Thesis On Triangulated Categories, Working title

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Abstract

Abstract goes here.

Sammendrag

Sammendraget skal inn her.

Introduction

This is an introduction, welcome!
Introduce notation which will be used in text. A list of notation and description would be nice, so that the reader might scroll back up if something is unclear.

Contents

Todo list

Chapter 1

Triangulated Categories

In this chapter the notion of a triangulated will be introduced. In litterature one are able to find two different types of triangulated categories, they are called algebraic and topological. How these categories differ is however quite subtle. Different properties of triangulated categories will be discussed. Localization categories will be explained as well as how triangulated subcategories gives rise to such localizations. Lastly an embedding of a triangulated category in an abelian will be shown to exist.

1.1 Definition and First Properties

Skriv om hva som skal være med i dette delkapittelet

In this section $\mathcal{T}$ denotes an additive category and $T : \mathcal{T} \to \mathcal{T}$ is an additive autoequivalence of $\mathcal{T}$, which is often called translation or suspension functor.

Definition 1.1.1. A candidate triangle is a collection (A, B, C, a, b, c) of objects $A, B, C \in T$ and morphisms $a : A \to B$, $b : B \to C$, $c : C \to TA$. These candidate triangles can be drawn as diagrams in the following way:

$$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA$$

A morphism between candidate triangles is a triple of morphism (α, β, γ), where $\alpha : A \to A'$, $\beta : B \to B'$ and $\gamma : C \to C'$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\alpha"] & B \arrow[r, "b"] \arrow[d, "\beta"] & C \arrow[r, "c"] \arrow[d, "\gamma"] & TA \arrow[d, "T\alpha"] \\
A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C \arrow[r, "c'"] & TA'
\end{tikzcd}$$

Dette avsnittet trengs å skrives om, det gir ikke lenger mening :(

The naming convention of the candidate triangles isn't standarized, some litteratures calls the candidate triangles for triangles instead; see [**keller**]. This name arises from an alternate description of the diagrams given above. To remove confusion about the domain or codomain of the arrows, one arrow of the triangle is decorated with "$_{T}|$". This decorator means that the functor T has to be applied to

Fotnote?

få fram tydeligere hvordan pilene ser ut the corresponding edge of the arrow. With this notation the c arrow points to TA, not A.

$$\begin{tikzcd} A \arrow[dr, "a"] & \\ & B \arrow[dl, "b"] \\ C \arrow[uu, "c", "T"'] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rrr, "\phi_a"] \arrow[dr, "a"] & & & A' \arrow[dl, "a'"'] \\ & B \arrow[r, "\phi_b"] \arrow[dl, "b"] & B' \arrow[dr, "b'"'] & \\ C \arrow[uu, dashed, "c", "T"'] \arrow[rrr, "\phi_c"] & & & C' \arrow[uu, dotted, "c'"', "T"] \end{tikzcd}$$

Kommer definisjonen av triangulerte kategorier for brått?

A triangulated category is an additive category together with a translation functor T and a triangulation class Δ consisting of candidate triangles. When a candidate triangle is an element of Δ it is usually called a triangle, an exact triangle or a distuingished triangle. Note that if the candidate triangles are referred to as triangles it is common to either call the elements of Δ for exact triangles or distuingished triangles. In this thesis the elements of Δ will be called for triangles.

Definition 1.1.2. A triangulation of an additive category $\mathcal{T}$ with translation T is a collection Δ of triangles consisting of candidate triangles in $\mathcal{T}$ satisfying the following axioms:

1. (TR1) Bookeeping axiom
   a. A candidate triangle isomorphic to a triangle is a triangle.
   b. Every morphism $a : A \to B$ can be embedded into a triangle (A, B, C, a, b, c).
   $$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA$$
   c. For every object A there is a triangle $(A, A, 0, id_A, 0, 0)$.
   $$A \xrightarrow{id_A} A \xrightarrow{0} 0 \xrightarrow{0} TA$$
2. (TR2) Rotation axiom
   For every triangle (A, B, C, a, b, c) there is a triangle $(B, C, TA, b, c, -Ta)$.
   $$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA \implies B \xrightarrow{b} C \xrightarrow{c} TA \xrightarrow{-Ta} TB$$
3. (TR3) Morphism axiom
   Given the two triangles (A, B, C, a, b, c) (1) and (A', B', C', a', b', c') (2)
   $$(1) \;\; A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA \quad (2) \;\; A' \xrightarrow{a'} B' \xrightarrow{b'} C' \xrightarrow{c'} TA'$$
   and morphisms $\phi_A : A \to A'$ and $\phi_B : B \to B'$ such that the square (1) commutes, then there is a morphism $\phi_C : C \to C'$ (not necessarily unique) such that (ϕ_A, ϕ_B, ϕ_C) is a morphism of triangles (2).
   $$(1) \;\; \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[d, "\phi_B"] \\ A' \arrow[r, "a'"] & B' \end{tikzcd} \qquad (2) \;\; \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[r, "b"] \arrow[d, "\phi_B"] & C \arrow[r, "c"] \arrow[d, dashed, "\phi_C"] & TA \arrow[d, "T\phi_A"] \\ A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & TA' \end{tikzcd}$$

4. (TR4) Octahedron axiom
   Given the triangles (A,B,C',a,x,x') (1), (B,C,A',b,y,y') (2) and $(A,C,B',b\circ a,z,z')$ (3)

$$
\begin{array}{ll}
(1) & A \xrightarrow{a} B \xrightarrow{x} C' \xrightarrow{x'} TA \\
(2) & B \xrightarrow{b} C \xrightarrow{y} A' \xrightarrow{y'} TB \\
(3) & A \xrightarrow{b\circ a} C \xrightarrow{z} B' \xrightarrow{z'} TA
\end{array}
$$

   then there exist morphisms $f : C' \to B'$ and $g : B' \to A'$, the following diagram commutes and the third row is a triangle.

$$
\begin{tikzcd}
T^{-1}B' \arrow[r, "T^{-1}z'"] \arrow[d, "T^{-1}g"] & A \arrow[r, equal, "id_A"] \arrow[d, "a"] & A \arrow[d, "b\circ a"] & & \\
T^{-1}A' \arrow[r, "T^{-1}y'"] & B \arrow[r, "b"] \arrow[d, "x"] & C \arrow[r, "y"] \arrow[d, "z"] & A' \arrow[r, "y'"] \arrow[d, equal, "id_{A'}"] & TB \arrow[d, "Tx'"] \\
& C' \arrow[r, "f"] \arrow[d, "x'"] & B' \arrow[r, "g"] \arrow[d, "z'"] & A' \arrow[r, "Ti\circ y'"] & TC' \\
& TA \arrow[r, equal, "id_{TA}"] & TA & &
\end{tikzcd}
$$

A triangulated category is denoted as $(\mathcal{T}, T, \Delta)$, where $\mathcal{T}$ is the additive category, T is the translation and Δ is the triangulation. When $\mathcal{T}$ is called a triangulated category, it should be understood as the triple given above.

Remark. The third object in a triangle is usually called cone, fiber or cofiber. These names are in use due to historic reasons, rather than portraying their functionality. The names weak kernel or weak cokernel would be better in the sense that it tells what the function of this object is. In this thesis it will either be referred to as cone, weak kernel or weak cokernel.

Remark. The rotation axiom has a dual, and it can be thought of as a shift in the opposite direction. The dual roation axiom goes as:

Given a triangle $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA$,

there is a triangle $T^{-1}C \xrightarrow{-T^{-1}c} A \xrightarrow{a} B \xrightarrow{b} C$

To be able to prove this, some more lemmata are needed.

Remark. By the previous remark one may see that the definition of a triangulated category is self dual. That is a category $\mathcal{T}$ is triangulated if and only if $\mathcal{T}^{op}$ is triangulated.

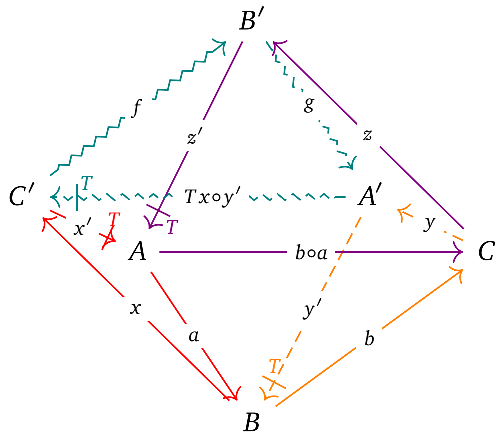
Remark. The final axiom is referred to as the octahedron axiom. By using the alternative description of the triangle diagram, it is possible to rewrite the diagram as an octahedron. The axiom can be restated as the following.

Given the triangles (A, B, C', a, x, x') (1), (B, C, A', b, y, y') (2) and $(A, C, B', b \circ a, z, z')$ (3)

$$(1)\quad \begin{tikzcd} A \arrow[dr, "a"] & \\ & B \arrow[dl, "x"] \\ C' \arrow[uu, "x'", "T"'] & \end{tikzcd} \qquad (2)\quad \begin{tikzcd} B \arrow[dr, "b"] & \\ & C \arrow[dl, "y"] \\ A' \arrow[uu, "y'", "T"'] & \end{tikzcd}$$

$$(3)\quad \begin{tikzcd} A \arrow[dr, "b\circ a"] & \\ & C \arrow[dl, "z"] \\ B' \arrow[uu, "z'", "T"'] & \end{tikzcd}$$

then there exists morphisms $f : C' \to B'$ and $g : B' \to A'$, the following diagram commutes and the squiggly teal back face is a triangle.



Proposition 1.1.1. *The axiom TR3 can be proven from TR1 and TR4.*

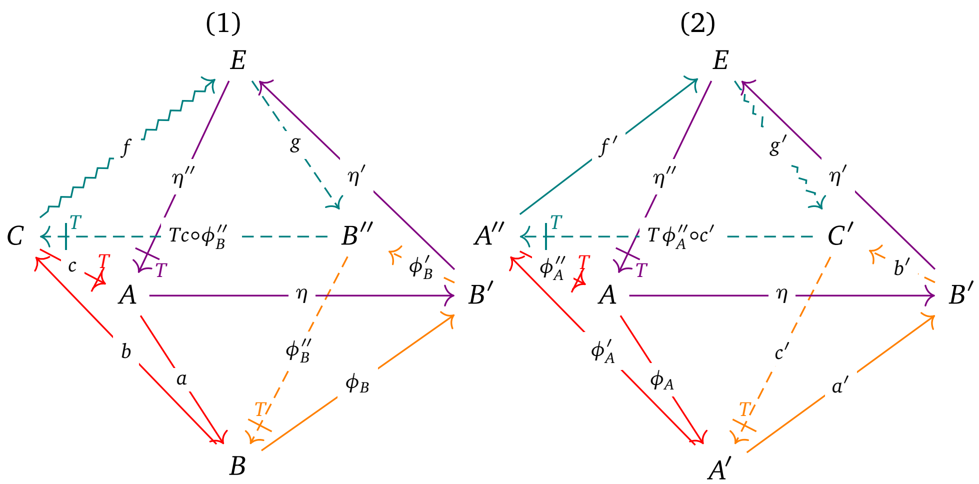
Proof. Suppose that there are two triangles and a commutative square as follows.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] \arrow[dr, "\eta"] & B \arrow[d, "\phi_B"] \\ A' \arrow[r, "a'"] & B' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[r, "b"] \arrow[d, "\phi_B"] & C \arrow[r, "c"] & TA \arrow[d, "T\phi_A"] \\ A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & TA' \end{tikzcd}$$

The upper and lower simplex of the square may be completed to two sets of triangles satisfying the condition of TR4. Applying the Octahedron axiom twice yields the diagrams as below.

$$(1)\quad \begin{tikzcd} A \arrow[dr, "a"] & \\ & B \arrow[dl, "b"] \\ C \arrow[uu, "c", "T"'] & \end{tikzcd} \qquad \begin{tikzcd} B \arrow[dr, "\phi_B"] & \\ & B' \arrow[dl, "\phi'_B"] \\ B'' \arrow[uu, "\phi''_B", "T"'] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[dr, "\eta"] & \\ & B' \arrow[dl, "\eta'"] \\ E \arrow[uu, "\eta''", "T"'] & \end{tikzcd}$$

(2)
$$\begin{tikzcd} A \arrow[dr, "\phi_A"] & \\ & A' \arrow[dl, "\phi'_A"] \\ A'' \arrow[uu, "T"', "\phi''_A"] & \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[dr, "a'"] & \\ & B' \arrow[dl, "b'"] \\ C' \arrow[uu, "T"', "c'"] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[dr, "\eta"] & \\ & B' \arrow[dl, "\eta'"] \\ E \arrow[uu, "T"', "\eta''"] & \end{tikzcd}$$



The teal squiggly lines at the back faces of each octahedra forms a morphism $g'f : C \to C'$. It remains to see that the morphism is a triangle morphism. Diagram chasing reveals that the following diagram is commutative, which is exactly the requirement for the collection $(\phi_A, \phi_B, g'f)$ to be a morphism of triangles.

$$\begin{tikzcd} B \arrow[r, "b"] \arrow[d, "\phi'_B"] & C \arrow[d, "f"] \arrow[dr, "c"] & \\ B' \arrow[r, "\eta'"] \arrow[dr, "b'"] & E \arrow[r, "\eta''"] \arrow[d, "g'"] & TA \arrow[d, "T\phi_A"] \\ & C' \arrow[r, "c'"] & TA' \end{tikzcd}$$

□

Lemma 1.1.2. *Let (A, B, C, a, b, c) be a triangle, then $b \circ a = 0$*

Proof. By TR2 the triangle (A, B, C, a, b, c) can be rotated to $(B, C, TA, b, c, -Ta)$.

$$\begin{tikzcd} A \arrow[dr, "a"] & \\ & B \arrow[dl, "b"] \\ C \arrow[uu, "T", "c"'] & \end{tikzcd} \Longrightarrow \begin{tikzcd} B \arrow[dr, "b"] & \\ & C \arrow[dl, "c"] \\ TA \arrow[uu, "T", "-Ta"'] & \end{tikzcd}$$

The triangle $(C, C, 0, id_C, 0, 0)$ exists by TR1 and TR3 states that there exists a morphism from TA to 0 making the diagram below commute.

$$\begin{tikzcd}
B \arrow[r, "b"] \arrow[d, "b"] & C \arrow[r, "c"] \arrow[d, "id_C"] & TA \arrow[r, "-Ta"] \arrow[d, dashed, "0"] & TB \arrow[d, "Tb"] \\
C \arrow[r, "id_C"] & C \arrow[r, "0"] & 0 \arrow[r, "0"] & TC
\end{tikzcd}$$

Thus $0 = Tb \circ -Ta = T(-ba) \implies b \circ a = 0$ as T is a translation. $\square$

Burde jeg introdusere triangulerte funktorer, eller går det greit å bare definere de brått?

Definition 1.1.3. An additive functor between triangulated categories $F : (\mathcal{T}, T, \Delta) \to (\mathcal{R}, R, \Gamma)$ is called triangulated if there exist a natural isomorphisms $\alpha : FT \to RF$ such that $F(\Delta) \subseteq \Gamma$.
A functor $F : \mathcal{T} \to \mathcal{R}$ is called a triangle-equivalence if it is triangulated and an equivalence of categories. In this case $\mathcal{T}$ and $\mathcal{R}$ are called triangle-equivalent.

Trenger kilder her

Remark. Some literatures refer to triangulated functors as exact functor.

Definition 1.1.4. Let $\mathcal{T}$ be a triangulated category and $\mathcal{A}$ be an abelian category. A covariant functor $H : \mathcal{T} \to \mathcal{A}$ is called homological if $\forall (A, B, C, a, b, c) : \Delta$ there is a long exact sequence in $\mathcal{A}$.

$$\begin{tikzcd}[row sep=small]
A \arrow[dr, "a"] & \\
& B \arrow[dl, "b"] \\
C \arrow[uu, "c"] &
\end{tikzcd}
\implies
\begin{aligned}
&\cdots \longrightarrow H(T^iA) \xrightarrow{H(T^ia)} H(T^iB) \xrightarrow{H(T^ib)} H(T^iC) \xrightarrow{H(T^ic)} \\
&\longrightarrow H(T^{i+1}A) \xrightarrow{H(T^{i+1}a)} H(T^{i+1}B) \xrightarrow{H(T^{i+1}b)} H(T^{i+1}C) \longrightarrow \cdots
\end{aligned}$$

Dually, a contravariant functor $H : \mathcal{T} \to \mathcal{A}$ is called cohomological if $\forall (A, B, C, a, b, c) : \Delta$ there is a long exact sequence in $\mathcal{A}$.

$$\begin{tikzcd}[row sep=small]
A \arrow[dr, "a"] & \\
& B \arrow[dl, "b"] \\
C \arrow[uu, "c"] &
\end{tikzcd}
\implies
\begin{aligned}
&\cdots \longleftarrow H(T^{i-1}A) \xleftarrow[H(T^{i-1}a)]{} H(T^{i-1}B) \xleftarrow[H(T^{i-1}b)]{} H(T^{i-1}C) \xleftarrow{H(T^ic)} \\
&\quad H(T^iA) \xleftarrow[H(T^ia)]{} H(T^iB) \xleftarrow[H(T^ib)]{} H(T^iC) \longleftarrow \cdots
\end{aligned}$$

Lemma 1.1.3. *Let $M : \mathcal{T}$ be any object of $\mathcal{T}$, then the represented functor $\mathcal{T}(M, \_)$ is homological and $\mathcal{T}(\_, M)$ is cohomological.*

Proof. Only the covariant case needs to be proved, as the contravariant case is dual. For $\mathcal{T}(M, \_)$ to be homological, it has to create long exact sequences for every triangle in Δ. Let $(A, B, C, a, b, c) : \Delta$ be a triangle, then sequences in Ab can be extracted for any $i : \mathbb{Z}$.

$$\begin{tikzcd}[row sep=small]
A \arrow[dr, "a"] & \\
& B \arrow[dl, "b"] \\
C \arrow[uu, "c"] &
\end{tikzcd}
\implies
\mathcal{T}(M, T^iA) \xrightarrow{T^ia_*} \mathcal{T}(M, T^iB) \xrightarrow{T^ib_*} \mathcal{T}(M, T^iC)$$

It is enough to prove that these types of diagrams are exact at B, as the other diagrams are obtained by the rotation axiom. Thus it remains to prove that $Im(T^i a_*) = Ker(T^i b_*)$. Since $ba = 0$ it follows that $Im(T^i a_*) \subseteq Ker(T^i b_*)$. Assume that $f : Ker(T^i b_*)$, that is $f : M \to T^i B$ such that $b_*(f) = 0$. Showing that f factors through $T^i A$ proves exactness, as this means that $Ker(T^i b_*) \subseteq Im(T^i a_*)$. Note that since T is a translation, it is necessarily a right adjoint to the inverse translation; $\mathcal{T}(M, T^i B) \simeq \mathcal{T}(T^{-i}M, B)$ and by this assertion it suffices to assume that $f : T^{-i}M \to B$ such that $b \circ f = 0$. By TR1 and TR2 there exists triangles $(T^{-i}M, 0, T^{-i+1}M, 0, 0, -T^{-i+1}id)$ and $(B, C, TA, b, c, -Ta)$.

$$\begin{tikzcd}
T^{-i}M \arrow[r, "0"] \arrow[d, "f"] & 0 \arrow[r, "0"] \arrow[d, "0"] & T^{-i+1}M \arrow[r, "-T^{-i+1}id"] \arrow[d, dashed, "g"] & T^{-i+1}M \arrow[d, "Tf"] \\
B \arrow[r, "b"] & C \arrow[r, "c"] & TA \arrow[r, "-Ta"] & TB
\end{tikzcd}$$

The left square commutes by the assumption, thus the morphism g exist by TR3 such that $-Ta \circ h = -Tf \circ T^{-i+1}id = -Tf \implies Ta \circ h = Tf$. This shows that $f = a \circ T^{-1}h$, asserting that f factors through A. $\square$

Her kan det også være veldig naturlig å skrive noe. Sammenligne med 5-lemma er vel det jeg tenker på.

Lemma 1.1.4. *Let $(\phi_A, \phi_B, \phi_C) : (A, B, C, a, b, c) \to (A', B', C', a', b', c')$ be a morphism of triangles. If 2 of the maps are isomorphisms, then the last one is an isomorphism as well.*

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\phi_A", "\wr"'] & B \arrow[r, "b"] \arrow[d, "\phi_B", "\wr"'] & C \arrow[r, "c"] \arrow[d, dashed, "\phi_C", "\wr"'] & TA \arrow[d, "T\phi_A", "\wr"'] \\
A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & TA'
\end{tikzcd}$$

Proof. Without loss of generality, assume that ϕ_A and ϕ_B are the isomorphisms. This can be done as the rotation axiom reduce the other cases to this case. Then the diagram depicted below exists.

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\phi_A", "\wr"'] & B \arrow[r, "b"] \arrow[d, "\phi_B", "\wr"'] & C \arrow[r, "c"] \arrow[d, "\phi_C"] & TA \arrow[d, "T\phi_A", "\wr"'] \\
A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & TA'
\end{tikzcd}$$

Applying the functor $\mathcal{T}(C', \_)$ to the diagram yields the following diagram in Ab:

$$\begin{tikzcd}
\mathcal{T}(C',A) \arrow[r, "a_*"] \arrow[d, "(\phi_A)_*", "\wr"'] & \mathcal{T}(C',B) \arrow[r, "b_*"] \arrow[d, "(phi_B)_*", "\wr"'] & \mathcal{T}(C',C) \arrow[r, "c_*"] \arrow[d, "(\phi_C)_*"] & \mathcal{T}(C',TA) \arrow[r, "Ta_*"] \arrow[d, "(\phi_T A)_*", "\wr"'] & \mathcal{T}(C',TB) \arrow[d, "(T\phi_B)_*", "\wr"'] \\
\mathcal{T}(C',A') \arrow[r, "a'_*"] & \mathcal{T}(C',B') \arrow[r, "b'_*"] & \mathcal{T}(C',C') \arrow[r, "c'_*"] & \mathcal{T}(C',TA') \arrow[r, "Ta_*"] & \mathcal{T}(C',TB)
\end{tikzcd}$$

By the 5-lemma, we get that $(\phi_C)_*$ is an isomorphisms, i.e. $(\phi_C)_*$ is both mono and epi. Thus for some unique s in $\mathcal{T}(C', C)$, $\phi_{C*}(s) = id_{C'}$.
By applying the functor $\mathcal{T}(\_, C)$ we get the diagram:

$$\begin{tikzcd}
\mathcal{T}(A,C) & \mathcal{T}(B,C) \arrow[l, "a^*"'] & \mathcal{T}(C,C) \arrow[l, "b^*"'] & \mathcal{T}(TA,C) \arrow[l, "c^*"'] & \mathcal{T}(TB,C) \arrow[l, "Ta^*"'] \\
\mathcal{T}(A',C) \arrow[u, "(\phi_A)^*", "\wr"'] & \mathcal{T}(B,C) \arrow[l, "a'^*"] \arrow[u, "(\phi_B)^*", "\wr"'] & \mathcal{T}(C',C) \arrow[l, "b'^*"] \arrow[u, "(\phi_C)^*"] & \mathcal{T}(TA',C) \arrow[l, "c'^*"] \arrow[u, "(\phi_T A)^*", "\wr"'] & \mathcal{T}(TB',C) \arrow[l, "Ta'^*"] \arrow[u, "(\phi_T B)^*", "\wr"']
\end{tikzcd}$$

Again, the 5-lemma asserts that $(\phi_C)^*$ is an isomorphisms, and by the same argument $id_C = s' \circ \phi_C$ for some unique s'. ϕ_C is both split mono and split epi, which means it is an isomorphism. □

Corollary 1.1.4.1. $(A, B, 0, a, 0, 0)$ *is a triangle if and only if a is an isomorphism.*

Proof. Assume that a is an isomorphism. Then it is seen that $(a, id_B, 0)$ is an isomorphism of triangles.

