \underline{Z}, \underline{x}, \underline{y}, \underline{z})$, the goal is to show that there is a triangle $(\underline{Y}, \underline{Z}, \underline{\mho X}, \underline{x}, \underline{y}, -\underline{\mho x})$. Let $I(X)$ and $I(Y)$ be injectives with inflations from X and Y respectively. Since $I(Y)$ is injective there is a unique map by the push-out property in (1).

$$(1)\quad \begin{tikzcd}
X \arrow[r, "x"] \arrow[d, tail, "i(X)"] & Y \arrow[d, "y"] \arrow[ddr, bend left, "i(Y)"] & \\
I(X) \arrow[r, "x'"] \arrow[drr, bend right, "f"] & Z \arrow[dr, dashed, "g"] & \\
 & & I(Y)
\end{tikzcd}
\qquad
(2)\quad \begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "i(X)"] & Y \arrow[d, "y"] \arrow[dr, "0"] & \\
I(X) \arrow[r, "x'"] \arrow[dr, "f"] & Z \arrow[r, "z"] \arrow[d, "g"] & \mho X \arrow[d, "\mho x"] \\
 & I(Y) \arrow[r, "p(Y)"] & \mho Y
\end{tikzcd}$$

From (2) one are able to use the push-out to see that the lower right square commutes, that is $p(Y)f\,i(X) = \mho xzyz = 0$. This is true as $p(Y)f\,i(X) = p(Y)gyx = p(Y)i(Y)x = 0$ by (1). Note that since z and $p(Y)$ are deflations with equal kernels, Proposition 2.1.3 says that $\left(\begin{pmatrix} g \\ z \end{pmatrix}, \begin{pmatrix} p(Y) & -\mho x \end{pmatrix}\right)$ is a conflation.
One is now able to find a commutative diagram and by Proposition 2.1.3 the upper left square is bicartesian.

$$\begin{tikzcd}
Y \arrow[r, tail, "y"] \arrow[d, tail, "i(Y)"] & Z \arrow[r, two heads, "z"] \arrow[d, tail, "\begin{pmatrix} g \\ z \end{pmatrix}"] & \mho X \arrow[d, equal] \\
I(Y) \arrow[r, tail, "\iota_1"'] \arrow[d, two heads, "p(Y)"] & I(Y) \oplus \mho X \arrow[r, two heads, "\pi_2"'] \arrow[d, two heads, "\begin{pmatrix} p(Y) & -\mho x \end{pmatrix}"] & \mho X. \\
\mho Y \arrow[r, equal] & \mho Y &
\end{tikzcd}$$

Thus $(\underline{Y},\underline{Z},\underline{\mho X},\underline{y},\underline{z},-\underline{\mho x})$ is a standard triangle.
(TR4) Suppose that there are three standard triangles where $\nu\upsilon=\omega$.

$$
\begin{tikzcd}
X \arrow[r, "\upsilon"] \arrow[d, "x"] & Y \arrow[d, "i"] \\
I(X) \arrow[r, "\bar\upsilon"] \arrow[d, "\bar x"] & Z' \arrow[d, "i'"] \\
\mho X \arrow[r, equal] & \mho X
\end{tikzcd}
\quad
\begin{tikzcd}
Y \arrow[r, "\nu"] \arrow[d, "y"] & Z \arrow[d, "j"] \\
I(Y) \arrow[r, "\bar\nu"] \arrow[d, "\bar y"] & X' \arrow[d, "j'"] \\
\mho Y \arrow[r, equal] & \mho Y
\end{tikzcd}
\quad
\begin{tikzcd}
X \arrow[r, "\omega"] \arrow[d, "x"] & Z \arrow[d, "k"] \\
I(X) \arrow[r, "\bar\omega"] \arrow[d, "\bar x"] & Y' \arrow[d, "k'"] \\
\mho X \arrow[r, equal] & \mho X
\end{tikzcd}
$$

By Noethers isomorphism lemma there is a conflation passing through on the right column and the middle square is bicartesian. z' exists by the injectivity of $I(Y)$ and that i is an inflation. z' is an inflation as y is an inflation, thus $\bar{z}'$ exists.

$$
\begin{tikzcd}
Y \arrow[r, tail, "i"] \arrow[d, equal] & Z' \arrow[r, two heads, "i'"] \arrow[d, tail, "z'"] & \mho X \arrow[d, tail, "s"] \\
Y \arrow[r, tail, "y"] & I(Y) \arrow[r, two heads, "\bar y"] \arrow[d, two heads, "\bar z'"] & \mho Y \arrow[d, two heads, "r"] \\
& \mho Z' \arrow[r, equal] & \mho Z'
\end{tikzcd}
$$

There is also a map $I_\upsilon : I(X)\to I(Y)$ induced by the maps between X and $I(Y)$. By using the following universal properties one may find the unique maps f and g.

$$
\begin{tikzcd}
X \arrow[r, "\upsilon"] \arrow[d, "x"] \arrow[rr, bend left, "\omega"] & Y \arrow[r, "\nu"] \arrow[d, "i"] & Z \arrow[dd, "k"] \\
I(X) \arrow[r, "\bar\upsilon"] \arrow[drr, bend right, "\bar\omega"] & Z' \arrow[dr, dashed, "f"] & \\
& & Y'
\end{tikzcd}
\quad
\begin{tikzcd}
X \arrow[r, "\omega"] \arrow[d, "x"] & Z \arrow[d, "k"] \arrow[ddr, bend left, "j"] & \\
I(X) \arrow[r, "\bar w"] \arrow[dr, bend right, "I_\upsilon"] & Y' \arrow[dr, dashed, "g"] & \\
& I(Y) \arrow[r, bend right, "\bar\nu"] & X'
\end{tikzcd}
$$

These maps can be arranged in the diagram below. It can be seen that middle square is a push-out, by using the fact that the upper left square and the larger rectangles are push-outs.

$$
\begin{tikzcd}
X \arrow[r, "\upsilon"] \arrow[d, "x"] & Y \arrow[r, "\nu"] \arrow[d, "i"] & Z \arrow[d, "k"] \\
I(X) \arrow[r, "\bar\upsilon"] \arrow[d, "\bar x"] \arrow[dr, "I_\upsilon"] & Z' \arrow[r, "f"] \arrow[d, "z'"] & Y' \arrow[d, "g"] \\
\mho X \arrow[dr, "s"] & I(Y) \arrow[r, "\bar\nu"] \arrow[d, "r"] & X' \arrow[d, "\mho i\circ j'"] \\
& \mho Y \arrow[r, "\mho i"] & \mho Z'
\end{tikzcd}
\quad
\begin{tikzcd}
Z' \arrow[r, "f"] \arrow[d, tail, "z'"] & Y' \arrow[d, tail, "g"] \\
I(Y) \arrow[r, "\bar\nu"] \arrow[d, two heads, "\bar z'"] & X' \arrow[d, two heads, "\mho i\circ j'"] \\
\mho Z' \arrow[r, equal] & \mho Z'
\end{tikzcd}
$$

Thus $(\underline{Z'}, \underline{Y'}, \underline{X'}, \underline{f}, \underline{g}, \underline{\mho i} \circ \underline{j'})$ is a triangle. $\square$

Remark. A more detailed and different proof may be found in [8] or [9].

This construction of triangles admits a close relation to conflations. If there is a conflation $(p : X \to Y, q : Y \to Z)$, then there is a triangle $(\underline{X}, \underline{Y}, \underline{Z}, \underline{p}, \underline{q}, -\underline{r})$ constructed as follows: Let $P : \mathcal{A}$ be a projective object with a deflation $\bar{p} : P \to Y$, then there exists a pullback (1), moreover the pullback square is bicartesian. By using TR2 one may find the triangle (2) as indicated in the diagram.

$$(1)\quad \begin{tikzcd} \Omega Z \arrow[r, two heads, "\Omega r"] \arrow[d, tail] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, tail, "p"] \\ P \arrow[r, two heads, "\bar{p}"] \arrow[d, two heads] \arrow[ur, phantom, "\lrcorner", very near end] & Y \arrow[d, two heads, "q"] \\ Z \arrow[r, equal] & Z \end{tikzcd} \qquad (2)\quad \underline{X} \xrightarrow{\underline{p}} \underline{Y} \xrightarrow{\underline{q}} \underline{Z} \xrightarrow{-\underline{r}} \underline{\mho X}$$

Remark. For any morphism $f : A \to B$ in $\mathcal{A}$, there is an inflation $\begin{pmatrix} f \\ -i \end{pmatrix} : A \to B \oplus I$ which is in the same equivalence class as f. Thus $\underline{f} = \underline{\begin{pmatrix} f \\ -i \end{pmatrix}}$, and any morphism in $\underline{\mathcal{A}}$ can be obtained from an inflation in $\mathcal{A}$.

2.3 Self-injective Algebras

The first example of a triangulated category is going to be derived from finite dimensional artin algebras. More specifically, let Λ be a self-injective finite dimensional artin R-algebra; that is ${}_\Lambda\Lambda$ is injective as left Λ-module, then the finitely generated projective objects coincide with the finitely generated injective objects.

Proposition 2.3.1. *If Λ is a self-injective finite dimensional artin R-algebra, then mod_Λ is a Frobenius category.*

To prove this statement we will need the following propositions.

Lemma 2.3.2. *The category mod_Λ has enough projectives*

Proof. Let $A : mod_\Lambda$, then A is finitely generated. This means there exists an epimorphism $p : R^n \to A$, where n is the number of generators of A. $\square$

Lemma 2.3.3. *Let R be an artin ring and $\mathfrak{r}$ denote the nilradical of R. Moreover, let J be the injective envelope of $R/\mathfrak{r}$, then functor $Hom_R(\_, J) : mod_\Lambda \to mod_{\Lambda^{op}}$ is a duality.*

Corollary 2.3.3.1. *The category mod_Λ has enough injectives*

Detailed proofs of these statements can be found in [10].

