On the Derived Category of Strongly Homotopy Associative Algebras

Thomas Wilskow Thorbjørnsen

November 30, 2022

Abstract

Fill inn abstract

Sammendrag

Fyll inn sammendraget

Acknowledgements

Thank the people in your life who has made this journey easier :D

Contents

Chapter 1

Bar and Cobar Construction

In Stasheffs papers [1] and [2], a strongly homotopy associative algebra, or A_∞-algebra, over a field is a graded vector space together with homogenous linear maps $m_n : A^{\otimes n} \to A$ of degree $2 - n$ satisfying some homotopical relations. This will be made precise later. We will regard m_2 to be a multiplication of A, but it is however not a priori associative. We choose m_3 to be a homotopy of m_2's associator. In this manner, we know that the homotopy of A is an associative algebra. The maps m_n corresponds uniquely to a map $m^c : BA \to \overline{A}[1]$, which extends to a coderivation $m^c : BA \to BA$ of the bar construction of A. By this relation we could instead define an A_∞-algebra to be a coalgebra on the form BA, and we will prefer to do so in this thesis.

In order to understand the bar construction, we will first study it on associative algebras. Given a differential graded coassociative coalgebra C and a differential graded associative algebra A, we say that a homogenous linear transformation $\alpha : C \to A$ is twisting if it satisfies the Maurer-Cartan equation;

$$\partial \alpha + \alpha \star \alpha = 0.$$

Let $\mathsf{Tw}(C, A)$ be the set of twisting morphisms. By considering it as a functor $\mathsf{Tw} : \mathsf{coAlg}_{\mathbb{K}}^{op} \times \mathsf{Alg}_{\mathbb{K}} \to Ab$ we want to show that it is represented in both arguments. Moreover, these representations give rise to an adjoint pair of functors, called the bar and cobar construction.

$$\mathsf{Alg}_{\mathbb{K},+}^{\bullet} \quad \overset{B}{\underset{\Omega}{\rightleftarrows}} \quad \top \quad \mathsf{Coalg}_{\mathbb{K},\mathsf{conil}}^{\bullet}$$

The bar and cobar construction will be the basis for our discussion of A_∞-algebras. As the bar construction can be used to define A_∞-algebras, we may easily dualize this to define A_∞-coalgebras in terms of the cobar construction. This chapter will follow the notions and progression presented in Loday and Vallete [3] to develop the theory for the bar-cobar adjunction.

1

1.1 Algebras and Coalgebras

1.1.1 Algebras

This section presents a review of associative algebras over a field $\mathbb{K}$. The main goal of this section is to establish some conventions before we look at how they are dualized to coalgebras. This collection of algebras, together with ring homomorphisms, form the category $\mathsf{Alg}_{\mathbb{K}}$ of associative algebras over $\mathbb{K}$.

Definition 1.1.1 ($\mathbb{K}$-Algebra)**.** Let $\mathbb{K}$ be a field with unit 1. A $\mathbb{K}$-algebra A, or an algebra A over $\mathbb{K}$, is a vector space with structure morphisms called multiplication and unit,

$$(\cdot_A) : A \otimes_{\mathbb{K}} A \to A$$
$$1_A : \mathbb{K} \to A,$$

satisfying the associativity and identity laws.

$$\text{(associativity)} \quad (a \cdot_A b) \cdot_A c = a \cdot_A (b \cdot_A c)$$
$$\text{(unitality)} \quad 1_A(1) \cdot_A a = a = a \cdot_A 1_A(1)$$

Whenever A does not posess a unit morphism, we will call A a non-unital algebra. In this case only the associativity law must hold.

By abuse of notation we will confuse the unit of $\mathbb{K}$ by the unit of A. Since 1_A is a ring homomorphism, this should not be a cause of confusion. However, when we want to use the unit as a morphism, we will stick to the 1_A notation. When there are no confusion, we will exchange the symbol $(\cdot_A)$ with words in A. $(\cdot_A)$ is then concatenation of variables.

Definition 1.1.2 (Algebra homomorphisms)**.** Let A and B be algebras. Then $f : A \to B$ is an algebra homomorphism if

1. f is $\mathbb{K}$-linear
2. $f(ab) = f(a)f(b)$
3. $f \circ 1_A = 1_B$

Whenever A and B are non-unital, we must drop the condition that f preserves units.

Definition 1.1.3 (Category of algebras)**.** We let $\mathsf{Alg}_{\mathbb{K}}$ denote the category of $\mathbb{K}$-algebras. Its objects consists of every algebra A, and the morphisms are algebra homomorphisms. The sets of morphisms between A and B are denoted as $\mathsf{Alg}_{\mathbb{K}}(A, B)$.

Let $\widehat{\mathsf{Alg}}_{\mathbb{K}}$ denote the category of non-unital algebras. It's objects consists of every non-unital algebra A, and the morphisms are non-unital algebra homomorphisms. The sets of morphisms between A and B are denoted as $\widehat{\mathsf{Alg}}_{\mathbb{K}}(A, B)$.

The cagtegory $(\mathsf{Mod}_{\mathbb{K}}, \otimes_{\mathbb{K}}, \mathbb{Z})$ is symmetric monoidal. Observe that given any algebra A in $\mathsf{Mod}_{\mathbb{K}}$, the triple $(A, (\cdot_A), 1_A)$ is a monoid. This gives us an isomorphism of categories, namely that $\mathsf{Alg}_{\mathbb{K}}$ is the category of monoids in $\mathsf{Mod}_{\mathbb{K}}$. Therefore, the algebra axioms are the following commutative diagrams:

$$
\begin{array}{ccc}
A \otimes_{\mathbb{K}} A \otimes_{\mathbb{K}} A & \xrightarrow{(\cdot_A) \otimes id_{\mathbb{K}}} & A \otimes_{\mathbb{K}} A \\
{\scriptstyle id_{\mathbb{K}} \otimes (\cdot_A)} \downarrow & & \downarrow {\scriptstyle (\cdot_A)} \\
A \otimes_{\mathbb{K}} A & \xrightarrow{\quad (\cdot_A) \quad} & A
\end{array}
\qquad
\begin{array}{ccc}
A \otimes_{\mathbb{K}} \mathbb{K} \xrightarrow{id_A \otimes 1_A} & A \otimes_{\mathbb{K}} A & \xleftarrow{1_A \otimes id_A} \mathbb{K} \otimes_{\mathbb{K}} A \\
{\scriptstyle \simeq} \searrow & \downarrow {\scriptstyle (\cdot_A)} & \swarrow {\scriptstyle \simeq} \\
& A &
\end{array}
$$

In any symmetric monoidal category $\mathcal{C}$ we may reformulate these defintions by using the monoidal structure more heavily. In section 3 we will introduce electronic circuits, which are inspired by some of the proofs found in [3]. These conventions will give us a graphical calculus of morphisms in $\mathcal{C}$.

We supply some examples of algebras one may encounter in nature.

Example 1.1.4. Let $\mathbb{K}$ be any field. The field is trivially an algebra over itself.

Example 1.1.5. The complex numbers $\mathbb{C}$ is an algebra over $\mathbb{R}$. This is true since it is a vector space over $\mathbb{R}$, and complex multiplication respects scalar multiplication.

Example 1.1.6. Let $\mathbb{K}$ be any field. The ring of n-dimensional matrices $M_n(\mathbb{K})$ is an algebra over $\mathbb{K}$. The multiplication is taken to be the matrix multiplication and the unit is the n-dimensional identity matrix.

Augmented algebras will be central to our discussion. An algebra A is augmented if it is equipped with an algebra homomorphism which splits the algebra into an ideal component and a unit component. We make this precise with the following definition

Definition 1.1.7 (Augmented algebras)**.** Let A be an algebra. It is called augmented if there is an algebra homomorphism $\varepsilon_A : A \to \mathbb{K}$.

Given this algebra homomorphism we know that it has to preserve the unit. Thus the kernel $\mathrm{Ker}\varepsilon_A \subseteq A$ is almost A, but without its unit. In the module category $Mod_{\mathbb{K}}$ the morphism ε_A is automatically a split-epimorphism, where the splitting is given by the unit 1_A. Thus as a module we have $A \simeq \overline{A} \oplus \mathbb{K}$, where $\overline{A} = \mathrm{Ker}\varepsilon_A$. $\overline{A}$ is called the augmentation ideal, or the reduced algebra of A.

A morphism $f : A \to B$ of augmented algebras is an algebra homomorphism, but with the added condition that it must preserve the augmentation; i.e. $\varepsilon_B \circ f = \varepsilon_A$. The collection of all augmented algebras over $\mathbb{K}$ together with the morphisms defines the category of augmented algebras over $\mathbb{K}$, $\mathsf{Alg}_{\mathbb{K},+}$.

Given an augmented algebra A, taking kernels of ε_A gives a functor $\overline{} : \mathsf{Alg}_{\mathbb{K},+} \to \widehat{\mathsf{Alg}}_{\mathbb{K}}$. This functor is well defined on morphisms of augmented algebras, as each morphism is required to

preserve the splitting. This functor has a quasi-inverse, given by the free augmentation $\_^+$: $\widehat{\mathsf{Alg}}_{\mathbb{K}} \to \mathsf{Alg}_{\mathbb{K},+}$. Given a non-unital algebra A, the free augmentation is defined as $A^+ = A \oplus \mathbb{K}$, where the multiplication is given by:

$$(a, k)(a', k') = (aa' + ak' + a'k, kk').$$

The unit is given by the element $(0, 1)$. We summarize this in the statement below.

Proposition 1.1.8. *The functors $\overline{}$ and $\_^+$ are quasi-inverse to each other.*

Proof. We show that the free augmentation functor is fully-faithfull and essentially surjective.

Let A and B be non-unital $\mathbb{K}$-algebras, and let $f, g : A \to B$ morphisms in $\widehat{\mathsf{Alg}}_{\mathbb{K}}$. Suppose that $f^+ = g^+$, then $f = \overline{f^+} = \overline{g^+} = g$. Now suppose that $h : A^+ \to B^+$, then $h = \overline{h}^+$.

Suppose that $A \in \mathsf{Alg}_{\mathbb{K},+}$. We want to show that $A \simeq \overline{A}^+$. As $\mathbb{K}$-modules, $A = \overline{A}^+$, so we propose that $id_A : A \to \overline{A}^+$ induces an isomorphism. To see that id_A is an algebra homomorphism is to see that the multiplication in A decomposes as $(a_1 + k)(a_2 + l) = (a_1 a_2 + a_1 l + k a_2) + kl$, where $a_1, a_2 \in \overline{A}$ and $k, l \in \mathbb{K}$. The second condition is equivalent to the existence of ε_A. id_A also preserves the augmentation as $\overline{A} \simeq \overline{\overline{A}^+}$. $\qquad\square$

There are many augmented algebras to encounter in nature. We will note some examples.

Example 1.1.9 (Group algebra). Pick any group G and any field $\mathbb{K}$. The group ring $K[G]$ is an augmented algebra where the augmentation $\varepsilon_{\mathbb{K}[G]} : \mathbb{K}[G] \to \mathbb{K}$ is given as

$$\varepsilon_{\mathbb{K}[G]}\Big(\sum_{g \in G} k_g g\Big) = \sum_{g \in G} k_g.$$

Among our most important example of algebras is the tensor algebra. This is also the free algebra over $\mathbb{K}$.

Example 1.1.10 (Tensor algebra). Let V be a $\mathbb{K}$-module. We define the tensor algebra $T(V)$ of V as the module

$$T(V) = \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \cdots.$$

The tensor algebra is then the algebra consisting of words in V. Given two words $v^1..v^i$ and $w^1...w^j$ in $T(V)$ we define the multiplication by the concatenation operation,

$$\nabla_{T(V)} : T(V) \otimes_{\mathbb{K}} T(V) \to T(V),$$
$$(v^1...v^i) \otimes (w^1...w^j) \mapsto v^1...v^i w^1...w^j.$$

The unit is given by including $\mathbb{K}$ into $T(V)$,

$$\upsilon_{T(V)} : \mathbb{K} \to T(V),$$
$$1 \mapsto 1.$$

Observe that the tensor algebra is augmented. The projection from $T(V)$ into $\mathbb{K}$ is an algebra homomorphism, so we may split the tensor algebra into its unit and its augmentation ideal $T(V) \simeq \mathbb{K} \oplus \overline{T}(V)$. We call $\overline{T}(V)$ the reduced tensor algebra.

Proposition 1.1.11 (Tensor algebras are free)**.** *The tensor algebras are the free algebras over the category of $\mathbb{K}$-modules, i.e. for any $\mathbb{K}$-module V there is a natural isomorphism $Hom_{\mathbb{K}}(V, A) \simeq Alg_{\mathbb{K}}(T(V), A)$.*

The reduced tensor algebra is the free non-unital algebra over the category of $\mathbb{K}$-modules, i.e. for any $\mathbb{K}$-module V there is a natural isomorphism $Hom_{\mathbb{K}}(V, A) \simeq \widehat{Alg}_{\mathbb{K}}(\overline{T}(V), A)$.

Proof. This proposition should be evident from the description of an algebra homomorphism from a tensor algebra. If $f : T(V) \to A$ is an algebra homomorphism, then f must satisfy the following conditions:

- Unitality: $f(1) = 1$
- Homomorphism property: Given $v, w \in V$, then $f(vw) = f(v) \cdot_A f(w)$

By induction, we see that f is completely determined by where it sends the elements of V. Thus restriction by the inclusion of V into $T(V)$ induces a bijection. $\square$

Modules

Just as for rings, every algebra A has a module category of its own. The definition is very much like that for rings.

Definition 1.1.12 (Modules)**.** Let A be an algebra over $\mathbb{K}$. A $\mathbb{K}$-module M is said to be a left (right) A-module if there exists a structure morphism $\mu_M : A \otimes_{\mathbb{K}} M \to M$ ($\mu_M : M \otimes_{\mathbb{K}} A \to M$) called multiplication. We require that μ_M is associative and preserves the unit of A; i.e. we have the commutative diagrams in $Mod_{\mathbb{K}}$,

$$
\begin{array}{ccc}
A \otimes_{\mathbb{K}} A \otimes_{\mathbb{K}} M & \xrightarrow{A \otimes \mu_M} & A \otimes_{\mathbb{K}} M \\
\downarrow{\cdot_A \otimes M} & & \downarrow{\mu_M} \\
A \otimes_{\mathbb{K}} M & \xrightarrow{\mu_M} & M
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{K} \otimes_{\mathbb{K}} M & \xrightarrow{1_A \otimes M} & A \otimes_{\mathbb{K}} M \\
& \searrow{\simeq} & \downarrow{\mu_M} \\
& & M
\end{array}
$$

Hvordan pu
tum???

Definition 1.1.13 (A-linear homomorphisms)**.** Let M, N be two left A-modules. A morphism $f : M \to N$ is called A-linear if it is $\mathbb{K}$-linear and for any a in A $f(am) = af(m)$.

The category of left A-modules is denoted as Mod_A, where the morphisms $\mathrm{Hom}_A(\_, \_)$ are A-linear. Likewise, the category of right A-modules is denoted as Mod^A. There is a free functor from the category of $\mathbb{K}$-modules to left A-modules.

Proposition 1.1.14. *Let M be a $\mathbb{K}$-module. The module $A \otimes_{\mathbb{K}} M$ is a left A-module. Moreover, it is the free left module over $\mathbb{K}$-modules, i.e. there is a natural isomorphism $Hom_{\mathbb{K}}(M, N) \simeq Hom_A(A \otimes_{\mathbb{K}} M, N)$.*

Proof. We define natural transformations in each direction and then show that they are inverses.

We define morphisms ϕ and ψ as

$$\phi : \mathsf{Hom}_A(A \otimes_{\mathbb{K}} M, N) \to \mathsf{Hom}_{\mathbb{K}}(M, N)$$
$$f \mapsto f \circ (1_A \otimes M),$$
$$\psi : \mathsf{Hom}_{\mathbb{K}}(M, N) \to \mathsf{Hom}_A(A \otimes_{\mathbb{K}} M, N)$$
$$g \mapsto \mu_N \circ (A \otimes g).$$

Pick an $f \in \mathsf{Hom}_A(A \otimes_{\mathbb{K}} M, N)$, then

$$\psi \circ \phi(f) = \mu_N \circ (A \otimes \phi(f)) = \mu_N \circ (A \otimes f(1_A \otimes M)) = f(A \otimes M) = f.$$

Pick a $g \in \mathsf{Hom}_{\mathbb{K}}(M, N)$, then

$$\phi \circ \psi(g) = \phi(\mu_N \circ (A \otimes g)) = \mu_N \circ (1_A \otimes g) = g.$$

$\square$

Corollary 1.1.14.1. *A as a left A-module is the free left A-module over $\mathbb{K}$; i.e. for any left A-module M, $M \simeq Hom_{\mathbb{K}}(\mathbb{K}, M) \simeq Hom_A(A, M)$*

Categorical structure

In order to fully understand the category of algebras, it is convenient to understand what some of the most basic limits and colimits looks like. Unfortunately, the category of algebras, does not have nice kernels and cokernels, so we will restrict our attention to augmented algebras.

The category of augmented algebras is pointed. The algebra $\mathbb{K}$ is both initial and terminal since we are forced to preserve the unit and counit.

Definition 1.1.15. Let A and B be augmented algebras. Their direct sum $A \oplus B$ is defined as the following limit:

$$
\begin{array}{ccc}
A \oplus B & \longrightarrow & B \\
\downarrow & & \downarrow{\scriptstyle \varepsilon_B} \\
A & \xrightarrow{\ \varepsilon_A\ } & \mathbb{K}
\end{array}
$$

Since $\mathbb{K}$ is terminal, then $A \oplus B$ is the product in $\mathsf{Alg}_{\mathbb{K},+}$. Since we may calculate this limit as a kernel, this is in fact a subobject of $A \oplus B$ in the sense of modules. One may observe the following relation between the direc sum and the ordinary direct sum.

Lemma 1.1.16. *Given augmented algebras A and B, then $A \oplus B \simeq (\overline{A} \oplus \overline{B})^+$, where on the right hand side we use the ordinary direct sum of the underlying modules.*

Proof. This is clear from the monadicity of the forgetful functor,

$$forget : \mathsf{Alg}_{\mathbb{K},+} \to \mathsf{Mod}_{\mathbb{K}}$$
$$A \mapsto \overline{A}.$$

$\square$

One should also observe that the injections $A \hookrightarrow A \oplus B$ and $B \hookrightarrow A \oplus B$ does not satisfy the universal property of the coproduct. Thus, the direct sum is no longer the coproduct in this category.

Definition 1.1.17. Given two augmented algebras A and B, the free product $A * B$ is defined as the following colimit:

$$
\begin{array}{ccc}
\mathbb{K} & \xrightarrow{\;v_A\;} & A \\
{\scriptstyle v_B}\downarrow & & \downarrow \\
B & \longrightarrow & A * B
\end{array}
$$

Like before, the free product is defined to be the coproduct. We will see that this is in fact a free product of augmented algebras. For commutative algebras, this case is very simple, and the coproduct of A and B is $A \otimes_{\mathbb{K}} B$. . We extend this idea to all augmented algebras, following the main idea of [4].

Lemma 1.1.18. *Given two augmented algebras A and B, then $A * B \simeq T(\overline{A} \oplus \overline{B})/I$ which is the tensor algebra over the direct sum of the underlying non-unital algebras. I is an ideal generated by elements $\langle a \otimes a' - a \cdot a', b \otimes b' - b \cdot b' \rangle$.*

Proof. We have naturally injective linear morphisms

$$\iota_A : A \hookrightarrow T(\overline{A} \oplus \overline{B})/I,$$
$$a \mapsto a,$$
$$1 \mapsto 1.$$

This is in fact a ring homomorphism since $\iota_A(aa') = aa' = a \otimes a' = \iota_A(a)\iota_A(a')$.

Suppose we have the following diagram.

$$A \overset{\iota_A}{\hookrightarrow} T(\overline{A} \oplus \overline{B})/I \overset{\iota_B}{\hookleftarrow} B$$

By functoriality we obtain a morphism $h = T(\overline{f} \oplus \overline{g}) : T(\overline{A} \oplus \overline{B}) \to T$. By unitality and augmentation property forces this to act as the identity on the respective identities. Clearly $f = h\iota_A$ and $g = h\iota_B$.

Assume there exists another $h' : T(\overline{A} \oplus \overline{B})/I \to T$ such that $f = h'\iota_A$ and $g = h'\iota_B$. Then $h = h'$ on $A \oplus B$ part of $T(\overline{A} \oplus \overline{B})/I$. Since h' is a ring morphism, $h = h'$ on all of $T(\overline{A} \oplus \overline{B})/I$. □

Every small limit in $\mathsf{Alg}_{\mathbb{K},+}$ is created by the forgetful functor, and the kernel is no exception to this.

Lemma 1.1.19. *Suppose that $f : A \to B$ is a morphism of augmented algebras. The kernel is isomorphic to* $\mathsf{Ker} f = (\overline{Ker} f)^+$.

Proof. This is clear from the monadicity of the forgetful functor. □

On the other hand, $\mathsf{Alg}_{\mathbb{K},+}$ is cocomplete as well. However, the colimits are not as simple to describe. In some cases, we are able to give a simple description of it. E.g. we know that the cokernel of a morphism $f : A \to B$ exists and is $\overline{B}/\overline{A}^+$ if A is an ideal of A. Thus A is the kernel of the cokernel morphism $g : B \to \overline{B}/\overline{A}^+$. Conversely if f is the kernel morphism of g, then A is an ideal of B. In other words we may think of an ideal as a kernel.

Given any morphism $f : A \to B$, we may consider its coimage-image factorization.

$$\mathsf{Ker} f \overset{0}{\longrightarrow} \mathsf{coIm} f \overset{\widetilde{f}}{\hookrightarrow} \mathsf{Im} f \overset{0}{\longrightarrow} \mathsf{coKer} f$$

It is clear that $\mathsf{Im} f$ is an ideal of B, thus $\mathsf{coKer} f \simeq \overline{B}/\overline{\mathsf{Im} f}^+$. The problem is that in the category of algebras we cannot be sure if $\widetilde{f}$ is an isomorphism, even if it is mono and epi. Thus the ordinary image may not be the categorical image. The image may then be defined as the smallest ideal of B such that $\mathsf{coIm} f \subseteq \mathsf{Im} f \subseteq B$. We say that f is regular whenever $\widetilde{f}$ is an isomorphism. In this case the image is then the same as the set-theoretic image, and

$$\mathsf{coKer} f \simeq \overline{B}/\overline{\mathsf{Im} f}^+.$$

1.1.2   Coalgebras

In this section we want to dualize the definitions given in the last section. As we will see, some strange artifacts from the structure of algebras appears in the dualization process. We would like to define a coalgebra as an algebra, but where all of the diagrams are reversed. For many purposes this dualization is good, but as we will observe, some finiteness conditions are necessary. The category of "ill-behaved" coalgebras will be denoted $\text{coAlg}_{\mathbb{K}}$, and we will look at this first.

Definition 1.1.20 ($\mathbb{K}$-Coalgebra)**.** Let $\mathbb{K}$ be a field. A coalgebra C over $\mathbb{K}$ is a $\mathbb{K}$-module with structure morphisms called comultiplication and counit,

$$(\Delta_C) : C \to C \otimes_{\mathbb{K}} C$$
$$\varepsilon_C : C \to \mathbb{K},$$

satisfying the coassociativity and coidentity laws.

$$\begin{aligned}\text{(coassociativity)} \quad & (\Delta_C \otimes id_C) \circ \Delta_C(c) = (id_C \otimes \Delta_C) \circ \Delta_C(c) \\ \text{(counitality)} \quad & (id_C \otimes \varepsilon_C) \circ \Delta_C(c) = c = (\varepsilon_C \otimes id_C) \circ \Delta_C(c)\end{aligned}$$

In the same way as for algebras, we say that a coalgebra is non-counital if it is without a counit.

Unlike algebras, coalgebras does not admit a single intuitive method for writing repeated application of the comulitplication. For instance, given an element $c \in C$, we may apply the comultiplication twice on c in two different ways:

$$\Delta^2_{C,(1)}(c) = (\Delta_C \otimes C)\Delta_C(c),$$
$$\Delta^2_{C,(2)}(c) = (C \otimes \Delta_C)\Delta_C(c).$$

One should immediately note that $\Delta^2_{C,(1)}(c) = \Delta^2_{C,(2)}(c)$ is coassociativity. Hence there is in fact a unique way to do repeated application of Δ_C on c. We denote the n-fold repeated application of Δ_C by Δ^n_C. Since the element $\Delta^n_C(c)$ represents a finite sum in $C^{\otimes n}$, one may use Sweedlers notation [3], as it is well-defined,

$$\Delta^n_C(c) = \sum c_{(1)} \otimes ... \otimes c_{(n)}.$$

Definition 1.1.21 (Coalgebra homomorphism)**.** Let C and D be coalgebras. Then $f : C \to D$ is a coalgebra morphism if

1. f is $\mathbb{K}$-linear
2. $(f \otimes f) \circ \Delta_C(c) = \Delta_D(f(c))$
3. $\varepsilon_D \circ f = \varepsilon_C$

Whenever C and D are non-counital, we only require 1. and 2. for a homomorphism of non-counital coalgebras.

Definition 1.1.22 (Category of coalgebras)**.** Let $\mathsf{coAlg}_{\mathbb{K}}$ denote the category of coalgebras. Its objects consists of coalgebras C, and the morphisms are coalgebra homomorphisms. The set of morphisms between C and D are denoted as $\mathsf{coAlg}_{\mathbb{K}}(C, D)$.

Let $\widehat{\mathsf{coAlg}}_{\mathbb{K}}$ denote the category of non-counital algebras. Its objects consists of non-counital algebras C, and the morphisms are non-uconital coalgebra homomorphisms. The set of morphisms between C and D are denoted as $\widehat{\mathsf{coAlg}}_{\mathbb{K}}(C, D)$.

At a first glance, coalgebras may seem weird and unnatural, but they appear many places in nature.

Example 1.1.23 ($\mathbb{K}$ as a coalgebra)**.** The field $\mathbb{K}$ can be given a coalgebra structure over itself. Since $\{1\}$ is a basis for $\mathbb{K}$ we define the structure morphisms as

$$\Delta_{\mathbb{K}}(1) = 1 \otimes 1$$
$$\varepsilon(1) = 1.$$

One may check that these morphisms are indeed coassociative and counital. Thus we may regard our field as either an algebra or a coalgebra over itself.

Example 1.1.24 ($\mathbb{K}[G]$ as a coalgebra)**.** The group algebra has a natural coalgebra structure. We may take duplication of group elements as the comultiplication, i.e.

$$\Delta_{\mathbb{K}[G]}(kg) = kg \otimes g.$$

Coincidentally we have already defined the counit, this is the augmentation $\varepsilon_{\mathbb{K}[G]}$ for the group algebra $\mathbb{K}[G]$. Recall that this was

$$\varepsilon_C(\sum k_g g) = \sum k_g.$$

One may see that these morphisms satisfy coassociativity and counitality.

Example 1.1.25 (The linear dual coalgebra)**.** Let M be any finite dimensional $\mathbb{K}$-module. There is a natural isomorphism $\xi : M^* \otimes_{\mathbb{K}} M^* \to (M \otimes_{\mathbb{K}} M)^*$, given on elementary tensors as

$$\xi(f \otimes g)(m \otimes n) = f(m)g(n).$$

Let A be a finite dimensional algebra, then its linear dual A^* is a coalgebra. The linear dual of the multiplication $(\cdot_A)$ is defined as

$$(\cdot_A)^* : A^* \to (A \otimes_{\mathbb{K}} A)^*.$$

We define the comulitplication of A^* as $\xi^{-1}(\cdot_A)^*$.

The counit of A^* is the morphism 1_A^*.

Before we state our main example, we will first introduce its essential structure.

Definition 1.1.26 (Coaugmented coalgebras)**.** Let C be a coalgebra. C is coaugumented if there is a coalgebra homomorphism $\eta_C : \mathbb{K} \to C$.

Similar to augmented algebras, each coaugmented coalgebra splits in the category $Mod_{\mathbb{K}}$. We first notice that given a coalgebra homomorphism f, the cokernel $\mathrm{Cok} f$ is also a coalgebra. This is some induced quotient structure. Given a coaugmentation $\eta_C : \mathbb{K} \to C$, we call $\mathrm{Cok} \eta_C = \overline{C}$ for the coaugmentation quotient or reduced coalgebra of C. Then we obtain the splitting $C \simeq \overline{C} \oplus \mathbb{K}$. The reduced coproduct, denoted $\overline{\Delta}_C$ may explicitly be given as

$$\overline{\Delta}_C(c) = \Delta_C(c) - 1 \otimes c - c \otimes 1.$$

Example 1.1.27 (Tensor Coalgebras). Let V be a $\mathbb{K}$-module. We define the tensor coalgebra $T^c(V)$ of V as the module

$$T^c(V) = \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \cdots.$$

Given a string $v^1...v^i$ in $T(V)$ we define the comultiplication by the deconcatenation operation,

$$\Delta_{T^c(V)} : T^c(V) \to T^c(V) \otimes_{\mathbb{K}} T^c(V)$$

$$v^1...v^i \mapsto 1 \otimes (v^1...v^i) + \left(\sum_{j=1}^{i-1}(v^1...v^j) \otimes (v^{j+1}...v^i)\right) + (v^1...v^i) \otimes 1.$$

The counit is given by projecting $T^c(V)$ onto $\mathbb{K}$,

$$\varepsilon_{T^c(V)} : T^c(V) \to \mathbb{K}$$
$$1 \mapsto 1$$
$$v^1...v^i \mapsto .$$

We observe that the tensor coalgebra is coaugmented. Its coaugmentation is given by the inclusion of $\mathbb{K}$ into $T^c(V)$. We can split $T^c(V) \simeq \mathbb{K} \oplus \overline{T}^c(V)$, where $\overline{T}^c(V)$ denotes the reduced tensor coalgebra.

Cofreeness does not come for free for the tensor coalgebra. Our problem is a mismatch in the behavior of algebras and coalgebras. The problem arises when we try to do evaluation. Suppose that A is an algebra and that we have n elements of A, i.e. an element of $A^{\otimes n}$. On this element we may apply the multiplication of A a maximum of n-times, we cannot multiply an element with nothing non-trivially. However, given an element in a coalgebra C, we may use the comultiplication on this element n times, $n + 1$ times and so on ad infinitum. In the coalgebra we may comultiply an element possibly an infinite amount of times. This is a property which is sometimes ill-behaved with our dualization of algebras to coalgebras.

However, when we dualized the tensor algebra to the tensor coalgebra we did not lose this property, that an element may only be comultiplied a finite number of times. This is due to the composition of $T^c(V)$ as a direct sum of $V^{\otimes n}$, i.e. any element is a finite sum of finite tensors.

This extra assumption we need for coalgebras will be called conilpotent. Let $C \simeq \mathbb{K} \oplus \overline{C}$ be a coaugmented coalgebra, we define the coradical filtration of C as a filtration $Fr_0 C \subseteq Fr_1 C \subseteq ... \subseteq Fr_r C \subseteq ...$ by the submodules:

$$Fr_0 C = \mathbb{K}$$
$$Fr_r C = \mathbb{K} \oplus \{c \in \overline{C} \mid \forall n \geqslant r, \overline{\Delta}_C(c) = 0\}.$$

Definition 1.1.28 (Conilpotent coalgebras)**.** Let C be a coaugmented coalgebra. We say that C is conilpotent if its coradical filtration is exhaustive, i.e.

$$\varinjlim_r Fr_r C \simeq C.$$

The full subcategory of conilpotent coalgebras will be denoted as $\mathsf{coAlg}_{\mathbb{K},\mathsf{conil}}$.

Proposition 1.1.29 (Conilpotent tensor coalgebra)**.** *Let V be a $\mathbb{K}$-module. The tensor coalgebra $T^c(V)$ is conilpotent.*

Proof. Let $v \in V$, then $\Delta_{T^c(V)}(v) = 1 \otimes v + v \otimes 1$ and $\overline{\Delta}_{T^c(V)}(v) = 0$. We then observe the following:

$$Fr_0 T^c(V) = \mathbb{K},$$
$$Fr_1 T^c(V) = \mathbb{K} \oplus V,$$
$$Fr_r T^c(V) = \bigoplus_{i \leqslant r} V^{\otimes i}.$$

This shows that the coradical filtration is exhaustive. $\qquad\square$

Proposition 1.1.30 (Cofree tensor coalgebra)**.** *The tensor coalgebra is the cofree conilpotent coalgebra over the category of $\mathbb{K}$-modules, i.e. for any $\mathbb{K}$-module V and any conilpotent coalgebra C there is a natural isomorphism $Hom_{\mathbb{K}}(\overline{C}, V) \simeq coAlg_{\mathbb{K},conil}(C, T^c(V))$.*

Proof. This proposition should be evident from the description of a coalgebra homomorphism into the a tensor coalgebra. If $g : C \to T^c(V)$ is a coalgebra homomorphism, then g must satisfy the following conditions:

1. (Coaugmentation) $g(1) = 1$,
2. (Counitality)   Given $c \in \overline{C}$ then $\varepsilon_{T^c(V)} \circ g(c) = 0$,
3. (Homomorphism property)   Given $c \in C$ then $\Delta_{T^c(V)}(g(c)) = (g \otimes g) \circ \Delta_C(c)$.

We will construct the maps for the isomorphism explicitly. If $g : C \to T^c(V)$ is a coalgebra homomorphism, then composing with projection gives a map $\pi \circ g : C \to V$. Note that $\pi \circ g(1) = 0$, so this is essentially a map $\pi \circ g : \overline{C} \to V$. For the other direction, let $\overline{g} : \overline{C} \to V$. We will then define g as

$$g = id_{\mathbb{K}} \oplus \sum_{i=1}^{\infty} (\otimes^i \overline{g}) \overline{\Delta}_C^{i-1}.$$

Observe that g is well defined, since convergence of the sum follows from conilpotency of C. One may then check that g is a coalgebra homomorphism, which yields the result. □

Comodules

Essential to our dualization are comodules, we provide a short definition.

Definition 1.1.31 (Comodules)**.** Let C be a coalgebra. A $\mathbb{K}$-module M is said to ba left (right) C-comodule if there exist a structure morphism $\omega_M : M \to C \otimes_{\mathbb{K}} M$ ($\omega_M : M \to M \otimes_{\mathbb{K}} C$) called comultiplication. We require that ω_M is coassociative with respect to the comultiplication of C and preserves the counit of C; i.e. we have the following commutative diagrams in $Mod_{\mathbb{K}}$,

$$
\begin{array}{ccc}
C \otimes_{\mathbb{K}} C \otimes_{\mathbb{K}} M \xleftarrow{C \otimes \omega_M} C \otimes_{\mathbb{K}} M & \qquad & \mathbb{K} \otimes_{\mathbb{K}} M \xleftarrow{\varepsilon_C \otimes M} C \otimes_{\mathbb{K}} M \\
\Delta_C \otimes M \uparrow \qquad \omega_M \uparrow & & \simeq \qquad \omega_M \uparrow \\
C \otimes_{\mathbb{K}} M \xleftarrow{\omega_M} M & & M
\end{array}
$$

punktum?

Definition 1.1.32 (C-colinear homomorphism)**.** Let M, N be two left C-comodules. A morphism $g : M \to N$ is called C-colinear if it is $\mathbb{K}$-linear and for any m in M, $\omega_N(g(m)) = (id_C \otimes g)\omega_M(m)$. In Sweedlers notation this looks like

$$\sum g(c)_{(1)} \otimes g(c)_{(2)} = \sum c_{(1)} \otimes g(c_{(2)}).$$

The category of left C-comodules is denoted as CoMod_C, where the morphisms $\mathsf{Hom}_C(\_, \_)$ are C-colinear. We would also like to restrict our attention to those C-comodules which are conilpotent, i.e. those comodules which have an exhaustive coradical filtration. The coradical filtration is defined analogously, as we only care for the $\mathbb{K}$-module structure. Notice that for conilpotent coalgebras this requirement is automatic. Likewise, the category of right C-comodules is denoted as CoMod^C.

Proposition 1.1.33. *Let M be a $\mathbb{K}$-module. The module $C \otimes_{\mathbb{K}} M$ is a left C-comodule. Moreover, it is the cofree left comodule over $\mathbb{K}$-modules, i.e. there is an isomorphism $Hom_{\mathbb{K}}(N, M) \simeq Hom_C(N, C \otimes_{\mathbb{K}} M)$.*

Proof. This is dual to proposition 1.1.14. We only give the isomorphism, its validity is clear.

$$
\begin{aligned}
\phi' : \mathsf{Hom}_C(N, C \otimes_{\mathbb{K}} M) &\to \mathsf{Hom}_{\mathbb{K}}(N, M) \\
f &\mapsto (\varepsilon_C \otimes M) \circ f, \\
\psi' : \mathsf{Hom}_{\mathbb{K}}(N, M) &\to \mathsf{Hom}_C(N, C \otimes_{\mathbb{K}} M) \\
g &\mapsto (C \otimes g) \circ \omega_N.
\end{aligned}
$$

□

Corollary 1.1.33.1. *C as a left C-comodule is the cofree C-comodule over $\mathbb{K}$; i.e. for any left C-comodule N, $N^* \simeq \mathrm{Hom}_{\mathbb{K}}(N, \mathbb{K}) \simeq \mathrm{Hom}_C(N, C)$.*

Categorical structure

Dual to the case of augmented algebras, conilpotent coalgebras have colimits which are easy to calculate, while the limits are a bit more difficult. For this discussion we will restrict our attention to $\mathrm{coAlg}_{\mathbb{K},\mathrm{conil}}$.

Like for augmented algebras, $\mathrm{coAlg}_{\mathbb{K},\mathrm{conil}}$ is a pointed category. The initial and terminal object is $\mathbb{K}$.

Definition 1.1.34. Let C and D be conilpotent coalgebras. Their direct sum $C \oplus D$ is defined as the following colimit:

$$
\begin{array}{ccc}
\mathbb{K} & \xrightarrow{\ \eta_C\ } & C \\
\downarrow{\scriptstyle \eta_D} & & \downarrow \\
D & \longrightarrow & C \oplus D
\end{array}
$$

In the same manner as before this is some abuse of notation. This direct sum will almost be the direct sum, except we have to fix the coaugmentation.

Lemma 1.1.35. *Given conilpotent coalgebras C and D, then $C \oplus D \simeq (\overline{C} \oplus \overline{D})^+$, where on the right hand side we use the ordinary direct sum of the underlying modules.*

Proof. This is clear from the comonadicity of the forgetful functor. □

Dually to before, the projection $C \oplus D \to C$ is not usually a coalgebra morphism.

Definition 1.1.36. Let C and D be two augmented algebras, the free product $C * D$ is defined as the following limit:

$$
\begin{array}{ccc}
C * D & \longrightarrow & C \\
\downarrow & & \downarrow{\scriptstyle \varepsilon_C} \\
D & \xrightarrow{\ \varepsilon_D\ } & \mathbb{K}
\end{array}
$$

This free product is a bit difficult to get a good grasp of. In the cocommutative case things are much simpler, and the free product is then $C * D \simeq C \otimes_{\mathbb{K}} D$.

Lemma 1.1.37. *Given to conilpotent coalgebras C and D, then $C * D \subseteq T^c(\overline{C} \oplus \overline{D})$ consists in words generated by letters in $\overline{C}$ or $\overline{D}$ on the form*

$$[\![c]\!] = \sum_{i=0}^{\infty} \Delta_C^i(c), \text{ and}$$

$$[\![d]\!] = \sum_{i=0}^{\infty} \Delta_D^i(d).$$

Proof. We define a projection $C * D \to C$ as the "identity" on the letters in C and 0 otherwise.

$$p_C : C * D \to C$$
$$[\![c]\!] \mapsto c$$
$$\_ \mapsto 0$$

By definition, p_C is a coalgebra morphism as

$$p_C^{\otimes 2}(\Delta_{T^c(\overline{C} \oplus \overline{D})}[\![c]\!]) = p_C^{\otimes 2}(\sum [\![c_{(1)}]\!] \otimes [\![c_{(2)}]\!]) = \sum c_{(1)} \otimes c_{(2)}.$$

The morphisms p_C and p_D defines a cone over C and D. It remains to check the universal property. Suppose that there are morphisms $f : T \to C$ and $g : T \to D$.

We define the morphism h as the following sum

$$h(t) = \sum_{i=1}^{\infty} [\![f(t_{(1)})]\!] \otimes [\![g(t_{(2)})]\!] \otimes \cdots \otimes [\![?(t_{(i)})]\!] + [\![g(t_{(1)})]\!] \otimes [\![f(t_{(2)})]\!] \otimes \cdots \otimes [\![?(t_{(i)})]\!],$$

where $?$ means either f or g, which is appropriate.

We see that this morphism have been constructed to be a coalgebra morphism, and every other coalgebra morphism have to be on this form. Thus h is unique. $\qquad\square$

Opposite to that of augmented algebras, every small colimit of conilpotent coalgebras is created by the forgetful functor.

Lemma 1.1.38. *Suppose that $f : C \to D$ is a morphism of augmented algebras. The cokernel is isomorphic to $\mathrm{coKer} f \simeq (\overline{\mathrm{coKer}} f)^+ \simeq \overline{D}/\overline{\mathrm{Im}} f^+$.*

Proof. This is clear from the comonadicity of the forgetful functor.   □

This time around we will instead have a problem calculating kernels. Let $f : C \to D$ be a morphism of coalgebras. The set $\{c \in C \mid f(c) = 0\}$ is not necessarily closed under comulitplication. We require that $f^{\otimes 2}(\Delta_C(c)) = f(c_{(1)}) \otimes f(c_{(2)}) = \Delta_D(f(c)) = \Delta_D(0) = 0$, but then only one of $f(c_{(1)})$ or $f(c_{(2)})$ has to be 0.

It is not that this does not always work. If f is a cokernel map, that is if $f : D \to \overline{D}/\overline{C}^+$, then $C = \{d \in D \mid f(d) = 0\}$. One may see that this is the case whenever f is epi and regular. Then this is no longer a problem for any regular morphism $f : C \to D$, as we can consider the morphism $\pi : C \to \mathrm{coIm} f$ instead. Since $\widetilde{f} : \mathrm{coIm} f \to \mathrm{Im} f$ is an isomorphism, we can use the set-theoretic description instead,

$$\mathrm{Ker} f = \{c \in C \mid f(c) = 0\}.$$

1.1.3   Electronic Circuits

Calculations involving both algebras and coalgebras tends to become convoluted and unmanagable. Since we want to study the interplay between algebras and coalgebras using other tools to write down equations can be handy. We will develop the graphical calculus briefly mentioned in [3]. This graphical calculus will consists of string diagrams, refered to as electronic circuits, which describes function compositions of tensors. Since we only care about the interplay of tensors, we may develop this graphical calculus in any closed symmetric monoidal category. Why do we want to introduce this abstraction? A closed symmetric monoidal category is a good category to model functions, or morphisms, which may take several variables in its argument. Moreover, in the next section we are going to switch categories. Thus the same notions may be applied and the differences are only seen in the implementations.

The definition of a closed symmetric monoidal category may be found in the appendix . We recall that it is a category $\mathcal{C}$ together with a bifunctor $\_ \otimes \_ : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ usually called tensor, and a unit object $Z \in \mathcal{C}$. We are also given four natural isomorphisms relating the functors and the unit to what they are meant to be:

$$
\begin{aligned}
\text{Associator} \quad & \alpha : (A \otimes B) \otimes C \to A \otimes (B \otimes C). \\
\text{Right unit} \quad & \rho : A \otimes Z \to A. \\
\text{Left unit} \quad & \lambda : Z \otimes A \to A. \\
\text{Braiding/Symmetry} \quad & \beta : A \otimes B \to B \otimes A.
\end{aligned}
$$

These natural isomorphisms are supposed to satisfy some laws as well. See the appendix for the full definition.

In an electronic circuit we want to rewrite equations, possibly involving tensors, into string diagrams. To illustrate with some simple examples, let $f : A \to B$, $g : B \to C$ and $h : D \to E$. We

may consider the composition

$$(g \otimes E) \circ (f \otimes h) : A \otimes D \to C \otimes E.$$

An electronic circuit is written from top to bottom and is composed of levels. This means that the first morphisms which we apply will be at the top, descending downwards with each function composition. We write each argument in the composition as its own string. Thus this example above will look like the circuit below. Notice how f and h are at the same level, indicating that they should be thought of as $f \otimes h$. Thus an $\otimes$ indicates a change of string, while a $\circ$ indicates a change of level.

Beware that when many tensors are used, we should really remember how each string is tensored. We may call adding tensors for horizontal composition and composition of morphism for vertical composition. Both of these have a choice in how we associate them, but both have unique choices up to isomorphism given by the associator.
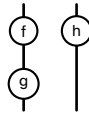
The true power of electronic circuit comes to light when we consider morphisms that in some sense "creates" or "destroys" strings. For example a morphism of 2 variables in some sense "destroys" a string by applying them to each other. Consider now a morphism $f : A \otimes B \to C$, we represent this morphism in an electronic circuit by using a converging fork. Likewise, "creation" of strings is seen as a diverging fork.

We may write the unit object Z without any strings in a circuit. By the right and left unit, an object is isomorphic to itself tensored with the unit. In this manner, whenever we have a morphism entering or exiting the unit Z we start a new string by using a sink or a source. For example, consider f as before and a morphism $g : Z \to A$, then we may write $f \circ (g \otimes B)$ as the circuit below. Notice that again this is only well-defined up to isomorphism by the right and left unit.
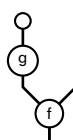
The final operation we have is braiding. Whenever we apply the braiding morphism on the tensors, we may simply denote this as interchanging the strings. For example, $\beta_{A,B} : A \otimes B \to B \otimes A$

is the circuit below. Notice that by naturality of β, we may move a braiding along the circuit. In this manner, if we have two braids, they may undo each other. A braid may also be moved to either end, and in this manner they may be ignored during calculations.

With the language of electronic circuits, we may now write down the axioms of an algebra or coalgebra by circuits. The axioms states the existence of morphisms, as well as commutativity of some diagrams. We may do this electronically. The structure maps of algebras and coalgebras will be given special notation, since we will use these often.

For convenience we will let $\mathcal{C} = \mathsf{Mod}_{\mathbb{K}}$. This is a closed symmetric monoidal category, with $\otimes_{\mathbb{K}}$ as $\otimes$. Recall that an algebra is a $\mathbb{K}$-module A together with maps $(\cdot_A) : A \otimes A \to A$ and $1_A : \mathbb{K} \to A$. These morphisms will be denoted electronically as the diagrams below.

We write the electronic laws for an algebra as how one would write equations. Associativity and unitality then becomes as follows.

Dually, given a coalgebra C we make the same notation, but everything will be turned upside down. The maps $\Delta_C : C \to C \otimes C$ and $\varepsilon_C : C \to \mathbb{K}$ will be denoted as the following electronic circuits.
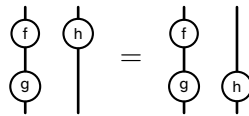
The electronic laws for C becomes the following diagrams.

Counitality

$$\text{(diagram)} \quad = \quad | \quad = \quad \text{(diagram)}$$

This notation will be adopted for our algebras and coalgebras when convenient. This will compactify the notation for comultiplication and make tensors more readable. Moreover, the intuiton for algebras is easier to dualize to coalgebras with this notation. One simply has to work out an algebra statement and then turn the diagram upside down to make it into a coalgebra statement.