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "a", "\wr"'] & B \arrow[r, "0"] \arrow[d, "id_B", "\wr"'] & 0 \arrow[r, "0"] \arrow[d, "0", "\wr"'] & TA \arrow[d, "Ta", "\wr"'] \\
B \arrow[r, "id_B"] & B \arrow[r, "0"] & 0 \arrow[r, "0"] & TB
\end{tikzcd}$$

Converesly, assume that $(A, B, 0, a, 0, 0)$ is a triangle. Then the same diagram as above can be constructed, and by the 2 out of 3 property, a has to be an isomorphism. □

Lemma 1.1.5. *For a triangle (A, B, C, a, b, c) the following are equivalent:*

$$\begin{tikzcd}
A \arrow[dr, "a"] & \\
& B \arrow[dl, "b"] \\
C \arrow[uu, "c", "T"'] &
\end{tikzcd}$$

- *a is split mono*
- *b is split epi*
- *$c = 0$*

Proof. The proof has two parts. First assume that a is split mono, then prove that b is split epi and $c = 0$. By duality, it is then known that b being split epi implies that a is split mono and $c = 0$. The final part is to assume that $c = 0$, and prove either a is split mono or b is split epi.
Assume that a is split mono, then there exist an a^{-1} such that $id_A = a^{-1}a$. Let $M : \mathcal{T}$ be any object, then there is a long exact sequence.

$$\begin{tikzcd}
\mathcal{T}(M, T^{-1}C) \arrow[r, "T^{-1}c_*"] & \mathcal{T}(M,A) \arrow[r, bend left, "a_*"] & \mathcal{T}(M,B) \arrow[l, bend left, dashed, "a_*^{-1}"] \arrow[r, "b_*"] & \mathcal{T}(M,C) \arrow[r, "c_*"] & \mathcal{T}(M,TA)
\end{tikzcd}$$

By assumption a_* is split mono, thus $T^{-1}c_* = 0$ and in particular $c = 0$. This implies that b_* is epi, making a split short exact sequence.

$$\begin{tikzcd}
0 \arrow[r, "0"] & \mathcal{T}(M,A) \arrow[r, bend left, "a_*"] & \mathcal{T}(M,B) \arrow[l, bend left, dashed, "a_*^{-1}"] \arrow[r, bend left, "b_*"] & \mathcal{T}(M,C) \arrow[l, bend left, dashed, "b_*^{-1}"] \arrow[r, "0"] & 0
\end{tikzcd}$$

This shows that b is split epi, completing the first part of the proof.
For the final part, assume that $c = 0$; construct the following triangles.

$$(1)\quad \begin{tikzcd} A \arrow[dr, "a"] & \\ & B \arrow[dl, "b"] \\ C \arrow[uu, "T"', "0"] & \end{tikzcd} \Longrightarrow \begin{tikzcd} C \arrow[dr, "0"] & \\ & TA \arrow[dl, "-Ta"] \\ TB \arrow[uu, "T"', "-Tb"] & \end{tikzcd}$$

$$(2)\quad \begin{tikzcd} A \arrow[dr, "id_A"] & \\ & A \arrow[dl, "0"] \\ 0 \arrow[uu, "T"', "0"] & \end{tikzcd} \Longrightarrow \begin{tikzcd} 0 \arrow[dr, "0"] & \\ & TA \arrow[dl, "-id_{TA}"] \\ TA \arrow[uu, "T"', "0"] & \end{tikzcd}$$

(1) is constructed by applying TR2 twice, while (2) is constructed with TR1 and TR2 twice. Observe that there is a commutative square between the triangles, allowing for TR3 to make a morphism of triangles.

Dette kan jeg nok utdype mer, slik at det ikke er så forvirrende

$$\begin{tikzcd} C \arrow[r, "0"] \arrow[d, "0"] & TA \arrow[r, "-Ta"] \arrow[d, equal, "id_{TA}"] & TB \arrow[r, "-Tb"] \arrow[d, dashed, "Ta^{-1}"] & TC \arrow[d, "0"] \\ 0 \arrow[r, "0"] & TA \arrow[r, equal, "-id_{TA}"] & TA \arrow[r, "0"] & 0 \end{tikzcd}$$

Thus $T(a^{-1}a) = id_{TA} = T(id_A) \implies id_A = a^{-1}a$, making a split mono. □

Lemma 1.1.6. *Given two triangles (A, B, C, a, b, c) and (A', B', C', a', b', c') the following are equivalent:*

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f"] & B \arrow[r, "b"] \arrow[d, "g"] & C \arrow[r, "c"] \arrow[d, "h"] & TA \arrow[d, "Tf"] \\ A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & TA' \end{tikzcd}$$

1. *(f, g, h) is a morphism of triangles*
2. *$\exists g : B \to B'$ such that $b'ga = 0$*

Moreover, if $\mathcal{T}(A, T^{-1}C') \simeq 0$, then f and h are unique.

Proof. 1. $\implies$ 2. The composition $b'ga = ba = 0$ shows the claim.
2. $\implies$ 1. The existence of f and h are evident from the long exact sequence of the bottom triangle at the covariant functor represented by A.

er det virkelig så evident?

$$\mathcal{T}(A, T^{-1}C') \xrightarrow{T^{-1}c'_*} \mathcal{T}(A, A') \xrightarrow{a'_*} \mathcal{T}(A, B') \xrightarrow{b'_*} \mathcal{T}(A, C')$$

The morphism $ga : \mathcal{T}(A, B')$ such that $b'ga = b'_*(ga) = 0$, thus $ga : Ker(b'_*)$. By exactness $\exists f : \mathcal{T}(A, A')$ such that $a'f = ga$, and by TR3 $\exists h : C \to C'$ such that (f, g, h) is a morphism of triangles. Now assume that $\mathcal{T}(A, T^{-1}C') \simeq 0$. Exactness determines that a'_* is a monomorphism, and f is then unique. Since T is a translation, we have that $\mathcal{T}(A, T^{-1}C') \simeq \mathcal{T}(TA, C')$. By using the functor $\mathcal{T}(\_, C')$ at the top triangle, we get that b^* is a monomorphism, and thus h is chosen uniquely. □

Hva som foregår her burde komme fram tydeligere

Lemma 1.1.7. ***Opposite Rotation Axiom;*** *TR2^{op}. If (A, B, C, a, b, c) is a triangle, then $(T^{-1}C, A, B, -T^{-1}c, a, b)$ is a triangle.*

Proof. Apply TR2 twice to construct the triangle below.

$$\begin{tikzcd} A \arrow[dr, "a"] & \\ & B \arrow[dl, "b"] \\ C \arrow[uu, "T"', "c"] & \end{tikzcd} \Longrightarrow \begin{tikzcd} C \arrow[dr, "c"] & \\ & TA \arrow[dl, "-Ta"] \\ TB \arrow[uu, "T"', "-Tb"] & \end{tikzcd}$$

The morphism $T^{-1}c$ has a triangle $(T^{-1}C, A, B', T^{-1}c, a', b')$ by TR1. Use TR3 to find a morphism between these associated candidate triangles.

$$\begin{tikzcd} C \arrow[r, "c"] \arrow[d, equal, "id_C"] & TA \arrow[r, "Ta'"] \arrow[d, equal, "id_{TA}"] & TB' \arrow[r, "Tb'"] \arrow[d, dashed, "h"] & TC \arrow[d, equal, "id_{TC}"] \\ C \arrow[r, "c"] & TA \arrow[r, "-Ta"] & TB \arrow[r, "-Tb"] & TC \end{tikzcd}$$

By the 2 out of 3 property it is seen that h is an isomorphism, so the triple $(id_{T^{-1}C}, id_A, T^{-1}h)$ is an isomorphism of candidate triangles, and by TR1, is an isomorphism of triangles, asserting that $(T^{-1}C, A, B, -T^{-1}c, a, b)$ is in fact a triangle. □

Lemma 1.1.8. *Let* (A, B, C, a, b, c) *and* (A', B', C', a', b', c') *be two triangles, then the direct sum of these triangles is a triangle.*

Hva er det jeg prøvde å si her???

Proof. Observe that for any functor $\mathcal{T}(K, \_)$ there is still a long exact sequence of Hom(ology)since $\mathcal{T}(K, A \oplus A') \simeq \mathcal{T}(K, A) \oplus \mathcal{T}(K, A')$. Thus for the direct sum of the triangles we have the following.

$$A \oplus A' \xrightarrow{\begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix}} B \oplus B' \xrightarrow{\begin{pmatrix} b & 0 \\ 0 & b' \end{pmatrix}} C \oplus C' \xrightarrow{\begin{pmatrix} c & 0 \\ 0 & c' \end{pmatrix}} TA \oplus TC$$

$$\Downarrow$$

$$\ldots \longrightarrow \mathcal{T}(K, A) \oplus \mathcal{T}(K, A') \longrightarrow \mathcal{T}(K, B) \oplus \mathcal{T}(K, B') \longrightarrow$$

$$\longrightarrow \mathcal{T}(K, C) \oplus \mathcal{T}(K, C') \to \mathcal{T}(K, TA) \oplus \mathcal{T}(K, TA') \longrightarrow \ldots$$

Thus the 2 out of 3 property holds for the direct sum. By TR1 there is a triangle

$$A \oplus A' \longrightarrow B \oplus B' \longrightarrow D \longrightarrow TA \oplus TA'$$

By TR3 there are morphisms from this triangle to to the direct summands. Adding these maps together there is a map from this triangle to direct sum, and by using the 2 out of 3 property this is an isomorphism of candidate triangles. Thus the direct sum is a triangle.

$$\begin{tikzcd} A \oplus A' \arrow[r] \arrow[d] & B \oplus B' \arrow[r] \arrow[d] & D \arrow[r] \arrow[d, dashed] & TA \oplus TA' \arrow[d] \\ A \arrow[r] & B \arrow[r] & C \arrow[r] & TA \end{tikzcd}$$

$$\&$$

$$\begin{tikzcd}
A \oplus A' \arrow[r] \arrow[d] & B \oplus B' \arrow[r] \arrow[d] & D \arrow[r] \arrow[d, dashed] & TA \oplus TA' \arrow[d] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & TA'
\end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd}
A \oplus A' \arrow[r] \arrow[d, equal] & B \oplus B' \arrow[r] \arrow[d, equal] & D \arrow[r] \arrow[d, dashed, "\wr"] & TA \oplus TA' \arrow[d, equal] \\
A \oplus A' \arrow[r] & B \oplus B' \arrow[r] & A'' \oplus B'' \arrow[r] & TA \oplus TA'
\end{tikzcd}$$

□

Lemma 1.1.9. *The direct summands of a triangle is a triangle.*

Proof. The proof can be found in [**neeman**] □

1.2 Mapping Cones, Homotopies and Contractibility

The observant reader might have seen that the Octahedron axiom have not yet been used once, other than for proving TR3. A lot of the theory proven for triangulated categories works without this axiom, and this motivates the definition of a pre-triangulated category.

Definition 1.2.1. A pre-triangulation of an additive category $\mathcal{T}$ with translation T is a collection Δ' of triangles consisting of candidate triangles in $\mathcal{T}$ satisfying TR1, TR2 and TR3.
The category $\mathcal{T}$ with the pre-triangulation Δ' is called a pre-triangulated category, and the candidate triangles in Δ' are called triangles. This notion of triangles will only be used in this subsection.

Kanskje få denne biten tidligere?

The main goal of this subsection is to see how we can find triangles, and when these are triangles. We will also look at triangulated functors and triangulated subcategories. For the rest of this subsection it is assumed that we work in a pre-triangulated category $\mathcal{T}$.

Definition 1.2.2. Let $\phi : (A, B, C, a, b, c) \to (A', B', C', a', b', c')$ be a morphism of candidate triangles.

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[r, "b"] \arrow[d, "\phi_B"] & C \arrow[r, "c"] \arrow[d, "\phi_C"] & TA \arrow[d, "T\phi_A"] \\
A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C \arrow[r, "c'"] & TA'
\end{tikzcd}$$

The mapping cone of ϕ is defined to be the candidate triangle below.

$$A' \oplus B \xrightarrow{\begin{pmatrix} -b & \phi_B \\ 0 & a' \end{pmatrix}} B' \oplus C \xrightarrow{\begin{pmatrix} -c & \phi_C \\ 0 & b' \end{pmatrix}} C' \oplus TA \xrightarrow{\begin{pmatrix} -Ta & T\phi_A \\ 0 & c' \end{pmatrix}} TA' \oplus TB$$

Definition 1.2.3. A morphism $\alpha : A, B, C, a, b, c) \to (A', B', C', a', b', c')$ between candidate triangles is called null-homotopic if it factors through a homotopy. A homotopy is defined to be a triple of maps Θ, Φ, Ψ in the following sense.

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\alpha_A"] & B \arrow[r, "b"] \arrow[d, "\alpha_B"] \arrow[dl, "\Theta"] & C \arrow[r, "c"] \arrow[d, "\alpha_C"] \arrow[dl, "\Phi"] & TA \arrow[d, "T\alpha_A"] \arrow[dl, "\Psi"] \\
A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C \arrow[r, "c'"] & TA'
\end{tikzcd}$$

It is required that $\alpha_A = \Theta a + T^{-1}(c'\Psi)$, $\alpha_B = \Phi b + a'\Theta$ and $\alpha_C = \Psi c + b'\Phi$ for the triple to be a homotopy. Two maps are called homotopic if their difference is null-homotopic

Lemma 1.2.1. *The mapping cone only depends on morphisms up to homotopy. I.e. if two maps are homotopic, their mapping cones are isomorphic.*

Proof. Suppose that (f, g, h) and (f', g', h') are two homotopic morphisms of triangles:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d] & B \arrow[r, "b"] \arrow[d] & C \arrow[r, "c"] \arrow[d] & TA \arrow[d] \\
A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & TA'
\end{tikzcd}$$

Let (Θ, Φ, Ψ) be the homotopy between the triangle morphisms. Then there is an isomorphism of triangles.

$$\begin{tikzcd}
A' \oplus B \arrow[r, "\begin{pmatrix} -b & g \\ 0 & a' \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & \Theta \\ 0 & 1 \end{pmatrix}"] & B' \oplus C \arrow[r, "\begin{pmatrix} -c & h \\ 0 & b' \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & \Phi \\ 0 & 1 \end{pmatrix}"] & C' \oplus TA \arrow[r, "\begin{pmatrix} -Ta & Tf \\ 0 & c' \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & \Psi \\ 0 & 1 \end{pmatrix}"] & TA' \oplus TB \arrow[d, "\begin{pmatrix} 1 & T\Theta \\ 0 & 1 \end{pmatrix}"] \\
A' \oplus B \arrow[r, "\begin{pmatrix} -b & g' \\ 0 & a' \end{pmatrix}"'] & B' \oplus C \arrow[r, "\begin{pmatrix} -c & h' \\ 0 & b' \end{pmatrix}"'] & C' \oplus TA \arrow[r, "\begin{pmatrix} -Ta & Tf' \\ 0 & c' \end{pmatrix}"'] & TA' \oplus TB
\end{tikzcd}$$

□

Lemma 1.2.2. *Let A denote the candidate triangle (A, A', A'') and B denote (B, B', B''). Suppose $\alpha, \beta : A \to B$ are two homotopic morphisms of candidate triangles. Then for any map $\gamma : \widetilde{A} \to A$ and any map $\delta : B \to \widetilde{B}$ the maps $\delta\alpha\gamma$ and $\delta\beta\gamma$ are homotopic as well.*

Proof. To prove this statement it is enough to prove that $\alpha\gamma$ is homotopic to $\beta\gamma$ du to the symmetry of the statement. The goal is then to show that $(\Theta\gamma', \Phi\gamma'', \Psi T\gamma)$ is the homotopy between these maps. This can be seen as

$$\alpha'\gamma' - \beta'\gamma' = (\alpha' - \beta')\gamma' = (b\Theta + \Phi a')\gamma' = b\Theta\gamma' + \Phi a'\gamma' = b(\Theta\gamma') + (\Phi\gamma'')\widetilde{a}'$$

. □

Definition 1.2.4. A candidate triangle A is called a contractible triangle if id_A is null-homotopic.

Remark. If A is a contractible triangle and $F : \mathcal{T} \to \mathcal{A}$ is an additive functor to an abelian category, then the identity of the cochain is null-homotopic as well.

$$... \longrightarrow F(A) \longrightarrow F(A') \longrightarrow F(A'') \longrightarrow F(TA) \longrightarrow ...$$

The homology of this sequence is therefore 0 everywhere, asserting that it is an exact sequence. The exactness of such sequences allow us to use the 2 out of 3 property on morphisms between contractible triangles.

Corollary 1.2.2.1. *If A is a contractible triangle, then any map in $\mathcal{T}(A,\_)$ or $\mathcal{T}(\_,A)$ is null-homotopic.*

Proof. By definition, being contractible is the same as the existence of a homotopy between the map and the zero map. If $id_A \sim 0 \implies f \circ id_A = f \sim f \circ 0 = 0$. So any map f is null-homotopic. □

Lemma 1.2.3. *A contractible triangle is a triangle.*

Proof. Let A be the contractible triangle (A, A', A''). Writing everything out, there is a homotopy between candidate triangles.

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "id_A"] & A' \arrow[r, "a'"] \arrow[d, "id_{A'}"] \arrow[dl, "\Theta"] & A'' \arrow[r, "a''"] \arrow[d, "id_{A''}"] \arrow[dl, "\Phi"] & TA \arrow[d, "id_{TA}"] \arrow[dl, "\Psi"] \\
A \arrow[r, "a"] & A' \arrow[r, "a'"] & A'' \arrow[r, "a''"] & TA
\end{tikzcd}$$

By using TR1 there is a triangle, and consequently, a long exact sequence.

$$A \xrightarrow{a} A' \xrightarrow{e} E \xrightarrow{e'} TA$$
$$\Downarrow$$
$$... \longrightarrow \mathcal{T}(TA, A) \longrightarrow \mathcal{T}(TA, A') \longrightarrow \mathcal{T}(TA, E) \xrightarrow{e'_*} \mathcal{T}(TA, TA) \xrightarrow{Ta_*} ...$$

Since the map $Ta \circ a''\Psi = 0$ and by exactness at $\mathcal{T}(TA, TA)$, the kernel $Ker Ta_* = Im e'_* \neq 0$. This shows that there is a map $\Psi' : \mathcal{T}(TA, E)$ such that $e'\Psi' = a''\Psi$, and the map $(id_A, id_{A'}, e\Theta + \Psi' a'')$ is a well defined map of candidate triangles. By the remark, we can use the 2 out of 3 properties to assert that the map found is an isomorphism, giving an isomorphism of triangles, showing that the contractible triangle is a triangle by Bookeeping. □

Corollary 1.2.3.1. *The mapping cone of the zero map between triangles is a triangle.*

Proof. The mapping cone of the zero map can be seen to be the direct sum of two triangles. Thus it is a triangle. □

Corollary 1.2.3.2. *The mapping cone of a null-homotopic map between triangles is a triangle.*

Remark. Suppose we have a morphism of triangles where one of the triangles are contractible, then the mapping cone is a triangle as well.

Skriv noe om hvordan dette har bygget opp til en annen variant av TR4.

Definition 1.2.5. A morphism of triangles will be called good if the mapping cone of the morphism is a triangle.

Theorem 1.2.4. *A pre-triangulated category $\mathcal{T}$ is triangulated if given two triangles (A,B,C,a,b,c) and (A',B',C',a',b',c') and diagram (1) commutes, then diagram (1) can be completed to diagram (2) such that ϕ is good.*

$$(1)\quad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[d, "\phi_B"] \\ A' \arrow[r, "a'"] & B' \end{tikzcd} \qquad (2)\quad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[r, "b"] \arrow[d, "\phi_B"] & C \arrow[r, "c"] \arrow[d, dashed, "\phi_C"] & TA \arrow[d, "T\phi_A"] \\ A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & TA' \end{tikzcd}$$

Utdyp hva som står her

Remark. This condition is equivalent to the Octahedron axiom.

Legg til kilder

Definition 1.2.6. A commutative square (1) is called homotopy cartesian if it arises from a triangle. That is, (2) is a triangle.

Skal jeg skrive også om Verdier sitt aksiom her? Jeg synes det kan være en gøy med en exposition av alt kaoset man kan finne :)

$$(1)\quad \begin{tikzcd} D \arrow[r] \arrow[d] \arrow[dr, phantom, "HO"] & A \arrow[d] \\ B \arrow[r] & C \end{tikzcd} \implies (2)\quad \begin{tikzcd} D \arrow[dr] & \\ & A \oplus B \arrow[dl] \\ C \arrow[uu, "T"] & \end{tikzcd}$$

Denne definisjonen er rar og er ikke presis, men jeg synes den er litt søt

Remark. A way to construct homotopy cartesian squares is with homotopy pullbacks. This is done by using TR1 on the following map to get a triangle.

$$\begin{tikzcd} & A \arrow[d, "a"] \\ B \arrow[r, "b"] & C \end{tikzcd} \implies A \oplus B \xrightarrow{\begin{pmatrix} a & b \end{pmatrix}} C$$

$$\implies A \oplus B \xrightarrow{\begin{pmatrix} a & b \end{pmatrix}} C \longrightarrow TD \longrightarrow TA \oplus TB \implies \begin{tikzcd} D \arrow[r] \arrow[d] \arrow[dr, phantom, "HO"] & A \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

Dually, one may use homotopy push-outs to get homotopy cartesian squares.

Remark. A remark about good maps and homotopy cartesian squares???

Lemma 1.2.5. *Suppose that there is a homotopy cartesian square (1). Then there are triangles and a triangle isomorphism as in (2).*

$$(1)\quad \begin{tikzcd} D \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "HO"] & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd} \quad (2)\quad \begin{tikzcd} D \arrow[r] \arrow[d, "f'"] & A \arrow[r] \arrow[d, "f"] & E \arrow[r] \arrow[d, equal] & TD \arrow[d, "Tf'"] \\ B \arrow[r] & C \arrow[r] & E \arrow[r] & TB \end{tikzcd}$$

Proof. There is a commutative square (1) which satisfies the requirements of the octahedron axiom (2), yielding a triangle (3).

$$(1)\quad \begin{tikzcd} D \arrow[r, "\begin{pmatrix} g' \\ f' \end{pmatrix}"] \arrow[d, equal] & A \oplus B \arrow[d, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] \\ D \arrow[r, "g'"] & A \end{tikzcd}$$

$$(2)\quad \begin{tikzcd}[column sep=small] D \arrow[dr, "\begin{pmatrix} g' \\ f' \end{pmatrix}"] & \\ & A \oplus B \arrow[dl, "\begin{pmatrix} f & g \end{pmatrix}"] \\ C \arrow[uu, "T", "0"'] & \end{tikzcd} \quad \begin{tikzcd}[column sep=small] A \oplus B \arrow[dr, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] & \\ & A \arrow[dl, "0"] \\ TB \arrow[uu, "T", "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"'] & \end{tikzcd} \quad \begin{tikzcd}[column sep=small] D \arrow[dr, "f'"] & \\ & A \arrow[dl] \\ E \arrow[uu, "T"] & \end{tikzcd}$$

$$(3)\quad C \longrightarrow E \longrightarrow TB \longrightarrow TC$$

This setup shows that there is a triangle with a commutative square as below.

$$\begin{tikzcd} A \oplus B \arrow[r, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} f & g \end{pmatrix}"] & A \arrow[d] \\ C \arrow[r] & E \end{tikzcd}$$

. Since $\begin{pmatrix} 1 & 0 \end{pmatrix}$ is a splitmono, there is a morphism of triangles proving the statement.

$$\begin{tikzcd} D \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, equal] & TD \arrow[d] \\ B \arrow[r] & C \arrow[r] & E \arrow[r] & TB \end{tikzcd}$$

□

1.3 Calculus of Fractions and the Verdier Quotient

Dette avsnittet er skrevet litt rart. Flyten kan bli mye bedre.

Localization is a method for adding formal inverses to a category. It is most notably known in commutative algebra where we can invert elements with respect to some ideal of the ring. The rational numbers can be shown to be a localization of the integers at every number except 0. The category gained from localizing at some set S of morphisms is the universal category where these morphisms are isomorphisms.

Definition 1.3.1. Let S be a collection of morphisms in the category $\mathcal{C}$. The Localization of $\mathcal{C}$ on S is the category $\mathcal{C}[S^{-1}]$ together with a functor $q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ such that:

- $\forall s : S | q(s)$ is an isomorphism
- Any functor $F : \mathcal{C} \to \mathcal{D}$ such that $\forall s : S\ F(s)$ is an isomorphism, then F factors through q. That is to say that there is a natural isomorphism $\eta : F \to F' \circ q$ so that $\mathcal{C}[S^{-1}]$ is the universal category where morphisms in S are isomorphisms.

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "q"'] & & \mathcal{D} \\
& S^{-1}\mathcal{C} \arrow[ur, dashed, "F'"'] &
\end{tikzcd}$$

Remark. Even though it is known that $\mathcal{C}$ is locally small, it is not clear a priori that the category $\mathcal{C}[S^{-1}]$ is again locally small.

Burde jeg endre på noe her. Kan det være greit å forklare hvorfor de er vanskelig å beskrive?

These categories are in generel pretty hard to describe. When the set of morphisms is a multiplicative system, there is a calculus of fractions description of these localization in the same style as for localizations of rings.