Proof. Suppose that Λ is self-injective. By the lemmas above it is known that mod_Λ has enough projectives and enough injectives. It remains to show that the class of injectives coincide with the projectives. Since every indecomposable Λ module is a summand of Λ up to isomorphism, it follows that they are injective. As they also are projective, the class of injectives and projectives coincide. $\square$

This shows that mod_Λ is a Frobenius category, thus $\underline{mod_\Lambda}$ is triangulated. The triangles in $\underline{mod_\Lambda}$ are the quotients of every short exact sequence in mod_Λ.

$$0 \longrightarrow X \overset{a}{\rightarrowtail} Y \overset{b}{\twoheadrightarrow} Z \longrightarrow 0 \implies \underline{X} \xrightarrow{\underline{a}} \underline{Y} \xrightarrow{\underline{b}} \underline{Z} \xrightarrow{\underline{c}} \underline{\mho X}$$

Proposition 2.3.4. *Let G be a group and R any commutative artin ring, then the group ring $R[G]$ is self-injective.*

Proposition 2.3.5. *Let K be a field, then $K[x]/(x^n)$ is self-injective.*

Proof. As $K[x]/(x^n)$ modules, there is only one indecomposable projective module up to isomorphism, that is $K[x]/(x^n)$. Since $K[x]/(x^n)$ is commutative, the duality functor is an automorphism of $mod_{K[x]/(x^n)}$, thus $Hom_K(K[x]/(x^n),K)$ is the indecomposable injective $K[x]/(x^n)$ module. As the duality functor preserves length the modules have equal length. By finding a monomorphism $i : K[x]/(x^n) \to Hom_K(K[x]/(x^n),K)$ we have that it is an isomorphism as the cokernel has length 0. The socle $soc(K[x]/x^n)$ is the simple module K, this means that the injective envelope of $K[x]/(x^n)$ is indecomposable, thus it is in the same isomorphism class as $Hom_K(K[x]/(x^n),K)$, proving that there is a monomorphism as stated. $\square$

In this particular case the triangles take on a somewhat special form, where repeatedly applying TR2 yields the same triangles after 6 iterations! This can be seen by calculating the triangles of the indecomposable modules. Every other triangle will be a direct sum of these.

Observe that every submodule of $K[x]/(x^n)$ is indecomposable, these make up the class of the indecomposable modules up to isomorphism. Further observe that the cosyzygy of any submodule is $\mho(x^k)/(x^n) \simeq (x^{n-k})/(x^n)$. The repetition of the triangles can be seen as the natural isomorphism $\mho^2(x^k)/(x^n) \simeq (x^{n-(n-k)})/(x^n) = (x^k)/(x^n)$.

To find the triangles, let $A, B : mod_{K[x]/(x^n)}$ and $T : A \to B$ be $K[x]/(x^n)$-linear. T is in the same equivalence class as $\begin{pmatrix} T \\ -i \end{pmatrix} : A \to B \oplus I$ with i as the injective envelope of A. Then there is a triangle as the diagram below.

$$\underline{A} \xrightarrow{\underline{T}} \underline{B} \longrightarrow Cok\underline{T} \oplus Ker\underline{\mho T} \longrightarrow \underline{\mho A}$$

Observe that $\underline{Cok\begin{pmatrix} f \\ -i \end{pmatrix}} \simeq Cok\underline{T} \oplus Ker\underline{\mho T}$, so the triangle above is in fact well-defined.

Lemma 2.3.6. *The category $Vect(K)$ is triangulated.*

Proof. Sketch. This follows immediately from the discussion above. Look at $mod_{K[x]/(x^2)}$, the indecomposable objects of this category are $K[x]/(x^2)$ and K up to isomorphism. As $K[x]/(x^2)$ is injective we have that K is the only indecomposable object of $\underline{mod_{K[x]/(x^2)}}$, thus every object is a direct summand K. Also observe that the cosyzygy is naturally isomorphic to the identity functor on the quotient. In order to be precise, one would need to show that there is an equivalence of categories $Vect(K) \simeq \underline{mod_{K[x]/(x^2)}}$. The triangles in $Vect(K)$ can then be seen as this three term repeating triangle.

$$V \xrightarrow{T} W \xrightarrow{\begin{pmatrix} \pi_T \\ 0 \end{pmatrix}} CokT \oplus KerT \xrightarrow{\begin{pmatrix} 0 & \iota_T \end{pmatrix}} V$$

□

2.4 The Homotopy Category

The next example of a triangulated category is the homotopy category. This category may be regarded as the prototype for triangulated categories. In order to define it, the category of chain complexes and homotopies must be defined first.

Definition 2.4.1. Let $\mathcal{A}$ be an additive category. Define $Ch(\mathcal{A})$ to be the category of diagrams in $\mathcal{A}$ on the form

$$\ldots \xrightarrow{d_{A^\bullet}^{-2}} A^{-1} \xrightarrow{d_{A^\bullet}^{-1}} A^0 \xrightarrow{d_{A^\bullet}^{0}} A^1 \xrightarrow{d_{A^\bullet}^{1}} \ldots$$

such that $d_{A^\bullet}^{i} \circ d_{A^\bullet}^{i-1} = 0$ for every $i : \{-\infty, \ldots, \infty\}$. These objects are referred to as (co)chain complexes and they are denoted as $A^\bullet$, and the maps in the objects are called differentials/(co)boundaries. A morphism $\phi^\bullet : A^\bullet \to B^\bullet$ between (co)chain complexes, also called chain map, is a collection of morphisms from $\mathcal{A}$, such that the morphisms commute with the differentials in the following manner:

$$\begin{tikzcd}
\ldots \arrow[r, "d_{A^\bullet}^{-2}"] & A^{-1} \arrow[r, "d_{A^\bullet}^{-1}"] \arrow[d, "\phi^{-1}"] & A^0 \arrow[r, "d_{A^\bullet}^{0}"] \arrow[d, "\phi^{0}"] & A^1 \arrow[r, "d_{A^\bullet}^{1}"] \arrow[d, "\phi^{1}"] & \ldots \\
\ldots \arrow[r, "d_{B^\bullet}^{-2}"] & B^{-1} \arrow[r, "d_{B^\bullet}^{-1}"] & B^0 \arrow[r, "d_{B^\bullet}^{0}"] & B^1 \arrow[r, "d_{B^\bullet}^{1}"] & \ldots
\end{tikzcd}$$

Remark. If $\mathcal{A}$ is abelian, then the category $Ch(\mathcal{A})$ is abelian. The kernels and cokernels of chain maps would be level-wise kernels and cokernels along the chain. Moreover, if $(\mathcal{A}, \mathcal{E})$ is an exact category, then $(Ch(\mathcal{A}), Ch(\mathcal{E}))$ will be exact as well, by using level-wise kernels and cokernels.

Remark. On the category of cochain complexes there is an additive autoequivalence called the translation functor. The functor is denoted as $(\_)[1] : Ch(\mathcal{A}) \to Ch(\mathcal{A})$ and it takes a complex $A^\bullet$ and shifts it one step to the left into $A^{\bullet+1}$. In fact there is a family of functors $A^\bullet[n] = A^{\bullet+n}$. Thus $(\_)[-1]$ is the quasi-inverse of $(\_)[1]$.

Definition 2.4.2. A chain map $f^\bullet : A^\bullet \to B^\bullet$ is called null-homotopic if there is a map $\varepsilon^\bullet : A^\bullet \to B^\bullet[-1]$ such that $f^\bullet = d_{B^\bullet}^{\bullet-1}\varepsilon^\bullet + \varepsilon^{\bullet+1}d_{A^\bullet}^\bullet$.

$$\begin{tikzcd}
\cdots \arrow[r, "d_{A^\bullet}^{-2}"] & A^{-1} \arrow[r, "d_{A^\bullet}^{-1}"] \arrow[d, "f^{-1}"] & A^0 \arrow[r, "d_{A^\bullet}^{0}"] \arrow[d, "f^0"] \arrow[dl, "\varepsilon^0"'] & A^1 \arrow[r, "d_{A^\bullet}^{1}"] \arrow[d, "f^1"] \arrow[dl, "\varepsilon^1"'] & \cdots \\
\cdots \arrow[r, "d_{B^\bullet}^{-2}"] & B^{-1} \arrow[r, "d_{B^\bullet}^{-1}"] & B^0 \arrow[r, "d_{B^\bullet}^{0}"] & B^1 \arrow[r, "d_{B^\bullet}^{1}"] & \cdots
\end{tikzcd}$$

$\varepsilon^\bullet$ is called the homotopy. Two chain maps $f^\bullet$ and $g^\bullet$ are said to be homotopic $f^\bullet \sim g^\bullet$ if their difference $f^\bullet - g^\bullet$ is null-homotopic.

