Thesis

Thomas Wilskow Thorbjørnsen

March 31, 2021

Contents

1 Introduction

This is an introduction, welcome!

Introduce notation which will be used in text. A list of notation and description would be nice, so that the reader might scroll back up if something is unclear.

2 Triangulated Categories

Probably introduce this section, what is happening and what will be done etc. I can maybe say something about algebraic triangulated categories and topological triangulated categories, and explaining the name cone, fiber and cofiber.

2.1 Definition and First Properties

In this section $\mathcal{T}$ denotes an additive category and $T : \mathcal{T} \to \mathcal{T}$ is an additive autoequivalence of $\mathcal{T}$, which is often called translation or suspension functor.

Definition 2.1. A candidate triangle is a collection (A, B, C, a, b, c) of objects $A, B, C \in T$ and morphisms $a : A \to B$, $b : B \to C$, $c : C \to TA$. These candidate triangles can be drawn as diagrams in the following way:

$$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA$$

A morphism between candidate triangles is a triple of morphism (α, β, γ), where $\alpha : A \to A'$, $\beta : B \to B'$ and $\gamma : C \to C'$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\alpha"] & B \arrow[r, "b"] \arrow[d, "\beta"] & C \arrow[r, "c"] \arrow[d, "\gamma"] & TA \arrow[d, "T\alpha"] \\
A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C \arrow[r, "c'"] & TA'
\end{tikzcd}$$

The naming convention of the candidate triangles isn't standarized, some literatures calls the candidate triangles for triangles instead; see [1]. This name arises from an alternate description of the diagrams given above. To remove confusion about the domain or codomain of the arrows, one arrow of the triangle is decorated with "$_T$—". This decorator means that the functor T has to be applied to the corresponding edge of the arrow. Thus the c arrow points to TA, not A.

$$\begin{tikzcd}
A \arrow[dr, "a"] & \\
& B \arrow[dl, "b"] \\
C \arrow[uu, "T", "c"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rrr, "\phi_a"] \arrow[dr, "a"] & & & A' \arrow[dl, "a'"'] \\
& B \arrow[r, "\phi_b"] \arrow[dl, "b"] & B' \arrow[dr, "b'"] & \\
C \arrow[uu, "T", "c"'] \arrow[rrr, "\phi_c"] & & & C' \arrow[uu, "T"', "c'"]
\end{tikzcd}$$

A triangulated category is an additive category together with a translation functor T and a triangulation Δ consisting of candidate triangles. When a candidate triangle is an element of Δ it is usually called a triangle, an exact triangles or just a triangle. Note that if the candidate triangles are referred to as triangles it is common to either call the elements of Δ for triangles or exact triangles. As this is not the case for this thesis these objects will be referred to as triangles.

Definition 2.2. A triangulation of an additive category $\mathcal{T}$ with translation T is a collection Δ of triangles consisting of candidate triangles in $\mathcal{T}$ satisfying the following axioms:

1. (TR1) Formation axiom

   (a) A candidate triangle isomorphic to a triangle is a triangle.

   (b) Every morphism $a : A \to B$ can be embedded into a triangle (A, B, C, a, b, c):

   $$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA$$

   (c) For every object A there is a triangle $(A, A, 0, id_A, 0, 0)$:

   $$A \xrightarrow{id_A} A \xrightarrow{0} 0 \xrightarrow{0} TA$$

2. (TR2) Rotation axiom

   For every triangle (A, B, C, a, b, c) there is a triangle (B, C, TA, b, c, Ta)

   $$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA \quad \Longrightarrow \quad B \xrightarrow{b} C \xrightarrow{c} TA \xrightarrow{-Ta} TB$$

3. (TR3) Morphism axiom

   Given the two triangles (A, B, C, a, b, c) (1) and (A', B', C', a', b', c') (2)

   $$(1) \quad A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA \quad (2) \quad A' \xrightarrow{a'} B' \xrightarrow{b'} C' \xrightarrow{c'} TA'$$

   and morphisms $\phi_A : A \to A'$ and $\phi_B : B \to B'$ such that the square (1) commutes, then there is a morphism $\phi_C : C \to C'$ (not necessarily unique) such that (ϕ_A, ϕ_B, ϕ_C) is a morphism of triangles (2).

   $$(1) \quad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[d, "\phi_B"] \\ A' \arrow[r, "a'"] & B' \end{tikzcd} \qquad (2) \quad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[r, "b"] \arrow[d, "\phi_B"] & C \arrow[r, "c"] \arrow[d, dashed, "\phi_C"] & TA \arrow[d, "T\phi_A"] \\ A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & TA' \end{tikzcd}$$

4. (TR4) Octahedron axiom

   Given the triangles (A, B, C', a, x, x') (1), (B, C, A', b, y, y') (2) and $(A, C, B', b \circ a, z, z')$ (3)

$$
\begin{aligned}
&(1)\quad A \xrightarrow{a} B \xrightarrow{x} C' \xrightarrow{x'} TA \\
&(2)\quad B \xrightarrow{b} C \xrightarrow{y} A' \xrightarrow{y'} TB \\
&(3)\quad A \xrightarrow{b\circ a} C \xrightarrow{z} B' \xrightarrow{z'} TA
\end{aligned}
$$

then there exist morphisms $f : C' \to B'$ and $g : B' \to A'$, the following diagram commutes and the third row is a triangle:

$$
\begin{tikzcd}
T^{-1}B' \arrow[r, "T^{-1}z'"] \arrow[d, "T^{-1}g"] & A \arrow[r, equal, "id_A"] \arrow[d, "a"] & A \arrow[d, "b\circ a"] & & \\
T^{-1}A' \arrow[r, "T^{-1}y'"] & B \arrow[r, "b"] \arrow[d, "x"] & C \arrow[r, "y"] \arrow[d, "z"] & A' \arrow[r, "y'"] \arrow[d, equal, "id_{A'}"] & TB \arrow[d, "Tx'"] \\
& C' \arrow[r, "f"] \arrow[d, "x'"] & B' \arrow[r, "g"] \arrow[d, "z'"] & A' \arrow[r, "Ti\circ y'"] & TC' \\
& TA \arrow[r, equal, "id_{TA}"] & TA & &
\end{tikzcd}
$$

A triangulated category is denoted as $(\mathcal{T}, T, \Delta)$, where $\mathcal{T}$ is the additive category, T is the translation and Δ is the triangulation. When $\mathcal{T}$ is called a triangulated category, it should be understanded as the triple given above.

Remark. The rotation axiom has a dual, and it can be thought of as a rotation in the opposite direction. This dual can be proved by the other axioms, so it is here omitted as an axiom. The dual roation axiom goes as:

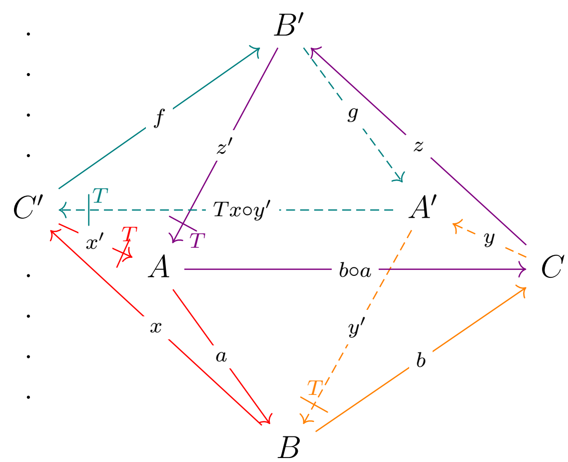
Given a triangle $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA$, there is a triangle $T^{-1}C \xrightarrow{-T^{-1}c} A \xrightarrow{a} B \xrightarrow{b} C$
To be able to prove this, some more lemmata are needed.

Remark. The final axiom is referred to as the octahedron axiom. By using the alternative description of the triangle diagram, it is possible to rewrite the diagram as an octahedron. The axiom can be restated as the following:

Given the triangles (A, B, C', a, x, x') (1), (B, C, A', b, y, y') (2)
and $(A, C, B', b \circ a, z, z')$ (3)

$$
(1)\;
\begin{tikzcd}
A \arrow[dr, "a"] & \\
& B \arrow[dl, "x"] \\
C' \arrow[uu, "T", "x'"'] &
\end{tikzcd}
\qquad
(2)\;
\begin{tikzcd}
B \arrow[dr, "b"] & \\
& C \arrow[dl, "y"] \\
A' \arrow[uu, "T", "y'"'] &
\end{tikzcd}
\qquad
(3)\;
\begin{tikzcd}
A \arrow[dr, "b\circ a"] & \\
& C \arrow[dl, "z"] \\
B' \arrow[uu, "T", "z'"'] &
\end{tikzcd}
$$

then there exists morphisms $f : C' \to B'$ and $g : B' \to A'$, the following diagram commutes and the teal back face is a triangle.



Lemma 2.1. *Let* (A, B, C, a, b, c) *be a triangle, then* $b \circ a = 0$

Proof. By TR2 the triangle (A, B, C, a, b, c) can be rotated to (B, C, TA, b, c, Ta).

$$
\begin{tikzcd}
A \arrow[dr, "a"] & \\
& B \arrow[dl, "b"] \\
C \arrow[uu, "c", "T"'] &
\end{tikzcd}
\implies
\begin{tikzcd}
B \arrow[dr, "b"] & \\
& C \arrow[dl, "c"] \\
TA \arrow[uu, "-Ta", "T"'] &
\end{tikzcd}
$$

The triangle exists $(C, C, 0, id_C, 0, 0)$ by TR1 and TR3 says there exists a morphism from TA to 0 making the diagram below commute.

$$
\begin{tikzcd}
B \arrow[r, "b"] \arrow[d, "b"] & C \arrow[r, "c"] \arrow[d, "id_C"] & TA \arrow[r, "-Ta"] \arrow[d, dashed, "0"] & TB \arrow[d, "Tb"] \\
C \arrow[r, "id_C"] & C \arrow[r, "0"] & 0 \arrow[r, "0"] & TC
\end{tikzcd}
$$

Thus $0 = Tb \circ -Ta = T(-ba) \implies b \circ a = 0$ as T is a translation. $\square$

Definition 2.3. An additive functor between triangulated categories $F : (\mathcal{T}, T, \Delta) \to (\mathcal{R}, R, \Gamma)$ is called exact or triangulated if there exist a natural isomorphisms $\alpha : FT \to RF$ such that $F(\Delta) \subseteq \Gamma$.

A functor $F : \mathcal{T} \to \mathcal{R}$ is called a triangle-equivalence if it is triangulated and an equivalence of categories. In this case $\mathcal{T}$ and $\mathcal{R}$ are called triangle-equivalent.