Previously we talked about braiding and how that relates to interchanging strings. In the same manner that we have a horizontal and vertical associator, we also have a vertical and horizontal braiding. The braiding we first encountered is the horizontal braiding. The vertical braiding refers then to function composition of tensors. This manifests in electronic circuits as sliding a morphism along a string. Whenever the given braiding of $\mathcal{C}$ is nice enough we can get away by ignoring it whenever we move a morphism along a string. For instance, look at the category of $\mathbb{K}$-modules where we may define the braiding on elementary tensors as $\beta(a \otimes b) = b \otimes a$. In this case the braiding is agnostic to how we move our morphisms along a string. This means that we have the following equality of circuits.

$$\text{(diagram)} \quad = \quad \text{(diagram)}$$
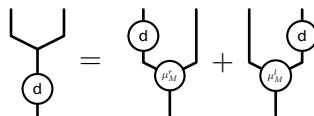
However, in nature we may encounter a braiding which is not nice enough. Here we should take a step back to figure out how we can move morphisms along strings, before we continue to use this graphical calculus of function composition. We will meet such a braiding soon.

1.1.4 Derivations and DG-Algebras

This section aims to define differential graded algebras and their modules. Given an algebra A, we may define a derivation as a map satisfying the Leibniz rule. In the dual case for a coalgebra we may define a coderivation as a map satisfying the Zinbiel rule, but for brevity we will refer to these maps as derivations. Once we get a grasp on how to make derivations we introduce graded algebras and modules, to equip these with derivations. This will give us the categories of differential graded algebras and chain complexes. Throughout this section we will also develop electronic circuits for these notions.

Definition 1.1.39 (Derivations and Coderivations)**.** Let M be an A-bimodule. A $\mathbb{K}$-linear morphism $d : A \to M$ is called a derivation if $d(ab) = d(a)b + ad(b)$, i.e. electronically,

$$\text{(diagram)} \quad = \quad \text{(diagram)} \quad + \quad \text{(diagram)}$$

punktum?

Let N be a C-bicomodule. A $\mathbb{K}$-linear morphism $d : N \to C$ is called a coderivation if $\Delta_C \circ d = (d \otimes id_C) \circ \omega_N^r + (id_C \otimes d) \circ \omega_N^l$, i.e. electronically,
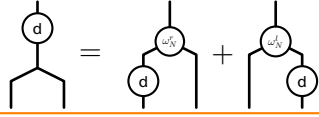
We remark that this translation between equations and electronic circuits is not at the same level of generalization. Due to this, the electronic circuit description has more advantages as it allows us to think with elements when we are really only dealing with morphisms. This will later be abused, to derive results which is e.g. independent of the given braiding on the category.

A useful fact about derivations is that they will always map the identity to 0. We obtain this from Leibniz rule as one would get $d(1) = 2d(1)$, and thus $d(1) = 0$.

Proposition 1.1.40. *Let V be a $\mathbb{K}$-module and M be a $T(V)$-bimodule. A $\mathbb{K}$-linear morphism $f : V \to M$ uniquely determines a derivation $d_f : T(V) \to M$, i.e. there is an isomorphism $Hom_\mathbb{K}(V, M) \simeq Der(T(V), M)$.*

Let N be a $T^c(V)$-bicomodule. A $\mathbb{K}$-linear morphism $g : M \to V$ uniquely determines a coderivation $d_g^c : N \to T^c(V)$, i.e. there is an isomorphism $Hom_\mathbb{K}(N, V) \simeq Coder(N, T^c(V))$.

Proof. Let $a_1 \otimes ... \otimes a_n$ be an elementary tensor of $T(V)$. We define a map $d_f : T(V) \to M$ as

$$d_f(a_1 \otimes ... \otimes a_n) = \sum_{i=1}^n a_1 ... f(a_i) ... a_n$$
$$d_f(1) = 0.$$

d_f is a derivation by defintion.

Restriction to V gives the natural isomorphism. Let $i : V \to T(V)$, then $i^* d_f = f$. Let $d : T(V) \to M$ be a derivation, then $d_{i*d} = d$. Suppose that $g : M \to N$ is a morphism between $T(V)$-bimodules, then naturality follows from bilinearity.

In the dual case, $d_g^c : N \to T^c(V)$ is a bit tricky to define. Let $\omega_N^l : N \to N \otimes T^c(V)$ and $\omega_N^r : N \to T^c(V) \otimes N$ denote the coactions on N. Since $T^c(V)$ is conilpotent, we get the same kind of finiteness restrictions on N. Define the reduced coactions as $\overline{\omega}_N^l = \omega_N^l - \_ \otimes 1$ and $\overline{\omega}_N^r = \omega_N^r - 1 \otimes \_$, this is well-defined by coassociativity. Observe that for any $n \in N$ there are k and $k' > 0$ such that $\overline{\omega}_N^{lk}(n) = 0$ and $\overline{\omega}_N^{rk'}(n) = 0$.

Let $n_{(k)}^{(i)}$ denote the extension of n by k coactions at position i, i.e.

$$n_{(k)}^{(i)} = \overline{\omega}_N^{r^i} \overline{\omega}_N^{l^{k-i}}(n).$$

The extension of n by k coactions is then the sum over every position i,

$$n_{(k)} = \sum_{i=0}^{k} n_{(k)}^{(i)}.$$

Observe that $n_{(0)} = n$. The grade of n may be thought of as the smallest k such that $n_{(k)}$ is zero. This grading gives us the coradical filtration of N, and it is exhaustive by the finiteness restrictions given above. With this notion, every element of N may be given a finite grade.

If $g : N \to V$ is a linear map, we may think of it as a map sending every element of N to an element of $T^c(V)$ of grade 1. To get a map which sends element of grade k to grade k, we must extend the morphism. Let $\pi : T^c(V) \to V$ be the linear projection and define $g_{(k)}^{(i)} = \pi \otimes ... \otimes g \otimes \pi$ as a morphism which of k tensors which is g at the i-th argument, but the projection otherwise. d_g^c is then defined as the sum over each coaction and coordinate,

$$d_g^c(n) = \sum_{k=0}^{\infty} \sum_{i=0}^{k} g_{(k)}^{(i)}(n_{(k)}^{(i)}).$$

Upon closer inspection we may observe that this is the dual construction of the derivation morphism. It is well-defined as the sum is finite by the finiteness restrictions. The map is a coderivation by duality, and the natural isomorphism is given by composition with the projection map π. □

We now wish to get into the territory of derivations as we know them from homological algebra. This is obtained by choosing an algebra and equipping it with a derivation. This will give us a universal derivation to work with when we want to define a differential on a graded module.

Definition 1.1.41 (Differential algebra)**.** Let A be an algebra. We say that A is a differential algebra if it is equipped with a derivation $d : A \to A$. Dually, a coalgebra C is called a differential coalgebra if it is equipped with at least one coderivation $d : C \to C$.

Definition 1.1.42 (A-derivation)**.** Let (A, d_A) be a differential algebra and M a left A-module. A $\mathbb{K}$-linear morphism $d_M : M \to M$ is called an A-derivation if $d_M(am) = d_A(a)m + ad_M(m)$, or electronically,

punktum?

Dually, given a differential coalgebra (C, d_C) and N a left C-comodule, a $\mathbb{K}$-linear morphism $d_N : N \to N$ is a coderivation if $\omega_N \circ d_N = (d_C \otimes id_N + id_C \otimes d_N) \circ \omega_N$, or electronically,

When there is no ambiguity, we will start to adopt writing the differential in electronic circuits as a triangle,

Proposition 1.1.43. *Let A be a differential algebra and M a $\mathbb{K}$-module. A $\mathbb{K}$-linear morphism $f : M \to A \otimes_{\mathbb{K}} M$ uniquely determines a derivation $d_f : A \otimes M \to A \otimes M$, i.e. there is an isomorphism $\text{Hom}_{\mathbb{K}}(M, A \otimes_{\mathbb{K}} M) \simeq \text{Der}(A \otimes_{\mathbb{K}} M)$. Moreover, d_f is given as $((\cdot_A) \otimes id_M) \circ (id_A \otimes f) + d_A \otimes id_M$.*

Dually, if C is a differential coalgebra and N is a $\mathbb{K}$-module, then a $\mathbb{K}$-linear morphism $g : C \otimes N \to N$ uniquely determines a coderivation $d_g : C \otimes_{\mathbb{K}} N \to C \otimes_{\mathbb{K}} N$. There is an isomorphism $\text{Hom}_{\mathbb{K}}(C \otimes_{\mathbb{K}} N, N) \simeq \text{Coder}(C \otimes_{\mathbb{K}} N)$, and d_g is given as $(id_C \otimes g) \circ (\Delta_C \otimes id_N) + d_C \otimes id_N$.

Proof. This is only proved in the case of an algebra. The case of a coalgebra is dual.

We first got to prove that the morphism $d_\_ : \text{Hom}_{\mathbb{K}}(M, A \otimes_{\mathbb{K}} M) \to \text{Der}(A \otimes_{\mathbb{K}} M)$ is well-defined. This means to check that for any morphism $f : M \to A \otimes_{\mathbb{K}} M$ the morphism d_f satisfies the Leibniz rule.

Assume that we have elements $a, b \in A$ and $m \in M$. Then $d_f(ab \otimes m) = d_f(a(b \otimes m))$. We now make an abuse of notation to write an equality between an element and a circuit. Recall that this really means that we have to think of a, b and m as generalized elements,

$$d_f(ab \otimes m) \;=\; \text{}$$

$$= d_A(a)b \otimes m + a d_f(b \otimes m).$$

Next we show that $d_\_$ has an inverse. This is given by "restriction to M", also known as

$$(1_A \otimes M)^* : \text{Hom}_{\mathbb{K}}(A \otimes_{\mathbb{K}} M, N) \to \text{Hom}_{\mathbb{K}}(M, N).$$

Let $f : M \to A \otimes_{\mathbb{K}} M$ be a linear map and $D : A \otimes_{\mathbb{K}} M \to A \otimes_{\mathbb{K}} M$ be a derivation, then a quick calculation verifies that $d_\_$ is inverse to restriction.

punktum?

$$d_f \circ (1_A \otimes M) = \quad \vcenter{\hbox{}} \;+\; \vcenter{\hbox{}} \quad = f$$

$$d_{D \circ (1_A \otimes M)} = \quad \vcenter{\hbox{}} \;+\; \vcenter{\hbox{}} \quad = D$$

Notice that in the last equation we use the Leibniz rule to get the equality to D.

$\square$

We are now going to slowly work our way towards the context of homological algebra. We say that a $\mathbb{K}$-module M^* admits a $\mathbb{Z}$-grading if it decomposes into either summands or factors

<div style="color:orange; border:1px solid orange;">Gir dette m
definere?</div>

$$M^* = \bigoplus_{z:\mathbb{Z}} M^z \text{ or } M^* = \prod_{z:\mathbb{Z}} M^z.$$

An element of $m \in M$ is said to be homogenous if it is properly contained in a single summand, i.e. $m \in M^n$. m is then said to have degree n. We say that a morphism of graded modules $f : M^* \to N^*$ is homogenous of degree n if it preserves the grading, i.e. $f(M^i) \subseteq N^{n+i}$. The degree of a homogenous element m or morphism f is denoted as $|m|$ or $|f|$.

There is a distinction between the ordinary and self-enriched category of graded modules. We are going to work with the self-enriched category, and its hom-objects is the graded module of homogenous morphism. We denote a factor in the grading as $\mathsf{Hom}_{\mathbb{K}}^w(M^*, N^*) = \{f : M^* \to N^* \mid f \text{ is homogenous and } |f| = w\}$, so the graded hom is

$$\mathsf{Hom}_{\mathbb{K}}^* = \prod_{w \in \mathbb{Z}} \mathsf{Hom}_{\mathbb{K}}^w.$$

The ordinary hom-objects are the hom-objects of the underlying modules, $\mathsf{Hom}_{\mathbb{K}}(M, N)$. This category is denoted as $\mathsf{Mod}_{\mathbb{K}}^*$. In general and whenever it makes sense, we denote $\mathcal{C}^*$ as the category of $\mathbb{Z}$-graded objects from $\mathcal{C}$.

The category $\mathsf{Mod}_{\mathbb{K}}^*$ is a closed symmetric monoidal category. The tensor is given by the following formula, using the ordinary tensor of $\mathsf{Mod}_{\mathbb{K}}$,

$$M^* \otimes N^* = \bigoplus_{n \in \mathbb{Z}} \bigoplus_{p \in \mathbb{Z}} M^p \otimes_{\mathbb{K}} N^q, \text{ where } q = n - p.$$

The associator of $\mathsf{Mod}_{\mathbb{K}}$ may be lifted to this tensor. The unit is the module $\mathbb{K}^* \simeq \mathbb{K}$ concentrated in degree 0. The right and left unit transformation may also be lifted from $\mathbb{K}$.

The category $\mathsf{Mod}_{\mathbb{K}}^*$ is closed. This means that our graded tensor fixed in one variable is left adjoint to the graded hom. We may obtain the graded hom as a right adjoint for the other variable as well by using the braiding which we will define later. Showing closedness is a routine

calculation by using the tensor-hom adjunction from $\mathsf{Mod}_{\mathbb{K}}$.

$$\mathsf{Hom}_{\mathbb{K}}^*(A^* \otimes B^*, C^*) = \prod_{w \in \mathbb{Z}} \prod_{n \in \mathbb{Z}} \mathsf{Hom}_{\mathbb{K}}^w(\bigoplus_{p \in \mathbb{Z}} A^p \otimes_{\mathbb{K}} B^{n-p}, C^n)$$

$$= \prod_{w \in \mathbb{Z}} \prod_{n \in \mathbb{Z}} \prod_{p \in \mathbb{Z}} \mathsf{Hom}_{\mathbb{K}}(A^p \otimes B^{n-(p+w)}, C^n) \simeq \prod_{w \in \mathbb{Z}} \prod_{n \in \mathbb{Z}} \prod_{p \in \mathbb{Z}} \mathsf{Hom}_{\mathbb{K}}(A^p, \mathsf{Hom}_{\mathbb{K}}(B^{n-(p+w)}, C^n))$$

$$\simeq \prod_{w \in \mathbb{Z}} \prod_{p \in \mathbb{Z}} \mathsf{Hom}_{\mathbb{K}}(A^p, \prod_{n \in \mathbb{Z}} \mathsf{Hom}_{\mathbb{K}}(B^{n-(p+w)}, C^n)) = \prod_{w \in \mathbb{Z}} \prod_{p \in \mathbb{Z}} \mathsf{Hom}_{\mathbb{K}}(A^p, \mathsf{Hom}_{\mathbb{K}}^{p+w}(B^*, C^*))$$

$$= \prod_{w \in \mathbb{Z}} \mathsf{Hom}_{\mathbb{K}}^w(A^*, \mathsf{Hom}_{\mathbb{K}}^*(B^*, C^*)) = \mathsf{Hom}_{\mathbb{K}}^*(A^*, \mathsf{Hom}_{\mathbb{K}}^*(B^*, C^*)).$$

We give a braiding on homogenous elementary tensors as

$$\beta(a \otimes b) = (-1)^{|a||b|} b \otimes a.$$

It is immediate that $\beta_{A,B}$ is inverse to $\beta_{B,A}$. Observe that this category also admits a braiding where we don't introduce a sign. This does however not work when we want to add differentials to our graded modules, so this is why we stick with this sign. This braiding is also commonly known as the Koszul sign convention.

Since $\mathsf{Mod}_{\mathbb{K}}^*$ is a closed symmetric monoidal category it admits electronic circuits. Thus the previous results which we have proved by electronic circuits also apply to this category, as the proof is identical in this language. One should note that the specific implementation may be different as vertical braiding may work different. We will now study the vertical braiding in more detail. By definition [5] the application of two homogenous morphisms $f : A \to A'$ and $g : B \to B'$ on elements $a \in A$ and $b \in B$ on tensors is

$$(f \otimes g)(a \otimes b) = (-1)^{|g||a|} f(a) \otimes g(b).$$

Viewing a and b as generalized elements again, we get Koszul sign rule on morphisms, i.e. given homogenous composable morphisms f, f', g, g' we get that

$$(f' \otimes g') \circ (f \otimes g) = (-1)^{|g'||f|}(f' \circ f) \otimes (g' \circ g).$$

Electronically we may represent this as a 2-string circuit where a morphism on the left wants to downwards pass a morphism on its right,

A good way of thinking about moving components in a circuit is that whenever we move a component downwards it has to pass over every component to the left of it on its current level, and every component to the right of it on the level below. Thus, in a 2-string component we may say

that if we move a component downwards to-, or completely past another component to its right, a sign is introduced. If we move a component upwards completely past another component to its left, we introduce a sign. In an n-string circuit it gets more complicated as the component may have to move past components on both the left and right side.

Unlike the other electronic equations in which we may substitute parts of an electronic circuit with other equal parts this does not however work a priori in this context. This is because how we defined levels. If we are given a 3-string circuit, then the formula changes, this is because we want to manipulate every element on a level simultanously. In the case of moving a left-most component downwards pass many components, we may regard them as a component on a single string. We will use this to prove a interchange of components on an n-string circuit formula.

Proposition 1.1.44. *Let $n \geqslant 1$ and suppose that we have $a_i \in A_i \to B_i$ and $b_i : B_i \to C_i$ for any $0 < i \leqslant n$. Then we get that*

$$(b_i \circ a_i) \otimes \cdots \otimes (b_n \circ a_n) = (-1)^s (b_1 \otimes \cdots \otimes b_n) \circ (a_1 \otimes \cdots \otimes a_n),$$

$$\text{where } s = \sum_{i=1}^{n} |b_i| (\sum_{1 \leqslant j < i} |a_j|).$$

Proof. We prove this by induction. If $n = 1$ this is true. $s = 0$ since the sum is empty, so $b_1 \circ a_1 = (-1)^s b_1 \circ a_1$.

Assume that the conclusion holds for $n - 1$, and that we have a_i and b_i as in the hypothesis. Let $s' = \sum_{i=1}^{n-1} |b_i| (\sum_{1 \leqslant j < i} |a_j|)$, then

$$s = s' + |b_n| (\sum_{i=1}^{n-1} |a_i|).$$

The conclusion follows from this calculation.

$$(b_1 \circ a_1) \otimes \cdots \otimes (b_n \circ a_n) = (-1)^{s'} ((b_1 \otimes \cdots \otimes b_{n-1}) \circ (a_1 \otimes \cdots \otimes a_{n-1})) \otimes (b_n \circ a_n)$$
$$= (-1)^{s' + |b_n| (\sum_{i=1}^{n-1} |a_i|)} (b_1 \otimes \cdots \otimes b_n) \circ (a_1 \otimes \cdots \otimes a_n).$$

$\square$

A final remark on this braiding is that it affects any scenario where we compose functions and they move past each other. Since function composition factors through this tensor, moving functions around is actually a braiding. An important example of this is the pre-composition functor. If f and g are homogenous and composable, then

$$f^*(g) = (-1)^{|f||g|} g \circ f.$$

The graphical calculus we have developed will be the same for any symmetric monoidal category where the braiding is similar. What this means will soon be evident when we add extra structure to the objects of $\mathrm{Mod}_{\mathbb{K}}^*$.

A graded $\mathbb{K}$-module $M^\bullet$ is called a cochain complex if it comes equipped with a differential $d_M : M^\bullet \to M^\bullet$. By a differential we mean a homogenous morphism of degree 1 such that $d_M^2 = 0$. Be cautious of bad notation, as d_M^2 might mean $d_M^2 = d_M \circ d_M$ and $d_M^2 : M^2 \to M^3$.

Given a cochain complex $M^\bullet$, we know by definition that the image of the differential lies inside the kernel of the differential. We denote this at the i'th coordinate as $B^i M \subseteq Z^i M$. $B^* M$ is the graded submodule of images, also called boundaries. $Z^* M$ is the graded submodule of kernels, also called cycles. The graded cohomology module $H^* M$ is defined as the quotient $Z^* M / B^* M$. A cochain complex is said to be exact if $H^* M \simeq 0$.

Cochain complexes are plentiful in nature.

Example 1.1.45 ($\mathbb{K}$ as a cochain complex). Let $\mathbb{K}^\bullet = (\mathbb{K}, 0)$ be the graded $\mathbb{K}$-module concentrated in degree 0 together with a 0 differential. This is trivially a cochain complex.

Example 1.1.46 (Trivial cochain complexes). Let M^* be a graded $\mathbb{K}$-module. Let $M^\bullet = (M^*, 0)$ be the same graded module with the 0 differential, this is also a cochain complex.

Example 1.1.47. We can create a chain complex as the following diagram.

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{K} \xrightarrow{id_{\mathbb{K}}} \mathbb{K} \longrightarrow 0 \longrightarrow \cdots$$

Example 1.1.48 (Cone of a chain map). Suppose that $f : A^\bullet \to B^\bullet$ is a homogenous morphism of degree 0 such that $f \circ d_A = d_B \circ f$. There is a cochain complex associated to f, yielding a short-exact sequence of cochain complexes. We define cone(f) at each degree by

$$\text{cone}(f)^n = A^{n+1} \oplus B^n,$$
$$d_{\text{cone}(f)}^n = \begin{pmatrix} d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{pmatrix}.$$

This gives us a short exact sequence,

$$B^\bullet \rightarrowtail \text{cone}(f) \twoheadrightarrow A^\bullet[1].$$

Example 1.1.49 (Normalized cochain complex). Let $A : \Delta^{op} \to \text{Mod}_{\mathbb{K}}$ be a simplicial $\mathbb{K}$-module. We define a collection of diagrams J^n as $J^0 = A_0$, and every other as

$$J^n = A_n \quad \begin{matrix} \xrightarrow{0} \\ \xrightarrow{d_1} \\ \vdots \\ \xrightarrow{d_n} \end{matrix} \quad A_{n-1}$$

A's normalized cochain complex is the complex given as

$$N A^{-n} = \varprojlim J^n.$$

In a complete pointed category, such as $\mathsf{Mod}_\mathbb{K}$, the limit is the same as the intersection of every kernel:

$$\varprojlim J^n = \bigcap_{i=1}^{n} \mathsf{Ker} d_i.$$

The differential of NA is given as d_0. Since we have turned the complex around, this is a morphism of degree 1. By taking the limit we force $d_0^2 = 0$ as well.

Example 1.1.50 (Associated cochain complex). Let $A : \Delta^{op} \to \mathsf{Mod}_\mathbb{K}$ be a simplicial $\mathbb{K}$-module. We define a differential as

$$d = \sum_{i=0}^{n} (-1)^i d_i.$$

Let $\mathsf{C}A$ be the complex given in each degree as

$$\mathsf{C}A^{-n} = A_n.$$

d defines a differential on $\mathsf{C}A$ of degree 1.

Example 1.1.51 (Singular chain complex with $\mathbb{K}$-coefficents). Let M be a topological space. There is a simplicial set defined as $\mathsf{Sing}(M) = \mathsf{Top}(\Delta-, M) : \Delta^{op} \to \mathsf{Set}$. Here $\Delta^{[n]}$ in Top refers to the topological standard n-simplex. We get a simplicial $\mathbb{K}$-module by creating the free one, $\mathbb{K}\mathsf{Sing}(M)$. By the above example defines a chain complex in $\mathsf{Mod}_\mathbb{K}$.

We make a distinction for some cochain complexes which is of particular interest.

Definition 1.1.52 (Almost free cochain complexes)**.** Suppose that $M^\bullet$ is a cochain complex. We say that $M^\bullet$ is almost free if the underlying graded module M^* is free, i.e. a tensor algebra.

Likewise, we say that $M^\bullet$ is almost cofree if M^* is cofree, i.e. a tensor coalgebra.

The category of cochain complexes will be denoted as $\mathsf{Mod}_\mathbb{K}^\bullet$. Note that this category is built upon $\mathsf{Mod}_\mathbb{K}^*$, and we inherit the braiding β. It can be shown that this is the only braiding which will make the tensors of this category well-defined. For this category we would like to entertain different collections of morphisms. This is because the morphisms which respects all of the structure and the morphisms which makes this category self-enriched are different. We will usually denote both of these categories as $\mathsf{Mod}_\mathbb{K}^\bullet$, but when we want to emphesize the structure preserving maps we will rather denote this as $\mathsf{Ch}(\mathbb{K})$.

When $A^\bullet$ and $B^\bullet$ are cochain complexes the graded $\mathbb{K}$-module $\mathsf{Hom}_\mathbb{K}^*(A^\bullet, B^\bullet)$ admits a derivative. Let $f : A^\bullet \to B^\bullet$ be any homogenous morphism, then the derivative-, or boundary of f is given by

$$\partial f = [d, f] = (d_{B*} + d_A^*)(f) = d_B \circ f - (-1)^{|f|} f \circ d_A.$$

We see that $|\partial| = |d_{B*} + d_A^*| = 1$, and

$$\partial^2 f = (d_{B*} + d_A^*)(d_B \circ f - (-1)^{|f|} f \circ d_A) = d_B^2 f + (-1)^{|f|} d_B f d_A - (-1)^{|f|} d_B f d_A - f d_A^2 = 0.$$

Thus, $\mathsf{Hom}_{\mathbb{K}}^{\bullet}(A^{\bullet}, B^{\bullet}) = (\mathsf{Hom}_{\mathbb{K}}^*(A^{\bullet}, B^{\bullet}), \partial)$ is a cochain complex. We endow $Mod_{\mathbb{K}}^{\bullet}$ with these hom-objects. In an electronic diagram we write ∂f as a sum of circuits,

$$\partial f = \begin{array}{c}\text{circuit} \end{array} + (-1)^{|f|} \begin{array}{c}\text{circuit}\end{array}$$

Notice how this construction of $\mathsf{Hom}_{\mathbb{K}}^{\bullet}$ is the same as the (product) total complex of an anti-commutative double complex. An anti-commutative double complex is a graded module of cochain complexes, together with a differential between the cochain complexes. These different differentials are supposed to anti-commute. An anti-commutative double complex may be illustrated in this manner.

$$
\begin{array}{ccccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \uparrow{\scriptstyle d_C^v} & & \uparrow{\scriptstyle d_C^v} & & \uparrow{\scriptstyle d_C^v} & \\
\cdots \xrightarrow{d_C^h} & C^{-1,1} & \xrightarrow{d_C^h} & C^{0,1} & \xrightarrow{d_C^h} & C^{1,1} & \xrightarrow{d_C^h} \cdots \\
 & \uparrow{\scriptstyle d_C^v} & & \uparrow{\scriptstyle d_C^v} & & \uparrow{\scriptstyle d_C^v} & \\
\cdots \xrightarrow{d_C^h} & C^{-1,0} & \xrightarrow{d_C^h} & C^{0,0} & \xrightarrow{d_C^h} & C^{1,0} & \xrightarrow{d_C^h} \cdots \\
 & \uparrow{\scriptstyle d_C^v} & & \uparrow{\scriptstyle d_C^v} & & \uparrow{\scriptstyle d_C^v} & \\
\cdots \xrightarrow{d_C^h} & C^{-1,-1} & \xrightarrow{d_C^h} & C^{0,-1} & \xrightarrow{d_C^h} & C^{1,-1} & \xrightarrow{d_C^h} \cdots \\
 & \uparrow{\scriptstyle d_C^v} & & \uparrow{\scriptstyle d_C^v} & & \uparrow{\scriptstyle d_C^v} & \\
 & \vdots & & \vdots & & \vdots &
\end{array}
$$

Another way of thinking of an anti-commutative double complex $C^{\bullet,\bullet}$ is that it is a bigraded $\mathbb{K}$-module with a vertical and horizontal differential such that $d_C^v \circ d_C^h = -d_C^h \circ d_C^v$.

Definition 1.1.53. Let $C^{\bullet,\bullet}$ be an anti-commutative double complex. We define the sum and product total complex. The differential at each $C^{p,q}$ is defined as $d_{\mathsf{Tot}C} = d_C^v + d_C^h$, and

$$\mathsf{Tot}^{\oplus}(C^{\bullet,\bullet}) = \bigoplus_{n \in \mathbb{Z}} \bigoplus_{p+q=n} C^{p,q},$$

$$\mathsf{Tot}^{\Pi}(C^{\bullet,\bullet}) = \prod_{n \in \mathbb{Z}} \prod_{p+q=n} C^{p,q}.$$

If $C^{\bullet,\bullet}$ is bounded, then $\mathsf{Tot}^{\oplus}(C^{\bullet,\bullet}) \simeq \mathsf{Tot}^{\Pi}(C^{\bullet,\bullet})$.

If we let $\text{Hom}_{\mathbb{K}}(A^\bullet, B^\bullet)^{\bullet,\bullet} = (\prod_{p,q\in\mathbb{Z}} \text{Hom}_{\mathbb{K}}(A^p, B^q), d_A^*, d_{B*})$, then it is clear that

$$\text{Hom}_{\mathbb{K}}^\bullet(A^\bullet, B^\bullet) = \text{Tot}^{\prod}(\text{Hom}_{\mathbb{K}}(A^\bullet, B^\bullet)^{\bullet,\bullet}).$$

From this we are able to deduce that $\text{Mod}_{\mathbb{K}}^\bullet$ is a closed symmetric monoidal category. The tensor may be deduced from the data of $\text{Hom}_{\mathbb{K}}^\bullet$. Define an anti-commutative double complex $(A^\bullet \otimes_{\mathbb{K}} B^\bullet)^{\bullet,\bullet} = (\bigoplus_{n\in\mathbb{Z}} \bigoplus_{p+q=n} A^p \otimes B^q, d_A \otimes B, A \otimes d_B)$, then the tensor is defined as

$$A^\bullet \otimes B^\bullet = \text{Tot}^\oplus((A^\bullet \otimes B^\bullet)^{\bullet,\bullet}).$$

It is clear that this tensor is left adjoint to $\text{Hom}_{\mathbb{K}}^\bullet$. All of the structure morphisms for a closed symmetric monoidal category are inherited from $\text{Mod}_{\mathbb{K}}^*$. This also means that $\text{Mod}_{\mathbb{K}}^\bullet$ employs the same electronic circuits as $\text{Mod}_{\mathbb{K}}^*$.

The category of cochain complexes with chain maps $\text{Ch}(\mathbb{K})$ is defined to have its hom-objects as $Z^0\text{Hom}_{\mathbb{K}}^\bullet(A^\bullet, B^\bullet)$. By abuse of notation we may write $\text{Ch}(\mathbb{K}) = Z^0\text{Mod}_{\mathbb{K}}^\bullet$. Notice that this condition means that the derivative of any morphism $f : A^\bullet \to B^\bullet$ in $\text{Ch}(\mathbb{K})$ is 0; i.e. that $\partial f = 0$, or $f \circ d_A = d_B \circ f$. We will call these morphisms for chain maps.

The homotopy category $\text{K}(\mathbb{K})$ is defined to be the quotient category of $\text{Ch}(\mathbb{K})$ at null-homotopic chain maps. Observe that $\text{K}(\mathbb{K}) = H^0\text{Mod}_{\mathbb{K}}^\bullet$. This is definitional, because the chain maps $f, g : A^\bullet \to B^\bullet$ are homotopic if there is a homogenous morphism $h : A^\bullet \to B^\bullet$ of degree -1 such that $\partial h = f - g$.

A chain map $f : A^\bullet \to B^\bullet$ induces homogenous morphisms of degree 0.

$$B^*f : B^*A \to B^*B$$
$$Z^*f : Z^*A \to Z^*B$$
$$H^*f : H^*A \to H^*B$$

We say that f is a quasi-isomorphism if H^*f is an isomorphism. This is equivalent to say that $\text{cone}(f)$ is exact.

A cochain complex $N^\bullet$ is said to be contractible if id_N is null-homotopic. Then it follows for any other cochain complexes $M^\bullet$ that $H^0\text{Hom}_{\mathbb{K}}^\bullet(M^\bullet, N^\bullet) \simeq 0$.

We define the shift functor $\_[n] : \text{Mod}_{\mathbb{K}}^\bullet \to \text{Mod}_{\mathbb{K}}^\bullet$ is defined on cochains $M^\bullet$ as

$$(M^\bullet, d_M)[n] = (M^\bullet[n], (-1)^n d_M).$$

With this definition, shifting is naturally isomorphic to tensoring. That is if $\mathbb{K}^\bullet[n]$ denotes the field concentrated in dimension $-n$, then

$$\mathbb{K}^\bullet[n] \otimes_{\mathbb{K}} M^\bullet \simeq M^\bullet[n] \simeq M^\bullet \otimes_{\mathbb{K}} \mathbb{K}^\bullet[n].$$

By writing out the total tensor product, one may see how the differential gets its sign. We usually call $\_[1]$ shifting, desuspension or looping; and $\_[-1]$ for inverse-shifting, suspension or delooping.

We are now ready to talk about algebras in $Mod_\mathbb{K}^\bullet$.

Definition 1.1.54 (Differential graded algebra)**.** $(A^\bullet, d_A)$ is a differential graded algebra if:

- $A^\bullet$ is a differential algebra in $\mathsf{Mod}_\mathbb{K}^\bullet$.
- The structure morphisms $(\cdot_A)$ and 1_A are chain maps.
- The derivation and differential coincides.

Example 1.1.55 (The unit). $\mathbb{K}^\bullet = (\mathbb{K}, 0)$ is a differential graded algebra in the trivial way. It is concentrated in degree 0 and the differential is the trivial derivation.

Example 1.1.56 (De Rham complex). Given a manifold M, the exterior algebra ΩM is a differential graded algebra. See Tu [6] for a thouough explanation.

Given a differential graded, or dg-algebra $A^\bullet$, we may form the category of left $A^\bullet$-modules, Mod_A.

Definition 1.1.57. $M^\bullet$ is a left $A^\bullet$-module if

- $M^\bullet$ is a cochain complex,
- there is a chain map $\mu_M : A^\bullet \otimes_\mathbb{K} M^\bullet$ satisfying associativity and unitality,
- d_M is an $A^\bullet$-derivation.

The hom-objects are defined analogously. We use $\mathsf{Hom}_{A^\bullet}^\bullet$ to denote the $\mathbb{K}$-linear cochain complex.

With this definition the categories $\mathsf{Mod}_{\mathbb{K}^\bullet}$ and $\mathsf{Mod}_\mathbb{K}^\bullet$ is the same category. This is because a chain complex already satisfy the first two bullet points by definition. Being a $\mathbb{K}^\bullet$-derivation is trivial, so every map satisfies this.

We also have the dual definition to obtain dg-coalgebras and their comodules.

Definition 1.1.58. $C^\bullet$ is a differential graded coalgebra if

- $C^\bullet$ is a differential coalgebra in $\mathsf{Mod}_\mathbb{K}^\bullet$,
- the structure morphisms Δ_C and ε_C are chain maps,
- the coderivation and differential coincides

Definition 1.1.59. $N^\bullet$ is a left $C^\bullet$-comodule if

- $N^\bullet$ is a cochain complex,
- there is a chain map $\omega_C : N^\bullet \to C^\bullet \otimes_\mathbb{K} N^\bullet$ satisfying coassociativity and counitalit,
- d_N is a $C^\bullet$-coderivation.

By these definitions we may extend proposition 1.1.43 to the category of cochain complexes.

Corollary 1.1.59.1. *Let $A^\bullet$ be a differential graded algebra and $M^\bullet$ a cochain complex. A homogenous $\mathbb{K}$-linear morphism $f : M \to A \otimes_\mathbb{K} M$ uniquely determines a derivation $d_f : A \otimes M \to A \otimes M$ of same degree, i.e. there is an isomorphism $Hom^*_\mathbb{K}(M^\bullet, A^\bullet\otimes_\mathbb{K}M^\bullet) \simeq Der^*(A^\bullet\otimes_\mathbb{K}M^\bullet)$. Moreover, d_f is given as $(\nabla_{A^\bullet}\otimes id_M)\circ(id_A\otimes f)+d_{A\otimes M}$.*

Dually, if $C^\bullet$ is a differential graded coalgebra and $N^\bullet$ is a cochain complex, then a homogenous $\mathbb{K}$-linear morphism $g : C^\bullet \otimes N^\bullet \to N^\bullet$ uniquely determines a coderivation $d_g : C^\bullet\otimes_\mathbb{K}N^\bullet \to C^\bullet\otimes_\mathbb{K}N^\bullet$. There is an isomorphism $Hom^*_\mathbb{K}(C^\bullet\otimes_\mathbb{K}N^\bullet, N^\bullet) \simeq Coder^*(C^\bullet\otimes_\mathbb{K}N^\bullet)$, and d_g is given as $(id_C \otimes g) \circ (\Delta_{C^\bullet} \otimes id_N) + d_{C\otimes N}$.

Proof. The same electronic circuits as in the proof of proposition 1.1.43 suffice to prove this statement. □

It is notable that this statement carries an additional two duals. That is we have the same result when we consider right modules as well. The same proof applies in these cases.

1.2 Cobar-Bar Adjunction

1.2.1 Convolution Algebras

Given a coalgebra C and an algebra A, we obtain a special product on the hom-object $Hom_\mathbb{K}(C, A)$ by twisting the comulitplication and multiplication together. The convolution algebra forms the backbone of our proof of the cobar-bar adjunction.

Let C be a coalgebra and A an algebra, then if $f, g : C \to A$ is a $\mathbb{K}$-linear morphism we may define $f \star g = (\cdot_A)(f \otimes g)\Delta_C$. This operation is called $\star$ convolution.

$$f \star g = \begin{array}{c}\boxed{}\end{array}$$

Proposition 1.2.1 (Convolution algebra)**.** *The $\mathbb{K}$-module $Hom_\mathbb{K}(C, A)$ is an associative algebra when equipped with convolution $\star : Hom_\mathbb{K}(C, A) \to Hom_\mathbb{K}(C, A)$. The unit is given by $1 \mapsto \upsilon_A\circ\varepsilon_C$.*

Proof. This proposition follows from (co)associativity and (co)unitality of (C) A.

$$(f \star g) \star h \quad = \quad \text{[diagram]} \quad = \quad \text{[diagram]} \quad = \quad \text{[diagram]} \quad = \quad f \star (g \star h)$$

$$(v_A \circ \varepsilon_C) \star f \quad = \quad \text{[diagram]} \quad = \quad \text{[diagram]} \quad = \quad \text{[diagram]} \quad = \quad f \star (v_A \circ \varepsilon_C)$$

$\square$

This proof does not rely on braiding and lifts to any closed symmetric monoidal category.

If we are given an algebra A, then it may be considered as a differential algebra with the the trivial derivation. That is, $(A, 0)$ is a differential algebra. For such structures, the set of A-derivations is precisely the set of A-linear morphisms. Dually, every coalgebra C may be thought of as a differential coalgebra.

We may apply a trivialization of proposition 1.1.43 to A and C considered as differential (co)algebra. When we look at the module $C \otimes_{\mathbb{K}} A$, it is both free over A on the right and cofree over C on the left. Consider a morphism $\alpha : C \to A$, then there are two ways to extend α to obtain a (co)derivation. Precomposing with C's comultiplication gives us a morphism from C to the free A-module $C \otimes_{\mathbb{K}} A$,

$$(id_C \otimes \alpha) \circ \Delta_C : C \to C \otimes_{\mathbb{K}} A.$$

Postcomposing with the multiplication of A gives us a morphism from to the cofree C-comodule $C \otimes_{\mathbb{K}} A$ to A,

$$(\cdot_A) \circ (\alpha \otimes id_A) : C \otimes_{\mathbb{K}} A \to A.$$

When we apply proposition 1.1.43 to both morphisms, it yields the same map. Therefore it is both a derivation and a coderivation, as

$$d_\alpha^r = (id_C \otimes (\cdot_A)) \circ (id_C \otimes \alpha \otimes id_A) \circ (\Delta_C \otimes id_A)$$

$$d_\alpha^r \quad = \quad \text{[diagram]}$$

One of our main tools will be this coderivation. In the ungraded case it determines a ring morphism with little to no complications.

Proposition 1.2.2. $d^r_{\_} : \mathrm{Hom}_{\mathbb{K}}(C, A) \to \mathrm{End}(C \otimes_{\mathbb{K}} A)$ *is a morphism of algebras. Moreover, if* $\alpha \star \alpha = 0$*, then* $(d^r_\alpha)^2 = 0$.

Proof. The proof quickly follows from (co)associativity and (co)unitality.

$$d^r_{\alpha \star \beta} \;=\; \boxed{\alpha\ \beta} \;=\; \boxed{\alpha\ \beta} \;=\; d^r_\alpha \circ d^r_\beta$$

$$d^r_{\upsilon_A \circ \varepsilon_C} \;=\; \;=\; \;=\; id_{C \otimes_{\mathbb{K}} A}$$

$\square$

This proof relies on braiding, so we will run into some problems when we try to lift this proposition to the graded case. We may observe that the above has no problem lifting, this is because the β has no morphisms of odd degrees to the right or over itself. However, the dual will introduce some signs when lifted.

Corollary 1.2.2.1. *Suppose that* C *and* A *are differential graded co/algebras.* $d^r_{\_} : \mathrm{Hom}^*_{\mathbb{K}}(C, A) \to \mathrm{End}^*(C \otimes_{\mathbb{K}} A)$ *extends to a homogenous ring morphism of degree* 0.

Suppose that C and A are differential graded (co)algebras. We want to expect that the differential ∂ makes $(\mathrm{Hom}^*_{\mathbb{K}}(C, A), \star)$ into a dg-algebra.

Proposition 1.2.3. *The convolution algebra* $(\mathrm{Hom}^*_{\mathbb{K}}(C, A), \star)$ *is a dg-algebra with differential* ∂.

Proof. We know that $(\mathrm{Hom}^*_{\mathbb{K}}(C, A), \star)$ is a convolution algebra and that $(\mathrm{Hom}^*_{\mathbb{K}}(C, A), \partial)$ is a chain complex. It remains to verify that the differential is compatible with the multiplication, i.e. $\partial(f \star g) = \partial f \star g + (-1)^{|f|} f \star \partial g$.

Let $f, g \in \mathrm{Hom}^*_{\mathbb{K}}(C, A)$ be two homogenous morphisms. The key property to arrive at the result is that the differential in a dg-(co)algebra is a (co)derivation. We denote the degree of $f \star g$ as $|f \star g| = |f| + |g| = d$. Then

$$\partial(f \star g) = \partial \; \text{[diagram]} \; = \; \text{[diagram]} \; -(-1)^d \; \text{[diagram]}$$

$$= \; \text{[diagram]} \; +(-1)^{|f|} \; \text{[diagram]} \; -(-1)^d((-1)^{|g|} \; \text{[diagram]} \; + \; \text{[diagram]})$$

$$= \; \text{[diagram]} \; -(-1)^{|f|} \; \text{[diagram]} \; +(-1)^{|f|}(\; \text{[diagram]} \; -(-1)^{|g|} \; \text{[diagram]})$$

$$= \; \text{[diagram]} \; +(-1)^{|f|} \; \text{[diagram]} \; = \; \partial(f) \star g + (-1)^{|f|} f \star \partial(g)$$

$\square$

Proposition 1.2.4. *The morphism* $d^r_{\_} : Hom^\bullet_{\mathbb{K}}(C, A) \to End^\bullet(C \otimes_{\mathbb{K}} A)$ *is a chain map.*

Proof. We already know from corollary 1.2.2.1 that $d^r_{\_}$ is a homogenous ring map. It remains to see that it commutes with the differentials. That is, $\bar{\partial}d^r_\alpha = d^r_{\partial\alpha}$. We write out each summand in ∂d^r_α,

$$d_{C \otimes_{\mathbb{K}} A} \circ d^r_\alpha = \; \text{[diagram]} \; + \; \text{[diagram]} \; +(-1)^{|\alpha|} \; \text{[diagram]}$$

$$d_\alpha^r \circ d_{C\otimes_{\mathbb{K}}A} = (-1)^{|\alpha|}$$ $$+$$ $$+$$

When α is of even degree, $\partial d_\alpha^r = d_{C\otimes_{\mathbb{K}}A} \circ d_\alpha^r - d_\alpha^r \circ d_{C\otimes_{\mathbb{K}}A}$. The outer summands cancel and we are left with

$$\partial d_\alpha^r = d_{d_A\alpha - \alpha d_C} = d_{\partial\alpha}.$$

When α is of odd degree, $\partial d_\alpha^r = d_{C\otimes_{\mathbb{K}}A} \circ d_\alpha^r + d_\alpha^r \circ d_{C\otimes_{\mathbb{K}}A}$. The outer summands cancel and we are left with

$$\partial d_\alpha^r = d_{d_A\alpha + \alpha d_C} = d_{\partial\alpha}.$$

$\square$

1.2.2 Twisting Morphisms

In this section we will define twisting morphisms from coalgebras to algebras. They are of importance as the bifunctor $\mathsf{Tw}(C, A)$ is represented in both arguments. To understand the elements of Tw we start this section be reviewing the Maurer-Cartan equation.

Suppose that C is a coaugmented dg-coalgebra and A is an augmented dg-algebra. We say that a morphism $\alpha \in \mathsf{Hom}_{\mathbb{K}}^*(C, A)$ is twisting if it is of degree 1, is 0 on the coaugmentation of C, is 0 on the augmentation of A and satisfies the Maurer-Cartan equation:

$$\partial\alpha + \alpha \star \alpha = 0.$$

We say that α is an element of $\mathsf{Tw}(C, A) \subset \mathsf{Hom}_{\mathbb{K}}^1(C, A) \subset \mathsf{Hom}_{\mathbb{K}}^*(C, A)$. Notice that these requirements means that $\mathsf{Im}\alpha|_{\overline{C}} \subseteq \overline{A}$. In light of proposition 1.2.2, every morphism between (coalgebras) algebras extend to a unique (co)derivation on the tensor product $C \otimes_{\mathbb{K}} A$. Let d_α^r denote this unique morphism. In the case of dg-coalgebras and dg-algebras we perturbate the total differential on the tensor with d_α^r, as in proposition 1.1.43. We call this derivation for the perturbated derivative,

$$d_\alpha = d_{C\otimes_{\mathbb{K}}A} + d_\alpha^r = d_C \otimes id_A + id_C \otimes d_A + d_\alpha^r.$$

Proposition 1.2.5. *Suppose that C is a dg-coalgebra and A is a dg-algebra, and $\alpha \in \mathsf{Hom}_{\mathbb{K}}^{-1}(C, A)$. The perturbated derivation satisfies the following relation.*

$$d_\alpha{}^2 = d_{\partial\alpha + \alpha\star\alpha}^r$$

Moreover, a morphism satisfies the Maurar-Cartan equation if and only if its associated perturbated derivative is a differential.

Proof. $d_\alpha{}^2 = d_{C\otimes_\mathbb{K} A} \circ d_\alpha^r + d_\alpha^r \circ d_{C\otimes_\mathbb{K} A} + d_\alpha^r{}^2$. By proposition 1.2.4 the result is immediate. $\square$

Corollary 1.2.5.1. *If $\alpha : C \to A$ is a twisting morphism, then $(C \otimes_\mathbb{K} A, d_\alpha^\bullet)$ is a chain complex which is also a left C-comodule and a right A-module. This is called the right twisted tensor product and is denoted as $C \otimes_\alpha A$.*

Normally $A \otimes C$ and $C \otimes A$ are isomorphic as modules. In general, it is not true that $C \otimes_\alpha A$ and $A \otimes_\alpha C$ are isomorphic, since we have to choose a particular side to perform the twisting. However, if A is commutative and C is cocommutative then they are isomorphic. To illustrate we realize the unique derivation above as a right derivative. The left derivative d_α^l is then defined analogously,

$d_\_^l : \mathsf{Hom}_\mathbb{K}^\bullet(C, A) \to \mathsf{End}^\bullet(C, A)$ does no longer define a ring morphism. Note that this still commutes with the differential, the problem lies in the ring homomorphism property. Observe that we get

$$d_{\alpha\star\beta}^l = (-1)^{|\alpha||\beta|} d_\beta^l \circ d_\alpha^l.$$

We summarize this in the next proposition.