Proposition 2.4.1. *There is an additive bifunctor $nullHom_{\mathcal{A}}(\_,\_) : Ch(\mathcal{A})^{op} \times Ch(\mathcal{A}) \to Ab$ mapping into the set of null-homotopic morphisms. The elements of $nullHom_{\mathcal{A}}(A^\bullet, B^\bullet)$ are pairs made of null-homotopic maps with their homotopy $(f^\bullet, \varepsilon^\bullet)$. This is an abelian group with the product group structure, that is $(f^\bullet, \varepsilon^\bullet) + (g^\bullet, \gamma^\bullet) = (f^\bullet + g^\bullet, \varepsilon^\bullet + \gamma^\bullet)$. The functor acts on morphisms almost the same way as the hom-functor. On a chain map $f^\bullet : B^\bullet \to C^\bullet$ define the covariant direction to be $f_*^\bullet = nullHom_{\mathcal{A}}(A^\bullet, f^\bullet) = \{(f^\bullet g^\bullet, f^{\bullet-1}\varepsilon^\bullet) | (g^\bullet, \varepsilon^\bullet) : nullHom_{\mathcal{A}}(A^\bullet, B^\bullet)\}$, and dually $f^*(g^\bullet, \varepsilon^\bullet) = (g^\bullet f^\bullet, \varepsilon^\bullet f^\bullet)$ in the contravariant direction.*

Proof. In order to prove the proposition, one must show that the assignment is in fact a functor and that it is additive as well. It suffices to show that $nullHom_{\mathcal{A}}(A^\bullet, \_)$ is an additive functor, as it will follow by duality that the there is an additive bifunctor as proposed.

Suppose that there is a chain map $f^\bullet : B^\bullet \to C^\bullet$, then $nullHom_{\mathcal{A}}(A^\bullet, \_)(f^\bullet) = f_*^\bullet$. Let $(g^\bullet, \varepsilon^\bullet) : nullHom_{\mathcal{A}}(A^\bullet, B^\bullet)$ be a null-homotopic chain map. By definition $f_*^\bullet(g^\bullet, \varepsilon^\bullet) = (f^\bullet g^\bullet, f^{\bullet-1}\varepsilon^\bullet)$. One may now see that $f^{\bullet-1}\varepsilon^\bullet$ is a homotopy by the following diagram. The commutativity of the lower left square shows the homotopy. It follows by functoriality from the Hom-functor that $nullHom_{\mathcal{A}}(A^\bullet, \_)$ is a functor.

$$\begin{tikzcd}
\cdots \arrow[r, "d_{A^\bullet}^{-2}"] & A^{-1} \arrow[r, "d_{A^\bullet}^{-1}"] \arrow[d, "g^{-1}"] & A^0 \arrow[r, "d_{A^\bullet}^{0}"] \arrow[d, "g^0"] \arrow[dl, "\varepsilon^0"'] & A^1 \arrow[r, "d_{A^\bullet}^{1}"] \arrow[d, "g^1"] \arrow[dl, "\varepsilon^1"'] & \cdots \\
\cdots \arrow[r, "d_{B^\bullet}^{-2}"] & B^{-1} \arrow[r, "d_{B^\bullet}^{-1}"] \arrow[d, "f^{-1}"] & B^0 \arrow[r, "d_{B^\bullet}^{0}"] \arrow[d, "f^0"] & B^1 \arrow[r, "d_{B^\bullet}^{1}"] \arrow[d, "f^1"] & \cdots \\
\cdots \arrow[r, "d_{C^\bullet}^{-2}"] & C^{-1} \arrow[r, "d_{C^\bullet}^{-1}"] & C^0 \arrow[r, "d_{C^\bullet}^{0}"] & C^1 \arrow[r, "d_{C^\bullet}^{1}"] & \cdots
\end{tikzcd}$$

Lastly, one must show that the functor is additive. This is the same as showing that the assignment
$nullHom_{\mathcal{A}}(A^\bullet, \_) : Hom_{Ch(\mathcal{A})}(B^\bullet, C^\bullet) \to Hom_{Ab}(nullHom_{\mathcal{A}}(A^\bullet, B^\bullet), nullHom_{\mathcal{A}}(A^\bullet, C^\bullet))$ is a group homomorphism. Let $f^\bullet, g^\bullet : B^\bullet \to C^\bullet$ be two chain maps, and $(h^\bullet, \varepsilon^\bullet)$:

$nullHom_{\mathcal{A}}(A^\bullet, B^\bullet)$. Then the following equation asserts the additivity:

$$\begin{aligned}
&(f^\bullet + g^\bullet)_*(h^\bullet, \varepsilon^\bullet) \\
&= ((f^\bullet + g^\bullet)h^\bullet, ((f^\bullet + g^\bullet)[-1])\varepsilon^\bullet) \\
&= (f^\bullet h^\bullet + g^\bullet h^\bullet, f^{\bullet -1}\varepsilon^\bullet + g^{\bullet -1}\varepsilon^\bullet) \\
&= (f^\bullet h^\bullet, f^{\bullet -1}\varepsilon^\bullet) + (g^\bullet h^\bullet, g^{\bullet -1}\varepsilon^\bullet) \\
&= f_*^\bullet(h^\bullet, \varepsilon^\bullet) + g_*^\bullet(h^\bullet, \varepsilon^\bullet)
\end{aligned}$$

□

Corollary 2.4.1.1. *The equivalence relation $\sim$ stated above is an additive congruence relation. The homotopy category is defined to be the quotient $K(\mathcal{A}) = Ch(\mathcal{A})/\sim$.*

The goal is to prove that the homotopy category is triangulated. This will be done by seeing that $Ch(\mathcal{A})$ admits an exact structure, which allows us to view it as a Frobenius category. By checking that the construction of $K(\mathcal{A})$ coincide with $\underline{Ch(\mathcal{A})}$ will prove that it is triangulated. This will be revealed by studying the representable nature of $nullHom_{\mathcal{A}}(\_, \_)$.

Definition 2.4.3. Let $f^\bullet : A^\bullet \to B^\bullet$ be a chain map. Define the object $cone(f^\bullet)$ to be the complex below.

$$\cdots \longrightarrow B^{-1} \oplus A^0 \xrightarrow{\begin{pmatrix} d_{B^\bullet}^{-1} & f^0 \\ 0 & -d_{A^\bullet}^0 \end{pmatrix}} B^0 \oplus A^1 \longrightarrow \cdots$$

Remark. For any chain map $f^\bullet : A^\bullet \to B^\bullet$ there is a short exact sequence.

$$B^\bullet \overset{\begin{pmatrix} 1 \\ 0 \end{pmatrix}^\bullet}{\rightarrowtail} cone(f^\bullet) \overset{\begin{pmatrix} 0 & -1^\bullet \end{pmatrix}^\bullet}{\twoheadrightarrow} A^\bullet[1]$$

Definition 2.4.4. An object $A^\bullet$ of $Ch(\mathcal{A})$ is called contractible if $id_{A^\bullet}^\bullet$ is null-homotopic.

Example. Let $A^\bullet$ be a complex, then $cone(id_{A^\bullet})$ is contractible. That is
$\left(\begin{pmatrix} id_{A^\bullet}^\bullet & 0 \\ 0 & id_{A^\bullet}^\bullet[1] \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ id_{A^\bullet}^\bullet & 0 \end{pmatrix}\right) : nullHom_{\mathcal{A}}(cone(id_{A^\bullet}^\bullet), cone(id_{A^\bullet}^\bullet))$

Proposition 2.4.2. *For any complex $A^\bullet$ there is a natural isomorphism $nullHom_{\mathcal{A}}(A^\bullet, \_) \simeq Hom_{Ch(\mathcal{A})}(cone(id_{A^\bullet}^\bullet), \_)$. This establish that $cone(id_{A^\bullet}^\bullet)$ is the universal contractible complex where null-homotopic morphisms from $A^\bullet$ factors through.*

Proof. This proof will construct two natural maps which are inverses. This is sufficient to prove the universal property by Yonedas lemma.
Let $construct_{(A^\bullet, \_)(B^\bullet)} : nullHom_{\mathcal{A}}(A^\bullet, B^\bullet) \to Hom_{Ch(\mathcal{A})}(cone(id_{A^\bullet}^\bullet), B^\bullet)$ and $destruct_{(A^\bullet, \_)(B^\bullet)} : Hom_{Ch(\mathcal{A})}(cone(id_{A^\bullet}^\bullet), B^\bullet) \to nullHom_{\mathcal{A}}(A^\bullet, B^\bullet)$ be two morphisms defined the following way. $construct_{(A^\bullet, \_)(B^\bullet)}(f^\bullet, \varepsilon^\bullet) = \begin{pmatrix} f^\bullet & \varepsilon^\bullet \end{pmatrix}$ and

$destruct_{(A^\bullet,\_)(B^\bullet)}\begin{pmatrix} f^\bullet & \varepsilon^\bullet \end{pmatrix} = (f^\bullet, \varepsilon^\bullet)$. These natural transformations are constructed such that they are inverses of each other. It remains to see that these maps are well defined. This will be done by showing that there is a chain map from the cone of the identity, if and only if there is a null-homotopic map from the object.

$$\begin{tikzcd}
\dots \arrow[r, "d^{-2}_{cone(id^\bullet_{A^\bullet})}"] & A^{-1} \oplus A^0 \arrow[r, "d^{-1}_{cone(id^\bullet_{A^\bullet})}"] \arrow[d, "\begin{pmatrix} f^{-1} & \varepsilon^0 \end{pmatrix}"] & A^0 \oplus A^1 \arrow[r, "d^{0}_{cone(id^\bullet_{A^\bullet})}"] \arrow[d, "\begin{pmatrix} f^{0} & \varepsilon^1 \end{pmatrix}"] & \dots \\
\dots \arrow[r, "d^{-2}_{B^\bullet}"] & B^{-1} \arrow[r, "d^{-1}_{B^\bullet}"] & B^0 \arrow[r, "d^{0}_{B^\bullet}"] & \dots
\end{tikzcd}$$

For $\begin{pmatrix} f^\bullet & \varepsilon^\bullet[1] \end{pmatrix}$ to be a chain map, the following conditions must hold, i.e. that the square commute.

$$\begin{pmatrix} f^0 & \varepsilon^1 \end{pmatrix} \begin{pmatrix} d^{-1}_{A^\bullet} & id^0_{A^\bullet} \\ 0 & -d^0_{A^\bullet} \end{pmatrix} = d^{-1}_{B^\bullet} \begin{pmatrix} f^{-1} & \varepsilon^0 \end{pmatrix}$$