Definition 2.4. Let $\mathcal{T}$ be a triangulated category and $\mathcal{A}$ be an abelian category. A covariant functor $H : \mathcal{T} \to \mathcal{A}$ is called a homological functor if $\forall (A, B, C, a, b, c) : \Delta$ there is a long exact sequence in $\mathcal{A}$.

$$\begin{tikzcd} A \arrow[dr, "a"] & \\ & B \arrow[dl, "b"] \\ C \arrow[uu, "c", "T"'] & \end{tikzcd} \implies \begin{tikzcd} \cdots \arrow[r] & H(T^iA) \arrow[r, "H(T^ia)"] & H(T^iB) \arrow[r, "H(T^ib)"] & H(T^iC) \arrow[dll, "H(T^ic)"] & \\ & H(T^{i+1}A) \arrow[r, "H(T^{i+1}a)"] & H(T^{i+1}B) \arrow[r, "H(T^{i+1}b)"] & H(T^{i+1}C) \arrow[r] & \cdots \end{tikzcd}$$

Dually, a contravariant functor $H : \mathcal{T} \to \mathcal{A}$ is called cohomological if $\forall (A, B, C, a, b, c) : \Delta$ there is a long exact sequence in $\mathcal{A}$.

$$\begin{tikzcd} A \arrow[dr, "a"] & \\ & B \arrow[dl, "b"] \\ C \arrow[uu, "c", "T"'] & \end{tikzcd} \implies \begin{tikzcd} \cdots & H(T^{i-1}A) \arrow[l] & H(T^{i-1}B) \arrow[l, "H(T^{i-1}a)"'] & H(T^{i-1}C) \arrow[l, "H(T^{i-1}b)"'] & \\ & H(T^iA) \arrow[urr, "H(T^ic)"] & H(T^iB) \arrow[l, "H(T^ia)"'] & H(T^iC) \arrow[l, "H(T^ib)"'] & \cdots \arrow[l] \end{tikzcd}$$

Lemma 2.2. *Let $M : \mathcal{T}$ be any object of $\mathcal{T}$, then the represented functors $\mathcal{T}(M, \_)$ is homological and $\mathcal{T}(\_, M)$ is cohomological.*

Proof. Only the covariant case needs to be proved, as the contravariant case is dual. For $\mathcal{T}(M, \_)$ to be homological, it has to create long exact sequences for every triangle in Δ. Let $(A, B, C, a, b, c) : \Delta$ be a triangle, then there can be extracted sequences in Ab for any $i : \mathbb{N}$.

$$\begin{tikzcd} A \arrow[dr, "a"] & \\ & B \arrow[dl, "b"] \\ C \arrow[uu, "c", "T"'] & \end{tikzcd} \implies \mathcal{T}(M, T^iA) \xrightarrow{T^ia_*} \mathcal{T}(M, T^iB) \xrightarrow{T^ib_*} \mathcal{T}(M, T^iC)$$

Observe that it is enough to prove that these types of diagrams are exact, as the other diagrams can be obtained by the rotation axiom, thus reducing it to same case.

The goal is then to prove that $Im(T^ia_*) = Ker(T^ib_*)$. Since $ba = 0$ it follows that $Im(T^ia_*) \subseteq Ker(T^ib_*)$. Assume that $f : Ker(T^ib_*)$, that is $f : M \to T^iB$ such that $b_*(f) = 0$. The current goal is to show that f factors through T^iA, as this means that $Ker(T^ib_*) \subseteq Im(T^ia_*)$. Note that since T is a translation, it is necessarily a right adjoint to the inverse translation; thus $\mathcal{T}(M, T^iB) \simeq \mathcal{T}(T^{-i}M, B)$, and by this assertion it suffices to assume that $f : T^{-i}M \to B$ such that $b \circ f = 0$. By TR1 and TR2 there exists triangles $(T^{-i}M, 0, T^{-i+1}M, 0, 0, -T^{-i+1}id)$ and $(B, C, TA, b, c, -Ta)$.

$$\begin{tikzcd} T^{-i}M \arrow[r, "0"] \arrow[d, "f"] & 0 \arrow[r, "0"] \arrow[d, "0"] & T^{-i+1}M \arrow[r, "-T^{-i+1}id"] \arrow[d, dashed, "g"] & T^{-i+1}M \arrow[d, "Tf"] \\ B \arrow[r, "b"] & C \arrow[r, "c"] & TA \arrow[r, "-Ta"] & TB \end{tikzcd}$$

The left square commutes by the assumption, thus the morphism g exist by TR3, such that $-Ta \circ h = -Tf \circ T^{-i+1}id = -Tf \implies Ta \circ h = Tf$, thus $f = a \circ T^{-1}h$ asserting that f factors through A. $\square$

Lemma 2.3. *Let* $(\phi_A, \phi_B, \phi_C) : (A, B, C, a, b, c) \to (A', B', C', a', b', c')$ *be a morphism of triangles. If 2 of the maps are isomorphisms, then the last one is an isomorphism as well.*

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\phi_A", "\wr"'] & B \arrow[r, "b"] \arrow[d, "\phi_B", "\wr"'] & C \arrow[r, "c"] \arrow[d, dashed, "\phi_C", "\wr"'] & TA \arrow[d, "T\phi_A", "\wr"'] \\
A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & TA'
\end{tikzcd}$$

Proof. Without loss of generality, assume that ϕ_A and ϕ_B are the isomorphisms. This can be done as the rotation axiom reduce the other cases to this case. Then we have the following diagram:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\phi_A", "\wr"'] & B \arrow[r, "b"] \arrow[d, "\phi_B", "\wr"'] & C \arrow[r, "c"] \arrow[d, "\phi_C"] & TA \arrow[d, "T\phi_A", "\wr"'] \\
A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & TA'
\end{tikzcd}$$

By applying the functor $\mathcal{T}(C', \_)$ we get the following diagram in Ab:

$$\begin{tikzcd}
\mathcal{T}(C', A) \arrow[r, "a_*"] \arrow[d, "(\phi_A)_*", "\wr"'] & \mathcal{T}(C', B) \arrow[r, "b_*"] \arrow[d, "(phi_B)_*", "\wr"'] & \mathcal{T}(C', C) \arrow[r, "c_*"] \arrow[d, "(\phi_C)_*"] & \mathcal{T}(C', TA) \arrow[r, "Ta_*"] \arrow[d, "(\phi_T A)_*", "\wr"'] & \mathcal{T}(C', TB) \arrow[d, "(T\phi_B)_*", "\wr"'] \\
\mathcal{T}(C', A') \arrow[r, "a'_*"] & \mathcal{T}(C', B') \arrow[r, "b'_*"] & \mathcal{T}(C', C') \arrow[r, "c'_*"] & \mathcal{T}(C', TA') \arrow[r, "Ta_*"] & \mathcal{T}(C', TB)
\end{tikzcd}$$

By the 5-lemma, we get that $(\phi_C)_*$ is an isomorphisms, i.e. $(\phi_C)_*$ is both mono and epi. Thus for some unique s in $\mathcal{T}(C', C)$, $\phi_{C*}(s) = id_{C'}$.

By applying the functor $\mathcal{T}(\_, C)$ we get the diagram:

$$\begin{tikzcd}
\mathcal{T}(A, C) & \mathcal{T}(B, C) \arrow[l, "a^*"'] & \mathcal{T}(C, C) \arrow[l, "b^*"'] & \mathcal{T}(TA, C) \arrow[l, "c^*"'] & \mathcal{T}(TB, C) \arrow[l, "Ta^*"'] \\
\mathcal{T}(A', C) \arrow[u, "(\phi_A)^*", "\wr"'] & \mathcal{T}(B, C) \arrow[l, "a'^*"'] \arrow[u, "(\phi_B)^*", "\wr"'] & \mathcal{T}(C', C) \arrow[l, "b'^*"'] \arrow[u, "(\phi_C)^*"] & \mathcal{T}(TA', C) \arrow[l, "c'^*"'] \arrow[u, "(\phi_T A)^*", "\wr"'] & \mathcal{T}(TB', C) \arrow[l, "Ta'^*"'] \arrow[u, "(\phi_T B)^*", "\wr"']
\end{tikzcd}$$

By the 5-lemma, we get that $(\phi_C)^*$ is an isomorphisms. By the same argument $id_C = s' \circ \phi_C$ for some unique s'. ϕ_C is both split mono and split epi, which means it is an isomorphism. $\square$

Corollary 2.3.1. $(A, B, 0, a, 0, 0)$ *is a triangle if and only if* a *is an isomorphism.*

Proof. Assume that a is an isomorphism. Then it is seen that $(a, id_B, 0)$ is an isomorphism of triangles.

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "a", "\wr"'] & B \arrow[r, "0"] \arrow[d, "id_B", "\wr"'] & 0 \arrow[r, "0"] \arrow[d, "0", "\wr"'] & TA \arrow[d, "Ta", "\wr"'] \\
B \arrow[r, "id_B"] & B \arrow[r, "0"] & 0 \arrow[r, "0"] & TB
\end{tikzcd}
$$

Converesly, assume that $(A, B, 0, a, 0, 0)$ is a triangle. Then the same diagram as above can be constructed, and by the 2 out of 3 property, a has to be an isomorphism. $\square$

Lemma 2.4. *For a triangle (A, B, C, a, b, c) the following are equivalent:*

$$
\begin{tikzcd}
A \arrow[dr, "a"] & \\
& B \arrow[dl, "b"] \\
C \arrow[uu, "c", "T"'] &
\end{tikzcd}
$$

- *a is split mono*
- *b is split epi*
- *$c = 0$*

Proof. The proof has two parts. First assume that a is split mono, and prove that b is split epi, and $c = 0$. By duality, it is then known that b being split epi implies that a is split mono and $c = 0$. The final part is to assume that $c = 0$, and prove either a is split mono or b is split epi.

Assume that a is split mono, then there exist an a^{-1} such that $id_A = a^{-1}a$. Let $M : \mathcal{T}$ be any object, then we can make a long exact sequence:

$$
\begin{tikzcd}
\mathcal{T}(M, T^{-1}C) \arrow[r, "T^{-1}c_*"] & \mathcal{T}(M, A) \arrow[r, bend left, "a_*"] & \mathcal{T}(M, B) \arrow[l, bend left, dashed, "a_*^{-1}"] \arrow[r, "b_*"] & \mathcal{T}(M, C) \arrow[r, "c_*"] & \mathcal{T}(M, TA)
\end{tikzcd}
$$

By assumption a_* is split mono, thus $T^{-1}c_* = 0$ and in particular $c = 0$. This implies that b_* is epi, making a split short exact sequence.

$$
\begin{tikzcd}
0 \arrow[r, "0"] & \mathcal{T}(M, A) \arrow[r, bend left, "a_*"] & \mathcal{T}(M, B) \arrow[l, bend left, dashed, "a_*^{-1}"] \arrow[r, bend left, "b_*"] & \mathcal{T}(M, C) \arrow[l, bend left, dashed, "b_*^{-1}"] \arrow[r, "0"] & 0
\end{tikzcd}
$$

This gives that b is split epi, completing the first part of the proof.