Proposition 1.2.6. *The morphism $d_\_^l : \mathsf{Hom}_\mathbb{K}^\bullet(C, A) \to \mathsf{End}^\bullet(C, A)$ is a skew chain map.*

Proof. This is clear by the previous discussion. $\square$

Functoriality of the right twisted tensor at the level of chain maps does not really work. To show where it may go wrong, pick two twisting morphisms $\alpha : C \to A$ and $\beta : C' \to A'$. Given a pair of morphisms $f : C \to C'$ and $g : A \to A'$, it is not clear if $f \otimes g$ will preserve the perturbated differential, and it should not be true in general.

It is however the case that the right twisted tensor product defines a tri-functor from the category of elements to chain complexes,

$$\_ \otimes_\_ \_ : \sum_{\mathsf{Coalg}\otimes\mathsf{Alg}} \mathsf{Tw} \to \mathsf{Mod}_C^A.$$

This means that any commutative square as below gets mapped to a morphism of its right twisted tensors. Here f is a morphism of coalgebras and g is a morphism of algebras,

$$
\begin{array}{ccc}
C \xrightarrow{\ \alpha\ } A & & C \otimes_\alpha A \\
\downarrow{\scriptstyle f} \quad \downarrow{\scriptstyle g} \quad \rightsquigarrow & & \downarrow{\scriptstyle f\otimes g} \\
C' \xrightarrow{\ \alpha'\ } A' & & C' \otimes_{\alpha'} A'
\end{array}
$$

The important property to obtain this is that f and g are morphisms in their respective categories. This allows us to collapse the different compositions to the same map, up to sign.

1.2.3 Bar and Cobar Construction

The bar construction was first formalized for augmented skew-commutative dg-rings by Eilenberg and Mac Lane [7]. The bar construction then served as a tool to calculate the homology of the Eilenberg-Mac Lane spaces. This construction was later dualized by Adams [8] to obtain the cobar construction. Its first purpose was to serve as a method for constructing an injective resolution in order to calculate the Cotor resolution [9]. With time, the bar-cobar construction have been subjected to much generalization, such as a fattened tensor product on simplicially enriched, tensored and cotensored categories [10]. We will mainly follow the work of [3] to obtain the bar and cobar construction. The approach which we are going to take is also slightly inspired by MacLanes [11] canonical resolutions of comonads.

For our purposes, the bar construction of an augmented algebra is a simplicial resolution as a cofree coalgebra structure. Given a dg-algebra, we will realize this as the total complex of its resoultion. Dually, the cobar construction of a conilpotent coalgebra is a cosimplicial resolution as a free algebra structure. We will see that these constructions defines an adjoint pair of functors.

An algebra A is a monoid in the monoidal category $(Mod_{\mathbb{K}}, \otimes_{\mathbb{K}}, \mathbb{K})$. By proposition B.1.5 we may think of A as an augmented cosimplicial object $A : \Delta_+ \to Mod_{\mathbb{K}}$. Notice that all of the cosimplical identities follow from associativity and unitality. If A is an augmented algebra, we may instead give it the structure of an augmented simplicial set. Let $d_0^0 = \varepsilon_A$ be the augmentation. We define $d_n^n = A^{\otimes n-1} \otimes \varepsilon_A$ and set $d_n^i = A^{i-1} \otimes (\cdot_A) \otimes A^{\otimes n-i-1}$. All the degeneracies are set to be the units, i.e. $s_n^i = A^{\otimes i} \otimes \upsilon_A \otimes A^{\otimes n-i-1}$. One may check that this structure defines an augmented simplicial object $A : \Delta_+^{op} \to Mod_{\mathbb{K}}$. Observe that the chain complex $\mathrm{C}A$ is exactly the Hochschild complex of A. We depict the simplicial object as the following diagram:

$$
\mathbb{K} \xleftarrow{\ \varepsilon_A\ } A \overset{(\cdot_A)}{\underset{A\otimes\varepsilon_A}{\Leftarrow}} A^{\otimes 2} \overset{(\cdot_A)}{\underset{A^{\otimes 2}\otimes\varepsilon_A}{\Lleftarrow}} A^{\otimes 3} \overset{(\cdot_A)}{\underset{A^{\otimes 4}\otimes\varepsilon_A}{\Lleftarrow}} \ ...
$$

$$
\mathbb{K} \qquad A \xrightarrow{\ s^1\ } A^{\otimes 2} \xrightarrow{\ s^i\ } A^{\otimes 3} \overset{s^i}{\Longrightarrow} \ ...
$$

The augmentation ideal $\overline{A}$ carries a natural semi-simplicial structure induced by A. By restricting each of the face maps $\overline{d}^i = d^i|_{\overline{A}} : \overline{A}^{\otimes n} \to \overline{A}^{\otimes n-1}$ we obtain the maps together with the simplical

identity 1. This is the non-unital Hochschild complex of A. We may depict the semi-simplical object as the following diagram:

$$\mathbb{K} \xleftarrow{\quad 0 \quad} \overline{A} \underset{0}{\overset{(\cdot_A)}{\Longleftarrow}} \overline{A}^{\otimes 2} \underset{0}{\overset{(\cdot_A)}{\Lleftarrow}} \overline{A}^{\otimes 3} \underset{0}{\overset{(\cdot_A)}{\Lleftarrow}} \dots$$

As graded modules, the chain complex $C\overline{A}$ is isomorphic to $T^c(\overline{A})$. Here we think of the grading $T^c(\overline{A})$ as starting at 0 and going down to negative degrees. Consider instead the looped non-unital algebra $\overline{A}[1]$. Every algebra may be considered as a graded algebra concentrated in degree 0, the shift functor then recontextualize the degree the algebra is concentrated in. This object $\overline{A}[1]$ is no longer an associative algebra. In order to understand this looped multiplication we will first consider $\mathbb{K}\{\omega\}$, where $|\omega| = -1$. We define a looped multiplication $(\cdot) : \mathbb{K}\{\omega\}^{\otimes 2} \to \mathbb{K}\{\omega\}$ as

$$\omega \cdot \omega = \omega.$$

Given an algebra A, the looped multiplication of $A[1]$ is defined as the composite

$$(\cdot_{A[1]}) = ((\cdot) \otimes (\cdot_A)) \circ (\mathbb{K}\{\omega\} \otimes \beta \otimes \overline{A}).$$

As an example, suppose that ωa_1 and ωa_2 are elements of $A[1]$, then their multiplication would look like

$$(\cdot_{A[1]})(\omega a_1 \otimes \omega a_2) = (-1)^{|a_1||\omega|}((\cdot) \otimes \cdot_A)(\omega^{\otimes 2} \otimes a_1 \otimes a_2) = (-1)^{|a_1|}\omega a_1 a_2.$$

Observe that the resulting morphism $(\cdot_{A[1]})$ is of degree 1.

Proposition 1.2.7. *Suppose that A is an augmented algebra. The differential $d_{\overline{A}[1]}$ is a coderivation for the cofree coalgebra $T^c(\overline{A}[1])$. Thus $(C\overline{A}[1], d_{\overline{A}[1]})$ is a dg-coalgebra.*

Proof. By injecting $\overline{A}[1]$ into $T^c(\overline{A}[1])$, we may think of $(\cdot_{\overline{A}[1]}) : \overline{A}[1]^{\otimes 2} \to T^c(\overline{A}[1])$ as a morphism into the tensor coalgebra. By using proposition 1.1.40, $(\cdot_{\overline{A}[1]})$ extends uniquely into a coderivation:

$$d^c_{\overline{A}[1]} = \sum_{n=0}^{\infty} \sum_{i=0}^{n} (\cdot_{\overline{A}[1]})^{(n)}_{(i)} = d_{\overline{A}[1]}.$$

$\square$

If (A, d_A) is an augmented dg-algebra, then A is a simplicial object of $Mod^{\bullet}_{\mathbb{K}}$. This also has an associated complex CA by taking the alternate sum of face maps. The complex CA may be seen to be the total complex of the double complex represented below.

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\ 0\ } & 0 & \xrightarrow{\ 0\ } & \mathbb{K} & \xrightarrow{\ 0\ } & 0 & \xrightarrow{\ 0\ } & \cdots \\
& & \uparrow{\scriptstyle\varepsilon_A} & & \uparrow{\scriptstyle\varepsilon_A} & & \uparrow{\scriptstyle\varepsilon_A} & & \\
\cdots & \xrightarrow{\ -d_A\ } & A^{-1} & \xrightarrow{\ -d_A\ } & A^0 & \xrightarrow{\ -d_A\ } & A^1 & \xrightarrow{\ -d_A\ } & \cdots \\
& & {\scriptstyle(\cdot_A)}\uparrow\uparrow{\scriptstyle A\otimes\varepsilon_A} & & {\scriptstyle(\cdot_A)}\uparrow\uparrow{\scriptstyle A\otimes\varepsilon_A} & & {\scriptstyle(\cdot_A)}\uparrow\uparrow{\scriptstyle A\otimes\varepsilon_A} & & \\
\cdots & \xrightarrow{\ d_{A\otimes 2}\ } & (A^{\otimes 2})^{-1} & \xrightarrow{\ d_{A\otimes 2}\ } & (A^{\otimes 2})^0 & \xrightarrow{\ d_{A\otimes 2}\ } & (A^{\otimes 2})^1 & \xrightarrow{\ d_{A\otimes 2}\ } & \cdots \\
& & \uparrow\uparrow\uparrow & & \uparrow\uparrow\uparrow & & \uparrow\uparrow\uparrow & & \\
& & \vdots & & \vdots\, x & & \vdots & &
\end{array}$$

For simplicity, we will write d_1 for the horizontal differential and d_2 for the vertical differential. CA is thus the total complex of the double complex above. Again we may instead consider the looped algebra $\overline{A}[1]$ to obtain a double complex similar as above. The following lemma states that this double complex is well-defined.

Proposition 1.2.8. *Let A an augmented dg-algebra. The bar complex BA is the total associated chain complex of the augmentation ideal $\overline{A}$. $(BA, d^\bullet_{BA})$ is the cofree conilpotent coalgebra equipped with $d^\bullet_{BA} = d_1 + d_2$ as coderivation.*

Proof. It is apparent that d_1 and d_2 are coderivations with respect to deconcatenation. Since the multiplication $(\cdot_A)$ is a chain map, we should have $d^\bullet_{BA}{}^2 = d_1 \circ d_2 + d_2 \circ d_1 = 0$. This will be shown for each element in $A^{\otimes 2}$, the result may then be extended to all of BA. Instead of decorating each a_i with an ω we will follow MacLanes notation and use brackets and bars, $\omega a_1 \otimes \omega a_2 = [a_1|a_2]$. This is how this coalgebra got its name as the bar construction.

$$\begin{aligned}
d_1 \circ d_2[a_1|a_2] &= (-1)^{|a_1|}d_1[a_1 a_2] = (-1)^{|a_1|}d_{A[1]}[a_1 a_2] \\
&= (-1)^{|a_1|+1}[d_A(a_1 a_2)] = (-1)^{|a_1|+1}([d_A(a_1)a_2] + (-1)^{|a_1|}[a_1 d_A(a_2)]) \\
&= (-1)^{|a_1|+1}[d_A(a_1)a_2] - [a_1 d_A(a_2)]
\end{aligned}$$

$$\begin{aligned}
d_2 \circ d_1[a_1|a_2] &= d_2 \circ (d_{A[1]} \otimes id_{A[1]} + id_{A[1]} \otimes d_{A[1]})[a_1 \otimes a_2] \\
&= -d_2 \circ ([d_A(a_1)|a_2] + (-1)^{|a_1|+1}[a_1|d_A(a_2)]) \\
&= (-1)^{|d_A(a_1)|+1}[d_A(a_1)a_2] + (-1)^{2|a_1|+2}[a_1 d_A(a_2)] \\
&= (-1)^{|a_1|}[d_A(a_1)a_2] + [a_1 d_A(a_2)] = -d_1 \circ d_2[a_1|a_2]
\end{aligned}$$

$\square$

Remark 1.2.9. For now we don't need to show that BA is a functor. This property follows from BA being the representing object of $\mathrm{Tw}(\_, A)$.

On the other hand, a coalgebra C is a comonoid in $Mod_{\mathbb{K}}$. By the dual of proposition B.1.5 we may think of it as an augmented simplicial object $C : (\Delta_+)^{op} \to Mod_{\mathbb{K}}$. Dually, all of the simplicial identities follows from coassociativity and counitality. A coaugmented coalgebra C may be given an augmented cosimplicial structure in the opposite way of algebras. We then get that the coaugmentation quotient $\overline{C}$ is a semi-cosimplicial object of $Mod_{\mathbb{K}}$. Observe that $\overline{C}$ has an associated chain complex like $\overline{A}$, but every arrow goes in the opposite direction.

$$\mathbb{K} \xrightarrow{v_C} C \underset{A \otimes v_C}{\overset{\Delta_C}{\rightrightarrows}} C^{\otimes 2} \underset{C^{\otimes 2} \otimes v_C}{\overset{\Delta_C}{\rightrightarrows}} C^{\otimes 3} \underset{C^{\otimes 4} \otimes v_C}{\overset{\Delta_C}{\rightrightarrows}} \ldots$$

$$\mathbb{K} \qquad C \overset{s_1}{\longleftarrow} C^{\otimes 2} \overset{s_i}{\Longleftarrow} C^{\otimes 3} \overset{s_i}{\Lleftarrow} \ldots$$

The cobar construction is made from the suspended coalgebra $C[-1]$. We may also denote suspension by tensoring with a formal generator s, such that $|s| = 1$. Then we have an isomorphism $C[-1] \simeq \mathbb{K}\{s\} \otimes C$. The cobar construction is realized as the free tensor algebra $T(\overline{C}[-1])$, where the comultiplication $\Delta_{\overline{C}[-1]}$ induces a derivation $d_{\overline{C}[-1]}$ by proposition 1.1.40.

Remark 1.2.10. As we have chosen to define $(\cdot_{A[1]})(a_1 \otimes a_2) = (-1)^{|a_1|}a_1 a_2$, we are forced by the linear dual to define $\Delta_{C[-1]}(c) = -(-1)^{|c_{(1)}|}c_{(1)} \otimes c_{(2)}$. Here we use the Sweedlers notation without sums to denote the comultiplication. Note that this really should be a sum of many different elementary tensors. Lastly, observe that this definition also agrees with Koszuls sign rule.

The associated cochain complex $\mathrm{C}C$ is the total complex of the double complex below. In the same manner as before, we want to study $C[-1]$ to obtain an analogous result to the bar construction.

Proposition 1.2.11. *Let C be a coaugmented dg-coalgebra. The cobar complex ΩC is the total associated chain complex of the suspended coaugmentation quotient $\overline{C}[-1]$. $(\Omega C, d_{\Omega C})$ is the free algebra equipped with the differential $d_{\Omega C} = d_1 + d_2$ as derivation.*

Proof. This proof is similar to the one given for the bar construction. □

Given a string of elements in the cobar $sc_1 \otimes \cdots$, we write it by using brackets and bars instead,

$$sc_1 \otimes sc_2 \otimes \cdots \otimes sc_n = \langle c_1 | c_2 | \cdots | c_n \rangle.$$

The bar and cobar construction defines an adjoint pair of functors. We want to show that for any conilpotent dg-coalgebra C, the object ΩC represents a functor in the category of augmented algebras. By Yoneda's lemma, Ω does truly define a functor.

Theorem 1.2.12. *Let C be a conilpotent dg-coalgebra and A an augmented dg-algebra. The functor Tw(C, A) is represented in both arguments, i.e.*

$$\mathit{Alg}^{\bullet}_{\mathbb{K},+}(\Omega C, A) \simeq \mathit{Tw}(C, A) \simeq \mathit{coAlg}^{\bullet}_{\mathbb{K},conil}(C, BA).$$

Proof. We will show that ΩC represents the set of twisting morphisms in the first argument. Showing that BA represents the second argument uses every dual proposition. Thus, it is necessary that C is conilpotent, in order to dualize the arguments.

Suppose that $f : \Omega C \to A$ is an augmented dg-algebra homomorphism. f is then a morphism of degree 0. By freeness, f is uniquely determined by a morphism $f \mid_{\overline{C}[-1]} : \overline{C}[-1] \to \overline{A}$ of degree 0, which corresponds to a morphism $f' : C \to A$ of degree 1 which is 0 on the augmentation and coaugmentation.

Since f is a morphism of chain complexes it commutes with the differential, i.e.

$$f \circ d_{\Omega C} = d_A \circ f$$
$$\Leftrightarrow \quad f \circ (d_1 + d_2) = d_A \circ f$$

By 1.1.11, to establish these conditions it is enough to consider the summand where $d_1 = -d_C$ and $d_2 = \overline{\Delta}_{C[-1]}$. Then the right hand side becomes $-f' \circ d_C - (-1)^{|f|}(\cdot_A)(f' \otimes f')\Delta_C$. This is equivalent to say that $-f' \circ d_C - f' \star f' = d_A \circ f'$. Thus f' is a twisting morphism as desired.

Since every step to establish that f' is a twisting morphism was a logic equivalence, we arrive at the desired conclusion. □

For our convenience, we will give these isomorphisms some names. Whenever $\tau : C \to A$ is a twisting morphism, the induced morphism of algebras is denoted $f_\tau : \Omega C \to A$ and the induced morphism of coalgebras is denoted $g_\tau : C \to BA$.

Remark 1.2.13. We could have defined a twisting morphism from any coalgebra C to algebra A. In this case we could have defined a twisting morphism as a morphism of degree 1 which satifies the Cartan-Maurer equation. This definition of twisting morphism is however not represented by the cobar and bar construction on augmented algebras. The subclass of twisting morphisms which also (co)restricts to twisting morphisms on its coaugmentation quotient and augmentation

ideal would be represented in this manner. This is simply because our definition requires that a twisting morphism on (co)augmented (co)algebras restricts to twisting morphisms in this new sense.

The cobar-bar adjunction consists of a composition with the augmentation ideal (quotient) and then the (co)free tensor (co)algebra. By reversing these operations we obtain another adjunction which is more or less the same adjunction. By abuse of language, we will call these functors for the bar and cobar construction as well, and they establish an adjoint pair between non-unital dg-algebras and reduced conilpotent dg-coalgebras. In other words, given a non-unital dg-algebra A and a reduced conilpotent dg-coalgebra C, $BA = \overline{T}^c(A[1])$ and $\Omega C = \overline{T}(C[-1])$.

$$\widehat{\mathsf{Alg}}^\bullet_{\mathbb{K}} \quad \underset{B}{\overset{\Omega}{\rightleftarrows}} \bot \quad \mathsf{coAlg}^\bullet_{\mathbb{K},\mathrm{conil},-}$$

Associated to this adjunction, we obtain universal elements, together with universal properties. Let A be an augmented dg-algebra, then the identity of the coalgebras $id_{BA} : BA \to BA$, the counit $\varepsilon_A : \Omega BA \to A$ and a twisting morphism $\pi_A : BA \to A$ are equivalent by the adjunction and representation. Dually, the identity of algebras $id_{\Omega C} : \Omega C \to \Omega C$, the unit $\eta_C : C \to B\Omega C$ and the twisting morphism $\iota_C : C \to \Omega C$ are equivalent. The morphisms π_A and ι_C are called the universal elements. We summarize their universal property in the following corollary.

Corollary 1.2.13.1. *Let A be an augmented dg-algebra, and C a conilpotent dg-coalgebra. Any twisting morphism $\alpha : C \to A$ factors uniquely through either π_A or ι_C.*

$$
\begin{array}{ccc}
 & \Omega C & \\
{\scriptstyle \iota_C}\nearrow & & \searrow {\scriptstyle g_\alpha} \\
C & \xrightarrow{\quad \alpha \quad} & A \\
{\scriptstyle f_\alpha}\searrow & & \nearrow {\scriptstyle \pi_A} \\
 & BA &
\end{array}
$$

Moreover, the morphism f_α is a morphism of dg-coalgebras, and g_α is a morphism of dg-algebras.

Definition 1.2.14 (Augmented Bar-Cobar construction)**.** Let A be an augmented dg-algebra. The (right) augmented bar construction is the right twisted tensor product $BA \otimes_{\pi_A} A$, where π_A is the universal twisting morphism.

Let C be a conilpotent dg-coalgebra. The (right) augmented cobar construction is the right twisted tensor product $C \otimes_{\iota_C} \Omega C$, where ι_C is the universal twisting morphism.

Remark 1.2.15. We could have defined the augmented bar-cobar construction as the left twisted tensor product. There is really no preference of handedness. Whenever we wish to be precise which handedness we will use it will be specified, e.g. the left augmented bar construction of A.

Proposition 1.2.16. *The augmentation ideal and quotient of the augmented bar and cobar construction are acyclic, i.e. $BA\overline{\otimes}_{\pi_A}A$ ($A\overline{\otimes}_{\pi_A}BA$) and $C\overline{\otimes}_{\iota_C}\Omega C$ ($\Omega C\overline{\otimes}_{\iota_C}C$) are acyclic.*

Proof. We will postpone this proof until chapter 3. This is a part of the fundamental theorem of twisting morphisms, and will not be relevant until then. $\square$

1.3   Strongly Homotopy Associative Algebras and Coalgebras

1.3.1   SHA-Algebras

We have seen from corollary 1.2.7 that any dg-algebra A defines a dg-coalgebra $T^c(A[1])$, the bar construction, with a coderivation m^c of degree -1. Does this however work in reverse? I.e. if A is a vector space such that the coalgebra $T^c(A[1])$ together with a coderivation m^c is a dg-coalgebra, is then A an algebra? The answer to this is no, but it leads to the definition of a strongly homotopy associative algebra.

Definition 1.3.1. An A_∞-algebra is a graded vector space A together with a differential $m : \overline{T}^c(A[1]) \to \overline{T}^c(A[1])$ that is a coderivation of degree 1.

The differential m induces structure morphisms on $A[1]$. By proposition 1.1.40, there is a natural bijection $\mathrm{Hom}_{\mathbb{K}}(\overline{T}^c(A[1]), A[1]) \simeq \mathrm{Coder}(\overline{T}^c(A[1]), \overline{T}^c(A[1]))$ given by the projection onto $A[1]$. Thus $m : \overline{T}^c(A[1]) \to \overline{T}^c(A[1])$ corresponds to maps $\widetilde{m}_n : A[1]^{\otimes n} \to A[1]$ of degree 1 for any $n \geqslant 1$. We define maps $m_n : A^{\otimes n} \to A$ by the composite $s\widetilde{m}_n\omega^{\otimes n}$. Since $\omega^{\otimes n}$ is of degree $-n$, $\widetilde{m}_n$ and s is of degree 1, we get that m_n is of degree $2 - n$.

$$
\begin{array}{ccc}
A^{\otimes n} & \xrightarrow{\ m_n\ } & A \\
{\scriptstyle \omega^{\otimes n}}\downarrow{\scriptstyle \simeq} & & {\scriptstyle s}\uparrow{\scriptstyle \simeq} \\
A[1]^{\otimes n} & \xrightarrow{\ \widetilde{m}_n\ } & A[1]
\end{array}
$$

Proposition 1.3.2. *An A_∞-algebra is equivalent to a graded vector space A together with homogenous morphisms $m_n : A^{\otimes n} \to A$ of degree $2 - n$. Moreover, the morphism must satisfy the following relations for any $n \geqslant 1$:*

$$(rel_n) \qquad \sum_{p+q+r=n} (-1)^{pq+r} m_{p+1+r} \circ (id^{\otimes p} \otimes m_q \otimes id^{\otimes r}) = 0$$

Remark 1.3.3. We make a more convenient notation for (rel_n), called partial composition $\circ_i$,

$$m_{p+1+r} \circ_{p+1} m_q = m_k \circ (id^{\otimes p} \otimes m_q \otimes id^{\otimes r})$$

$$(\text{rel}_n) \qquad \sum_{p+q+r=n} (-1)^{pq+r} m_{p+1+r} \circ_{p+1} m_q = 0.$$

Before starting with the proof, we will need a lemma for checking whether a coderivation $m : T^c(A) \to T^c(A)$ is a differential.

Lemma 1.3.4. *Let* $m : T^c(A) \to T^c(A)$ *be a coderivation, and denote* $m_n = m|_{A^{\otimes n}}$. *m is a differential if and only if the following relations are satisfied,*

$$\sum_{p+q+r=n} m_{p+1+r} \circ_{p+1} m_q = 0.$$

Proof. By proposition 1.1.40 we may write $m = \sum_{n=0}^{\infty} \sum_{i=0}^{n} m_{(n)}^{(i)}$. By using partial composition, we rewrite its n'th component as,

$$m_n = \sum_{q=1}^{n} \sum_{p=1}^{n} id^{\otimes(n-q)} \circ_p m_q = \sum_{p+q+r=n} id^{\otimes(p+1+r)} \circ_{p+1} m_q.$$

For m^2 we denote it's n'th component as m_n^2. Let $\pi : T^c(A) \to A$ denote the projection onto A. Observe the following:

$$m_n^2 = m \circ m_n = m \circ \sum_{p+q+r=n} id^{\otimes(p+1+r)} \circ_{p+1} m_q = \sum_{p+q+r=n} m \circ_{p+1} m_q,$$

$$\pi m_n^2 = \pi \sum_{p+q+r=n} m \circ_{p+1} m_q = \sum_{p+q+r=n} m_{p+1+r} \circ_{p+1} m_q.$$

By 1.1.43, every coderivation is uniquely determined by π, we get that $m^2 = 0$ if and only if

$$\sum_{p+q+r=n} m_{p+1+r} \circ_{p+1} m_q = 0.$$

$\square$

Proof of proposition 1.3.2. Let (A, m) be an A_∞-algebra. We denote the n'th component of m as $\tilde{m}_n$. The n'th components thus define maps $m_n : A^{\otimes n} \to A$ as $m_n = s\tilde{m}_n \omega^{\otimes n}$.

By the above lemma we know that the n'th component of m^2 is,

$$\sum_{p+q+r=n} \tilde{m}_{p+1+r} \circ_{p+1} \tilde{m}_q$$

$$= \sum_{p+q+r=n} \omega m_{p+1+r} s^{\otimes(p+1+r)} \circ_{p+1} \omega m_q s^{\otimes q} = \sum_{p+q+r=n} (-1)^{pq+r} \omega m_{p+1+r} \circ_{p+1} m_q s^{\otimes n}.$$

The last equation is given by applying proposition 1.1.44 twice. In other words, we want to find a parity $p = p_1 + p_2$ which determines the sign above. To get p_1 we start with moving the s on the left,

$$s^{\otimes p+1+r} \circ (id^{\otimes p} \otimes \omega m_q s^{\otimes q} \otimes id^{\otimes r}) = (-1)^{p_1} (s^{\otimes q} \otimes m_q s^{\otimes q} \otimes s^{\otimes r}).$$

By proposition 1.1.44,

$$p_1 = \sum_{i=1}^{n} \sum_{1 \leqslant j < i} (\text{if } j = p+1 \text{ then } 1 \text{ otherwise } 0) = r.$$

In the next step we separate the s on the right,

$$(id^{\otimes p} \otimes m_q \otimes id^{\otimes r}) \circ s^{\otimes n} = (-1)^{p_2} (s^{\otimes q} \otimes m_q s^{\otimes q} \otimes s^{\otimes r}).$$

We calculate p_2 to be,

$$p_2 = (2 - q) \sum_{1 \leqslant j < p+1} 1 = 2p - qp.$$

Thus the parity of p is $p = 2p - qp + r = pq + r$ modulo 2.

Since suspension and loop are isomorphisms, we get that $m^2 = 0$ if and only if (rel_n) are 0 for every $n \geqslant 1$, i.e.

$$\sum_{p+q+r=n} (-1)^{pq+r} m_{p+1+r} \circ_{p+1} m_q = 0.$$

$\square$

Given an A_∞ algebra A, we may either think of it as a differential tensor coalgebra $\overline{T}^c(A[1])$ with differential $m : \overline{T}^c(A[1]) \to \overline{T}^c(A[1])$, or as a graded vector space with morphisms $m_n : A^{\otimes n} \to A$ satisfying (rel_n). We will calculate (rel_n) for $n = 1, 2, 3$:

(rel_1) $m_1 \circ m_1 = 0$

(rel_2) $m_1 \circ m_2 - m_2 \circ_1 m_1 - m_2 \circ_2 m_1 = 0$

(rel_3) $m_1 \circ m_3 - m_2 \circ_1 m_2 + m_2 \circ_2 m_2 + m_3 \circ_1 m_1 + m_3 \circ_2 m_1 + m_3 \circ_3 m_1 = 0$

We see that (rel_1) states that m_1 should be a differential. Thus we may think of (A, m_1) as a chain complex. Furthermore, (rel_2) says that $m_2 : (A^{\otimes 2}, m_1 \otimes id_A + id_A \otimes m_1) \to (A, m_1)$ is a morphism of chain complexes. Lastly, (rel_3) gives us a homotopy for the associator of m_2, namely m_3. Thus we may regard (A, m_1, m_2) as an algebra which is associative up to the homotopy m_3. Regarding A as a chain complex instead we obtain our final definition of an A_∞-algebra.

Proposition 1.3.5. *Suppose that* (A, d) *is a chain complex, and that there exists morphisms* $m_n : A^{\otimes n} \to A$ *of degree* $2 - n$ *for any* $n \geqslant 2$. *A is an* A_∞*-algebra if and only it satisfies the following relations:*

$$(\textbf{rel'}_n) \qquad \partial(m_n) = - \sum_{\substack{n=p+q+r \\ k=p+1+r \\ k>1, q>1}} (-1)^{pq+r} m_k \circ_{p+1} m_q$$

We define the homotopy of an A_∞-algebra to be the homology of the chain complex (A, m_1). Since $\partial(m_3) = m_2 \circ_1 m_2 - m_2 \circ_2 m_2$, we get that m_2 is associative in homology. Thus for any A_∞-algebra A, the homotopy HA is an associative algebra. The operadic homology of A is defined as the homology of $(T^c(A[1]), m)$, which is the non-unital augmented Hochschild homology of A.

Example 1.3.6. Suppose that V is a cochain complex with differential d. Then V is an A_∞-algebra with trivial multiplication. In other words $m^1 = d$ and $m^i = 0$ for any $i > 1$.

Example 1.3.7. Suppose that A is a dg-algebra. Then A is an A_∞-algebra where $m^1 = d, m^2 = (\cdot)$ and $m^i = 0$ for any $i > 2$.

Example 1.3.8. Let A be a connected weight-graded algebra over $\mathbb{K}$. We may then think of $\mathbb{K}$ as an A-module with trivial action, i.e. as the quotient $A/A^{(i)}{}_{(i \geqslant 1)}$, then $Ext_A^*(\mathbb{K}, \mathbb{K})$ is an A_∞-algebra. This was shown by Lu, Palmari, Wu and Zhang [12].

Next we want to understand the category of A_∞-algebras. A morphism between A_∞-algebras is called an ∞-morphism. We define such an ∞-morphism $f : A \rightsquigarrow B$ between A_∞-algebras as associated dg-coalgebra homomorphism $Bf : (\overline{T}^c(A[1]), m^A) \to (\overline{T}^c(B[1]), m^B)$. Here Bf is purely formal, we will make sense of this soon.

Proposition 1.3.9. *Let* A, B *be two* A_∞*-algebras. A collection of morphisms* $f_n : A^{\otimes n} \to B$ *of degree* $1 - n$ *for any* $n \geqslant 1$ *defines an* ∞*-morphism* $f : A \rightsquigarrow B$ *if and only if* f_1 *is a morphism of chain complexes and for any* $n \geqslant 2$ *the following relations are satisfied:*

$$(\textbf{rel}_n) \qquad \partial(f_n) = \sum_{\substack{p+1+r=k \\ p+q+r=n}} (-1)^{pq+r} f_k \circ_{p+1} m_q^A - \sum_{\substack{k \geqslant 2 \\ i_1 + \dots + i_k = n}} (-1)^e m_k^B \circ (f_{i_1} \otimes f_{i_2} \otimes \dots \otimes f_{i_k}),$$

where e *is given as*

$$e = \sum_{l=1}^k (1 - i_l) \sum_{1 \leqslant m < l} i_m.$$

Proof. Establishing the shape of this equation is immediate by the universal property of cofree coalgebras. The parity e is obtained by factorizing out the s on the right.

$$(f_{i_1} \otimes \dots \otimes f_{i_k}) \circ s^{\otimes n} = (-1)^e (f_{i_1} s^{\otimes i_1} \otimes \dots \otimes f_{i_k} s^{\otimes i_k}).$$

By proposition 1.1.44, we arrive at the conclusion,

$$e = \sum_{l=1}^{k} |f_{i_l}| \sum_{1 \leqslant m < l} |s^{\otimes i_m}| = \sum_{l=1}^{k} (1 - i_l) \sum_{1 \leqslant m < l} i_m$$

$$\square$$

Since the composition of two dg-coalgebra homomorphisms is again a dg-coalgebra homomorphism, we get that the composition of two ∞-morphisms is again an ∞-morphism. More explicitly if $f : A \rightsquigarrow B$ and $g : B \rightsquigarrow C$ are two ∞-morphisms, then their composition is defined as

$$(fg)_n = \sum_{r} \sum_{i_1 + \ldots + i_r = n} (-1)^e g_r(f_{i_1} \otimes \ldots \otimes f_{i_r}).$$

Here e denotes the same parity as above.

Definition 1.3.10. An ∞-morphism $f : A \rightsquigarrow B$ is called strict if $f_n = 0$ for any $n \geqslant 2$.

Definition 1.3.11. Alg_∞ denotes the category of A_∞-algebras, and the morphisms in this category are the ∞-morphisms.

Observe that we may extend the bar construction to $B : \mathsf{Alg}_\infty \to \mathsf{CoAlg}^\bullet_{\mathbb{K},\mathrm{conil}}$ to a fully fatihful functor. This may be done explicitly by using proposition 1.1.40. The subcategory of the essential image is the full subcategory of every almost cofree dg-coalgebra. Notice that the bar construction on the category of dg-algebras is a non-full injection into the category of A_∞-algebras. This inclusion gives us a recontextualization of a dg-algebra as an A_∞-algebra.

A quasi-isomorphism between A_∞-algebras is called an ∞-quasi-isomorphism. Given an ∞-morphism $f : A \rightsquigarrow B$, then we say that it is an ∞-quasi-isomorphism if f_1 is a quasi-isomorphism. If we wanted to be more stringent with this definition, we would want an ∞-quasi-isomorphism to be an ∞-morphism which is a quasi-isomorphism of dg-coalgebras. We will later see that these definitions are equivalent.

A homotopy between two A_∞-algebras is a homotopy between the dg-coalgebras they define. We may however trace this definition back along the quasi-inverse of the bar construction to get a new definition in terms of many morphisms. Let $f, g : A \rightsquigarrow B$ be two ∞-morphisms, we say that $f - g$ is null-homotopic if there is a collection of morphisms $h_n : A^{\otimes n} \to B$ of degree n such that the following relations are satisfied for any $n \geqslant 1$:

$$f_n - g_n = \sum (-1)^s m_{r+1+t}^B \circ (f_{i_1} \otimes \ldots \otimes f_{i_r} \otimes h_k \otimes g_{j_1} \otimes \ldots \otimes g_{j_t}) + \sum (-1)^{j+kl} h_i \circ_{j+1} m_k^A.$$

Where s is some constant depending on t, r and k. More details may be found in [13].

As in the same case for algebras, there is also a notion of unital A_∞-algebras and augmented A_∞-algebras. For this discussion it is important to note that the field $\mathbb{K}$ is also an A_∞-algebra. This algebra will serve as the initial algebra, in the same way as it does for ordinary algebras.

Definition 1.3.12. A strictly unital A_∞-algebra is an A_∞-algebra A together with a unit morphism $v_A : \mathbb{K} \to A$ of degree 0 such that the following are satisfied:

- $m_1 \circ v_A = 0$.
- $m_2(id_A \otimes v_A) = id_A = m_2(v_A \otimes id_A)$.
- $m_i \circ_k v_A = 0$ for any $i \geqslant 3$ and $1 \leqslant k < i$.

A strictly unital ∞-morphism $f : A \rightsquigarrow B$ between strictly unital A_∞-algebras is a morphism which preserves the unit. This means that $f_1 v_A = v_B$ and $f_i \circ_k v_A = 0$ for any $i \geqslant 2$ and $1 \leqslant k < i$. The collection of strictly unital A_∞-algebras and strictly unital ∞-morphisms form a non-full subcategory of A_∞-algbras. A strict ∞-morphism which is unital at the level of chain complexes is automatically stricly unital. Strict unital will then mean strict and strictly unital. Note that $\mathbb{K}$ is strictly unital where the unit is $id_\mathbb{K}$.

Definition 1.3.13. An augmented A_∞-algebra is a strictly unital A_∞-algebra A together with a strict unital morphism $\varepsilon_A : A \to \mathbb{K}$. The ∞-morphism ε_A is called the augmentation of A.

The collection of augmented A_∞-algebras and strictly unital morphism is the category of augmented A_∞-algebras, denoted as $\mathsf{Alg}_{\infty,+}$. As in the same way for algebras, there is an equivalence of categories $\mathsf{Alg}_\infty \simeq \mathsf{Alg}_{\infty,+}$. The augmentation ideal, or the reduced A_∞-algebra is the kernel of the augmentation ε_A. A priori, it does not make sense to talk about this limit as we do not know if it exists. However, we will see in section 2.3.3 that such morphisms does in fact have a kernel. This defines a functor, $\overline{\_} : \mathsf{Alg}_{\infty,+} \to \mathsf{Alg}_\infty$, where $Ker\varepsilon_A = \overline{A}$. The quasi-inverse to this functor is given by free augmentations. Given an A_∞-algebra A, we may construct the A_∞-algebra $A \oplus \mathbb{K}$. The structure morphisms are given by m_i^A, but there is now a unit $v_{A\oplus\mathbb{K}}$. Thus we get that $m_1(1) = 0$, $m_2(a \otimes 1) = a$ and $m_i \circ_k 1 = 0$ in the same manner. We obtain a functor $\_^+ : \mathsf{Alg}_\infty \to \mathsf{Alg}_{\infty,+}$, where $A \oplus \mathbb{K} = A^+$.

1.3.2 A_∞-Coalgebras

Dual to A_∞-algebras we got conilpotent A_∞-coalgebras. Here we instead ask ourselves if the cobar construction has some converse, i.e. if C is a graded vector space such that $T(C[-1])$ together with a derivation m is a dg-algebra, is then C a coalgebra? Again, the answer to this is no, but we do obtain a definition for conilpotent A_∞-coalgebras.

Definition 1.3.14. A graded vector space C is called a conilpotent A_∞-coalgebra if it is a dg-algebra of the form $(\overline{T}(C[-1]), d)$ where d is a derivation of degree 1.

Remark 1.3.15. For the rest of this thesis, an A_∞-coalgebra should be understood as a conilpotent A_∞-coalgebra unless otherwise specified.

Corollary 1.3.15.1. *C is an A_∞-coalgebra with differential d then there is a chain complex (C, d^1), where d^1 is of degree 1, and together with morphisms $d^n : C \to C^{\otimes n}$ such that d*

uniquely determines each d^i for any $i > 0$. Conversely, if the morphisms d^i satisfy (rel)$_n$, then they uniquely determine a d such that C is an A_∞-coalgebra,

$$(\textit{rel}_n) \textit{ is} \qquad \sum_{p+q+r=n} (-1)^{pq+r} d^{p+1+q} \circ^{op}_{p+1} d^q = 0$$

A morphism of A_∞-coalgebras would be defined in the same manner, but opposite. An ∞-morphism $f : C \rightsquigarrow D$ is then either a morphism $\widetilde{f} : (T(C[-1]), m^C) \to (T(D[-1]), m^D)$ of dg-algebras; or equivalently it is a collection of morphisms $f_n : C \to D^{\otimes n}$ of degree $1 - n$ such that f_1 is a morphism of chain complexes, and for any $n \geqslant 2$ the following relations are satisfied:

$$(\mathsf{rel}_n) \qquad \partial(f_n) = \sum_{\substack{p+1+r=k \\ p+q+r=n}} (-1)^{pq+r} f_k \circ^{op}_{p+1} m^D_q - \sum_{\substack{k \geqslant 2 \\ i_1+\ldots+i_k=n}} (-1)^e m^C_k \circ^{op} (f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_k}),$$

where e is given as

$$e = \sum_{l=1}^{k} (1 - i_l) \sum_{1 \leqslant m < l} i_m.$$

We denote coAlg_∞ as the category of A_∞-coalgebras. In the same manner, the cobar construction extends to this category and gives us an identification of A_∞-coalgebras and a subcategory of dg-algebras. This subcategory consists of every dg-algebra that is isomorphic, as an algebra, to a free tensor algebra. Lastly, every dg-coalgebra is an A_∞-coalgebra by letting every morphism $m^i = 0$ where $i > 2$. This gives a non-full inclusion.

Chapter 2

Homotopy Theory of Algebras

Quillen envisioned a more general approach to homotopy theory, which he dubbed homotopical algebra. A homotopy theory was first enclosed by the structure of a model category, and now we mostly consider closed model categories. Many of the results from classical homotopy theory was recovered in this new setting of model categories. The theorem which we are most concerned about is Whitehead's theorem:

Theorem 2.0.1 (Whitehead's Theorem)**.** *Let X and Y be two CW-complexes. If $f : X \to Y$ is a weak equivalence, then it is also a homotopy equivalence. I.e. there exists a morphism $g : Y \to X$ such that $gf \sim id_X$ and $fg \sim id_Y$.*

If we endow a Quillen model category onto the category Top, we get that a space X is bifibrant if and only if it is a CW-complex. The natural generalization is then to not ask of X to be a CW-complex, but a bifibrant object.

Theorem 2.0.2 (Generalized Whiteheads Theorem, [Proposition 1.2.8 14, p. 11])**.** *Let $\mathcal{C}$ be a model category. Suppose that X and Y are bifibrant objects of $\mathcal{C}$, and that there is a weak-equivalence $f : X \to Y$. Then f is also a homotopy equivalence, i.e. there exists a morphism $g : Y \to X$ such that $gf \sim id_X$ and $fg \sim id_Y$.*

The category of differential graded algebras employs such a model category, and here we let the weak equivalences be quasi-isomorphisms. On the other hand the category of differential graded coalgebras has a model structure where the weak equivalences are the maps which are sent to quasi-isomorphism by the cobar construction. Moreover, in this case the bar and cobar construction defines a Quillen equivalence between these model structures. As we will see, a dg-coalgebra will be bifibrant exactly when it is an A_∞-algebra. Thus, by Whiteheads theorem, quasi-isomorphisms lift to homotopy equivalences. In this case the derived category of A_∞-algebras is equivalent to the homotopy category of A_∞-algebras.

We will conclude this chapter by looking at the category of algebras as a subcategory of A_∞-algebras. The derived category may then be expressed as the homotopy category of A_∞-algebras,

restricted to algebras.

2.1   Model categories

In this section we will define Quillens model category. As one may see, there is a plethora of semantically different definitions of model categories, however they are all made to be equivalent under good enough conditions. The difference mostly comes down to preference. In this thesis we will use the definitions from Mark Hoveys book "Model Categories" [14]. We will then go on to prove the basic results known about model categories, its associated homotopy category and Quillen functors between model categories.

Before we state the definition of a model category we need some preliminary definitions. For this section, let $\mathcal{C}$ be a category.

Definition 2.1.1 (Retract)**.** A morphism $f : A \to B$ in $\mathcal{C}$ is a retract of a morphism $g : C \to D$ if it fits in a commutative diagram on the form

$$
\begin{array}{ccccc}
 & & \xrightarrow{\quad id_A \quad} & & \\
A & \longrightarrow & C & \longrightarrow & A \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
B & \longrightarrow & D & \longrightarrow & B \\
 & & \xrightarrow[\quad id_B \quad]{} & &
\end{array}
$$

Definition 2.1.2 (Functorial factorization)**.** A pair of functors $\alpha, \beta : \mathcal{C}^{\to} \to \mathcal{C}^{\to}$ is called a functorial factorization if for any morphism $f \in \mathsf{Mor}(\mathcal{C})$, there is a factorization $f = \beta(f) \circ \alpha(f)$. We will use the notation $f_\alpha = \alpha(f)$ and $f_\beta = \beta(f)$. The functorial factorization may be depicted by the following commutative diagram:

$$
\begin{array}{ccc}
A & \xrightarrow{\quad f \quad} & B \\
 & {\scriptstyle f_\alpha}\searrow \quad \nearrow {\scriptstyle f_\beta} & \\
 & C &
\end{array}
$$

Definition 2.1.3 (Lifting properties)**.** Suppose that the morphisms $i : A \to B$ and $p : C \to D$ fits inside a commutative square. i is said to have the left lifting property with resptect to p, or p has the right lifting property with respect to i, if there is an $h : B \to C$ such that the two triangles commute.

$$
\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow{\scriptstyle i} & {\scriptstyle h}\nearrow & \downarrow{\scriptstyle p} \\
B & \longrightarrow & D
\end{array}
$$

Remark 2.1.4. We will call the left lifting property LLP and the right lifting property RLP.

Definition 2.1.5 (Wide subcategory)**.** A subcategory $\mathcal{W} \subset \mathcal{C}$ is called wide if $\mathcal{W}$ has every object $\mathcal{C}$ has. In particular, $\mathcal{W}$ is a subcategory having every identity morphism.

2.1.1 Model categories

Definition 2.1.6 (Model category)**.** Let $\mathcal{C}$ be a category with all finite limits and colimits. $\mathcal{C}$ admits a model structure if there are three wide subcategories each defining a class of morphisms:

- Ac $\subset \mathrm{Mor}(\mathcal{C})$ are called weak equivalences
- Cof $\subset \mathrm{Mor}(\mathcal{C})$ are called cofibrations
- Fib $\subset \mathrm{Mor}(\mathcal{C})$ are called fibrations

In addition we call morphisms in Cof $\cap$ Ac for acyclic cofibrations and Fib $\cap$ Ac for acyclic fibrations. Moreover, $\mathcal{C}$ has two functorial factorizations (α, β) and (γ, δ). The following axioms should be satisfied:

MC1 The class of weak equivalences satisfy the 2-out-of-3 property, i.e. if f and g are composable morphisms such that 2 out of f, g and gf are weak equivalences, then so is the third.

MC2 The three classes Ac, Cof and Fib are retraction closed, i.e. if f is a retraction of g, and g is either a weak-equivalence, cofibration or fibration, then so is f.

MC3 The class of cofibrations have the left lifting property with respect to acyclic fibrations, and fibrations have the right lifting property with respect to acyclic cofibrations.

MC4 Given any morphism f, f_α is a cofibration, f_β is an acyclic fibration, f_γ is an acyclic cofibration and f_δ is a fibration.

Remark 2.1.7. The class Ac has every isomorphism. This is because every isomorphism is a retract of some identity morphism.

Remark 2.1.8. The type of category which has been introduced above was first called a closed model category by Quillen [15]. In his sense, a model category does not require to have neither finite limits nor finite colimits. In our case, we will explicitly state whenever a model category is non-closed, i.e. does not have every finite limit or colimit.

A model category $\mathcal{C}$ is now defined to be a category equipped with a particular model structure. Notice that a category may admit several model structures. We will postpone examples until sufficient theory have been developed. For more topological examples, we refer to Dwyer–Spalinski [16] and Hovey [14].

An interesting and a not so non-trivial property of model categories is that giving all three classes Ac, Cof and Fib is redundant. Given the class of weak equivalences and either cofibrations or fibrations, the model structure is determined. Thus the classes of fibrations are determined by

acyclic cofibrations and cofibrations are determined by fibrations. The next two results will show this.