Definition 1.3.2. A set S of morphisms in a category $\mathcal{C}$ is called right multiplicative if it satisfies the following conditions:

- S is closed under composition, i.e. if $f, g : S$ are composable then $gf : S$. Every identity morphism in $\mathcal{C}$ is in S.
- (Right Ore condition) If $t : X \to Y$ is a morphism in S, then $\forall g : Z \to Y$ there is a commutative square (1) such that $f : W \to X$ and $s : W \to Z$ exists and $s : S$ as well.

$$(1) \quad \begin{tikzcd}
W \arrow[r, dashed, "f"] \arrow[d, dashed, "s"] & X \arrow[d, "t"] \\
Z \arrow[r, "g"] & Y
\end{tikzcd}$$

- (Left cancellation) Suppose $f, g : X \to Y$ are parallell morphisms in $\mathcal{C}$, then 1. $\implies$ 2.:
  1. $sf = sg$ for som $s : S$ starting at Y
  2. $ft = gt$ for som $t : S$ ending at X

Remark. The previous definition has a dual statement. A set S of morphisms is left multiplicative if it satisfies:

- S is closed under composition, i.e. if $f, g : S$ are composable then $gf : S$. Every identity morphism in $\mathcal{C}$ is in S.
- (Left Ore condition) If $s : Y \to Z$ is a morphism in S, then $\forall f : Y \to X$ there is a commutative square (1) such that $g : Z \to W$ and $t : X \to W$ exists and $t : S$ as well.

$$(1)\quad \begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "s"] & X \arrow[d, dashed, "t"] \\ Z \arrow[r, dashed, "g"] & W \end{tikzcd}$$

- (Right cancellation) Suppose $f, g : X \to Y$ are parallell morphisms in $\mathcal{C}$, then 1. $\implies$ 2.:
  1. $ft = gt$ for som $t : S$ ending at X
  2. $sf = sg$ for som $s : S$ starting at Y

If S is both right multiplicative and left multiplicative then it is called multiplicative.

Prototype. Let R be a commutative integral domain, ... (look at Bacharaya and how they define the field of fractions, or ask Andreas if he have any good literature on this topic) Fjerne dette?

As with the definition of localization of rings, localization of a category $\mathcal{C}$ at a multiplicative system will be defined with fractions. That is the morphisms will be "fractions" of morphisms. These morphisms will be described as diagrams over spans for right multiplicative systems (or dually cospans for left multiplicative systems), together with an equivalence relation.

Definition 1.3.3. A span is a diagram of the form:

$$\cdot \longleftarrow \cdot \longrightarrow \cdot$$

Definition 1.3.4. Let S be a right multiplicative system of morphisms in a category $\mathcal{C}$. Given a morphism $s : Y \to X$ in S and a morphism $t : Y \to Z$, define the right fraction of s and t to be the span of the morphisms. That is s and t fit in the diagram below.

$$X \xleftarrow[s]{} Y \xrightarrow{t} Z$$

Right fractions are denoted as ts^{-1}. Let $\sim$ be the equivalence relation of right fractions given by the diagram (1) such that $ts^{-1} \sim t's'^{-1}$ if and only if $\exists w, w' : \mathcal{C}$ making the diagram commute and that the middle row is a right fraction.

$$\begin{tikzcd} & Y \arrow[dl, "s"'] \arrow[dr, "t"] & \\ X & W \arrow[l] \arrow[r] \arrow[u, "w"] \arrow[d, "w'"] & Z \\ & Y' \arrow[ul, "s'"] \arrow[ur, "t'"'] & \end{tikzcd}$$

Dually, define left fractions as diagrams over cospans such that if $t : S$, then there is a left fraction $t^{-1}s$ as the diagram below.

$$X \xrightarrow{s} Y \xleftarrow[t]{} Z$$

The equivalence relation $\sim$ is given by the diagram in the same manner as above.

$$\begin{tikzcd}
 & Y \arrow[d, "w"] & \\
X \arrow[ur, "s"] \arrow[r] \arrow[dr, "s'"'] & W & Z \arrow[ul, "t"'] \arrow[l] \arrow[dl, "t'"] \\
 & Y' \arrow[u, "w'"] &
\end{tikzcd}$$

Proposition 1.3.1. *Suppose that S is a right multiplicative system, then the relation stated above is in fact an equivalence relation.*

Proof. An equivalence relation is proven by showing that $\sim$ is reflexive, symmetric and transitive.

- (Reflexive) Let fs^{-1} be a right fraction. Then diagram (1) shows that $fs^{-1} \sim fs^{-1}$.

$$(1) \quad \begin{tikzcd}
 & W \arrow[dl, "s"'] \arrow[dr, "f"] \arrow[d, equal] & \\
X & W \arrow[l, "s"] \arrow[r, "f"] & Y \\
 & W \arrow[ul, "s"] \arrow[ur, "f"'] \arrow[u, equal] &
\end{tikzcd}$$

- (Symmetric) Let fs^{-1} and gt^{-1} be two right fractions such that $fs^{-1} \sim gt^{-1}$, that is diagram (2) commute. Due to inherent symmetric nature of the diagram it follows that $gt^{-1} \sim fs^{-1}$.

$$(2) \quad \begin{tikzcd}
 & W \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & \widetilde{W} \arrow[l] \arrow[r] \arrow[u, "w"] \arrow[d, "w'"] & Y \\
 & W' \arrow[ul, "t"] \arrow[ur, "g"'] &
\end{tikzcd}
\implies
\begin{tikzcd}
 & W' \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & \widetilde{W} \arrow[l] \arrow[r] \arrow[u, "w'"] \arrow[d, "w"] & Y \\
 & W \arrow[ul, "s"] \arrow[ur, "f"'] &
\end{tikzcd}$$

- (Transitive) Suppose that there are three right fractions fs^{-1}, gt^{-1} and hu^{-1} such that $fs^{-1} \sim gt^{-1}$ and $gt^{-1} \sim hu^{-1}$. This may be written as diagram (3) and (4).

$$(3) \quad \begin{tikzcd}
 & W' \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & \widetilde{W} \arrow[l] \arrow[r] \arrow[u, "w'"] \arrow[d, "\widetilde{w'}"] & Y \\
 & W \arrow[ul, "t"] \arrow[ur, "g"'] &
\end{tikzcd}
\quad (4) \quad
\begin{tikzcd}
 & W \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & \widetilde{\widetilde{W}} \arrow[l] \arrow[r] \arrow[u, "\widetilde{w''}"] \arrow[d, "w''"] & Y \\
 & W'' \arrow[ul, "u"] \arrow[ur, "h"'] &
\end{tikzcd}$$

«««< HEAD Diagram (5) may be created by using the Ore condition on the maps $\widetilde{w'}$ and $\widetilde{w''}$. Since both morphisms are assumed to be in S, it follows that both $\widetilde{w'}$ and $\widetilde{w''}$ are in S as well. Diagram (6) then shows that $fs^{-1} \sim hu^{-1}$.

(5) ======= Two new maps may be created by applying the Ore condition on $\widetilde{w'}$ and $\widetilde{w''}$. Since both morphisms are assumed to be in S, it follows that both $\widetilde{w'}$ and $\widetilde{w''}$ are in S. The following diagram then shows that $fs^{-1} \sim hu^{-1}$. A simple diagram chase shows that it is commutative.

»»»> 153477f27ab47a8db5d0ea7cda8e781b36875c7d

$$\begin{tikzcd}
\widetilde{\widetilde{\widetilde{W}}} \arrow[r, "\widetilde{\widetilde{w''}}"] \arrow[d, "\widetilde{\widetilde{w'}}"] & \widetilde{\widetilde{W}} \arrow[d, "\widetilde{w''}"] \\
\widetilde{W} \arrow[r, "\widetilde{w'}"] & W
\end{tikzcd} \tag{6}$$

$$\begin{tikzcd}
& W' \arrow[ddl, "s"'] \arrow[ddr, "f"] & \\
& \widetilde{W} \arrow[u, "\widetilde{w'}"'] \arrow[dl] \arrow[dr] & \\
X & \widetilde{\widetilde{\widetilde{W}}} \arrow[l] \arrow[r] \arrow[u, "\widetilde{\widetilde{w'}}"'] \arrow[d, "\widetilde{\widetilde{w''}}"] & Y \\
& \widetilde{\widetilde{W}} \arrow[ul] \arrow[ur, "h"'] \arrow[d, "\widetilde{w''}"] & \\
& W'' \arrow[uul, "u"] \arrow[uur] &
\end{tikzcd}$$

□

Definition 1.3.5. Let S be a multiplicate system in a category $\mathcal{C}$. Given two right fractions fs^{-1} and gt^{-1}

$$X \xleftarrow[s]{} W \xrightarrow{f} Y \quad \& \quad Y \xleftarrow[t]{} W' \xrightarrow{g} Z$$

«««< HEAD the composition of the fractions are defined to be $gt^{-1} \circ fs^{-1}$. The Ore condition describes how this composition should be defined, ======= Define the composite of these fractions $gt^{-1} \circ fs^{-1}$ by the Ore condition: »»»> 153477f27ab47a8db5d0ea7cda8e781b36875c7d

$$\begin{tikzcd}
& \widetilde{W} \arrow[r, "h"] \arrow[d, "u"] & W' \arrow[r, "g"] \arrow[d, "t"] & Z \\
X & W \arrow[l, "s"] \arrow[r, "f"] & Y &
\end{tikzcd}$$

the composite is the right fraction $gt^{-1} \circ fs^{-1} = gh(su)^{-1}$.

Proposition 1.3.2. *The composition of right fractions is well-defined up to equivalence.*

Proof. «««< HEAD In order to prove that the composite is well-defined one must prove that the composite is independent from the different options of morphisms provided by the right Ore condition, and that it is therefore independent from the choice of right fraction. There will only be presented a proof for that the choice of Ore maps is independent, as the other other case is analogous.
Suppose there are two right fractions fs^{-1} and gt^{-1} as indicated by the diagrams. ======= In order to prove that the composite is well-defined it is needed to prove that the composite is independent from the different choices of the right Ore condition, and that it is independent from choice of right fraction. There will only be presented a proof for that the choice of Ore maps is independent, as the other two cases are analogous.
Suppose there are two right fractions fs^{-1} and gt^{-1} as indicated from the diagrams. »»»> 153477f27ab47a8db5d0ea7cda8e781b36875c7d

Har spurt Håkon om hjelp her.

$$X \xleftarrow[s]{} W_1 \xrightarrow{f} Y \quad \& \quad Y \xleftarrow[t]{} W_2 \xrightarrow{g} Z$$

«««< HEAD Further suppose that there are at least two different choices for the morphisms provided by the right Ore condition, for example $(\widetilde{W}, \widetilde{s}, \widetilde{f})$ and $(\widehat{W}, \widehat{s}, \widehat{g})$. The two compositions may be drawn as the diagrams below. ======= Further suppose that there are at least two different choices for the maps gained by the right Ore condition. That is for example $(\widetilde{W}, \widetilde{s}, \widetilde{f})$ and $(\widehat{W}, \widehat{s}, \widehat{g})$. The two compositions may be drawn as the diagrams below. »»»> 153477f27ab47a8db5d0ea7cda8e781b36875c7d

Dette er et incosist format

$$\begin{tikzcd}
 & \widetilde{W} \arrow[r, "\widetilde{g}"] \arrow[d, "\widetilde{s}"] & W_2 \arrow[r, "g"] \arrow[d, "t"] & Z \\
X & W_1 \arrow[l, "s"] \arrow[r, "f"] & Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \widehat{W} \arrow[r, "\widehat{f}"] \arrow[d, "\widehat{s}"] & W_2 \arrow[r, "g"] \arrow[d, "t"] & Z \\
X & W_1 \arrow[l, "s"] \arrow[r, "f"] & Y &
\end{tikzcd}$$

«««< HEAD By combining the diagrams at W_1 using the right Ore condition again one may find $(W, \widetilde{w}, \widehat{w})$ fitting in the diagram below. ======= Combining the diagrams at W_1 by using the right Ore condition, the objects and maps $(W, \widetilde{w}, \widehat{w})$ exists as in the diagram below. »»»> 153477f27ab47a8db5d0ea7cda8e781b36875c7d

Er dette inkonsekvent?

$$\begin{tikzcd}
\bar{W} \arrow[dr, dashed, "\xi"] & & & & \\
 & W \arrow[r, "\widehat{w}"] \arrow[d, "\widetilde{w}"] & \widehat{W} \arrow[r, "\widehat{f}"] \arrow[d] & W_2 \arrow[r, "g"] \arrow[d, "t"] & Z \\
 & \widetilde{W} \arrow[r, "\widetilde{s}"] \arrow[urr, "\widetilde{g}"] & W_1 \arrow[r, "f"] \arrow[d, "s"] & Y & \\
 & & X & &
\end{tikzcd}$$

Observe that the three squares commute, as by the definition of right Ore condition. Thus it follows that $s\widetilde{s}\widetilde{w} = s\widehat{s}\widehat{w}$, and that $t\widehat{f}\widehat{w} = t\widetilde{g}\widetilde{w}$. As $t : S$ one may use right cancellation to find a $\xi : \bar{W} \to W$ such that $\widehat{f}\widehat{w}\xi = \widetilde{g}\widetilde{w}\xi \implies g\widehat{f}\widehat{w}\xi = g\widetilde{g}\widetilde{w}\xi$. Thus the equivalence relation diagram commutes.

$$\begin{tikzcd}
 & \widehat{W} \arrow[dl, "s\widehat{s}"'] \arrow[dr, "g\widehat{f}"] & \\
X & \bar{W} \arrow[l] \arrow[r] \arrow[u, "\widehat{w}\xi"'] \arrow[d, "\widetilde{w}\xi"] & Z \\
 & \widetilde{W} \arrow[ul, "s\widetilde{s}"] \arrow[ur, "g\widetilde{g}"'] &
\end{tikzcd}$$

□

Proposition 1.3.3. *The composition of right fractions is associative.*

Proof. Let fs^{-1}, gt^{-1} and hu^{-1} be right fractions as in the diagrams below.

Kommer det fram at dette er en proof sketch?

$$A \xleftarrow[s]{} X \xrightarrow{f} B \quad , \quad B \xleftarrow[t]{} Y \xrightarrow{g} C \quad \& \quad C \xleftarrow[u]{} Z \xrightarrow{h} D$$

There are two different ways of calculating the compostion. Every morphism in S will be marked blue.

$\underline{hu^{-1} \circ (gt^{-1} \circ fs^{-1})}$

$$\begin{tikzcd}
W \arrow[rr] \arrow[d] & & Z \arrow[r] \arrow[d] & D \\
V \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & C & \\
X \arrow[r] \arrow[d] & B & & \\
A & & &
\end{tikzcd}$$

$\underline{(hu^{-1} \circ gt^{-1}) \circ fs^{-1}}$

$$\begin{tikzcd}
V' \arrow[r] \arrow[dd] & W' \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & D \\
 & Y \arrow[r] \arrow[d] & C & \\
X \arrow[r] \arrow[d] & B & & \\
A & & &
\end{tikzcd}$$

To be able to find a relation between these diagrams create another diagram with the right Ore condition.

$$\begin{tikzcd}
T \arrow[r, dashed] \arrow[d, dashed] & V' \arrow[d] \\
W \arrow[r] & X
\end{tikzcd}$$

To finish the proof, one would need to show that the maps to A and D commute. The maps to A commute right out of the bat, by the right Ore condition. To prove that the maps to D commute, first apply right cancellation on the maps to B, then on the maps to C.

□

Definition 1.3.6. Let S be a right multiplicative system in a category $\mathcal{C}$. Define a category $\mathfrak{r}S^{-1}\mathcal{C}$ to have objects $\mathfrak{Obr}S^{-1}\mathcal{C} = \mathfrak{Ob}\mathcal{C}$ and morphisms $\mathfrak{Arr}S^{-1}\mathcal{C} = \{\text{right fractions of } S\}/\sim$. This means that the morphisms $\mathfrak{r}S^{-1}\mathcal{C}(X,Y)$ are spans in $\mathcal{C}$ where one of the maps are in S up to equivalence.

$$X \longleftarrow A \longrightarrow Y$$

This is well-defined by the previous results and the identity morphisms are the right fractions of the form:

$$X =\!= X =\!= X$$

Remark. Dually there is a category $\mathfrak{l}S^{-1}\mathcal{C}$ for a left multiplicative system S in a category $\mathcal{C}$. It is defined in the same manner as $\mathfrak{r}S^{-1}\mathcal{C}$, but with left fractions instead.

Det virker som at det som foregår her er galt. Jeg må først skaffe oversikt over hva som skjer

Remark. Given that S is right multiplicative, A right fraction from the object A to the object B can be described with a special kind of diagram. Let $A \downarrow S$ be the comma category of arrows from S starting at A and let $\delta : A \downarrow S \to \mathcal{C}$ be the forgetful functor, sending each arrow to its codomain. A morphism in $A \downarrow S$ from the objects (b, B') to (c, C') is a morphism $t : B' \to C'$ such that $b = ct$. We see that there is a correspondance between right fractions and elements in components of diagrams over $A \downarrow S$ such as $\mathcal{C}(\delta b, B) = \mathcal{C}(B', B)$ and right fractions. That is, let $f : \delta(b, B) \to B'$, then f can be regarded as fb^{-1}. A morphism from $\mathcal{C}(\delta c, B)$ to $\mathcal{C}(\delta b, B)$ is a morphism induced by a morphism from (b, B') to (c, C') in $A \downarrow S$. By the equivalence relation above we want to fractions fb^{-1} and gc^{-1} to be identified if there exists morphisms from $\mathcal{C}(\delta d, B)$ with maps $b' : (d, D') \to (b, B')$ and $c' : (d, D') \to (c, C')$ in $A \downarrow S$ such that $b' * f = c' * g$. This would be the same as saying that the right fractions are the coequalizer of the diagram $\mathcal{C}(\delta b, B) \coprod \mathcal{C}(\delta c, B) \rightrightarrows \mathcal{C}(\delta d, B)$. This observation motivates that the right fractions from A to B is described as the colimit of the functor $\mathcal{C}(\delta\_, B) : A \downarrow S \to SET$. Dually, if S is left multiplicative we get that the left fractions from A to B can be described as the colimit of the functor $\mathcal{C}(A, \rho\_) : S \downarrow B \to SET$. More details can be found in [**zisman**] and [**weibel**].

To ensure us that these categories $\mathfrak{r}S^{-1}\mathcal{C}$ does indeed exist there are many different criteria which we can place upon our assumptions. A natural restriction is to ensure that the colimits above exists as sets

Dette må også fikses etter at det over er fikset!!!

Definition 1.3.7. A multiplicative system S in a locally small category $\mathcal{C}$ is called locally small on the right if for every object $X : \mathcal{C}$ there is a set S_X of morphisms from S such that for every morphism $f : X_1 \to X$ in S there is a morphism $f' : X' \to X$ in S_X factoring thorugh f.
The dual of this definition will be called a locally small multiplicative system on the left. If it is both locally small on the left and the right, we will simply call it locally small.

Remark. If S is a left multiplicative system, then $S \downarrow A$ is a filtered category for every object A. Dually, if S is right multiplicative then $A \downarrow S$ is cofiltered.

Remark. Equipped with this notion we are now able to prove that the localizations exists as locally small categories. Locally small right multiplicative systems allows us to prove that the classes $\mathfrak{r}S^{-1}\mathcal{C}(X,Y)$ are sets. This can be seen as we can regard S_X as a small category. By using the right Ore condition and left cancellation we can extend S_X such that it is again cofiltered and admits the same colimit, i.e. $\varinjlim \mathcal{C}(\delta\_,B) : A \downarrow S \to Set \simeq \varinjlim \mathcal{C}(\delta\_,B) : S_A \to Set$.

Theorem 1.3.4. ***Gabriel-Zisman****. Let S be a locally small right multiplicative system of morphisms in a category $\mathcal{C}$. Then the category $\mathfrak{r}S^{-1}\mathcal{C}$ exists and it is the localization of $\mathcal{C}$ on S. This mean that there is an equivalence of categories $\mathcal{C}[S^{-1}] \simeq \mathfrak{r}S^{-1}\mathcal{C}$ together with a functor $q : \mathcal{C} \to \mathfrak{r}S^{-1}\mathcal{C}$ sending a morphism $f : X \to Y$ to the right fraction $f id_X^{-1}$.*

Proof. To prove the theorem one must show that q is a functor, and that it is universal. Suppose that $f : X \to Y$ and $g : Y \to Z$ are morphisms in $\mathcal{C}$. Then $q(gf) = (gf)id_X^{-1}$ and $q(g)q(f) = (gid_Y^{-1}) \circ (f id_X^{-1})$. Choose the compostion to be defined by the diagram below.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \\
X \arrow[r, "f"] \arrow[d, equal] & Y & \\
X & &
\end{tikzcd}$$

Observe that $(gid_Y^{-1}) \circ (f id_X^{-1}) = (gf)id_X^{-1}$, asserting the functoriality of q.
To see that q is universal let $\mathcal{D}$ be a category where every morphism of S is an isomorphism, and suppose there is a functor $F : \mathcal{C} \to \mathcal{D}$. Define a functor $\mathfrak{r}S^{-1}F : \mathfrak{r}S^{-1}\mathcal{C} \to \mathcal{D}$ by $\mathfrak{r}S^{-1}F(fs^{-1}) = F(f)F(s)^{-1}$. One may see that $F = \mathfrak{r}S^{-1}F \circ q$, it remains to show that it is well-defined. Suppose $fs^{-1} = gt^{-1}$, that means there is a diagram in $\mathcal{C}$ with the blue arrows in S.

$$\begin{tikzcd}
& W' \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & W \arrow[u, "w'"'] \arrow[d, "w''"] & Y \\
& W'' \arrow[ul, "t"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Thus there is a relationship in $\mathcal{D}$ such that $F(t) = F(sw')F(w'')^{-1}$ and $F(g) = F(fw')F(w'')^{-1}$. This again shows that

$$\begin{aligned}
&\mathfrak{r}S^{-1}F(gt^{-1}) = F(g)F(t)^{-1} \\
&\quad = F(fw')F(w'')^{-1}(F(fw')F(w'')^{-1})^{-1} = F(fw')F(w'')^{-1}F(w'')F(sw')^{-1} \\
&\qquad = F(f)F(w')F(w')^{-1}F(s)^{-1} = F(f)F(s)^{-1} = \mathfrak{r}S^{-1}F(fs^{-1})
\end{aligned}$$

It follows that $\mathfrak{r}S^{-1}F$ is well-defined and is unique by construction. $\square$

Corollary 1.3.4.1. *If S is a locally small left multiplicative system instead then $\mathfrak{l}S^{-1}\mathcal{C}$ is the localization of $\mathcal{C}$ on S.*
If moreover S is a locally small multiplicative system, then there is an equivalence of categories $\mathfrak{r}S^{-1}\mathcal{C} \simeq \mathfrak{l}S^{-1}\mathcal{C}$.

Proof. The first statement is dual to the theorem.
To see the other statement, note that both $\mathfrak{r}S^{-1}\mathcal{C}$ and $\mathfrak{l}S^{-1}\mathcal{C}$ are the universal categories where the morphisms of S are isomorphisms. Thus it follows that these categories have to be equivalent. $\square$

Remark. Since righthandedness of lefthandedness of the multiplicative system S doesn't affect the localization, one simply call the localization of a (left/right) multiplicative system for $S^{-1}\mathcal{C}$.

Remark. A morphisms $f : \mathcal{C}(X,Y)$ will be invertible in the localized category if it is in the same equivalence class as the identity, both id_X and id_Y. This forces a morphism f to be invertible in $S^{-1}\mathcal{C}$ if and only if there is $g,h : S$ such that $fg, hf : S$.

Proposition 1.3.5. *Let $\mathcal{C}$ be a category, and S a right multiplicative set of morphisms. The cannonical functor $q : \mathcal{C} \to S^{-1}\mathcal{C}$ commutes with finite limits.*

Proof. Let $T : \mathcal{D} \to \mathcal{C}$ be a diagram over a finite category $\mathcal{D}$. Then for any object $A : S^{-1}\mathcal{C}$ one may find the following equation.

$$\begin{aligned} S^{-1}\mathcal{C}(qA, q(\varprojlim T\_) &\simeq \varinjlim \mathcal{C}(\delta\_, \varprojlim T\_)) \\ &\simeq \varinjlim \varprojlim \mathcal{C}(\delta\_, T\_) \simeq \varprojlim \varinjlim \mathcal{C}(\delta\_, T\_) \simeq \varprojlim S^{-1}\mathcal{C}(qA, q(T\_)) \end{aligned}$$

The first isomorphism is given by the remark, the second is given by the representative nature of finite limits and the third isomorphism is given by that filtered colimits commute with finite limits. The colimits are filtered by the remark that S_A is cofiltered and that the functor $\mathcal{C}(\_, A)$ is contravariant. $\square$

I am not quite sure yet how this argument proves the statement I want to prove, but that can be figure out later. I also don't know the proof for why filtered colimits commute with finite colimits, but it is in Riehls book.