For the next part, assume that $c = 0$; then we can construct the following triangles.

$$(1)\quad \begin{tikzcd} A \arrow[dr, "a"] & \\ & B \arrow[dl, "b"] \\ C \arrow[uu, "0", "T"'] & \end{tikzcd} \implies \begin{tikzcd} C \arrow[dr, "0"] & \\ & TA \arrow[dl, "-Ta"] \\ TB \arrow[uu, "-Tb", "T"'] & \end{tikzcd}$$

$$(2)\quad \begin{tikzcd} A \arrow[dr, "id_A"] & \\ & A \arrow[dl, "0"] \\ 0 \arrow[uu, "0", "T"'] & \end{tikzcd} \implies \begin{tikzcd} 0 \arrow[dr, "0"] & \\ & TA \arrow[dl, "-id_{TA}"] \\ TA \arrow[uu, "0", "T"'] & \end{tikzcd}$$

(1) is constructed by applying TR2 twice, while (2) is constructed with TR1 and TR2 twice. Observe that there is a commutative square between the triangles, allowing for TR3 to make a morphism of triangles.

$$\begin{tikzcd} C \arrow[r, "0"] \arrow[d, "0"] & TA \arrow[r, "-Ta"] \arrow[d, equal, "id_{TA}"] & TB \arrow[r, "-Tb"] \arrow[d, dashed, "Ta^{-1}"] & TC \arrow[d, "0"] \\ 0 \arrow[r, "0"] & TA \arrow[r, equal, "-id_{TA}"] & TA \arrow[r, "0"] & 0 \end{tikzcd}$$

Thus $T(a^{-1}a) = id_{TA} = T(id_A) \implies id_A = a^{-1}a$, making a split mono. □

Lemma 2.5. *Given two triangles (A, B, C, a, b, c) and (A', B', C', a', b', c') the following are equivalent:*

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f"] & B \arrow[r, "b"] \arrow[d, "g"] & C \arrow[r, "c"] \arrow[d, "h"] & TA \arrow[d, "Tf"] \\ A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & TA' \end{tikzcd}$$

1. *(f, g, h) is a morphism of triangles*
2. *$\exists g : B \to B'$ such that $b'ga = 0$*

Moreover, if $\mathcal{T}(A, T^{-1}C') \simeq 0$, then f and h are unique.

Proof. 1. $\implies$ 2. as the composition $ba = 0 = b'a'$, so assume 2. The existence of f and h is evident from the long exact sequence of the bottom triangle at the functor represented by A.

$$\mathcal{T}(A, T^{-1}C') \xrightarrow{T^{-1}c'_*} \mathcal{T}(A, A') \xrightarrow{a'_*} \mathcal{T}(A, B') \xrightarrow{b'_*} \mathcal{T}(A, C')$$

The morphism $ga : \mathcal{T}(A, B')$ such that $b'ga = b'_*(ga) = 0$, thus $ga : Ker(b'_*)$. By exactness $\exists f : \mathcal{T}(A, A')$ such that $a'f = ga$, and by TR3 $\exists h : C \to C'$ such that (f, g, h) is a morphism of triangles. Now assume that $\mathcal{T}(A, T^{-1}C') \simeq 0$. Exactness determines that a'_* is a monomorphism, and f is then unique. Since T is a translation, we have that

$\mathcal{T}(A, T^{-1}C') \simeq \mathcal{T}(TA, C')$. By using the functor $\mathcal{T}(\_, C')$ at the top triangle, we get that b^* is a monomorphism, and thus h is chosen uniquely. □

Lemma 2.6. *If (A, B, C, a, b, c) is a triangle, then $(T^{-1}C, A, B, T^{-1}c, a, b)$ is a triangle.*

Proof. By TR2 we can construct a triangle.

$$\begin{tikzcd} A \arrow[dr, "a"] & & & C \arrow[dr, "c"] & \\ & B \arrow[dl, "b"] & \Longrightarrow & & TA \arrow[dl, "Ta"] \\ C \arrow[uu, "c", "T"'] & & & TB \arrow[uu, "Tb", "T"'] & \end{tikzcd}$$

By TR1 we can create a triangle $(T^{-1}C, A, B', T^{-1}c, a', b')$, and then use TR3 to find a morphism.

$$\begin{tikzcd} C \arrow[r, "c"] \arrow[d, equal, "id_C"] & TA \arrow[r, "Ta'"] \arrow[d, equal, "id_{TA}"] & TB' \arrow[r, "Tb'"] \arrow[d, dashed, "h"] & TC \arrow[d, equal, "id_{TC}"] \\ C \arrow[r, "c"] & TA \arrow[r, "Ta"] & TB \arrow[r, "Tb"] & TC \end{tikzcd}$$

By the 2 out of 3 property it is seen that h is an isomorphism, so the triple $(id_{T^{-1}C}, id_A, T^{-1}h)$ is an isomorphism of candidate triangles, and by TR1, is an isomorphism of triangles, asserting that $(T^{-1}C, A, B, T^{-1}c, a, b)$ is in fact a triangle. □

2.2 Mapping Cones, Homotopies and Contractibility

Remark. The observant reader might have seen that the Octahedron axiom have not jet been used once! A lot of the theory proven for triangulated categories work without this axiom, and this motivates the definition of a pre-triangulated category. This odd axiom might seem to be pointless, but it is one of the most fundamental axioms, as TR3 can be proven with TR1, TR2 and TR4; see [need sources here].

Definition 2.5. A pre-triangulation of an additive category $\mathcal{T}$ with translation T is a collection Δ' of triangles consisting of candidate triangles in $\mathcal{T}$ satisfying the following axioms:

1. (TR1) Formation axiom

   (a) A candidate triangle isomorphic to a triangle is a triangle.

   (b) Every morphism $a : A \to B$ can be embedded into a triangle (A, B, C, a, b, c):

   $$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA$$

   (c) For every object A there is a triangle $(A, A, 0, id_A, 0, 0)$:

$$A \xrightarrow{id_A} A \xrightarrow{0} 0 \xrightarrow{0} TA$$

2. (TR2) Rotation axiom

   For every triangle (A, B, C, a, b, c) there is a triangle (B, C, TA, b, c, Ta)

$$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA \implies B \xrightarrow{b} C \xrightarrow{c} TA \xrightarrow{-Ta} TB$$

3. (TR3) Morphism axiom

   Given the two triangles (A, B, C, a, b, c) (1) and (A', B', C', a', b', c') (2)

$$(1) \quad A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} TA \quad (2) \quad A' \xrightarrow{a'} B' \xrightarrow{b'} C' \xrightarrow{c'} TA'$$

   and morphisms $\phi_A : A \to A'$ and $\phi_B : B \to B'$ such that the square (1) commutes, then there is a morphism $\phi_C : C \to C'$ (not necessarily unique) such that (ϕ_A, ϕ_B, ϕ_C) is a morphism of triangles (2).

$$(1) \quad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[d, "\phi_B"] \\ A' \arrow[r, "a'"] & B' \end{tikzcd} \qquad (2) \quad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[r, "b"] \arrow[d, "\phi_B"] & C \arrow[r, "c"] \arrow[d, dashed, "\phi_C"] & TA \arrow[d, "T\phi_A"] \\ A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & TA' \end{tikzcd}$$

The category $\mathcal{T}$ with the pre-triangulation Δ' is called a pre-triangulated category, and the candidate triangles in Δ' are called triangles. We will only use this notion of triangles in this subsection.

The main goal of this subsection is to see how we can find triangles, and when these are triangles. We will also look at triangulated functors and triangulated subcategories. For the rest of this subsection it is assumed that we work in a pre-triangulated category $\mathcal{T}$.

Definition 2.6. Suppose there is a morphism of candidate triangles $\phi : (A, B, C, a, b, c) \to (A', B', C', a', b', c')$.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[r, "b"] \arrow[d, "\phi_B"] & C \arrow[r, "c"] \arrow[d, "\phi_C"] & TA \arrow[d, "T\phi_A"] \\ A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C \arrow[r, "c'"] & TA' \end{tikzcd}$$

We define the mapping cone to be the candidate triangle:

$$B \oplus A' \xrightarrow{\begin{pmatrix} -b & 0 \\ \phi_B & a' \end{pmatrix}} C \oplus B' \xrightarrow{\begin{pmatrix} -c & 0 \\ \phi_C & b' \end{pmatrix}} TA \oplus C' \xrightarrow{\begin{pmatrix} -Ta & 0 \\ T\phi_A & c' \end{pmatrix}} TB \oplus TA'$$

Definition 2.7. A morphism $\alpha : A, B, C, a, b, c) \to (A', B', C', a', b', c')$ between candidate triangles is called null-homotopic if it factors through a homotopy. That is, there exists maps Θ, Φ, Ψ in the following diagram:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\alpha_A"] & B \arrow[r, "b"] \arrow[d, "\alpha_B"] \arrow[dl, "\Theta"] & C \arrow[r, "c"] \arrow[d, "\alpha_C"] \arrow[dl, "\Phi"] & TA \arrow[d, "T\alpha_A"] \arrow[dl, "\Psi"] \\
A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C \arrow[r, "c'"] & TA'
\end{tikzcd}$$

such that $\alpha_A = \Theta a + T^{-1}(c'\Psi)$, $\alpha_B = \Phi b + a'\Theta$ and $\alpha_C = \Psi c + b'\Phi$. Two maps are called homotopic if their difference is null-homotopic

Lemma 2.7. *The mapping cone only depends on morphisms up to homotopy. I.e. if two maps are homotopic, their mapping cones are isomorphic.*

Proof. □

Lemma 2.8. *Suppose $\alpha, \beta : A \to B$ are two homotopic morphisms of candidate triangles. Then for any map $\gamma : A' \to A$ and any map $\delta : B \to B'$ the maps $\delta\alpha\gamma$ and $\delta\beta\gamma$ are homotopic as well.*

Proof. □

Definition 2.8. A candidate triangle A is called a contractible triangle if id_A is null-homotopic.

Lemma 2.9. *If A is a contractible triangle, then any map in $\mathcal{T}(A, \_)$ or $\mathcal{T}(\_, A)$ is null-homotopic.*

Proof. □

Lemma 2.10. *A contractible triangle is a triangle.*

Proof. □

Corollary 2.10.1. *The mapping cone of the zero map between triangles is a triangle.*

Proof. □

Corollary 2.10.2. *The mapping cone of a null-homotopic map between triangles is a triangle.*

Remark. Suppose we have a morphism of triangles where one of the triangles are contractible, then the mapping cone is a triangle as well.

Definition 2.9. A morphism of triangles will be called good if the mapping cone of the morphism is a triangle.

Theorem 2.11. *A pre-triangulated category $\mathcal{T}$ is triangulated if given two triangles (A, B, C, a, b, c) and (A', B', C', a', b', c') and diagram (1) commutes, then diagram (1) can be completed to diagram (2) such that ϕ is good. That is the mapping cone of ϕ is a triangle.*

$$
(1)\quad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[d, "\phi_B"] \\ A' \arrow[r, "a'"] & B' \end{tikzcd}
\qquad
(2)\quad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "\phi_A"] & B \arrow[r, "b"] \arrow[d, "\phi_B"] & C \arrow[r, "c"] \arrow[d, dashed, "\phi_C"] & TA \arrow[d, "T\phi_A"] \\ A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' \arrow[r, "c'"] & TA' \end{tikzcd}
$$

Remark. This condition is equivalent to the Octahedron axiom.

2.3 Calculus of Fractions and Localizations of Triangulated Categories

Localization is a method for adding formal inverses to a category. It is most notably known in commutative algebra where we can invert elements with respect to some ideal of the ring. The rational numbers can be shown to be a localization of the integers at every number except 0. The category gained from localizing at some set S of morphisms is the universal category where these morphisms are isomorphisms.