Lemma 2.1.9 (The retract argument)**.** *Let $\mathcal{C}$ be a category. Suppose there is a factorization $f = pi$ and that f has LLP with respect to p, then f is a retract of i. Dually, if f has RLP to i, then it is a retract of p.*

Proof. We assume that $f : A \to C$ has LLP with respect to $p : B \to C$. Then we may find a lift $r : C \to B$, which realize f as a retract of i.

$$
\begin{array}{ccc}
A \xrightarrow{\ i\ } B & & A \xlongequal{\quad} A \xlongequal{\quad} A \\
\Big\downarrow{\scriptstyle f} \ \nearrow{\scriptstyle r} \ \Big\downarrow{\scriptstyle p} \implies & & \Big\downarrow{\scriptstyle f} \qquad \Big\downarrow{\scriptstyle i} \qquad \Big\downarrow{\scriptstyle f} \\
C \xlongequal{\quad} C & & C \xrightarrow{\ r\ } B \xrightarrow{\ p\ } C
\end{array}
$$

$\square$

Proposition 2.1.10. *Let $\mathcal{C}$ be a model category. A morphism f is a cofibration (acyclic cofibration) if and only if f has LLP with respect acyclic fibrations (fibrations). Dually, f is a fibration (acyclic fibration) if and only if it has RLP with respect to acyclic cofibrations (cofibrations).*

Proof. Assume that f is a cofibration. By MC3, we know that f has LLP with respect to acyclic fibrations. Assume instead that f has LLP with respect to every acyclic fibration. By MC4 we factor $f = f_\alpha \circ f_\beta$, where f_α is a cofibration and f_β is an acyclic fibration. Since we assume f to have LLP with respect to f_β, by Lemma 2.1.9 we know that f is a retract of f_α. Thus by MC2, we know that f is a cofibration. $\square$

Corollary 2.1.10.1. *Let $\mathcal{C}$ be a model category. (Acyclic) Cofibrations are stable under pushouts, i.e. if f is an (acyclic) cofibration, then f' is an (acyclic) cofibration.*

$$
\begin{array}{ccc}
A & \longrightarrow & C \\
\Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f'} \\
B & \longrightarrow & D
\end{array}
$$

Dually, fibrations are stable under pullbacks.

Proof. Consider the diagram

$$
\begin{array}{ccccc}
A & \longrightarrow & C & \longrightarrow & E \\
\Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f'} & & \Big\downarrow{\scriptstyle g} \\
B & \longrightarrow & D & \longrightarrow & F
\end{array}
$$

where the left hand square is a pushout. Then f has LLP to g if and only if f' has LLP to g by the universal property of the pushout. It follows by Proposition 2.1.10 that f' is a cofibration.   □

Since we assume that every model category $\mathcal{C}$ admits finite limits and colimits, we know that it has both an initial and a terminal object. We let $\varnothing$ denote the initial object and $*$ denote the terminal object.

Definition 2.1.11 (Cofibrant, fibrant and bifibrant objects)**.** Let $\mathcal{C}$ be a model category. An object X is called cofibrant if the unique morphism $\varnothing \to X$ is a cofibration. Dually, X is called fibrant if the unique morphism $X \to *$ is fibrant. If X is both cofibrant and fibrant, we call it bifibrant.

There is no reason for every object to be either cofibrant or fibrant. However, we may see that every object is weakly equivalent to an object which is either fibrant or cofibrant. In this case we can think of X and Y being weakly equivalent if there is a weak equivalence $f : X \to Y$. We will make it precise what it means for two objects to be weakly equivalent later.

Construction 2.1.12. Let X be an object of a model category $\mathcal{C}$. The morphism $i : \varnothing \to X$ has a functorial factorization $i = i_\beta \circ i_\alpha$, where $i_\alpha : \varnothing \to QX$ is a cofibration and $i_\beta : QX \to X$ is an acyclic fibration. By definition QX is cofibrant and weakly equivalent to X.

$Q : \mathcal{C} \to \mathcal{C}$ defines a functor called the cofibrant replacement. To see this we first look at the slice category $\varnothing/\mathcal{c}$. The objects are morphisms $f : \varnothing \to X$ for any object X in $\mathcal{C}$, while morphisms are commutative triangles. We first observe that $\varnothing/\mathcal{c} \subset \mathcal{C}^{\to}$ is a subcategory of the arrow category. Thus (α, β) may be interpreted as functors on the slice category to the arrow category. Moreover, since every arrow $f : \varnothing \to X$ is unique, we observe that this category is equivalent to $\mathcal{C}$. Thus (α, β) may be interpreted as functors on $\mathcal{C}$ into arrows. We define Q as the composition $Q = \mathrm{tar} \circ \alpha$.

Dually, we get a fibrant replacement R by dualizing the above argument.

We collect the following properties

Lemma 2.1.13. *The cofibrant replacement Q and fibrant replacement R preserves weak equivalences.*

Proof. Suppose there is a weak equivalence $f : X \to Y$. Then there is a commutative square

$$
\begin{array}{ccc}
QX & \xrightarrow{\ Qf\ } & QY \\
\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\
X & \xrightarrow[\sim]{\ f\ } & Y
\end{array}
$$

where every morphism is a weak equivalence by the 2-out-of-3 property.   □

Lemma 2.1.14 (Ken Brown's lemma)**.** *Let $\mathcal{C}$ be a model category and $\mathcal{D}$ be a category with weak equivalences satisfying the 2-out-of-3 property. If $F : \mathcal{C} \to \mathcal{D}$ is a functor sending acyclic cofibrations between cofibrant objects to weak equivalences, then F takes all weak equivalences between cofibrant objects to weak equivalences. Dually, if F takes all acyclic fibrations between fibrant objects to weak equivalences, then F takes all weak equivalences between fibrant objects to weak equivalences.*

Proof. Suppose that A and B are cofibrant objects and that $f : A \to B$ is a weak equivalence. Using the universal property of the coproduct we define the map $(f, id_B) = p : A \coprod B \to B$. p has a functorial factorization into a cofibration and acyclic fibration, $p = p_\beta \circ p_\alpha$. We recollect the maps in the following pushout diagram:

$$
\begin{array}{ccc}
\varnothing & \xrightarrow{\quad\quad} & B \\
\downarrow & & \downarrow{\scriptstyle i_2} \\
A & \xrightarrow{\;\; i_1 \;\;} & A \coprod B
\end{array}
$$

with maps id_B, p_α, p_β, f and object C, to B.
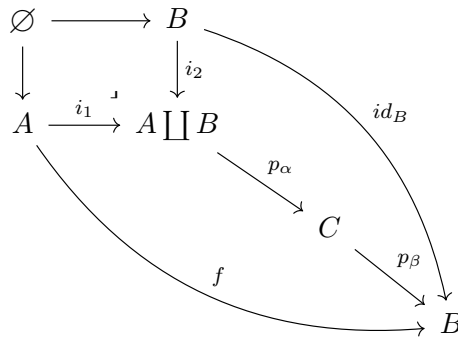
By Corollary 2.1.10.1, both i_1 and i_2 are cofibrations. Since f, id_B and p_β are weak equivalences, so are $p_\alpha \circ i_1$ and $p_\alpha \circ i_2$ by MC2. Moreover, they are acyclic cofibrations.

Assume that $F : \mathcal{C} \to \mathcal{D}$ is a functor as described above. Then by assumption, $F(p_\alpha \circ i_1)$ and $F(p_\alpha \circ i_2)$ are weak equivalences. Since a functor sends identity to identity, we also know that $F(id_B)$ is a weak equivelnce. Thus by the 2-out-of-3 property $F(p_\beta)$ is a weak equivalence, as $F(p_\beta) \circ F(p_\alpha \circ i_2) = id_{F(B)}$. Again, by 2-out-of-3 property $F(f)$ is a weak equivalence, as $F(f) = F(p_\beta) \circ F(p_\alpha \circ i_1)$. $\qquad\square$

2.1.2 Homotopy category

Homotopy theory at its most abstract is the study of categories and functors up to weak equivalences. Here, a weak equivalence may be anything, but most commonly it is a weak equivalence in topological homotopy or a quasi-isomorphism in homological algebra. The biggest concern when dealing with such concepts is to make a functor well-defined when these chosen weak equivalences are inverted. To this end, there is a construction to amend these problems, known as derived functors. We define a homotopical category in the sense of Riehl [17].

Definition 2.1.15 (Homotopical Category)**.** Let $\mathcal{C}$ be a category. $\mathcal{C}$ is homotopical if there is a wide subcategory constituting a class of morphisms known as weak equivalences, $\mathsf{Ac} \subset Mor\mathcal{C}$.

The weak equivalences should satisfy the 2-**out-of-**6 **property**, i.e. given three composable morphisms f, g and h, if gf and hg are weak equivalences, then so are f, g, h and hgf.

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B \\
 & \searrow\ {\scriptstyle gf} & \downarrow {\scriptstyle g} \quad {\scriptstyle hg} \searrow \\
 & & C \xrightarrow{\ h\ } D
\end{array}
$$
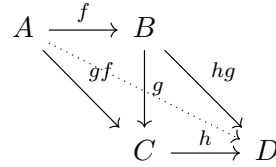
Remark 2.1.16. Notice that the 2-out-of-6 property is stronger than the 2-out-of-3 property. To see this, let either f, g or h be the identity, and then conclude with the 2-out-of-3 property.

Remark 2.1.17. The collection of weak equivalences contains every isomorphism. To see this pick an isomorphism f and f^{-1}, then the compositions are the identity on the domain and codomain, which are assumed to be in Ac.

Given such a homotopical category $\mathcal{C}$, we want to invert every weak equivalence and create the homotopy category of $\mathcal{C}$. This construction is developed in Gabriel and Zisman [18] called calculus of fractions. This method essentially tries to mimic localization for commutative rings in a category theoretic fashion. We will not give an account of existence or construction of localizations, however the reader is encouraged to read in Gabriel–Zisman.

Definition 2.1.18. Let $\mathcal{C}$ be a homotopical category. Its homotopy category is Ho$\mathcal{C} = \mathcal{C}[\text{Ac}^{-1}]$, together with a localization functor $q : \mathcal{C} \to$ Ho$\mathcal{C}$. The localization is determined by the following universal property: If $F : \mathcal{C} \to \mathcal{D}$ is a functor sending weak equivalences to isomorphisms, then it uniquely factors through the homotopy category up to a unique natural isomorphism η.

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ \ \ F\ \ \ } & \mathcal{D} \\
 {\scriptstyle q}\searrow & {\scriptstyle \eta} \Downarrow \ {\scriptstyle F'} \nearrow & \\
 & \text{Ho}\mathcal{C} &
\end{array}
$$
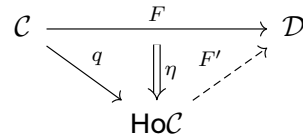
Definition 2.1.19. Suppose that $\mathcal{C}$ is a homotopical category. Two objects of $\mathcal{C}$ are said to be weakly equivalent if they are isomorphic in Ho$\mathcal{C}$. I.e. X and Y are weakly equivalent if there is some zig-zag relation between the objects, consisting only of weak equivalences.

$$
\begin{array}{ccccccc}
X & & \cdots & & & & Y \\
\ \searrow{\scriptstyle \sim} & \swarrow{\scriptstyle \sim} & & \searrow{\scriptstyle \sim} & & \swarrow{\scriptstyle \sim} & \\
 & X' & & & Y' & &
\end{array}
$$
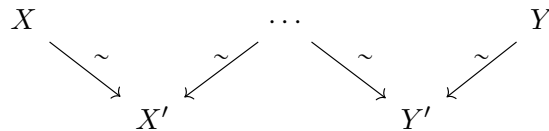
Remark 2.1.20. A renown problem with localizations is that even if $\mathcal{C}$ is a locally small category, any localization $\mathcal{C}[S^{-1}]$ does not need to be. Thus, without a good theory of classes or higher universes, we cannot in general ensure that the localization still exists as a locally small category.

We see from the definition of the homotopy category that a functor F admits a lift F' from the homotopy category whenever weak equivalences are sent to isomorphisms. Moreover, if we have a functor F between homotopical categories which preserves weak equivalences, it then induces a functor between the homotopy categories.

Definition 2.1.21 (Homotopical functors)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ between homotopical categories is homotopical if it preserves weak equivalences. Moreover, there is a lift of functors as in the following diagram where η is a natural isomorphism.

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ F\ } & \mathcal{D} \\
{\scriptstyle q_{\mathcal{C}}}\big\downarrow & \overset{\eta}{\diagup} & \big\downarrow{\scriptstyle q_{\mathcal{D}}} \\
\mathsf{Ho}\mathcal{C} & \dashrightarrow[F'] & \mathsf{Ho}\mathcal{D}
\end{array}
$$

Derived functors come into play whenever we want to make a lift of non-homotopical functors. These lifts will be the closest approximation which we can make functorial. The general exposition of derived functors is beyond the scope of this thesis, but an account of it may be found in [17]. As we will see, model categories is a nice environment to work with these concepts. Firstly we will amend the problem with localizations, where the homotopy category may not exists. Secondly, we will obtain a simple description of some important derived functors.

Proposition 2.1.22. *Any model category $\mathcal{C}$ is a homotopical category.*

Proof. To show that a model category is homotopical it suffices to show that Ac satisfies the 2-out-of-6 property. Assume there are 3 composable morphisms f, g, h such that $gf, hg \in$ Ac. By the 2-out-of-3 property for Ac it is enough to show that at least one of f, g, h, fgh is a weak equivalence to deduce that every other morphism is a weak equivalence.

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B \\
& {\scriptstyle gf}\diagdown \quad \big\downarrow{\scriptstyle g} \quad \diagup{\scriptstyle hg} & \\
& C \xrightarrow{\ h\ } & D
\end{array}
$$

To be able to use the model structure, we will first show that we may assume f, g to be cofibrant and g, h to be fibrant. We know by MC4 that f, g, gf may be factored into a cofibration composed with an acyclic fibration, e.g. $f = f_\beta f_\alpha$. Since gf is a weak equivalence, so is $(gf)_\alpha$ by the 2-out-of-3 property.

$$
\begin{array}{ccccccc}
A \xrightarrow{\ f\ } B && B \xrightarrow{\ g\ } C && A \xrightarrow{\ gf\ } C \\
{\scriptstyle f_\alpha}\searrow \quad \nearrow{\scriptstyle f_\beta} && {\scriptstyle g_\alpha}\searrow \quad \nearrow{\scriptstyle g_\beta} && {\scriptstyle (gf)_\alpha}\searrow \quad \nearrow{\scriptstyle (gf)_\beta} \\
B' && C' && C''
\end{array}
$$

Notice that the "cofibrant approximation" of the map from A to C either goes through C' or C''. We conjoin these by taking the pullback. Since acyclic fibrations are stable under pullbacks, we get a pullback square where every morphism is an acyclic fibration. Thus the map $A \to \widetilde{C}$ is a weak equivalence by 2-out-of-3.

$$
\begin{array}{ccc}
A & \xrightarrow{\;g_\alpha f\;} & \\
 & \overset{t}{\dashrightarrow} & \\
(gf)_\alpha & \widetilde{C} \xrightarrow{\ulcorner} C' & \\
 & \downarrow & \downarrow g_\beta \\
 & C'' \xrightarrow{(gf)_\beta} C &
\end{array}
$$

To replace f with f_α we must lift the composition into our "new" C, which is $\widetilde{C}$. This is done by using MC3, as f_α is a cofibration and the pullback square above consists entirely of acyclic fibrations.

$$
\begin{array}{ccc}
A & \longrightarrow & \widetilde{C} \\
\downarrow{\scriptstyle f_\alpha} & \overset{s}{\nearrow} & \downarrow \\
B' & \longrightarrow & C
\end{array}
$$

To summarize we have the following diagram, where every squiggly arrow is a weak equivalence.

$$
\begin{array}{ccccc}
A & \xrightarrow{f_\alpha} & B' & \xrightarrow{\ \sim\ } & B \\
 & {\scriptstyle t}\searrow_{\sim} & \downarrow{\scriptstyle s} & & \downarrow \quad \searrow_{\sim} \\
 & & \widetilde{C} & \xrightarrow[\sim]{} & C & \longrightarrow D
\end{array}
$$

We now wish to promote the arrow $s : B' \to \widetilde{C}$ into a cofibration. We do this by factoring both s and t with $MC4$. Notice that s_β, t_β and t_α are weak equivalences.

$$
\begin{array}{ccc}
B' \xrightarrow{\ \ s\ \ } \widetilde{C} & \qquad & A \xrightarrow{\ \ t\ \ } \widetilde{C} \\
{\scriptstyle s_\alpha}\searrow \quad \nearrow {\scriptstyle s_\beta} & & {\scriptstyle t_\alpha}\searrow \quad \nearrow {\scriptstyle t_\beta} \\
\widetilde{C'} & & \widetilde{C''}
\end{array}
$$

To obtain our final factorization we use RLP of s_β on t_α.

$$
\begin{array}{ccc}
 & & B' \\
 & \nearrow^{f_\alpha} & \downarrow s_\alpha \\
A & \longrightarrow & \widetilde{C}' \\
\downarrow t_\alpha & \overset{u}{\nearrow} & \downarrow s_\beta \\
\widetilde{C}'' & \underset{t_\beta}{\longrightarrow} & \widetilde{C}
\end{array}
$$

Since the bottom square only consists of weak equivalences, u has to be a weak equivalence by the 2-out-of-3 property. In this manner we may transform our diagram into the following diagram

$$
\begin{array}{ccccc}
A & \xrightarrow{f_\alpha} & B' & \overset{\sim}{\longrightarrow} & B \\
 & \searrow_{\sim}^{ut_\alpha} & \downarrow s_\alpha & & \searrow^{\sim} \\
 & & \widetilde{C}' & \underset{\sim}{\longrightarrow} & C & \longrightarrow & D
\end{array}
$$

We have now obtained a factorization of gf which are two cofibrations followed by an acyclic fibration, in such a manner that it is compatible with the original diagram. The dual to this claim is that we may also factor hg into two fibrations which is preceeded by an acyclic cofibration. In other words, we may assume without loss of generality that f and g are cofibrations, and that g and h are fibrations.

In this case, to show the 2-out-of-6 property it is enough to show that g is an isomorphism. Consider the diagram below with lifts i and j, these exists since we assume gf and hg to be weak equivalences.

$$
\begin{array}{ccccc}
A & \xrightarrow{f} & B & \xrightarrow{id_B} & B \\
\downarrow{gf} & \overset{i}{\nearrow} & \downarrow{g} & \overset{j}{\nearrow} & \downarrow{hg} \\
C & \underset{id_C}{\longrightarrow} & C & \underset{h}{\longrightarrow} & D
\end{array}
$$

Since the diagram is commutative, we get that $i = j$, and that g is both split mono and split epi, with i as its splitting. $\qquad\square$

Since every model category is homotopical, it also has an associated homotopy category Ho$\mathcal{C}$. Let $\mathcal{C}_c$, $\mathcal{C}_f$ and $\mathcal{C}_{cf}$ denote the full subcategories consisting of cofibrant, fibrant and bifibrant objects respectively.

Proposition 2.1.23. *Let $\mathcal{C}$ be a model category. The following categories are equivalent:*

- *Ho$\mathcal{C}$,*
- *Ho$\mathcal{C}_c$,*
- *Ho$\mathcal{C}_f$,*

- *Ho$\mathcal{C}_{cf}$.*

Proof. We only show that Ho$\mathcal{C} \simeq$ Ho$\mathcal{C}_c$, the other arguments are similar. The inclusion $i : \mathcal{C}_c \to \mathcal{C}$ clearly preserves weak equivalences, thus i is homotopical and admits a lift. Moreover, since the cofibrant replacement is also homotopical, it also has a lift.

$$
\begin{array}{ccc}
\mathcal{C}_c & \xrightarrow{\ i\ } & \mathcal{C} \\
\downarrow & \text{Ho } Q & \downarrow \\
\text{Ho}\mathcal{C}_c & & \mathcal{C} \\
& \text{Ho } i &
\end{array}
$$

It is clear that Ho Q is the quasi-inverse of Ho i. $\hfill\square$

As of now we still don't see how model categories will fix the size issues. To do this we will develop the notion of homotopy equivalence, $\sim$. On the subcategory of bifibrant objects $\mathcal{C}_{cf}$, this homotopy equivalence will be a congruence relation. This, together with the fact that there is an equivalence of categories Ho$\mathcal{C}_{cf} \simeq \mathcal{C}_{cf}/\sim$, is enough to solve the size issues.

Definition 2.1.24 (Cylinder and path objects)**.** Let $\mathcal{C}$ be a model category. Given an object X, a cylinder object $X \wedge I$ is a factorization of the codiagonal map $i : X \coprod X \to X$, such that p_0 is a cofibration and that p_1 is a weak equivalence.

$$
\begin{array}{ccc}
X \coprod X & \xrightarrow{\ \ \ \ i\ \ \ \ } & X \\
& {\scriptstyle p_0} \searrow \quad \nearrow {\scriptstyle p_1} & \\
& X \wedge I &
\end{array}
$$

Dually, a path object X^I is a factorization of the diagonal map $i : X \to X \prod X$, such that p_0 is a weak equivalence and that p_1 is a fibration.

$$
\begin{array}{ccc}
X & \xrightarrow{\ \ \ \ i\ \ \ \ } & X \prod X \\
& {\scriptstyle p_0} \searrow \quad \nearrow {\scriptstyle p_1} & \\
& X^I &
\end{array}
$$

Remark 2.1.25. Even though we have written $X \wedge I$ suggestively to be a functor, it is not. There may be many choices for a cylinder object. However, by using the functorial factorization from MC4, we get a canonical choice of a cylinder object, as it factors every map into a cofibration and an acyclic fibration. By letting the cylinder object be this object, we do indeed obtain a functor.

Proposition 2.1.26. *Let $\mathcal{C}$ be a model category and X an object of $\mathcal{C}$. Given two cylinder objects $X \wedge I$ and $X \wedge I'$, then they are weakly equivalent.*

Proof. It is enough to show that there is a weak equivalence from any cylinder object into one specified cylinder object. This is in fact true for the functorial cylinder object $X \wedge I$, as the morphism p_1 is an acyclic fibration, which enables a lift which is a weak equivalence by the 2-out-of-3 property.

$$
\begin{array}{ccc}
X \coprod X & \xrightarrow{p_0} & X \wedge I \\
\downarrow{\scriptstyle p_0'} & \nearrow & \downarrow{\scriptstyle p_1} \\
X \wedge I' & \xrightarrow{p_1'} & X
\end{array}
$$

$\square$

Definition 2.1.27 (Homotopy equivalence)**.** Let $f, g : X \to Y$. A left homotopy between f and g is a morphism $H : X \wedge I \to Y$ such that $H i_0 = f$ and $H i_1 = g$. We say that f and g are left homotopic if a left homotopy exists, and it is denoted $f \overset{l}{\sim} g$.

A right homotopy between f and g is a morphism $H : X \to Y^I$ such that $i_0 H = f$ and $i_1 H = g$. We say that f and g are right homotopic if a right homotopy exists, and it is denoted $f \overset{r}{\sim} g$.

f and g are said to be homotopic if they are both left and right homotopic, it is denoted $f \sim g$. f is said to be a homotopy equivalence, if it has a homotopy inverse $h : Y \to X$, such that $hf \sim id_X$ and $fh \sim id_Y$.

It is important to note that homotopy equivalence is not a priori an equivalence relation. This is amended by taking both fibrant and cofibrant replacements. We see this in the following propositions.

Proposition 2.1.28. *Let $\mathcal{C}$ be a model category, and $f, g : X \to Y$ be morphisms. We have the following:*

1. *If $f \overset{l}{\sim} g$ and $h : Y \to Z$, then $hf \overset{l}{\sim} hg$.*
2. *If Y is fibrant, $f \overset{l}{\sim} g$ and $h : W \to X$, then $fh \overset{l}{\sim} gh$.*
3. *If X is cofibrant, then left homotopy is an equivalence relation on $\mathcal{C}(X, Y)$.*
4. *If X is cofibrant and $f \overset{l}{\sim} g$, then $f \overset{r}{\sim} g$.*

Proof. (1.) Assume that $f \overset{l}{\sim} g$ and $h : Y \to Z$. Let $H : X \wedge I \to Y$ denote the left homotopy between f and g. The left homotopy between hf and hg is given as hH.

(2.) Assume that Y is fibrant, $f \overset{l}{\sim} g$ and that $h : W \to X$. Let $H : X \wedge I \to Y$ be a left homotopy. We construct a new cylinder object for the homotopy. Factor $p_1 : X \wedge I \to X$ as $q_1 \circ q_0$ where $q_0 : X \wedge I \to X \wedge I'$ is an acyclic cofibration and $q_1 : X \wedge I' \to X$ is a fibration. By the 2-out-of-3 property q_1 is an acyclic fibration, as p_1 and q_0 are weak equivalences. $X \wedge I'$ is a cylinder object as $q_0 \circ p_0$ is a cofibration and q_1 is a weak equivalence. Since we assume Y to be fibrant we lift the left homotopy $H : X \wedge I \to Y$ to the left homotopy $H' : X \wedge I' \to Y$ with the following diagram:

$$
\begin{array}{ccc}
X \wedge I & \xrightarrow{\ H\ } & Y \\
{\scriptstyle q_0}\big\downarrow & {\scriptstyle H'}\nearrow & \big\downarrow \\
X \wedge I' & \longrightarrow & *
\end{array}
$$

We let WI be a cylinder object for W, where $p_0' : W \sqcup W \to QI$ is a cofibration. We can find an appropriate homotopy needed with LLP of q_1 against p_0', as done in the diagram below.

$$
\begin{array}{ccc}
W \coprod W & \xrightarrow{q_0 p_0 (h \coprod h)} & X \wedge I' \\
{\scriptstyle p_0'}\big\downarrow & {\scriptstyle k}\nearrow & \big\downarrow{\scriptstyle q_1} \\
W \wedge I & \xrightarrow[\ hp_1'\ ]{} & X
\end{array}
$$

The morphism $H'k$ is the desired left homotopy witnessing $fh \overset{l}{\sim} gh$.

(3.) Assume that X is cofibrant. First observe that a left homotopy is reflexive and symmetric. We must show that in this case it is also transitive. Thus, assume that $f, g, h : X \to Y$ and that $H : X \wedge I \to Y$ is a left homotopy witnessing $f \overset{l}{\sim} g$ and that $H' : X \wedge I' \to Y$ is a left homotopy

witnessing $g \overset{l}{\sim} h$. We first observe that $i_0 : X \to X \wedge I$ is a weak equivalence, as $id_X = p_1 i_0$ where id_X and p_1 are weak equivalences. Since X is assumed to be cofibrant, we see that $X \coprod X$ is cofibrant by the following pushout:

$$\begin{array}{ccc} \varnothing & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle inr} \\ X & \overset{inl}{\longrightarrow} & X \coprod X \end{array}$$

Moreover, both inl and inr are cofibrations. This shows that i_0 is a cofibration as $i_0 = p_0 \circ inr$ is a composition of two cofibrations. i_0 is thus an acyclic cofibration. We define an almost cylinder object C by the pushout of i_1 and i_0'. We define the maps t and H'' by using the universal property in the following manner:

$$\begin{array}{ccc} X & \overset{i_1}{\longrightarrow} & X \wedge I \\ \downarrow{\scriptstyle i_0'} & & \downarrow \quad\quad {\scriptstyle p_1} \\ X \wedge I' & \longrightarrow & C \\ & {\scriptstyle p_1'} \quad\quad {\scriptstyle t} & \downarrow \\ & & X \end{array} \qquad \begin{array}{ccc} X & \overset{i_1}{\longrightarrow} & X \wedge I \\ \downarrow{\scriptstyle i_0'} & & \downarrow \quad\quad {\scriptstyle H} \\ X \wedge I' & \longrightarrow & C \\ & {\scriptstyle H'} \quad\quad {\scriptstyle H''} & \downarrow \\ & & Y \end{array}$$

Observe that there is a factorization of the codiagonal map $X \coprod X \overset{s}{\to} C \overset{t}{\to} X$. However, s may not be a cofibration, so we replace C with the cylinder object $X \wedge I''$ such that we have the factorization $X \coprod X \overset{s_\alpha}{\to} X \wedge I'' \overset{ts_\beta}{\to} X$. The morphism $H'' s_\beta$ is then our required homotopy for $f \overset{l}{\sim} g$.

(4.) Suppose that X is cofibrant and that $H : X \wedge I \to Y$ is a left homotopy for $f \overset{l}{\sim} g$. Pick a path object for Y, such that we have the factorization $Y \overset{q_0}{\to} Y^I \overset{q_1}{\to} Y \coprod Y$ where q_0 is a weak equivalence and q_1 is a fibration. Again, as X is cofibrant we get that i_0 is an acyclic cofibration, so we have the following lift of the homotopy:

$$\begin{array}{ccc} X & \overset{q_0 f}{\longrightarrow} & Y^I \\ \downarrow{\scriptstyle i_0} & {\scriptstyle J} \nearrow & \downarrow{\scriptstyle q_1} \\ X \wedge I & \overset{(f p_1, H)}{\longrightarrow} & Y \coprod Y \end{array}$$

The right homotopy is given by injecting away from f, i.e. $H' = J i_1$. $\qquad \square$

Corollary 2.1.28.1. *We collect the dual results of the above proposition, and thus have the following.*

1. *If $f \overset{r}{\sim} g$ and $h : W \to X$, then $fh \overset{r}{\sim} gh$.*
2. *If X is cofibrant, $f \overset{r}{\sim} g$ and $h : Y \to Z$, then $hf \overset{r}{\sim} hg$.*
3. *If Y is fibrant, then left homotopy is an equivalence relation on $\mathcal{C}(X,Y)$.*
4. *If Y is fibrant and $f \overset{r}{\sim} g$, then $f \overset{l}{\sim} g$.*

Corollary 2.1.28.2. *Homotopy is a congruence relation on $\mathcal{C}_{cf}$. Thus the category $\mathcal{C}_{cf}/\sim$ is well-defined, exists and inverts every homotopy equivalence.*

Lemma 2.1.29 (Weird Whitehead)**.** *Let $\mathcal{C}$ be a model category. Suppose that C is cofibrant and $h : X \to Y$ is an acyclic fibration or a weak equivalence between fibrant objects, then h induces an isomorphism:*

$$\mathcal{C}(C,X)/\overset{l}{\sim} \xrightarrow{\ \overset{h_*}{\cong}\ } \mathcal{C}(C,Y)/\overset{l}{\sim}$$

Dually, if X is fibrant and $h : C \to D$ is an acyclic cofibration or a weak equivalence between cofibrant objects, then h induces an isomorphism:

$$\mathcal{C}(D,X)/\overset{r}{\sim} \xrightarrow{\ \overset{h^*}{\cong}\ } \mathcal{C}(C,X)/\overset{r}{\sim}$$

Proof. We assume $\mathcal{C}$ to be cofibrant and $h : X \to Y$ to ba an acyclic fibration. We first prove that h is surjective. Let $f : C \to Y$. By RLP of h there is a morphism $f' : C \to X$ such that $f = hf'$.

$$
\begin{array}{ccc}
\varnothing & \longrightarrow & X \\
\downarrow & \nearrow^{f'} & \downarrow{\scriptstyle h} \\
C & \xrightarrow{\ f\ } & Y
\end{array}
$$

To show injectivity we assume $f, g : C \to X$ such that $hf \overset{l}{\sim} hg$, in particular there is a left homotopy $H : C \wedge I \to Y$. Remember that since C is cofibrant, the map p_0 is a cofibration. We find a left homotopy $H : C \wedge I \to X$ witnessing $f \overset{l}{\sim} g$ by the following lift.

$$
\begin{array}{ccc}
C \coprod C & \xrightarrow{\ f+g\ } & X \\
\downarrow{\scriptstyle p_0} & \nearrow^{H'} & \downarrow{\scriptstyle h} \\
C \wedge I & \xrightarrow{\ H\ } & Y
\end{array}
$$

If we instead assume that both X and Y are fibrant, then the functor $\mathcal{C}(C,\_)/\overset{l}{\sim}$ sends acyclic fibrations to isomorphisms by Corollary 2.1.28.1. Ken Brown's lemma, Lemma 2.1.14, tells us then that $\mathcal{C}(C,\_)/\overset{l}{\sim}$ sends weak equivalences between fibrant objects to isomorphisms. $\qquad\square$

Theorem 2.1.30 (Generalized Whitehead's theorem)**.** *Let $\mathcal{C}$ be a model category. Suppose that $f : X \to Y$ is a morphism of bifibrant objects, then f is a weak equivalence if and only if f is a homotopy equivalence.*

Proof. Suppose first that f is a weak equivalence. Pick a bifibrant object A, then by Lemma 2.1.29 $f_* : \mathcal{C}(A, X)/{\sim} \to \mathcal{C}(A, Y)/{\sim}$ is an isomorphism. Letting $A = Y$ we know that there is a morphism $g : Y \to X$, such that $f_* g = fg \sim id_Y$. Furthermore, by Proposition 2.1.28, since X is bifibrant composing on the right preserves homotopy equivalence, e.g. $fgf \sim f$. By letting $A = X$, we get that $f_* gf = fgf \sim f = f_* id_X$, thus $gf \sim id_X$.

For the opposite direction, assume that f is a homotopy equivalence. We factor f into an acyclic cofibration f_γ and a fibration f_δ, i.e. $X \xrightarrow{f_\gamma} Z \xrightarrow{f_\delta} Y$. Observe that Z is bifibrant as X and Y is, in particular, f_γ is a weak equivalence of bifibrant objects, so it is a homotopy equivalence.

It is enough to show that f_δ is a weak equivalence. Let g be the homotopy inverse of f, and $H : Y \wedge I \to Y$ is a left homotopy witnessing $fg \sim id_Y$. Since Y is bifibrant, the following square has a lift.

$$
\begin{array}{ccc}
Y & \xrightarrow{\;f_\gamma g\;} & Z \\
{\scriptstyle i_0}\downarrow & \overset{H'}{\nearrow} & \downarrow{\scriptstyle f_\delta} \\
Y \wedge I & \xrightarrow{\;H\;} & Y
\end{array}
$$

Let $h = H' i_1$, then by definition we know that $f_\delta H' i_1 = id_Y$. Moreover, H is a left homotopy witnessing $f_\gamma g \sim h$. Let $g' : Z \to X$ be the homotopy inverse of f_γ. We have the following relations $f_\delta \sim f_\delta f_\gamma g' \sim fg'$, and $hf_\delta \sim (f_\gamma g)(fg') \sim f_\gamma g' \sim id_Z$. Let $H'' : Z \wedge I \to Z$ be a left homotopy witnessing this homotopy. Since Z is bifibrant, i_0 and i_1 are weak equivalences. By the 2-out-of-3 property H'' and hf_δ are weak equivalences. Since $f_\delta h = id_Y$, it follows that f_δ is a retract of $f_\delta h$, and is thus a weak equivalence. $\qquad\square$

Corollary 2.1.30.1. *The category $\mathcal{C}_{cf}/{\sim}$ satisfies the universal property of the localization of $\mathcal{C}_{cf}$ by the weak equivalences. I.e. there is a categorical equivalence $Ho\mathcal{C}_{cf} \simeq \mathcal{C}_{cf}/{\sim}$.*

Proof. By generalized Whitehead's theorem, Theorem 2.1.30 weak equivalences and homotopy equivalences coincide. The corollary follows steadily from both the universal property of the localization category and the quotient category. $\qquad\square$

We collect the results from above in the following theorem.

Theorem 2.1.31 (Fundamental theorem of model categories)**.** *Let $\mathcal{C}$ be a model category and denote $q : \mathcal{C} \to Ho\mathcal{C}$ the localization functor. Let X and Y be objects of $\mathcal{C}$.*

    1. There is an equivalence of categories $Ho\mathcal{C} \simeq \mathcal{C}_{cf}/{\sim}$.

2. *There are natural isomorphisms $\mathcal{C}_{cf}/{\sim}(QRX, QRY) \simeq \mathsf{Ho}\mathcal{C}(X,Y) \simeq \mathcal{C}_{cf}/{\sim}(RQX, RQY)$. Additionally, $\mathsf{Ho}\mathcal{C}(X,Y) \simeq \mathcal{C}_{cf}/{\sim}(QX, RY)$.*
3. *The localization q identifies left or right homotopic morphisms.*
4. *A morphism $f : X \to Y$ is a weak equivalence if and only if qf is an isomorphism.*

Proof. This is clear by the results above. □

2.1.3 Quillen adjoints

We now want to study morphisms, or certain functors, between model categories. Like in the case of homotopical functors we want these morphisms to induce a functor between the homotopy categories. However, we also want them to respect the cofibration and fibration structure, not just weak equivalences. In this way we will instead look towards derived functors to be able to define this extension to the homotopy category. We recall the definition of a total (left/right) derived functor. In the case of model categories, we get a simple description for some of these derived functors which are of special interest.

Definition 2.1.32 (Total derived functors)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be homotopical categories, and $F : \mathcal{C} \to \mathcal{D}$ a functor. Whenever it exists, a total left derived functor of F, is a functor $\mathbb{L}F : \mathsf{Ho}\mathcal{C} \to \mathsf{Ho}\mathcal{D}$ with a natural transformation $\varepsilon : \mathbb{L}F \circ q \Rightarrow q \circ F$ satisfying the universal property: If $G : \mathsf{Ho}\mathcal{C} \to \mathsf{Ho}\mathcal{D}$ is a functor and there is a natural transformation $\alpha : G \circ q \Rightarrow q \circ F$, then it factors uniquely up to unique isomorphism through ε.

Dually, whenever it exists, a total right derived functor of F, is a functor $\mathbb{R}F : \mathsf{Ho}\mathcal{C} \to \mathsf{Ho}\mathcal{D}$ with a natural transformation $\eta : q \circ F \Rightarrow \mathbb{R}F \circ q$ having the opposite universal property.

Definition 2.1.33 (Deformation)**.** A left (right) deformation on a homotopical category $\mathcal{C}$ is an endofunctor Q together with a natural weak equivalence $q : Q \Rightarrow Id_{\mathcal{C}}$ ($q : Id_{\mathcal{C}} \Rightarrow Q$).

A left (right) deformation on a functor $F : \mathcal{C} \to \mathcal{D}$ between homotopical categories, is a left (right) deformation Q on $\mathcal{C}$ such that weak equivalences in the image of Q is preserved by F.

Remark 2.1.34 (Cofibrant and fibrant replacement). If $\mathcal{C}$ is a model category, then we have a left and a right deformation. The cofibrant replacement Q defines a left deformation, and the fibrant replacement defines a right deformation. Notice that this is only due to the fact that the factorization system is functorial.

Proposition 2.1.35. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between homotopical categories. If F has a left deformation Q, then the total left derived functor $\mathbb{L}F$ exists. Moreover, the functor FQ is homotopical, and $\mathbb{L}F$ is the unique extension of FQ.*

Proof. Since we already have a candidate for the derived functor, the proof must just check that it has the universal property. A proof may be found in Riehl [**207**]Riehl16. □

Remark 2.1.36. There is a somewhat weaker statement by Dwayer and Spalinski [**111**]Dwyer95. If we instead ask for functors F which have the cofibrant replacement Q (fibrant replacement R) as a left (right) deformation we may make this proof more explicit.

Equipped with the above proposition and remark, it makes sense to define Quillen functors as left and right Quillen functors. A left Quillen functor should be left deformable by the cofibrant replacement. Moreover, for the composition of two left Quillen functors to make sense, we also need weak equivalences between cofibrant objects to be mapped to weak equivalences between cofibrant objects. We make the following definition.

Definition 2.1.37 (Quillen adjunction). Let $\mathcal{C}$ and $\mathcal{D}$ be model categories.

1. A left Quillen functor is a functor $F : \mathcal{C} \to \mathcal{D}$ such that it preserves cofibrations and acyclic cofibrations.
2. A right Quillen functor is a functor $F : \mathcal{C} \to \mathcal{D}$ such that it preserves fibrations and acyclic fibrations.
3. Suppose that (F, U) is an adjunction where $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to U. (F, U) is called a Quillen adjunction if F is a left Quillen functor and U is a right Quillen functor.

Remark 2.1.38. By Ken Brown's lemma, Lemma 2.1.14, we see that a left Quillen functor F is left deformable to the cofibrant replacement functor Q. Thus the total left derived functor exists and is given by $\mathbb{L}F = \mathrm{Ho}FQ$.

In order to eliminate the choice of left or right derivedness, we will think of a morphism of model categories as a Quillen adjunction. The direction of the arrow can be chosen to be along either the left or right adjoints, we make the convention of following the left adjoint functors. We summarize the following properties.

Lemma 2.1.39. *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories, and suppose there is an adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$. The following are equivalent:*

1. *(F, U) is a Quillen adjunction.*
2. *F is a left Quillen functor.*

3. U *is a right Quillen functor.*

Proof. This follows from naturality of the adjunction. I.e. any square in $\mathcal{C}$, with the right side from $\mathcal{D}$ is commutative if and only if any square in $\mathcal{D}$ with the left side from $\mathcal{C}$ is commutative. Now, f has LLP with respect to Ug if and only if Ff has LLP with respect to g.

$$
\begin{array}{ccc}
A \xrightarrow{\ k\ } UX & & FA \xrightarrow{\ k^T\ } X \\
f\downarrow\ \ \ _h\ \nearrow\ \ \downarrow Ug & \rightsquigarrow & Ff\downarrow\ \ \ _{h^T}\ \nearrow\ \ \downarrow g \\
B \xrightarrow{\ l\ } UY & & FB \xrightarrow{\ l^T\ } Y
\end{array}
$$

$\square$

Remark 2.1.40. In the proof above we say that h^T is the transpose of h along the unique natural isomorphism witnessing the adjunction between F and U. With this notion, $(h^T)^T = h$.

Proposition 2.1.41. *Suppose that $(F, U) : \mathcal{C} \to \mathcal{D}$ is a Quillen adjunction. The functors $\mathbb{L}F : Ho\mathcal{C} \to Ho\mathcal{D}$ and $\mathbb{R}U : Ho\mathcal{D} \to Ho\mathcal{C}$ forms an adjoint pair.*

Proof. We must show that $Ho\mathcal{D}(\mathbb{L}FX, Y) \simeq Ho\mathcal{D}(X, \mathbb{R}UY)$. By using the fundamental theorem of model categories, Theorem 2.1.31, we have the following isomorphisms: $Ho\mathcal{D}(\mathbb{L}FX, Y) \simeq {}^{\mathcal{C}(FQX, RY)}/\sim$ and $Ho\mathcal{D}(X, \mathbb{R}UY) \simeq {}^{\mathcal{D}(QX, URY)}/\sim$. In other words, if we assume X to be cofibrant, and Y to be fibrant, we must show that the adjunction preserves homotopy equivalences.

We show it for one direction. Suppose that the morphisms $f, g : FA \to B$ are homotopic, witnessed by a right homotopy $H : FA \to B^I$. Since we assume U to preserve products, fibrations and weak equivalences between fibrant objects, $U(B^I)$ is a path object for UB. Thus the transpose $H^T : A \to U(B^I)$ is the desired homotopy witnessing $f^T \sim g^T$ $\square$

Definition 2.1.42 (Quillen equivalence)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be model categories, and $(F, U) : \mathcal{C} \to \mathcal{D}$ be a Quillen adjunction. (F, U) is called a Quillen equivalence if for any cofibrant X in $\mathcal{C}$, fibrant Y in $\mathcal{D}$ such that any morphism $f : FX \to Y$ is a weak equivalence if and only if its transpose $f^T : X \to UY$ is a weak equivalence.

Proposition 2.1.43. *Suppose that $(F, U) : \mathcal{C} \to \mathcal{D}$ is a Quillen adjunction. The following are equivalent:*

1. *(F, U) is a Quillen equivalence.*
2. *Let $\eta : Id_{\mathcal{C}} \Rightarrow UF$ denote the unit, and $\varepsilon : FU \Rightarrow Id_{\mathcal{D}}$ denote the counit. The composite $Ur_F\eta : Id_{\mathcal{C}_c} \Rightarrow URF|_{\mathcal{C}_c}$, and $\varepsilon_{FQU}Fq_U : FQU|_{\mathcal{D}_f} \Rightarrow Id_{\mathcal{D}_f}$ are natural weak equivalences.*
3. *The derived adjunction $(\mathbb{L}F, \mathbb{R}U)$ is an equivalence of categories.*

Proof. Firstly observe that $2. \implies 3.$ by definition. Secondly observe that equivalences both preserves and reflects isomorphisms, from this we get $3. \implies 1.$. We now show $1. \implies 2.$.

Pick X in $\mathcal{C}$ such that X is cofibrant. Since (F, U) is assumed to be a Quillen adjunction we know that FX is still cofibrant. The fibrant replacement $r_{FX} : FX \to RFX$ gives us a weak equivalence. Furthermore, since (F, U) is assumed to be a Quillen equivalence, its transpose $r_{FX}^T : X \to URFX$ is a weak equivalence. Unwinding the definition of the transpose we get that $r_{FX}^T = Ur_{rFX} \circ \eta_X$.

$\square$

We have the following refinement.

Corollary 2.1.43.1. *Suppose that* $(F, U) : \mathcal{C} \to \mathcal{D}$ *is a Quillen adjunction. The following are equivalent:*

1. (F, U) *is a Quillen equivalence.*
2. F *reflects weak equivalences between cofibrant objects, and* $\varepsilon_{FQU} F_{qU} : FQU|_{\mathcal{D}_f} \Rightarrow Id_{\mathcal{D}_f}$ *is a natural weak equivalence.*
3. U *reflects weak equivalences between fibrant objects, and* $U_{rF}\eta : Id_{\mathcal{C}_c} \Rightarrow URF|\mathcal{C}_c$ *is a natural weak equivalence.*

Proof. We start by showing $1. \implies 2.$ and $3..$ We already know that the derived unit and counit are isomorphism in homotopy, so we only need to show that F (U) reflects weak equivalences between cofibrant (fibrant) objects. Suppose that $Ff : FX \to FY$ is a weak equivalence between cofibrant objects. Since F preserves weak equivalences between cofibrant objects, we get that FQf is a weak equivalence, or that $\mathbb{L}Ff$ is an isomorphism. By assumption, $\mathbb{L}F$ is an equivalence of categories, so f is a weak equivalence as needed.

We will show $2. \implies 1.$, the case $3. \implies 1.$ is dual. We assume that the counit map is an isomorphism in homotopy. By assumption, the derived unit $\mathbb{L}\eta$ is split-mono on the image of $\mathbb{L}F$. Moreover, the derived counit $\mathbb{R}\varepsilon$ is assumed to be an isomorphism, in particular the derived unit $\mathbb{L}F\mathbb{L}\eta$ is an isomorphism. Unpacking this, we have a morphism, call it $\eta'_X : FQX \to FQURFQX$, which is a weak equivalence. Since F and Q reflects weak equivalences, we get that $\eta_X : X \to URFQX$ is a weak equivalence. $\square$

2.2   Model structures on Algebraic Categories

In order to understand ∞-quasi-isomorphism of strongly homotopy associative algebras we will study different homotopy theories of various categories. Munkholm [19] successfully showed that the derived category of augmented algebras is equivalent to the derived category of augmented algebras equipped with ∞-morphisms. To be more precise, he showed that certain subcategories of augmented algebras had this property. Lefevre-Hasagawas phd thesis [13] builds upon this identification, but with help of further devolpment within the field. We will follow the approach of Lefevre-Hasegawa, by comparing the model structure for algebras and coalgebras,

2.2.1 DG-Algebras as a Model Category

Bousefield and Gugenheim [20] proved that the category of commutative dg-algebras had a model structure whenever the base field was a field of characteristic 0. In a joint project, Jardine's paper from 1997 [21] shows that this construction may be extended to dg-algebras over any commutative ring. On the other hand, Munkholm expanded on the ideas from Bousfield and Gugenheim to get an identification of derived categories. Also, Hinich's paper from 1997 [22] details another method to obtain the model category which we want. We will follow the approach of Hinich, as it will be usefull later on. Notice that where Hinich use theory of algebraic operads to show that the category of algebras is a model category, we will however give a more explicit formulation.