By calculating the matrix, it is a chain map if the following conditions are met.

$$f^0 d^{-1}_{A^\bullet} = d^{-1}_{A^\bullet} f^{-1}$$
$$f^0 = d^{-1}_{B^\bullet} \varepsilon^0 + \varepsilon^1 d^0_{A^\bullet}$$

Thus, a morphism is a chain map from the identity cone if and only if it is a null-homotopic chain map, which proves that there is a natural isomorphism as stated. □

Remark. The identity cone is universal with respect to homotopies. A null-homotopic chain map $f^\bullet : A^\bullet \to B^\bullet$ might admit several factorization through the identity cone. The factorizations are only unique when there is a homotopy witnessing the null-homotopy property.

$$\begin{tikzcd}
A^\bullet \arrow[rr, "(f^\bullet, \varepsilon^\bullet)"] \arrow[dr, swap, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & & B^\bullet \\
& cone(id^\bullet_{A^\bullet}) \arrow[ur, dashed, swap, "\begin{pmatrix} f^\bullet & \varepsilon^\bullet[1] \end{pmatrix}"] &
\end{tikzcd}$$

Corollary 2.4.2.1. *The contravariant functor $nullHom_{\mathcal{A}}(\_, B^\bullet)$ is represented by $cone(id^\bullet_{B^\bullet})[-1]$. Thus there is a factorization of null-homotopic maps which ends in $B^\bullet$ as follows.*

$$\begin{tikzcd}
A^\bullet \arrow[rr, "(f^\bullet, \varepsilon^\bullet)"] \arrow[dr, dashed, swap, "\begin{pmatrix} \varepsilon^\bullet \\ f^\bullet \end{pmatrix}"] & & B^\bullet \\
& cone(id^\bullet_{B^\bullet})[-1] \arrow[ur, swap, "\begin{pmatrix} 0 & -1^\bullet \end{pmatrix}"] &
\end{tikzcd}$$

Lemma 2.4.3. *$f^\bullet$ is null-homotopic if and only if $f^\bullet$ factors through a contractible object.*

Proof. Suppose that $f^\bullet$ is null-homotopic, then by the universal property of null-homotopy, it factors through the identity cone. Conversely, suppose that $f^\bullet : A^\bullet \to C^\bullet$ factors through a contractible object $B^\bullet$ as $g^\bullet h^\bullet$. Then $f^\bullet = g^\bullet h^\bullet = g^\bullet id^\bullet_{B^\bullet} h^\bullet$. $id^\bullet_{B^\bullet}$ is null-homotopic and homotopy equivalence is a congruence relation shows that $f^\bullet$ is null-homotopic. □

By the example in 3.1, any additive category $\mathcal{A}$ admits an exact category $\mathcal{A}, \mathcal{E}$, where $\mathcal{E} = \{\text{Split short-exact sequences}\}$. Then there is an exact category $(Ch(\mathcal{A}), Ch(\mathcal{E}))$, where $Ch(\mathcal{E}) = \{\text{level-wise split short-exact sequences}\}$. This exact structure has enough projectives and injectives which also coincide. Instead of using level-wise split short-exact sequences, there is a more specific description of this exact structure.

Proposition 2.4.4. *The exact structure $Ch(\mathcal{E})$ are diagrams on the form as below, where $r^\bullet : A^\bullet \to B^\bullet$ is a chain map.*

$$B^\bullet \overset{\begin{pmatrix}1\\0\end{pmatrix}}{\rightarrowtail} cone(r^\bullet) \overset{\begin{pmatrix}0 & -1^\bullet\end{pmatrix}}{\twoheadrightarrow} A^\bullet[1]$$

Proof. Suppose that there is a conflation $(i^\bullet : Q^\bullet \to R^\bullet, p^\bullet : R^\bullet \to P^\bullet)$ in $Ch(\mathcal{A})$. The goal is to realize the object $R^\bullet$ as a cone of some map. Since the conflation is level-wise split one get that in the following diagram $R^i \simeq Q^i \oplus P^i$.

$$\begin{tikzcd}
 & Q^1 \arrow[rr, "i^1"] & & R^1 \arrow[rr, "p^1"] & & P^1 \\
Q^0 \arrow[ur, "d^0_{Q^\bullet}"] \arrow[rr, "i^0"] & & R^0 \arrow[ur, "d^0_{R^\bullet}"] \arrow[rr, "p^0"] & & P^0 \arrow[ur, "d^0_{P^\bullet}"] &
\end{tikzcd}$$

Commutativity of the squares may be rewritten as.

$$d^0_{R^\bullet} i^0 = i^1 d^0_{Q^\bullet} \iff \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} d^0_{Q^\bullet} \\ 0 \end{pmatrix}$$

$$d^0_{P^\bullet} p^0 = p^1 d^0_{R^\bullet} \iff \begin{pmatrix}0 & d^0_{P^\bullet}\end{pmatrix} = \begin{pmatrix}0 & -1\end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Thus $a = d^0_{R^\bullet}$, $d = -d^0_{P^\bullet}$ and $c = 0$. The map $b : P^0 \to Q^1$ induces a map $b'^\bullet : P^\bullet[1] \to Q^\bullet$. This is a chain map by the following calculation.

$$\begin{pmatrix} d^1_{Q^\bullet} & b^1 \\ 0 & d^1_{P^\bullet} \end{pmatrix}\begin{pmatrix} d^0_{Q^\bullet} & b^0 \\ 0 & -d^0_{P^\bullet} \end{pmatrix} = \begin{pmatrix} 0 & d^1_{Q^\bullet} b^0 - b^1 d^0_{P^\bullet} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$b^\bullet$ is a chain map and thus $R^\bullet = cone(b^\bullet)$. □

To show that $(Ch(\mathcal{A}), Ch(\mathcal{E}))$ is a Frobenius category, one must show that every projective object is contractible. The case of every injective object is contractible will follow from duality, as there is a covariant and contravariant representation of null-homotopies.

Proposition 2.4.5. *An object $P^\bullet$ is projective if and only if it is contractible.*

Proof. Suppose that $P^\bullet$ is projective, then it can be found in a conflation over $cone(id^\bullet_{P^\bullet})[-1]$. By the contravariant universal property of null-homotopies, the identity map is null-homotopic as described by the diagram below.

$$\begin{tikzcd}
P^\bullet[-1] \arrow[r, tail] & cone(id^\bullet_{P^\bullet})[-1] \arrow[r, two heads] & P^\bullet \arrow[d, equal] \\
 & & P^\bullet \arrow[ul, dashed]
\end{tikzcd}$$

Conversely, suppose that $P^\bullet$ is contractible, then one may se that $P^\bullet$ is projective if and only if $cone(id^\bullet_{P^\bullet})$ is projective by the following diagram.

$$\begin{tikzcd}
P^\bullet[-1] \arrow[r, tail] & cone(id^\bullet_{P^\bullet})[-1] \arrow[r, two heads] & P^\bullet \arrow[d, equal] \\
 & cone(id^\bullet_{P^\bullet}) \arrow[u, dashed] \arrow[ur, dashed] & P^\bullet \arrow[l, tail]
\end{tikzcd}$$

It is enough to show that every identity cone is projective, to show that every contractible is projective. This is shown if the functor $Hom_{Ch(\mathcal{A})}(cone(id^\bullet_{P^\bullet}), \_) : Ch(\mathcal{A}) \to Ab$ is an exact functor, which is the same as saying that every conflation gets mapped to short-exact sequences. Suppose further that there is a morphism $p^\bullet : cone(\beta^\bullet) \to B^\bullet[1]$, where $\beta^\bullet : B^\bullet \to C^\bullet$. To show exactness, one must show that $Hom_{Ch(\mathcal{A})}(cone(id^\bullet_{P^\bullet}), p^\bullet)$ is a surjection.

First observe that there is an isomorphism $Hom_{Ch(\mathcal{A})}(cone(id^\bullet_{P^\bullet}), p^\bullet) \simeq nullHom_{\mathcal{A}}(P^\bullet, p^\bullet)$. Suppose that $(f^\bullet, \varepsilon^\bullet) : nullHom_{\mathcal{A}}(P^\bullet, B^\bullet)$. Then there is a null-homotopic chain map $(f'^\bullet, \varepsilon'^\bullet) = (\begin{pmatrix} -\beta^{\bullet-1}\varepsilon^\bullet \\ f^\bullet \end{pmatrix}, \begin{pmatrix} 0 \\ (-1)^{\bullet+1}\varepsilon^\bullet \end{pmatrix}) : nullHom_{\mathcal{A}}(P^\bullet, cone(\beta^\bullet)[-1])$ such that $p^\bullet_*(f'^\bullet, \varepsilon'^\bullet) = (f^\bullet, \varepsilon^\bullet)$. A diagram chase suffices to check that this is a chain map and the the proposed homotopy is in fact a homotopy. □

Corollary 2.4.5.1. *The class of contractible objects is precisely the class of projectives and the class of injectives, making $(Ch(\mathcal{A}), Ch(\mathcal{E}))$ a Frobenius category. The stable Frobenius category is equivalent with the homotopy category, i.e.* $\underline{Ch(\mathcal{A})} = K(\mathcal{A})$.