Proposition 1.3.6. *Let $\mathcal{C}$ be a category with a zero. That is an object which is both initial and terminal. Suppose that S is a right multiplicative system, then $q0$ is a zero object in $S^{-1}\mathcal{C}$.*

Proof. The claim that $q0$ is initial follows from that initial is a limit of a diagram over the empty category. To see that $q0$ is terminal one have to prove that every right fraction of the form $0f^{-1}$ is equivalent to $0id_A^{-1}$, where A is the codomain of f. This fact can be seen with the diagram below.

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, "f"] \arrow[dr, "0"] & \\ A \arrow[r, equal] & A \arrow[r, "0"] & 0 \end{tikzcd}$$

□

Proposition 1.3.7. *If $\mathcal{A}$ is an additive category and S is a right multiplicative system, then $S^{-1}\mathcal{A}$ is additive as well.*

Proof. From the previous propositions it is known that $q0$ is the zero object and that $q(A\times B)\simeq qA\times qB$. By proving that there is an addition induced by $\mathcal{A}$ and that q preserves this addition one obtains that the product is the biproduct induced by the maps in $\mathcal{A}$.
Suppose that there are fractions $fs^{-1}, gt^{-1} : S^{-1}\mathcal{C}(A,B)$. Define their addition by using the right Ore condition to find new morphisms f', g' and u such that $fs^{-1} = f'u^{-1}$ and $gt^{-1} = g'u^{-1}$.

$$fs^{-1} + gt^{-1} = (f' + g')u^{-1}$$

To prove that this is an addition one must prove that it is well defined; associativity, inverses and commutativity will be inherited from $\mathcal{A}$. Let $\bar{f}$, $\bar{g}$ and v be another choice provided by the right Ore condition. To summarize, the equations $\bar{f}v^{-1} = fs^{-1} = f'u^{-1}$ and $\bar{g}v^{-1} = gt^{-1} = g'u^{-1}$ have been established. In order to prove well-definedness, one must show that $(\bar{f}+\bar{g})v^{-1} - (f'+g')u^{-1} = 0$. By definition $(\bar{f}+\bar{g})v^{-1} - (f'+g')u^{-1} = \bar{f}v^{-1} - f'u^{-1} + \bar{g}v^{-1} - g'u^{-1}$. Proving that the whole sum is 0, is the same as proving that $\bar{f}v^{-1} + (-f')u^{-1} = (\bar{\bar{f}} - f'')w^{-1} = 0$. This can be done by writing out the diagrams after repeatedly applying the right Ore condition.

$$\begin{tikzcd}
\cdot \arrow[dr, dashed] \arrow[drrr, dashed, bend left, "p"] \arrow[dddr, dashed, bend right, "p"'] & & & & \\
& \cdot \arrow[r] \arrow[d] \arrow[dr, "w"] & \cdot \arrow[r] \arrow[d, "u"] & \cdot \arrow[r, "f"] \arrow[dl, "s"] & B \\
& \cdot \arrow[r, "v"] \arrow[d] & A & & \\
& \cdot \arrow[ur, "s"] \arrow[d, "f"] & & & \\
& B & & &
\end{tikzcd}$$

The line to the bottom represents $\bar{\bar{f}}$ and the line to the right represents f''. Using left cancellation on the common morphism s into A one obtains the morphism p, which relates the two fractions and make the sum go to zero.
It remains to show that $q : \mathcal{C} \to S^{-1}\mathcal{C}$ respects addition. Assume that $f, g : \mathcal{C}(X,Y)$, then

$$q(f+g) = (f+g)id_X^{-1} = f\,id_X^{-1} + g\,id_X^{-1} = qf + qg.$$

□

Corollary 1.3.7.1. *If $\mathcal{A}$ is abelian and S is a multiplicative system, then $S^{-1}\mathcal{A}$ is abelian as well.*

Kan skrive noe her for å løsne overgangen fra lokalisering til lokalisering av triangulerte kategorier

Definition 1.3.8. A triangulated functor $F : \mathcal{T} \to \mathcal{S}$ between two triangulated categories $(\mathcal{T}, T, \Delta_{\mathcal{T}}$ and $(\mathcal{S}, S, \Delta_{\mathcal{S}})$, is an additive functor along with a natural isomorphism $\phi_X : F(T(X)) \to S(F(X))$ such that $F(\Delta_{\mathcal{T}}) \subseteq \Delta_{\mathcal{S}}$. This means that for every triangle in $\mathcal{T}$ there is a triangle in $\mathcal{S}$.

Jeg må fikse funktor notasjonen min. Den er vanskelig å lese...

$$\begin{tikzcd} A \arrow[dr, "a"] & & & F(A) \arrow[dr, "F(a)"] & \\ & B \arrow[dl, "b"] & \Longrightarrow & & F(B) \arrow[dl, "F(b)"] \\ C \arrow[uu, "c", "T"'] & & & F(C) \arrow[uu, "F(c)"', "T"] & \end{tikzcd}$$

Definition 1.3.9. A triangulated subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$ is a full additive subcategory such that the inclusion functor is triangulated.

Definition 1.3.10. Let $F : \mathcal{S} \to \mathcal{T}$ be a triangulated functor. The kernel of F is defined to be the full subcategory $Ker(F)$ of $\mathcal{S}$ such that every object in $Ker(F)$ gets mapped to 0 by F. That is, $Ker(F)$ is the class of objects $\{K : \mathcal{S} | F(K) \simeq 0\}$.

Lemma 1.3.8. *The kernel of a triangulated functor $F : \mathcal{C} \to D$ is a triangulated subcategory.*

Proof. Let $X : KerF$, since F is a triangulated functor $CX : KerF$ as $F(CX) = D(FX) = D0 = 0$. As F is triangulated, one has that every triangle maps to a triangle. Let $X, Y : KerF$, then:

$$\begin{tikzcd} X \arrow[dr] & & & 0 \arrow[dr] & \\ & Y \arrow[dl] & \Longrightarrow & & 0 \arrow[dl] \\ Z \arrow[uu, "T"] & & & F(Z) \arrow[uu, "T"] & \end{tikzcd}$$

By TR3 and the 2 out of 3 property $F(Z) \simeq 0 \implies Z : KerF$. Thus $KerF$ is a triangulated subcategory of $\mathcal{C}$. □

Definition 1.3.11. A subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$ is called thick if it contains all the direct summands of its objects.

Lemma 1.3.9. *The kernel of a triangulated functor $F : \mathcal{C} \to \mathcal{D}$ is thick.*

Proof. Let $X \oplus Y : KerF$, since F is additive one may see that $0 \simeq F(X \oplus Y) \simeq F(X) \oplus F(Y)$, but then there is a splitmono $F(X) \to 0 \implies F(X) \simeq 0 \simeq F(Y)$. □

Lemma 1.3.10. *Let $F : \mathcal{C} \to \mathcal{D}$ be a triangulated functor. Suppose that $f : X \to Y$ is a morphism such that $F(f)$ is an isomorphism. Then the cone of f is in $KerF$.*

Proof. There is an isomorphism of triangles in $\mathcal{D}$, showing that the cone of f is in $KerF$.

$$\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, equal] & FY \arrow[r] \arrow[d, equal] & F(cone(f)) \arrow[r] \arrow[d, dashed, "\wr"] & FTX \arrow[d, equal] \\
FX \arrow[r, "Ff"] & FY \arrow[r] & 0 \arrow[r] & FTX
\end{tikzcd}$$

□

The goal for the rest of this section is to prove that there is a localization at any triangulated subcategory $\mathcal{S} \subseteq \mathcal{C}$. This localization will yield a functor $q : \mathcal{C} \to \mathcal{C}/\mathcal{S}$ such that $\mathcal{S} \subseteq Kerq$. There is a set of morphism $Mor_{\mathcal{S}}$ related to $\mathcal{S}$ such that this set is multiplicative.

Definition 1.3.12. Let $\mathcal{C}$ be a triangulated category and $\mathcal{S} \subseteq \mathcal{C}$ be a triangulated subcategory. Define the collection $Mor_{\mathcal{S}}$ to be a collection of morphisms in $\mathcal{C}$ such that for any $f : Mor_{\mathcal{S}}$ there is a triangle with $C : \mathcal{S}$.

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow TA$$

Remark. Every isomorphism is in $Mor_{\mathcal{S}}$. This is because isomorphisms are found in triangles $(A, B, 0, f, 0, 0)$ and $0 : \mathcal{S}$ for any triangulated subcategory.

Lemma 1.3.11. *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms. If any two of the morphisms f, g and gf are in $Mor_{\mathcal{S}}$ then so is the third.*

Proof. We are able to find three triangles in $\mathcal{C}$.

(1)
$$\begin{tikzcd}
X \arrow[dr, "f"] & \\
& Y \arrow[dl] \\
Z' \arrow[uu, "T"] &
\end{tikzcd}$$

(2)
$$\begin{tikzcd}
Y \arrow[dr, "g"] & \\
& Z \arrow[dl] \\
X' \arrow[uu, "T"] &
\end{tikzcd}$$

(3)
$$\begin{tikzcd}
A \arrow[dr, "g \circ f"] & \\
& C \arrow[dl] \\
Y' \arrow[uu, "T"] &
\end{tikzcd}$$

By the Octahedron axiom there exist another triangle in $\mathcal{C}$:

$$Z' \longrightarrow X' \longrightarrow Y' \longrightarrow TZ'$$

Note that f is in $Mor_{\mathcal{S}}$ if and only if $Z' : S$. WLOG assume that f and g is in $Mor_{\mathcal{S}}$, this can be done by the rotation axiom. Thus one may find the triangle in $\mathcal{S}$ by TR1 (Z', X', Y'') proving that $Y' \simeq Y''$.

$$\begin{tikzcd}
Z' \arrow[r] \arrow[d, equal] & X' \arrow[r] \arrow[d, equal] & Y' \arrow[r] \arrow[d, "\wr"] & TZ' \arrow[d, equal] \\
Z' \arrow[r] & X' \arrow[r] & Y'' \arrow[r] & TZ'
\end{tikzcd}$$

To see that gf is in $Mor_{\mathcal{S}}$ one can construct the triangle below with the isomorphism given above.

$$A \qquad C \qquad Y'' \qquad TA$$

□

Proposition 1.3.12. *Let $\mathcal{S} \subseteq \mathcal{C}$ be a triangulated subcategory, then $Mor_{\mathcal{S}}$ satisfies the Ore condition.*

Proof. To prove that a system satisfies the Ore condition there has to be a proof for both right and left condition. Luckily, the arguments presented here can be dualized to give a proof for the other condition. Thus there will only be presented a proof for the right Ore condition. Let $f : A \to C$ be in $Mor_{\mathcal{S}}$ and $g : B \to C$ in $\mathcal{C}$. Then one may form a homotopy pullback creating a homotopy cartesian square as below.

$$\begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd} \implies \begin{tikzcd} D \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "HO"] & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

By Lemma 1.2.5 there are triangles along this homotopy cartesian square identifying the cones. Since the cone of f is assumed to be in $\mathcal{S}$, the cone of f' is also in $\mathcal{S}$. This proves that $f' : Mor_{\mathcal{S}}$. □

Proposition 1.3.13. *For any parallell morphism $f, g : X \to Y$ in $\mathcal{C}$ the following are equivalent:*

1. *$sf = sg$ for some $s : Mor_{\mathcal{S}}$ starting at Y.*
2. *$ft = gt$ for some $t : Mor_{\mathcal{S}}$ ending at X.*
3. *$f - g$ factors through an object $C : \mathcal{S}$.*

Proof. (1. $\iff$ 3.): Suppose that there exists an $s : Y \to Z$ such that $s(f - g) = 0$. By TR1 there is a triangle $Y \xrightarrow{s} Z \xrightarrow{Ts'} TC \longrightarrow TY$ and a long exact sequence.

$$\mathcal{C}(X, C) \xrightarrow{s'_*} \mathcal{C}(X, Y) \xrightarrow{s_*} \mathcal{T}(X, Z)$$

$$p \xmapsto{s'_*} f - g \xmapsto{s_*} 0$$

Since $s(f - g) = 0$ there exists a $p : \mathcal{C}(X, C)$ such that $f - g = s'_* p$. By definition, $s : Mor_{\mathcal{S}} \iff C : \mathcal{S}$, but $s : Mor_{\mathcal{S}} \implies f - g$ factors through C, and vice versa. (2. $\iff$ 3.): This argument is dual. □

This has shown that $Mor_{\mathcal{S}}$ is a multiplicative system, and Theorem 1.3.4 say that the localization exists given that $Mor_{\mathcal{S}}$ is locally small. The category $Mor_{\mathcal{S}}^{-1}$ will be denoted as $\mathcal{C}/\mathcal{S}$ and it is called the Verdier quotient. As $\mathcal{C}$ is additive, it is known that $\mathcal{C}/\mathcal{S}$ is additive as well by Proposition 2.20. The remaining part is to show that $\mathcal{C}/\mathcal{S}$ is triangulated and that localization functor $q : \mathcal{C} \to \mathcal{C}/\mathcal{S}$ is a triangulated functor.

Theorem 1.3.14. *Let $\mathcal{S} \subseteq \mathcal{C}$ be triangulated categories. Then the Verdier quotient $\mathcal{C}/\mathcal{S}$ together with the functor $q : \mathcal{C} \to \mathcal{C}/\mathcal{S}$ is the universal triangulated category where morphisms in $Mor_{\mathcal{S}}$ are isomorphisms.*

Proof. The triangulation on $\mathcal{C}/\mathcal{S}$ is defined as the following. Let $C/S : \mathcal{C}/\mathcal{S} \to \mathcal{C}/\mathcal{S}$ be the additive autoequivalence defined by $C/S(A) = C(A)$. Since $q : \mathcal{C} \to \mathcal{C}/\mathcal{S}$ maps every object to itself it follows that $q(C(A)) \simeq C/S(A) = C/S(q(A))$, and define $\Delta_{\mathcal{C}/\mathcal{S}} \supseteq q(\Delta_{\mathcal{C}})$ such that $\Delta_{\mathcal{C}/\mathcal{S}}$ has every isomorphism class of $q(\Delta_{\mathcal{C}})$.

$$\begin{tikzcd} qX \arrow[dr] & & & X \arrow[dr] & \\ & qY \arrow[dl] & \Longleftarrow & & Y \arrow[dl] \\ qZ \arrow[uu, "T"] & & & Z \arrow[uu, "T"] & \end{tikzcd}$$

Then by definition q is triangulated if the category $\mathcal{C}/\mathcal{S}$ is triangulated. By definition, the triangles are closed under isomorphisms, $(X, X, 0, id_X, 0, 0)$ is a triangle, and TR2 holds. Thus it remains to show TR1 and TR4 (TR3 is implied by the other axioms). To prove TR1, let $fs^{-1} : \mathcal{C}/\mathcal{S}(qW, qY)$. Expand $f : \mathcal{C}(X, Y)$ to a triangle in $\mathcal{C}$ with TR1, it will induce a triangle in $\mathcal{C}/\mathcal{S}$.

$$qX \xrightarrow{fid_X^{-1}} qY \xrightarrow{gid_Y^{-1}} qZ \xrightarrow{hid_Z^{-1}} qTX$$

There is an isomorphism to the following candidate triangle from the induced triangle, proving TR1.

$$\begin{tikzcd} qX \arrow[r, "fid_X^{-1}"] \arrow[d, "sid_X^{-1}", "\wr"'] & qY \arrow[r, "gid_Y^{-1}"] \arrow[d, equal] & qZ \arrow[r, "hid_Z^{-1}"] \arrow[d, equal] & qTX \arrow[d, "(Ts)id_{Tx}^{-1}", "\wr"'] \\ qW \arrow[r, "fs^{-1}"] & qY \arrow[r, "gid_Y^{-1}"] & qZ \arrow[r, "(Ts)hid_Z^{-1}"] & qTW \end{tikzcd}$$

To show the Octahedron axiom, suppose that there are three triangles in $\mathcal{C}/\mathcal{S}$. By construction, these triangles can be chosen such that only the first map is a fraction up to isomorphism of triangles.

$$(1)\ \begin{tikzcd} Z \arrow[r, "t'"] & X \arrow[dl, "s"'] \arrow[dd, "a"] \\ A \arrow[dr, "as^{-1}"] & \\ & B \arrow[dl, "x"] \\ C' \arrow[uu, "x'", "T"'] & \end{tikzcd} \quad (2)\ \begin{tikzcd} & Y \arrow[dl, "t"'] \arrow[dd, "b"] \\ B \arrow[dr, "bt^{-1}"] & \\ & C \arrow[dl, "y"] \\ A' \arrow[uu, "y'", "T"'] & \end{tikzcd} \quad (3)\ \begin{tikzcd} & Z \arrow[dl, "st'"'] \arrow[dd, "ba'"] \\ A \arrow[dr, "b \circ a"] & \\ & C \arrow[dl, "z"] \\ B' \arrow[uu, "z'", "T"'] & \end{tikzcd}$$

This is possible, as when composing the fractions from A to B and B to C one may find an object Z as in the diagram by using the Ore condition. To illustrate with triangle (1), there is a correspondance of triangles in $\mathcal{C}/\mathcal{S}$ and $\mathcal{C}$ by the following isomorphism.

$$\begin{tikzcd}
Z \arrow[r, "at'"] \arrow[d, "t", "\cong"'] & B \arrow[r] \arrow[d, equal] & Z' \arrow[r] \arrow[d, dashed, "\wr"] & TZ \arrow[d, equal] \\
X \arrow[r, "a"] \arrow[d, "s"] & B \arrow[r] & C' \arrow[r] & TX \arrow[d, "Ts"] \\
A & & & TA
\end{tikzcd}$$

The result of the octahedron axiom follows as one instead consider the triangles found by the composition of morphisms as below.

$$\begin{tikzcd}
Z \arrow[d, "f'"] \arrow[dr, "bf'"] & \\
Y \arrow[r, "b"] & C
\end{tikzcd}$$

$\square$

Proposition 1.3.15. *Let $\mathcal{S} \subseteq \mathcal{C}$ be triangulated categories. If* $0 : X \to 0$ *is an isomorphism in $\mathcal{C}/\mathcal{S}$, then there is an object Y such that $X \oplus Y : \mathcal{S}$.*

Proof. If $0 : X \to 0$ is invertible, then there exist a map $0 : 0 \to Y$, such that $0 : X \to Y$ is in $Mor_{\mathcal{S}}$. By definition $X \oplus Y$ is in $\mathcal{S}$. $\square$

This proposition shows that the kernel of $q : \mathcal{C} \to \mathcal{C}/\mathcal{S}$ is the smallest thick subcategory of $\mathcal{C}$ such that $\mathcal{C}/Kerq$ is the universal category where every morphism in $Mor_{\mathcal{S}}$ is an isomorphism. For this reason $\widehat{\mathcal{S}} = Kerq$ is called the thick closure of $\mathcal{S}$.

1.4 Universal Homological Embedding

Do Yoneda embedding into functor categories.

Chapter 2

Exact Categories

I can maybe write some of the history of the development of the idea of exact categories.

2.1 Definitions and First Properties

This section will focus on defining what an exact category is and its elementary properties. The axiom dubbed as "the obscure axiom" will be proved and hopefully be seen to not be as obscure as its name suggest. Some "short" variants of some homological diagram lemmata will also be proved.
To start with exact categories one should first take a look towards the abelian categories first. Short exact sequences are of great interest, and they can be characterized with two morphisms $p : A \to B$ and $q : B \to C$ such that p is the kernel of q and q is the cokernel of p. This leads to the first definition.

Definition 2.1.1. Let $\mathcal{A}$ be an additive category. A kernel-cokernel pair is a pair of maps (p, q) such that p is the kernel of q and q is the cokernel of p. A morphism of kernel-cokernel pairs (p, q) and (p', q') is a triple (f, g, h) such that the following diagram commutes. An isomorphism of a kernel-cokernel pair is a triple in which each morphism is an isomorphism.

$$\begin{tikzcd}
A \arrow[r, tail, "p"] \arrow[d, two heads, "f"] & B \arrow[r, "q"] \arrow[d, "g"] & C \arrow[d, "h"] \\
A' \arrow[r, tail, "p'"] & B' \arrow[r, two heads, "q'"] & C'
\end{tikzcd}$$

Lemma 2.1.1. *Let (p, q) be a kernel-cokernel pair, then the image and coimage of p exists and are isomorphic. I.e. this diagram exists, such that the left square is a push-out and the right square is a pull-back:*

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[dr, "0"'] & A \arrow[r, tail, "p"] \arrow[d, two heads] & B \arrow[r, two heads, "q"] & C \\
& Coim(p) \arrow[r, tail, two heads, "iso"] & Im(p) \arrow[u, tail] \arrow[ur, "0"] &
\end{tikzcd}$$

Proof. Since (p,q) is a kernel-cokernel pair one may see that the first simplex is bicartesian and the second simplex is a push-out.

$$
\begin{tikzcd}
A \arrow[r, tail, "p"] \arrow[dr, "0"'] & B \arrow[d, two heads, "q"] \\
 & C
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r, "0"] \arrow[dr, "0"'] & A \arrow[d, equal] \\
 & A
\end{tikzcd}
$$

Thus $Im(p) = Coim(p) = A$, asserting the isomorphism as the identity in the diagram.

$$
\begin{tikzcd}
0 \arrow[r, "0"] \arrow[dr, "0"'] & A \arrow[r, tail, "p"] \arrow[d, equal] & B \arrow[r, two heads, "q"] & C \\
 & A \arrow[r, equal] & A \arrow[u, tail, "p"] \arrow[ur, "0"'] &
\end{tikzcd}
$$

□

Corollary 2.1.1.1. *Suppose that (p,q) is a kernel-cokernel pair. If p is an epimorphism, then p is an isomorphism.*

Definition 2.1.2. An exact structure for an additive category $\mathcal{A}$ is a class $\mathcal{E}$ of kernel-cokernel pairs which are closed under isomorphisms. A pair $(p,q) : \mathcal{E}$ is called a conflation, here p is called an inflation and q is called a deflation. $(\mathcal{A},\mathcal{E})$ is called exact when the following axioms holds:

- (QE0) $\forall A : \mathcal{A}$, id_A is both an inflation and a deflation.
- (QE1) Both inflations and deflations are closed under composition.
- (QE2) The push-out of an inflation is an inflation.
- (QE2^{op}) The pull-back of a deflation is a deflation.

An exact category is the additive category $\mathcal{A}$ together with an exact structure $\mathcal{E}$.

Remark. Decorated arrows will be used when writing diagrams to indicate that a morphism is either an inflation or a deflation. A tail with a circle means inflation: $A \rightarrowtail\!\!\circ\!\!\rightarrow B$. Double heads with a circle means deflation: $A \multimap\!\!\twoheadrightarrow B$. $(QE2^{*})$ axioms can now be written as the diagrams below.

$$
\begin{tikzcd}
A \arrow[r, tail, "\circ" marking] \arrow[d] & B \arrow[d] \\
C \arrow[r, tail, "\circ" marking] & D \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, two heads, "\circ" marking] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d] \\
C \arrow[r, two heads, "\circ" marking] & D
\end{tikzcd}
$$

Remark. In litterature, inflations are also referred to as admissable monomorphisms, and deflations are referred to as admissable epimorphisms while conflations are also called short exact sequences.

Remark. Observe that the axioms for an exact structure is self dual. This allows for reasoning with duality, as a category has an exact structure $(\mathcal{A},\mathcal{E})$ if and only if $(\mathcal{A}^{op},\mathcal{E}^{op})$ has an exact structure.

Remark. For any category $\mathcal{C}$, there is a category $\mathcal{C}^{\rightarrow} = \mathcal{C} \downarrow \mathcal{C}$ consisting of arrows and $\mathcal{C}^{\rightarrow\rightarrow} = \mathcal{C} \downarrow \mathcal{C} \downarrow \mathcal{C}$ consisting of pairs of composable arrows. If $\mathcal{A}$ is additive, then $\mathcal{A}^{\rightarrow}$ and $\mathcal{A}^{\rightarrow\rightarrow}$ are additive as well. It can be seen that $\mathcal{E}$ may be considered as an extension closed additive subcategory of $\mathcal{A}^{\rightarrow\rightarrow}$.

Har jeg lyst til å beholde disse eksemplene? Føler de faller litt flatt...