Let $\mathbb{K}$ be a field, and $\mathcal{C}$ be a category such that there is an adjunction $F : \mathcal{C} \rightleftharpoons Ch(\mathbb{K}) : \#$, where F is left adjoint to #. Furthermore, suppose that $\mathcal{C}$ satsifies the 2 conditions:

(H0) $\mathcal{C}$ admits finite limits and every small colimit. The functor # commutes with filtered colimits.
 (H1) Let M be the complex below, concentrated in 0 and 1.

$$ ... \longrightarrow 0 \longrightarrow \mathbb{K} \xrightarrow{\ id\ } \mathbb{K} \longrightarrow 0 \longrightarrow ... $$

For any $d \in \mathbb{Z}$ and for any $A \in \mathcal{C}$ the injection $A \to A \coprod F(M[d])$ induces a quasi-isomorphism $A^{\#} \to (A \coprod F(M[d]))^{\#}$.

With this adjunction in mind, we define weak equivalences, fibrations and cofibrations as follows: Let $f \in \mathcal{C}$ be a morphism

- $f \in \mathsf{Ac}$ if $f^{\#}$ is a quasi-isomorphism.
- $f \in \mathsf{Fib}$ if $f^{\#}$ is surjective on each component.
- $f \in \mathsf{Cof}$ if f has LLP to acyclic fibrations.

Theorem 2.2.1. *The category $\mathcal{C}$ equipped with the weak equivalences, fibrations and cofibrations as defined above is a model category.*

Before we show this theorem we need to understand the cofibrations better. Let $A \in \mathcal{C}$, $M \in Ch(\mathbb{K})$ and $\alpha : M \to A^{\#}$ a morphism in $Ch(\mathbb{K})$. We define a functor

$$ h_{A,\alpha}(B) = \{(f,t) \mid f \in \mathcal{C}(A,B), t \in \mathsf{Hom}_{\mathbb{K}}^{-1}(M, B^{\#}) \text{ s.t. } \partial t = f^{\#} \circ \alpha\}. $$

Note that t is not a morphism of chain maps. This is a homogenous morphism of degree -1. The differential then promotes this morphism to a chain map, and t is thus a homotopy for the comoposite $f^{\#} \circ \alpha$.

This functor is represented by an object of $\mathcal{C}$. We define this representing object $A\langle M, \alpha \rangle$ as the pushout:

$$
\begin{array}{ccc}
F(A^{\#}) & \xrightarrow{\ \varepsilon_A\ } & A \\
\downarrow & & \downarrow{\scriptstyle a} \\
F(\mathsf{cone}(\alpha)) & \xrightarrow{\ e\ } & A\langle M, \alpha\rangle
\end{array}
$$

Let $i : M[1] \to \mathsf{cone}(\alpha)$ be a homogenous morphism which is the injection when considered as graded modules. Notice that we have a pair of morphisms $(a, e^T i) \in h_{A,\alpha}(A\langle M, \alpha\rangle)$.

Proposition 2.2.2. *The functor $h_{A,\alpha}$ is represented by $A\langle M, \alpha\rangle$, i.e. $h_{A,\alpha} \simeq \mathcal{C}(A\langle M, \alpha\rangle, \_)$ is a natural isomorphism. Moreover, the pair $(a, e^T i)$ is the universal element of the functor $h_{A,\alpha}$, i.e. the natural isomorphism is induced by this element under Yoneda's lemma.*

Proof. Let $(f, t) \in h_{A,\alpha}(B)$ for some $B \in \mathcal{C}$. The condition that $\partial t = f^{\#}\alpha$ is equivalent to say that $f^{\#}$ extends to a morphism $f' : \mathsf{cone}(\alpha) \to B^{\#}$ along t, i.e. $f' = \begin{pmatrix} f^{\#} & t \end{pmatrix}$. This concludes the isomorphism part, as being an element (f, t) is equivalent to the existence of the diagram below, where $\tilde{f}$ is uniquely determined.

$$
\begin{array}{ccc}
F(A^{\#}) & \xrightarrow{\ \varepsilon_A\ } & A \\
\downarrow & & \downarrow{\scriptstyle a} \quad\searrow^{f} \\
F(\mathsf{cone}(\alpha)) & \xrightarrow{\ e\ } & A\langle M, \alpha\rangle \\
 & \searrow_{f'} & \quad\dashrightarrow^{\tilde{f}} \\
 & & B
\end{array}
$$

To obtain naturality, we use the adjunction to observe that the element $(a, e^T i)$ is in fact universal. $\qquad\square$

We are now in a position to explicitly find some important cofibrations. We collect these morphisms into the "standard" cofibrations.

Definition 2.2.3. Let $f : A \to B$ be a morphism in $\mathcal{C}$. Suppose that f factors as a transfinite composition of morphisms on the form $A_i \to A_i\langle M, \alpha\rangle$, i.e. f factors into the diagram below, where $A_{i+1} = A_i\langle M, \alpha\rangle$.

$$
A \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \dots \longrightarrow B
$$

- If every such M is a complex consisting of free $\mathbb{K}$-modules and has a 0-differential, we call f a standard cofibration.
- If every such M is a contractible complex and $\alpha = 0$, we call f a standard acyclic cofibration.

Proposition 2.2.4. *Every standard cofibration is a cofibration, and every standard acyclic cofibration is an acyclic cofibration. We will see that these morphisms in some sense generate every (acyclic) cofibrations.*

Proof. First observe that every standard cofibration may be made iteratively from the chain complexes $\mathbb{K}[n]$, and likewise, every standard acyclic cofibration may be made iteratively from M as in $H1$.

We first prove that if $M \simeq \mathbb{K}[n]$, and $\alpha : M \to A^{\#}$ is any map, then the map $A \to A\langle M, \alpha\rangle$ is a cofibration. This amounts to show that it has LLP to every acyclic fibration. Suppose that $h : B \to C$ is an acyclic fibration and that there is a commutative square as below.

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle h} \\
A\langle M, \alpha\rangle & \xrightarrow{\ g\ } & C
\end{array}
$$

By the universal property of $h_{A,\alpha}$, Proposition 2.2.2, it suffices to find a pair (f', t') such that $f : A \to B$, $t' : M \to B^{\#}$ is homogenous of degree -1, that $\partial t = f^{\#}\alpha$ and that h induces a morphism $h : (f', t') \to g$. We see that we are forced to choose $f' = f$ as $hf = ga$. By the existence of g, there exists a $t : M \to C^{\#}$ such that $\partial t = g^{\#}a^{\#}\alpha = h^{\#}f^{\#}\alpha$. Since h is an acyclic fibration $h^{\#}$ is a surjective quasi-isomorphism. Since $M \simeq \mathbb{K}[n]$, the morphism t is really an element of $(C^{\#})^{n-1}$. By surjectivity of $h^{\#}$ there is an element u of $(B^{\#})^{n-1}$ such that $h^{\#}(u) = t$. Moreover, the difference $h^{\#}(\partial u - f^{\#}\alpha) = 0$, so $\partial u - f^{\#}\alpha$ factors through the kernel $\mathrm{Ker}h^{\#}$, which is assumed to be acyclic. This element is furthermore a cycle, so by acyclicity there is another element u' such that $\partial u' = \partial u - f^{\#}\alpha$. We may now see that $(f, u - u')$ is our desired factorization.

Secondly we prove that if M is as in $H1$ and $\alpha = 0$, then the map $A \to A\langle M, \alpha\rangle$ is an acyclic cofibration. This amounts to show that it has LLP to every fibration.

Suppose that $h : B \to C$ is a fibration and that there is a commutative square as below.

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle h} \\
A\langle M, \alpha\rangle & \xrightarrow{\ g\ } & C
\end{array}
$$

We will again use 2.2.2, so it suffices to find a t' such that $\partial t' = f^{\#}\alpha = 0$. By g, let $t : M \to C^{\#}$ such that $\partial t = g^{\#}a^{\#}\alpha = h^{\#}f^{\#}\alpha = 0$. Since $h^{\#}$ is surjective t admits a lift $t' : M \to B^{\#}$ such that $t' = h^{\#}t$. The result follows as $\partial t' = \partial h^{\#}t = h^{\#}\partial t = 0$.

$\square$

In light of the above proposition we would like to make some more convenient notation. If $M \simeq \mathbb{K}[n]$ and $\alpha : M \to Z^n(A^{\#})$, s.t. $\alpha(1) = a$, we write $A\langle M, \alpha \rangle$ as $A\langle T; dT = a \rangle$ instead. Hinich calls this "adding a variable to kill a cycle". If M is the acyclic complex as below and $\alpha = 0$, we write $A\langle T, S; dT = S \rangle$. This could be thought of "adding a variable and a cycle to kill itself".

$$ ... \longrightarrow 0 \longrightarrow \mathbb{K} \xrightarrow{id} \mathbb{K} \longrightarrow 0 \longrightarrow ... $$

proof of theorem. **MC1** and **MC2** are satisfied. By definition we also have the first part of **MC3**. We start by checking **MC4**.

Let $f : A \to B$ be a morphism in $\mathcal{C}$. Given any $b \in B^{\#}$, let $C_b = A\langle T_b, S_b; dT_b = S_b \rangle$. We define $g_b : C_b \to B$ by the conditions that it acts on A as f, $g_b^{\#}(T_b) = b$ and $g_b^{\#}(S_b) = db$. Iterating this construction for every $b \in B$, we obtain an object C, such that the injection $A \to C$ is an acyclic standard cofibration, and the map $g : C \to B$ is a fibration. This gives us a factorization $f = f_\delta \circ f_\gamma$, where f_γ is the injection and $f_\delta = g$.

To obtain the other factorization we want to make a standard cofibration. We already know that the map $A \to C$ is a standard cofibration, so let $C_0 = C$. From here on, we will make each C_i inductively, such that $\varinjlim C_i$ has the factorization property which we desire. Notice that from C_0 there is a morphism $g_0 : C_0 \to B$, which is surjective and surjective on every kernel. This morphism may fail to be a quasi-isomorphism, so it is not an acyclic fibration.

To construct C_1 we assign to every pair of elements (c, b), such that $c \in ZC_0^{\#}$ and $g_0^{\#}(c) = db$, a variable to kill a cycle. If (c, b) is such a pair, then we add a variable T such that $dT = c$ and $g_1^{\#}(T) = b$. C_1 is then the complex where each cycle c has been killed by adding a variable T. Now if we suppose that we have constructed C_i, then C_{i+1} is constructed in the same manner, by adding a variable to kill each cycle which is a boundary in the image.

When adding a variable, we have also updated the morphism g_i by letting $g_{i+1}^{\#}(T) = b$. Thus in each step we have also made a new morphism g_{i+1}. If g denote the morphism at the colimit, then it is clear that this is still a fibration and it has also become a quasi-isomorphism. This is because every cycle which failed to be in the homology of B have been killed.

It remains to check the last part of **MC3**. Suppose that $f : A \to B$ is an acyclic cofibration. By **MC4**, we know that it factors as $f = f_\delta \circ f_\gamma$, where f_δ is an acyclic fibration and f_γ is a standard acyclic fibration. We thus obtain that f is a retract of f_γ by the commutative diagram below.

$$
\begin{array}{ccc}
A & \xrightarrow{f_\gamma} & C \\
\downarrow{\scriptstyle f} & \nearrow & \downarrow{\scriptstyle f_\delta} \\
B & = & B
\end{array}
$$

$\square$

The following corollary will concretize what it means that the standard cofibrations generate every cofibration. This corollary is really a step used within in the proof.

Corollary 2.2.4.1. *Any (acyclic) cofibration is a retract of a standard (acyclic) cofibration.*

We may immedietly apply this theorem to some familiar examples.

Corollary 2.2.4.2. *Let A be a dg-algebra over the field $\mathbb{K}$. The category Mod_A of left modules is a model category.*

sketch of proof. We establish the adjunction by letting $F = A \otimes_{\mathbb{K}} \_$. H0 is satisfied as this category is bicomplete, and filtered colimits may be thought of as unions of sets. Moreover, since mod_A is an Abelian category, the forgetful functor # commutes with coproducts, or direct sums, which makes H1 trivially satisfied. $\square$

Corollary 2.2.4.3. *The categories $Alg_{\mathbb{K}}^{\bullet}$ ($Alg_{\mathbb{K},+}^{\bullet}$) are model categories.*

Proof. We establish the adjunction by letting $F = T(M)$, the tensor algebra of a cochain complex. For the same reasons as above, H0 is trivially satisfied.

Given a cochain complex $N^{\bullet}$, we may consider the free dg-algebra $T(N^{\bullet})$. In this case the coproduct $A * T(N^{\bullet})$ has an easier description. We define a complex

$$A[N^{\bullet}] = A \oplus (A \otimes N^{\bullet} \otimes A) \oplus (A \otimes N^{\bullet} \otimes A \otimes N^{\bullet} \otimes A) \oplus \cdots.$$

The differential on $A[N^{\bullet}]$ is the differential induced by the tensor product. We define a multiplication on $A[N^{\bullet}]$ by the following formula

$$(a_1 \otimes \cdots \otimes a_i) \cdot (a_1' \otimes \cdots \otimes a_j') = a_1 \otimes \cdots \otimes a_i a_1' \otimes \cdots a_j'.$$

Let $i : A \to A[N^{\bullet}]$ denote the inclusion, and $\iota : T(N^{\bullet}) \to A[N^{\bullet}]$ is defined by interspersing the $N^{\bullet}$ tensors with 1s. I.e. $\iota(n_1 \otimes \cdots \otimes n_j) = 1 \otimes n_1 \otimes 1 \otimes \cdots \otimes 1 \otimes n_j \otimes 1$.

To define a map $f : A[N^{\bullet}] \to T$ it is enough by the ring homomorphism property to define a map $g : A \to T$ and a map $h : T(N^{\bullet}) \to T$. We observe that this choice of g and h is unique for any f, establishing the universal property. I.e. $A[N^{\bullet}] \simeq A * T(N^{\bullet})$.

To see that the map $i^{\#} : A^{\#} \to A[M^{\bullet}]^{\#}$ is a quasi-isomorphism, it is enough to see that contractible complexes are stable under tensoring. Given a contractible complex $C^{\bullet}$, then there is a homotopy $h : C^{\bullet} \to C^{\bullet}$ such that $\partial h = id_C$. Observe that $id_N \otimes h : N^{\bullet} \otimes C^{\bullet} \to N^{\bullet} \otimes C^{\bullet}$ is a homotopy witnessing $id_{N^{\bullet} \otimes C^{\bullet}} \sim 0$. Since M is contractible, we know that the homology of the inclusion is $H^* i = id_{H*A}$. This shows H1. $\square$

We summarize the last result:

The category of augmented dg-algebras $\mathsf{Alg}_{\mathbb{K},+}^{\bullet}$ is a model category. Let $f : X \to Y$ be a homomorphism of augmented algebras.

- $f \in \mathsf{Ac}$ if $f^{\#}$ is a quasi-isomorphism.
- $f \in \mathsf{Fib}$ if $f^{\#}$ is an epimorphism (surjective onto every component).
- $f \in \mathsf{Cof}$ if f has LLP with respect to to every acyclic fibration.

The category of augmented dg-algebras has a zero object. This is the stalk of $\mathbb{K}$. We see that every object is fibrant, as the augmentation map is preserved by the forgetful functor and it is by definition a split-epimorphism.

2.2.2 A Model Structure on DG-Coalgebras

We now want to equip the category of dg-coalgebras with a suitable model structure. This model structure should be suitable in the sense that it give rise to the same homotopy theory as dg-algebras. The bar-cobar construction will be crucial in this construction, as it is in fact a Quillen adjunction. To this end we will follow the setup as presented by Lefevre-Hasegawa [13]. His method is a modification of Hinich's paper [23].

Let $f : C \to D$ be a morphism of coalgebras, the category of dg-coalgebras will be equipped with the three following classes of morphisms:

- $f \in \mathsf{Ac}$ if Ωf is a quasi-isomorphism.
- $f \in \mathsf{Fib}$ if f has RLP with respect to every acyclic cofibration.
- $f \in \mathsf{Cof}$ if $f^{\#}$ is a monomorphism (injective in every component).

To see that these classes of morphisms does indeed define a model structure, we will get a better description for a subclass of the weak equivalences. This can only be done by calculating homologies, since f is a weak equivalence if and only if $H^* \mathrm{cone}(\Omega f) \simeq 0$. Using spectral sequences to calculate these homologies is not crucial, but it gives us a method to tackle the kinds of problems which we will face.

Definition 2.2.5. A filtered chain map $f : M \to N$ of filtered complexes M and N is a filtered quasi-isomorphism if $\mathrm{gr} f : \mathrm{gr} M \to \mathrm{gr} N$ is a quasi-isomorphism of the associated graded complexes.

Lemma 2.2.6. *Let $f : C \to C'$ be a graded quasi-isomorphism between conilpotent dg-coalgebras, then $\Omega f : \Omega C \to \Omega C'$ is a quasi-isomorphism.*

Proof. We do this by considering a spectral sequence. Endow C with a grading (as a vector space) induced by the coradical filtration, i.e. $c \in C$ has degree $|c| = n$ if n is the smallest

number such that $\overline{\Delta}^n c = 0$. We define a filtration on ΩC by

$$F_p \Omega C = \{\langle c_1 | \cdots | c_n \rangle \mid |c_1| + \ldots + |c_n| \leqslant p\}$$

Since C is a dg-coalgebra, the coradical filtration respects the differential. In other words, $F_p \Omega C$ is still a chain complex, which is a subcomplex of ΩC. This filtration is clearly bounded below and exhaustive. Thus by the classical convergence theorem of spectral sequences, [Theorem 5.5.1 24, p. 135], the spectral sequence converges to the homology $E\Omega C \Rightarrow H^* \Omega C$.

By definition, the 0'th page is defined as

$$E^0_{p,q} \Omega C = {}^{(F_p \Omega C)_{p+q}} \big/ {}_{(F_{p-1} \Omega C)_{p+q}}.$$

Furthermore, notice that at this page we have the following isomorphism $E^0_{p,q} \Omega C \simeq (\Omega \mathrm{gr} C)^{(p)}_{p+q}$, where $(\Omega \mathrm{gr} C)^{(p)} = \{\langle c_1 | \cdots | c_n \rangle \mid |c_1| + \ldots + |c_n| = p\}$.

Evaluating f at the 0'th page would look like $E^0 \Omega f \simeq \Omega \mathrm{gr} f$. By the mapping lemma, [Exercise 5.2.3 24, p. 125], it is enough to check that $\Omega \mathrm{gr} f$ is a quasi-isomorphism to see that Ωf is a quasi-isomorphism. We show that $\Omega \mathrm{gr} f$ is a quasi-isomorphism by inspecting every cochain complex $E^0_{p,\bullet} \Omega C$.

Define a filtration G_k on $E^0_{p,\bullet} \Omega C$ as

$$G_k = \{\langle c_1 | \cdots | c_n \rangle \mid n \geqslant -k\}.$$

We see that $G_0 = E^0_{p,\bullet} \Omega C$ by definition and $G_{-p-1} \simeq 0$ on the coaugmentation quotient $\overline{C}$. By the classical convergence theorem of spectral sequences, this defines a spectral sequence such that $EG \Rightarrow H^* E^0_{p,\bullet} \Omega C$.

To see that $\Omega \mathrm{gr} f$ is a quasi-isomorphism, it is now enough to see that $E^0 G f$ is a quasi-isomorphism for any p. Notice that $E^0_{l,\bullet} G \subseteq (\mathrm{gr} C[-1])^{\otimes l}$ where the total grading is p. Since f is a filtered quasi-isomorphisms and by the Kunneth-formula, theorem 3.6.3 [Theorem 3.6.3 24, p. 88], it follows that $E^0 G f$ is a quasi-isomorphism. $\qquad \square$

This proof will serve as a template for how we want to approach many of the proofs we will encounter. The lemma says that to show that f is a weak equivalence it is sufficient to show that f is a filtered quasi-isomorphism. However, to show that f is a filtered quasi-isomorphism we first need a good filtering, and once we have a filtering we look at its spectral sequence. If we still have a problem, we look at complexes within a page on a spectral sequence, and the mapping lemma says that it is enough to verify that a morphism becomes a quasi-isomorphism on any page to see that it is a quasi-isomorphism. On these complexes we may define new filtrations in order to calculate the next page. We will informally call this technique for an iterated spectral sequence argument.

For completeness we include the following statement.

Lemma 2.2.7. *Let $f : A \to A'$ be a quasi-isomorphism between dg-algebras, then $Bf : BA \to BA'$ is both a filtered and unfiltered quasi-isomorphism.*

Proof. Notice that the homology of BA may calculated from the double complex used to define BA. In fact, at the 0'th page of the canonical spectral sequence we have $E^0_{p,\bullet}f \simeq f^{\otimes p}$. It follows that f is a quasi-isomorphism on the 0'th page from the Kunneth formula, [Theorem 3.6.3 24, p. 88]. $\square$

Let A (C) be a filtered dg-algebra (coalgebra). Given an element $a \in A$ $(c \in C)$ we say that its filtered degree f-deg(a) (f-deg(c)) is the smallest number such that $a \in F_{\text{f-deg}(a)}A$ $(c \in F_{\text{f-deg}(c)}C)$ but not $a \in F_{\text{f-deg}(a)-1}A$ $(c \in F_{\text{f-deg}(c)-1}C)$. There is then an associated filtration on the bar (cobar) construction of this complex, defined as

$$F_p BA = \{[a_1 \mid \cdots \mid a_n] \mid \sum \text{f-deg}(a_i) \leqslant p\}$$
$$(F_p \Omega C = \{\langle c_1 \mid \cdots \mid c_n \rangle \mid \sum \text{f-deg}(c_i) \leqslant p\}).$$

We will call this the induced filtration on the bar or cobar construction.

Proposition 2.2.8. *Let A be an augmented dg-algebra and C a conilpotent dg-coalgebra. The counit $\varepsilon_A : \Omega BA \to A$ a quasi-isomorphism. The unit $\eta_C : C \to B\Omega C$ is a filtered quasi-isomorphism, moreover $\Omega \eta_C$ is a quasi-isomorphism.*

The following proof is due to [13], but with corrections given by [25]. Some minor modifications are given to the proof as it resembles a proof given earlier.

Proof. We start by showing that the counit is a quasi-isomorphism. Define the following filtration for A.

$$F_0 A = \mathbb{K}$$
$$F_1 A = A$$
$$F_p A = F_1 A$$

We clearly see that this filtration endows A with the structure of a filtered dg-algebra. For ΩBA we will use the induced filtration from the coradical filtration of BA.

The counit acts on ΩBA as tensorwise projection, followed by multiplication in A. This morphism respects the filtration, so it is a filtered morphism. Notice that both of these filtrations are bounded below, and exhaustive, so the classical convergence theorem of spectral sequences apply.

Let $E_r \Omega BA$ and $E_r A$ be the spectral sequences given by these filtrations. We have that $E_1^p \Omega BA \simeq \text{gr}_p \Omega BA$ and $E_1^p A \simeq \text{gr}_p A$. For $p = 1$ both complexes are isomorphic to the same complex, $\overline{A}$. Moreover, $E_1^1 \varepsilon_A = id_{\overline{A}}$. Whenever $p \neq 1$ we get that $E_1^p A \simeq 0$, so it remains to show that $E_1^p \Omega BA \simeq \text{gr}_p \Omega BA$ is contractible for any $p \geqslant 2$.

The differential of ΩBA is generated by three actions: the differential on A, the multiplication on A and the comultiplication on BA. With the induced filtration on ΩBA, we see that the multiplication on A is the only action which maps $F_p \Omega BA \to F_{p-1} \Omega BA$. Thus this action is 0 in the associated graded and the spectral sequence.

There is a homotopy of the identity given as $r : \mathsf{gr}_i \Omega BA \to \mathsf{gr}_i \Omega BA$, which is 0 except if there is an element on the form $\langle [a] \mid [\cdots] \mid [\cdots] \rangle$. In this case r is

$$r\langle [a] \mid [\cdots] \mid \cdots \rangle = (-1)^{|a|+1}\langle [a \mid [\cdots] \mid \cdots \rangle$$

We will show that this is a homotopy by induction on i.

Let $i = 2$. Then there are two cases we must handle, either an element is on the form $\langle [a_1] \mid [a_2] \rangle$ or $\langle [a_1 \mid a_2] \rangle$. We consider the latter case first. If we apply r to this element we are returned 0.

$$(r \circ d_{\Omega BA} + d_{\Omega BA} \circ r)\langle [a_1 \mid a_2] \rangle = r(-1)^{|a_1|+1}\langle [a_1] \mid [a_2] \rangle = \langle [a_1 \mid a_2] \rangle$$

Then we treat the former case

$$
\begin{aligned}
(r \circ d_{\Omega BA} + &d_{\Omega BA} \circ r)\langle [a_1] \mid [a_2] \rangle \\
&= r\langle [d_A a_1] \mid [a_2] \rangle + (-1)^{|a_1|}r\langle [a_1] \mid [d_A a_2] \rangle + d_{\Omega BA}(-1)^{|a_1|+1}\langle [a_1 \mid a_2] \rangle \\
&= (-1)^{|a_1|}\langle [d_A a_1 \mid a_2] \rangle - \langle [a_1 \mid d_A a_2] \rangle + \langle [a_1] \mid [a_2] \rangle \\
&\qquad + (-1)^{|a|+1}\langle [d_A a_1 \mid a_2] \rangle + \langle [a_1 \mid d_A a_2] \rangle = \langle [a_1] \mid [a_2] \rangle.
\end{aligned}
$$

We see that this is in fact a homotopy, which makes $id_{\mathsf{gr}_2 \Omega BA}$ null-homotopic.

To extend this argument by induction, we will observe that the terms where the differential is applied will have opposite signs, such that they cancel. Then the result follows for any i, by observing that the tensors far enough out to the right is not affected by r.

If C is a dg-coalgebra, we use the same technique as in Lemma 2.2.6. Consider the filtration on $B\Omega C$ given as

$$F_p B\Omega C = \{[\langle sc_{1,1} \mid \cdots \mid sc_{1,n_1} \rangle \mid \cdots \mid \langle sc_{m,1} \mid \cdots \mid sc_{m,n_m} \rangle] \mid |c_{1,1}| + \cdots + |c_{m,n_m}| \leqslant p\}.$$

This filtration is both bounded below and exhaustive, so the classical convergence theorem says that the associated spectral sequence converges. If we denote this sequence as EF, then $EF \implies H^*B\Omega C$. Let EC denote the spectral sequence associated to C. Since C is conilpotent $EC \implies H^*C$. The unit $\eta_C : C \to B\Omega C$ is now a map acting on EC^0 as the identity, sending each element in $EC^0_{p,q}$ to itself in $EF^0_{p,q}$.

On each row $EF^0_{p,\bullet}$ we make another filtration called G.

$$G_k EF^0_{p,\bullet} = \{[\langle ... \rangle_1 \mid ... \mid \langle ... \rangle_n] \mid n \geqslant -k\}$$

Similarly as in Lemma 2.2.6, this filtration is bounded below and exhaustive, so we may again apply the classical convergence theorem to obtain a spectral sequence E_pG such that $E_pG \implies H^*EF^0_{p,\bullet} \simeq EF^1_{p,\bullet}$. Since the unit acts as the identity on EC^0, it descends to a morphism $\mathrm{gr}_pC \to E_pG^0_{k,\bullet}$ which is the identity when $k = -1$ and 0 otherwise. Notice that every string of length $\geqslant 2$ is not hit by this morphism. However, by employing r as above we may show that these summands are contractible. This shows that the unit is an isomorphism in homology.

$\square$

Lemma 2.2.9. *Let $f : C \to D$ be a morphism of dg-coalgebras, then:*

- *if f is a cofibration, then Ωf is a standard cofibration.*
- *if f is a weak equivalence, then Ωf is as well.*

Almost dually, let $f : A \to B$ be a morphism of dg-algebras, then:

- *if f is a fibration, then Bf is a fibration.*
- *if f is a weak equivalence, then Bf is as well.*

Proof. First suppose that $f : C \to D$ is a cofibration. We define a filtration on D as the sum of the image of f and the coradical filtration on D: $D_i = Imf + Fr_iD$. f being a cofibration ensures us that $D_0 \simeq C$. Since D is conilpotent we know that $D \simeq \varinjlim D_i$, and since Ω commutes with colimits there is a sequence of algebras $\Omega C \to \Omega D_1 \to ... \to \Omega D$. It is enough to show that each morphism $\Omega D_i \to \Omega D_{i+1}$ is a standard cofibration. The quotient coalgebra D_{i+1}/D_i only has a trivial comultiplication, thus every element is primitive. This means that as a cochain complex D_{i+1} is constructed from D_i by attaching possibly very many copies of $\mathbb{K}$. We treat the case when there is only one such $\mathbb{K}$, here $D_{i+1} \simeq D_i \oplus \mathbb{K}\{x\}$ where $dx = y$ for some $y \in D_i$. We observe that this is exactly the condition for that the morphism $\Omega D_i \to \Omega D_{i+1}$ is a standard cofibration.

If f is a weak equivalence, then Ωf is a quasi-isomorphism by definition.

By Lemma **??**]lem: Quill-adj, or adjointness more specifically, B preserving fibrations is a consequence of Ω preserving cofibrations.

It remains to show that if $f : A \to B$ is a quasi-isomorphism, then Bf is a weak equivalence. Now, Bf is a weak equivalence if and only if ΩBf is a quasi-isomorphism. By Proposition 2.2.8, the counit $A \to \Omega BA$ is a quasi-isomorphism, so Bf is a weak equivalence by 2-out-of-3 property.

$$
\begin{array}{ccc}
A & \xrightarrow{\ \ f\ \ } & B \\
{\scriptstyle \varepsilon_A}\big\uparrow & & {\scriptstyle \varepsilon_B}\big\uparrow \\
\Omega BA & \xrightarrow{\ \Omega Bf\ } & \Omega BB
\end{array}
$$

$\square$

We will need one more technical lemma.

Lemma 2.2.10 (Key lemma)**.** *Let A be a dg-algebra, D a dg-coalgebra and $p : A \to \Omega D$ a fibration of algebras. The projection morphism $* : BA \prod_{B\Omega D} D \to BA$ is an acyclic cofibration.*

$$
\begin{array}{ccc}
BA *_{B\Omega D} D & \longrightarrow & D \\
\big\downarrow{\scriptstyle \pi} & \ulcorner & \big\downarrow{\scriptstyle \eta_D} \\
BA & \xrightarrow[\ Bp\ ]{} \!\!\!\!\twoheadrightarrow & B\Omega D
\end{array}
$$

This proof has a troubled past. In [13], Lefevre-Hasegawa made a proof which was a straight-forward modification of Hinich's proof from [23]. However, this translation does not behave as well as one would like. Keller points out that this method may work in some cases, but fails in its full generality [26]. The proof presented here will be a modification of Vallette's proof of "A technical lemma" [27, Appendix B].

Proof. π being a cofibration is immediate by Corollary 2.1.10.1.

To see that π is a weak equivalence it is enough to show that it is a filtered quasi-isomorphism by Lemma 2.2.6. Since we assume p to be a fibration onto a quasi-free algebra we may realize the algebra A as the following extension.

$$
\begin{array}{ccccc}
\cdots & \lhook\joinrel\longrightarrow & \mathsf{cone}(d') & \xrightarrow{\ p\ }\!\!\!\!\twoheadrightarrow & \Omega D \\
& & & & \\
\mathsf{Ker}(p)[1] & \lhook\joinrel\longrightarrow & \mathsf{cone}(d')[1] & \longrightarrow\!\!\!\!\twoheadrightarrow & \cdots
\end{array}
$$

with dashed arrow labelled d'.

As graded modules, $A \simeq \mathsf{cone}(d') \simeq \mathsf{Ker}(p) \oplus \Omega D$. We denote $K = \mathsf{Ker}(p)$, so that the differential of A is then the differential coming from

$$
\begin{aligned}
d_K &: K \to K, \\
d_{\Omega D} &: \Omega D \to \Omega D \text{ and} \\
d' &: \Omega D \to K.
\end{aligned}
$$

In the category $\mathsf{Alg}^{\bullet}_{\mathbb{K},+}$, $\oplus$ is the product. Since $B : \mathsf{Alg}^{\bullet}_{\mathbb{K},+} \to \mathsf{coAlg}^{\bullet}_{\mathbb{K},conil}$ is right adjoint, it necessarily preserves products. Thus

$$
\begin{aligned}
BA &\simeq B(K \oplus \Omega D) \simeq BK * B\Omega D \text{ and} \\
BA *_{B\Omega D} D &\simeq BK * D.
\end{aligned}
$$

Using this identification of the underlying graded modules we may identify the morphism π with $id_{BK} * \eta_D$. If the differential of BA was not perturbated by d', then we could have appealed to the morphism π being a filtered quasi-isomorphism to arrive at the conclusion. Instead, we will employ some smart filtrations onto BA and $BA *_{B\Omega D} D$.

Since BK is quasi-free, by the comonadic presentation of D, we are able to obtain an identification of graded modules, $BK * D \subseteq T^c(\overline{K}[1] \oplus \overline{D})$. Likewise, since both BK and $B\Omega D$ are quasi-free, we realize the product as $BK * B\Omega D \simeq T^c(\overline{K}[1] \oplus (\overline{\Omega}D)[1])$.

With this description we define filtrations as

$$F_n(BA *_{B\Omega D} D) \subseteq F_n(T^c(\overline{K}[1] \oplus \overline{D})) = \bigoplus_{k=0}^{\infty} \sum_{\substack{n_1+\cdots+n_k \\ \leqslant n}} \bigotimes_{i=1}^{k} (\overline{K}[1] \oplus Fr_{n_i}\overline{D}) \text{ and}$$

$$F_n(BA) = F_n(T^c(\overline{K}[1] \oplus (\overline{\Omega}D)[1])) = \bigoplus_{k=0}^{\infty} \sum_{\substack{n_1+\cdots+n_k \\ \leqslant n}} \bigotimes_{i=1}^{k} (\overline{K}[1] \oplus \widetilde{Fr}_{n_i}(\overline{\Omega}D)[1]).$$

Here Fr and $\widetilde{Fr}$ respectively refers to the coradical and induced coradical filtration. This filtration is made to be agnostic towards K, in other words morphisms into K are a priori filtered. Thus the part of the differential coming from d_K and d' are filtered. Likewise, the coradical filtration preserves the part of the differential coming from $d_{\Omega D}$. The differential coming from the multiplication of K and ΩD is seen to be of -1 filtered degree. It is clear that $\eta_{\overline{D}}$ preserves this filtration as it acts like the identity.

The associated graded component reduces to the associated graded of D and $B\Omega D$. This can be seen as if we lower the degree of a n_i by 1, then this component lands in the lower degree of the filtration. By cocontinuity of the tensor, we may then move the associated graded into each variable. The sum then handles every other component.

$$\mathsf{gr}_n(BA *_{B\Omega D} D) \simeq BK * \mathsf{gr}_n D$$
$$\mathsf{gr}_n(BA) \simeq BK * B\Omega\mathsf{gr}_n D$$

In the same manner, the morphism π then acts on each element as $id_{BK} * \mathsf{gr}(\eta_D)$.

These filtrations are clearly bounded below. Since D and $B\Omega D$ are both conilpotent dg-coalgebras, the filtrations are then exhaustive as well. By the classical convergence theorem of filtered spectral sequences we obtain spectral seqences $E(BA *_{B\Omega D} D) \implies \mathsf{H}^*(BA *_{B\Omega D} D)$ and $E(BA) \implies \mathsf{H}^*(BA)$. We want to show that the morphism of spectral sequences $id_{BK} *_{B\Omega D} \mathsf{gr}\eta_D : E(BA *_{B\Omega D} D) \to E(BA)$ eventually becomes a quasi-isomorphism. This will in fact happen on the first page.

To obtain this at the first page we will define another spectral sequence $\widetilde{E}$ such that $\widetilde{E} \implies E_1$.

We start by defining new filtrations,

$$\widetilde{F}_n(BK * \mathsf{gr}D) \subseteq \bigoplus_{k=0}^{\infty} \sum_{\substack{n_1+\cdots+n_k+k \\ \leqslant n}} \bigotimes_{i=1}^{k} (\overline{K}[1] \oplus \mathsf{gr}_{n_i}\overline{D}) \text{ and}$$

$$\widetilde{F}_n(BK * B\Omega\mathsf{gr}D) = \bigoplus_{k=0}^{\infty} \sum_{\substack{n_1+\cdots+n_k \\ \leqslant n}} \bigotimes_{i=1}^{k} (\overline{K}[1] \oplus (\bigoplus_{t=1}^{\infty} \sum_{\substack{m_1+\cdots+m_t+t \\ \leqslant n_i}} \bigotimes_{j=1}^{t} \mathsf{gr}_{m_j}\overline{D}[-1])[1]).$$

Again, these filtrations are agnostic towards K, so both parts of the differential which comes from d_K and d' are filtered. The part of the differential which comes from d_D naturally goes from $\mathsf{gr}_{n_i}\overline{D}$ to itself. The differential coming from the multiplication has already been dealt with, so these filtrations respects our differential. The morphism $id_{BK} * \mathsf{gr}(\eta_D)$ also preserves this filtration, as it acts like the identity on elements. In other words, the first filtered object is naturally a subobject of the second filtered object by identifying the elements d with $[\langle d \rangle]$.

At the 0'th page of $\widetilde{E}$ we want to show that the part of the differential coming from d' acts like 0. This is the same to say that $\mathrm{Im}d' \mid_{F_n} \subseteq F_{n-1}$. We calculate the 0'th page of the double spectral sequence as below.

$$\widetilde{E}_0^{-n}(BK * \mathsf{gr}D)[-n] \subseteq \mathsf{gr}_n(BK * \mathsf{gr}D) \simeq \bigoplus_{k=0}^{\infty} \sum_{\substack{n_1+\cdots+n_k+k \\ =n}} \bigotimes_{j=1}^{k} (\overline{K}[1] \oplus \mathsf{gr}_{n_i}\overline{D})$$

$$\widetilde{E}_0^{-n}(BK * B\Omega\mathsf{gr}D)[-n] = \mathsf{gr}_n(BK * B\Omega\mathsf{gr}D)$$

$$\simeq \bigoplus_{k=0}^{\infty} \sum_{\substack{n_1+\cdots+n_k \\ =n}} \bigotimes_{i=1}^{k} (\overline{K}[1] \oplus (\bigoplus_{t=1}^{\infty} \sum_{\substack{m_1+\cdots+m_t+t \\ =n_i}} \bigotimes_{j=1}^{t} \mathsf{gr}_{m_j}\overline{D}[-1])[1])$$

We now pick an element $([k_1]+d_1)\otimes\cdots\otimes([k_k]+d_k) \in \mathsf{gr}_n(BK*\mathsf{gr}D)$. Then $|d_1|+\cdots+|d_k|+k = n$. The differential coming from d' is the alternate sum of d' at each tensor argument. We illustrate what happens at the i'th argument.

$$\widetilde{d}'(([k_1] + d_1) \otimes \cdots \otimes ([k_i] + d_i) \otimes \cdots \otimes ([k_k] + d_k))$$
$$=([k_1] + d_1) \otimes \cdots \otimes ([k_i] + d'(d_i)) \otimes \cdots \otimes ([k_k] + d_k)$$

Since $|[k] + d'(d_i)| = 0$, the total degree of this element goes down at least 1 if $d_i \neq 0$. If $d_i = 0$, then $d'(d_i) = 0$ anyway. In this manner, this morphism does not survive at the $\widetilde{E}_0$ page. Likewise, given an element on the form $[k_1 + \langle d_{1,1} \mid \cdots \mid d_{1,t_1} \rangle \mid \cdots \mid k_k + \langle d_{k,1} \mid \cdots \mid d_{k,t_k} \rangle]$, then $|d'(\langle d_{i,1} \mid \cdots \mid d_{i,t_i} \rangle)| = 0$. So the phenomenon occurs at the other spectra sequence as well.

In this way $\mathsf{gr}(id_{BK} * \mathsf{gr}\eta_D)$, is in fact a quasi-isomorphism between the sequences $\widetilde{E}(BK * \mathsf{gr}D) \to \widetilde{E}(BK * B\Omega\mathsf{gr}D)$ just as Lemma 2.2.8. By the classical convergence theorem this assembles into a quasi-isomorphism on the E_1 page of the previous spectral sequences, showing that π is a filtered quasi-isomorphism. $\qquad\square$

Theorem 2.2.11. *The category coAlg$^\bullet_{\mathbb{K},conil}$ is a model category with the classes Ac, Fib and Cof as defined above.*

Proof. The axioms **MC1** and **MC2** are immediate. Also, fibrations having RLP with respect to acyclic cofibrations is by definition.

We show **MC4** first. Let $f : C \to D$ be a morphism of coalgebras. There is a factorization $\Omega f = pi$ of morphisms between algebras, where i is a cofibration, p is a fibration and at least one of i and p are quasi-isomorphisms. Applying bar we get a factorization $B\Omega f = BiBp$, where Bp is a fibration and at least one of Bi and Bp are weak equivalences.

$$\Omega C \xrightarrow{\Omega f} \Omega D \qquad\qquad B\Omega C \xrightarrow{B\Omega F} B\Omega D$$

We construct a pullback with Bp and η_D. By Lemma 2.2.10 the morphism π is an acyclic cofibration. We collect our morphisms in a big diagram. The dashed arrow exists since the rightmost square is a pullback.

$$BA\textstyle\prod_{B\Omega D} D$$

First notice that q is a fibration, since fibrations are stable under pullbacks. j is a cofibration, or a monomorphism, as the composition $Bi \circ \eta_C$ is a monomorphism. Thus it remains to see that if Bi (Bp) is a weak equivalence, then j (q) is as well. This is evident from the 2-out-of-3 property, as η is a natural weak equivalence, π is a weak equivalence and Bi (Bp) is a weak equivalence.

We now show **MC3**. Suppose that there is a square as below, where i is a cofibration and t is an acyclic cofibration.

$$\begin{array}{ccc} E & \longrightarrow & C \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle t} \\ F & \longrightarrow & D \end{array}$$

We can factor t as $t = qj$ by **MC4**. Notice that t is a retract of q, i.e. there is a commutative diagram as below.

$$\begin{array}{ccc} C & = = = = & C \\ \downarrow{\scriptstyle j} & \nearrow & \downarrow{\scriptstyle t} \\ BA \textstyle\prod_{B\Omega A} D & \xrightarrow{q} & D \end{array}$$

So in order to find a lift to C, we may instead find a lift to $BA \prod_{B\Omega D} D$. Since p is an acyclic fibration by construction and Ωi is a cofibration by Lemma 2.2.9, there is a lift $h : \Omega E \to A$ of algebras. We obtain our desired lift from the bar-cobar adjunction and the universal property of the pullback.

$$\begin{array}{ccccccc} E & \longrightarrow & BA\prod_{B\Omega D} D & \xrightarrow{\pi} & BA & & \Omega E \longrightarrow A \\ \downarrow{\scriptstyle i} & \nearrow & \downarrow{\scriptstyle q} \quad {\scriptstyle h^T} & & \downarrow{\scriptstyle Bp} \quad\leftsquigarrow & & \downarrow{\scriptstyle \Omega i} \; {\scriptstyle h} \;\; \downarrow{\scriptstyle p} \\ F & \longrightarrow & D & \xrightarrow{\eta_D} & B\Omega D & & \Omega F \longrightarrow \Omega D \end{array}$$

$\square$

We restate the corollary of the adjunction.

Corollary 2.2.11.1. *The bar-cobar construction* $B : Alg^{\bullet}_{\mathbb{K},+} \rightleftarrows coAlg^{\bullet}_{\mathbb{K},conil} : \Omega$ *as a Quillen equivalence.*

Proof. We first observe that (B, Ω) is a Quillen adjunction by Lemma 2.2.9. Moreover, since the unit and counit are weak equivalences by Proposition 2.2.8, it follows by either Proposition 2.1.43 or its Corollary 2.1.43.1 that (B, Ω) is a Quillen equivalence. $\square$

2.2.3 Homotopy theory of A_∞-algebras

This section aims to finalize the discussion of the homotopy theory of A_∞-algebras. We will look at the homotopy invertibility of every strongly homotopy associative quasi-isomorphism, and its relation to ordinary associative algebras. This discussion will end with mentioning different results which gives a clearer description of fibrations, cofibrations and homotopy equivalences.

This section follows Lefevre-Hasegawa [13]. Before we get to the main theorem, we start by discussing a non-closed model structure on the category of Alg_∞.

Let $f : A \rightsquigarrow B$ be a morphism between A_∞-algebras, the category of A_∞-algebras will be equipped with the three following classes of morphisms:

- $f \in \mathsf{Ac}$ if f is an ∞-quasi-isomorphism, i.e. f_1 is a quasi-isomorphism.
- $f \in \mathsf{Fib}$ if f_1 is an epimorphism.
- $f \in \mathsf{Cof}$ if f_1 is a monomorphism.

This category does not make a model category in the sense of a closed model category, as we are lacking many finite limits. It does however come quite close to be such a category.

Theorem 2.2.12. *The category Alg$_\infty$ equipped with the three classes as defined above satisfies:*

a *The axioms* **MC1** *through* **MC4**.
b *Given a diagram as below, where p is a fibration, then its limit exists.*

$$
\begin{array}{ccc}
 & & A \\
 & & \downarrow{\scriptstyle p} \\
B & \longrightarrow & C
\end{array}
$$

We will first prove one lemma which technique we will reuse. A second lemma used to simplify the theorem will be stated without proof.

Lemma 2.2.13. *let A be an A_∞-algebra, and K a contractible complex considered as an A_∞-algebra. If $g : (A, m_1^A) \to (K, m_1^K)$ is a cochain map, then it extends to an ∞-morphism $f : A \rightsquigarrow K$.*

Proof. We construct each f_i inductively. The case $i = 1$ is degenerate as we have assumed $f_1 = g$.

Assume that we have already constructed f_1 through f_n. We observe that the sum below is a cycle of $\mathrm{Hom}_\mathbb{K}^*(A, K)$.

$$
\sum_{\substack{p+1+r=k \\ p+q+r=n}} (-1)^{pq+r} f_k \circ_{p+1} m_q^A - \sum_{\substack{k \geqslant 2 \\ i_1+\ldots+i_k=n}} (-1)^e m_k^B \circ (f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_k})
$$

Thus since K is contractible, $\mathrm{Hom}_\mathbb{K}^*(A, K)$ is acyclic and there exists some morphism f_{n+1} such that $\partial(fn+1)$ is the sum above. This says that this extension does in fact satisfy (rel_{n+1}). $\square$

Lemma 2.2.14. *Let $j : A \rightsquigarrow D$ be a cofibration of A_∞-algberas, then there is an isomorphism $k : D \rightsquigarrow D'$ such that the composition $k \circ j : A \rightsquigarrow D'$ is a strict morphism of A_∞-algebras.*

Dually, if $j : A \rightsquigarrow D$ is a fibration, then there is an isomorphism $l : A' \rightsquigarrow A$ such that the composition $j \circ l : A' \rightsquigarrow D$ is a strict morphism of A_∞-algebras.