Definition 2.10. Let S be a collection of morphisms in the category $\mathcal{C}$. We say that the Localization of $\mathcal{C}$ on $\mathcal{S}$ is the category $\mathcal{C}[S^{-1}]$ together with a functor $q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ such that:

- $\forall s : S | q(s)$ is an isomorphism
- For any functor $F : \mathcal{C} \to \mathcal{D}$ such that $\forall s : S$ such that $F(s)$ is an isomorphism, then F factors through q. That is to say that there is a natural isomorphism $\eta : F \to F' \circ q$ so that $\mathcal{C}[S^{-1}]$ is the universal category where morphisms in S are isomorphisms.

$$
\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "q"'] & & \mathcal{D} \\ & S^{-1}\mathcal{C} \arrow[ur, dashed, "F'"'] & \end{tikzcd}
$$

Remark. Even though if we know that $\mathcal{C}$ is locally small, then we cannot be sure that the category $\mathcal{C}[S^{-1}]$ is again locally small.

These categories are in generel pretty hard to describe. When the set of morphisms is what we call a multiplicative system, we get the same calculus of fractions description of these localization the same style as for localizations of rings.

Definition 2.11. A set S of morphisms in a category $\mathcal{C}$ is called right multiplicative if it satisfies the following conditions:

- S is closed under composition, i.e. if $f, g : S$ are composable then $gf : S$. Every identity morphism in $\mathcal{C}$ is in S.
- (Right Ore condition) If $t : X \to Y$ is a morphism in S, then $\forall g : Z \to Y$ there is a commutative square (1) such that $f : W \to X$ and $s : W \to Z$ exists and $s : S$ as well.

$$(1) \quad \begin{tikzcd} W \arrow[r, dashed, "f"] \arrow[d, dashed, "s"] & X \arrow[d, "t"] \\ Z \arrow[r, "g"] & Y \end{tikzcd}$$

- (Left cancellation) Suppose $f, g : X \to Y$ are parallell morphisms in $\mathcal{C}$, then 1. $\implies$ 2.:
    1. $sf = sg$ for som $s : S$ starting at Y
    2. $ft = gt$ for som $t : S$ ending at X

Remark. The previous definition has an obvious dual statement. We say that a set S of morphisms is a left multiplicative system if it satisfies:

- S is closed under composition, i.e. if $f, g : S$ are composable then $gf : S$. Every identity morphism in $\mathcal{C}$ is in S.
- (Left Ore condition) If $s : Y \to Z$ is a morphism in S, then $\forall f : Y \to X$ there is a commutative square (1) such that $g : Z \to W$ and $t : X \to W$ exists and $t : S$ as well.

$$(1) \quad \begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "s"] & X \arrow[d, dashed, "t"] \\ Z \arrow[r, dashed, "g"] & W \end{tikzcd}$$

- (Right cancellation) Suppose $f, g : X \to Y$ are parallell morphisms in $\mathcal{C}$, then 1. $\implies$ 2.:
    1. $ft = gt$ for som $t : S$ ending at X
    2. $sf = sg$ for som $s : S$ starting at Y

If S is both right multiplicative and left multiplicative then we just say that it is multiplicative.

Prototype. Let R be a commutative integral domain, ... (look at Bacharaya and how they define the field of fractions, or ask Andreas if he have any good literature on this topic)

As with the definition of localization of rings, localization of a category $\mathcal{C}$ at a multiplicative system will be defined with fractions. That is the morphisms will be "fractions" of morphisms. These morphisms will be described as diagrams over spans for right multiplicative systems (or dually cospans for left multiplicative systems), together with an equivalence relation.

Definition 2.12. A span is a diagram of the form:

$$\cdot \longleftarrow \cdot \longrightarrow \cdot$$

Definition 2.13. Let S be a right multiplicative system of morphisms in a category $\mathcal{C}$. Given a morphism $s : Y \to X$ in S and a morphism $t : Y \to Z$ we define the right fraction of s and t to be the span of the morphisms. That is s and t fit in the diagram:

$$X \xleftarrow[s]{} Y \xrightarrow{t} Z$$

We denote the right fraction as ts^{-1}. Let $\sim$ be the equivalence relation of right fractions given by the diagram (1) such that $ts^{-1} \sim t's'^{-1}$ if and only if $\exists w, w' : \mathcal{C}$ making the diagram commute and that the middle row is a right fraction.

$$\begin{tikzcd}
& Y \arrow[dl, "s"'] \arrow[dr, "t"] & \\
X & W \arrow[l] \arrow[r] \arrow[u, "w"] \arrow[d, "w'"] & Z \\
& Y' \arrow[ul, "s'"] \arrow[ur, "t'"'] &
\end{tikzcd}$$

Dually, we define left fractions as diagrams over cospans such that if $t : S$ we get a left fraction $t^{-1}s$ as the diagram:

$$X \xrightarrow{s} Y \xleftarrow[t]{} Z$$

The equivalence relation $\sim$ is given by the diagram in the same manner as above.

$$\begin{tikzcd}
& Y \arrow[d, "w"] & \\
X \arrow[ur, "s"] \arrow[r] \arrow[dr, "s'"] & W & Z \arrow[l] \arrow[ul, "t"'] \arrow[dl, "t'"] \\
& Y' \arrow[u, "w'"] &
\end{tikzcd}$$

Proposition 2.12. *Suppose that S is a right multiplicative system, then the relation stated above is in fact an equivalence relation.*

Proof. We will need to prove that $\sim$ is reflexive, symmetric and transitive.

- (Reflexive) Let fs^{-1} be a right fraction. Then $fs^{-1} \sim fs^{-1}$ by the diagram:

$$\begin{tikzcd}
& W \arrow[dl, "s"'] \arrow[d, equal] \arrow[dr, "f"] & \\
X & W \arrow[l, "s"] \arrow[r, "f"] & Y \\
& W \arrow[ul, "s"] \arrow[u, equal] \arrow[ur, "f"'] &
\end{tikzcd}$$

- (Symmetric) Let fs^{-1} and gt^{-1} be two right fractions such that $fs^{-1} \sim gt^{-1}$, that is the diagram commute. Due to inherent symmetric nature of the diagram it follows that $gt^{-1} \sim fs^{-1}$.

$$\begin{tikzcd}
& W \arrow[dl, "s"'] \arrow[dr, "f"] & & & & W' \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & \widetilde{W} \arrow[l] \arrow[r] \arrow[u, "w"] \arrow[d, "w'"'] & Y & \Longrightarrow & X & \widetilde{W} \arrow[l] \arrow[r] \arrow[u, "w'"] \arrow[d, "w"'] & Y \\
& W' \arrow[ul, "t"] \arrow[ur, "g"'] & & & & W \arrow[ul, "s"] \arrow[ur, "f"'] &
\end{tikzcd}$$

- (Transitive) Suppose that there are three right fractions fs^{-1}, gt^{-1} and hu^{-1} such that $fs^{-1} \sim gt^{-1}$ and $gt^{-1} \sim hu^{-1}$. That is, diagrammically speaking:

$$\begin{tikzcd}
& W' \arrow[dl, "s"'] \arrow[dr, "f"] & & & W \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & \widetilde{W} \arrow[l] \arrow[r] \arrow[u, "w'"] \arrow[d, "\widetilde{w'}"'] & Y & X & \widetilde{\widetilde{W}} \arrow[l] \arrow[r] \arrow[u, "\widetilde{w''}"] \arrow[d, "w''"'] & Y \\
& W \arrow[ul, "t"] \arrow[ur, "g"'] & & & W'' \arrow[ul, "u"] \arrow[ur, "h"'] &
\end{tikzcd}$$

By the Ore condition we can create two new maps from $\widetilde{w'}$ and $\widetilde{w''}$. Since both morphisms are assumed to be in S, then we get that both $\widetilde{w'}$ and $\widetilde{w''}$ are in S. The following diagram then shows that $fs^{-1} \sim hu^{-1}$. A simple diagram chase shows that it is commutative.

$$
\begin{tikzcd}
\widetilde{\widetilde{\widetilde{W}}} \arrow[r, "\widetilde{\widetilde{w''}}"] \arrow[d, "\widetilde{\widetilde{w'}}"] & \widetilde{\widetilde{\widetilde{W}}} \arrow[d, "\widetilde{w''}"] \\
\widetilde{W} \arrow[r, "\widetilde{w'}"] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
& W' \arrow[ddl, "s"'] \arrow[ddr, "f"] & \\
& \widetilde{W} \arrow[u, "\widetilde{w'}"] \arrow[dl] \arrow[dr] & \\
X & \widetilde{\widetilde{\widetilde{W}}} \arrow[l] \arrow[r] \arrow[u, "\widetilde{\widetilde{w'}}"] \arrow[d, "\widetilde{\widetilde{w''}}"] & Y \\
& \widetilde{\widetilde{W}} \arrow[ul] \arrow[ur, "h"] \arrow[d, "\widetilde{w''}"] & \\
& W'' \arrow[uul, "u"] \arrow[uur] &
\end{tikzcd}
$$

$\square$

Definition 2.14. Let S be a multiplicate system in a category $\mathcal{C}$. Given two right fractions fs^{-1} and gt^{-1} in the diagrams:

$$
X \xleftarrow[s]{} W \xrightarrow{f} Y \quad \& \quad Y \xleftarrow[t]{} W' \xrightarrow{g} Z
$$

we can define the composite of these fractions $gt^{-1} \circ fs^{-1}$ by the Ore condition:

$$
\begin{tikzcd}
& \widetilde{W} \arrow[r, "h"] \arrow[d, "u"] & W' \arrow[r, "g"] \arrow[d, "t"] & Z \\
X & W \arrow[l, "s"] \arrow[r, "f"] & Y &
\end{tikzcd}
$$

The composite is then the right fraction $gt^{-1} \circ fs^{-1} = gh(su)^{-1}$.

Proposition 2.13. *The composition of right fractions is well-defined up to equivalence.*

Proof. In order to prove that the composite is well-defined we need to prove that the composite is independent from the different choices of the right Ore condition and that it is independent from choice of right fraction. There will only be presented a proof for that the choice of Ore maps is independent, as the other two cases are analogous.