Example. Any abelian category is exact with every short-exact sequence as the exact structure.

Example. Any additive category is exact with every split short-exact sequence as the exact structure.

Lemma 2.1.2. *The map* $0 : 0 \to A$ *is an inflation. Dually, the map* $0 : A \to 0$ *is a deflation.*

Proof. Consider the diagram $0 \overset{0}{\rightarrowtail} A \overset{id_A}{\twoheadrightarrow} A$. The left morphism is the kernel of the right morphism making a kernel-cokernel pair $(0, id_A)$. The identity id_A is assumed to be a deflation, implying that the pair is a conflation. □

Remark. It can be seen that isomorphisms are deflations. Let $f : A \to B$ be an isomorphism, then there are two kernel-cokernel pairs: $(0, id_A)$ and $(0, f)$. Between these there is an isomorphism which is the triple $(0, id_A, f^{-1})$. As the conflations are closed under isomorphism, $(0, f)$ is a conflation, making f into a deflation. By dualizing this argument, f is also an inflation.

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[d, "0"] & A \arrow[r, tail, two heads, "f"] \arrow[d, "id_A"] & B \arrow[d, tail, two heads, "f^{-1}"] \\
0 \arrow[r, tail, "0"] & A \arrow[r, two heads, "id_A"] & A
\end{tikzcd}$$

Corollary 2.1.2.1. *A kernel-cokernel pair* (i, p) *found as a split short-exact sequence (1) is a conflation.*

$$(1) \quad A \overset{i}{\rightarrowtail} A \oplus B \overset{p}{\twoheadrightarrow} B$$

Proof. In a category with an initial object the coproduct can be thought of as the push-out with the initial in the upper right corner. This can be assembled into push-out (1). By the lemma the zero morphisms are inflations, asserting that i and i' are inflations by (QE2). Thus there are conflations (i, p) and (i', p').

$$(1) \quad \begin{tikzcd}
0 \arrow[r, tail, "0"] \arrow[d, tail, "0"] & A \arrow[d, tail, "i"] \\
B \arrow[r, tail, "i'"] & A \oplus B
\end{tikzcd}$$

□

Corollary 2.1.2.2. *The direct sum of conflations is a conflation. I.e. there is a diagram:*

$$\begin{array}{c} A \overset{i}{\rightarrowtail} B \overset{p}{\twoheadrightarrow} C \quad \& \quad A' \overset{i'}{\rightarrowtail} B' \overset{p'}{\twoheadrightarrow} C' \\ \Downarrow \\ A \oplus A' \overset{i \oplus i'}{\rightarrowtail} B \oplus B' \overset{p \oplus p'}{\twoheadrightarrow} C \oplus C' \end{array}$$

Proof. Start by only considering the conflation (i, p). For any $D : \mathcal{A}$ there is a conflation $(i \oplus id_D, p \oplus 0)$, drawn as the diagram.

$$A \oplus D \overset{\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}}{\rightarrowtail} B \oplus D \overset{\begin{pmatrix} p & 0 \end{pmatrix}}{\twoheadrightarrow} C$$

As kernels and cokernels are preserved by direct sums, this pair is in fact a kernel-cokernel pair. The epimorphism is a deflation as it can be factored by the deflations:

$$B \oplus D \overset{\begin{pmatrix} 1 & 0 \end{pmatrix}}{\twoheadrightarrow} B \overset{p}{\twoheadrightarrow} C$$

Thus it is seen that $(i \oplus id_D, p \oplus 0)$ is a conflation, and dually $(i \oplus 0, p \oplus id_D)$ is also a conflation. To finish off the proof it is seen that the morphism $i \oplus i'$ factors as $i \oplus id_{A'} \circ id_A \oplus i'$, asserting that it is an inflation by (QE1). By dualizing the argument, one get that the direct sum of conflations is a conflation. □

Definition 2.1.3. A square is bicartesian if it is both a pull-back and a push-out.

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \; \lrcorner"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

Proposition 2.1.3. *The following statements are equivalent:*

1. *The square (1) is a push-out.*
2. *The sequence (2) is a conflation.*
3. *The square (1) is bicartesian.*
4. *The square (1) is a part of the commutative diagram (3)*

$$(1)\ \begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[d, "g"] \\ C \arrow[r, tail, "j"] & D \end{tikzcd} \quad (2)\ A \overset{\begin{pmatrix} i \\ -f \end{pmatrix}}{\rightarrowtail} B \oplus C \overset{\begin{pmatrix} g & j \end{pmatrix}}{\twoheadrightarrow} D \quad (3)\ \begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[r, two heads, "p"] \arrow[d, "g"] & E \arrow[d, equal] \\ C \arrow[r, tail, "j"] & D \arrow[r, two heads, "q"] & E \end{tikzcd}$$

Before the proof for this proposition there will be presented a useful lemma, which will be proved first.

Lemma 2.1.4. *Assume that there is a commutative square (1) and an associatied sequence (2). (1) is a push-out square if and only if* $\begin{pmatrix} p & q \end{pmatrix}$ *is the cokernel of the morphism* $\begin{pmatrix} i \\ -j \end{pmatrix}$

$$
\text{(1)}\quad
\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, "p"] \\
C \arrow[r, "q"] & D
\end{tikzcd}
\qquad
\text{(2)}\quad
A \xrightarrow{\begin{pmatrix} i \\ -j \end{pmatrix}} B \oplus C \xrightarrow{\begin{pmatrix} p & q \end{pmatrix}} D
$$

Proof. For any test object T and two maps $t_1 : B \to T$ and $t_2 : C \to T$, one may construct the diagrams for the universal properties of both the cokernel and the push-out. It is seen that these diagrams are equivalent, proving the lemma.

Hva prøver jeg å si her?

$$
\begin{tikzcd}
A \arrow[r, "\begin{pmatrix} i \\ -f \end{pmatrix}"] \arrow[dr, "0"'] & B \oplus C \arrow[d, "\begin{pmatrix} g & j \end{pmatrix}"] \arrow[dr, bend left, "\begin{pmatrix} t_1 & t_2 \end{pmatrix}"] & \\
& D \arrow[r, dashed, "t'"] & T
\end{tikzcd}
\quad \Leftrightarrow \quad
\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[ddr, bend left, "t_1"] & \\
C \arrow[r, "j"] \arrow[drr, bend right, "t_2"'] & D \arrow[dr, dashed, "t'"] & \\
& & T
\end{tikzcd}
$$

$\square$

Corollary 2.1.4.1. *For the same diagrams (1) and (2) as above the dual statement is also true. (1) is a pull-back square if and only if* $\begin{pmatrix} i \\ -j \end{pmatrix}$ *is kernel of the morphism* $\begin{pmatrix} p & q \end{pmatrix}$*. Thus it follows that (1) is bicartesian (i.e. both a pull-back and a push-out) if and only if the morphisms make a kernel-cokernel pair.*

Proof. **of Proposition 3.3** 1. $\Rightarrow$ 2.: By the previous lemma it is known that $\begin{pmatrix} g & j \end{pmatrix}$ is the cokernel of $\begin{pmatrix} i \\ -j \end{pmatrix}$. Thus proving that $\begin{pmatrix} i \\ -j \end{pmatrix}$ is an inflation, will prove that the pair is a conflation.
Observe that the morphism $\begin{pmatrix} i \\ -f \end{pmatrix}$ can be factored through the sequence.

$$
A \overset{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\rightarrowtail} A \oplus C \underset{\cong}{\overset{\begin{pmatrix} 1 & 0 \\ -f & 1 \end{pmatrix}}{\rightarrowtail\!\!\!\!\twoheadrightarrow}} A \oplus C \overset{\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}}{\rightarrowtail} B \oplus C
$$

By corollary 3.2.1 the first map is an inflation, as the second map is an isomorphism it is also an inflation and the last map is the direct sum of two inflations. Thus the composite of all these maps is an inflation by (QE1), proving the first implication.
2. $\Rightarrow$ 3.: This follows from corollary 3.4.1.
3 $\Rightarrow$ 1.: This is by definition.
1. $\Rightarrow$ 4.: Let p be the cokernel of i, then form the diagram below using the pushout property.

$$
\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[d, "g"] \arrow[ddr, bend left, two heads, "p"] & \\
C \arrow[r, tail, "j"] \arrow[drr, bend right, "0"] & D \arrow[dr, dashed, two heads, "p'"] & \\
 & & T
\end{tikzcd}
$$

p' is an epimorphism as $p = p'g$ is epi. To prove that p' is the cokernel of j let T' be another test object with a map $t' : D \to T'$ such that $0 = t'j$. By doing some diagram chases one may see that $0 = t'jf = t'gi$, thus by the universal property of p the morphism $t'g$ factors through T such that $t'g = tp$ for some unique t. This shows that $t'g = tp'g = tp$, and $t'j = tp'j = 0$. Since t' is the unique morphism satisfying this equation we demand that $t' = tp'$. t is also unique, for if there exist another map h such that $tp' = hp'$, then $t = h$ as p' is epic. The unique existence proves the universal property, and p' is the cokernel of j.

$$
\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[r, two heads, "p"] \arrow[d, "g"] & T \arrow[d, equal] \\
C \arrow[r, tail, "j"] \arrow[drr, bend right, "0"] & D \arrow[r, dashed, two heads, "p'"] \arrow[dr, "t'"] & T \arrow[d, dashed, "t"] \\
 & & T'
\end{tikzcd}
$$

Hva skjer egentlig her?

4. $\Rightarrow$ 2.: Start by taking the pullback of p and p' using $(QE2^{op})$. The diagrams below are determined by using the dual statement of the last implication.

$$
\begin{tikzcd}
 & A \arrow[r, equal] \arrow[d, tail, "i'"] & A \arrow[d, tail, "i"] \\
C \arrow[r, tail, "j'"] \arrow[d, equal] & E \arrow[r, two heads, "q'"] \arrow[d, two heads, "q"] & B \arrow[d, two heads, "p"] \arrow[dl, dashed, "g"'] \\
C \arrow[r, tail, "j"] & D \arrow[r, two heads, "p'"] & T
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
B \arrow[drr, bend left, equal] \arrow[dr, dashed, "k"] \arrow[ddr, bend right, "g"'] & & \\
 & E \arrow[r, two heads, "q'"] \arrow[d, two heads, "q"] & B \arrow[d, two heads, "p"] \\
 & D \arrow[r, two heads, "p'"] & T
\end{tikzcd}
$$

From these diagrams one can see that q' is a split-epimorphism. The composite $q'(id_E - kq') = q' - q'kq' = q' - q' = 0$ as q' is split-epi, so $(id_E - kq')$ factors over j' as in the following diagram.

$$
\begin{tikzcd}
E \arrow[dr, dashed, "l"] \arrow[drr, bend left, "id_E - kq'"] & & \\
 & C \arrow[r, tail, "j'"] \arrow[dr, "0"] & E \arrow[d, two heads, "q'"] \\
 & & B
\end{tikzcd}
$$

From these diagrams one may find three different equations:

- $0 = k - k = k - kq'k = (id_E - kq')k = j'lk \implies lk = 0$ as j' is monic

- $j'lj' = (id_E - kq')j' = j' \implies lj' = id_C$ as j' is monic
- $jli' = (qj')li' = q(id_E - kq')i' = -(qk)(q'i') = -gi = -jf \implies li' = -f$ as j is monic

The morphisms $\begin{pmatrix} k & j' \end{pmatrix}$ and $\begin{pmatrix} q' \\ l \end{pmatrix}$ are inverses:

- $\begin{pmatrix} k & j' \end{pmatrix}\begin{pmatrix} q' \\ l \end{pmatrix} = kq' + j'l = kq' + id_E - kq' = id_E$
- $\begin{pmatrix} q' \\ l \end{pmatrix}\begin{pmatrix} k & j' \end{pmatrix} = \begin{pmatrix} q'k & q'j' \\ lk & lj' \end{pmatrix} = \begin{pmatrix} id_B & 0 \\ 0 & id_C \end{pmatrix}$

Thus there is an isomorphism of kernel-cokernel pairs $(id_A, , \begin{pmatrix} q' \\ l \end{pmatrix} \begin{pmatrix} k & j' \end{pmatrix})$,

from $(\begin{pmatrix} i \\ -f \end{pmatrix}, \begin{pmatrix} f' & i' \end{pmatrix})$ to (i', q). This proves 2. ☐

Corollary 2.1.4.2. *The pull-back of an inflation along a deflation is an inflation.*

$$\begin{tikzcd}
A \arrow[r, tail, "i'"] \arrow[d, two heads, "e'"] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d, two heads, "e"] \\
C \arrow[r, tail, "i"] & D
\end{tikzcd}$$

Proof. By (QE2) this pullback exists, as there is a deflation in the pullback. Extend the diagram by adding the deflation of the inflation in the following manner.

$$\begin{tikzcd}
 & T \arrow[dl, dashed, "t'"'] \arrow[d, "t"] \arrow[dr, bend left, "0"] & \\
A \arrow[r, tail, "i'"] \arrow[d, two heads, "e'"] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[r, two heads, "pe"] \arrow[d, two heads, "e"] \arrow[dl, dashed, "t''"'] & C \arrow[d, equal] \\
C \arrow[r, tail, "i"] & D \arrow[r, two heads, "p"] & C
\end{tikzcd}$$

pe is a deflation by (QE1), and i' is a mono as a limit of a mono is a mono. The goal is to prove that i' is the kernel of pe. Let T be a test object such that $pet = 0$, then if follows that te factorizes over i, such that we can apply the universal property of the pullback to factorize te over i'. Uniqueness of t' is achieved with i' being monic. This proves that (i', pe) is a conflation. ☐

Theorem 2.1.5. ***The obscure axiom.*** *Assume that $i : A \to B$ is a morphism with a cokernel. If there is a morphism $j : B \to C$ such that ji is an inflation, then i is an inflation.*

Proof. ☐

Proof. Let $p : B \to D$ be the cokernel of i. Start by forming the push-out of i and ji.

$$\begin{tikzcd}
A \arrow[r, tail, "ji"] \arrow[d, "i"] & C \arrow[d] \\
B \arrow[r, tail] & E \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

By proposition 3.3 $\begin{pmatrix} i \\ ji \end{pmatrix}$ is an inflation. $\begin{pmatrix} i \\ 0 \end{pmatrix} = \begin{pmatrix} id_B & 0 \\ -j & id_C \end{pmatrix} \begin{pmatrix} i \\ ji \end{pmatrix}$, this is an inflation by (QE1) as the 2x2 matrix is an isomorphism. Observe that the cokernel of this map is $\begin{pmatrix} k & 0 \\ 0 & id_C \end{pmatrix}$. The final trick will be to show that there is a pullback square, and then use (QE2) to say that k is a deflation.

$$\begin{tikzcd}
T \arrow[drr, bend left, "t_1"] \arrow[dr, dashed, "t"] \arrow[ddr, bend right, "\begin{pmatrix} t_2 \\ 0 \end{pmatrix}"'] & & \\
& B \arrow[r, "k"] \arrow[d, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & D \arrow[d, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] \\
& B \oplus C \arrow[r, two heads, "\begin{pmatrix} k & 0 \\ 0 & id_C \end{pmatrix}"'] & D \oplus C
\end{tikzcd}$$

Note that setting $t = t_2$ one get the universal property. This is well defined as $kt_2 = t_1$ by assumption, thus $kt = t_1$. This is what is needed to prove that the square is a pullback, proving the obscure axiom. □

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Remark. Write a bit about the dual of the obscure axiom.

Lemma 2.1.6. *Let (p,q) and (p',q') be the conflations:*

- (p,q): $A \rightarrowtail^{p} B \xrightarrow{q} \twoheadrightarrow C$
- (p',q'): $A' \rightarrowtail^{p'} B' \xrightarrow{q'} \twoheadrightarrow C'$

A morphism of the conflations $(f,g,h) : (p,q) \to (p',q')$ factors through the conflation $A \rightarrowtail D \twoheadrightarrow C'$ such that the following diagram exists, where $g = g_2 g_1$.

$$\begin{tikzcd}
A \arrow[r, tail, "p"] \arrow[d, "f"] \arrow[dr, phantom, "\ulcorner", very near end] & B \arrow[r, two heads, "q"] \arrow[d, "g_1"] & C \arrow[d, equal] \\
A' \arrow[r, tail] \arrow[d, equal] & D \arrow[r, two heads] \arrow[d, "g_2"] \arrow[dr, phantom, "\ulcorner", very near start] & C \arrow[d, "h"] \\
A' \arrow[r, tail, "p'"] & B' \arrow[r, two heads, "q'"] & C'
\end{tikzcd}$$

Proof. Observe that the upper part of the diagram is made by taking a push-out of p and f, where the right part is gained from proposition 3.3. Combine the upper part with the lower part using the push-out property.

$$
\begin{tikzcd}
A \arrow[r, tail, "p"] \arrow[d, "f"'] & B \arrow[d, "g_1"] \arrow[ddr, bend left, "g"] & \\
A' \arrow[r, tail] \arrow[drr, bend right, "p'"'] & D \arrow[dr, dashed, "g_2"] & \\
& & B'
\end{tikzcd}
\implies
\begin{tikzcd}
A \arrow[r, tail, "p"] \arrow[d, "f"'] & B \arrow[r, two heads, "q"] \arrow[d, "g_1"] & C \arrow[d, equal] \\
A' \arrow[r, tail, "a"] \arrow[d, equal] & D \arrow[r, two heads, "c"] \arrow[d, "g_2"] & C \arrow[d, "h"] \\
A' \arrow[r, tail, "p'"] & B' \arrow[r, two heads, "q'"] & C'
\end{tikzcd}
$$

It remains to show that the lower right square is commutative, then use the dual of proposition 3.3 to see that the square is bicartesian. Note that $q = cg_1$ by prop 3.3 thus $q'g_2g_1 = q'g = hq = hcg_1$. Uniqueness of the push-out property asserts that $hc = q'g_2$. □

Corollary 2.1.6.1. ***The short five lemma***. *Suppose that there is a morphism of conflations (f, g, h) as above. If f and h are isomorphisms, then g is an isomorphism.*

Proof. Since f is an isomorphism it is at least an inflation, thus g_1 is an inflation by (QE2). As colimits preserve epis, g_1 is also an epimorphism. Corollary 3.1.1 states that g_1 is an iso, and dually that g_2 is an iso. Since isomorphisms are closed under composition it follows that g is an isomorphism.

$$
\begin{tikzcd}
A \arrow[r, tail, "p"] \arrow[d, tail, "f", "\wr"'] & B \arrow[r, two heads, "q"] \arrow[d, tail, "g_1", "\wr"'] & C \arrow[d, equal] \\
A' \arrow[r, tail] \arrow[d, equal] & D \arrow[r, two heads] \arrow[d, two heads, "g_2", "\wr"'] & C \arrow[d, two heads, "h", "\wr"'] \\
A' \arrow[r, tail, "p'"] & B' \arrow[r, two heads, "q'"] & C'
\end{tikzcd}
$$

□

Lemma 2.1.7. ***Noethers isomorphism lemma***. *Suppose there is a diagram with rows as conflations and the first column as a conflation. Then the final column is also a conflation.*

$$
\begin{tikzcd}
A \arrow[r, tail] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d, tail] & X \arrow[d, tail, dashed] \\
A \arrow[r, tail] & C \arrow[r, two heads] \arrow[d, two heads] & Y \arrow[d, two heads, dashed] \\
& Z \arrow[r, equal] & Z
\end{tikzcd}
$$

Proof. Assume that only the solid part of the diagram above exists. By the universal property of cokernels, the upper dashed map exists, and by the dual of proposition 3.3 the square is bicartesian. This infers that the upper dashed map is an inflation, and since the square is a push-out it follows that the lower dashed map exists such that the final column is a conflation by proposition 3.3. □

2.2 The Stable Frobenis Category

Definition 2.2.1. Let $(\mathcal{A},\mathcal{E})$ and $(\mathcal{A}',\mathcal{E}')$ be two exact categories. A functor $F : (\mathcal{A},\mathcal{E}) \to (\mathcal{A}',\mathcal{E}')$ is called exact if it is additive and $F(\mathcal{E}) \subseteq \mathcal{E}'$. That is to say that conflations gets mapped to conflations. One speaks of a reflective exact functor for whenever the pair (Fp, Fq) is a conflation, then (p,q) is a conflation.

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Definition 2.2.2. Let $(\mathcal{A},\mathcal{E})$ be an exact category. An object $P : \mathcal{A}$ is called projective if $\mathcal{A}(P,\_) : (\mathcal{A},\mathcal{E}) \to \mathbf{Ab}$ is an exact functor. Objects $I : \mathcal{A}$ are called injective whenever $\mathcal{A}(\_,I) : \mathcal{A}^{op} \to \mathbf{Ab}$ is an exact functor.

Remark. In the case of exact functors $F : (\mathcal{A},\mathcal{E}) \to \mathbf{Ab}$, one generally speaks of a functor which maps conflations to short exact sequences.

Remark. The hom-functor is called left-exact. This means that conflations gets mapped to sequences which is only exact in the first two terms.

Proposition 2.2.1. *Let $(\mathcal{A},\mathcal{E})$ be an exact category. $P : \mathcal{A}$ is projective if and only if for every deflation $q : A \to B$ and morphism $f : P \to B$ there is a morphism $f' : P \to A$ rendering the diagram below commutative.*

$$\begin{tikzcd} & P \arrow[d, "f"] \arrow[dl, dashed, "f'"'] \\ A \arrow[r, two heads, "q"'] & B \end{tikzcd}$$

Proof. Suppose that P is projective, then $\mathcal{A}(P,\_)$ is an exact functor. Let $(p : A \to B, q : B \to C)$ be a conflation, then there is a short exact sequence.

$$0 \xrightarrow{0} \mathcal{A}(P,A) \xrightarrow{p_*} \mathcal{A}(P,B) \xrightarrow{q_*} \mathcal{A}(P,C) \xrightarrow{0} 0$$

Pick $f : \mathcal{A}(P,C)$, since q_* is a surjection there exists an $f' : \mathcal{A}(P,B)$ such that $pf' = f$. Now, suppose that P has the property described by the diagram in the proposition and that $(p : A \to B, q : B \to C)$ is a conflation, then there is an exact sequence in $\mathbf{Ab}$ by $\mathcal{A}(P,\_)$.

$$0 \xrightarrow{0} \mathcal{A}(P,A) \xrightarrow{p_*} \mathcal{A}(P,B) \xrightarrow{q_*} \mathcal{A}(P,C)$$

To see that q_* is a surjection, let $f : P \to C$. As q is a deflation there exists an $f' : P \to B$ such that $q_*(f') = f$. Thus the sequence above is short exact and P is projective. □

Corollary 2.2.1.1. *Let P be projective, then if $q : A \to P$ is a deflation, it is split-epi.*

Corollary 2.2.1.2. *Two objects P and Q are projective if and only if $P \oplus Q$ is projective.*

Corollary 2.2.1.3. *$I : \mathcal{A}$ is injective if and only if for every inflation $p : B \to A$ and morphism $g : B \to I$ there is a morphism $g' : A \to I$ rendering the diagram below commutative.*

$$\begin{tikzcd} & I \\ A \arrow[ur, dashed, "g'"] & B \arrow[l, tail, "p"] \arrow[u, "f"'] \end{tikzcd}$$

Definition 2.2.3. A category $(\mathcal{A}, \mathcal{E})$ has enough projective objects if for any object $A : \mathcal{A}$ there is a projective object P along with a deflation $q : P \to A$. Dually, it has enough injective objects if for any object $A : \mathcal{A}$ there is an injective object I along with an inflation $p : A \to I$.

Definition 2.2.4. An exact category is called a Frobenius category if it has enough projective and injective objects and the class of projective objects coincide with the injective objects.

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From the Frobenius categories, one can construct the stable Frobenius category where every morphism factoring through an injective will be identified. These stable categories will give rise to a class of categories called the algebraic triangulated categories. In order to construct the stable Frobenius category we will use the construction of the quotient category.

Definition 2.2.5. A congruence relation $\sim$ on a category $\mathcal{C}$ is a relation on the hom-sets, such that:

1. $\forall A, B : \mathcal{C}$ the relation $\sim_{A,B}$ is an equivalence relation.
2. Given that $f, f' : A \to B$ is related ($f \sim f'$) and morphisms $g : A' \to A$ and $h : B \to B'$, then $hfg \sim hf'g$.