Proof. The proof may be found in [Lemma 1.3.3.3 13, p. 44]. □

Proof of Theorem 2.2.12. We start by showing (b). Suppose that we have a diagram of A_∞-algebras, such that g_1 is an epimorphism.

$$
\begin{array}{ccc}
 & & A \\
 & & \downarrow g \\
A' & \xrightarrow{\;f\;} & A''
\end{array}
$$

First notice that as dg-coalgebras, this pullback exists and defines a new dg-coalgebra $BA \prod_{BA''} BA'$.

Since g_1 is an epimorphism, $A[1]$ as a graded vector space splits into $A''[1] \oplus K$, where $K = \mathsf{Ker} g_1$. The pullback is then naturally identified with $BA \prod_{BA''} BA' \simeq \overline{T}^c(K) \prod \overline{T}^c(A'[1])$ as graded vector spaces. Since the cofree coalgebra is right adjoint to forget, it commutes with products and we get, $\overline{T}^c(A'[1]) \prod \overline{T}^c(K) \simeq \overline{T}^c(A'[1] \oplus K)$. Thus the pullback is isomorphic to a cofree coalgebra as a graded coalgebra, i.e. it is an A_∞-algebra.

We now prove (a). **MC1** and **MC2** are immediate, so we will not prove them.

We start by proving **MC3**. Suppose that there is a square of A_∞-algebras as below, where j is a cofibration and q is a fibration.

$$
\begin{array}{ccc}
A & \xrightarrow{\;f\;} & B \\
\downarrow j & & \downarrow q \\
C & \xrightarrow{\;g\;} & D
\end{array}
$$

By Lemma 2.2.14, we may assume assume that both j and q are strict morphisms. We now assume that q is an ∞-quasi-isomorphism, the proof will be analogous if j is an ∞-quasi-isomorphism instead.

Our goal is to construct a lifting in this diagram inductively. Having a lift means finding an ∞-morphism $a : C \rightsquigarrow B$, such that the following hold for any $n \geqslant 1$:

- a satisfy (rel_n).
- $a_n \circ j_1 = f_n$.
- $q_1 \circ a_n = g_n$.

We start by showing there is such an a_1. Consider the diagram below of chain complexes over $\mathbb{K}$.

$$
\begin{array}{ccc}
A & \xrightarrow{\ f_1\ } & B \\
{\scriptstyle j_1}\Big\downarrow & \nearrow{\scriptstyle a_1} & \Big\downarrow{\scriptstyle q_1} \\
C & \xrightarrow{\ g_1\ } & D
\end{array}
$$

The lift exists since the category $\mathsf{Ch}(\mathbb{K})$ is a model category. Here j_1 is a cofibration, while q_1 is an acyclic fibration, so the lift a_1 exists.

We now wish to extend this. Suppose that we have been able to create morphisms a_1 up to a_n, all satisfying the above points. A naive solution to make a_{n+1} is
$b = f_{n+1}r^{\otimes n+1} + sg_{n+1} - sq_1 f_{n+1}r^{\otimes n+1}$, where $r : C \to A$ is a splitting of j_1 and $s : D \to B$ is a splitting of q_1. Notice that this morphism satisfies the two last points by definition. We will augment b to get an a_{n+1} which also satisfies (rel_{n+1}).

For our own convenience, let $-c(f_1, ..., f_n)$ denote the right hand side of (rel_{n+1}) formula. Since both j and q are strict ∞-morphisms we get the following identites:

$$
(\partial b + c(a_1, ..., a_n)) \circ j_1 = \partial(b \circ j_1) + c(a_1 \circ j_1, ..., a_n \circ j_1) = \partial f_{n+1} + c(f_1, ..., f_n) = 0
$$
$$
q_1 \circ (\partial b + c(a_1, ..., a_n)) = \partial(q_1 \circ b) + c(q_1 \circ a_1, ..., q_1 \circ a_n) = \partial g_{n+1} + c(g_1, ..., g_n) = 0
$$

We thus obtain that the cycle $\partial b + c(a_1, ..., a_n)$ factors thorugh the cokernel of j and the kernel of q. Let us say that it factors like the diagram below:

$$
C \xrightarrow{\ p\ } \mathsf{Cok}j_1 \xrightarrow{\ c'\ } \mathsf{Ker}q_1 \xrightarrow{\ i\ } D
$$

Now, c' is a morphism between two A_∞-algebras. Since q is assumed to be an ∞-quasi-isomorphism, it follows that $\mathsf{Ker}q_1$ is acyclic. Since c' is a cycle, it necessarilly have to be in the image of the differential. Let h be a morphism such that $\partial h = c'$, and define $a_{n+1} = b - i \circ h \circ p$. One may check that this morphism satisfies all three properties.

We will now show **MC4**. Since the two properties have a similar proof, we will only show one direction. Let $f : A \rightsquigarrow B$ be an ∞-morphism. Let $C = \mathsf{cone}(id_{B[-1]})$. The complex C may be considered as an A_∞-algebra. Let $j : A \rightsquigarrow A \prod C$ be the morphism induced by id_A and $0 : A \to C$. The canonical projection $q_1 : A \oplus C \to B$ gives a lift of the following diagram.

$$
\begin{array}{ccc}
A & \xrightarrow{\ f_1\ } & B \\
{\scriptstyle j_1}\Big\downarrow & {\scriptstyle q_1}\nearrow & \Big\downarrow \\
A \oplus C & \xrightarrow{\hspace{2em}} & 0
\end{array}
$$

Since we have a morphism of chain complexes, lodged between an acyclic cofibration and a fibration we use the same technique as above to construe an ∞-morphism $q : A \prod C \to B$. q is

a fibration by construction. The morphism f may be factored as $f = q \circ j$, where j is an acyclic cofibration and q is a fibration. $\qquad\square$

With this model structure we are finally able to characterize the fibrant and cofibrant conilpotent dg-coalgebras.

Proposition 2.2.15. *Let C be a conilpotent dg-coalgebra. Then C is cofibrant, and C is fibrant if and only if there is a cochain complex V, such that $C \simeq T^c(V)$ as complexes.*

Proof. To see that C is cofibrant is the same as to verify that the map $\mathbb{K} \to C$ is a monomorphism, but this is clear.

We start by assuming that C is fibrant. Then there is a lift in the square below, making the unit split-mono.

$$
\begin{array}{ccc}
C & \xequal & C \\
{\scriptstyle \eta_C}\downarrow & {\scriptstyle r}\nearrow & \downarrow{\scriptstyle \varepsilon_C} \\
B\Omega C & \xrightarrow{\varepsilon_{B\Omega C}} & \mathbb{K}
\end{array}
$$

Consider the morphism $p_1^C : C \to Fr_1C$ which is defined as $p_1^C = Fr_1r \circ p_1 \circ \eta_C$, where $p_1 : B\Omega C \to Fr_1B\Omega C$ is the canonical projection on the filtration induced by the coradical filtration on C. Clearly, r makes p_1 into a universal arrow in the category of conilpotent coalgebras, so $C \simeq T^c(\overline{Fr_1C})$.

Now, assume that C is isomorphic to $T^c(V)$ as coalgebras for some cochain complex V. Note that, by definition, C is an A_∞-algebra. By definition, we have a commutative square of A_∞-algebras. Since every A_∞-algebra is bifibrant, we know that this diagram has a lift, exhibiting C as a retract of $B\Omega C$.

$$
\begin{array}{ccc}
C & \xequal & C \\
\downarrow & \nearrow & \downarrow \\
B\Omega C & \xrightarrow{} & \mathbb{K}
\end{array}
$$

We know that ΩC is fibrant, since the map $\Omega C \to \mathbb{K}$ is epi. By Lemma 2.2.9, we know that the bar construction preserves fibrations, so $B\Omega C$ is fibrant. Thus C is fibrant as well. $\qquad\square$

The model structure of A_∞-algebras is compatible with the model structure of conilpotent dg-coalgebras in the following sense. If $f : A \rightsquigarrow B$ is an ∞-morphism, we denote its dg-coalgebra counterpart as $Bf : BA \to BB$. Remember that the bar construction is extended such that it is an equivalence of categories on its image. We use this to realize Alg_∞ as a subcategory of $\mathrm{coAlg}_\mathbb{K}$

to essentially obtain two different model structure on this category. The following proposition tells us that these structures do not differ.

Lemma 2.2.16. *Let A and B be A_∞-algebras. Suppose that $f : A \rightsquigarrow B$ is an ∞-morphism and $Bf : BA \to BB$ is a filtered quasi-isomorphism, then f is an ∞-quasi-isomorphism.*

Proof. Given $Bf : BA \to BB$, we may reconstruct $f_i = s \circ \pi_{B[1]} Bf \circ (\omega \circ \iota_A)^{\otimes i}$.

We know that the unit η_{BA} is a filtered quasi-isomorphism from Proposition 2.2.8. The inverse of the bar construction restricts this morphism to the first filtered degree, together with some shift; $B^{-1}\eta_{BA} : A \to \Omega BA$. This is then again a quasi-isomorphism by assumption. □

Proposition 2.2.17. *Let $f : A \rightsquigarrow A'$ be an ∞-morphism. Then we have the following:*

- *f is an ∞-quasi-isomorphism if and only if Bf is a weak equivalence.*
- *f_1 is a monomorphism if and only if Bf is a cofibration.*
- *f_1 is an epimorphism if and only if Bf is a fibration.*

Proof. Suppose that $f : A \to A'$ is an ∞-quasi-isomorphism. By the Kunneth theorem, $Bf : BA \to BA'$ is a filtered quasi-isomorphism.

Suppose that $Bf : BA \to BA'$ is a weak equivalence. Then $B\Omega Bf : B\Omega BA \to B\Omega BA'$ is a filtered quasi-isomorphism. By Proposition 2.2.8, we know that η_{BA} and $\eta_{BA'}$ are both filtered quasi-isomorphism. By Lemma 2.2.16, we get that the ∞-morphisms ΩBf, $B^{-1}\eta_{BA}$ and $B^{-1}\eta_{BA'}$ are ∞-quasi-isomorphisms. By the 2-out-of-3 property we get that f has to be as well.

The cofibrations of $\mathrm{coAlg}^\bullet_{\mathbb{K}, conil}$ are monomorphisms. Since B is an equivalence of categories it necessarilly have to preserve and reflect monomorphisms.

Suppose that Bf is a fibration. Then it has RLP to acyclic cofibrations Bg. By the previous points we know that g_1 is a quasi-isomorphism and a monomorphism, in particular f has RLP to g.

Suppose that f_1 is an epimorphism, and that there exists morphism fitting inside a commutative diagram as below.

$$
\begin{array}{ccc}
BC & \xrightarrow{\ Bh\ } & BA \\
{\scriptstyle Bg}\downarrow & & \downarrow{\scriptstyle Bf} \\
BD & \xrightarrow{\ Bi\ } & BB
\end{array}
$$

Assume that Bg is an acyclic cofibration. We want to show that Bf has RLP to Bg, then Bf has to be a fibration. Notice that BA and BA' are fibrant, so the terminal morphism is a fibration. We find the lifting by considering the following diagram.

$$
\begin{array}{ccc}
BC & \xrightarrow{\;Bh\;} & BA \\
& \searrow{\scriptstyle B\eta_{BA}} & \\
\downarrow{\scriptstyle Bg} & B\Omega BC & \uparrow{\scriptstyle Bf} \\
& {\scriptstyle B\Omega Bg} & \\
BD & \xrightarrow{\;Bi\;} & BB \\
& \searrow{\scriptstyle B\eta_{BD}} & \\
& B\Omega BD &
\end{array}
$$

$\square$

2.3   The Homotopy Category of Alg$_\infty$

With the results we have establish, we are now ready to talk about homotopies in Alg$_\infty$.

Theorem 2.3.1. *In the category Alg$_\infty$ we have the following:*

- *Homotopy equivalence is an equivalence relation.*
- *A morphism is an ∞-quasi-isomorphism if and only if it is a homotopy equivalence.*
- *By abuse of notation, let Alg$_\mathbb{K} \subseteq$ Alg$_\infty$ be the full subcategory consisting of dg-algebras considered as A_∞-algebras. Alg$_\mathbb{K}$ has an induced homotopy equivalence from Alg$_\infty$, and the inclusion Alg$_\mathbb{K} \to$ Alg$_\mathbb{K} \subset$ Alg$_\infty$ induces an equivalence in homotopy Alg$[Qis^{-1}] \simeq$ $^{Alg_\mathbb{K}}/\sim$.*

Proof. The first point is observed from Corollary 2.1.28.2, and the second point is Whitehead's theorem, Theorem 2.1.30.

To see the final point, observe that the inclusion functor is given by the bar construction B. By Corollary 2.2.11.1, we know that the bar construction induces an equivalence on the homotopy categories, i.e. HoAlg $\simeq$ HocoAlg. Moreover, we know that by Theorem 2.1.31 that HocoAlg $\simeq$ $^{Alg_\infty}/\sim$. Notice that the image of B is Alg$_\mathbb{K}$, so in homotopy, we get that the image $^{Alg_\mathbb{K}}/\sim$ is equivalent to the essentiall image HoAlg$_\infty$. $\square$

Homotopy equivalence in the algebraic sense and the model categorical sense differ in how they are defined. In homological algebra, two morphisms are called homotopic if their difference is a boundary of some homotopy, i.e. $f - g = \partial h$. In the model categorical sense, a homotopy is a morphism through either a cylinder or a path object. The following proposition tells us how they differ in the category of conilpotent dg-coalgebras.

Må fikses!

Proposition 2.3.2. *Let C and D be two conilpotent dg-coalgebras, where $f, g : C \to D$ are two morphisms. Then:*

- *If $f - g$ is null homotopic, then they are left homotopic.*
- *If D is fibrant, then $f - g$ is null homotopic if and only if f and g are left homotopic.*

This tells us that in general, these concepts are usually not the same. However, we know that the subcategory of bifibrant objects in this category is exactly the subcategory Alg_∞. Thus for A_∞-algebras homological homotopy is the same as model categorical homotopy. In this sense we obtain that the homotopy category in the homological sense is equivalent to the derived category.

Chapter 3

Derived Categories of Strongly Homotopy Associative Algebras

In this chapter we wish to study the derived categories of A_∞-algebras. This category lies at the heart of homological algebra, so it is only natural to ask how this category looks like in the case of an A_∞-algebra. In the last chapter we studied the relationship between the category of dg-algebras and dg-coalgebras to understand how quasi-isomorphisms between A_∞-algebras worked. In this chapter we will instead study the relationship between module and comodule categories in order to understand how quasi-isomorphisms between A_∞-modules will work. Twisting morphisms $\alpha : C \to A$ will reappear, and they allow us to study the relationship between Mod^A and coMod^C.

From twisting morphisms we obtain functors $L_\alpha : \mathsf{coMod}^C \to \mathsf{Mod}^A$ and $R_\alpha : \mathsf{Mod}^A \to \mathsf{coMod}^C$, which creates an adjoint pair of functors. Whenever the twisting morphism α is acyclic, this adjoint pair will in fact become a Quillen Equivalence.

We wish to reuse all of the methods we have gained and acquired thorughout this thesis. The first part of this chapter will mostly be reformulations and recontextualizations of previous definitions, concepts and techniques.

3.1 Twisting Morphisms

Twisting morphisms were already introduced in chapter 1. There, they were used mostly to be represented by the bar and cobar construction. Now we want twisting morphisms and twisting tensors to play a bigger role. In order to define the functors L_α and R_α, the choice of a given morphism will be crucial.

3.1.1 Twisted Tensor Products

Let A be an augmented dg-algebra, C a conilpotent dg-coalgebra and $\alpha : C \to A$ a twisting morphism. The right (left) twisted tensor product is the complex $C \otimes_\alpha A$ ($A \otimes_\alpha C$) together with the differential $d_\alpha^\bullet = d_{C \otimes A}^\bullet + d_\alpha^r$. The perturbation is defined as

$$d_\alpha^r = (\nabla_A \otimes id_C) \circ (id_A \otimes \alpha \otimes id_C) \circ (id_A \otimes \Delta_C).$$

If M is a right A-module and N is a left C-comodule then the tensor product $M \otimes_\mathbb{K} N$ exists and is a $\mathbb{K}$-module with differential $d_{M \otimes N}$. We may define a perturbation to this differential as

$$d_\alpha^r = (\mu_M \otimes id_N) \circ (id_M \otimes \alpha \otimes id_N) \circ (id_M \otimes \nu_N).$$

By using the same line of thought as in Proposition 1.2.5, there is a twisted tensor product $M \otimes_\alpha N$ with differential $d_\alpha^\bullet = d_{M \otimes N} + d_\alpha^r$.

Remark 3.1.1. Koszul's sign rule forces us to define the differential of the left twisted tensor product as $d_\alpha^\bullet = d_{N \otimes M} - d_\alpha^l$. This is due to the skewness of left twisted tensor products.

Definition 3.1.2. Suppose that $M \in \mathsf{Mod}^A$ ($M \in \mathsf{Mod}_A$) and $N \in \mathsf{coMod}_C$ ($N \in \mathsf{coMod}^C$), then the left (right) twisted tensor product is the $\mathbb{K}$-module $M \otimes_\alpha N$ ($N \otimes_\alpha M$).

In this setting, right handedness and left handedness for the twisted tensor product is clearer, as we only have an action or coaction from one of the chosen sides. Trying to force the other handedness on the twisted tensors would just be ill-defined.

Definition 3.1.3. Let A be an augmented dg-algebra and C a conilpotent dg-coalgebgra, such that there is a twisting morphism $\alpha : C \to A$. Given a linear map $f : N \to M$ between a right C-comodule N and a right A-module M we say that it is an α right twisted linear homomorphism, or just an α twisted morphism, if it satisfies

$$\partial f - f \star \alpha = 0.$$

This definition essentially describes a functor $\mathsf{Tw}_\alpha^r : \mathsf{coMod}^C \times \mathsf{Mod}^A \to \mathsf{Mod}_\mathbb{K}$, which is the collection of right twisted linear homomorphisms between a comodule and module.

Suppose that $\alpha : C \to A$ is a twisting morphism. Define the functor $L_\alpha = \otimes_\alpha A : \mathsf{coMod}^C \to \mathsf{Mod}^A$ as an arbitrary right twisted tensor product with A. This functor does indeed hit Mod^A by using the free right A-module structure on A. Likewise, we define a functor $R_\alpha = \otimes_\alpha C : \mathsf{Mod}^A \to \mathsf{coMod}^C$ as an arbitrary left twisted tensor product with C. This does also hit right C-comodules by using the cofree right C-comodule structure on C.

Proposition 3.1.4. *Suppose that $\alpha : C \to A$ is a twisting morphism. The functor L_α and R_α form an adjoint pair of categories.*

$$\mathsf{coMod}^C \quad \perp \quad \mathsf{Mod}^A$$

with L_α the top arrow and R_α the bottom arrow.

Proof. This proof breaks down to showing $\mathsf{coMod}^C(N, L_\alpha(M)) \simeq \mathsf{Tw}_\alpha(N, M) \simeq \mathsf{Mod}^A(R_\alpha(N), M)$. This is a rutine calculation, much like the proof for 1.2.12. □

Let A be a dg-algebra, and M a right A-module. Recall that by the Cobar-Bar adjunction, Theorem 1.2.12, there exists a universal twisting morphism $\pi_A : BA \to A$. We define the bar construction of M as $B_A M = R_{\pi_A} M = M \otimes_{\pi_A} BA$. Likewise, given a conilpotent dg-coalgebra C and N a right C-comodule we define the cobar construction as $\Omega_C N = L_{\iota_C} N = N \otimes_{\iota_C} \Omega C$. In these cases we obtain adjunctions $\Omega_{BA} \dashv B_A$ and $\Omega_C \dashv B_{\Omega C}$.

Let A and B be two algebras and $f : A \to B$ is an algebra morphism. Then f induces a functor between the module categories by restriction: $f^* : \mathsf{Mod}^B \to \mathsf{Mod}^A$. Since A and B considered as algebroids are small, and the category of abelian groups is cocomplete, the left Kan extension (induction) along this functor exists.

$$\mathsf{Mod}^B \xrightarrow{f^*} \mathsf{Mod}^A$$

with $f_!$ the top arrow.

Dually, if C and D are two coalgebras and $g : C \to D$ is an coalgebra morphism. Then g induces a functor between the module categories by composing: $g_* : \mathsf{coMod}^C \to \mathsf{coMod}^D$. Since C and D considered as coalgebroids are small, and the category of abelian groups is complete, the right Kan extension (co-induction) along this functor exists.

$$\mathsf{coMod}^C \xrightarrow{g_*} \mathsf{coMod}^D$$

with $g^!$ the bottom arrow.

Lemma 3.1.5. *Let $\tau : C \to A$ be a twisting morphism. The adjunction (L_τ, R_τ) factors as $(f_{\tau!}, f_\tau^*) \circ (L_{\iota_C}, R_{\iota_C})$ or $(L_{\pi_A}, R_{\pi_A}) \circ (g_{\tau*}, g_\tau^!)$.*

Proof. This follows from Corollary 1.2.13.1, that is $\tau = f_\tau \circ \iota_C = \pi_A \circ g_\tau$. □

Definition 3.1.6. A twisting morphism $f : C \to A$ is called acyclic if the counit of the adjunction $L_\alpha \dashv R_\alpha$ is a pointwise quasi-isomorphism.

Lemma 3.1.7. *Let A be an augmented dg-algebra and C a conilpotent dg-coalgebra. The universal twisting morphisms π_A and ι_C are acyclic.*

Proof. We start with π_A. Recall that π_A is constructed as the twisting morphism corresponding to id_{BA}. This morphism is thus given as the projection onto the first dimension of BA, that is:

$$\pi_A sa = a$$
$$\pi_A(sa \otimes ...) = 0$$

We say that π_A is acyclic if the counit $\varepsilon : L_{\pi_A} R_{\pi_A} \Rightarrow Id_{\mathsf{Mod}^A}$ at each object M is a quasi-isomorphism.

For each M in Mod^A, $L_{\pi_A} R_{\pi_A} M = M \otimes_{\pi_A} BA \otimes_{\pi_A} A$. We may split up the differential into two summands, d_v and d_h. d_v is the ordinary differential on the tensor product, while $d_h = (-d^l_{\pi_A} \otimes A) + M \otimes d_2 \otimes A + d^r_{\pi_A}$. Since $(d_v + d_h)^2 = 0$ and $d_v^2 = 0$ we can observe that $d_v d_h = -d_h d_v$ and $d_h^2 = 0$. This is evident as d_v changes the homological degree while d_h does not, so in order for both of the first equations to hold, the last two must hold as well. We almost obtain a double complex.

It is clear that the total complex of this "double complex" is in fact $L_{\pi_A} R_{\pi_A} M$. Moreover, the counit induces an augmentation to this complex resolution of M, denoted as $\mathsf{cone}(\varepsilon_M)$.

To see that this is in fact a resolution we define a morphism $h : \mathsf{cone}(\varepsilon_M) \to \mathsf{cone}(\varepsilon_M)$ of degree -1. It works by the following formula:

$$h(m \otimes (sa_1 \otimes ... \otimes sa_n) \otimes a) = m \otimes (sa_1 \otimes ... \otimes sa_n \otimes sa) \otimes 1$$

It is clear that $id_{\mathsf{cone}(\varepsilon_M)} = d_h h - h d_h$ and $d_v h = h d_v$. Thus to see that the cone is acyclic we let $c \in \mathsf{cone}(\varepsilon_M)$ be a cycle, that is $(d_v + d_h)(c) = 0$. Our goal is to show that $h(c)$ is a preimage of c along $d_v + d_h$.

$$(d_v + d_h) \circ h(c) = d_v \circ h(c) + d_h \circ h(c) = h \circ d_v(c) + c + h \circ d_h(c) = h \circ (d_v + d_h)(c) + c = c$$

Next up we show that ι_C is acyclic. Equip C with its coradical filtration, then this induces a filtration $F_p \Omega C$. We will freely use $|\_|$ to denote the filtered degree of every element. This may lead to ambiguity, but we wish to use this notation for clarity.

$$Fr_p C = \{c \mid |c| \leqslant p\}$$
$$f_p \Omega C = \{\langle c_1 \mid \cdots \mid c_n \rangle \mid |c_1| + \cdots + |c_n| \leqslant p\}$$

Let $M \in \mathsf{Mod}^{\Omega C}$, we equip this module with a trivial filtration,

$$F_p M = M.$$

M's associated graded is then quite trivial, $\mathsf{gr}_0 M \simeq M$ and every other $\simeq 0$.

All of these three filtrations together induces a filtration on $L_{\iota_C} R_{\iota_C} M$,

$$F_p L_{\iota_C} R_{\iota_C} M = \{ m \otimes c \otimes \langle c_1 \mid \cdots \mid c_n \rangle \mid |m| + |c| + |c_1| + \cdots + |c_n| \leqslant p \}.$$

We calculate the associated graded of this module.

$$\mathsf{gr}_0 L_{\iota_C} R_{\iota_C} M \simeq M$$
$$\mathsf{gr}_p L_{\iota_C} R_{\iota_C} M \simeq \bigoplus_{i_1 + i_2 = p} M \otimes \mathsf{gr}_{i_1} C \otimes \mathsf{gr}_{i_2} \Omega C$$

The graded counit $\mathsf{gr}_p \varepsilon : \mathsf{gr}_p L_{\iota_C} R_{\iota_C} M \to \mathsf{gr}_p M$ becomes the identity on M when $p = 0$. To see that $\mathsf{gr}\varepsilon$ is a quasi-isomorphism, it is enough to show that $\mathsf{gr}_p L_{\iota_C} R_{\iota_C} M$ is acyclic for every $p \geqslant 1$.

Consider the graded differential component $\mathsf{gr}_p d_{\iota_C}^l$ when it acts as a morphism $\mathsf{gr}_{i_1} C \otimes \mathsf{gr}_{i_2} \Omega C \to \mathsf{gr}_{i_1 + i_2} \Omega C$. This can be considered as a morphism

$$\rho : \bigoplus_{i_1 + i_2 = p} \mathsf{gr}_{i_1} C[-1] \otimes \mathsf{gr}_{i_2} \Omega C \to \mathsf{gr}_p \Omega C,$$

which is an isomorphism by reversing the operation.

$$\rho(sc \otimes \langle \cdots \rangle) = \langle c \mid \cdots \rangle,$$
$$\rho^{-1}(\langle c \mid \cdots \rangle) = sc \otimes \langle \cdots \rangle.$$

Since ρ is an isomorphism, $\mathsf{cone}(\rho)$ is then acyclic. By construction, we have that $\mathsf{cone}(\rho) \simeq \mathsf{gr}_p L_{\iota_C} R_{\iota_C} M$.

$\square$

3.1.2 Model Structure on Module Categories

Let A be an augmented dg-algebra, then we know that Mod^A is a model category. By Corollary 2.2.4.2 we have a model structure where the fibrations, cofibrations and weak equivalences are given as follows:

- $f \in \mathsf{Ac}$ is a weak equivalence if f is a quasi-isomorphism,
- $f \in \mathsf{Fib}$ is a fibration if $f^{\#}$ is an epimorphism,
- $f \in \mathsf{Cof}$ is a cofibration if it has LLP to acyclic fibrations.

Every object in this category is fibrant as the morphism $0 : M \to 0$ is always an epimorphism.

3.1.3   Model Structure on Comodule Categories

Unless stated otherwise, for this section we fix A to be an augmented dg-algebra, C as a conilpotent dg-coalgebra and $\tau : C \to A$ as an acyclic twisting morphism. We endow $\mathsf{coMod}^C_{\mathsf{conil}}$ with three classes of morphisms:

- $f \in \mathsf{Ac}$ is a weak equivalence if $L_\tau f$ is a quasi-isomorphism.
- $f \in \mathsf{Cof}$ is a cofibration if $f^\#$ is a monomorphism.
- $f \in \mathsf{Fib}$ is a fibration if it har RLP to acyclic cofibrations.

Theorem 3.1.8. *The category $\mathsf{coMod}^C_{\mathsf{conil}}$ with the three classes as above form a model category. Every object is cofibrant, and those objects which is a direct summand of $R_\tau M$ for some $M \in \mathsf{Mod}^A$ are fibrant. The adjoint pair (L_τ, R_τ) is a Quilllen equivalence.*

We will call this model structure for the canonical model structure on $\mathsf{coMod}^C_{\mathsf{conil}}$. Under the hypothesis of this theorem, we may observe that every object of $\mathsf{coMod}^C_{\mathsf{conil}}$ is cofibrant. Since every $M \in \mathsf{Mod}^A$ is fibrant, and R_τ preserves fibrant objects we known that $R_\tau M$ is fibrant as well. Clearly every direct summand of $R_\tau M$ is fibrant. If $N \in \mathsf{coMod}^C_{\mathsf{conil}}$ is fibrant, then it is a direct summand of $R_\tau L_\tau N$. This shows that the bifibrant objects of $\mathsf{coMod}^C_{\mathsf{conil}}$ is exactly the thick image of R_τ.

To be able to prove this we will need some lemmata. This proof is essentially the same as the case for dg-coalgebras. The main difference is to show independence of the choice of twisting morphisms τ. To this end we must establish the relationship between graded quasi-isomorphisms and weak equivalences, as well as a technical lemma.

Recall that given a coaugmented coalgebra C we have a filtration called the coradical filtration, defined as $Fr_i C = \mathsf{Ker}(\bar{\Delta}_C)^i$. If N is a right C-comodule we may define the coradical filtration of N as $Fr_i N = \mathsf{Ker}(\bar{\omega}_N^i)$. This filtration is admissable, meaning it is exhaustive and $Fr_0 N = 0$.

Lemma 3.1.9. *Let C be a conilpotent dg-coalgebra, M and N be right C-comodules. Then any graded quasi-isomorphism $f : M \to N$ is a weak equivalence.*

Proof. This proof is identical to Lemma 2.2.6. □

Lemma 3.1.10. *Let M and N be two objects of Mod^A. The functor R_τ sends a quasi-isomorphism $f : M \to N$ to a weak equivalence $R_\tau f : R_\tau M \to r_\tau N$.*

The unit of the adjunction $\eta : \mathsf{Id}_{\mathsf{coMod}^C} \to R_\tau L_\tau$ is a pointwise weak equivalence.

Proof. $R_\tau f$ is a weak equivalence if $L_\tau R_\tau f$ is a quasi-isomorphism. By naturality of the counit we have the following commutative diagram.

$$
\begin{array}{ccc}
M & \xleftarrow{\;\varepsilon_M\;} & L_\tau R_\tau M \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle L_\tau R_\tau f} \\
N & \xleftarrow{\;\varepsilon_N\;} & L_\tau R_\tau N
\end{array}
$$

By assumption we know that all three of f, ε_M and ε_N are quasi-isomorphisms. It follows by the 2-out-of-3 property that $L_\tau R_\tau f$ is a quasi-isomorphism as well.

To show that $\eta : \mathrm{Id}_{\mathsf{coMod}} \to L_\tau R_\tau$ is a pointwise weak equivalence, we must show that $L\eta$ is a pointwise quasi-isomorphism. Since L_τ is left adjoint to R_τ we know that η is split on the image of L_τ, i.e.

$$
\varepsilon_{L_\tau} \circ L_\tau \eta = id_{L_\tau}
$$

Since we know that the natural isomorphisms ε and id are pointwise quasi-isomorphisms, we get by the 2-out-of-3 property that $L\eta$ is a pointwise quasi-isomorphism as well. $\qquad\square$

Lemma 3.1.11. *The functor L_τ preserves cofibrations and sends weak-equivalences to quasi-isomorphisms.*

Proof. This proof is essentially the same as Lemma 2.2.9. $\qquad\square$

With the above lemmata, we have now established that the adjunction (L_τ, R_τ) form a Quillen equivalence if coMod^C is a model category.

Lemma 3.1.12 ([Lemma 2.2.2.9 13, p. 74]). *Let M be a right A-module and N a right C-comodule. Let $p : M \to L_\tau N$ be a fibration of modules. The projection $j : R_\tau M \prod_{R_\tau L_\tau N} N \to R_\tau M$ is an acyclic cofibration of comodules.*

Proof. Let $K = \mathrm{Ker}\, p$. Then since R_τ is a right adjoint, it preserves kernels, so $R_\tau K \simeq \mathrm{Ker} R_\tau p$. Consider the pullback-square with the horizontal kernels

$$
\begin{array}{ccccc}
R_\tau K & \rightarrowtail & RM \prod_{R_\tau L_\tau N} N & \twoheadrightarrow & N \\
\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \eta_N} \\
R_\tau K & \rightarrowtail & R_\tau M & \longrightarrow & R_\tau L_\tau N
\end{array}
$$

Since $L_\tau N$ is a quasi-free module, we get that $M \simeq K \oplus L_\tau N$ as a graded module. In other words, the short exact sequences above are split when considered as graded sequences. If we apply L_τ this sequence, then L_τ has to preserve exactness at the graded level, since it is additive. Thus we obtain a morphism of exact sequences, and $L_\tau j$ is a quasi-isomorphism by 5-Lemma.

$$L_\tau R_\tau K \rightarrowtail L_\tau(RM \textstyle\prod_{R_\tau L_\tau N} N) \twoheadrightarrow L_\tau N$$

$$\downarrow \simeq \qquad\qquad \downarrow j \qquad\qquad\qquad \downarrow \eta_N$$

$$L_\tau R_\tau K \rightarrowtail L_\tau R_\tau M \longrightarrow L_\tau R_\tau L_\tau N$$

$\square$

Proof of Theorem 3.1.8. With the above lemmata established, this proof is identical to the proof of Theorem 2.2.11. $\qquad\square$

3.1.4 Triangulation of Homotopy Categories

In this section we will show that the homotopy categories are triangulated. If we look at the category Mod^A we will observe that the category HoMod^A is our beloved derived category $\mathcal{D}(A)$. It is not quite the same for the cateogry coMod^C. Here we want $\mathsf{HocoMod}^C$ to be equivalent to the derived category of a ring, so we will see that the derived category is a further localization of $\mathsf{HocoMod}^C$.

Furthermore, by employing the theory of triangulated categories we will show that the model structure on coMod^C is independent on the choice of acyclic twisting morphism. This breaks down to show that every acyclic twisting morphism induce an equivalence between derived categories, as done by Bernhard Keller in [28].

Mod^A is a an abelian category, where we employ the maximal exact structure $\mathcal{E}'$ consisting of short exact sequences in Mod^A. This translates to short exact sequences which is short exact in each degree. However, this category also has an exact structure $\mathcal{E}$ which makes Mod^A into a Frobenius category, which we will now describe.

Let $f : M \to N$, be a chain map from M to N. Then $\mathcal{E}$ contains a conflation on the form:

$$N \rightarrowtail \mathsf{cone}(f) \twoheadrightarrow M[1]$$

We define $\mathcal{E}$ to be the smallest exact structure on Mod^A which contains every conflation arising from a chain map f. Observe that these conflations are exactly the short exact sequences of Mod^A such that they are split when regarded as graded modules, i.e. forgetting the differential. Thus the smallest such $\mathcal{E}$ is exactly the collection of every conflation arising from a chain map f.

Recall that an object M is projective (injective) if the represented functor $\mathsf{Mod}^A(M, \_)$ ($\mathsf{Mod}^A(\_, M)$) is exact. For the category $(\mathsf{Mod}^A, \mathcal{E})$

Proposition 3.1.13. *Let M be an object of Mod^A. The following are equivalent:*

- M is $\mathcal{E}$-projective
- M is $\mathcal{E}$-injective
- M is contractible

Proof. This is a well known statement from literature. See Krause [29], Happel [30] or Buehler [31] for an account of this result. □

To see that $(\mathsf{Mod}^A, \mathcal{E})$ has both enough projectives and injectives we consider the following conflation:

$$M \rightarrowtail \mathsf{cone}(id_M) \twoheadrightarrow M[1]$$

The complex $\mathsf{cone}(id_M)$ is contractible for any complex M. By letting M vary we can find an inflation or deflation from the identity cone to or from any complex. This concludes that $(\mathsf{Mod}^A, \mathcal{E})$ is a Frobenius category.

Let $\underline{\mathsf{Mod}}^A$ denote the injectively stable module category. Let $I(M, N)$ denote the set of chain maps from M to N which factors through an injective object. We define the injectively stable category as the quotient of abelian groups $\underline{\mathsf{Mod}}^A(M, N) = {}^{\mathsf{Mod}^A(M,N)}/_{I(M,N)}$.

Theorem 3.1.14. *Suppose that $(\mathcal{C}, \mathcal{E})$ is a Frobenius category, then the injectively stable category $\underline{\mathcal{C}}$ is triangulated. The additive auto-equivalence is given by cozyzygy, and the standard triangles is the image of the conflations into the quotient.*

Proof. This is well known in literature. An account for it may also be found in Krause [29], Happel [30] or Buehler [31]. □

We thus obtain a triangulated category $\underline{\mathsf{Mod}}^A$ associated to the Frobenius pair $(\mathsf{Mod}^A, \mathcal{E})$. This category is commonly denoted as $K(A)$, and we will do this as well. Notice that with the structure given by $\mathcal{E}$, the cozyzygy is defined by the shift functor $[1]$. Every standard triangle is also on the form:

$$M \xrightarrow{f} N \longrightarrow \mathsf{cone}(f) \longrightarrow M[1]$$

To define the derived category $D(A)$ of A we will consider the localization of $K(A)$ at the quasi-isomorphisms, $D(A) = K(A)[\mathsf{Qis}^{-1}]$. To see that the derived category is still triangulated we may realize it as a Verdier quotient of $K(A)$.

Proposition 3.1.15. *The derived category of A is equivalent to the Verdier quotient ${}^{K(A)}/_{\mathsf{Ac}}$, where Ac denotes the image of acyclic objects in $K(A)$.*

Proof. A proof may be found in Buehler [31]. $\square$

There is another way of telling the story of the derived category $D(A)$. That is to directly localize it at the quasi-isomorphisms. We may directly see that $D(A) \simeq \mathsf{Mod}^A[\mathsf{Qis}^{-1}]$ which we know is HoMod^A by definition. This gives us our first important identification.

Theorem 3.1.16. *The homotopy category of Mod^A is triangulated, and moreover it is the derived category $D(A)$.*

Proof. Follows from discussion above. $\square$

The triangulated construction for the category $\mathsf{HocoMod}^C$ closely resembles that of HoMod^A. We start by studying the Frobenius pair $(\mathsf{coMod}^C, \mathcal{E})$, where $\mathcal{E}$ is the same exact structure. Notice that this exact structure only takes the underlying category of chain complexes into account, so this follows from the above description.

We define the injectively stable category $\underline{\mathsf{coMod}}^C = K(C)$ in the same manner. The standard triangles and the additive auto-equivalence stays the same.

At this point things start to differ. The definition for the homotopy category $\mathsf{HocoMod}^C$ is $\mathsf{coMod}^C[\mathsf{Ac}^{-1}]$, here Ac denotes the class of weak equivalences in coMod^C. By abuse of notation we also let $\mathsf{Ac} \subset K(C)$ be the collection of objects which are cones of weak equivalences. This sub-category can be characterized by being the preimage of acyclic objects $\mathsf{Ac} \subset K(A)$ along $L_\tau : \mathsf{coMod}^C \to \mathsf{Mod}^A$. To see this look at the image of the triangle where the cone is in Ac. This identification suffices to show that $\mathsf{Ac} \subset K(C)$ is a triangulated subcategory. In this manner $\mathsf{HocoMod}^C$ is the category $K(C)/_{Ac}$, which is a triangulated category.

Remark 3.1.17. We may show that $\mathsf{Ac} \subset K(C)$ is a subcategory of acyclic objects. In this manner we get that $D(C) \simeq \mathsf{HocoMod}^C[\mathsf{Qis}^{-1}]$. This is done in Lefevre-Hasegawa as [Proposition 1.3.5.1 13, p. 51] [Lemma 2.2.2.11 13, p. 75]. This follows from the fact that we have an equivalence of categories $\mathsf{coMod}^C[\mathsf{fQis}^{-1}] \simeq \mathsf{HocoMod}^C$, here fQis means the collection of filtered quasi-isomorphisms. Since every filtered quasi-isomorphism is in fact a quasi-isomorphism by a spectral sequence argument we get the inclusion of triangulated subcategories $\langle \mathsf{cone}(\mathsf{fQis}) \rangle \subseteq \langle \mathsf{cone}(\mathsf{Qis}) \rangle \subseteq K(C)$.

Let $\tau : C \to A$ and $\upsilon : C \to A'$ be two acyclic twisting morphisms. These independently defines two different model structures on coMod^C by the adjunctions (L_τ, R_τ) and (L_υ, R_υ). By Lemma 3.1.5 we have the identification $(L_\tau, R_\tau) = (f_{\tau!}, f_\tau^*)(L_{\iota_C}, R_{\iota_C}) = (f_{\tau!}L_{\iota_C}, R_{\iota_C}f_\tau^*)$, and likewise for υ. In order to show that τ and υ defines equivalent model structures on coMod^C, it is enough that both define the same structure as ι_C. By symmetry, we may assume that $\upsilon = \iota_C$. From Lemma 3.1.7, we know that ι_C is acyclic, so this assumption is well-founded.

Since we already know that (L_τ, R_τ) and $(L_{\iota_C}, R_{\iota_C})$ are Quillen equivalences, it remains to show that $(f_{\tau!}, f_\tau^*)$ is a Quillen equivalence. This is shown if f_τ^* is a right Quillen functor, and that it induces a triangle equivalence between $D(A)$ and $D(\Omega C)$.

We know that f_τ^* preserves fibrations (epimorphisms). This is because on morphisms, this functor acts as the identity. It only changes the ring action, so epimorphisms stay as epimorphisms. It remains to see that quasi-isomorphisms are preserved. We will show this by identifying the derived categories. This follows the methods given by Keller in [28]. We will however simplify this discussion by restricting our attention solely to dg-algebras.

Let A be a dg-algebra. A is then free in the enriched sense; i.e. for any right A-module M, $\mathrm{Hom}_A^\bullet(A, M) \simeq M$. Recall that P is projective if it is a direct summand of A^n for some $n \in \mathbb{N}$.

Given a right bounded complex M, we know how to construct a projective resolution $p : pM \to M$. Associated to this resoultion there is a triangle in $K(\mathbb{K})$ consisting of the complexes M, pM and aM, where aM is an acyclic complex.

$$M \xrightarrow{\ p\ } pM \longrightarrow aM \longrightarrow M[1]$$

In this sense we obtain an identification $M \simeq pM$ in $D(\mathbb{K})^-$. By following Kellers construction we are able to weaken this identification to all of $D(\mathbb{K})$, by weakening the structure of the projective resolution. In Kellers paper he calls these, complexes of property (P). We will however refer to them as homotopy projective complexes, since these complexes are built up from projective complexes in a manner respecting homotopy colimits.

Definition 3.1.18. Let P be a complex of Mod^A. We say that P is homotopy projective if there exists a complex P', a homotopy equivalence $P \simeq P'$ and a filtration of P'.

$$0 = F_0 \subseteq F_1 \subseteq ... \subseteq F_n \subseteq ... \subseteq P'$$

The filtration should satisfy these properties:

   F1 P' is the colimit of the filtration.
   F2 Each inclusion $i_n : F_n \subseteq F_{n+1}$ is split as graded modules.
   F3 The quotient F_{n+1}/F_n is projective.

Remark 3.1.19. The properties $F1$ and $F2$ may be reformulated to require that P' should be the homotopy colimit of the filtration. Thus there is a canonical triangle in $K(A)$:

$$\bigoplus F_n \xrightarrow{\ \Phi\ } \bigoplus F_n \longrightarrow P' \longrightarrow \bigoplus F_p[1]$$

Φ is given as the unique morphism which acts as the identity and the inclusion on each summand of $\bigoplus F_p$:

$$\Phi_n = \begin{pmatrix} id_{F_n} \\ -i_n \end{pmatrix}$$

In the definition of a homotopy projective complex we have required that each quotient is strictly projective. If only this was true, then these objects would be ill-behaved in the homotopy category. We can weaken this assumption to (F3'), the quotient F_{n+1}/F_n is homotopy equivalent to a projective complex.

Lemma 3.1.20. *If P is the colimit of a filtration admitting (F2) and (F3'), then P is homotopy projective.*

Proof. Let $\{F_n\}$ denote the filtration on P. Showing that P is homotopy projective is the same as finding a homotopy equivalence to a complex P', such that P' is the homotopy colimit of a filtration admitting (F3).

Suppose that $F_{n+1}/F_n \simeq Q_{n+1}$, where each Q_{n+1} is projective. We wish to inductively define a filtration $\{F'_n\}$ which has (F2) and (F3) and a pointwise homotopy equivalence of filtrations $f : \{F_n\} \to \{F'_n\}$. The object P' is then defined to be the (homotopy) colimit of this new filtration.

Define $F'_0 = Q_0$, and let $f_0 : F_0 \to F'_0$ be the projection onto Q_0. By assumption f_0 is a homotopy equivalence and we have a commutative square where the vertical arrows are homotopy equivalences. Moreover, each horizontal arrow splits as a graded arrow.

$$
\begin{array}{ccc}
0 & \xrightarrow{0} & F_0 \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f_0} \\
0 & \xrightarrow{0} & Q_0
\end{array}
$$

Suppose that we have been able to constructed this filtration up to F'_p. By using our known homotopy equivalences there is an isomorphism of Ext groups:

$$\mathsf{Ext}_A\big(F_p/F_{p-1}, F_{p-1}\big) \simeq \mathsf{Ext}_A(Q_p, F'_{p-1})$$

Given the triangle consisting of F_{p-1}, F_p and F_p/F_{p-1} there is an assosiated triangle with the morphisms as follows:

$$
\begin{array}{ccccccc}
F_{p-1} & \longrightarrow & F_p & \longrightarrow & F_p/F_{p-1}1111 & \longrightarrow & F_{p-1}[1] \\
\downarrow{\scriptstyle f_{p-1}} & & \downarrow & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle f_{p-1}[1]} \\
F'_{p-1} & \longrightarrow & F'_p & \longrightarrow & Q_p & \longrightarrow & F'_{p-1}
\end{array}
$$

By the morphism axiom there is a morphism $f_p : F_p \to F'_p$ which is also a homotopy equivalence by the 2-out-of-3 property.

This defines a filtration $\{F'_p\}$, with (F3) and P' as its homotopy colimit. To see that P is homotopy equivalent to P' we use the maps f_p constructed to obtain a homotopy equivalence by the morphism axiom and the 2-out-of-3 property.

$$
\begin{array}{ccccccc}
\bigoplus F_p & \xrightarrow{\ \Phi\ } & \bigoplus F_p & \longrightarrow & P & \longrightarrow & \bigoplus F_p[1] \\
\downarrow{\scriptstyle \oplus f_p} & & \downarrow{\scriptstyle \oplus f_p} & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \oplus f_p[1]} \\
\bigoplus F'_p & \xrightarrow{\ \Phi'\ } & \bigoplus F'_p & \longrightarrow & P' & \longrightarrow & \bigoplus F'_p[1]
\end{array}
$$

$\square$

The projective complexes are the complexes which are generated by the free module A in the sense that they are all in the smallest thick triangulated subcategory of $K(A)$ containing A. By definition, we may see that the homotopy projective complexes are the complexes in the smallest thick triangulated subcategory of $K(A)$ which is closed under well-ordered homotopy colimits and contains $K(A)$. By devissage we may extend fully-fatihfullness of functors on the set $\{A\}$ to the class of homotopy projective objects.