Suppose we have two right fractions fs^{-1} and gt^{-1} as indicated from the diagrams.

$$
X \xleftarrow[s]{} W_1 \xrightarrow{f} Y \quad \& \quad Y \xleftarrow[t]{} W_2 \xrightarrow{g} Z
$$

Further suppose that there are at least two different choices for the maps gained by the right Ore condition. That is for example $(\widetilde{W}, \widetilde{s}, \widetilde{f})$ and $(\widehat{W}, \widehat{s}, \widehat{g})$. We can draw the two compositions as:

$$
\begin{tikzcd}
 & \widetilde{W} \arrow[r, "\widetilde{g}"] \arrow[d, "\widetilde{s}"] & W_2 \arrow[r, "g"] \arrow[d, "t"] & Z \\
X & W_1 \arrow[l, "s"'] \arrow[r, "f"] & Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \widehat{W} \arrow[r, "\widehat{f}"] \arrow[d, "\widehat{s}"] & W_2 \arrow[r, "g"] \arrow[d, "t"] & Z \\
X & W_1 \arrow[l, "s"'] \arrow[r, "f"] & Y &
\end{tikzcd}
$$

By combining the diagrams at W_1 by using the right Ore condition again we can find $(W, \widetilde{w}, \widehat{w})$ as in the diagram below.

$$
\begin{tikzcd}
\bar{W} \arrow[dr, dashed, "\xi"] & & & & \\
 & W \arrow[r, "\widehat{w}"] \arrow[d, "\widetilde{w}"] & \widehat{W} \arrow[r, "\widehat{f}"] \arrow[d, "\widehat{s}"] & W_2 \arrow[r, "g"] \arrow[d, "t"] & Z \\
 & \widetilde{W} \arrow[r, "\widetilde{s}"] \arrow[urr, "\widetilde{g}"] & W_1 \arrow[r, "f"] \arrow[d, "s"] & Y & \\
 & & X & &
\end{tikzcd}
$$

We see that the three squares commute, as by the definition of right Ore condition. Thus we have that $s\widetilde{s}\widetilde{w} = s\widehat{s}\widehat{w}$. As the three squares commute we get that $t\widehat{f}\widehat{w} = t\widetilde{g}\widetilde{w}$. As $t : S$ we can use right cancellation to find a $\xi : \bar{W} \to W$ such that $\widehat{f}\widehat{w}\xi = \widetilde{g}\widetilde{w}\xi \implies g\widehat{f}\widehat{w}\xi = g\widetilde{g}\widetilde{w}\xi$. Thus the equivalence relation diagram commutes:

$$
\begin{tikzcd}
 & \widehat{W} \arrow[dl, "s\widehat{s}"'] \arrow[dr, "g\widehat{f}"] & \\
X & \bar{W} \arrow[l] \arrow[r] \arrow[u, "\widehat{w}\xi"'] \arrow[d, "\widetilde{w}\xi"] & Z \\
 & \widetilde{W} \arrow[ul, "s\widetilde{s}"] \arrow[ur, "g\widetilde{g}"'] &
\end{tikzcd}
$$

□

Proposition 2.14. *The composition of right fractions is associative.*

Proof. Let fs^{-1}, gt^{-1} and hu^{-1} be right fractions as in the diagrams below:

$$
A \xleftarrow[s]{} X \xrightarrow{f} B \quad , \quad B \xleftarrow[t]{} Y \xrightarrow{g} C \quad \& \quad C \xleftarrow[u]{} Z \xrightarrow{h} D
$$

There are two different ways of calculating the compostion. Every morphism in S will be marked blue.

$\underline{hu^{-1} \circ (gt^{-1} \circ fs^{-1})}$

$$\begin{tikzcd}
W \arrow[rr] \arrow[d] & & Z \arrow[r] \arrow[d] & D \\
V \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & C & \\
X \arrow[r] \arrow[d] & B & & \\
A & & &
\end{tikzcd}$$

$\underline{(hu^{-1} \circ gt^{-1}) \circ fs^{-1}}$

$$\begin{tikzcd}
V' \arrow[r] \arrow[dd] & W' \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & D \\
 & Y \arrow[r] \arrow[d] & C & \\
X \arrow[r] \arrow[d] & B & & \\
A & & &
\end{tikzcd}$$

To be able to find a relation between these diagrams we create another diagram with the right Ore condition.

$$\begin{tikzcd}
T \arrow[r, dashed] \arrow[d, dashed] & V' \arrow[d] \\
W \arrow[r] & X
\end{tikzcd}$$

To finish the proof, one would need to show that the maps to A and D commute. The maps to A commute right out of the bat, by the right Ore condition. To prove that the maps to D commute, first apply right cancellation on the maps to B, then on the maps to C.

□

Definition 2.15. Let S be a right multiplicative system in a category $\mathcal{C}$. We define a category $\mathfrak{r}S^{-1}\mathcal{C}$ to have objects $\mathfrak{Ob}\mathfrak{r}S^{-1}\mathcal{C} = \mathfrak{Ob}\mathcal{C}$ and morphisms $\mathfrak{Mor}\mathfrak{r}S^{-1}\mathcal{C} = \{$right fractions of $S\}/\sim$. This means that the morphisms $\mathfrak{r}S^{-1}\mathcal{C}(X, Y)$ are spans in $\mathcal{C}$ where one of the maps are in S up to equivalence.

$$X \longleftarrow A \longrightarrow Y$$

This is well-defined by the previous results and the identity morphisms are the right fractions of the form:

$$X =\!= X =\!= X$$

Remark. Dually there is a category $\mathfrak{l}S^{-1}\mathcal{C}$ for a left multiplicative system S in a category $\mathcal{C}$. It is defined in the same manner as $\mathfrak{r}S^{-1}\mathcal{C}$, but with left fractions instead.

Remark. Given that S is right multiplicative, A right fraction from the object A to the object B can be described with a special kind of diagram. Let $A \downarrow S$ be the comma category of arrows from S ending in A and let $\delta : A \downarrow S \to \mathcal{C}$ be the forgetful functor, sending each arrow to its domain. A morphism in $A \downarrow S$ from the objects (b, B') to (c, C') is a morphism $t : B' \to C'$ such that $b = ct$. We see that there is a correspondance between right fractions and elements in components of diagrams over $A \downarrow S$ such as $\mathcal{C}(\delta b, B) = \mathcal{C}(B', B)$ and right fractions. That is, let $f : \delta\lfloor, \mathcal{B}$, then f can be regarded as fb^{-1}. A morphism from $\mathcal{C}(\delta c, B)$ to $\mathcal{C}(\delta b, B)$ is a morphism induced by a morphism from (b, B') to (c, C') in $A \downarrow S$. By the equivalence relation above we want to fractions fb^{-1} and gc^{-1} to be identified if there exists morphisms from $\mathcal{C}(\delta d, B)$ with maps $b' : (d, D') \to (b, B')$ and $c' : (d, D') \to (c, C')$ in $A \downarrow S$ such that $b' * f = c' * g$. This would be the same as saying that the right fractions are the coequalizer of the diagram $\mathcal{C}(\delta b, B) \coprod \mathcal{C}(\delta c, B) \rightrightarrows \mathcal{C}(\delta d, B)$. This observation motivates that the right fractions from A to B is described as the colimit of the functor $\mathcal{C}(\delta_{\_}, B) : A \downarrow S \to SET$. Dually, if S is left multiplicative we get that the left fractions from A to B can be described as the colimit of the functor $\mathcal{C}(A, \rho_{\_}) : S \downarrow B \to SET$. More details can be found in [2] and [3].

To ensure us that these categories $\mathfrak{r}S^{-1}\mathcal{C}$ does indeed exist there are many different criteria which we can place upon our assumptions. A natural restriction is to ensure that the colimits above exists as sets

Definition 2.16. A multiplicative system S in a locally small category $\mathcal{C}$ is called locally small on the right if for every object $X : \mathcal{C}$ there is a set S_X of morphisms from S such that for every morphism $f : X_1 \to X$ in S there is a morphism $f' : X' \to X$ in S_X factoring thorugh f.

The dual of this definition will be called a locally small multiplicative system on the left. If it is both locally small on the left and the right, we will simply call it locally small.

Remark. If S is a left multiplicative system, then $S \downarrow A$ is a filtered category for every object A. Dually, if S is right multiplicative then $A \downarrow S$ is cofiltered.

Remark. Equipped with this notion we are now able to prove that the localizations exists as locally small categories. Locally small right multiplicative systems allows us to prove that the classes $\mathfrak{r}S^{-1}\mathcal{C}(X, Y)$ are sets. This can be seen as we can regard S_X as a small category. By using the right Ore condition and left cancellation we can extend S_X such that it is again cofiltered and admits the same colimit, i.e.
$\varinjlim \mathcal{C}(\delta_{\_}, B) : A \downarrow S \to Set \simeq \varinjlim \mathcal{C}(\delta_{\_}, B) : S_A \to Set.$

Theorem 2.15. *(Gabriel-Zisman) Let S be a locally small right multiplicative system of morphisms in a category $\mathcal{C}$. Then the category $\mathfrak{r}S^{-1}\mathcal{C}$ exists and is the localization of $\mathcal{C}$ on S. That is there is an equivalence of categories $\mathcal{C}[S^{-1}] \simeq \mathfrak{r}S^{-1}\mathcal{C}$ together with a functor $q : \mathcal{C} \to \mathfrak{r}S^{-1}\mathcal{C}$ sending a morphism $f : X \to Y$ to the right fraction fid_X^{-1}.*

Proof. To prove the theorem we have to show that q is a functor, and that it is universal. Suppose that $f : X \to Y$ and $g : Y \to Z$ are morphisms in $\mathcal{C}$. Then $q(gf) = (gf)id_X^{-1}$ and $q(g)q(f) = (gid_Y^{-1}) \circ (fid_X^{-1})$. We can then choose the compostion to be defined by the diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \\
X \arrow[r, "f"] \arrow[d, equal] & Y & \\
X & &
\end{tikzcd}$$

Thus we can see that $(gid_Y^{-1}) \circ (fid_X^{-1}) = (gf)id_X^{-1}$ asserting the functoriality of q.

To see that q is universal let $\mathcal{D}$ be a category where every morphism of S is an isomorphism, and suppose there is a functor $F : \mathcal{C} \to \mathcal{D}$. We can define a functor $\mathfrak{r}S^{-1}F : \mathfrak{r}S^{-1}\mathcal{C} \to \mathcal{D}$ by $\mathfrak{r}S^{-1}F(fs^{-1}) = F(f)F(s)^{-1}$. One can see that $F = \mathfrak{r}S^{-1}F \circ q$, it remains to show that it is well-defined. Suppose $fs^{-1} = gt^{-1}$, that means there is a diagram in $\mathcal{C}$ with the blue arrows in S:

$$\begin{tikzcd}
& W' \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & W \arrow[u, "w'"'] \arrow[d, "w''"] & Y \\
& W'' \arrow[ul, "t"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Thus in $\mathcal{D}$ we have that $F(t) = F(sw')F(w'')^{-1}$ and $F(g) = F(fw')F(w'')^{-1}$, and this again gives us that

$$\begin{aligned}
\mathfrak{r}S^{-1}F(gt^{-1}) &= F(g)F(t)^{-1} \\
&= F(fw')F(w'')^{-1}(F(fw')F(w'')^{-1})^{-1} = F(fw')F(w'')^{-1}F(w'')F(sw')^{-1} \\
&= F(f)F(w')F(w')^{-1}F(s)^{-1} = F(f)F(s)^{-1} = \mathfrak{r}S^{-1}F(fs^{-1})
\end{aligned}$$

Thus $\mathfrak{r}S^{-1}F$ is well-defined and is unique by construction. □

Corollary 2.15.1. *If S is a locally small left multiplicative system instead then $\mathfrak{l}S^{-1}\mathcal{C}$ is the localization of $\mathcal{C}$ on S.*

If moreover S is a locally small multiplicative system, then there is an equivalence of categories $\mathfrak{r}S^{-1}\mathcal{C} \simeq \mathfrak{l}S^{-1}\mathcal{C}$.

Proof. The first statement is dual to the theorem.

To see the other statement, note that both $\mathfrak{r}S^{-1}\mathcal{C}$ and $\mathfrak{l}S^{-1}\mathcal{C}$ are the universal categories where the morphisms of S are isomorphisms. Thus it follows that these categories have to be equivalent. □

Remark. Since righthandedness of lefthandedness of the multiplicative system S doesn't affect the localization we can simply call the localization of a (left/right) multiplicative system for $S^{-1}\mathcal{C}$.