Corollary 2.4.5.2. *The homotopy category $K(\mathcal{A})$ is triangulated.*

Since the identity cones are injective, one may verify that the cosyzygy functor is the shift functor ($\mho\_ = \_[1]$). The standard triangles in $K(\mathcal{A})$ are therefore the candidate triangles on the form below.

$$A^\bullet \xrightarrow{f^\bullet} B^\bullet \longrightarrow cone(f^\bullet) \longrightarrow A^\bullet[1]$$

Chapter 3

Derived Categories

3.1 Idempotent Completeness and Krull-Schmidt Categories

This section will introduce the concepts of idempotent complete categories, weakly idempotent complete categories and Krull-Schmidt categories. These are extra conditions which may be put onto an exact category. The conditions to be introduced are three different levels of strengthening, where weakly idempotent completeness is the weakest and Krull-Schmidt is the strongest condition. The concepts introduced in this section is based of the ideas from [11], [12] and [10]

Definition 3.1.1. An idempotent complete category is an additive category where every idempotent split. That is, if there is an idempotent $p : A \to A$ ($p^2 = p$), and there is an isomorphism $A \simeq I \oplus K$ such that $p \simeq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Every idempotent in an idempotent complete category admits an analysis. That is the idempotent $p : A \to A$ has a kernel, cokernel, image and coimage. In fact, the kernel is isomorphic to the cokernel, and the image is canonically isomorphic to the coimage. As p is isomorphic to the matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ one may observe that the inclusion $\iota_1 : I \to A$ is the kernel of p, while the projection $\pi_1 : A \to I$ is the cokernel. Similarly the maps $\iota_2 : K \to A$ and $\pi_2 : A \to K$ are the kernel and cokernel of the map $1 - p$ respectively. Using the fact that p splits we are able to construct the following analysis.

$$
\begin{tikzcd}
& A \arrow[rr, "p"] \arrow[dr, two heads, "\pi_2"] & & A \arrow[dr, two heads, "\pi_1"] & \\
I \arrow[ur, tail, "\iota_1"] & & K \arrow[ur, tail, "\iota_2"] & & I
\end{tikzcd}
$$

Remark. Assuming that every idempotent in an additive category $\mathcal{A}$ has a kernel is sufficient for $\mathcal{A}$ to be idempotent complete. The limits and colimits as described above may be found with the idempotents p and $1 - p$.

Every additive category $\mathcal{A}$ has a fully faithful embedding $i_{\mathcal{A}} : \mathcal{A} \to \widehat{\mathcal{A}}$ into an idempotent complete category $\widehat{\mathcal{A}}$. This completion satisfies the universal property in which if there is a functor $F : \mathcal{A} \to \mathcal{B}$ which sends every idempotent p in $\mathcal{A}$ to a splitting idempotent, then the functor factors through the idempotent complete category $\widehat{\mathcal{A}}$.

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "F"] \arrow[dr, "i_{\mathcal{A}}"'] & & \mathcal{B} \\ & \widehat{\mathcal{A}} \arrow[ur, "\widehat{F}"'] & \end{tikzcd}$$

One may define this completion $\widehat{\mathcal{A}}$ to be the category with objects (A, p), where A is an object of $\mathcal{A}$ and $p : A \to A$ is an idempotent. A morphism $\widehat{f} : (A, p) \to (B, q)$ is defined as the morphism $\widehat{f} = q \circ f \circ p$ for some morphism $f : A \to B$. The injection functor is defined as $i_{\mathcal{A}}(A) = (A, id_A)$. More on this injection can be found in [11].
Many of the useful theorems needed to describe the triangulated subcategory needed for the construction of the derived category will arise from the weaker condition of weakly split idempotents.

Lemma 3.1.1. *The following are equivalent in an additive category:*

1. *Every split-epi has a kernel*
2. *Every split-mono has a cokernel*

Proof. It suffices to prove that (1.) $\implies$ (2.), as the other claim is dual. Suppose that $g : B \to A$ is split-epi with $f : A \to B$ as the corresponding split-mono such that $gf = id_A$. Since g is split-epi it has a kernel $h : C \to B$.

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \arrow[r, bend left, dashed, "i"] & C \arrow[l, bend left, "h"] \end{tikzcd}$$

Looking at the map $id_B - fg$, one may see that $g(id_B - fg) = g - gfg = g - g = 0$, thus $id_B - fg$ factors over the kernel of h as indicated by the dashed arrow.
h is split-mono as $hih = (id_B - fg)h = h - fgh = h$. As h is mono from being a kernel, it follows that $ih = id_C$. B is the biproduct $B \simeq A \oplus C$ as $id_B - fg = hi \iff id_B = fg + hi$. This in turn implies that i is the cokernel of f. $\square$

This lemma is at the core of weakly idempotent complete categories.

Definition 3.1.2. An additive category $\mathcal{A}$ is weakly idempotent complete if it satisfies either of the conditions of Lemma 4.1.

Corollary 3.1.1.1. *Let $(\mathcal{A}, \mathcal{E})$ be an exact category, then the following are equivalent:*

1. *The category $\mathcal{A}$ is weakly idempotent complete*
2. *Every split-mono is an inflation*
3. *Every split-epi is a deflation*

With the notion of a weakly idempotent complete category, the Obscure axiom can be strengthened into Hellers cancellation axiom.

Proposition 3.1.2. ***Hellers cancellation axiom*** *For an exact category $(\mathcal{A}, \mathcal{E})$ the following are equivalent:*

1. *$\mathcal{A}$ is weakly idempotent complete*
2. *Let $f : A \to B$ and $g : B \to C$ be two morphisms in $\mathcal{A}$. Then if $gf : A \to C$ is a deflation, then g is.*

Proof. Suppose (1.). Let $f : A \to B$ and $g : B \to C$ be morphisms such that their composition $gf : A \to C$ is a deflation. Since gf is a deflation, the pullback square exists.

$$\begin{tikzcd}
A \arrow[drr, bend left, "f"] \arrow[dr, dashed, "f'"] \arrow[ddr, bend right, "id_A"'] & & \\
& B' \arrow[r, two heads, "f'"] \arrow[d, "h"] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d, "g"] \\
& A \arrow[r, two heads, "gf"] & C
\end{tikzcd}$$

By using the universal property,one may see that g' is split-mono, hence it admits an inflation $h' : A' \to B'$. The claim is that $hh' : A' \to B$ is the kernel of g. If the claim is true, the Obscure axiom yields that g is a deflation.
To show that hh' is the kernel one must show the universal property. Let $t : T \to B$ be a test object, such that $gt = 0$.

$$\begin{tikzcd}
T \arrow[drr, bend left, "t"] \arrow[dr, dashed, "t'"] & & \\
& A' \arrow[r, "hh'"] \arrow[dr, "0"] & B \arrow[d, "g"] \\
& & C
\end{tikzcd}$$

It is known that t' exists as t factors through B' with t'', by the pull-back property. As $g't'' = 0$, t'' factors through A' using the fact that h' is the kernel of g', this proves the claim.
For the other direction, suppose (2.) instead and let $gf = id_A$, gf is a deflation and g is split-epi. By the assumption, g is a deflation, so it has a kernel. $\square$

Lastly, suppose that $\mathcal{A}$ is an idempotent complete category and that there are some idempotents over an object A. These idempotents admits a description of A as a direct sum of kernels and cokernels. There is, however, no guarantee that these decompositions are unique. To fix this, define the following category.

Definition 3.1.3. Let $\mathcal{A}$ be an additive category. An object A is called indecomposable if the endomorphism ring of A is local.
An object is called decomposable if it is not indecomposable.

Definition 3.1.4. An additive category $\mathcal{A}$ is called Krull-Schmidt if any object A decomposes into a finite direct sum of indecomposable objects.

Having that each indecomposable object is local is enough for the following proposition to hold.

Proposition 3.1.3. *Every decomposition in a Krull-Schmidt category is unique up to isomorphism*

As being Krull-Schmidt admits decomposition whenever an endomorphism ring is not local implies a connection to idempotent completeness. That is whenever there is an idempotent over an object, this idempotent give rise to two comaximal ideals for the endomorphism ring. This gives us the decomposition which is required for the idempotent to split. Moreover, there is a deeper connection with being Krull-Schmidt and idempotent complete.

Definition 3.1.5. Let R be a ring. We say that R is semiperfect if R as a module over itself admits a decomposition ${}_RR \simeq P_1 \oplus P_2 \oplus \ldots \oplus P_n$ such that each P_i has a local endomorphism ring.

Remark. For a ring R the following conditions are equivalent:

- The category mod_R is a Krull-Schmidt category
- R is semiperfect
- Every simple R-module has a projective cover
- Every finitely generated R-module has a projective cover

Thus any of these conditions can be taken to be the definition of semiperfect.

With this definition we are able to state the following proposition, which says whenever an idempotent complete category is Krull-Schmidt.

Proposition 3.1.4. *Let $\mathcal{A}$ be an additive category, then the following are equivalent:*

1. *$\mathcal{A}$ is Krull-Schmidt*
2. *$\mathcal{A}$ is idempotent complete and every endomorphism ring are semiperfect.*

Example. Let Λ be any artin R-algebra, then mod_Λ is a Krull-Schmidt category. As an example, the category of finitely generated real vector spaces is Krull-Schmidt. Every vector space is a finite direct summand of the only indecomposable vector space $\mathbb{R}$.

More details and examples of Krull-Schmidt categories may be found in Henning Krause notes ([12]).

3.2 Normal Morphisms and Long Exact Sequences

Definition 3.2.1. Let $(\mathcal{A}, \mathcal{E})$ be an exact category. A morphism $f : A \to B$ is called normal if it has a deflation-inflation factorization. They will be drawn as in the following diagram.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, two heads] & & B \\ & I \arrow[ur, tail] & \end{tikzcd}$$

Remark. A monomorphism is normal if and only if it is an inflation. Dually, an epimorphism is normal if and only if it is a deflation.

Remark. In general the composition gf of two normal morphisms f and g are not normal. However, if g is a deflation, the composition can be seen to normal, as deflations are closed under composition. One may also observe that an exact category is abelian if and only if normal morphisms are closed under composition.