Lemma 3.1.21 (Devissage)**.** *Let $F : \mathcal{T} \to \mathcal{U}$ be a triangulated functor between triangulated categories. Suppose $S \subseteq \mathcal{T}$ is a class of objects closed under shift, and denote $\langle S \rangle$ for the smallest thick triangulated subcategory (closed under well-ordered homotopy colimits). If $F|_S$ is fully faithful, then $F|_{\langle S \rangle}$ is fully faithful as well.*

Proof. This is straightforward by using the Yoneda embeddings and 5-lemma. More details may be found in [29]. To get closed under homotopy colimits we also need that F commutes with infinite direct sums, and that the set $\{S\}$ only contains small objects. $\square$

Lemma 3.1.22. *Supppose we have F and S as above. If $F|_S = 0$, then it is 0 on all of $\langle S \rangle$.*

Proof. The same argument as above, except we have to squeeze out zeros from exact sequences. $\square$

The final ingredient to construct a homotopy projective resolution for our complexes is the acyclic assembly lemma [24].

Lemma 3.1.23 (acyclic assembly)**.** *Suppose that C is a double complex of R-modules. Then $Tot^{\oplus}C$ is acyclic if either:*

- *C is a lower half-plane complex with exact rows.*
- *C is a left half-plane complex with exact columns.*

Proof. This is proposition 2.7.3 in [24]. We omit the proof as the next proof is in some sense very similar. □

Corollary 3.1.23.1. *Suppose that C is a double complex of R-modules such that every column is exact and that the kernels along the rows give rise to exact columns, then $Tot^{\oplus}C$ is acyclic.*

Proof. We want to realize the images along the rows as the coimage along the horizontal differential. Write $Z^n(C)$ for the n-th horizontal kernel and $B^n(C)$ for the n-th horizontal image. We have a short exact sequence of complexes:

$$Z^n(C)^* \longrightarrow C^{n,*} \longrightarrow B^n(C)^*$$

Given that $C^{n,*}$ is acyclic we get that $Z^n(C)^*$ is acyclic if and only if $B^n(C)^*$ is acyclic.

Assuming that all of these three constructions are acyclic we make a filtration on C. Let $F_n C^{p,*} = C$ if $p \in [-n, n-1]$, $F_n C^{n,*} = Z^n C$ and $F_n C^{p,*} = 0$ otherwise.

This filtration is bounded below and exhaustive as colimits commute with colimits.

$$Tot^{\oplus}C = Tot^{\oplus} \varinjlim F_n C \simeq \varinjlim Tot^{\oplus} F_n C$$

We should be a bit careful here as the total complex is not really a coproduct, but since coproducts and cokernels are calculated pointwise we obtain the commutativity.

Now we apply the classical convergence theorem to the filtration to obtain a converging spectral sequence $EF_2 C \implies H^*(Tot^{\oplus}C)$. But since we assume each column to be exact in the filtration, the second page is 0, so $H^*(Tot^{\oplus}C) \simeq 0$ as desired. □

Theorem 3.1.24. *Suppose that P is homotopy projective, N is acyclic. Then $K(A)(P, N) \simeq 0$.*

Given any module M, there is a homotopy projective object pM and an acyclic object aM giving rise to a triangle in $K(A)$.

$$pM \longrightarrow M \longrightarrow aM \longrightarrow pM[1]$$

Proof. We assume that $P \simeq A$. By a devissage argument we may extend the isomorphism to all homotopy projective P.

$$K(A)(A, N) \simeq H^0 \mathsf{Hom}_A^{\bullet}(A, N) \simeq H^0 N \simeq 0$$

We want to construct two complexes pM and aM by taking total complexes. We show that aM is acyclic by using Corollary 3.1.23.1. To use it we will construct an exact sequence of complexes

satisfying the assumptions. As described by MacLane [32], there is an exact structure $\mathcal{E}$ on Mod^R such that the collections on conflations are the short exact sequences such that the kernel functor is exact.

$$L \rightarrowtail^{f} M \xrightarrow{g} N$$

$$Z^*L \rightarrowtail^{Z^*f} Z^*M \xrightarrow{Z^*g} Z^*N$$

Since limits commute with limits, the kernel functor preserves any limit. Thus the kernel is left exact, and its only obstruction for exactness is to preserve cokernels. We may thus characterize the conflations by inflations and deflations, which are monomorphisms and epimorphisms which are preserved by the kernel functor. Mac Lane calls these deflations for proper epimorphisms instead.

We want to construct $\mathcal{E}$-projectives be on the form of homotopy projective complexes. $A[-n]$ is $\mathcal{E}$-projective by the following isomorphism.

$$Z^0\text{Hom}^\bullet_A(A[-n], M) \simeq M^n$$

Define the trivialization $\text{triv}M$ of M be the underlying graded module M endowed with a trivial differential. This trivial differential is in some sense the inclusion of graded modules into chain complexes. Thus we have the following isomorphism on hom-sets:

$$Z^i\text{Hom}^\bullet_A(\text{triv}M, \text{triv}N) \simeq \text{Hom}^i_A(M, N)$$

triv is then well-defined as a functor as every morphism between chain complexes uniquely defines a morphism between their trivializations. By using the isomorphisms from Keller [28] 2.2. we get that:

$$Z^0\text{Hom}^\bullet_A(\text{cone}(id_{\text{triv}A}), M) \simeq Z^0\text{Hom}^\bullet_A(\text{cone}(id_{\text{triv}A[-1]})[1], M)$$
$$\simeq \text{Hom}^*_A(\text{triv}A, \text{triv}M[-1])^0 \simeq \text{Hom}^*_A(A, M)^{-1} \simeq M^{-1}.$$

This shows that if P is homotopy projective, then P and $\text{cone}(id_{\text{triv}P})$ are $\mathcal{E}$-projective. To see that there are enough $\mathcal{E}$-projectives pick an arbitrary module M. Since we know that there are enough projectives, let P be a projective such that there is an epimorphism $p : P \to M$. We don't know if this morphism is a deflation, so pick another projective Q such that there is an epimorphism $q : Q \to Z^*M$. Since Z^*M has a trivial differential we know that $d_Q q = 0$. Thus this morphism extends to $q' = \begin{bmatrix} q & 0 \end{bmatrix} : \text{cone}(id_{\text{triv}Q}) \to M$ such that Z^*q' is an epimorphism. The morphism $\begin{bmatrix} p & q' \end{bmatrix} : P \oplus \text{cone}(id_{\text{triv}Q}) \to M$ is thus a deflation. $P' = P \oplus \text{cone}(id_{\text{triv}Q})$ shows that we have enough projectives. Moreover every $\text{cone}(id_{\text{triv}Q})$ is contractible, so $P' \simeq P$ in $K(A)$.

Since we have enough $\mathcal{E}$-projective, we may construct a $\mathcal{E}$-projective resolution $P'^{*,*}$ of M in the standard way. See Keller [33] for details. Such resolutions are then double complexes, and the augmented resolution below is $\mathcal{E}$-acyclic.

$$\ldots \longrightarrow P_1' \longrightarrow P_0' \longrightarrow\!\!\!\!\!\rightarrow M \xrightarrow{\ 0\ } 0$$

Having an $\mathcal{E}$-acyclic resolution means that each row is exact, and taking kernels along the columns preserve exactness of the rows.

Denote the augmentation of $P'^{*,*}$ by $m : P'^{,*} \to M$. We define the complexes $pM = Tot^{\oplus}(P'^{*,*})$ and $aM = Tot^{\oplus}(\mathsf{cone}(m))$.

pM carries a natural filtration $F_n pM$ from the double complex structure. Let $F_n pM$ be the truncated complex:

$$\ldots \longrightarrow 0 \longrightarrow P'^{n,*} \longrightarrow \ldots \longrightarrow P'^{1,*} \longrightarrow P'^{0,*} \longrightarrow 0 \longrightarrow \ldots$$

The filtration $F_n pM$ satisfies F1 and F2 by construction. The quotients $F_{n+1}pM/_{f_n pM} \simeq P_n'$ which is homotopy equivalent to a projective. By Lemma 3.1.20 pM is homotopy projective.

The complex $\mathsf{cone}(m)$ satisfies the conditions for Corollary 3.1.23.1, thus aM is acyclic and we have a triangle in $K(A)$ as desired. $\qquad\square$

Corollary 3.1.24.1. *Let M be an erbitrary module. If P is homotopy projective, then $K(A)(P, M) \simeq K(A)(P, pM)$. If N is acyclic, then $K(A)(M, N) \simeq (aM, N)$.*

a and p are well-defined functors which commutes with infinite direct sums.

Corollary 3.1.24.2. *Let $\langle A \rangle$ denote the smallest thick triangulated subcategory of $D(A)$ which is closed under homotopy colimits and contains $\{A\}$. Then $D(A) \simeq \langle A \rangle$.*

Corollary 3.1.24.3. *Suppose that $f : A \to B$ is a dg-algebra homomorphism and a quasi-isomorphism between the dg-algebras, then $D(A) \simeq D(B)$.*

Proof. f endows B with both a left and right A-module structure. We will think of B as a left A-module and right B module. There is then a natural hom-tensor adjunction between the differential graded enriched categories.

$$\mathsf{Mod}^A \underset{f^*}{\overset{f_!}{\rightleftarrows}} \mathsf{Mod}^B \qquad \perp$$

The restriction functor f^* can naturally be identified with the hom functor $\mathrm{Hom}^\bullet_A(B, \_)$, and then it is evident to realize $f_!$ as $\_ \otimes_A B$. In this way, $f_!(A) \simeq B$, so $f_! : \mathrm{Hom}^\bullet_A(A, A) \to \mathrm{Hom}^\bullet_B(B, B)$ is given by f. Since we assume f to be a quasi-isomorphism, it follows that $\mathbb{L}f_! : D(A) \to D(B)$ is fully faithful on $\{A\}$.

By devissage the functor $\mathbb{L}f_!$ is fully-faithful on all of $D(A)$, since $D(A) \simeq \langle A \rangle$. As $f_!$ hits all of $D(B)$'s generators, $\mathbb{L}f_!$ is essentially surjective as well. $\qquad\square$

Remark 3.1.25. We have ignored smallness conditions for objects. This technique does not always work, since it depends on some unstated isomorphisms which we have implicitly used. We have these since the objects A and B are small. This detail is given more care in Keller [28].

With this result we are able to show that HoMod^A and $\mathrm{HoMod}^{\Omega C}$ are equivalent. Since we assumed the morphism $\tau : C \to A$ to be acyclic, we would expect the morphism $f_\tau : \Omega C \to A$ to be a quasi-isomorphism. If this is the case, we know that $D(\Omega C) \simeq D(A)$.

3.1.5 The Fundamental Theorem of Twisting Morphisms

In this section we aim to finish what we started the previous section. We will prove a characterization for the acyclic twisting morphisms.

Theorem 3.1.26 (Fundamental Theorem of Twisting Morphisms). *Let $\tau : C \to A$ be a twisting morphism between augmented objects. The following are equivalent:*

1. *τ is acyclic, i.e. the natural transformation $\varepsilon : L_\tau R_\tau \implies Id_{\mathsf{Mod}^A}$ is a pointwise quasi-isomorphism.*
2. *The unit transformation $\eta : Id_{\mathsf{coMod}^C} \implies R_\tau L_\tau$ is a pointwise weak equivalence.*
3. *The counit at A is a quasi-isomorphism, i.e. $\varepsilon_A : L_\tau R_\tau A \to A$ is a quasi-isomorphism.*
4. *The unit at $\mathbb{K}$ is a weak-equivalence, i.e. the algebra unit v_A and coaugmentation v_C assembles into a weak-equivalence: $v_A \otimes v_C : \mathbb{K} \to A \otimes_\tau C$.*
5. *The morphism of algebras $f_\tau : \Omega C \to A$ is a quasi-isomorphism.*
6. *The morphism of coalgebras $g_\tau : C \to BA$ is a weak-equivalence.*

Proof. Notice that 1. is equivalent to 2. since $\mathbb{L}L$ and $\mathbb{R}R$ are quasi-inverse. 3. is a special case of 1. and 4. is a special case of 2. Observe that 5. and 6. are equivalent since the cobar-bar-adjunction is a Quillen equivalence, which is Corollary 2.2.11.1.

We show 3. $\implies$ 1. Let $\mathcal{T} \subseteq D(A)$ be the full subcategory consisting of objects M where ε_M is a quasi-isomorphism. This subcategory is by assumption non-empty and contains A. By the 5-lemma, making triangles (and smallness of A), this subcategory contains the smallest thick triangulated subcategory closed under homotopy colimits which contains A. We know this to be all of $D(A)$.

To show 4. implies 5. we consider the twisting morphism ι_C. Since ι_C is acyclic we know that the counit at A is a quasi-isomorphism.

$$L_{\iota_C} R_{\iota_C} f_\tau^* A \to f_\tau^* A$$

By assumption the unit morphism $\eta_{\mathbb{K}} : \mathbb{K} \to A \otimes_\tau C$ is a weak equivalence, so the morphism $L_{\iota_C} \eta_{\mathbb{K}} : \Omega C \to L_{\iota_C} R_\tau A = L_{\iota_C} R_{\iota_C} f_\tau^* A$ is a quasi-isomorphism. Let ε' denote the counit of $L_{\iota_C} \dashv R_{\iota_C}$, then we see that $f_\tau = \varepsilon'_A \circ L_{\iota_C} \eta_{\mathbb{K}}$, so f_τ is a quasi-isomorphism by the 2-out-of-3 property.

It remains to show that 5. implies 1. Let the counit of $f_{\tau*} \dashv f_\tau^*$ be denoted as $\tilde{\varepsilon}$. Since f_τ is a quasi-isomorphism, f_τ^* descends to an equivalence between the derived categories, which is Corollary 3.1.24.3. Thus $\tilde{\varepsilon} : f_{\tau!} f_\tau^* \implies \mathrm{Id}$ is a pointwise quasi-isomorphism. Observe that the counit factors as

$$\varepsilon = \tilde{\varepsilon} \circ f_{\tau!} \varepsilon'_{f_\tau^*}$$

By the 2-out-of-3 property it follows that ε is a quasi-isomorphism. $\square$

Corollary 3.1.26.1. *There is one canonical model structure on coMod^C defined by the acyclic twisting morphisms $\tau : C \to A$, for any algebra A. I.e. each acyclic twisting morphism defines the same model structure for coMod^C.*

Proof. Apply the fundamental theorem of twisting morphisms to the discussion of the last section. $\square$

3.2 Polydules

3.2.1 The Bar Construction

In Section 1.3 we saw that we could extend the domain of the bar construction to obtain an equivalence of categories. This converse led us to the definition of an A_∞-algebra, as well as recognizing them as quasi-free dg-coalgebras. By employing the adjunction $(L_\tau, R_\tau) : \mathsf{coMod}^C \rightleftharpoons \mathsf{Mod}^A$ we can do something similar for modules.

Let A be an augmented dg-algebra. The bar construction of A gives us a universal adjunction $(L_{\pi_A}, R_{\pi_A}) : \mathsf{coMod}^{BA} \rightleftharpoons \mathsf{Mod}^A$. We will call $R_{\pi_A}(\_[1]) = \_[1] \otimes_{\pi_A} BA$ for B_A, the bar construction on Mod^A. In this manner every A-module M give rise to a quasi-free BA-comodule $B_A M$, but does the converse of this construction work?

Let us first look at what B_A does to an A-module M. $B_A M$ is the dg-comodule which as a graded comodule is the free comodule $M[1] \otimes BA$. The differential of $B_A M$ is given by the A-module

structure of M. That is, every elementary element m' of $B_A M$ is an element of M together with a finite string of elements of A.

$$m' = [m \mid\mid a_1 \mid ... \mid a_n]$$

The differential acts on m' by using the differential of $d_{M[1]\otimes BA}$ and multiplication from the right.

$$d_{B_A M}(m') = d_{M[1]\otimes BA}(m') + (-1)^{|m|+|a|}[m \cdot a_1 \mid\mid a_2 \mid ... \mid a_n]$$

By using delooping, we see that in turn that $d_{B_A M}$ defines an A-module structure for M. We may decompose $B_A M$ as:

$$B_A M = M[1] \oplus M[1] \otimes \bar{A} \oplus M[1] \otimes \bar{A}^{\otimes 2} \oplus ...$$

Let $\pi_M : R_{\pi_A} M \to M$ be the linear map which kills anything not on the form $[m]$. We denote $(d_{B_A M})_i$ by $d_{B_A M} \circ \iota_i$, where $\iota_i : M[-1] \otimes \bar{A}^{\otimes i-1} \hookrightarrow B_A M$. Proposition 1.1.43 tells us that we may recover the structure of M from the differential $d_{B_A M}$. This is done by conjugating the components of $d_{B_A M}$ with desuspension and applying projections appropriately. We recover the maps as:

1. The differential of M is $d_M = s \circ \pi_{M[1]} \circ (d_{B_A M})_1 \omega$
2. The right multiplication from A is $\mu_M = s \circ \pi_{M[-1]} \circ (d_{B_A M})_2 \circ \omega^{\otimes 2}$
3. For $i \geqslant 3$ we have $0 = s \circ \pi_{M[1]} \circ (d_{B_A M})_i \circ \omega^{\otimes i}$

Now, let $\widetilde{N}$ be a quasi-free BA-comodule. That is, $\widetilde{N} = N[1] \otimes BA$ as a graded comodule. We would now like that N carries an A-module structure. Unfortunately, this does not happen in general. However, like in the case of algebras, this defines a notion of A_∞-modules to the algebra A. If we try to recover the same structure we obtain the following structure morphisms for N:

$$\text{A differential of degree } 1: m_1 = d_N = s \circ \pi_N (d_{\widetilde{N}})_1 \circ \omega$$

$$\text{A 2-ary operation of degree } 0: m_2 = s \circ \pi_N (d_{\widetilde{N}})_2 \circ \omega^{\otimes 2}$$

$$\text{A 3-ary operation of degree } -1: m_3 = s \circ \pi_N (d_{\widetilde{N}})_3 \circ \omega^{\otimes 3}$$

$$\text{A 4-ary operation of degree } -2: ...$$

Let $\widetilde{m}_i$ be the looped versions of the m_i. Then the sum $\sum \widetilde{m}_i : \widetilde{N} \to N[1]$ extends to $d_{B_A N}$ by Proposition 1.1.43, i.e.

$$d_{B_A N} = \left(\sum \widetilde{m}_i \otimes id_{BA}\right)(id_N \otimes \Delta_{BA}) + N[1] \otimes d_{BA}.$$

Since $d_{B_A N}^2 = 0$ we get the relations (rel_n) as defined in Section 1.3 imposed on the morphisms m_i. We summarize this in the next definition.

Definition 3.2.1 (A-polydule)**.** Let A be a dg-algebra and M be a graded $\mathbb{K}$-module. We say that M is a right A-polydule if there are morphisms

$$m_i : M \otimes A^{\otimes i-1} \to M \tag{3.1}$$

of degree $|m_i| = 2 - i$ for any $i \geqslant 1$. Furthermore, the morphisms should satisfy the relations

$$(rel_n) \qquad \partial(m_n) = - \sum_{\substack{n=p+q+r \\ k=p+1+r \\ k>1, q>1}} (-1)^{pq+r} m_k \circ_{p+1} m_q^?,$$

where $m_q^?$ is meant as either m_q or m_q^A, that which is appropriate.

A left A-polydule is defined analogously. If M is an A-polydule, then it has the structure of an A-module where associativity is only well-defined up to strong homotopy. In other words, m_3 is a homotopy for the associator for m_2, m_4 is a homotopy for the associator of m_3 and so on.

The category of A-polydules is denoted as Mod_∞^A. We have defined its objects in correspondance to the bar construction, thus every object has been uniquely defined from a quasi-free $B(A^+)$-comodule. Likewise, we will uniquely define every morphism to come from $B(A^+)$-comodule morphisms. In this manner B_{A^+} defines a fully faithful functor $B_{A^+} : \mathsf{Mod}_\infty^A \to \mathsf{coMod}^{B(A^+)}$ which is an isomorphism on the full subcategory of quasi-free $B(A^+)$-comodules.

Definition 3.2.2 (∞-morphisms)**.** Let A be a dg-algebra, and let M and N be two right A-polydules. We say that $f : M \rightsquigarrow N$ is an ∞-morphism if there are morphisms

$$f_i : M \otimes A^{\otimes i-1} \to N$$

of degree $|f_i| = 1 - i$ for any $i \geqslant 1$. Furthermore, the morphism should satisfy the relations

$$(rel_n) \qquad \sum_{p+q+r=n} (-1)^{pq+r} f_{p+1+r} \circ_{p+1} m_q^M = \sum_{p+q=n} m_{p+1}^N \circ_1 f_q$$

Suppose that we have the A-polydules M, N and P. If $f : M \rightsquigarrow N$ and $g : N \rightsquigarrow P$ are ∞-morphisms, then their composition is defined as

$$(gf)_n = \sum_{p+q=n} g_{p+1} \circ_1 f_q.$$

To illustrate what the bar construction does, suppose that $f : M \rightsquigarrow N$ is an ∞-morphism. The bar construction on f is then defined as
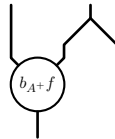
where $b_{A^+} f = \sum s \circ f_i \circ \omega^{\otimes i}$.

There is a natural inclusion on objects $i : \mathsf{Mod}^A \to \mathsf{Mod}_\infty^A$. This functor acts as the identity on each object, letting every higher $m_i = 0$:

$$i : \mathsf{Mod}^A \to \mathsf{Mod}_\infty^A,$$
$$(M, d_M, \mu_M) \mapsto (M, d_M, \mu_M, 0, 0, \cdots).$$

Suppose that $f : M \to N$ is a morphism between the A-modules M and N. Then this defines an ∞-morphism $i \circ f : M \rightsquigarrow N$, such that $if_1 = f$ and $if_n = 0$ for every $n \geqslant 2$. Thus $i : \mathsf{Mod}^A \to \mathsf{Mod}_\infty^A$ is a functor.

Definition 3.2.3 (strict ∞-morphisms)**.** Let $f : M \rightsquigarrow N$ be an ∞-morphism. We say that it is strict if $f_i = 0$ for every $i \geqslant 2$.

The category $\mathsf{Mod}_{\infty,strict}^A$ is the non-full subcategory of Mod_∞^A such that every ∞-morphism are strict.

We will give some examples of A-polydules given an augmented algebra A. ⟞ !!!

3.2.2 Polydules of SHA-algebras

In the last section we developed the notion of a polydule to a augmented and ordinary algebras. We extend this notion to any A_∞-algebra.

Suppose that A is an A_∞-algebra. Recall the bar construction BA, this is a quasi-free coalgebra on the form

$$BA = \bigoplus_{i=1}^\infty A[1]^{\otimes i},$$

where the differential comes from the $m_i : A^{\otimes i} \to A$. In order to define the A-polydules, we will consider the quasi-free comodules in coMod^{BA}. This will be completely analogous to how it worked for ordinary dg-algbras.

Definition 3.2.4 (A-polydule)**.** Let A be an A_∞-algebra, and M a graded $\mathbb{K}$-module. We say that M is a right A-polydule if there exists morphisms

$$m_i : M \otimes A^{\otimes i-1} \to M,$$

where the degree $|m_i| = 2 - i$ for any $i \geqslant 1$. Furthermore, the morphisms should satisfy the relations

$$(rel_n) \qquad \partial(m_n) = - \sum_{\substack{n=p+q+r \\ k=p+1+r \\ k>1, q>1}} (-1)^{pq+r} m_k \circ_{p+1} m_q.$$

Definition 3.2.5 (∞-morphisms)**.** Let A be an A_∞-algebra, and let M and N be two right A-polydules. We say that $f : M \rightsquigarrow N$ is an ∞-morphism if there are morphisms

$$f_i : M \otimes A^{\otimes i-1} \to N$$

of degree $|f_i| = 1 - i$ for any $i \geqslant 1$. Furthermore, the morphism should satisfy the relations

$$(rel_n) \qquad \sum_{p+q+r=n} (-1)^{pq+r} f_{p+1+r} \circ_{p+1} m_q^M = \sum_{p+q=n} m_{p+1}^N \circ_1 f_q$$

Definition 3.2.6. Let A be an A_∞-algebra. The category Mod_∞^A has A-polydules as objects and ∞-morphisms as morphisms.

The quasi-isomorphisms in Mod_∞^A are the ∞-morphisms f such that f_1 is a quasi-isomorphism.

Remark 3.2.7. The isomorphisms of Mod_∞^A are the ∞-morphisms f where f_1 is an isomorphism.

We say that an ∞-morphism is strict if $f_i = 0$ for any $i \geqslant 2$. The category $\mathsf{Mod}_{\infty,strict}^A$ is the non-full subcategory of Mod_∞^A restricted to strict ∞-morphisms.

Suppose now that A is instead a strictly unital A_∞-algebra, see Definition (1.3.12). We may define strictly unital A-polydules as an A-polydule M such that

$$m_2^M \circ (id_M \otimes v_A) = id_M$$
$$\forall i \geqslant 3 \quad m_i^M \circ (id_M \otimes ... \otimes v_A \otimes ... \otimes id_A) = 0$$

An ∞-morphism $f : M \rightsquigarrow N$ is strictly unital if

$$\forall i > 2 \quad f_i(id_M \otimes ... \otimes v_A \otimes ... \otimes id_A) = 0$$

We define the categories of strictly unital polydules with stricly unital morphisms suMod_∞^A and $\mathsf{suMod}_{\infty,strict}^A$. These categories are non-full subcategories of Mod_∞^A.

Given an augmented A_∞-algebra A, see Definition (1.3.13), we obtain an equivalence of categories. Recall that the categories Alg_∞ and $\mathsf{Alg}_{\infty,+}$ were equivalent by taking the kernel of augmentation and applying the free augmentation as its quasi-inverse. In the same manner, given a strictly unital A-polydule M, then it defines a strictly unital $\bar{A}$-polydule $\bar{M}$ by restricting the structure maps to $\bar{A}^{\otimes n}$. This defines an equivalence of categories.

We may call its quasi-inverse for the free strict unitization. This takes an $\overline{A}$-polydule M and turns it into a strictly unital A-polydule by defining the structure morphism as 0 on the unit.

The reduced bar construction allows us to translate an A-polydule M to a quasi-free BA-comodule. We let $\overline{B}_A M = M[1] \otimes BA$, together with the differential coming from each $m_n : M \otimes A^{\otimes n-1} \to M$

$$d_{\overline{B}_A M} = (\sum \widetilde{m}_i \otimes id_{BA})(id_{M[1]} \otimes \Delta_{BA}) + id_{M[1]} \otimes d_{BA} = d_m + id_{M[1]} \otimes d_{BA}.$$

Likewise, we may take a quasi-free BA-comodule to obtain an A-polydule by doing the reverse bar construction, as it is done in Proposition 1.1.43.

We will mostly restrict our attention to augmented A_∞-algebras. The reason for this is that if A is an arbitrary A_∞-algebra, then studying Mod_∞^A would be the same as studying $\mathsf{suMod}_\infty^{A^+}$. We extend the bar construction along this equivalence to a fully faithful functor $B_A : \mathsf{suMod}_\infty^A \to \mathsf{coMod}^{B\overline{A}}$. By abuse of equivalence we may write $B_{A^+} : \mathsf{Mod}_\infty^A \to \mathsf{coMod}^{BA}$.

We may also lift homotopies between quasi-free BA-comodules and A-polydules. A homotopy $B_{A^+}h : B_{A^+}M \to B_{A^+}M$ is a morphism of degree -1. Thus the collection $h_n : M \otimes A^{\otimes n-1} \to N$ has morphisms of degree $-i$. Moreover, $h : M \rightsquigarrow N$ defines a homotopy of $f, g : M \rightsquigarrow N$ if we have

$$f_n - g_n = \sum_{p+q} (-1)^p m_{p+1}^N \circ_1 h_q - \sum_{p+q+r=n} (-1)^{pq+r} h_{p+1+r} \circ_{p+1} m_q^M$$

We say that a homotopy is strictly unital if it is a strictly unital ∞-morphism.

3.2.3 Universal Enveloping Algebra

Given any augmented A_∞-algebra A, there is a universal enveloping algebra UA. This algebra is universal in the sense that given any augmented algebra A' and an ∞-morphism $A' \to A$, then this factors through UA by an algebra map $A' \to UA$. By the cobar-bar adjunction there is essentially only one way to define this algebra.

Definition 3.2.8. Let A be an A_∞-algebra. The universal enveloping algebra is the algebra defined as ΩBA.

Remark 3.2.9. In this definition we have used the extended bar construction to A_∞-algebras and the cobar construction on dg-coalgebras.

Lemma 3.2.10. *There is an isomorphism of categories* $i : \mathsf{Mod}^{UA} \to \mathsf{suMod}_{\infty,strict}^A$ *given by delooping.*

Proof. This is immediate by the definition of a UA-module. To have a UA-module $M[1]$, we must have structure maps $m_i^M : M \otimes A^{\otimes i-1} \to M$ of degree $2 - i$ for any $i \geqslant 2$. Unwinding this definition and using the adjunction data establishes this isomorphism. $\qquad\square$

We can generalize the universal enveloping algebra to the case of A_∞-algebras. This construction is very non-trivial and requires to use the notion of the universal enveloping algebra relative to an operad. The necessary definitions may be found in Kriz and May's paper "Operads, algebras, modules and motives" [34].

Given an A_∞-algebra, we will denote its universal enveloping algebra UA. We have the following proposition due to Kriz and May.

Proposition 3.2.11 ([Proposition 4.10 34, p. 19])**.** *Let A be an A_∞-algebra. There is an equivalence of categories*

$$i : Mod^{UA} \to suMod^A_{\infty, strict}.$$

With the established equivalences we can now pull the model structure on Mod^{UA} onto $\mathsf{suMod}^A_{\infty,\text{strict}}$. Recall that this is the model structure as defined in Theorem 2.2.1.

3.2.4   Bipolydules

For ordinary algebras A and A', an A-A'-bimodule M may serve as a kind of morphism from Mod^A to $\mathsf{Mod}^{A'}$. This is used in conjunction with the tensor product to form the correct functors. We will now look at this idea for A_∞-algebras.

Definition 3.2.12 (A-A'-Bipolydule)**.** Suppose that A and A' are A_∞-algebras, and that M is a graded $\mathbb{K}$-module. M is an A-A'-bipolydule if there are morphisms

$$m_{i,j} : A^{\otimes i} \otimes M \otimes A'^{\otimes j} \to M,$$

such that the degree $|m_{i,j}| = 1 - i - j$ for any $i, j \geqslant 0$. Furthermore, the morphisms should satisfy the relations

$$(rel_n) \qquad \sum_{\substack{n=p+q+r \\ p+1+r=s+t \\ q=u+v \\ s,t,u,v \geqslant 0}} (-1)^{pq+r} m_{s,t} \circ_{p+1} m_{u,v} = 0$$

Definition 3.2.13 (Strictly Unital A-A'-Bipolydule)**.** Suppose that A and A' are stricly unital A_∞-algebras, and that M is an A-A'-bipolydule. We say that M is strictly unital if

$$m_{i,j}(id^{\otimes p} \otimes \upsilon_? \otimes id^{\otimes q}) = 0;$$

where $?$ is either A or A', $p \neq i$ and $(i, j) \neq (0, 1)$ nor $(i, j) \neq (1, 0)$. Lastly,

$$m_{1,0}(\upsilon_A \otimes id_M) = m_{0,1}(id_M \otimes \upsilon_{A'}) = id_M.$$

A morphism of bipolydules is a bit more complicated than for normal right polydules. This is because the left module structure induces some more signs.

Definition 3.2.14 (∞-morphisms)**.** Let A and A' be two A_∞-algebras and let M and N be two A-A'-bipolydules. An ∞-morphism $f : M \rightsquigarrow N$ is a collection of morphisms

$$f_{i,j} : A^{\otimes i} \otimes M \otimes A'^{\otimes j} \to N,$$

where the degree $|f_{i,j}| = -i - j$ for any $i, , \geqslant 0$. Furthermore, the morphisms should satisfy the following relations

$$(rel_n) \qquad \sum_{\substack{n=p+q+r \\ q=s+t}} (-1)^{p(-s-t)} m_{p,q} \circ_{p+1} f_{s,t} = \sum_{n=p+q+r} (-1)^{pq+r} f_{p,r} \circ_{p+1} m_q^?,$$

where $m_q^?$ is meant to mean the appropriate structure morphism.

This definition is well-defined. If $m_q^?$ is supposed to mean $m_{q_1,q_2} : A^{\otimes q_1} \otimes M \otimes B^{\otimes q_2} \to M$, then q_1 and q_2 are not uniquely determined. However, the sum will span every possibility of q_1 and q_2.

We say that an ∞-morphism is strict if $f_{0,0}$ is the only non-zero component.

The polydules assemble into categories $\mathsf{Mod}_{A,\infty}^{A'}$, $\mathsf{Mod}_{A,\infty,strict}^{A'}$, $\mathsf{suMod}_{A,\infty}^{A'}$ and $\mathsf{suMod}_{A,\infty,strict}^{A'}$ like in the usual sense. These definitions may seem somewhat more complicated. However, they almost reduce to the ordinary case by considering the category $\mathsf{coMod}^{BA^{op} \otimes BA'}$. We may derive a 2-sided bar-construction $B_{A+-A'+} : \mathsf{Mod}_{A,\infty}^{A'} \to \mathsf{coMod}_{BA}^{BA'}$. However, we know that $\mathsf{coMod}_{BA}^{BA'} \simeq \mathsf{coMod}^{BA^{op} \otimes BA'}$. In this manner we may argue about bipolydules with the techniques we have developed for comodules.

3.2.5   A Tensor and Hom on Mod_∞^A

In order to understand the category Mod_∞^A better we would like to construct a tensor product and a hom-functor on it. In its most generality, the tensor will be a bifunctor:

$$\_ \otimes_{A'}^\infty \_ : \mathsf{Mod}_{A,\infty}^{A'} \otimes \mathsf{Mod}_{A',\infty}^{A''} \to \mathsf{Mod}_{A,\infty}^{A''}.$$

In the usual sense, given a bipolydule $M \in \mathsf{Mod}_{A,\infty}^{A'}$, then it will act as a morphism

$$\_ \otimes_A^\infty M : \mathsf{Mod}_\infty^A \to \mathsf{Mod}_\infty^{A'}.$$

In particular, this functor will be a left adjoint to its corresponding hom-functor. In its most general form, the hom functor will be a bifunctor:

$$\mathsf{Hom}_{A'}^\infty : \mathsf{Mod}_{A,\infty}^{A'} \otimes \mathsf{Mod}_{A',\infty}^{A''} \to \mathsf{Mod}_{A,\infty}^{A''}.$$

We start by describing the tensor product in the simplest case. Let A be an A_∞-algebra, and let M and N be a right and left A-polydule respectively. We define $M \otimes_A^\infty N$ as a cochain complex

$$M \otimes_A^\infty N = M \otimes T^c(A[1]) \otimes N.$$

Its structure comes from the cotensor product of quasi-free coalgebras. Consider instead the right and left BA dg-comodules $B_{A^+}M = M[1] \otimes BA$ and $B_{A^+}N = BA \otimes N[1]$.

$$B_{A^+}M \,\square_{BA}\, B_{A^+}N = \mathsf{Ker}(\omega^r_{B_{A^+}M} \otimes B_{A^+}N - B_{A^+}M \otimes \omega^l_{B_{A^+}N})$$

Then $B_{A^+}M \,\square_{BA}\, B_{A^+}N$ is a $\mathbb{K}$ dg-module. By taking the cotensor, we restrict our attention to solely those parts of this tensor in which comulitplication from the left is the same as comultiplication from the right. An element may then be seen to be of the form

$$[m \,||\, a_1 \mid \cdots \mid a_n] \otimes [n]$$
$$+[m \,||\, a_1 \mid \cdots \mid a_{n-1}] \otimes [a_n \,||\, n]$$
$$+\cdots$$
$$+[m \,||\, a_1] \otimes [a_2 \mid \cdots \mid a_n \,||\, n]$$
$$+[m] \otimes [a_1 \mid \cdots \mid a_n \,||\, n].$$

There is an evident isomorphism to $M[1] \otimes BA \otimes N[1]$ by sending each of the elements above to the elements

$$[m \,||\, a_1 \mid \cdots \mid a_n \,||\, n].$$

Its differential is induced by the restriction of the differential on the cochain-complex $B_{A^+}M \otimes B_{A^+}N$. Since $d_{B_{A^+}M \otimes B_{A^+}N}$ is well-defined on each element in $B_{A^+}M \otimes B_{A^+}N$, the restricted differential $d_{mM} \otimes id_{N[1]} + id_{M[1]} \otimes d_{BA} \otimes id_{N[1]} + id_{M[1]} \otimes d_{mN}$ on $M[1] \otimes BA \otimes N[1]$ is well defined as well.

Definition 3.2.15 (The tensor product). Let A be an A_∞-algebra, and let M and N be respectively a right and a left A-polydule. The tensor $M \otimes_A^\infty N = M \otimes BA \otimes N$ is a cochain complex with differential

$$-(s \otimes id_{BA} \otimes s)(d_{mM} \otimes id_{N[1]} + id_{M[1]} \otimes d_{BA} \otimes id_{N[1]} + id_{M[1]} \otimes d_{mN})(\omega \otimes id_{BA} \otimes \omega).$$

An element of $M \otimes_A^\infty N$ may be written on the form

$$m[a_1 \mid \cdots \mid a_n]n.$$

Given A-polydules M, M', N and N' and ∞-morphisms $f : M \rightsquigarrow M'$ and $g : N \rightsquigarrow N'$, we define $f \otimes_A^\infty g$ as

$$f \otimes_A^\infty g(m[a_1 \mid \cdots \mid a_n]n) = \sum_{p+q+r=n+2} (-1)^s f_p(m, a_1, \cdots)[\cdots]g_r(\cdots, a_n, n),$$

where s is the appropriate sign derived from Koszul's sign rule. Note that as a $\mathbb{K}$-polydule, this morphism is a strict ∞-morphism. This fact will not change, even in the more general cases.

We will extend this tensor to bipolydules. Suppose that N now has the structure of an A-A'-bipolydule. The cotensor $B_{A^+}M \,\square_{BA}\, B_{A^+-A'^+}N \simeq (B_{A^+}M \,\square_{BA}\, B_{A^+}N) \otimes T^c(A'[1])$ as graded

comodules. When we thus recover the structure morphisms, we may recover them at $T^c(A'[1])$. In other words, $m_{0,n} : N \otimes A'^{\otimes n-1} \to N$ induces morphisms $m_n : M \otimes_A^\infty N \otimes A'^{\otimes n-1} \to M \otimes_A^\infty N$. Thus, given a bipolydule such as N we obtain a functor

$$\_ \otimes_A^\infty N : \mathsf{Mod}_\infty^A \to \mathsf{Mod}_\infty^{A'}.$$

We will now describe the hom functor in the simplest case. Let A be an A_∞-algebra, and let M and N be right A-polydules. We define $\mathsf{Hom}_A^\infty(M, N)$ as a cochain complex

$$\mathsf{Hom}_A^\infty(M, N) = \mathsf{Hom}_{BA}^\bullet(B_{A^+}M, B_{A^+}N).$$

Its differential is the usual hom differential, i.e. given $f \in \mathsf{Hom}_{BA}^\bullet(B_{A^+}M, B_{A^+}N)$ then

$$\partial f = d_{B_{A^+}N} \circ f - (-1)^{|f|} f \circ d_{B_{A^+}M}.$$

Functoriality is given by post- and pre-composition in the usual sense for dg-comodules. If we are given ∞-morphisms, we will rather consider the dg-comodule counterpart and define functoriality purely through that. Because of this, when we regard this as $\mathbb{K}$-polydule, post- and pre-composition is a strict ∞-morphism. This will also hold in the more general case which we will soon get to.

In order to be able to get to a more complicated case, we first need a new way to encode the data of an A-polydule. The $\mathbb{K}$-module $\mathsf{Hom}_{BA}(B_{A^+}M, B_{A^+}N)$ carries a natural bimodule structure. There are actions on $\mathsf{Hom}_{BA}(B_{A^+}M, B_{A^+}N)$ on the right from the dg-endomorphism algebra $\mathsf{End}(B_{A^+}M)$, and on the left from $\mathsf{End}(B_{A^+}N)$ by composition. If we consider these dg-algebras as A_∞-algebras, then we may give $\mathsf{Hom}_{BA}(B_{A^+}M, B_{A^+}N)$ the structure of a bipolydule. The following lemma connects representations of A_∞-algebras to A-polydules.

Lemma 3.2.16 (Representation lemma, [Lemme 5.3.0.1 13, p. 140]). *Let A be an A_∞-algebra, and let M be a graded $\mathbb{K}$-module. The following are equivalent:*

- *There is an ∞-morphism of A_∞-algebras $\phi : A \rightsquigarrow \mathsf{End}(M)$,*
- *M is a left A-polydule.*

Proof. We will only establish the bijection map. A proof of well-definedness may be found in the original source.

The bijection is given the transpose of tensor. Notice that as $\mathbb{K}$-linear morphisms we have the following bijections

$$\mathsf{Hom}_{\mathbb{K}}(A^{\otimes n-1}, \mathsf{End}(M)) \simeq \mathsf{Hom}_{\mathbb{K}}(A^{\otimes n-1} \otimes M, M).$$

Thus if $\phi : A \to \mathsf{End}(M)$ is an ∞-morphism, then we may define

$$m_n : A^{\otimes n-1} \otimes M \to M$$
$$(a_1 \otimes \cdots \otimes a_{n-1}) \otimes m \mapsto \phi(a_1 \otimes \cdots \otimes a_{n-1})(m).$$

On the other hand, if we have structure morphisms $m_n : A^{\otimes n-1} \otimes M \to M$, then we may define ϕ by uncurrying:

$$\phi_n : A^{\otimes n} \to \mathsf{End}(M),$$
$$a_1 \otimes \cdots \otimes a_n \mapsto (m \mapsto m_{n+1}(a_1 \otimes \cdots \otimes a_n \otimes m)).$$

$\square$

Remark 3.2.17. This lemma is well-known and even holds in many other aspects. One may for example recognize this as in representation theory of finite groups. A more general account on this lemma may be found as [Proposition 5.2.2. 3, p. 139].

Corollary 3.2.17.1. *Let A and A' be two A_∞-algebras, and let M be an A-A'-bipolydule. Then there is an A_∞-morphism $\phi : A \rightsquigarrow \mathsf{End}(B_{A'+}M)$. In particular, any $\mathsf{End}(B_{A'+}M)$-modules is an A-polydule.*

Proof. By Lemma 3.2.16 we obtain the ∞-morphism $\phi : A \rightsquigarrow \mathsf{End}(B_{A'+}M)$ by transposing the structure morphisms

$$m_{i,j} : A^{\otimes i} \otimes M \otimes A'^{\otimes j} \to M.$$

In other words,

$$\phi_n : A^{\otimes n} \to \mathsf{End}(B_{A'+}M),$$
$$a_1 \otimes \cdots \otimes a_n \mapsto ($$
$$[m \,||\, a'_1 \,|\, \cdots \,|\, a'_l] \mapsto d_{B_{A^+-A'+}M} \circ (\omega^{\otimes n} \otimes id_{M[1]} \otimes id_{A'[1]}^{\otimes l})(a_1 \otimes \cdots \otimes a_n \otimes [m \,||\, a'_1 \,|\, \cdots \,|\, a'_l])).$$

$\square$

We are now ready to describe the hom-functor. Suppose that A and A' are A_∞-algebras, and that M is an A-A'-polydule and N a right A'-polydule. We define the A-polydule

$$\mathsf{Hom}_{A'}^\infty(M, N) = \mathsf{Hom}_{BA'}^\bullet(B_{A'+}M, B_{A'+}N),$$

with structure map $\phi : A \rightsquigarrow \mathsf{End}(B_{A'+}M)$ defined by the above corollary. In this way we obtain a functor

$$\mathsf{Hom}_{A'}^\infty(M, \_) : \mathsf{Mod}_\infty^{A'} \to \mathsf{Mod}_\infty^A.$$

Lemma 3.2.18 (Hom-Tensor adjunction, [Lemme 4.1.1.4 13, p. 115])**.** *Let A and A' be two A_∞-algebras and M an A-A'-bipolydule. There is an adjoint pair of functors*

$$
\begin{array}{ccc}
 & \xrightarrow{\;\_\otimes_A^\infty M\;} & \\
\mathit{Mod}_\infty^A & \perp & \mathit{Mod}_\infty^{A'} \\
 & \xleftarrow[\mathit{Hom}_{A'}^\infty(M,\_)]{} &
\end{array}
$$

Proof. We establish the natural bijection. We refer to [13, Lemme 4.1.1.4] to see that it is well-defined.

Consider an ∞-morphism $f : L \otimes_A^\infty M \rightsquigarrow R$ of right A'-polydules. By consider the bar construction of A', this morphism is in correspondance with $B_{A'+}f : L \otimes_A^\infty B_{A'+}M \to B_{A'+}R$. Through the ordinary tensor-hom adjunction we get a correspondance $f_i^T : L \otimes A^{\otimes i} \to \mathrm{Hom}_{BA'}(B_{A'+}M, B_{A'+}R)$.

$\square$

3.2.6 Homologically Unital SHA-Algebras and Polydules

In this section we will define the notion of homologically unital A_∞-algebras and polydules. These notions will be weaker than strictly unitary objects, but their definition may be easier to work with. As we will see, these notions almost coincide up to homotopy. This section will be given without proofs.

If A is an A_∞-algebra, or M is an A-polydule, we will use H^*A and H^*M to denote their homology. Note that H^*A is an associative algebra, as m_i for $i \geqslant 3$ are homotopies, witnessing associativity of H^*m_2. In the same manner, H^*M, becomes a H^*A-module, by considering $\mathrm{H}^*m_2^M$.

Definition 3.2.19 (Homologically unital A_∞-algebra)**.** Let A be an A_∞-algebra. A morphism $v_A : \mathbb{K} \to A$ is called a homological unit, if $\mathrm{H}^*v_A : \mathbb{K} \to \mathrm{H}^*A$ is a unit in homology. We say that A equipped with a homological unit v_A is a homologically unital A_∞-algebra.

An ∞-morphism $f : A \rightsquigarrow A'$ is homologically unital if it preserves the unit in homology, i.e. $\mathrm{H}^*f : \mathrm{H}^*A \to \mathrm{H}^*A'$ is also a morphism of graded algebras.

Given two ∞-morphisms $f, f' : A \rightsquigarrow A'$, then they are homotopically unital if there is a homotopy $h : A \rightsquigarrow A'$ between f and f' which is strictly unital with respect to the homological unit v_A.

We let suAlg_∞ denote the non-full subcategory of strictly unital A_∞-algebras with strictly unital ∞-morphisms, huAlg_∞ denote the non-full subcategory of homologically unital A_∞-algebras with homologically unital ∞-morphism, and uAlg_∞ denote the full subcategory of strictly unital A_∞-algebras with ∞-morphisms. Note that if A is a strictly unital A_∞-algebra, then it is also homologically unital. Thus we see that $\mathrm{suAlg}_\infty \subseteq \mathrm{huAlg}_\infty$.

In order to obtain a stronger relationship between homologically unital A_∞-algebras and strictly unital A_∞-algebras we need the notion of a minimal model.

Definition 3.2.20 (Minimal SHA-algebra/polydule)**.** Let A be an A_∞-algebra, and M an A-polydule. We say that A is minimal if $m_1^A = 0$, and likewise M is minimal if $m_1^M = 0$

Definition 3.2.21 (Minimal model)**.** Let A and A' be A_∞-algebras. We say that an ∞-quasi-isomorphism $f : A' \rightsquigarrow A$, is a minimal model of A.