Proposition 2.16. *Let $\mathcal{C}$ be a category, and S a right multiplicative set of morphisms. The cannonical functor $q : \mathcal{C} \to S^{-1}\mathcal{C}$ commutes with finite limits.*

Proof. Let $T : \mathcal{D} \to \mathcal{C}$ be a diagram over a finite category $\mathcal{D}$. Then for any object $A : S^{-1}\mathcal{C}$ we have the following equation.

$$\begin{aligned} S^{-1}\mathcal{C}(qA, q(\varprojlim T_{\_}) &\simeq \varinjlim \mathcal{C}(\delta_{\_}, \varprojlim T_{\_})) \\ &\simeq \varinjlim \varprojlim \mathcal{C}(\delta_{\_}, T_{\_}) \simeq \varprojlim \varinjlim \mathcal{C}(\delta_{\_}, T_{\_}) \simeq \varprojlim S^{-1}\mathcal{C}(qA, q(T_{\_})) \end{aligned}$$

The first isomorphism is by the remark, the second is by the representative nature of limits and the third isomorphism is from the fact that filtered colimits commute with finite limits. The colimits are filtered by the remark that S_A is cofiltered and that the functor $\mathcal{C}(,A)$ is contravariant. $\square$

Remark. I am not quite sure yet how this argument proves the statement I want to prove, but that can be figure out later. I also don't know the proof for why filtered colimits commute with finite colimits, but it is in Riehls book.

Proposition 2.17. *Let $\mathcal{C}$ be a category with a zero. That is an object which is both initial and terminal. Suppose that S is a right multiplicative system, then $q0$ is a zero object in $S^{-1}\mathcal{C}$.*

Proof. The claim that $q0$ is initial follows from that initial is a limit of a diagram over the empty category. To see that $q0$ is terminal we only need to prove that every right fraction is equivalent with $0id_A^{-1}$. The following diagram proves this:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, "f"] \arrow[dr, "0"] & \\ A \arrow[r, equal] & A \arrow[r, "0"] & 0 \end{tikzcd}$$

$\square$

Proposition 2.18. *If $\mathcal{A}$ is an additive category and S is a right multiplicative system, then $S^{-1}\mathcal{A}$ is additive as well.*

Proof. From the previous propositions we know that $q0$ is the zero object and that $q(A \times B) \simeq qA \times qB$. By proving that there is an addition induced by $\mathcal{A}$ and that q preserves this addition we get that the product is the biproduct induced by the maps in $\mathcal{A}$.

Suppose that we have the fractions $fs^{-1}, gt^{-1} : S^{-1}\mathcal{C}(A, B)$. We define their addition by using the Ore condition to find new morphisms f, g' and u such that $fs^{-1} = f'u^{-1}$ and $gt^{-1} = g'u^{-1}$. Then the addition is defined as follows:

$$fs^{-1} + gt^{-1} = (f' + g')u^{-1}$$

To prove that this is an addition we should prove that it is well defined, associativity, inverses and commutativity will be inherited from $\mathcal{A}$. Let v be another extension. That is $\bar{f}v^{-1} = fs^{-1} = f'u^{-1}$ and $\bar{g}v^{-1} = gt^{-1} = g'u^{-1}$, so our goal is to prove that $(\bar{f}+\bar{g})v^{-1} - (f'+g')u^{-1} = 0$. By definition we have that $(\bar{f}+\bar{g})v^{-1} - (f'+g')u^{-1} = \bar{f}v^{-1} - f'u^{-1} + \bar{g}v^{-1} - g'u^{-1}$. So to prove that the whole sum is 0 is the same to prove as $\bar{f}v^{-1} + (-f')u^{-1} = (\bar{\bar{f}} - f'')w^{-1} = 0$. This can be done by writing out the diagrams after repeatedly applying right Ore condition.

$$\begin{tikzcd}
\cdot \arrow[dr, dashed] \arrow[drrr, dashed, bend left, "p"] \arrow[dddr, dashed, bend right, "p"'] & & & & \\
& \cdot \arrow[r] \arrow[d] \arrow[dr, "w"] & \cdot \arrow[r] \arrow[d, "u"] & \cdot \arrow[r, "f"] \arrow[dl, "s"] & B \\
& \cdot \arrow[r, "v"] \arrow[d] & A & & \\
& \cdot \arrow[ur, "s"] \arrow[d, "f"] & & & \\
& B & & &
\end{tikzcd}$$

The line to the bottom represents $\bar{\bar{f}}$ and the line to the right represents f''. By using left cancellation on the common morphism s into A one obtains the morphism p which relates the two fractions and makes the sum go to zero. $\square$

Corollary 2.18.1. *If $\mathcal{A}$ is abelian and S is a multiplicative system, then $S^{-1}\mathcal{A}$ is abelian as well.*

Definition 2.17. A triangulated functor $F : \mathcal{T} \to \mathcal{S}$ between two triangulated categories $(\mathcal{T}, T, \Delta_{\mathcal{T}}$ and $(\mathcal{S}, S, \Delta_{\mathcal{S}})$, is an additive functor along with a natural isomorphism $\phi_X : F(T(X)) \to S(F(X))$ such that $F(\Delta_{\mathcal{T}}) \subseteq \Delta_{\mathcal{S}}$. This means that for every triangle in $\mathcal{T}$ there is a triangle in $\mathcal{S}$.

$$\begin{tikzcd}
A \arrow[dr, "a"] & \\
& B \arrow[dl, "b"] \\
C \arrow[uu, "c", "T"'] &
\end{tikzcd}
\qquad \Longrightarrow \qquad
\begin{tikzcd}
F(A) \arrow[dr, "F(a)"] & \\
& F(B) \arrow[dl, "F(b)"] \\
F(C) \arrow[uu, "F(c)"', "T"] &
\end{tikzcd}$$

Definition 2.18. A triangulated subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$ is a full additive subcategory closed under isomorphisms such that the inclusion functor is triangulated.

Definition 2.19. Let $\mathcal{T}$ be a triangulated category and $\mathcal{S} \subseteq \mathcal{T}$ be a triangulated subcategory. Define a collection $Mor_{\mathcal{S}}$ to be a collection of morphisms in $\mathcal{T}$ such that for any $f : Mor_{\mathcal{S}}$ there is a triangle with $C : \mathcal{S}$.

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow TA$$

Remark. Every isomorphism is in $Mor_{\mathcal{S}}$. That is because isomorphisms are found in triangles $(A, B, 0, f, 0, 0)$ and $0 : \mathcal{S}$ for any triangulated subcategory.

Lemma 2.19. *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms. If any two of the morphisms f, g and gf are in $Mor_{\mathcal{S}}$ then so is the third.*

Proof. We are able to find three triangles in $\mathcal{T}$.

$$(1)\ \begin{tikzcd} X \arrow[dr, "f"] & \\ & Y \arrow[dl] \\ Z' \arrow[uu, "T"] & \end{tikzcd} \qquad (2)\ \begin{tikzcd} Y \arrow[dr, "g"] & \\ & Z \arrow[dl] \\ X' \arrow[uu, "T"] & \end{tikzcd} \qquad (3)\ \begin{tikzcd} A \arrow[dr, "g\circ f"] & \\ & C \arrow[dl] \\ Y' \arrow[uu, "T"] & \end{tikzcd}$$

By the Octahedron axiom there exist another triangle in $\mathcal{T}$:

$$Z' \longrightarrow X' \longrightarrow Y' \longrightarrow TZ'$$

Note that f is in $Mor_{\mathcal{S}}$ if and only if $Z' : S$. WLOG assume that f and g is in $Mor_{\mathcal{S}}$, this can be done by the rotation axiom. Thus we can find the triangle (Z', X', Y'') in $\mathcal{S}$ proving that Y' is in $\mathcal{S}$ by the following diagram:

$$\begin{tikzcd} Z' \arrow[r] \arrow[d, equal] & X' \arrow[r] \arrow[d, equal] & Y' \arrow[r] \arrow[d, "\wr"] & TZ' \arrow[d, equal] \\ Z' \arrow[r] & X' \arrow[r] & Y'' \arrow[r] & TZ' \end{tikzcd}$$

□

Definition 2.20. Let $F : \mathcal{S} \to \mathcal{T}$ be a triangulated functor. The kernel of F is defined to be the full subcategory $Ker(F)$ of $\mathcal{S}$ such that every object in $Ker(F)$ gets mapped to 0 by F. That is, $Ker(F)$ is the class of objects $\{K : \mathcal{S} | F(K) \simeq 0\}$.

Lemma 2.20. *The kernel of a triangulated functor F is a triangulated subcategory.*

Proof. □

Definition 2.21. A subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$ is called thick if it contains all the direct summands of its objects.

Lemma 2.21. *The kernel of a triangulated functor F is thick.*

Proof. $\square$

2.4 Discussion of Triangulations

Do Yoneda embedding into functor categories.

3 Exact Categories

I can maybe write some of the history of the development of the idea of exact categories.

3.1 Definitions and First Properties

In this section we will focus on defining what an exact category is and the first elementary properties. We will prove the axiom dubbed as "the obscure axiom" and motivate that it is not as obscure as its name suggest. Some "short" variants of some homological diagram lemmas will also be proved.

To start with the exact categories we will first take a look towards the abelian ones first. Short exact sequences are of great interest, and they can be characterized with two morphisms $p : A \to B$ and $q : B \to C$ such that p is the kernel of q and q is the cokernel of p. This leads to the first definition.

Definition 3.1. A kernel-cokernel pair is a pair of maps (p, q) such that p is the kernel of q and q is the cokernel of p. A morphism of kernel-cokernel pairs (p, q) and (p', q') is a triple (f, g, h) such that the following diagram commutes. An isomorphisms is a triple in which each morphism is an isomorphism.

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "f"] & B \arrow[r, "q"] \arrow[d, "g"] & C \arrow[d, "h"] \\ A' \arrow[r, "p'"] & B' \arrow[r, "q'"] & C' \end{tikzcd}$$

Lemma 3.1. *Let (p, q) be a kernel-cokernel pair, then the image and coimage of p exists and are isomorphic. I.e. this diagram exists, such that the left square is a push-out and the right square is a pull-back:*

$$\begin{tikzcd} 0 \arrow[r, "0"] \arrow[dr, "0"'] & A \arrow[rr, "p"] \arrow[d] & & B \arrow[r, "q"] & C \\ & Coim(p) \arrow[rr, "iso"] & & Im(p) \arrow[u] \arrow[ur, "0"'] & \end{tikzcd}$$

Proof. Since (p, q) is a kernel-cokernel pair we have that the first square is bicartesian and the second square is a push-out.

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[dr, "0"'] & B \arrow[d, "q"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, "0"] \arrow[dr, "0"'] & A \arrow[d, equal] \\ & A \end{tikzcd}$$

Thus $Im(p) = Coim(p) = A$, asserting the isomorphism as the identity in the diagram.