Lemma 3.2.1. ***Hellers factorization lemma***. *The factorization of normal morphisms are unique up to unique isomorphisms.*

Proof. Suppose that a normal morphism admits two different factorization. That means there exists a commutative diagram as follows.

$$\begin{tikzcd} A \arrow[r, two heads, "p"] \arrow[d, two heads, "q"] & I \arrow[d, tail, "i"] \arrow[dl, dashed, bend left, "\phi"] \\ I' \arrow[r, tail, "j"] \arrow[ur, dashed, "\phi'"] & B \end{tikzcd}$$

By assumption $ip = jq$, thus $jq \circ Ker(p) = 0$. $q \circ Ker(p) = 0$ as j is mono, thus there exists a morphism $\phi : I \to I'$ uniquely such that $q = \phi p$. Now $ip = jq = j\phi p$, and as p is epi it follows that $i = j\phi$. Reiterating the argument, but with $Ker(q)$ instead, there exists a ϕ' uniquely such that $p = \phi' q$ and $j = i\phi'$. Thus $i = j\phi = i\phi'\phi$, and since i is mono it follows that $id_I = \phi'\phi$; dually $Id_{I'} = \phi\phi'$. $\square$

Remark. Du to Hellers factorization axiom one may see that normal morphisms admits analysis.

$$\begin{tikzcd} & A \arrow[rr, "f"] \arrow[dr, two heads, "p"] & & B \arrow[dr, two heads, "Cok(i)"] & \\ K \arrow[ur, tail, "Ker(p)"] & & I \arrow[ur, tail, "i"] & & C \end{tikzcd}$$

Observe that the object I coincide with the image and coimage of f. This object will then be referred to the image of f. As a consequence of this unique factorization, a normal morphism is iso if and only if it is mono and epi.

Definition 3.2.2. A sequence of normal morphisms is exact if the inflation of the factorization together with the consecutive deflation forms a conflation. That is there are conflations between morphisms as in the following diagram. The conflation pairs are highlighted with different colors.

$$\begin{tikzcd}
\cdots \arrow[rr, "f_{-2}"] \arrow[dr, "p_{-2}"'] & & A_{-1} \arrow[rr, "f_{-1}"] \arrow[dr, "p_{-1}"'] & & A_0 \arrow[rr, "f_0"] \arrow[dr, "p_0"'] & & A_1 \arrow[rr, "f_1"] \arrow[dr, "p_1"'] & & \cdots \\
& I_{-2} \arrow[ur, "i_{-2}"'] & & I_{-1} \arrow[ur, "i_{-1}"'] & & I_0 \arrow[ur, "i_0"'] & & I_1 \arrow[ur, "i_1"'] &
\end{tikzcd}$$

A morphism of exact sequences is the same as a morphism of sequences. That is a collection of morphisms $(..., \phi_{-1}, \phi_0, \phi_1, ...)$ such that the squares in the diagram commute.

$$\begin{tikzcd}
\cdots \arrow[r, "a_{-2}"] & A_{-1} \arrow[r, "a_{-1}"] \arrow[d, "\phi_{-1}"] & A_0 \arrow[r, "a_0"] \arrow[d, "\phi_0"] & A_1 \arrow[r, "a_1"] \arrow[d, "\phi_1"] & \cdots \\
\cdots \arrow[r, "b_{-2}"] & B_{-1} \arrow[r, "b_{-1}"] & B_0 \arrow[r, "b_0"] & B_1 \arrow[r, "b_1"] & \cdots
\end{tikzcd}$$

Remark. An exact sequence of normal morphisms is called short exact if it consists of morphisms on the form $(, 0, i, p, 0,)$, i.e. as in the following diagram.

$$\begin{tikzcd}
0 \arrow[rr, "0"] \arrow[dr, equal] & & A \arrow[rr, "i"] \arrow[dr, equal] & & B \arrow[rr, "p"] \arrow[dr, "p"'] & & C \arrow[rr, "0"] \arrow[dr, "0"'] & & 0 \\
& 0 \arrow[ur, "0"] & & A \arrow[ur, "i"'] & & C \arrow[ur, equal] & & 0 \arrow[ur, equal] &
\end{tikzcd}$$

Observe how conflations are exactly the class of short exact sequences.

This definition admits properties which mimics properties from homological algebra.

Lemma 3.2.2. ***5 Lemma****. Given two 5 term exact sequences and a morphism between them as in the diagram. Then ϕ is an isomorphism as well.*

$$\begin{tikzcd}
A_0 \arrow[r, "a_0"] \arrow[d, "\simeq"] & A_1 \arrow[r, "a_1"] \arrow[d, "\simeq"] & A_2 \arrow[r, "a_2"] \arrow[d, "\phi"] & A_3 \arrow[r, "a_3"] \arrow[d, "\simeq"] & A_4 \arrow[d, "\simeq"] \\
B_0 \arrow[r, "b_0"] & B_1 \arrow[r, "b_1"] & B_2 \arrow[r, "b_2"] & B_3 \arrow[r, "b_3"] & B_4
\end{tikzcd}$$

Lemma 3.2.3. ***Kernel-Cokernel sequence****. Let $(\mathcal{A}, \mathcal{E})$ be an exact category which is weakly idempotent complete. Suppose that there are composable normal morphism f and g such that gf is normal as well. Then there exists an exact sequence.*

$$Ker(f) \longrightarrow Ker(gf) \longrightarrow Ker(h) \longrightarrow Cok(f) \longrightarrow Cok(gf) \longrightarrow Cok(g)$$

Remark. If $(\mathcal{A}, \mathcal{E})$ is an exact category, then one may show that the category $\mathcal{A}$ admits Kernel-Cokernel sequences if and only if it is weakly idempotent complete.

The Kernel-Cokernel sequence enables one to prove that the snake lemma holds in weakly idempotent complete categories.

Corollary 3.2.3.1. ***Snake Lemma.*** *Let $(\mathcal{A}, \mathcal{E})$ be a weakly idempotent complete category. Suppose there is a diagram in $\mathcal{A}$ having exact rows.*

$$\begin{tikzcd}
& A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' &
\end{tikzcd}$$

Then there is an exact sequence.

$$Ker(f) \longrightarrow Ker(g) \longrightarrow Ker(h) \overset{\delta}{\dashrightarrow} Cok(f) \longrightarrow Cok(g) \longrightarrow Cok(h)$$

3.3 Homology and Derived Categories

This section aims to provide a construction for derived categories and discuss the underlying assumptions. In the case where $\mathcal{A}$ is abelian, the derived category is constructed by localizing at quasi-isomorphisms. However, in the context of exact categories, how much of the theory transfers? For abelian groups, homology is defined to be the quotient of the kernel of a map by the image of the preceding map. For exact categories the discussion is a bit more complex. Consider the usual construction of homology, when does the homology exists?.

$$\begin{tikzcd}
\cdots \arrow[r, "d^{-2}_{A^\bullet}"] & A_{-1} \arrow[r, "d^{-1}_{A^\bullet}"] \arrow[d, two heads, "p"] & A_0 \arrow[r, "d^{0}_{A^\bullet}"] & A_1 \arrow[r, "d^{1}_{A^\bullet}"] & \cdots \\
& Im(d^{-1}_{A^\bullet}) \arrow[r, dashed, "h"] \arrow[ur, tail, "\iota"] & Ker(d^0_{A^\bullet}) \arrow[r, dotted, "?"] \arrow[u, tail, "\kappa"] & H^0(A^\bullet) &
\end{tikzcd}$$

The complex must admit an analysis at each differential, so assume that the complex only contains normal morphisms. By looking at the 0-th homology one can find a condition for when the homology exists. Using the fact that $d^0_{A^\bullet}\iota = 0$, there is an unique morphism h, such that $\iota = \kappa h$ by the universal property. The 0-th homology exists whenever the morphism h has a cokernel, and then h satisfies the assumption of the Obscure axiom, making h an inflation. One way to not break this condition is to assume that $\mathcal{A}$ is weakly idempotent complete. By Hellers cancellation axiom it is known that h is an inflation, which then proves the existence of the cokernel. However, this only allows the construction of quasi-isomorphisms at weakly idempotent complete categories.

Recall that a quasi-isomorphism is a chain map $f^\bullet : A^\bullet \to B^\bullet$ such that $H^*(f^\bullet) : H^*(A^\bullet) \to H^*(B^\bullet)$ is an isomorphism in homology. In the abelian case, suppose that $f^\bullet : A^\bullet \to B^\bullet$ is a quasi-isomorphism. Consider the standard triangle
$A^\bullet \xrightarrow{f^\bullet} B^\bullet \longrightarrow cone(f^\bullet) \longrightarrow A^\bullet[1]$ in the homotopy category $K(\mathcal{A})$, then $f^\bullet$ becomes an isomorphism in the derived category $D(\mathcal{A})$. This shows that there is a quasi-isomorphism between $0^\bullet$ and $cone(f^\bullet)$ by corollary 1.1.4.1. It follows that $cone(f^\bullet)$ is an exact sequence, and this motivate to the following definition.

Definition 3.3.1. Let $(\mathcal{A}, \mathcal{E})$ be an exact category. Define the category $Ac(\mathcal{A}) \subset K(\mathcal{A})$ to be the full category whose objects are exact sequences.

The exact complexes are also referred to as acyclic complexes. Note that this subcategory is not in general either thick or closed under isomorphisms. To be able to show that it is a triangulated subcategory, it suffices to show that the mapping cone of two acyclic complexes is again acyclic.