Proposition 2.2.2. *Let $\mathcal{C}$ be a category and $\sim$ be a congruence relation. Then there is a universal category $\mathcal{C}/\sim$ together with a functor $q : \mathcal{C} \to \mathcal{C}/\sim$ such that morphisms $f, g : A \to B$ are identified if $f \sim g$. Universality means that if there is a functor $H : \mathcal{C} \to \mathcal{D}$ such that $Hf = Hg$ for any f, g iff $\sim g$, then H factors uniquely through $\mathcal{C}/\sim$.*

Proof. Define the category $\mathcal{C}/\sim$ to have the same objects as $\mathcal{C}$, and define $\mathcal{C}/\sim(A, B) = \mathcal{C}(A, B)/\sim_{a,b}$. This definition is well defined as $\sim$ is a congruence relation. A sketch of this proof can be found in [**Mac71**]. □

Remark. Any functor gives rise to a congruence relation. That is, if $F : \mathcal{C} \to \mathcal{D}$ is a functor, then there is a congruence relation $\sim$ defined as follows: $\forall A, B : \mathcal{C}$ and $f, g : A \to B$, we define $f \sim_{A,B} g \iff Ff = Fg$. This is a congruence as equlity within $\mathcal{D}$ gives rise to an equivalence relation, and functoriality gives the congruence.

Remark. For any relation $\sim$ the universal category $\mathcal{C}/\sim$ exists. As in the case for the Verdier quotient, $\mathcal{C}/\sim$ is the same as the quotient category of the smallest congruence relation having the same relations as $\sim$.

If $\mathcal{A}$ is an additive category, the quotient categories which respects the additive structures are of interest. That is to say that the functor $q : \mathcal{A} \to \mathcal{C}/\sim$ is additive and the equivalence relation $\sim$ should respect the additive structure. Then a quotient category is additive if $f \sim f'$ and $g \sim g'$, then $f + g \sim f' + g'$. This leads to the following definition.

Definition 2.2.6. Let $\mathcal{A}$ be an additive category. $\mathcal{I}$ is an ideal of $\mathcal{A}$ if:

1. (subgroup) for every abelian group $\mathcal{A}(A, B)$ there is a subgroup $\mathcal{I}(A, B) \subseteq \mathcal{A}(A, B)$.
2. (absorption) For every $g : A' \to A$, $h : B \to B'$ and $f : \mathcal{I}(A, B)$ we have that $hfg : \mathcal{I}(A', B')$

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This is equivalent of saying that the equivalence relation gained from $\mathcal{I}(A, B)$ ($f \sim g \iff f - g : \mathcal{I}(A, B)$) is a congruence relation.

Corollary 2.2.2.1. *Let $\mathcal{A}$ be an additive category and $\mathcal{I}$ be an ideal of $\mathcal{A}$, then $\mathcal{A}/\mathcal{I}$ is an additive category.*

Let $\mathcal{A}$ be a Frobenius category. Define the ideal $\mathcal{I}$ as the subgroups of every morphism factoring through injective objects.

Proposition 2.2.3. *For any Frobenius category $\mathcal{A}$ the ideal $\mathcal{I}$ is well defined and $\underline{\mathcal{A}} = \mathcal{A}/\mathcal{I}$ is the stable Frobenius category.*

Proof. To prove this one must show that $\mathcal{I}(A, B)$ is a subgroup for any $A, B : \mathcal{A}$, and that it is absorptive. First observe that $0 : \mathcal{I}(A, B)$. Let $f, g : \mathcal{I}(A, B)$. Since $\mathcal{A}$ has enough injectives there exists an injective object with an inflation from A.

$$\begin{tikzcd}
 & J_1 \arrow[dr, "f_2"] & \\
A \arrow[ur, "f_1"] \arrow[r, tail, "i"] \arrow[dr, "g_1"'] & I \arrow[u, dashed, "f_1'"] \arrow[d, dashed, "g_1'"] & B \\
 & J_2 \arrow[ur, "g_2"'] &
\end{tikzcd}$$

$f - g = f_2 \circ f_1 - g_2 \circ g_1 = (f_2 \circ f_1' - g_2 \circ g_1') \circ i$. Thus $f - g$ factors through an injective, and $\mathcal{I}(A, B)$ is a subgroup. To see that it is absorptive is to see that if f factors through an injective, then gf factors through an injective as well. $\square$

Objects in the stable Frobenius category is denoted as $\underline{X}$ and morphisms are denoted as $\underline{f}$. That is the functor $q : \mathcal{A} \to \underline{\mathcal{A}}$ is defined as $q(X) = \underline{X}$ and $q(f) = \underline{f}$. One important property is that in the stable Frobenius category, taking syzygies or cozysigies is a functor.

Definition 2.2.7. A syzygy of an object X, if it exists, is denoted ΩX. The syzygy is defined to be the kernel object of a deflation $p : P \to X$, where P is projective. A cosyzygy, denoted as $\mho X$ is defined to be the cokernel of an inflation $i : X \to I$, where I is injective.

Remark. Note that this choice is not necessarily unique up to isomorphism. Thus syzygies and cosyzygies are not in general functors.

Lemma 2.2.4. *Let $\mathcal{A}$ be a Frobenius category and suppose that there are two conflations with injectives as below. Then $\underline{X'} \simeq \underline{X''}$.*

$$X \overset{i}{\rightarrowtail} I \overset{i'}{\rightarrowtail} X'$$

$$X \overset{j}{\rightarrowtail} J \overset{j'}{\rightarrowtail} X''$$

Proof. Observe that there are morphisms in the diagram as I and J are injective.

$$\begin{tikzcd}
X \arrow[r, tail, "i"] \arrow[d, equal] \arrow[dr, "j"] & I \arrow[d, dashed, "f"] \\
X \arrow[r, tail, "j"] & J
\end{tikzcd}$$

The commutative diagram below is created by the cokernel property.

$$\begin{tikzcd}
X \arrow[r, tail, "i"] \arrow[d, equal] & I \arrow[r, two heads, "i'"] \arrow[d, "f"] & X' \arrow[d, dashed, "g"] \\
X \arrow[r, tail, "j"] \arrow[d, equal] & J \arrow[r, two heads, "j'"] \arrow[d, "f'"] & X'' \arrow[d, dashed, "g'"] \\
X \arrow[r, tail, "i"] & I \arrow[r, two heads, "i'"] & X'
\end{tikzcd}$$

A diagram chase shows that $i - f'fi = (id_I - f'f) \circ i = 0$. This means that $(f'f - id_I)$ factors through X', i.e. there exists $h : X' \to I$ and $f'f = hi' + id_I$. Diagram chasing also reveals that $g'gi' = i'f'f = i'(hi' + id_I) = i'hi' + i' = (i'h + id_{X'})i'$. As i' is an epi one obtains that $g'g = i'h + id_{X'} \implies \underline{g'g} = id_{\underline{X'}}$ as $i'h$ factors through I. $\underline{gg'} = id_{\underline{X''}}$ is dual. $\square$

Corollary 2.2.4.1. *Cosyzygy is a well defined functor $\mho$: $\underline{\mathcal{A}} \to \underline{\mathcal{A}}$*

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{A}$. Then the following diagrams representing the different choices of syzygies.

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"] & I \arrow[r, "p"] \arrow[d, dashed] & \mho X \arrow[d, dashed, "\mho f"] \\
Y \arrow[r, "j"] & J \arrow[r, "q"] & \mho Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "i'"] \arrow[d, "f"] & I' \arrow[r, "p'"] \arrow[d, dashed] & \mho' X \arrow[d, dashed, "\mho' f"] \\
Y \arrow[r, "j'"] & J' \arrow[r, "q'"] & \mho' Y
\end{tikzcd}$$

By the previous proof there are maps between the diagrams making an almost commutative diagram where all the outer squares commute.

$$\begin{tikzcd}
X \arrow[rr, tail, "i"] \arrow[dr, equal, "\alpha(X)"'] \arrow[dd, "f"'] & & I \arrow[rr, two heads, "p"] \arrow[dr, "\beta(X)"] \arrow[dd, dashed, "If"] & & \mho X \arrow[dr, "\gamma(X)"] \arrow[dd, dashed, "\mho f"] \arrow[dddl, dashed, "\chi"] & \\
& X \arrow[rr, tail, "i'"] \arrow[dd, "f"] & & I' \arrow[rr, two heads, "p'"] \arrow[dd, dashed, "I'f"] & & \mho' X \arrow[dd, dashed, "\mho' f"] \\
Y \arrow[rr, tail, "j"] \arrow[dr, equal, "\alpha(Y)"'] & & J \arrow[rr, two heads, "q"] \arrow[dr, "\beta(Y)"] & & \mho Y \arrow[dr, "\gamma(Y)"] & \\
& Y \arrow[rr, tail, "j'"] & & J' \arrow[rr, two heads, "q'"] & & \mho' Y
\end{tikzcd}$$

To see that the definition of the cosyzygy is well defined is to show that the inner squares is commutative in the quotient category, i.e. that the diagram commutes in the quotient.

Observe that the left inner square commutes by definition, and that the central inner square commutes in the quotient as every morphism gets related to 0. Thus it remains to show that $\underline{\gamma(Y)}\circ\underline{\mho f}=\underline{\mho' f}\circ\underline{\gamma(X)}$, which is the same as to say that $\gamma(Y)\circ\mho f - \mho' f \circ \gamma(X)$ factors over an injective.

By doing a diagram chase in the left cube one may find the following equation $(I'f \circ \beta(X) - \beta(Y)\circ If)i = 0$. This means that the map $I'f \circ \beta(X) - \beta(Y)\circ If$ factors through the cokernel of i as χp. By chasing the right cube one may assert the equation $q'\chi p = (\gamma(Y)\circ\mho f - \mho' f \circ \gamma(X))p$, thus $q'\chi = \gamma(Y)\circ\mho f - \mho' f \circ \gamma(X)$. □

Corollary 2.2.4.2. *Cosyzygy $\mho$ is an autoequivalence with syzygy Ω as inverse.*

Proof. The goal is to show that there is a natural isomorphisms $\Omega\mho \simeq Id_{\underline{\mathcal{A}}}$ and $\mho\Omega \simeq Id_{\underline{\mathcal{A}}}$. As these are inverse operations one have that taking syzygy then cosyzygy is the same as taking cosyzygy then syzygy in $\mathcal{A}^{op}$. Let $X : \mathcal{A}$, the goal is to show that the following diagram gives a natural isomorphism at the rightmost arrow in $\underline{\mathcal{A}}$.

$$\begin{tikzcd}
\Omega X \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d, dashed] & X \arrow[d, dashed] \\
\Omega X \arrow[r] & I \arrow[r] & \mho\Omega X
\end{tikzcd}$$

Observe that this case is identical as the one previous proved. This shows that there is a natural isomorphism from X to $\mho\Omega$X. □

Bedre ord?

Remark. A subtle, but important point is that the category $\mathcal{A}$ has enough projectives and injectives. This enables us to find the syzygies and cozysigies. It is also important that the projectives are the same as the injectives for this construction to give the isomorphisms as well.

Dette kommer for brått

The triangulation of $\underline{\mathcal{A}}$ will be defined as the set of candidate triangles in $\underline{\mathcal{A}}$ called standard triangles. Let $\underline{x} : \underline{X} \to \underline{Y}$ be a morphism, then by (QE2) there is a pushout in $\mathcal{A}$. Moreover, by Proposition 2.1.3 (y, z) is a conflation.

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, tail, "i"] & Y \arrow[d, tail, "y"] \arrow[ddr, bend left, "0"] & \\ I(X) \arrow[r, "x'"] \arrow[drr, bend right, two heads, "p"'] & Z \arrow[dr, dashed, two heads, "z"] & \\ & & \mho X \end{tikzcd}$$

The set of standard triangles will be of the form $(\underline{X}, \underline{Y}, \underline{Z}, \underline{x}, \underline{y}, \underline{z})$. Thus a triangle $(A, B, C, a, b, c) : \Delta_{\underline{\mathcal{A}}}$ if and only if (A, B, C, a, b, c) is isomorphic to a standard triangle.

Proposition 2.2.5. *$\Delta_{\underline{\mathcal{A}}}$ is a triangulation of $\underline{\mathcal{A}}$.*

Proof. I will give a sketch of how to prove this proposition. Most of the details will be omitted, and they can be found in [**happel**] or [**May01**]. The proof is structured in 3 different parts, namely showing TR1, TR2 and TR4. Note that TR1 is satisfied by definition of $\Delta_{\underline{\mathcal{A}}}$. To see this observe that the following diagram is a pushout.

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, tail, "i"] & X \arrow[d, tail, "i"] \\ I(X) \arrow[r, equal] & I(X) \end{tikzcd}$$

Thus $(\underline{X}, \underline{X}, \underline{0}, id_{\underline{X}}, \underline{0}, \underline{0})$ is a standard triangle.
(TR2) Consider the standard triangle $(\underline{X}, \underline{Y}, \underline{Z}, \underline{x}, \underline{y}, \underline{z})$, the goal is to show that there is a triangle $(\underline{Y}, \underline{Z}, \underline{\mho X}, \underline{x}, \underline{y}, -\underline{\mho x})$. Let $I(X)$ and $I(Y)$ be injectives with inflations from X and Y respectively. Since $I(Y)$ is injective there is a unique map by the pushout property in (1).

$$(1)\quad \begin{tikzcd} X \arrow[r, "x"] \arrow[d, tail, "i(X)"] & Y \arrow[d, "y"] \arrow[ddr, bend left, "i(Y)"] & \\ I(X) \arrow[r, "x'"] \arrow[drr, bend right, "f"'] & Z \arrow[dr, dashed, "g"] & \\ & & I(Y) \end{tikzcd} \qquad (2)\quad \begin{tikzcd} X \arrow[r, "x"] \arrow[d, "i(X)"] & Y \arrow[d, "y"] \arrow[dr, "0"] & \\ I(X) \arrow[r, "x'"] \arrow[dr, "f"'] & Z \arrow[r, "z"] \arrow[d, "g"] & \mho X \arrow[d, "\mho x"] \\ & I(Y) \arrow[r, "p(Y)"] & \mho Y \end{tikzcd}$$

From (2) one are able to use the pushout to see that the lower right square commutes, that is $p(Y)f i(X) = \mho xzyz = 0$. This is true as $p(Y)f i(X) = p(Y)gyx = p(Y)i(Y)x = 0$ by (1). Note that since z and $p(Y)$ are deflations with equal kernels, Proposition 3.3 says that $\left(\begin{pmatrix} g \\ z \end{pmatrix}, \begin{pmatrix} p(Y) & -\mho x \end{pmatrix}\right)$ is a conflation.
One is now able to find a commutative diagram and by Proposition 3.3, say that the upper left square is bicartesian.

$$\begin{tikzcd} Y \arrow[r, tail, "y"] \arrow[d, tail, "i(Y)"] & Z \arrow[r, two heads, "z"] \arrow[d, tail, "\begin{pmatrix} g \\ z \end{pmatrix}"] & \mho X \arrow[d, equal] \\ I(Y) \arrow[r, tail, "\iota_1"'] \arrow[d, two heads, "p(Y)"] & I(Y) \oplus \mho X \arrow[r, two heads, "\pi_2"'] \arrow[d, two heads, "\begin{pmatrix} p(Y) & -\mho x \end{pmatrix}"] & \mho X. \\ \mho Y \arrow[r, equal] & \mho Y & \end{tikzcd}$$

Thus $(\underline{Y}, \underline{Z}, \underline{\mho X}, \underline{y}, \underline{z}, -\underline{\mho x})$ is a standard triangle.

(TR4) Suppose that there are three standard triangles where $\nu\upsilon = \omega$.

$$
\begin{tikzcd}
X \arrow[r, "\upsilon"] \arrow[d, "x"] & Y \arrow[d, "i"] \\
I(X) \arrow[r, "\bar{\upsilon}"] \arrow[d, "\bar{x}"] & Z' \arrow[d, "i'"] \\
\mho X \arrow[r, equal] & \mho X
\end{tikzcd}
\quad
\begin{tikzcd}
Y \arrow[r, "\nu"] \arrow[d, "y"] & Z \arrow[d, "j"] \\
I(Y) \arrow[r, "\bar{\nu}"] \arrow[d, "\bar{y}"] & X' \arrow[d, "j'"] \\
\mho Y \arrow[r, equal] & \mho Y
\end{tikzcd}
\quad
\begin{tikzcd}
X \arrow[r, "\omega"] \arrow[d, "x"] & Z \arrow[d, "k"] \\
I(X) \arrow[r, "\bar{\omega}"] \arrow[d, "\bar{x}"] & Y' \arrow[d, "k'"] \\
\mho X \arrow[r, equal] & \mho X
\end{tikzcd}
$$

Er dette riktig?

By Noethers isomorphism lemma there is a conflation passing through on the right column and the middle square is bicartesian. z' exists by the injectivity of $I(Y)$ and that i is an inflation. z' is an inflation as y is an inflation, thus $\bar{z}'$ exists.

$$
\begin{tikzcd}
Y \arrow[r, tail, "i"] \arrow[d, equal] & Z' \arrow[r, two heads, "i'"] \arrow[d, tail, "z'"] & \mho X \arrow[d, tail, "s"] \\
Y \arrow[r, tail, "y"] & I(Y) \arrow[r, two heads, "\bar{y}"] \arrow[d, two heads, "\bar{z}'"] & \mho Y \arrow[d, two heads, dashed, "r"] \\
& \mho Z' \arrow[r, equal] & \mho Z'
\end{tikzcd}
$$

There is also a map $I_\upsilon : I(X) \to I(Y)$ induced by the maps between X and $I(Y)$. By using the following universal properties one may find the unique maps f and g.

$$
\begin{tikzcd}
X \arrow[r, "\upsilon"] \arrow[rr, bend left, "\omega"] \arrow[d, "x"] & Y \arrow[r, "\nu"] \arrow[d, "i"] & Z \arrow[dd, "k"] \\
I(X) \arrow[r, "\bar{\upsilon}"] \arrow[drr, bend right, "\bar{\omega}"'] & Z' \arrow[dr, dashed, "f"] & \\
& & Y'
\end{tikzcd}
\quad
\begin{tikzcd}
X \arrow[r, "\omega"] \arrow[d, "x"] & Z \arrow[d, "k"] \arrow[ddr, bend left, "j"] & \\
I(X) \arrow[r, "\bar{w}"] \arrow[dr, bend right, "I_\upsilon"'] & Y' \arrow[dr, dashed, "g"] & \\
& I(Y) \arrow[r, bend right, "\bar{\nu}"'] & X'
\end{tikzcd}
$$

These maps can be arranged in the diagram below. It can be seen that middle square is a pushout, by using the fact that the upper left square and the larger rectangles are pushouts.

$$
\begin{tikzcd}
X \arrow[r, "\upsilon"] \arrow[d, "x"] & Y \arrow[r, "\nu"] \arrow[d, "i"] & Z \arrow[d, "k"] \\
I(X) \arrow[r, "\bar{\upsilon}"] \arrow[d, "\bar{x}"] \arrow[dr, "I_{\upsilon}"] & Z' \arrow[r, "f"] \arrow[d, "z'"] & Y' \arrow[d, "\bar{g}"] \\
\mho X \arrow[dr, "s"] & I(Y) \arrow[r, "\bar{\nu}"] \arrow[d, "r"] & X' \arrow[d, "\mho i \circ j'"] \\
& \mho Y \arrow[r, "\mho i"] & \mho Z'
\end{tikzcd}
\qquad
\begin{tikzcd}
Z' \arrow[r, "f"] \arrow[d, tail, "z'"] & Y' \arrow[d, tail, "g"] \\
I(Y) \arrow[r, "\bar{\nu}"] \arrow[d, two heads, "\bar{z}'"] & X' \arrow[d, two heads, "\mho i \circ j'"] \\
\mho Z' \arrow[r, equal] & \mho Z'
\end{tikzcd}
$$

Thus $(\underline{Z'}, \underline{Y'}, \underline{X'}, \underline{f}, \underline{g}, \underline{\mho i \circ j'})$ is a triangle. □

Remark. A more detailed and different proof may be found in [**Hol12**] or [**Mat20**].

This construction of triangles admits a close relation to conflations. If there is a conflation $(p : X \to Y, q : Y \to Z)$, then there is a triangle $(\underline{X}, \underline{Y}, \underline{Z}, \underline{p}, \underline{q}, -\underline{r})$ constructed as follows: Let $P : \mathcal{A}$ be a projective object with a deflation $\bar{p} : P \to Y$, then there exists a pullback (1), moreover the pullback square is bicartesian. By using TR2 one may find the triangle (2) as indicated in the diagram.

$$
(1)\quad
\begin{tikzcd}
\Omega Z \arrow[r, two heads, "\Omega r"] \arrow[d, tail] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, tail, "p"] \\
P \arrow[r, two heads, "\bar{p}"] \arrow[d, two heads] & Y \arrow[d, two heads, "q"] \\
Z \arrow[r, equal] & Z
\end{tikzcd}
\qquad
(2)\quad
\underline{X} \xrightarrow{\underline{p}} \underline{Y} \xrightarrow{\underline{q}} \underline{Z} \xrightarrow{-\underline{r}} \underline{\mho X}
$$

Remark. For any morphism $f : A \to B$ in $\mathcal{A}$, there is an inflation $\begin{pmatrix} f \\ -i \end{pmatrix} : A \to B \oplus I$ which is in the same equivalence class as f. Thus $\underline{f} = \underline{\begin{pmatrix} f \\ -i \end{pmatrix}}$, and any morphism in $\underline{\mathcal{A}}$ can be obtained from an inflation in $\mathcal{A}$.

2.3 Self-injective Algebras

The first example of a triangulated category is going to be derived from finite dimensional artin algebras. More specifically, let Λ be a self-injective finite dimensional artin R-algebra; that is ${}_{\Lambda}\Lambda$ is injective as left Λ-module, then the finitely generated projective objects coincide with the finitely generated injective objects.

Proposition 2.3.1. *If Λ is a self-injective finite dimensional artin R-algebra, then mod_{Λ} is a Frobenius category.*

To prove this statement we will need the following propositions.

Lemma 2.3.2. *The category mod_{Λ} has enough projectives*

Proof. Let $A : mod_\Lambda$, then A is finitely generated. This means there exists an epimorphism $p : R^n \to A$, where n is the number of generators of A. □

Lemma 2.3.3. *Let R be an artin ring and $\mathfrak{r}$ denote the nilradical of R. Moreover, let J be the injective envelope of $R/\mathfrak{r}$, then functor $Hom_R(\_, J) : mod_\Lambda \to mod_{\Lambda^{op}}$ is a duality.*

Corollary 2.3.3.1. *The category mod_Λ has enough injectives*

Detailed proofs of these statements can be found in [**Rei95**].

Proof. Suppose that Λ is self-injective. By the lemmas above it is known that mod_Λ has enough projectives and enough injectives. It remains to show that the class of injectives coincide with the projectives. Since every indecomposable Λ module is a summand of Λ up to isomorphism, it follows that they are injective. As they also are projective, the class of injectives and projectives coincide. □

This shows that mod_Λ is a Frobenius category, thus $\underline{mod_\Lambda}$ is triangulated. The triangles in $\underline{mod_\Lambda}$ are the quotients of every short exact sequence in mod_Λ.

$$0 \longrightarrow X \overset{a}{\rightarrowtail} Y \overset{b}{\twoheadrightarrow} Z \longrightarrow 0 \implies \underline{X} \xrightarrow{\underline{a}} \underline{Y} \xrightarrow{\underline{b}} \underline{Z} \xrightarrow{\underline{c}} \underline{\mho X}$$

Proposition 2.3.4. *Let G be a group and R any commutative artin ring, then the group ring $R[G]$ is self-injective.*

Proposition 2.3.5. *Let K be a field, then $K[x]/(x^n)$ is self-injective.*

Proof. As $K[x]/(x^n)$ modules, there is only one indecomposable projective module up to isomorphism, that is $K[x]/(x^n)$. Since $K[x]/(x^n)$ is commutative, the duality functor is an automorphism of $mod_{K[x]/(x^n)}$, thus $Hom_K(K[x]/(x^n), K)$ is the indecomposable injective $K[x]/(x^n)$ module. As the duality functor preserves length the modules have equal length. By finding a monomorphism $i : K[x]/(x^n) \to Hom_K(K[x]/(x^n), K)$ we have that it is an isomorphism as the cokernel has length 0. The socle $soc(K[x]/x^n)$ is the simple module K, this means that the injective envelope of $K[x]/(x^n)$ is indecomposable, thus it is in the same isomorphism class as $Hom_K(K[x]/(x^n), K)$, proving that there is a monomorphism as stated. □

In this particular case the triangles take on a somewhat special form, where repeatedly applying TR2 yields the same triangles after 6 iterations! This can be seen by calculating the triangles of the indecomposable modules. Every other triangle will be a direct sum of these.
Observe that every submodule of $K[x]/(x^n)$ is indecomposable, these make up the class of the indecomposable modules up to isomorphism. Further observe that the cosyzygy of any submodule is $\mho(x^k)/(x^n) \simeq (x^{n-k})/(x^n)$. The repetition of the triangles can be seen as the natural isomorphism $\mho^2(x^k)/(x^n) \simeq (x^{n-(n-k)})/(x^n) = (x^k)/(x^n)$.