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[dr, "0"] & A \arrow[r, "p"] \arrow[d, equal] & B \arrow[r, "q"] & C \\
& A \arrow[r, equal] & A \arrow[u, "p"] \arrow[ur, "0"] &
\end{tikzcd}$$

□

Corollary 3.1.1. *Suppose that* (p, q) *is a kernel-cokernel pair. If* p *is an epimorphism, then* p *is an isomorphism.*

Definition 3.2. An exact structure for an additive category $\mathcal{A}$ is a class $\mathcal{E}$ of kernel-cokernel pairs which are closed under isomorphisms. A pair $(p, q) : \mathcal{E}$ is called a conflation, here p is called an inflation and q is called a deflation. $(\mathcal{A}, \mathcal{E})$ is called exact when the following axioms holds:

- (QE0) $\forall A : \mathcal{A}\ id_A$ is both an inflation and a deflation.
- (QE1) Both inflations and deflations are closed under composition.
- (QE2) The push-out of an inflation is an inflation.
- (QE2^{op}) The pull-back of a deflation is a deflation.

An exact category is the additive category $\mathcal{A}$ together with an exact structure $\mathcal{E}$.

Remark. When writing diagrams we use decorated arrows to indicate that a morphism is either an inflation or a deflation. A tail with a circle means inflation: $A \overset{\circ}{\rightarrowtail} B$. Double heads with a circle means deflation: $A \overset{\circ}{\twoheadrightarrow} B$. We can now rewrite the $(QE2^*)$ axioms as:

$$\begin{tikzcd}
A \arrow[r, tail, "\circ" description] \arrow[d] & B \arrow[d] & A \arrow[r, two heads, "\circ" description] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d] \\
C \arrow[r, tail, "\circ" description] \arrow[ur, phantom, "\lrcorner", very near end] & D & C \arrow[r, two heads, "\circ" description] & D
\end{tikzcd}$$

Remark. Inflations are also called admissable monomorphisms, deflations are also called admissable epimorphisms and conflations are also called short exact sequences.

Remark. The axioms for an exact category is made in such a way that $\mathcal{E}$ is an exact structure for $\mathcal{A}$ if and only if $\mathcal{E}^{op}$ is an exact structure for $\mathcal{A}^{op}$.

Lemma 3.2. *The map* $0 : 0 \to A$ *is an inflation. Dually, the map* $0 : A \to 0$ *is a deflation.*

Proof. Consider the diagram $0 \overset{0}{\rightarrowtail} A \overset{id_A}{\twoheadrightarrow} A$. The left morphism is the kernel of the right morphism making a kernel-cokernel pair $(0, id_A)$. The identity id_A is assumed to be a deflation, implying that the pair is a conflation. □

Remark. It can be seen that isomorphisms are deflations. Let $f : A \to B$ be an isomorphism, then there are two kernel-cokernel pairs: $(0, id_A)$ and $(0, f)$. Between these there is an isomorphism which is the triple $(0, id_A, f^{-1})$. As the conflations are closed under isomorphism, $(0, f)$ is a conflation, making f into a deflation. By dualizing this argument f is also an inflation.

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[d, "0"] & A \arrow[r, "f"] \arrow[d, "id_A"] & B \arrow[d, "f^{-1}"] \\
0 \arrow[r, tail, "0"] & A \arrow[r, two heads, "id_A"] & A
\end{tikzcd}$$

Corollary 3.2.1. *A kernel-cokernel pair (i, p) found as a split short-exact sequence (1) is a conflation.*

$$\text{(1)} \quad A \overset{i}{\rightarrowtail} A \oplus B \overset{p}{\twoheadrightarrow} B$$

Proof. In a category with an initial object the coproduct can be thought of as the push-out with the initial in the upper right corner. This can be assembled into push-out (1). By the lemma the zero morphisms are inflations, asserting that i and i' are inflations by (QE2). Thus there are conflations (i, p) and (i', p').

$$\text{(1)} \quad \begin{tikzcd}
0 \arrow[r, tail, "0"] \arrow[d, tail, "0"] & A \arrow[d, tail, "i"] \\
B \arrow[r, tail, "i'"] & A \oplus B
\end{tikzcd}$$

∎

Corollary 3.2.2. *The direct sum of conflations is a conflation. I.e. there is a diagram:*

$$A \overset{i}{\rightarrowtail} B \overset{p}{\twoheadrightarrow} C \quad \& \quad A' \overset{i'}{\rightarrowtail} B' \overset{p'}{\twoheadrightarrow} C'$$
$$\Downarrow$$
$$A \oplus A' \overset{i \oplus i'}{\rightarrowtail} B \oplus B' \overset{p \oplus p'}{\twoheadrightarrow} C \oplus C'$$

Proof. We start with only considering the conflation (i, p). $\forall D$ there is a conflation $(i \oplus id_D, p \oplus 0)$, drawn as the diagram.

$$A \oplus D \xrightarrow{\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}} B \oplus D \xrightarrow{\begin{pmatrix} p & 0 \end{pmatrix}} C$$

As kernels and cokernels are preserved by direct sums, this pair is in fact a kernel-cokernel pair. The epimorphism is a deflation as it can be factored by the deflations:

$$B \oplus D \xrightarrow{\begin{pmatrix} 1 & 0 \end{pmatrix}} B \xrightarrow{p} C$$

Thus it is seen that $(i \oplus id_D, p \oplus 0)$ is a conflation, and dually $(i \oplus 0, p \oplus id_D)$ is also a conflation. To finish off the proof it is seen that the morphism $i \oplus i'$ factors as $i \oplus id_{A'} \circ id_A \oplus i'$ asserting that it is an inflation by (QE1). By the dual argument we then get that the direct sum of conflations is a conflation. $\square$

Definition 3.3. A square is bicartesian if it is both a pull-back and a push-out.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

Proposition 3.3. *The following statements are equivalent:*

1. *The square (1) is a push-out.*
2. *The sequence (2) is a conflation.*
3. *The square (1) is bicartesian.*
4. *The square (1) is a part of the commutative diagram (3)*

$$(1)\ \begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[d, "g"] \\ C \arrow[r, tail, "j"] & D \end{tikzcd} \quad (2)\ A \xrightarrow{\begin{pmatrix} i \\ -f \end{pmatrix}} B \oplus C \xrightarrow{\begin{pmatrix} g & j \end{pmatrix}} D \quad (3)\ \begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[r, two heads, "p"] \arrow[d, "g"] & E \arrow[d, equal] \\ C \arrow[r, tail, "j"] & D \arrow[r, two heads, "q"] & E \end{tikzcd}$$

Before the proof for this proposition will be presented a useful lemma will be proved first.

Lemma 3.4. *Assume that there is a commutative square (1) and an associatied sequence (2). (1) is a push-out square if and only if* $\begin{pmatrix} p & q \end{pmatrix}$ *is the cokernel of the morphism* $\begin{pmatrix} i \\ -j \end{pmatrix}$

$$(1)\ \begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"] & B \arrow[d, "p"] \\ C \arrow[r, "q"] & D \end{tikzcd} \quad (2)\ A \xrightarrow{\begin{pmatrix} i \\ -j \end{pmatrix}} B \oplus C \xrightarrow{\begin{pmatrix} p & q \end{pmatrix}} D$$

Proof. For any test object T and two maps $t_1 : B \to T$ and $t_2 : C \to T$, we can construct the diagrams for the universal properties of both the cokernel and the push-out. It is seen that these diagrams are equivalent, proving the lemma.

$$
\begin{tikzcd}
A \arrow[r, "\begin{pmatrix} i \\ -f \end{pmatrix}"] \arrow[dr, "0"'] & B \oplus C \arrow[d, two heads, "\begin{pmatrix} g & j \end{pmatrix}"] \arrow[dr, bend left, "\begin{pmatrix} t_1 & t_2 \end{pmatrix}"] & \\
& D \arrow[r, dashed, "t'"] & T
\end{tikzcd}
\quad \Leftrightarrow \quad
\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[d, "g"] \arrow[ddr, bend left, "t_1"] & \\
C \arrow[r, "j"] \arrow[drr, bend right, "t_2"'] & D \arrow[dr, dashed, "t'"] & \\
& & T
\end{tikzcd}
$$

$\square$

Corollary 3.4.1. *For the same diagrams (1) and (2) as above the dual statement is also true. (1) is a pull-back square if and only if* $\begin{pmatrix} i \\ -j \end{pmatrix}$ *is kernel of the morphism* $\begin{pmatrix} p & q \end{pmatrix}$. *Thus we have that (1) is bicartesian (i.e. both a pull-back and a push-out) if and only if the morphisms make a kernel-cokernel pair.*

Proof. **of Proposition 3.3** 1. $\Rightarrow$ 2.: By the previous lemma we know that $\begin{pmatrix} g & j \end{pmatrix}$ is the cokernel of $\begin{pmatrix} i \\ -j \end{pmatrix}$. Thus proving that $\begin{pmatrix} i \\ -j \end{pmatrix}$ is an inflation, will prove that the pair is a conflation.

Observe that the morphism $\begin{pmatrix} i \\ -f \end{pmatrix}$ can be factored through the sequence.

$$
\begin{tikzcd}
A \arrow[r, tail, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & A \oplus C \arrow[r, tail, two heads, "\begin{pmatrix} 1 & 0 \\ -f & 1 \end{pmatrix}", "\cong"'] & A \oplus C \arrow[r, tail, "\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}"] & B \oplus C
\end{tikzcd}
$$

By corollary 3.2.1 the first map is an inflation, as the second map is an isomorphism it is also an inflation and the last map is the direct sum of two inflations. Thus the composite of all these maps surely is an inflation by (QE1), proving the first implication.

2. $\Rightarrow$ 3.: This follows from corollary 3.4.1.

3 $\Rightarrow$ 1.: This is by definition.

1. $\Rightarrow$ 4.: Let p be the cokernel of i, then we can form the diagram below.

$$
\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[d, "g"] \arrow[ddr, bend left, two heads, "p"] & \\
C \arrow[r, tail, "j"] \arrow[drr, bend right, "0"'] & D \arrow[dr, dashed, two heads, "p'"] & \\
& & T
\end{tikzcd}
$$

p' is an epimorphism as $p = p'g$ is epi. To prove that p' is the cokernel of j we let T' be another test object with a map $t' : D \to T'$ such that $0 = t'j$. By doing some diagram chases we have that $0 = t'jf = t'gi$, thus by the universal property of p the morphism $t'g$ factors through T such that $t'g = tp$ for some unique t. By rearranging we have that $t'g = tp'g = tp$, and $t'j = tp'j = 0$, thus since t' is the unique morphism satisfying this equation we demand that $t' = tp'$. t is also unique, for if there exist another map h such that $tp' = hp'$, then $t = h$ as p' is epic. The unique existence proves the universal property, and p' is the cokernel of j.

$$\begin{tikzcd}
A \arrow[r, tail, "i"] \arrow[d, "f"] & B \arrow[r, two heads, "p"] \arrow[d, "g"] & T \arrow[d, equal] \\
C \arrow[r, tail, "j"] \arrow[rrd, bend right, "0"] \arrow[ur, phantom, "\lrcorner"] & D \arrow[r, dashed, two heads, "p'"] \arrow[rd, "t'"] & T \arrow[d, dashed, "t"] \\
 & & T'
\end{tikzcd}$$

4. $\Rightarrow$ 2.: We start by taking the pullback of p and p' using $(QE2^{op})$, and determines the diagram with the dual statement of the last implication.