Lemma 3.3.1. *Let $f^\bullet : A^\bullet \to B^\bullet$ be a chain map between acyclic chain complexes, then $cone(f^\bullet)$ is acyclic as well.*

Proof. Suppose that $A^\bullet$ and $B^\bullet$ are acyclic chain complexes and that $f^\bullet : A^\bullet \to B^\bullet$ is a chain map. The chain map factorizes through the images/kernels, as in the diagram below.

$$\begin{tikzcd}
A^{-1} \arrow[rr, "d^{-1}_{A^\bullet}"] \arrow[dddd, "f^{-1}"'] \arrow[dr, two heads] & & A^0 \arrow[rr, "d^0_{A^\bullet}"] \arrow[dddd, "f^0"'] \arrow[dr, two heads] & & A^1 \arrow[dddd, "f^1"'] \\
& Im(d^{-1}_{A^\bullet}) \arrow[ur, tail] \arrow[dd, dashed, "f^{-1\prime}"] & & Im(d^0_{A^\bullet}) \arrow[ur, tail] \arrow[dd, dashed, "f^{0\prime}"] & \\
& & & & \\
& Im(d^{-1}_{B^\bullet}) \arrow[dr, tail] & & Im(d^0_{B^\bullet}) \arrow[dr, tail] & \\
B^{-1} \arrow[rr, "d^{-1}_{B^\bullet}"] \arrow[ur, two heads] & & B^0 \arrow[rr, "d^0_{B^\bullet}"] \arrow[ur, two heads] & & B^1
\end{tikzcd}$$

This shows that the chain map may be promoted to a morphism of conflations $(f^{-1\prime}, f^0, f^{0\prime})$. By lemma 2.1.6 this morphism factors through another conflation in the following manner.

$$\begin{tikzcd}
Im(d^{-1}_{A^\bullet}) \arrow[r, tail] \arrow[d] & A^0 \arrow[r, two heads] \arrow[d] & Im(d^0_{A^\bullet}) \arrow[d, equal] \\
Im(d^{-1}_{B^\bullet}) \arrow[r, tail] \arrow[d, equal] & C^0 \arrow[r, two heads] \arrow[d] & Im(d^0_{A^\bullet}) \arrow[d] \\
Im(d^{-1}_{B^\bullet}) \arrow[r, tail] & B^0 \arrow[r, two heads] & Im(d^0_{B^\bullet})
\end{tikzcd}$$

These factorizations exists at every index over the chain complexes. In this way one may find the diagram below. Note that two bicartesian squares may be connected to make a bicartesian rectangle.

$$
\begin{tikzcd}
A^{-1} \arrow[rr, "d^{-1}_{A^\bullet}"] \arrow[dr, two heads] \arrow[dd] & & A^0 \arrow[rr, "d^{0}_{A^\bullet}"] \arrow[dr, two heads] \arrow[dd] & & A^1 \arrow[dd] \\
& Im(d^{-1}_{A^\bullet}) \arrow[ur, tail] \arrow[dd, dashed, "f^{-1\prime}"] & & Im(d^{0}_{A^\bullet}) \arrow[ur, tail] \arrow[dd, dashed, "f^{0\prime}"] & \\
C^{-1} \arrow[ur, two heads] \arrow[dd] & & C^0 \arrow[ur, two heads] \arrow[dd] & & C^1 \arrow[dd] \\
& Im(d^{-1}_{B^\bullet}) \arrow[ur, tail] \arrow[dr, tail] & & Im(d^{0}_{B^\bullet}) \arrow[ur, tail] \arrow[dr, tail] & \\
B^{-1} \arrow[rr, "d^{-1}_{B^\bullet}"] \arrow[ur, two heads] & & B^0 \arrow[rr, "d^{0}_{B^\bullet}"] \arrow[ur, two heads] & & B^1
\end{tikzcd}
$$

By proposition 2.1.3 there are conflations, along the cone of $f^\bullet$.

$$
\begin{tikzcd}
C^0 \arrow[r, two heads] \arrow[d] & Im(d^0_{A^\bullet}) \arrow[r, tail] \arrow[d, dotted, "f^{0\prime}"] & A^1 \arrow[d] \\
B^0 \arrow[r, two heads] & Im(d^0_{B^\bullet}) \arrow[r, tail] & C^1
\end{tikzcd}
\implies
\begin{tikzcd}
C^0 \arrow[r, tail] & A^1 \oplus B^0 \arrow[r, two heads] & C^1
\end{tikzcd}
$$

By Hellers cancellation axiom, combining these sequences creates a long exact sequence. This shows that $cone(f^\bullet)$ is acyclic.

$$
\begin{tikzcd}
\dots \arrow[r, "d^{-2}_{cone(f^\bullet)}"] & A^0 \oplus B^{-1} \arrow[rr, "d^{-1}_{cone(f^\bullet)}"] \arrow[dr, two heads] & & A^1 \oplus B^0 \arrow[rr, "d^{0}_{cone(f^\bullet)}"] \arrow[dr, two heads] & & A^2 \oplus B^1 \arrow[r, "d^{1}_{cone(f^\bullet)}"] & \dots \\
& & C^{-1} \arrow[ur, tail] & & C^0 \arrow[ur, tail] & &
\end{tikzcd}
$$

□

Corollary 3.3.1.1. *$Ac(\mathcal{A})$ is a triangulated subcategory of $K(\mathcal{A})$.*

Since $Ac(\mathcal{A})$ is triangulated, it makes sense to talk about the class of morphisms $Mor_{Ac(\mathcal{A})}$. By definition, a morphism $f^\bullet$ is in $Mor_{Ac(\mathcal{A})}$ if and only if $cone(f^\bullet)$ is in $Ac(\mathcal{A})$. Therefore it makes sense to say that the class of morphisms $Mor_{Ac(\mathcal{A})}$ may be regarded as quasi-isomorphisms.

Corollary 3.3.1.2. *The derived category is defined by the Verdier quotient $D(\mathcal{A}) = K(\mathcal{A})/Ac(\mathcal{A})$ whenever it exists, and it is triangulated.*

Remark. The derived category exists whenever $Mor_{Ac(\mathcal{A})}$ is a locally small multiplicative system.

As stated, it is not true a priori that $Ac(\mathcal{A})$ is either thick or closed under isomorphisms. When this is not true, it might happen that $C^\bullet \simeq A^\bullet \oplus B^\bullet$ where $C^\bullet$ is acyclic, but neither $A^\bullet$ or $B^\bullet$ need not be acyclic. However, the kernel of localization $\widehat{Ac(\mathcal{A})}$ contain all of these objects. In this way $A^\bullet$ and $B^\bullet$ will be related in the derived category, even though they are not quasi-isomorphic. The following lemma and corollary says whenever this is not a problem.

Lemma 3.3.2. *The following are equivalent:*

1. *Every null-homotopic chain complex is acyclic*
2. *The category $\mathcal{A}$ is idempotent complete*
3. *The subcategory $Ac(\mathcal{A})$ is closed under isomorphisms*

Corollary 3.3.2.1. *The subcategory $Ac(\mathcal{A})$ is thick if and only if $\mathcal{A}$ is idempotent complete.*

In order to weaken the conditions above, one can set boundedness conditions on the chain complexes. A chain complex is called left bounded if there is some $m : \mathbb{N}$ such that for any $n : \mathbb{N}$ and $n \leq m < 0$ it is true that $A^n = 0$. Likewise, right bounded complexes are the defined for $n \geq m > 0$ such that $A^n = 0$. A chain complex is called bounded if it is both left bounded and right bounded.

Definition 3.3.2. The category $K(\mathcal{A})^+$,$K(\mathcal{A})^-$ and $K(\mathcal{A})^\flat$ are the homotopy categories of left bounded, right bounded and bounded respectively. $Ac(\mathcal{A})^* \subset K(\mathcal{A})^*$ for $* : \{+,-,\flat\}$ will be the subcategory of acyclic chain complexes satisfying the correct boundedness condition.
Similarly one defines the (left/right) bounded derived category as $D^*(\mathcal{A}) = K^*(\mathcal{A})/Ac(\mathcal{A})^*$.

Lemma 3.3.3. *The following are equivalent:*

1. *The subcategories $Ac(\mathcal{A})^* \subset K(\mathcal{A})^*$ for $* : \{+,-\}$ are thick*
2. *The subcategory $Ac(\mathcal{A})^\flat$ is thick*
3. *The category $\mathcal{A}$ is weakly idempotent complete*

3.4 The Way Forward

At the end of this thesis, some topics involving derived categories will be looked. This is a sneak peak into the theory of derived functors and Auslander-Reiten triangles. This section is based on [6] and [13].