To find the triangles, let $A, B : mod_{K[x]/(x^n)}$ and $T : A \to B$ be $K[x]/(x^n)$-linear. T is in the same equivalence class as $\begin{pmatrix} T \\ -i \end{pmatrix} : A \to B \oplus I$ with i as the injective envelope of A. Then there is a triangle as the diagram below.

$$\underline{A} \xrightarrow{T} \underline{B} \longrightarrow Cok\underline{T} \oplus Ker\underline{\mho T} \longrightarrow \underline{\mho A}$$

Observe that $Cok\underline{\begin{pmatrix} f \\ -i \end{pmatrix}} \simeq Cok\underline{T} \oplus Ker\underline{\mho T}$, so the triangle above is in fact well-defined.

Lemma 2.3.6. *The category $Vect(K)$ is triangulated.*

Proof. This follows immediately from the discussion above. Look at $mod_{K[x]/(x^2)}$, the indecomposable objects of this category are $K[x]/(x^2)$ and K up to isomorphism. As $K[x]/(x^2)$ is injective we have that K is the only indecomposable object of $\underline{mod}_{K[x]/(x^2)}$, thus every object is a direct summand K. Also observe that the cosyzygy is naturally isomorphic to the identity functor on the quotient. In order to be precise, one would need to show that there is an equivalence of categories $Vect(K) \simeq mod_{K[x]/(x^2)}$. The triangles in $Vect(K)$ can then be seen as this three term repeating triangle.

fremhev at dette er en sketch av et bevis

$$V \xrightarrow{T} W \xrightarrow{\begin{pmatrix} \pi_T \\ 0 \end{pmatrix}} CokT \oplus KerT \xrightarrow{\begin{pmatrix} 0 & \iota_T \end{pmatrix}} V$$

□

Fjern dette?

Suppose now that Λ is not a self-injective artin R-algebra, then there is an extension to this ring which makes it self injective.

Definition 2.3.1. Let Λ be an artin R-algebra. There is an associated product called the semi-direct proudct with this ring and it's dual. Define $\Lambda \rtimes Hom_R(\Lambda, J)$ to have the elements (λ, ϕ), where $\lambda : \Lambda$ and $\phi : \Lambda \to J$. Let (λ, ϕ) and (μ, ψ) be two elements in this set. Set $(\lambda, \phi) + (\mu, \psi) = (\lambda + \mu, \phi + \psi)$ and $(\lambda, \phi)(\mu, \psi) = (\lambda\mu, \lambda \cdot \psi + \phi \cdot \mu)$, where $\lambda\phi(\chi)\mu = \phi(\lambda\chi\mu)$.

Proposition 2.3.7. *The trivial extension $\Lambda \rtimes Hom_R(\Lambda, J)$ is self-injective.*

Proof. HELP (jeg er kaskje litt dramatisk her, for jeg har ikke prøvd å bevise det enda.) □

2.4 The Homotopy Category

The next example of a triangulated category is something which is referred to as *the* triangulated category. The homotopy category is the category which may be regarded as the prototype for every triangulated category. It can be defined with the following category.

Gir denne dramatikken så mye mening?

Definition 2.4.1. Let $\mathcal{A}$ be an additive category. Define $Ch(\mathcal{A})$ to be the category of diagrams in $\mathcal{A}$ on the form

$$\dots \xrightarrow{d_{A^\bullet}^{-2}} A^{-1} \xrightarrow{d_{A^\bullet}^{-1}} A^0 \xrightarrow{d_{A^\bullet}^{0}} A^1 \xrightarrow{d_{A^\bullet}^{1}} \dots$$

such that $d_{A^\bullet}^{i} \circ d_{A^\bullet}^{i-1} = 0$ for every $i : \{-\infty, ..., \infty\}$. These objects are reffered to as (co)chain complexes and they are denoted as $A^\bullet$, and the maps in the objects are called differentials/(co)boundaries. A morphism $\phi^\bullet : A^\bullet \to B^\bullet$ between (co)chain complexes, also called chain map, is a collection of morphisms from $\mathcal{A}$, such that the morphisms commute with the differentials in the following manner:

$$\begin{tikzcd}
\dots \arrow[r, "d_{A^\bullet}^{-2}"] & A^{-1} \arrow[r, "d_{A^\bullet}^{-1}"] \arrow[d, "\phi^{-1}"] & A^0 \arrow[r, "d_{A^\bullet}^{0}"] \arrow[d, "\phi^{0}"] & A^1 \arrow[r, "d_{A^\bullet}^{1}"] \arrow[d, "\phi^{1}"] & \dots \\
\dots \arrow[r, "d_{B^\bullet}^{-2}"] & B^{-1} \arrow[r, "d_{B^\bullet}^{-1}"] & B^0 \arrow[r, "d_{B^\bullet}^{0}"] & B^1 \arrow[r, "d_{B^\bullet}^{1}"] & \dots
\end{tikzcd}$$

Remark. If $\mathcal{A}$ is abelian, then the category $Ch(\mathcal{A})$ is abelian. The kernels and cokernels of chain maps would be levelwise kernels and cokernels along the chain. Moreover, if $(\mathcal{A}, \mathcal{E})$ is an exact category, then $(Ch(\mathcal{A}), Ch(\mathcal{E}))$ will be exact as well, by using levelwise kernels and cokernels.

Remark. On the category of cochain complexes there is an additive autoequivalence called the translation functor. The functor is denoted as $(\_)[1] : Ch(\mathcal{A}) \to Ch(\mathcal{A})$ and it takes a complex $A^\bullet$ and shifts it one step to the left into $A^{\bullet+1}$. In fact there is a family of functors $A^\bullet[n] = A^{\bullet+n}$. Thus $(\_)[-1]$ is the quasi-inverse of $(\_)[1]$.

Definition 2.4.2. A chain map $f^\bullet : A^\bullet \to B^\bullet$ is called null-homotopic if there is a map $\varepsilon^\bullet : A^\bullet \to B^\bullet[-1]$ such that $f^\bullet = d_{B^\bullet}^{\bullet-1}\varepsilon^\bullet + \varepsilon^{\bullet+1} d_{A^\bullet}^{\bullet}$.

$$\begin{tikzcd}
\dots \arrow[r, "d_{A^\bullet}^{-2}"] & A^{-1} \arrow[r, "d_{A^\bullet}^{-1}"] \arrow[d, "f^{-1}"] & A^0 \arrow[r, "d_{A^\bullet}^{0}"] \arrow[d, "f^{0}"] \arrow[dl, "\varepsilon^0"'] & A^1 \arrow[r, "d_{A^\bullet}^{1}"] \arrow[d, "f^{1}"] \arrow[dl, "\varepsilon^1"'] & \dots \\
\dots \arrow[r, "d_{B^\bullet}^{-2}"] & B^{-1} \arrow[r, "d_{B^\bullet}^{-1}"] & B^0 \arrow[r, "d_{B^\bullet}^{0}"] & B^1 \arrow[r, "d_{B^\bullet}^{1}"] & \dots
\end{tikzcd}$$

$\varepsilon^\bullet$ is called the homotopy. Two chain maps $f^\bullet$ and $g^\bullet$ are said to be homotopic $f^\bullet \sim g^\bullet$ if their difference $f^\bullet - g^\bullet$ is null-homotopic.

Proposition 2.4.1. *There is an additive bifunctor* $nullHom_{\mathcal{A}}(\_,\_) : Ch(\mathcal{A})^{op} \times Ch(\mathcal{A}) \to Ab$ *mapping into the set of null-homotopic morphisms. The elements of* $nullHom_{\mathcal{A}}(A^\bullet, B^\bullet)$ *are pairs made of null-homotopic maps with their homotopy* $(f^\bullet, \varepsilon^\bullet)$. *This is an abelian group with the product group structure, that is* $(f^\bullet, \varepsilon^\bullet) +$

$(g^\bullet, \gamma^\bullet) = (f^\bullet + g^\bullet, \varepsilon^\bullet + \gamma^\bullet)$. *The functor acts on morphisms almost the same way as the hom-functor.* Skriv ut hva dette betyr konkret

Proof. In order to prove the proposition, one must show that the assignment is in fact a functor and that it is additive as well. It suffices to show that $nullHom_{\mathcal{A}}(A^\bullet, \_)$ is an additive functor, as it will follow by duality that the there is an additive bi-functor as proposed.
Suppose that there is a chain map $f^\bullet : B^\bullet \to C^\bullet$, then $nullHom_{\mathcal{A}}(A^\bullet, \_)(f^\bullet) = f_*^\bullet$. Let $(g^\bullet, \varepsilon^\bullet) : nullHom_{\mathcal{A}}(A^\bullet, B^\bullet)$ be a null-homotopic chain map. By definition $f_*^\bullet(g^\bullet, \varepsilon^\bullet) = (f^\bullet g^\bullet, f^{\bullet-1}\varepsilon^\bullet)$. One may now see that $f^{\bullet-1}\varepsilon^\bullet$ is a homotopy by the following diagram. The commutativity of the lower left square shows the homotopy. It follows by functoriality from the Hom-functor that $nullHom_{\mathcal{A}}(A^\bullet, \_)$ is a functor.

$$\begin{tikzcd}
\dots \arrow[r, "d_{A^\bullet}^{-2}"] & A^{-1} \arrow[r, "d_{A^\bullet}^{-1}"] \arrow[d, "g^{-1}"] & A^0 \arrow[r, "d_{A^\bullet}^{0}"] \arrow[d, "g^0"] \arrow[dl, "\varepsilon^0"'] & A^1 \arrow[r, "d_{A^\bullet}^{1}"] \arrow[d, "g^1"] \arrow[dl, "\varepsilon^1"'] & \dots \\
\dots \arrow[r, "d_{B^\bullet}^{-2}"] & B^{-1} \arrow[r, "d_{B^\bullet}^{-1}"] \arrow[d, "f^{-1}"] & B^0 \arrow[r, "d_{B^\bullet}^{0}"] \arrow[d, "f^0"] & B^1 \arrow[r, "d_{B^\bullet}^{1}"] \arrow[d, "f^1"] & \dots \\
\dots \arrow[r, "d_{C^\bullet}^{-2}"] & C^{-1} \arrow[r, "d_{C^\bullet}^{-1}"] & C^0 \arrow[r, "d_{C^\bullet}^{0}"] & C^1 \arrow[r, "d_{C^\bullet}^{1}"] & \dots
\end{tikzcd}$$

The last thing to show is that it is additive. This is the same as showing that the assignment $nullHom_{\mathcal{A}}(A^\bullet, \_) : Hom_{Ch(\mathcal{A})}(B^\bullet, C^\bullet) \to Hom_{Ab}(nullHom_{\mathcal{A}}(A^\bullet, B^\bullet), nullHom_{\mathcal{A}}(A^\bullet, C^\bullet))$ is a group homomorphism. Let $f^\bullet, g^\bullet : B^\bullet \to C^\bullet$ be two chain maps, and $(h^\bullet, \varepsilon^\bullet) : nullHom_{\mathcal{A}}(A^\bullet, B^\bullet)$. Then the following equation asserts the additivity:

$$\begin{aligned}
&(f^\bullet + g^\bullet)_*(h^\bullet, \varepsilon^\bullet) \\
=\,&((f^\bullet + g^\bullet)h^\bullet, ((f^\bullet + g^\bullet)[-1])\varepsilon^\bullet) \\
=\,&(f^\bullet h^\bullet + g^\bullet h^\bullet, f^{\bullet-1}\varepsilon^\bullet + g^{\bullet-1}\varepsilon^\bullet) \\
=\,&(f^\bullet h^\bullet, f^{\bullet-1}\varepsilon^\bullet) + (g^\bullet h^\bullet, g^{\bullet-1}\varepsilon^\bullet) \\
=\,&f_*^\bullet(h^\bullet, \varepsilon^\bullet) + g_*^\bullet(h^\bullet, \varepsilon^\bullet)
\end{aligned}$$

□

Corollary 2.4.1.1. *The equivalence relation $\sim$ stated above is an additive congruence relation. The homotopy category is defined to be the quotient $K(\mathcal{A}) = Ch(\mathcal{A})/\sim$.*

Skriv om dette avsnittet

The goal for the rest of this section is to prove that the homotopy category is triangulated. This will be done by putting an exact structure onto $Ch(\mathcal{A})$, which turns it into a Frobenius category, such that the stable Frobenius category and the Homotopy category will be equivalent. This will be seen by studying the representable nature of $nullHom_{\mathcal{A}}(\_, \_)$.

Definition 2.4.3. Let $f^\bullet : A^\bullet \to B^\bullet$ be a chain map. Define the object $cone(f^\bullet)$ to be the complex below.

$$\cdots \longrightarrow B^{-1} \oplus A^0 \xrightarrow[\begin{pmatrix} d_{B^\bullet}^{-1} & f^0 \\ 0 & -d_{A^\bullet}^0 \end{pmatrix}]{} B^0 \oplus A^1 \longrightarrow \cdots$$

Remark. For any chain map $f^\bullet : A^\bullet \to B^\bullet$ there is a short exact sequence.

$$B^\bullet \overset{\begin{pmatrix} 1 \\ 0 \end{pmatrix}^\bullet}{\rightarrowtail} cone(f^\bullet) \overset{\begin{pmatrix} 0 & -1^\bullet \end{pmatrix}^\bullet}{\twoheadrightarrow} A^\bullet[1]$$

Definition 2.4.4. An object $A^\bullet$ of $Ch(\mathcal{A})$ is called contractible if $id_{A^\bullet}^\bullet$ is null-homotopic.

Dette ser ikke pent ut

Example. Let $A^\bullet$ be a complex, then $cone(id_{A^\bullet})$ is contractible. That is
$\left(\begin{pmatrix} id_{A^\bullet}^\bullet & 0 \\ 0 & id_{A^\bullet}^\bullet[1] \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ id_{A^\bullet}^\bullet & 0 \end{pmatrix}\right) : nullHom_{\mathcal{A}}(cone(id_{A^\bullet}^\bullet), cone(id_{A^\bullet}^\bullet))$

Proposition 2.4.2. *For any complex $A^\bullet$ there is a natural isomorphism $nullHom_{\mathcal{A}}(A^\bullet, \_) \simeq Hom_{Ch(\mathcal{A})}(cone(id_{A^\bullet}^\bullet), \_)$. This establish that $cone(id_{A^\bullet}^\bullet)$ is the universal contractible complex where null-homotopic morphisms from $A^\bullet$ factors through.*

Proof. This proof will construct two natural maps which are inverses. This is sufficient to prove the universal property by Yonedas lemma.
Let $construct_{(A^\bullet,\_)(B^\bullet)} : nullHom_{\mathcal{A}}(A^\bullet, B^\bullet) \to Hom_{Ch(\mathcal{A})}(cone(id_{A^\bullet}^\bullet), B^\bullet)$ and $destruct_{(A^\bullet,\_)(B^\bullet)} : Hom_{Ch(\mathcal{A})}(cone(id_{A^\bullet}^\bullet), B^\bullet) \to nullHom_{\mathcal{A}}(A^\bullet, B^\bullet)$ be two morphisms defined the following way. $construct_{(A^\bullet,\_)(B^\bullet)}(f^\bullet, \varepsilon^\bullet) = \begin{pmatrix} f^\bullet & \varepsilon^\bullet \end{pmatrix}$ and $destruct_{(A^\bullet,\_)(B^\bullet)} \begin{pmatrix} f^\bullet & \varepsilon^\bullet \end{pmatrix} = (f^\bullet, \varepsilon^\bullet)$. These natural transformations are constructed such that they are inverses of each other. It remains to see that these maps are well defined. This will be done by showing that there is a chain map from the cone of the identity, if and only if there is a null-homotopic map from the object.

$$\begin{tikzcd}
\cdots \arrow[r, "d^{-2}_{cone(id_{A^\bullet}^\bullet)}"] & A^{-1} \oplus A^0 \arrow[r, "d^{-1}_{cone(id_{A^\bullet}^\bullet)}"] \arrow[d, "\begin{pmatrix} f^{-1} & \varepsilon^0 \end{pmatrix}"] & A^0 \oplus A^1 \arrow[r, "d^{0}_{cone(id_{A^\bullet}^\bullet)}"] \arrow[d, "\begin{pmatrix} f^{0} & \varepsilon^1 \end{pmatrix}"] & \cdots \\
\cdots \arrow[r, "d^{-2}_{B^\bullet}"] & B^{-1} \arrow[r, "d^{-1}_{B^\bullet}"] & B^0 \arrow[r, "d^{0}_{B^\bullet}"] & \cdots
\end{tikzcd}$$

For $\begin{pmatrix} f^\bullet & \varepsilon^\bullet[1] \end{pmatrix}$ to be a chain map, the following conditions must hold, i.e. that the square commute.

$$\begin{pmatrix} f^0 & \varepsilon^1 \end{pmatrix} \begin{pmatrix} d_{A^\bullet}^{-1} & id_{A^\bullet}^0 \\ 0 & -d_{A^\bullet}^0 \end{pmatrix} = d_{B^\bullet}^{-1} \begin{pmatrix} f^{-1} & \varepsilon^0 \end{pmatrix}$$

By calculating the matrix, it is a chain map if the following conditions are met.

$$f^0 d_{A^\bullet}^{-1} = d_{A^\bullet}^{-1} f^{-1}$$
$$f^0 = d_{B^\bullet}^{-1} \varepsilon^0 + \varepsilon^1 d_{A^\bullet}^0$$

Thus, a morphism is a chain map from the identity cone if and only if it is a null-homotopic chain map, which proves that there is a natural isomorphism as stated. □

Remark. The identity cone is universal with respect to homotopies. A null-homotopic chain map $f^\bullet : A^\bullet \to B^\bullet$ might admit several factorization through the identity cone. The factorizations are only unique when there is a homotopy witnessing the null-homotopy property.

$$\begin{tikzcd}
A^\bullet \arrow[rr, "(f^\bullet{,}\varepsilon^\bullet)"] \arrow[dr, "\begin{pmatrix}1\\0\end{pmatrix}"'] & & B^\bullet \\
& cone(id^\bullet_{A^\bullet}) \arrow[ur, dashed, "\begin{pmatrix}f^\bullet & \varepsilon^\bullet[1]\end{pmatrix}"'] &
\end{tikzcd}$$

Corollary 2.4.2.1. *The contravariant functor* $nullHom_{\mathcal{A}}(\_, B^\bullet)$ *is represented by* $cone(id^\bullet_{B^\bullet})[-1]$*. Thus there is a factorization of null-homotopic maps which ends in* $B^\bullet$ *as follows.*

$$\begin{tikzcd}
A^\bullet \arrow[rr, "(f^\bullet{,}\varepsilon^\bullet)"] \arrow[dr, dashed, "\begin{pmatrix}\varepsilon^\bullet\\f^\bullet\end{pmatrix}"'] & & B^\bullet \\
& cone(id^\bullet_{B^\bullet})[-1] \arrow[ur, "\begin{pmatrix}0 & -1^\bullet\end{pmatrix}"'] &
\end{tikzcd}$$

Lemma 2.4.3. $f^\bullet$ *is null-homotopic if and only if* $f^\bullet$ *factors through a contractible object.*

Proof. Suppose that $f^\bullet$ is null-homotopic, then by the universal property of null-homotopy, it factors through the identity cone. Converesly, suppose that $f^\bullet : A^\bullet \to C^\bullet$ factors thorugh a contractible object $B^\bullet$ as $g^\bullet h^\bullet$. Then $f^\bullet = g^\bullet h^\bullet = g^\bullet id^\bullet_{B^\bullet} h^\bullet$. $id^\bullet_{B^\bullet}$ is null-homotopic and homotopy equivalence is a congruence relation shows that $f^\bullet$ is null-homotopic. □

By the example in 3.1, any additive category $\mathcal{A}$ admits an exact category $\mathcal{A}, \mathcal{E}$, where $\mathcal{E} = \{$Split short-exact sequences$\}$. Then there is an exact category $(Ch(\mathcal{A}), Ch(\mathcal{E}))$, where $Ch(\mathcal{E}) = \{$Levelwise split short-exact sequences$\}$. This exact structure has enough projectives and injectives which also coincide. Instead of using levelwise split short-exact sequences, there is a more specific description of this exact structure.

Proposition 2.4.4. *The exact structure* $Ch(\mathcal{E})$ *are diagrams on the form as below, where* $r^\bullet : A^\bullet \to B^\bullet$ *is a chain map.*

$$\begin{tikzcd}
B^\bullet \arrow[r, tail, "\begin{pmatrix}1\\0\end{pmatrix}"] & cone(r^\bullet) \arrow[r, two heads, "\begin{pmatrix}0 & -1^\bullet\end{pmatrix}"] & A^\bullet[1]
\end{tikzcd}$$

Proof. Suppose that there is a conflation $(i^\bullet : Q^\bullet \to R^\bullet, p^\bullet : R^\bullet \to P^\bullet)$ in $Ch(\mathcal{A})$. The goal is to realize the object $R^\bullet$ as a cone of some map. Since the conflation is levelwise split one get that in the following diagram $R^i \simeq Q^i \oplus P^i$.

$$\begin{tikzcd}
 & Q^1 \arrow[rr, "i^1"] & & R^1 \arrow[rr, "p^1"] & & P^1 \\
Q^0 \arrow[ur, "d^0_{Q^\bullet}"] \arrow[rr, "i^0"] & & R^0 \arrow[ur, "d^0_{R^\bullet}"] \arrow[rr, "p^0"] & & P^0 \arrow[ur, "d^0_{P^\bullet}"] &
\end{tikzcd}$$

Commutativity of the squares may be rewritten as.

$$d^0_{R^\bullet} i^0 = i^1 d^0_{Q^\bullet} \iff \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} d^0_{Q^\bullet} \\ 0 \end{pmatrix}$$
$$d^0_{P^\bullet} p^0 = p^1 d^0_{R^\bullet} \iff \begin{pmatrix} 0 & d^0_{P^\bullet} \end{pmatrix} = \begin{pmatrix} 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Thus $a = d^0_{R^\bullet}$, $d = -d^0_{P^\bullet}$ and $c = 0$. The map $b : P^0 \to Q^1$ induces a map $b'^\bullet : P^\bullet[1] \to Q^\bullet$. This is a chain map by the following calculation.

$$\begin{pmatrix} d^1_{Q^\bullet} & b^1 \\ 0 & d^1_{P^\bullet} \end{pmatrix}\begin{pmatrix} d^0_{Q^\bullet} & b^0 \\ 0 & -d^0_{P^\bullet} \end{pmatrix} = \begin{pmatrix} 0 & d^1_{Q^\bullet} b^0 - b^1 d^0_{P^\bullet} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$b^\bullet$ is a chain map and thus $R^\bullet = cone(b^\bullet)$. □

To show that $(Ch(\mathcal{A}), Ch(\mathcal{E}))$ is a Frobenius category, one must show that every projective object is contractible. The case of every injective object is contractible will follow from duality, as there is a covariant and contravariant representation of null-homotopies.

Proposition 2.4.5. *An object $P^\bullet$ is projective if and only if it is contractible.*

Proof. Suppose that $P^\bullet$ is projective, then it can be found in a conflation over $cone(id^\bullet_{P^\bullet})[-1]$. By the contravariant universal property of null-homotopies, the identity map is null-homotopic as described by the diagram below.

$$\begin{tikzcd}
P^\bullet[-1] \arrow[r, tail] & cone(id^\bullet_{P^\bullet})[-1] \arrow[r, two heads] & P^\bullet \arrow[d, equal] \\
 & & P^\bullet \arrow[ul, dashed]
\end{tikzcd}$$

Converesly, suppose that $P^\bullet$ is contractible, then one may se that $P^\bullet$ is projective if and only if $cone(id^\bullet_{P^\bullet})$ is projective by the following diagram.

$$\begin{tikzcd}
P^\bullet[-1] \arrow[r, tail] & cone(id^\bullet_{P^\bullet})[-1] \arrow[r, two heads] & P^\bullet \arrow[d, equal] \\
 & cone(id^\bullet_{P^\bullet}) \arrow[u, dashed] \arrow[ur, dashed] & P^\bullet \arrow[l, tail]
\end{tikzcd}$$

It is enough to show that every identity cone is projective, to show that every contractible is projective. This is shown if the functor
$Hom_{Ch(\mathcal{A})}(cone(id^{\bullet}_{P^{\bullet}}), \_) : Ch(\mathcal{A}) \to Ab$ is an exact functor, which is the same as saying that every conflation gets mapped to short-exact sequences. Suppose further that there is a morphism $p^{\bullet} : cone(\beta^{\bullet}) \to B^{\bullet}[1]$, where $\beta^{\bullet} : B^{\bullet} \to C^{\bullet}$. To show exactness, one must show that $Hom_{Ch(\mathcal{A})}(cone(id^{\bullet}_{P^{\bullet}}), p^{\bullet})$ is a surjection.
First observe that there is an isomorphism $Hom_{Ch(\mathcal{A})}(cone(id^{\bullet}_{P^{\bullet}}), p^{\bullet}) \simeq nullHom_{\mathcal{A}}(P^{\bullet}, B^{\bullet})$.