$$\begin{tikzcd}
 & A \arrow[r, equal] \arrow[d, tail, "i'"] & A \arrow[d, tail, "i"] \\
C \arrow[r, tail, "j'"] \arrow[d, equal] & E \arrow[r, two heads, "q'"] \arrow[d, two heads, "q"] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d, two heads, "p"] \arrow[dl, dashed, "g"] \\
C \arrow[r, tail, "j"] & D \arrow[r, two heads, "p'"] & T
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
B \arrow[dr, dashed, "k"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, "g"] & & \\
 & E \arrow[r, two heads, "q'"] \arrow[d, two heads, "q"] & B \arrow[d, two heads, "p"] \\
 & D \arrow[r, two heads, "p'"] & T
\end{tikzcd}$$

From these diagrams we can deduce that q' is a split-epimorphism. The composite $q'(id_E - kq') = q' - q'kq' = q' - q' = 0$ as q' is split-epi, so $(id_E - kq')$ factors over j' as in the following diagram.

$$\begin{tikzcd}
E \arrow[dr, dashed, "l"] \arrow[drr, bend left, "id_E - kq'"] & & \\
 & C \arrow[r, tail, "j'"] \arrow[dr, "0"] & E \arrow[d, two heads, "q'"] \\
 & & B
\end{tikzcd}$$

From these diagrams we can extract three different equations:

- $0 = k - k = k - kq'k = (id_E - kq')k = j'lk \implies lk = 0$ as j' is monic
- $j'lj' = (id_E - kq')j' = j' \implies lj' = id_C$ as j' is monic

- $jli' = (qj')li' = q(id_E - kq')i' = -(qk)(q'i') = -gi = -jf \implies li' = -f$ as j is monic

The morphisms $\begin{pmatrix} k & j' \end{pmatrix}$ and $\begin{pmatrix} q' \\ l \end{pmatrix}$ are inverses:

- $\begin{pmatrix} k & j' \end{pmatrix} \begin{pmatrix} q' \\ l \end{pmatrix} = kq' + j'l = kq' + id_E - kq' = id_E$
- $\begin{pmatrix} q' \\ l \end{pmatrix} \begin{pmatrix} k & j' \end{pmatrix} = \begin{pmatrix} q'k & q'j' \\ lk & lj' \end{pmatrix} = \begin{pmatrix} id_B & 0 \\ 0 & id_C \end{pmatrix}$

Thus we have an isomorphism of kernel-cokernel pairs $(id_A, , \begin{pmatrix} q' \\ l \end{pmatrix} \begin{pmatrix} k & j' \end{pmatrix})$,
from $(\begin{pmatrix} i \\ -f \end{pmatrix}, \begin{pmatrix} f' & i' \end{pmatrix})$ to (i', q). This proves 2. $\square$

Corollary 3.4.2. *The pull-back of an inflation along a deflation is an inflation.*

$$\begin{tikzcd}
A \arrow[r, tail, "i'"] \arrow[d, two heads, "e'"] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d, two heads, "e"] \\
C \arrow[r, tail, "i"] & D
\end{tikzcd}$$

Proof. By (QE2) this pullback exists, as there is a deflation in the pullback. Extend the diagram by adding the deflation of the inflation in the following manner.

$$\begin{tikzcd}
 & T \arrow[dl, dashed, bend right, "t'"'] \arrow[d, "t"] \arrow[dr, bend left, "0"] & \\
A \arrow[r, tail, "i'"] \arrow[d, two heads, "e'"] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[r, two heads, "pe"] \arrow[d, two heads, "e"] \arrow[dl, dashed, "t''"'] & C \arrow[d, equal] \\
C \arrow[r, tail, "i"] & D \arrow[r, two heads, "p"] & C
\end{tikzcd}$$

pe is a deflation by (QE1), and i' is a mono as a limit of a mono is a mono. Our goal is to prove that i' is the kernel of pe. Let T be a test object such that $pet = 0$. Then we have that te factorizes over i, such that we can apply the universal property of the pullback to factorize te over i'. Uniqueness of t' is achieved with i' being monic. This proves that (i', pe) is a conflation. $\square$

Theorem 3.5. ***The obscure axiom.*** *Assume that $i : A \to B$ is a morphism with a cokernel. If there is a morphism $j : B \to C$ such that ji is an inflation, then i is an inflation.*

Proof. ☐

Proof. Let $p : B \to D$ be the cokernel of i. We start the proof with forming the push-out of i and ji.

$$\begin{tikzcd} A \arrow[r, tail, "ji"] \arrow[d, "i"] & C \arrow[d] \\ B \arrow[r, tail] \arrow[ur, phantom, "\lrcorner"] & E \end{tikzcd}$$

By proposition 3.3 we get that $\begin{pmatrix} i \\ ji \end{pmatrix}$ is an inflation. $\begin{pmatrix} i \\ 0 \end{pmatrix} = \begin{pmatrix} id_B & 0 \\ -j & id_C \end{pmatrix} \begin{pmatrix} i \\ ji \end{pmatrix}$, this is an inflation by (QE1) as the 2x2 matrix is an isomorphism. Observer that the cokernel of this map is $\begin{pmatrix} k & 0 \\ 0 & id_C \end{pmatrix}$. Our final trich will be to show that there is a pullback square, and then use (QE2) to say that k is a deflation.

$$\begin{tikzcd} T \arrow[dr, dashed, "t"] \arrow[drr, bend left, "t_1"] \arrow[ddr, bend right, "\begin{pmatrix} t_2 \\ 0 \end{pmatrix}"'] & & \\ & B \arrow[r, "k"] \arrow[d, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & D \arrow[d, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] \\ & B \oplus C \arrow[r, two heads, "\begin{pmatrix} k & 0 \\ 0 & id_C \end{pmatrix}"'] & D \oplus C \end{tikzcd}$$

Note that setting $t = t_2$ we get the universal property. This is well defined as $kt_2 = t_1$ by assumption, thus $kt = t_1$. This is what we need to prove that the square is a pullback, proving the obscure axiom. ☐

Remark. Write a bit about the dual of the obscure axiom.

Lemma 3.6. *Let (p, q) and (p', q') be the conflations:*

- (p, q)*:* $A \overset{p}{\rightarrowtail} B \overset{q}{\twoheadrightarrow} C$
- (p', q')*:* $A' \overset{p'}{\rightarrowtail} B' \overset{q'}{\twoheadrightarrow} C'$

A morphism of the conflations $(f, g, h) : (p, q) \to (p', q')$ factors through the conflation $A \rightarrowtail D \twoheadrightarrow C'$ *such that we have the following diagram where $g = g_2 g_1$.*

$$
\begin{tikzcd}
A \arrow[r, tail, "p"] \arrow[d, "f"] & B \arrow[r, two heads, "q"] \arrow[d, "g_1"] & C \arrow[d, equal] \\
A' \arrow[r, tail] \arrow[d, equal] & D \arrow[r, two heads] \arrow[d, "g_2"] & C \arrow[d, "h"] \\
A' \arrow[r, tail, "p'"] & B' \arrow[r, two heads, "q'"] & C'
\end{tikzcd}
$$

Proof. Observe that the upper part of the diagram is made by taking a push-out of p and f, where the right part is gained from proposition 3.3. Next we will combine the upper part with the lower part using the push-out property.

$$
\begin{tikzcd}
A \arrow[r, tail, "p"] \arrow[d, "f"] & B \arrow[d, "g_1"] \arrow[ddr, bend left, "g"] & \\
A' \arrow[r, tail] \arrow[drr, tail, bend right, "p'"] & D \arrow[dr, dashed, "g_2"] & \\
& & B'
\end{tikzcd}
\implies
\begin{tikzcd}
A \arrow[r, tail, "p"] \arrow[d, "f"] & B \arrow[r, two heads, "q"] \arrow[d, "g_1"] & C \arrow[d, equal] \\
A' \arrow[r, tail, "a"] \arrow[d, equal] & D \arrow[r, two heads, "c"] \arrow[d, "g_2"] & C \arrow[d, "h"] \\
A' \arrow[r, tail, "p'"] & B' \arrow[r, two heads, "q'"] & C'
\end{tikzcd}
$$

It remains to show that the lower right square is commutative, then we can apply the dual of proposition 3.3 to see that the square is bicartesian. Note that $q = cg_1$ by prop 3.3 thus $q'g_2g_1 = q'g = hq = hcg_1$. By uniqueness of the push-out property we have that $hc = q'g_2$. □

Corollary 3.6.1. ***The short five lemma***. *Suppose that there is a morphism of conflations (f, g, h) as above. If f and h are isomorphisms, then g is an isomorphism.*

Proof. Since f is an isomorphism it is at least an inflation, thus g_1 is an inflation by (QE2). As colimits preserve epis, g_1 is also an epimorphism. by corollary 3.1.1 we know that g_1 is an iso, and dually that g_2 is an iso. Since isomorphisms are closed under composition we have that g is an isomorphism.

$$
\begin{tikzcd}
A \arrow[r, tail, "p"] \arrow[d, tail, "f", "\wr"'] & B \arrow[r, two heads, "q"] \arrow[d, tail, "g_1", "\wr"'] & C \arrow[d, equal] \\
A' \arrow[r, tail] \arrow[d, equal] & D \arrow[r, two heads] \arrow[d, two heads, "g_2", "\wr"'] & C \arrow[d, two heads, "h", "\wr"'] \\
A' \arrow[r, tail, "p'"] & B' \arrow[r, two heads, "q'"] & C'
\end{tikzcd}
$$

□

Lemma 3.7. ***Noethers isomorphism lemma***. *Suppose there is a diagram with rows as conflations and the first column as a conflation. Then the final column is also a conflation.*

$$
\begin{tikzcd}
A \arrow[r, tail] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, "\ulcorner" very near start, "\lrcorner" very near end] & X \arrow[d, tail, dashed] \\
A \arrow[r, tail] & C \arrow[r, two heads] \arrow[d, two heads] & Y \arrow[d, two heads, dashed] \\
 & Z \arrow[r, equal] & Z
\end{tikzcd}
$$

Proof. Assume that we only have the solid part of the diagram above. By the universal property of cokernels, the upper dashed map exists, and by the dual of proposition 3.3 the square is bicartesian. We can state that the upper dashed map is an inflation, and since the square is a push-out we get that the lower dashed map exists such that the final column is a conflation by proposition 3.3. □

3.2 The Frobenis Category

3.3 Self-injective Algebras

3.4 The Homotopy Category

4 The Derived Category

4.1 Admissable Morphisms

4.2 Homology and Long Exact Sequences

4.3 The Derived Category

4.4 If time, derived functors as well

5 Auslander-Reiten Triangles

5.1 Krull-Schmidt Categories

5.2 Definition and First Properties

5.3 Description of Derived Categories

References

[1] Bernhard Keller. Chain complexes and stable categories. *Manuscripta Mathematica*, 67(1):379–417, 1990.

[2] Peter Gabriel and Michel Zisman. *Calculus of Fractions and Homotopy Theory*. Springer-Verlag, 1967.

[3] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge University Press, 1994.

[4] Dietr Happel. *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras*. Cambridge University press, 1988.

[5] Emily Riehl. *Category Theory in Context*. Dover Publications, 2016.

[6] Theo Bhuler. Exact categories. *Expositiones Mathematicae*, 28(1):1–69, 2010.

[7] Amnon Neeman. *Triangulated Categories*. Princeton University Press, 2001.

[8] Saunders MacLane. *Categories for the Working Mathematician*. Springer-Verlag, 1971.