3.4.1 Derived Functors

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are abelian categories, and that there is an additive functor $F : \mathcal{A} \to \mathcal{B}$ between the categories. This functor can be lifted to an additive functor between the chain complex categories and a triangulated functor between the homotopy categories by applying the functor level-wise. Unfortunately, when localizing down to the derived categories there are no simple lift. As there is no way of assuring that a quasi-isomorphism gets mapped to another quasi-isomorphism, the lift will not be well-defined. This is however true whenever $F(Ac(\mathcal{A})) \subseteq Ac(\mathcal{B})$, which is the same as F being an exact functor. As known in the classical sense of derived functors, is that they admit a left and right construction using projective and injective resolutions respectively.

$$\begin{tikzcd} K(\mathcal{A}) \arrow[r, "F"] \arrow[d, "q_{\mathcal{A}}"] & K(\mathcal{B}) \arrow[d, "q_{\mathcal{B}}"] \\ D(\mathcal{A}) \arrow[r, dotted, "?F"] & D(\mathcal{B}) \end{tikzcd}$$

For a more general discussion, suppose that $\mathcal{T}$ and $\mathcal{S}$ are triangulated categories and that there is a triangulated subcategories $\mathcal{M} \subseteq \mathcal{T}$ such that the Verdier quotient exists. When does a triangulated functor $F : \mathcal{T} \to \mathcal{S}$ induce a functor $?F : \mathcal{T}/\mathcal{M} \to \mathcal{S}$? Similar to above, F admits a lift if every morphism in $Mor_{\mathcal{M}}$ gets inverted. This functor is induced by the universal property of $\mathcal{T}/\mathcal{M}$

$$\begin{tikzcd} \mathcal{T} \arrow[rr, "F"] \arrow[dr, "q"] & & \mathcal{S} \\ & \mathcal{T}/\mathcal{M} \arrow[ur, dotted, "?F"] & \end{tikzcd}$$

In order to answer the question about derived functors, this thesis will follow Delignes approach on how to construct right derived functors. Suppose there is an object $A : \mathcal{T}/\mathcal{M}$ and a functor $F : \mathcal{T} \to \mathcal{S}$, then let $\mathbf{r}F(\_,A) : \mathcal{S}^{op} \to Ab$ be a contravariant functor sending objects $B : \mathcal{S}$ to groups of semi left fractions $(f : \mathcal{S}(B, FC) | g : \mathcal{T}(A, C))$ such that $g : Mor_{\mathcal{M}}$. The caveat is that the functor sends objects of $\mathcal{T}/\mathcal{M}$ and $\mathcal{S}$ to arrows in $\mathcal{T}$ and $\mathcal{S}$. This is well-defined since $q(A) = A$ by definition and no choice of arrows need to be specified by the functor. Thus the localization is hidden from the functor at the element level. To illustrate how these elements look like the semi left fraction is to the left, and the equivalence relation on this set is given by the diagram on the right. In the same manner as before, blue arrows denote arrows located in Mor_S.

$$\begin{tikzcd} B \arrow[r, "f"] & FC \\ C & A \arrow[l, "g"'] \end{tikzcd} \qquad \begin{tikzcd} & FC' \arrow[d, "Fu"] \\ B \arrow[ur, "f'"] \arrow[r, "f'''"] \arrow[dr, "f''"'] & FC''' \\ & FC'' \arrow[u, "Fv"'] \end{tikzcd} \qquad \begin{tikzcd} C' \arrow[d, "u"] & \\ C''' & A \arrow[ul, "g'"'] \arrow[l, "g'''"] \arrow[dl, "g''"] \\ C'' \arrow[u, "v"] & \end{tikzcd}$$

The aim is to construct a functor $\mathbb{R}F : \mathcal{U} \to \mathcal{S}$ on a subcategory $\mathcal{U}$ of $\mathcal{T}/\mathcal{M}$. The functor $\mathbb{R}F$ is said to be defined at A if the functor $\mathbf{r}F(\_,A)$ is represented, i.e. $\mathcal{S}(\_, \mathbb{R}FA) \simeq \mathbf{r}F(\_,A)$. This isomorphism also defines a canonical morphism $can_r : F \implies \mathbb{R}F \circ q$ given by the identity semi left fraction on every object $A : \mathcal{U}$.

$$FA = FA \qquad A = A \iff FA \xrightarrow{can} \mathbb{R}FA$$

In order to show that $\mathbb{R}F$ is a functor, one must show that $\mathbf{r}F(\_,A)$ is functorial at the index A. That is, suppose there is a left fraction $t^{-1}s : A \to A'$ in $\mathcal{T}/\mathcal{M}$, then for $(f|g)$ as described above one may define $\mathbf{r}F(t^{-1}s)(f|g) = (Fs' \circ f | g't)$. g' and s' are defined with the right Ore condition on $Mor_{\mathcal{M}}$, this may be seen with the diagram below.

$$\begin{tikzcd}
FC'' & & & C'' & & \\
FC' \arrow[u, "Fs'"] & & C' \arrow[ur, "s'"] & & A'' \arrow[ul, "g'"'] & \\
B \arrow[u, "f"] & & & A \arrow[ul, "g"] \arrow[ur, "s"'] & & A' \arrow[ul, "t"]
\end{tikzcd}$$

Functoriality of $\mathbf{r}F(\_,\_)$ shows that there is a functor $\mathbb{R}F$, where the action on maps are defined with the representative isomorphism.

$$\begin{tikzcd}
\mathcal{S}(_,\mathbb{R}FA) \arrow[r, "\simeq"] \arrow[d, "\mathbb{R}F(t^{-1}s)"] & \mathbf{r}F(_,A) \arrow[d, "\mathbf{r}F(t^{-1}s)"] \\
\mathcal{S}(_,\mathbb{R}FA') \arrow[r, "\simeq"] & \mathbf{r}F(_,A')
\end{tikzcd}$$

If $\mathcal{U}$ is the full subcategory of objects which are defined for $\mathbb{R}F$, then there is a triangulated functor $\mathbb{R}F : \mathcal{U} \to \mathcal{S}$. $\mathcal{U}$ is in fact triangulated as F is a triangulated functor, so it commutes with the autoequivalence $\Sigma_{\mathcal{T}}$. Moreover, if $I : q^{-1}\mathcal{U} \to \mathcal{T}$ is the inclusion from the preimage of $\mathcal{U}$ into $\mathcal{T}$, then the canonical morphism induces a morphism of triangulated functors $can : FI \implies \mathbb{R}FI$.

Dually, one may define left derived functors using semi right fractions. As the set $Mor_{\mathcal{M}}$ is multiplicative, this doesn't make any problems. If the functor F is in fact exact in the sense that F sends morphisms in $Mor_{\mathcal{M}}$ to isomorphisms, then $\mathbb{R}F \simeq \mathbb{L}F$. If one would define derived functors between abelian categories $\mathcal{A}$ and $\mathcal{B}$, then the classical versions may be retrieved by taking homology, i.e. $\mathbb{R}^nF = H^n(\mathbb{R}F)$ and $\mathbb{L}_nF = H^{-n}(\mathbb{L}F)$.

Suppose that the categories $\mathcal{A}$ and $\mathcal{B}$ have exact structures and that there is an additive functor $F : \mathcal{A} \to \mathcal{B}$, then there is a connection to the classical derived functors. Let $\mathcal{A}^I$ be the full subcategory of injective objects, since every conflation in $\mathcal{A}^I$ splits, $F|_{\mathcal{A}^I} : \mathcal{A}^I \to \mathcal{B}$ is an exact functor. Thus F induced on $K(\mathcal{A})$ preserves quasi-isomorphisms of left bounded injective complexes. In this manner there is a functor $\mathbb{R}F : D^+(\mathcal{A}^I) \to D^+(\mathcal{B})$, and moreover this functor extends to the subcategory $\mathcal{U} \subseteq D^+(\mathcal{A})$ where every object in $\mathcal{U}$ admits an injective resolution.

3.4.2 Auslander-Reiten Triangles

For the rest of this thesis it will be assumed that $\mathcal{A}$ is an additive $\mathbb{K}$-linear category, that is it is enriched over $Mod\mathbb{K}$, and it is also Krull-Schmidt. With these assumptions, one may be able to define a special set of triangles whenever $\mathcal{A}$ has a triangulation. As $\mathcal{A}$ is Krull-Schmidt, the notion of indecomposable objects exists, which motivates the definition of triangles which acts as indecomposable triangles.

Definition 3.4.1 (Auslander-Reiten triangles)**.** A triangle (X,Y,Z,u,v,w) is called an Auslander-Reiten triangle if it satisfies the following conditions:

1. (AR1) X and Z are indecomposable
2. (AR2) $w \neq 0$
3. (AR3) If a morphism $f : X \to W$ is not split mono, then there is a morphism $f' : Y \to W$, such that $f = f'u$

Auslander-Reiten triangles will be abbreviated to AR-triangles

A category $\mathcal{A}$ has enough AR-triangles if for every indecomposable object, it is a part of a triangle satisfying the conditions above. One may show that the axioms for AR-triangles are self dual, that is (X, Y, Z, u, v, w) is an AR-triangle if and only if $(X^{op}, Y^{op}, Z^{op}, u^{op}, v^{op}, w^{op})$ is a triangle. Note that (AR2) forces the morphisms u and v to not be split. In order to see that these triangles represent irreducibility, the following definition is needed.

Definition 3.4.2 (Irreducible morphisms)**.** A morphism $f : X \to Y$ is called irreducible if f is neither split mono nor split epi, and for any factorization $f = f_2 f_1$ either f_1 is split mono or f_2 is split epi.

Proposition 3.4.1. *Let* (X, Y, Z, u, v, w) *be an AR-triangle, then the following hold:*

1. *Z is unique up to isomorphism of triangles*
2. *u and v are irreducible morphisms*
3. *If $f : X \to X_1$ is irreducible, then there is a split epi $f' : Y \to X_1$ such that $f = f'u$.*

The motivation for studying AR-triangles comes from their similarity with Auslander-Reiten sequences. An Auslander-Reiten sequence is a short exact sequence over a finite-dimensional $\mathbb{K}$-algebra, which is not split, and satisfying (AR1) and (AR3). Thus one may observe that the definition of AR-triangles are exactly the stable Auslander-Reiten sequences. If A is a finite-dimensional $\mathbb{K}$-algebra, then one have the following results.

Theorem 3.4.2. *If A is a finite-dimensional $\mathbb{K}$-algebra of finite global dimension, then the derived category $D^\flat(A)$ has enough AR-triangles.*

Proposition 3.4.3. *The following are equivalent:*

1. *(X, Y, Z, u, v, w) is an AR-triangle in $D^\flat(A)$*
2. *$inj.dim(X) \leq 1$ and $proj.dimZ \leq 1$*
3. *$Hom_A(I, X) = 0$ for any injective I and $Hom_A(Z, P) = 0$ for any projective P*

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