Dette avsnittet ser ikke pent ut

Suppose that $(f^{\bullet}, \varepsilon^{\bullet}) : nullHom_{\mathcal{A}}(P^{\bullet}, B^{\bullet})$. Then there is a null-homotopic chain map $(f'^{\bullet}, \varepsilon'^{\bullet}) = (\begin{pmatrix} -\beta^{\bullet-1}\varepsilon^{\bullet} \\ f^{\bullet} \end{pmatrix}, \begin{pmatrix} 0 \\ (-1)^{\bullet+1}\varepsilon^{\bullet} \end{pmatrix}) : nullHom_{\mathcal{A}}(P^{\bullet}, cone(\beta^{\bullet})[-1])$ such that
$p^{\bullet}_{*}(f'^{\bullet}, \varepsilon'^{\bullet}) = (f^{\bullet}, \varepsilon^{\bullet})$. A diagram chase suffices to check that this is a chain map and the the proposed homotopy is in fact a homotopy. □

Corollary 2.4.5.1. *The class of contractible objects is precisly the class of projectives and the class of injectives, making $(Ch(\mathcal{A}), Ch(\mathcal{E}))$ a Frobenius category. The stable Frobenius category is equivalent with the homotopy category, i.e. $\underline{Ch(\mathcal{A})} = K(\mathcal{A})$.*

Corollary 2.4.5.2. *The homotopy category $K(\mathcal{A})$ is triangulated.*

Since the identity cones are injective, one may verify that the cosyzygy functor is the shift functor ($\mho\_ = \_[1]$). The standard triangles in $K(\mathcal{A})$ are therefore the candidate triangles on the form below.

$$A^{\bullet} \xrightarrow{f^{\bullet}} B^{\bullet} \longrightarrow cone(f^{\bullet}) \longrightarrow A^{\bullet}[1]$$

Chapter 3

The Derived Category

3.1 Idempotent Completeness and Krull-Schmidt Categories

This section is based of the ideas from [**buhler**], [**Kra12**] and [**Rei95**]. At first the ideas of idempotent complete categories will be introduced, the we will look at a weakening and a strengthening of this condition, known as weak idempotent complete and Krull-Schmidt.

Omformuler

Definition 3.1.1. An idempotent complete category is an additive category where every idempotent split. That is, if there is an idempotent $p : A \to A$ ($p^2 = p$), and there is an isomorphism $A \simeq I \oplus K$ such that $p \simeq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Every idempotent in an idempotent complete category admits an anlysis. That is the idempotent $p : A \to A$ has a kernel, cokernel, image and coimage. In fact, the kernel is isomorphic to the cokerenl, and the image is cannonically isomorphic to the coimage. As p is isomorphic to the matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ one may observe that the inclusion $\iota_1 : I \to A$ is the kernel of p, while the projection $\pi_1 : A \to I$ is the cokernel. Similarly the maps $\iota_2 : K \to A$ and $\pi_2 : A \to K$ are the kernel and cokernel of the map $1 - p$ respectively. Using the fact that p splits we are able to construct the following analysis.

Hva er en analyse, jeg bruker dette tidligere

$$\begin{tikzcd}
& A \arrow[rr, "p"] \arrow[dr, two heads, "\pi_2"] & & A \arrow[dr, two heads, "\pi_1"] & \\
I \arrow[ur, tail, "\iota_1"] & & K \arrow[ur, tail, "\iota_2"] & & I
\end{tikzcd}$$

Remark. Assuming that every idempotent in an additive category $\mathcal{A}$ has a kernel is sufficient for $\mathcal{A}$ to be idempotent complete. The limits and colimits as described above may be found with the idempotents p and $1 - p$.

Every additive category $\mathcal{A}$ has a fully faithful embedding $i_{\mathcal{A}} : \mathcal{A} \to \widehat{\mathcal{A}}$ into an idempotent complete category $\widehat{\mathcal{A}}$. This completion satisfies the universal property in which if there is a functor $F : \mathcal{A} \to \mathcal{B}$ which sends every idempotent p in $\mathcal{A}$ to a splitting idempotent, then the functor factors through the idempotent complete category $\widehat{\mathcal{A}}$.

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "F"] \arrow[dr, "i_{\mathcal{A}}"'] & & \mathcal{B} \\ & \widehat{\mathcal{A}} \arrow[ur, "\widehat{F}"'] & \end{tikzcd}$$

One may define this completion $\widehat{\mathcal{A}}$ to be the category with objects (A, p), where A is an object of $\mathcal{A}$ and $p : A \to A$ is an idempotent. A morphism $\widehat{f} : (A, p) \to (B, q)$ is defined as the morphism $\widehat{f} = q \circ f \circ p$ for some morphism $f : A \to B$. The injection functor is defined as $i_{\mathcal{A}}(A) = (A, id_A)$. More on this injection can be found in [**buhler**].
Many of the useful theorems needed to describe the triangulated subcategory needed for the construction of the derived category will arise from the weaker condition of weakly split idempotents.

Lemma 3.1.1. *The following are equivalent in an additive category:*

1. *Every split-epi has a kernel*
2. *Every split-mono has a cokernel*

Proof. It suffices to prove that (1.) $\implies$ (2.), as the other claim is dual. Suppose that $g : B \to A$ is split-epi with $f : A \to B$ as the corresponding split-mono such that $gf = id_A$. Since g is split-epi it has a kernel $h : C \to B$.

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \arrow[r, bend left, dashed, "i"] & C \arrow[l, bend left, "h"] \end{tikzcd}$$

Looking at the map $id_B - fg$, one may see that $g(id_B - fg) = g - gfg = g - g = 0$, thus $id_B - fg$ factors over the kernel of h as indicated by the dashed arrow.
h is split-mono as $hih = (id_B - fg)h = h - fgh = h$. As h is mono from being a kernel, it follows that $ih = id_C$. $B \simeq A \oplus C$ as $id_B - fg = hi \iff id_B = fg + hi$. This in turn implies that i is the cokernel of f. $\square$

Finnes det en bedre formulering her?

This lemma is at the core of weakly idempotent complete categories.

Definition 3.1.2. An additive category $\mathcal{A}$ is weakly idempotent complete if it satisfies either of the conditions of Lemma 4.1.

Corollary 3.1.1.1. *Let $(\mathcal{A}, \mathcal{E})$ be an exact category, then the following are equivalent:*

1. *The category $\mathcal{A}$ is weakly idempotent complete*
2. *Every split-mono is an inflation*
3. *Every split-epi is a deflation*

With the notion of a weakly idempotent complete category, the Obscure axiom can be strengthened into Hellers cancellation axiom.

Proposition 3.1.2. ***Hellers cancellation axiom*** *For an exact category* $(\mathcal{A}, \mathcal{E})$ *the following are equivalent:*

1. $\mathcal{A}$ *is weakly idempotent complete*
2. *Let* $f : A \to B$ *and* $g : B \to C$ *be two morphisms in* $\mathcal{A}$*. Then if* $gf : A \to C$ *is a deflation, then* g *is.*

Proof. Suppose (1.). Let $f : A \to B$ and $g : B \to C$ be morphisms such that their composition $gf : A \to C$ is a deflation. Since gf is a deflation, the pullback square exists.

$$\begin{tikzcd}
A \arrow[drr, bend left, "f"] \arrow[dr, dashed, "f'"] \arrow[ddr, bend right, "id_A"'] & & \\
& B' \arrow[r, two heads, "f'"] \arrow[d, "h"] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d, "g"] \\
& A \arrow[r, two heads, "gf"] & C
\end{tikzcd}$$

By using the universal property,one may see that g' is split-mono, hence it admits an inflation $h' : A' \to B'$. The claim is that $hh' : A' \to B$ is the kernel of g. If the claim is true, the Obscure axiom yields that g is a deflation.
To show that hh' is the kernel one must show the universal property. Let $t : T \to B$ be a test object, such that $gt = 0$.

$$\begin{tikzcd}
T \arrow[drr, bend left, "t"] \arrow[dr, dashed, "t'"] & & \\
& A' \arrow[r, "hh'"] \arrow[dr, "0"] & B \arrow[d, "g"] \\
& & C
\end{tikzcd}$$

It is known that t' exists as t factors through B' with t'', by the pull-back property. As $g't'' = 0$, t'' factors through A' using the fact that h' is the kernel of g', this proves the claim.
For the other direction, suppose (2.) instead and let $gf = id_A$, gf is a deflation and g is split-epi. By the assumption, g is a deflation, so it has a kernel. □

Omformulering

For the final part, if $\mathcal{A}$ is an idempotent complete category and there are idempotents over an object A, then these idempotents admits a description of A as a direct sum of kernels and cokernels. There is, however, no guarantee that these decompositions are unique. To fix this, define the following category. These results will not be proved, and the reader is instead reffered to Henning Krause ([**Kra12**]) and Auslander, Reiten and Smalø ([**Rei95**]).

Definition 3.1.3. Let $\mathcal{A}$ be an additive category. An object A is called indecomposable if the endomorphism ring of A is local.
An object is called decomposable if it is not indecomposable.

Definition 3.1.4. An additive category $\mathcal{A}$ is called Krull-Schmidt if any object A decomposes into a finite direct sum of indecomposable objects.

Having that each indecomposable object is local is enough for the following proposition to hold.

Proposition 3.1.3. *Every decomposition in a Krull-Schmidt category is unique up to isomorphism*

As being Krull-Schmidt admits decomposition whenever an endomorphism ring is not local implies a connection to idempotent completeness. That is whenever there is an idempotent over an object, this idempotent give rise to two comaximal ideals for the endomorphism ring. This gives us the decomposition which is required for the idempotent to split. Moreover, there is a deeper connection with being Krull-Schmidt and idempotent complete.

Definition 3.1.5. Let R be a ring. We say that R is semiperfect if R as a module over itself admits a decomposition ${}_RR \simeq P_1 \oplus P_2 \oplus \ldots \oplus P_n$ such that each P_i has a local endomorphism ring.

Remark. For a ring R the following conditions are equivalent:

- The category mod_R is a Krull-Schmidt category
- R is semiperfect
- Every simple R-module has a projective cover
- Every finitely generated R-module has a projective cover

Thus any of these conditions can be taken to be the definition of semiperfect.

With this definition we are able to state the following proposition, which says whenever an idempotent complete category is Krull-Schmidt.

Proposition 3.1.4. *Let $\mathcal{A}$ be an additive category, then the following are equivalent:*

1. *$\mathcal{A}$ is Krull-Schmidt*
2. *$\mathcal{A}$ is idempotent complete and every endomorphism ring are semiperfect.*

Example. Let Λ be any artin R-algebra, then mod_Λ is a Krull-Schmidt category. As an example, the category of finitely generated real vector spaces is Krull-Schmidt. Every vector space is a finite direct summand of the only indecomposable vector space $\mathbb{R}$.

More details and examples of Krull-Schmidt categories may be found in Henning Krause notes ([**Kra12**]).

3.2 Admissable Morphisms, Homology and Long Exact Sequences

Definition 3.2.1. Let $(\mathcal{A}, \mathcal{E})$ be an exact category. A morphism $f : A \to B$ is called normal if it has a deflation-inflation factorization. They will be drawn as in the following diagram.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, two heads] & & B \\ & I \arrow[ur, tail] & \end{tikzcd}$$

Remark. A monomorphism is normal if and only if it is an inflation. Dually, an epimorphism is normal if and only if it is a deflation.

Remark. In general the composition gf of two normal morphisms f and g are not normal. However, if g is a deflation, the composition can be seen to normal, as deflations are closed under composition. One may also observe that an exact category is abelian if and only if normal morphisms are closed under composition.

Lemma 3.2.1. ***Hellers factorization lemma****. The factorization of normal morphisms are unique up to unique isomorphisms.*

Proof. Suppose that a normal morphism admits two differnt factorization. That means there exists a commutative diagram as follows.

$$\begin{tikzcd} A \arrow[r, two heads, "p"] \arrow[d, two heads, "q"] & I \arrow[d, tail, "i"] \arrow[dl, dashed, bend left, "\phi"] \\ I' \arrow[r, tail, "j"] \arrow[ur, dashed, "\phi'"] & B \end{tikzcd}$$

By assumption $ip = jq$, thus $jq \circ Ker(p) = 0$. $q \circ Ker(p) = 0$ as j is mono, thus there exists a morphism $\phi : I \to I'$ uniquely such that $q = \phi p$. Now $ip = jq = j\phi p$, and as p is epi it follows that $i = j\phi$. Reiterating the argument, but with $Ker(q)$ instead, there exists a ϕ' uniquely such that $p = \phi' q$ and $j = i\phi'$. Thus $i = j\phi = i\phi'\phi$, and since i is mono it follows that $id_I = \phi'\phi$; dually $Id_{I'} = \phi\phi'$. □

Er dette ordet stavet riktig?

Remark. Du to Hellers factorization axiom one may see that normal morphisms admits analysis.

$$\begin{tikzcd} & A \arrow[rr, "f"] \arrow[dr, two heads, "p"] & & B \arrow[dr, two heads, "Cok(i)"] & \\ K \arrow[ur, tail, "Ker(p)"] & & I \arrow[ur, tail, "i"] & & C \end{tikzcd}$$

Observe that the object I coincide with the image and coimage of f. This object will then be reffered to the image of f. As a consequence of this unique factorization, a normal morphism is iso if and only if it is mono and epi.

Definition 3.2.2. A sequence of normal morphisms is exact if the inflation of the factorization together with the consecutive deflation forms a conflation. That is there are conflations between morphisms as in the following diagram. The conflation pairs are highlighted with different colors.

$$\begin{tikzcd}
\cdots \arrow[rr, "f_{-2}"] \arrow[dr, "p_{-2}"'] & & A_{-1} \arrow[rr, "f_{-1}"] \arrow[dr, "p_{-1}"'] & & A_0 \arrow[rr, "f_0"] \arrow[dr, "p_0"'] & & A_1 \arrow[rr, "f_1"] \arrow[dr, "p_1"'] & & \cdots \\
& I_{-2} \arrow[ur, "i_{-2}"'] & & I_{-1} \arrow[ur, "i_{-1}"'] & & I_0 \arrow[ur, "i_0"'] & & I_1 \arrow[ur, "i_1"'] &
\end{tikzcd}$$

A morphism of exact sequences is the same as a morphism of sequences. That is a collection of morphisms $(..., \phi_{-1}, \phi_0, \phi_1, ...)$ such that the squares in the diagram commute.

$$\begin{tikzcd}
\cdots \arrow[r, "a_{-2}"] & A_{-1} \arrow[r, "a_{-1}"] \arrow[d, "\phi_{-1}"] & A_0 \arrow[r, "a_0"] \arrow[d, "\phi_0"] & A_1 \arrow[r, "a_1"] \arrow[d, "\phi_1"] & \cdots \\
\cdots \arrow[r, "b_{-2}"] & B_{-1} \arrow[r, "b_{-1}"] & B_0 \arrow[r, "b_0"] & B_1 \arrow[r, "b_1"] & \cdots
\end{tikzcd}$$

Remark. An exact sequence of normal morphisms is called short exact if it consists of morphisms on the form $(, 0, i, p, 0,)$, i.e. as in the following diagram.

$$\begin{tikzcd}
0 \arrow[rr, "0"] \arrow[dr, equal] & & A \arrow[rr, "i"] \arrow[dr, equal] & & B \arrow[rr, "p"] \arrow[dr, "p"'] & & C \arrow[rr, "0"] \arrow[dr, "0"'] & & 0 \\
& 0 \arrow[ur, "0"] & & A \arrow[ur, "i"'] & & C \arrow[ur, equal] & & 0 \arrow[ur, equal] &
\end{tikzcd}$$

Observe how conflations are exactly the class of short exact sequences.

This definition admits properties which mimics properties from homologicial algebra.

Lemma 3.2.2. ***5 Lemma****. Given two 5 term exact sequences and a morphism between them as in the diagram. Then ϕ is an isomorphism as well.*

$$\begin{tikzcd}
A_0 \arrow[r, "a_0"] \arrow[d, "\simeq"] & A_1 \arrow[r, "a_1"] \arrow[d, "\simeq"] & A_2 \arrow[r, "a_2"] \arrow[d, "\phi"] & A_3 \arrow[r, "a_3"] \arrow[d, "\simeq"] & A_4 \arrow[d, "\simeq"] \\
B_0 \arrow[r, "b_0"] & B_1 \arrow[r, "b_1"] & B_2 \arrow[r, "b_2"] & B_3 \arrow[r, "b_3"] & B_4
\end{tikzcd}$$

Lemma 3.2.3. ***Kernel-Cokernel sequence****. Let $(\mathcal{A}, \mathcal{E})$ be an exact category which is weakly idempotent complete. Suppose that there are composable normal morphism f and g such that gf is normal as well. Then there exists an exact sequence.*

$$Ker(f) \longrightarrow Ker(gf) \longrightarrow Ker(h) \longrightarrow Cok(f) \longrightarrow Cok(gf) \longrightarrow Cok(g)$$

Remark. If $(\mathcal{A}, \mathcal{E})$ is an exact category, then one may show that the category $\mathcal{A}$ admits Kernel-Cokernel sequences if and only if it is weakly idempotent complete.

The Kernel-Cokernel sequence enables one to prove that the snake lemma holds in weakly idempotent complete categories.

Corollary 3.2.3.1. ***Snake Lemma.*** *Let $(\mathcal{A}, \mathcal{E})$ be a weakly idempotent complete category. Suppose there is a diagram in $\mathcal{A}$ having exact rows.*

$$\begin{tikzcd}
 & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' &
\end{tikzcd}$$

Then there is an exact sequence.

$$\begin{tikzcd}
Ker(f) \arrow[r] & Ker(g) \arrow[r] & Ker(h) \arrow[r, dashed, "\delta"] & Cok(f) \arrow[r] & Cok(g) \arrow[r] & Cok(h)
\end{tikzcd}$$

Merkelig formulert

The action which defines homological algebra is to find the homology of cochain complexes. For abelian groups, homology is defined to be the quotient of the kernel of a map by the image of the preceeding map. We would like to find a similar definition for exact categories. For this discussion, let $(Ch(\mathcal{A}), Ch(\mathcal{E}))$ be an exact category. In order for a complex to have kernels and images, it is required for the complex to consist only of normal morphisms. This will allow for the existence of analyses of every differential. Let $A^\bullet$ be such a complex, then when does the homology exists?

$$\begin{tikzcd}
\dots \arrow[r, "d^{-2}_{A^\bullet}"] & A_{-1} \arrow[r, "d^{-1}_{A^\bullet}"] \arrow[d, two heads, "p"] & A_0 \arrow[r, "d^{0}_{A^\bullet}"] & A_1 \arrow[r, "d^{1}_{A^\bullet}"] & \dots \\
 & Im(d^{-1}_{A^\bullet}) \arrow[ur, tail, "\iota"] \arrow[r, dashed, "h"] & Ker(d^0_{A^\bullet}) \arrow[u, "\kappa"] \arrow[r, dotted, "?"] & H^0(A^\bullet) &
\end{tikzcd}$$

By looking at the 0-th homology one can find a condition for when the homology exists. Using the fact that $d^0_{A^\bullet}\iota = 0$, there is an unique morphism h, such that $\iota = \kappa h$ by the universal property. The 0-th homology exists whenever the morphism h has a cokernel, and then h satisfies the assumption of the Obscure axiom, making h and inflation. One way to circumvent this assumption is to assume that $\mathcal{A}$ is weakly idempotent complete. Then by Hellers cancellation axiom it is known that h is an inflation, which then proves the existence of the cokernel.

3.3 The Derived Category/Derived Functors

Rar temasetning

In homological algebra, we want to localize the homotopy category over quasi-isomorphisms. A quasi-isomorphism is a chain map $f^\bullet : A^\bullet \to B^\bullet$ such that $H^*(f^\bullet) : H^*(A^\bullet) \to H^*(B^\bullet)$ is an isomorphism in homology. One may observe by the long exact sequence in homology of short exact sequence that a chain map like $f^\bullet$ is a quasi-isomorphism if and only if $cone(f^\bullet) \simeq 0$. Moreover, the cone is isomorphic to 0 in homology if it is an exact sequence. This motivates the definition of the subcategory of acyclic complexes.

Definition 3.3.1. Let $(\mathcal{A}, \mathcal{E})$ be an exact category. We define the category $Ac(\mathcal{A}) \subset K(\mathcal{A})$ to be the full category whose objects are exact sequences.

The exact complexes are also referred to as acyclic complexes. Note that this subcategory is not in general closed under isomorphisms. To be able to show that it is a triangulated subcategory, it suffices to show that the mapping cone of two acyclic complexes is again acyclic.

Lemma 3.3.1. *Let $f^\bullet : A^\bullet \to B^\bullet$ be a chain map between acyclic chain complexes. Then the mapping cone $cone(f^\bullet)$ is acyclic as well.*

Proof. Write diagrams here □

Since $Ac(\mathcal{A})$ is triangulated, it makes sense to talk about the class of morphisms $Mor_{Ac(\mathcal{A})}$. Observe that in the case of the category $\mathcal{A}$ being abelian, one may observe that this is the class of quasi-isomorphic chain maps. Therefore the class of morphisms $Mor_{Ac(\mathcal{A})}$ may be regarded as quasi-isomorphisms.

Definition 3.3.2. The derived category is the Verdier quotient $D(\mathcal{A}) = K(\mathcal{A})/Ac(\mathcal{A})$ whenever it exists.

Merkelig avsnitt

Given some conditions on the category $(\mathcal{A}, \mathcal{E})$ one is able to get a nice description of the derived category. That is we want the category of acyclic chain complexes to be closed under isomorphisms. The following proposition tells us whener this is true.

Lemma 3.3.2. *The following are equivalent:*

1. *Every null-homotopic chain complex is acyclic*
2. *The category $\mathcal{A}$ is idempotent complete*
3. *The subcategory $Ac(\mathcal{A})$ is closed under isomorphisms*

Proof. □

Corollary 3.3.2.1. *The subcategory $Ac(\mathcal{A})$ is thick if and only if $\mathcal{A}$ is idempotent complete.*

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We may also set boundedness conditions on the chain complex to get weaker assumptions to get nice acyclic chain complexes. We say that a chain complex is called left bounded if there is some m such that for any n $n \leq m < 0$ we have that $A^n = 0$. Likewise, right bounded complexes are the defined for $n \geq m > 0$ we have that $A^n = 0$. A chain complex is called bounded if it is both left bounded and right bounded.

Definition 3.3.3. The category $K(\mathcal{A})^+$,$K(\mathcal{A})^-$ and $K(\mathcal{A})^\flat$ are the homotopy categories of left bounded, right bounded and bounded respectively. $Ac(\mathcal{A})^* \subset K(\mathcal{A})^*$ for $* : \{+,-,\flat\}$ will be the subcategory of acyclic chain complexes satisfying the correct boundedness condition.

Lemma 3.3.3. *The following are equivalent:*

1. *The subcategories $Ac(\mathcal{A})^* \subset K(\mathcal{A})^*$ for $* : \{+,-\}$ are thick*
2. *The subcategory $Ac(\mathcal{A})^\flat$*
3. *The category $\mathcal{A}$ is weakly idempotent complete*

3.4 Auslander-Reiten triangles

For the rest of this thesis it will be assumed that $\mathcal{A}$ is an additive $\mathbb{K}$-linear category, which is also Krull-Schmidt. With these assumptions, one may be able to define a special set of triangles whenever $\mathcal{A}$ has a triangulation. As $\mathcal{A}$ is Krull-Schmidt, the notion of indecomposable objects exists, which motivates the definition of triangles which acts as indecomposable triangles.

What is a $\mathbb{K}$-linear category?

Definition 3.4.1. Define an Auslander-Reiten triangle

Proposition 3.4.1. *Selfduality of AR-triangles*

Definition 3.4.2. Define irreducible morphisms

Proposition 3.4.2. *irreducible morphisms have the following properties:*

3.5 Description of Derived Categories

Add something to the introduction about derived categories. The goal is to describe the derived category of a fairly simple algebra.

Bibliography