Theorem 3.2.22 ([Corollaire 1.4.1.4 13, p. 54])**.** *Let A be an A_∞-algebra. The injection from the homology H^*A into A is a minimal model of A.*

Proof. We will only construct the first component of this injection.

Since $\mathsf{Mod}^{\mathbb{K}}$ is semi-simple, A splits naturally as $A \simeq \mathsf{H}^*A \oplus K$. By definition K is acyclic, and the inclusion $\mathsf{H}^*A \to A$ is a quasi-isomorphism.   □

We now state the following relationship between homologically unital and strictly unital A_∞-algebras.

Theorem 3.2.23 ([Theoreme 3.2.1.1 13, p. 99])**.** *Any minimal homologically unital A_∞-algebra is isomorphic to a minimal strictly unital A_∞-algebra.*

Corollary 3.2.23.1 (Unital strictification of A_∞-algebras, [Corollaire 3.2.1.2 13, p. 99])**.** *Any homologically unital A_∞-algebra is homotopy equivalent to a stricly unital A_∞-algebra.*

Proof. This is obtained by combining the two above theorems, i.e. Theorem 3.2.22 and Theorem 3.2.23.   □

Theorem 3.2.24 (Unital strictification of ∞-morphisms, [Theoreme 3.2.2.1 13, p. 103])**.** *An homologically unital ∞-morphism of strictly unital minimal A_∞-algebras is homotopic to an strictly unital ∞-morphism.*

Theorem 3.2.25 (Unital strictification of homotopies, [Theoreme 3.2.3.1 13, p. 104])**.** *Let A and A' be two minimal strictly unital A_∞-algebras. Let $f, g : A \rightsquigarrow A'$ be strictly unital ∞-morphisms which are homotopic, then there is a stricly unital homotopy witnessing them being homotopic.*

Corollary 3.2.25.1. *Let A and A' be two A_∞-algebra, and let $f : A \rightsquigarrow A'$ be a strictly unital homotopy equivalence. Then there is a strictly unital homotopy equivalence $g : A' \rightsquigarrow A'$, with strictly unital homotopies witnessing that g is the homotopy inverse of f.*

With the above results we learn that the homotopic information of strictly unital A_∞-algebras is essentially controlled by strictly unital ∞-morphism. In other words the non-full inclusion $\mathsf{suAlg}_\infty \to \mathsf{uAlg}_\infty$ induces an equivalence of categories

$$\mathsf{suAlg}_\infty \big/ \sim \; \simeq \; \mathsf{uAlg}_\infty \big/ \sim.$$

We also get that the unital strictification of homologically unital A_∞-algebras induces an equivalence

$$\mathsf{huAlg}_\infty \big/ \sim \; \simeq \; \mathsf{suAlg}_\infty \big/ \sim.$$

We also have similar results for polydules.

Definition 3.2.26. Let A be a homologically unital A_∞-algebra, and let M be an A-polydule. We say that M is homologically unital if H^*M is a unital H^*A-module.

Let M and N be two homologically unital A-polydules, and $f : M \rightsquigarrow N$ be an ∞-morphism. We say that $f : M \rightsquigarrow N$ is homologically unital if $\mathsf{H}^*f_1 : \mathsf{H}^*M \to \mathsf{H}^*N$ is a H^*A-linear morphism.

We denote the category of homologically unital A-polydules with homologically unital ∞-morphisms by huMod_∞^A. This is a non-full subcategory of Mod_∞^A. Recall that we also have suMod_∞^A, the category of strictly unital A-polydules with strictly unital ∞-morphism. Let uMod_∞^A denote the full subcategory of Mod_∞^A consisting of strictly unital A-polydules. We have the same kind of results as for A_∞-algebras.

Theorem 3.2.27 (Unital strictification of A-polydules, [Theoreme 3.3.1.2 13, p. 109])**.** *Let A be a strictly unital A_∞-algebra. Any minimal homologically unital A-polydule is isomorphic to a strictly unital A-polydule.*

Corollary 3.2.27.1 ([Corollaire 3.3.1.3 13, p. 109])**.** *Let A be a minimal strictly unital A_∞-algebra. Any homologically unital A-polydule is homotopy equivalent to a strictly unital A-polydule.*

Theorem 3.2.28 (Unital strictification of ∞-morphisms, [Theoreme 3.3.1.4 13, p. 109])**.** *Let A be a strictly unital A_∞-algebra, and let M and N be minimal strictly unital A-polydules. Any ∞-morphism $f : M \rightsquigarrow N$ is homotopic to a strictly unital ∞-morphism.*

Theorem 3.2.29 (Unital strictification of homotopies, [Theoreme 3.3.1.5 13, p. 109])**.** *Let A be a strictly unital A_∞-algebra, and let M and N be minimal strictly unital A-polydules. Let $f, g : M \rightsquigarrow N$ be homotopic ∞-morhpisms, then there is a strictly unital homotopy between f and g.*

Proposition 3.2.30 (Minimal models, [Proposition 3.3.1.7 13, p. 109])**.** *Let A be a strictly unital A_∞-algebra, and let M be a strictly unital A-polydule. Then there is a minimal strictly unital A-polydule N together with a strictly unital minimal model $f : N \rightsquigarrow M$. In particular f_1 is a quasi-isomorphism.*

Suppose that A is a minimal strictly unital A_∞-akgebra. With the above results, we are now able to deduce that the non-full inclusion $\mathsf{suMod}_\infty^A \to \mathsf{uMod}_\infty^A$ induces an equivalence

$$\mathsf{suMod}_\infty^A \big/ \sim \; \simeq \; \mathsf{uMod}_\infty^A \big/ \sim,$$

and the non-full inclusion $\mathsf{huMod}_\infty^A \to \mathsf{suMod}_\infty^A$ induces an equivalence

$$\mathsf{huMod}_\infty^A \big/ \sim \; \simeq \; \mathsf{suMod}_\infty^A \big/ \sim.$$

3.2.7 H-Unitary SHA-Algebras and Polydules

In this section we will define notions which will help us to calculate homologies. We will start with defining a twisting morphism between an augmented A_∞-algebra and a conilpotent dg-coalgebra. For the second part we will define H-unitary A_∞-algebras and polydules.

Definition 3.2.31. Let A be an augmented A_∞-algebra, and let C be a conilpotent dg-coalgebra. $\tau : C \to A$ is a twisting morphism if it is of degree 1, it is 0 on the augmentation and the coaugmentation and

$$\sum_{i \geqslant 1} m_i \otimes (\tau^{\otimes i}) \otimes \Delta_C^i = 0.$$

Let M be an A-polydule, and N a C-comodule. Given a twisting morphism $\tau : C \to A$, we define the twisted tensor products

$$\_ \otimes_\tau C : \mathsf{Mod}_\infty^A \to \mathsf{coMod}^C,$$
$$\_ \otimes_\tau A : \mathsf{coMod}^C \to \mathsf{Mod}_\infty^A.$$

The perturbations are given as

$$d_\tau^r = \sum_{i=1}^\infty (m_i \otimes C)(M \otimes \tau^{\otimes i-1} \otimes C)(M \otimes \Delta_C^i),$$
$$d_\tau^l = \sum_{i=1}^\infty (N \otimes m_i)(N \otimes \tau^{\otimes i-1} \otimes A)(\nu_N^i \otimes A).$$

We define the perturbated differential of the cochain complexes $M \otimes C$ and $N \otimes A$ as

$$d_\tau^\bullet = d_{M \otimes C} + d_\tau^r, \text{ and}$$
$$d_\tau^\bullet = d_{N \otimes A} - d_\tau^l.$$

Definition 3.2.32 (Twisted tensor products)**.** Let A be an augmented A_∞-algebra, let C be a conilpotent dg-coalgebra and let $\tau : C \to A$ be a twisting morphism. Given an A-polydule M (a C-comodule N) we define the right (left) twisted tensor product as $M \otimes_\tau C$ ($N \otimes_\tau A$) together with the perturbated differential $d_\tau^\bullet$.

Pick an augmented A_∞-algebra A. The morphism

$$\tau = i \circ s \circ \pi_1 : B\overline{A} \to A$$

is a twisting morphism. Here $\pi_1 : B\overline{A} \to \overline{A}[1]$ is the projection onto first component, and $i : \overline{A} \to A$ is the inclusion.

Lemma 3.2.33. *The morphism* $\varepsilon_{B\overline{A}} \otimes_\tau \varepsilon_A : B\overline{A} \otimes_\tau A \to \mathbb{K}$ *is a quasi-isomorphism.*

Proof. We have already seen this in Lemma 3.1.7. $\qquad\square$

Twisting morphisms will be important in understanding H-unitary A_∞-algebras and polydules.

Definition 3.2.34. Let A be an A_∞-algebra. We say that A is H-unitary if the bar construction BA is acyclic.

Lemma 3.2.35. *Let* A *be a minimal strictly unital* A_∞-*algebra, then it is H-unitary.*

Proof. The unit map $id_BA \otimes \upsilon_A[1] : BA \to BA$ is a morphism of degree -1, and is a homotopy of the identity. $\qquad\square$

Corollary 3.2.35.1. *Any homologically unital A_∞-algebra is H-unitary.*

Proof. Pick any homologically unital A_∞-algebra A. By Corollary 3.2.23.1, there exists a strictly unital A_∞-algebra A' and an ∞-quasi-isomorphism $f : A' \rightsquigarrow A$. Applying the bar construction yields a quasi-isomorphism $Bf : BA' \to BA$. By Lemma 3.2.35, BA' is acyclic, so BA has to be acyclic as well. $\qquad\square$

We have the same kind of relationships between polydules.

Definition 3.2.36. Let A be an augmented srictly unital A_∞-algebra. Any A-polydule M is H-unitary if $B_A M$ is acyclic.

Lemma 3.2.37. *Let A be a strictly unital A_∞-algebra. An A^+-polydule M is H-unitary if and only if it is homologically unital as an A-polydule.*

Proof. Suppose first that M is a homologically unital A-polydule. Then by Corollary 3.2.27.1, there is a strictly unital A-polydule M' together with an ∞-quasi-isomorphism $M' \rightsquigarrow M$. It is enough to show that $B_{A^+} M'$ is acyclic. The unit v_A defines a homotopy of the identity

$$id_{B_{A^+} M'} \otimes v_A[1] : B_{A^+} M' \to B_{A^+} M'.$$

For the other direction, suppose instead that M is a H-unitary A^+-polydule. First note that we have an exact sequence

$$0 \longrightarrow A \longrightarrow A^+ \longrightarrow \mathbb{K} \longrightarrow 0$$

Recall that $\tau = i \circ s \circ \pi_1 : BA \to A^+$. This sequence induces an exact sequence on the twisted tensors

$$0 \longrightarrow M \otimes_\tau BA \otimes_\tau A \longrightarrow M \otimes_\tau BA \otimes_\tau A^+ \longrightarrow M \otimes_\tau BA \otimes_\tau \mathbb{K} \longrightarrow 0$$

By assumption $M \otimes_\tau BA \otimes_\tau \mathbb{K} \simeq (M[1] \otimes_\tau BA)[-1] \simeq (B_{A^+} M)[-1]$ which is acyclic by assumption. Thus $M \otimes_\tau BA \otimes_\tau A$ is quasi-isomorphic to $M \otimes_\tau BA \otimes A^+$. By Lemma 3.2.33, $M \otimes_\tau BA \otimes A^+ \simeq M \otimes_\tau \mathbb{K} \simeq M$. Thus $M \simeq M \otimes_\tau BA \otimes_\tau A$, which is a strictly unital right A-polydule by freeness. $\qquad\square$

3.3   The Derived Category $D_\infty A$

3.3.1   The Derived Category of Augmented SHA-Algebras

In this section we wish to define the derived category of strictly unital polydules of an augmented A_∞-algebra. If [Qis] denote the class of ∞-quasi-isomorphisms, we want the derived category to be the localization at ∞-quasi-isomorphisms, e.g.

$$\mathcal{D}_\infty A = \mathsf{suMod}_\infty^A[\mathsf{Qis}^{-1}].$$

Like in the case of algebras, we may understand the quasi-isomorphisms better. The category suMod_∞^A is not complete, but we may in the same sense as before give it a model structure without limits. Within this structure we already know that every object is cofibrant, and the goal is to show that every object is fibrant as well. This will allow us to lift every ∞-quasi-isomorphism to a homotopy equivalence. With this we may see that the localization from $K_\infty A \to D_\infty A$ is given by the identity.

Within the category suMod_∞^A we define three classes of morphisms:

- $f \in Ac$ is a weak equivalence if f_1 is a quasi-isomorphism,
- $f \in Cof$ is a cofibration if f_1 is a monomorphism,
- $f \in Fib$ is a fibration if f_1 is an epimorphism,

Theorem 3.3.1. *The category suMod_∞^A is a model category without enough limits. Moreover, every object are bifibrant.*

Proof. This is more or less identical to the proof of Theorem 2.3.1. □

Corollary 3.3.1.1. *Homotopy equivalence defined in suMod_∞^A is an equivalence relation, and every ∞-quasi-isomorphism is a homotopy equivalence.*

If A is an ordinary associative augmented algebra, then Mod^A fully-faithfully embeds into $\mathsf{suMod}_{\infty,strict}^A$. This inclusion defines an equivalence

$$DA \simeq \mathsf{suMod}_{\infty,strict}^A/\sim_\infty.$$

We now want this model structure on suMod_∞^A to respect the model structure on the category $\mathsf{coMod}_{conil}^{B\overline{A}}$. In other words, we want the functor $B_A : \mathsf{suMod}_\infty^A \to \mathsf{coMod}_{conil}^{B\overline{A}}$ to preserve and reflect the model structure of both categories.

Lemma 3.3.2. *Let M be an object of suMod_∞^A. The unit $B_A M \to R_{\iota_{B\overline{A}}} L_{\iota_{B\overline{A}}} B_A M$ is a quasi-isomorphism on the primitive elements.*

Proof. This proof uses the same trick as lemma 3.1.7. Equip M, the trivial filtraton, $B\overline{A}$ the coradical filtration and $\Omega B\overline{A} = UA$ the induced filtration.

$$F_p M = M,$$
$$Fr_p B\overline{A} = \{[a_1 \mid \cdots \mid a_n] \mid n \leqslant p\},$$
$$F_p UA = \{\langle [a_{1_1} \mid \cdots \mid a_{n_1}] \mid \cdots \mid [a_{1_k} \mid \cdots \mid a_{n_k}]\rangle \mid n_1 + \cdots + n_k \leqslant p\}.$$

We see that $\mathrm{gr}_0 M[1] \simeq M[1]$ and otherwise $\simeq 0$. In the same way, $\mathrm{gr}_0 \eta$ acts as the identity on $M[1]$. By the similar lemma, we know that each $\mathrm{gr}_p M[1] \otimes B\overline{A} \otimes UA$ is acyclic for $p \geqslant 1$. Thus $\mathrm{gr}\eta$, is a filtered quasi-isomorphism on the primitives.

$\square$

Proposition 3.3.3. *Let M and M' be objects of suMod_∞^A, together with an ∞-morphism $f : M \to M'$.*

- *f is an ∞-quasi-isomorphism if and only if $B_A f$ is a weak equivalence.*
- *f is a fibration if and only if $B_A f$ is a fibration.*
- *f is a cofibration if and only if $B_A f$ is a cofibration.*

Proof. Recall from theorem 3.1.8 that the morphism $\iota_{B\overline{A}} : B\overline{A} \to UA$ is an acyclic twisting morphism. Thus the adjoint pair $(L_{\iota_{BA}}, R_{\iota_{BA}})$ defines a Quillen equivalence.

We show only the first bullet point. The last two are identical to the proof of proposition 2.2.17.

If f_1 is a quasi-isomorphism, then $B_A f$ is a filtered quasi-isomorphism by definition. So suppose that $B_A f$ is a weak equivalence instead. The unit transformation gives us a natural square.

$$
\begin{array}{ccc}
B_A M & \longrightarrow & R_{\iota_{B\overline{A}}} L_{\iota_{B\overline{A}}} B_A M \\
\downarrow{\scriptstyle B_A f} & & \downarrow{\scriptstyle R_{\iota_{B\overline{A}}} L_{\iota_{B\overline{A}}} B_A f} \\
B_A M' & \longrightarrow & R_{\iota_{B\overline{A}}} L_{\iota_{B\overline{A}}} B_A M'
\end{array}
$$

In this case $R_{\iota_{B\overline{A}}} = B_A i$, so this diagram is in the image of B_A. Since B_A is fully-faithful, we consider this diagram in suMod_∞^A instead.

$$
\begin{array}{ccc}
M & \longrightarrow & iL_{\iota_{B\overline{A}}} BM \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle iL_{\iota_{B\overline{A}}} Bf} \\
M' & \longrightarrow & iL_{\iota_{B\overline{A}}} BM'
\end{array}
$$

Since $B_A f$ is a weak equivalence, $iL_{\iota_{B\overline{A}}}B_A f$ is an ∞-quasi-isomorphism by definition. By the above lemma, the horizontal maps are ∞-quasi-isomorphisms. Thus by the 2-out-of-3 property, f is an ∞-quasi-isomorphism as well. $\qquad\square$

Associated to each augmented A_∞-algebra there is also a homotopy category. Since homotopy equivalence $\sim_\infty$ in suMod^A_∞ defines a congruence relation we may construct the homotopy category $K_\infty A$.

Corollary 3.3.3.1. *The localization $K_\infty A \to D_\infty A$ is given by the identity. Moreover, $K_\infty A = D_\infty A$.*

Remark 3.3.4. The name homotopy category comes from homological algebra and has a priori nothing to do with the homotopy category $\mathsf{Ho}(\mathsf{suMod}^A_\infty)$. However, in this particular case these naming conventions coincide.

Lemma 3.3.5. *The composition $J : Mod^{UA} \to suMod^A_{\infty,strict} \to suMod^A_\infty$ given by $J = \iota \circ i$, induces an equivalence of categories:*

$$DUA \simeq D_\infty A.$$

Proof. Consider the commutative square:

$$
\begin{array}{ccc}
\mathsf{Mod}^{UA} & \xrightarrow{\ i\ } & \mathsf{suMod}^A_{\infty,strict} \\
{\scriptstyle R_{\iota_{B\overline{A}}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \iota} \\
\mathsf{coMod}^{B\overline{A}} & \xleftarrow[\ B_A\ ]{} & \mathsf{suMod}^A_\infty
\end{array}
$$

Since the three functors $R_{\iota_{B\overline{A}}}$, i and B_A all induces equivalences on the derived categories, then ι has to as well. $\qquad\square$

To summarize, we have established an equivalence between 4 different categories:

- $D_\infty A$, derived category of A,
- $\mathsf{suMod}^A_{\infty,strict}[Qis^{-1}]$ derived category of A with only strict morphisms,
- $DB\overline{A}$, derived category of $B\overline{A}$ as a dg-coalgebra,
- DUA, derived category of the universal enveloping algebra of A.

In this sense, we may also see that in the derived category, all of the higher homotopic data of each morphism have been collapsed by the homotopy.

The triangulated structure on $D_\infty A$ may be lifted along these equivalences making them triangulated as well. Note that $R_{\iota_{B\overline{A}}}$ is already triangulated, and there is only one way of forcing the triangulated structure on suMod^A_∞. Since suMod^A_∞ isn't complete it isn't easy to obtain a

description of the triangles along any ∞-morphism f. However, this problem does not appear in $\mathsf{suMod}^A_{\infty,strict}$, so one should think of only strict morphisms instead, but in this case we are already working in the category Mod^{UA}.

3.3.2 The Derived Category of Strictly Unital SHA-Algebras

In this section we will generalize the construction of the derived category to any strictly unital A_∞-algebra. Consider the strictly unital A_∞-algebra A. If we look at the augmented algebra A^+, then the augmentation $\varepsilon_A : A^+ \to \mathbb{K}$ gives $\mathbb{K}$ the structure of an A^+-polydule. We construct the following functor

$$\_ \otimes^\infty_{A^+} \mathbb{K} : \mathsf{Mod}^{A^+}_\infty \to \mathsf{Mod}^\infty_\mathbb{K}.$$

We may observe that this functor maps strictly unital objects into strictly unital objects

$$\_ \otimes^\infty_{A^+} \mathbb{K} : \mathsf{uMod}^{A^+}_\infty \to \mathsf{uMod}^\infty_\mathbb{K}.$$

The derived category $D_\infty A^+$ is equivalent to $\mathsf{uMod}^{A^+}_\infty / \sim$. Since the functor above preserves ∞-quasi-isomorphisms, it induces a functor between the derived categories

$$\_ \otimes^\infty_{A^+} \mathbb{K} : D_\infty A^+ \to D_\infty \mathbb{K}.$$

Definition 3.3.6. Let A be an A_∞-algebra. We define the derived category as the kernel

$$D_\infty A = \mathsf{Ker}(\_ \otimes^\infty_{A^+} \mathbb{K} : D_\infty A^+ \to D_\infty \mathbb{K}).$$

Theorem 3.3.7. *Let A and A' be two A_∞-algebras, and let $f : A \to A'$ be an ∞-quasi-isomorphism. The restriction*

$$f^* : Mod^{A'}_\infty \to Mod^A_\infty$$

induces an equivalence on the derived categories

$$f^* : D_\infty A' \to D_\infty A.$$

Proof. We have already seen a variant of this. Consider the diagram

$$
\begin{array}{ccccc}
D^{A'}_\infty & \rightarrowtail & D_\infty A'^+ & \longrightarrow & D_\infty \mathbb{K} \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle (f^+)^*} & & \downarrow{\scriptstyle \simeq} \\
D_\infty A & \rightarrowtail & D_\infty A^+ & \longrightarrow & D_\infty \mathbb{K}
\end{array}
$$

By Lemma 3.3.5, we have a commutative square

$$DU(A'^+) \xrightarrow{\simeq} D_\infty A'^+$$

$$\simeq \Big\downarrow U((f^+)^*) \qquad \Big\downarrow (f^+)^*$$

$$DU(A^+) \xrightarrow{\simeq} D_\infty A^+$$

Since $U((f^+)^*)$ is an equivalence by Corollary 3.1.24.3, then $((f^+)^*)$ is an equivalence as well. By the first diagram, f^* has to be an equivalence by the kernel property. $\square$

A usefull property of the ∞-tensor is that it behaves like the ordinary tensor up to homotopy.

Lemma 3.3.8. *Let A be an A_∞-algebra. Let M be a strictly unital A-polydule. On the category uMod$_\infty^A$ we have the following:*

- *There is an ∞-quasi-isomorphism $M \otimes_A^\infty A \rightsquigarrow M$,*
- *and there is an ∞-quasi-isomorphism $M \rightsquigarrow$ Hom$_A^\infty(A, M)$.*

Proof. Since the second point is the transpose of the first point, we will only prove that $M \otimes_A^\infty A \rightsquigarrow M$ is an ∞-quasi-isomorphism.

We define the multiplication morphism compenentwise

$$g_{i,j} : M \otimes_A^\infty A \to M,$$

$$m \otimes [a_1 \mid \cdots \mid a_j] \otimes a \otimes a_1' \otimes \cdots \otimes a_{i-1}' \mapsto m_{1+j+1+i-1}(m, a_1, \cdots, a_j, a, a_1', \cdots, a_i'),$$

so that $g_i = \sum_{j=1}^\infty g_{i,j}$.

To see that g defines an ∞-quasi-isomorphism we calculate the homology of cone(g_1).

One may observe that the morphism

$$id \otimes v_A[1] \otimes id : M \otimes (A[1])^{\otimes i} \otimes A \to M \otimes (A[1])^{\otimes i+1} \otimes A$$

induces a homotopy between $id_{\mathsf{cone}(g_1)}$ and 0, so g_1 is a quasi-isomorphism. $\square$

We are now going to define other categories which will look very similar to the derived category in the augmented case. It is also true that these categories will be equivalent to the derived category in the strictly unital case.

Definition 3.3.9 (Compactly generated triangulated category)**.** Let A be a strictly unital A_∞-algebra. We let $\langle A \rangle$ denote the smallest thick triangulated subcategory category of $D_\infty A^+$ containing A and which is closed under infinite sums.

Definition 3.3.10 (Homotopy category)**.** Let A be a strictly unital A_∞-algebra. Let the homotopy category be

$$K_\infty A = \mathsf{suMod}_\infty^A / \sim,$$

where $\sim$ is homotopy equivalence.

A priori, we are not sure if the congruence relation generated by homotopy equivalence is strictly greater than just homotopy equivalence. However, by considering the restriction map

$$r = \begin{pmatrix} id_A & v_A \end{pmatrix} : A^+ \to A,$$

we obtain a faithfull functor

$$r^* : \mathsf{suMod}_\infty^A \to \mathsf{suMod}_\infty^{A^+},$$

which respects homotopy equivalences. This functor also induces a fully-faithfull functor

$$r^*/\sim \; : K_\infty A \to K_\infty A^+. \tag{3.2}$$

Since homotopy equivalence is a congruence relation in the latter category, it necessarily have to be that in former category.

Theorem 3.3.11. *Let A be a strictly unital A_∞-algebra. The following categories are equivalent:*

- $D_\infty A$
- $\langle A \rangle$
- $K_\infty A$
- *$\mathsf{suMod}_\infty^A[Qis^{-1}]$*
- *$Ho(\mathsf{suMod}_{\infty,strict}^A)$*

Proof of $D_\infty A \simeq \langle A \rangle$. We start by showing that $D_\infty A \simeq \langle A \rangle$. In order to see this, we would like to have an exact sequence of triangulated categories

$$\langle A \rangle \rightarrowtail D_\infty A^+ \longrightarrow\!\!\!\!\!\rightarrow D_\infty \mathbb{K}$$

By [Proposition 3.2.8 29, p. 81] it suffices to show that for any A^+-polydule M, in the triangle

$$M \otimes_{A^+}^\infty A \longrightarrow M \longrightarrow M \otimes_{A^+}^\infty \mathbb{K} \longrightarrow (M \otimes_{A^+}^\infty A)[1]$$

the objects $M \otimes_{A^+}^\infty A \in \langle A \rangle$ and $M \otimes_{A^+}^\infty \mathbb{K}$ are $\langle A \rangle$-local. An object of $M \in D_\infty A^+$ is said to be $\langle A \rangle$-local if for any $L \in \langle A \rangle$

$$D_\infty A^+(L, M) = 0.$$

We start by observing that $M \otimes_{A^+} A = M \otimes BA^+ \otimes A$, so $M \otimes_{A^+}^\infty A$ is in fact contained in $\langle A \rangle$.

To see that $M \otimes_{A^+} \mathbb{K}$ is $\langle A \rangle$-local, we start by considering the following triangle

$$A \otimes_{A^+}^\infty \mathbb{K} \longrightarrow A^+ \otimes_{A^+}^\infty \mathbb{K} \longrightarrow \mathbb{K} \otimes_{A^+}^\infty \mathbb{K} \longrightarrow (A \otimes_{A^+}^\infty \mathbb{K})[1]$$

By assumption A is strictly unital, so it is homologically unital as well, even considered as an A-polydule. By Lemma 3.2.37, A is H-unitary as an A^+-polydule. Notice that $A \otimes_{A^+} \mathbb{K} = A \otimes BA^+ \otimes \mathbb{K} \simeq B_{A^+}A$. Since A is H-unitary, we get that $A \otimes_{A^+}^{\infty} \mathbb{K}$ is acyclic. Moreover, by thickness, any $L \in \langle A \rangle$ has the property that

$$L \otimes_{A^+}^{\infty} \mathbb{K} \simeq 0.$$

By acyclicity of $A \otimes_{A^+}^{\infty} \mathbb{K}$, we obtain an ∞-quasi-isomorphism

$$A^+ \otimes_{A^+}^{\infty} \mathbb{K} \to \mathbb{K} \otimes_{A^+}^{\infty} \mathbb{K}. \tag{3.3}$$

If we consider the projection

$$A^+ \otimes_{A^+} \mathbb{K} \to \mathbb{K},$$

we see that this is an ∞-quasi-isomorphism, since the cone is the bar construction of A^+. BA^+ is acyclic, as A^+ is strictly unital and thus H-unitary.

By composing these morphisms in the derived category $D_\infty A^+$, we get an isomorphism

$$\mathbb{K} \to \mathbb{K} \otimes_{A^+}^{\infty} \mathbb{K}. \tag{3.4}$$

Now, pick an arbitrary morphism $f : L \to M \otimes_{A^+}^{\infty} \mathbb{K}$. We have the following commutative diagram

$$
\begin{array}{ccc}
L & \xrightarrow{\ \ f\ \ } & M \otimes_{A^+}^{\infty} \mathbb{K} \\
\downarrow & & \downarrow{\scriptstyle \simeq} \\
L \otimes_{A^+}^{\infty} \mathbb{K} & \longrightarrow & M \otimes_{A^+}^{\infty} \mathbb{K} \otimes_{A^+}^{\infty} \mathbb{K}
\end{array}
$$

As $L \otimes_{A^+}^{\infty} \mathbb{K} \simeq 0$, the morphism f factors 0. Thus $f = 0$. $\qquad\square$

Proof of $D_\infty A \simeq K_\infty A$. Let M be an A^+-polydule. We evaluate $M \otimes_{A^+}^{\infty} \mathbb{K} = M \otimes BA^+ = B_{A^+}M$. In other words, M is H-unitary if and only if $M \otimes_{A^+}^{\infty} \mathbb{K}$ is acyclic. By definition, $D_\infty A$ is thus made up of every H-unitary A^+-polydules. By Lemma 3.2.37, we know that $D_\infty A$ is then formed by the homologically unital A-polydules. By Corollary 3.2.27.1, every such A-polydule is ∞-quasi-isomorphic to a strictly unital A-polydule.

For the augmented A_∞-algebra A^+ we know already that $K_\infty A^+ \simeq D_\infty A^+$. Thus $K_\infty A$ is exactly the kernel in the following diagram

$$K_\infty A \rightarrowtail K_\infty A^+ \twoheadrightarrow D_\infty \mathbb{K}$$

as the inclusion sends strictly unital polydules to H-unitary polydules. $\qquad\square$

Proof of $K_\infty A \simeq \text{suMod}_\infty^A[\text{Qis}^{-1}]$. Since there is a fully-faithfull functor

$$K_\infty A \rightarrowtail K_\infty A^+$$

it follows that every ∞-quasi-isomorphism in suMod_∞^A is a homotopy equivalence. Thus $\text{suMod}_\infty^A[\text{Qis}^{-1}] \simeq K_\infty A$. $\qquad\square$

We prove the final statement first in the case of ordinary associative algebras.

Lemma 3.3.12. *Let A be a differential graded algebra. The inclusion $i : \text{Mod}^A \to \text{suMod}_\infty^A$ induces an equivalence of categories*

$$DA \simeq \text{suMod}_\infty^A[\text{Qis}^{-1}],$$

where the inverse is given by $\_ \otimes_A^\infty A$.

Proof. Let M be an A-polydule, then we already know that there is an ∞-quasi-isomorphism $M \otimes_A^\infty A \rightsquigarrow M$.

Let instead M be an A-module. Then we can consider it as an A-polydule by letting the higher multiplication $m_i = 0$ for any $i \geqslant 3$. Thus we see that the ∞-morphism g defined as in Lemma 3.3.8 is a strict morphism. In other words $g = g_1$ defines a morphism of algebras. We have already seen that it is the component g_1 which is a quasi-isomorphism, so there is a quasi-isomorphism of modules $i(M) \otimes_A^\infty A \to M$.

Thus we have proved that in the derived categories $D_\infty A$ and DA composing the functors are isomorphic to applying the identity functors. Thus we get an equivalence

$$DA \simeq D_\infty A.$$

$\qquad\square$

Before the last proof we will need some technical lemmata.

Lemma 3.3.13 ([Proposition 7.5.0.2 13, p. 171])**.** *Let A be a strictly unital A_∞-algebra, then there is a dg-algebra A' and a strictly unital acyclic cofibration*

$$A \rightsquigarrow A'.$$

Lemma 3.3.14. *[Proposition 3.2.4.5 13, p. 106] Let A and A' be two strictly unital A_∞-algebras. If $i : A \rightsquigarrow A'$ is a strictly unital acyclic cofibration, then there is a strictly unital acyclic fibration $p : A' \to A$, such that $p \circ i = id_A$ and $i \circ p \sim id_{A'}$.*

Lemma 3.3.15. *[Lemme 4.1.3.15 13, p. 128] Let A and B be two unital differential graded algebras. Let $f, f' : A \to B$ be two morphisms of algebras, such that they are right homotopic $f \sim_r f'$. The restriction functors*

$$f^*, f'^* : Mod^B \to Mod^A \tag{3.5}$$

induces equivalent functors on the derived category

$$f^* \simeq f'^* : DB \to DA.$$

Proof of $suMod_\infty^A[Qis^{-1}] \simeq Ho(suMod_{\infty,strict}^A)$. Assume first that A is a differential graded associative algebra. We have the following chain of faithfull inclusions

$$\mathsf{Mod}^A \rightarrowtail \mathsf{suMod}_{\infty,\mathsf{strict}}^A \rightarrowtail \mathsf{suMod}_\infty^A.$$

By lemma 3.3.12, the composition is an equivalence on the derived categories, and then necessarily essentially surjective and fully-faithfull. The last inclusion is by definition essentially surjective, and also fully-faithfull on the derived categories. In this manner all three categories are equivalent.

We will now suppose that A is an A_∞-algebra. By Lemma 3.3.13, there exists a dg-algebra A' and an acyclic cofibration

$$p : A \rightsquigarrow A'.$$

By Lemma 3.3.14, there also exists an acyclic fibration $q : A' \rightsquigarrow A$, splitting p as $q \circ p = id_A$ and $p \circ q \sim id_{A'}$.

If we are using the model structures on $\mathsf{suMod}_{\infty,\mathsf{strict}}^A$ and $\mathsf{suMod}_{\infty,\mathsf{strict}}^{A'}$ induced by the universal enveloping algebras, the morphisms p and q induces functors

$$\mathsf{Ho}(p^*) : \mathsf{Ho}(\mathsf{suMod}_{\infty,\mathsf{strict}}^{A'}) \to \mathsf{Ho}(\mathsf{suMod}_{\infty,\mathsf{strict}}^A) \text{ and,}$$
$$\mathsf{Ho}(q^*) : \mathsf{Ho}(\mathsf{suMod}_{\infty,\mathsf{strict}}^A) \to \mathsf{Ho}(\mathsf{suMod}_{\infty,\mathsf{strict}}^{A'}).$$

If we have that

$$\mathsf{Ho}(p^*)\mathsf{Ho}(q^*) \simeq \mathsf{Id}_{\mathsf{Ho}(\mathsf{suMod}_{\infty,\mathsf{strict}}^A)} \text{ and}$$
$$\mathsf{Ho}(q^*)\mathsf{Ho}(p^*) \simeq \mathsf{Id}_{\mathsf{Ho}(\mathsf{suMod}_{\infty,\mathsf{strict}}^{A'})},$$

then we would be done. This is because $p^* : D_\infty A' \to D_\infty A$ induces an equivalence by Theorem 3.3.7. Thus we may consider the following commutative diagram

$$\mathsf{Ho}(\mathsf{suMod}^A_{\infty,\mathsf{strict}}) \xrightarrow{\ \simeq\ } \mathsf{Ho}(\mathsf{suMod}^{A'}_{\infty,\mathsf{strict}})$$
$$\Big\downarrow \qquad\qquad\qquad\qquad\qquad \Big\downarrow{\simeq}$$
$$D_\infty A \xleftarrow{\qquad\simeq\qquad} D_\infty A'$$

Here the equivalence on the right hand side is given by the case for ordinary algebras treated earlier. Finally, by previous results we know that $D_\infty A \simeq \mathsf{suMod}^A_\infty[\mathsf{Qis}^{-1}]$.

To see that we have the equivalences as claimed, we first note that the first one is automatic by the equation $q \circ p = id_A$. We must show that $p \circ q$ is isomorphic to the identity on $\mathsf{Ho}(\mathsf{suMod}^{A'}_{\infty,\mathsf{strict}})$. By the earlier argument, proving this will be the same as proving that $p \circ q$ induces an equivalence on DA'. Since $p \circ q$ is homotopic to $id_{A'}$ we get by Lemma 3.3.15 that they induce the same functor

$$(p \circ q)^* \simeq \mathsf{Id}_{DA'} : DA' \to DA'.$$

$\square$

Bibliography

[1] J. D. Stasheff, "Homotopy associativity of h-spaces. i," *Transactions of the American Mathematical Society*, vol. 108, pp. 275–292, 1963.

[2] J. D. Stasheff, "Homotopy associativity of h-spaces. ii," *Transactions of the American Mathematical Society*, vol. 108, pp. 293–312, 1963.

[3] J.-L. Loday and B. Vallette, *Algebraic Operads*. Springer Verlag, 2012.

[4] T. Aambø, "On formal dg-algebras," M.S. thesis, Norwegian University of Science and Technology, 2021.

[5] G. M. Kelly, "Basic concepts of enriched category theory," *Elements of ∞-Category Theory*, 2005.

[6] L. W. Tu, "An introduction to manifolds," 2007.

[7] S. Eilenberg and S. Mac Lane, "On the groups h((π,n), i," *Annals of Mathematics*, vol. 58, pp. 55–106, 1953.

[8] J. Adams, "On the cobar construction," *PNAS*, vol. 42, pp. 409–412, 1956.

[9] S. Eilenberg and J. C. Moore, "Homology and fibrations. i. coalgebras, cotensor product and its derived functors," *Commentarii mathematici Helvetici*, vol. 40, pp. 199–236, 1965/66.

[10] E. Riehl, *Categorical Homotopy Theory*. Cambridge University Press, 2014.

[11] S. MacLane, *Categories for the working mathematician* (Graduate Texts in Mathematics, Vol. 5). Springer-Verlag, New York-Berlin, 1971, pp. ix+262.

[12] D. .-. Lu, J. H. Palmieri, Q. .-. Wu, and J. J. Zhang, *A-infinity structure on ext-algebras*, 2006. DOI: `10.48550/ARXIV.MATH/0606144`. [Online]. Available: `https://arxiv.org/abs/math/0606144`.

[13] K. Lefevre-Hasegawa, "Sur les a [infini]-catégories," *arXiv: Category Theory*, 2003.

[14] M. Hovey, *Model Categories*. American Mathematical Society, 1999.

[15] D. G. Quillen, *Homotopical Algebra*, A. Dold, Heidelberg, and B. Eckmann, Eds. Springer-Verlag, 1967.

[16] W. G. Dwyer and J. Spalinsky, "Homotopy theories and model categories," in *Handbook of Algebraic Topology*, I. M. James, Ed. Elsevier Science, 1995, ch. 2, pp. 73–126.

[17] E. Riehl, *Category Theory in Context*. Dover Publications, 2016.

[18] P. Gabriel and M. Zisman, *Calculus of Fractions and Homotopy Theory*. Springer-Verlag, 1967, pp. 6–20.

[19] H. J. Munkholm, "Dga algebras as a quillen model category and relations to shm maps," *Journal of Pure and Applied Algebra*, vol. 13, pp. 221–232, 1978.

[20] A. K. Bousfield and V. K. A. M. Gugenheim, "On pl de rham theory and rational homotopy type," *Memoirs of the American Mathematical Society*, vol. 8, no. 179, 1976.

[21] J. F. Jardine, "A closed model structure for differential graded algebras," *Fields Institute Communications*, vol. 17, pp. 55–58, 1997.

[22] V. Hinich, "Homology algebra of homotopy algebras," *Communications in Algebra*, vol. 25(10), pp. 2391–3323, 1997.

[23] V. Hinich, "Dg coalgebras as formal stacks," *Journal of Pure and Applied Algebra*, vol. 162, no. 2, pp. 209–250, 2001, ISSN: 0022-4049. DOI: `https://doi.org/10.1016/S0022-4049(00)00121-3`. [Online]. Available: `https://www.sciencedirect.com/science/article/pii/S0022404900001213`.

[24] C. A. Weibel, *An Introduction to Homological Algebra* (Cambridge Studies in Advanced Mathematics). Cambridge University Press, 1994. DOI: `10.1017/CBO9781139644136`.

[25] B. Keller, *Corrections to 'sur les a-infini categories'*, https://webusers.imj-prg.fr/ bernhard.keller/lefevre/TheseFinale/corrainf.pdf, 2005.

[26] B. Keller, "A-infinity algebras, modules and functor categories," in *Trends in representation theory of algebras and related topics*, ser. Contemp. Math. Vol. 406, Amer. Math. Soc., Providence, RI, 2006, pp. 67–93. DOI: `10.1090/conm/406/07654`. [Online]. Available: `https://doi.org/10.1090/conm/406/07654`.

[27] B. Vallette, "Homotopy theory of homotopy algebras," *Annales de l'Institut Fourier*, vol. 70, no. 2, pp. 683–738, 2020, ISSN: 0373-0956.

[28] B. Keller, "Deriving DG categories," *Ann. Sci. École Norm. Sup. (4)*, vol. 27, no. 1, pp. 63–102, 1994, ISSN: 0012-9593. [Online]. Available: `http://www.numdam.org/item?id=ASENS_1994_4_27_1_63_0`.

[29] H. Krause, *Homological Theory of Representations -Draft Version of a Book Project-*. Cambridge University Press, Aug. 2021. [Online]. Available: `https://www.math.uni-bielefeld.de/~hkrause/HomTheRep.pdf`.

[30] D. Happel, *Triangulated Categories in the Representation of Finite Dimensional Algebras* (London Mathematical Society Lecture Note Series). Cambridge University Press, 1988. DOI: `10.1017/CBO9780511629228`.

[31] T. Bühler, "Exact categories," *Expositiones Mathematicae*, vol. 28, no. 1, pp. 1–69, 2010, ISSN: 0723-0869. DOI: `https://doi.org/10.1016/j.exmath.2009.04.004`. [Online]. Available: `https://www.sciencedirect.com/science/article/pii/S0723086909000395`.

[32] S. Mac Lane, *Homology* (Classics in Mathematics), 1st ed. Springer Berlin, 1994. DOI: `https://doi.org/10.1007/978-3-642-62029-4`.

[33] B. Keller, "Chain complexes and stable categories," *manuscripta mathematica*, vol. 67, pp. 379–417, 1990.

[34] I. Kriz and J. P. May, *Operads, algebras, modules and motives*. Société mathématique de France, 1995.

Appendix A

Monads

A.1 Monads and Categories of Algebras

Definition A.1.1 (Monad)**.** Let $\mathcal{C}$ be a category. We say that an endofunctor $T : \mathcal{C} \to \mathcal{C}$ together with

- a multiplication $\mu : M \circ M \Rightarrow M$
- and a unit $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow M$

is a monad, if the following diagrams commute

$$
\begin{array}{ccc}
M \circ M \circ M & \xrightarrow{M(\mu)} & M \circ M \\
\downarrow{\scriptstyle \mu_M} & & \downarrow{\scriptstyle \mu} \\
M \circ M & \xrightarrow{\quad \mu \quad} & M
\end{array}
\qquad
\begin{array}{ccccc}
M \circ \mathrm{Id}_{\mathcal{C}} & \xrightarrow{M(\eta)} & M \circ M & \xleftarrow{\eta_M} & \mathrm{Id}_{\mathcal{C}} \circ M \\
 & \searrow & \downarrow{\scriptstyle \mu} & \swarrow & \\
 & & M & &
\end{array}
$$

In other words, a monad is a monoid in the category of endofunctors, $(T, \mu, \eta) \in (\mathrm{End}\mathcal{C}, \circ, \mathrm{Id}_{\mathcal{C}})$.

Lemma A.1.2 (Monads from adjunctions, [Lemma 5.1.3. 10, p. 155])**.** *Given an adjunction $F \dashv G : \mathcal{C} \to \mathcal{D}$ and*

- *a unit $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$*
- *and a counit $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$,*

there is an associated monad (T, μ, η). Let $T = GF$, together with

- *a multiplication given by the counit $\mu = G(\varepsilon_F) : T \circ T \Rightarrow T$*

- *and the unit $\eta : Id_{\mathcal{C}} \Rightarrow T$,*

is a monad on $\mathcal{C}$.

Given any monad $(T : \mathcal{C} \to \mathcal{C}, \mu, \eta)$, we say that an object $M \in \mathcal{C}$ is a T-algebra if there exists a morphism $m : T(M) \to M$ such that the following diagrams commute

$$
\begin{array}{ccc}
T \circ T(M) & \xrightarrow{T(m)} & T(M) \\
\downarrow{\varepsilon_M} & & \downarrow{m} \\
T(M) & \xrightarrow{m} & M
\end{array}
\qquad
\begin{array}{ccc}
M & \xrightarrow{\eta_M} & T(M) \\
& \diagdown & \downarrow{m} \\
& & M
\end{array}
$$

If M and N are two T-algebras, then we say that a morphism $f : M \to N$ is a T-algebra morphism if the following diagram commute

$$
\begin{array}{ccc}
T(M) & \xrightarrow{T(f)} & T(N) \\
\downarrow{m} & & \downarrow{n} \\
M & \xrightarrow{f} & N
\end{array}
$$

Definition A.1.3 (Eilenberg-Moore category)**.** The Eilenberg-Moore category, or the category of algebras $\mathcal{C}^T$ is the category having

- objects as M as T-algebras
- and morphisms $f : M \to N$ as T-algebra morphisms.

There is a free functor from $\mathcal{C}$ to T-algebras

$$
F^T : \mathcal{C} \to \mathcal{C}^T,
$$
$$
M \mapsto (T(M), \mu_M).
$$

By forgetting the T-algebra structure we obtain a forgetful functor

$$
U^T : \mathcal{C}^T \to \mathcal{C},
$$
$$
(M, m) \mapsto M.
$$

The next lemma justifies calling these functors for free and forgetful.

Lemma A.1.4 (Adjunctions from monads, [Lemma 5.2.8 10, p. 162])**.** *Given any monad (T, μ, η) : $\mathcal{C} \to \mathcal{C}$, then the pair of functors F^T and U^T defines an adjunction*

$$
F^T \dashv U^T : \mathcal{C} \to \mathcal{C}^T.
$$

Definition A.1.5 (Free T-algebra)**.** (M, m) is a free T-algebra if there is an object $N \in \mathcal{C}$ and an isomorphism $(M, m) \simeq F^T(N)$.

In the category of algebras $\mathcal{C}^T$ we may approximate every T-algebra M by free T-algebras. This means that we may construct a canonical free resoulution of any T-algebra M.

Proposition A.1.6 (Free resolutions, [Proposition 5.4.3 1Ø, p. 169])**.** *Given any T-algebra M, then*

$$((T \circ T)(M), \mu_{TM}) \underset{\mu_M}{\overset{Tm}{\rightrightarrows}} (TM, \mu_M) \xrightarrow{\ m\ } (M, m)$$

is a colimit diagram in $\mathcal{C}^T$.

It is useful to recognize when a category is a category of some algebra. Then every object is generated by every free objects, which may arise from a simpler category.

Definition A.1.7 (Monadicity)**.** Suppose that there is an adjunction $F \vdash G : \mathcal{C} \to \mathcal{D}$ and that $T = GF$. We say that the adjunction, or $G : \mathcal{D} \to \mathcal{C}$, is monadic if there exists an equivalence of categories $K : \mathcal{D} \to \mathcal{C}^T$ such that there are natural isomorphisms $G \simeq U^T \circ K$ and $F^T \simeq K \circ F$.

Many of the categories which we consider are monadic.

Example A.1.8 (Ab is monadic over Set, [Corollary 5.5.3 1Ø, p. 174])**.** Consider the adjoint pair of functors $\mathbb{Z}_{\_} \dashv \text{forget} : \text{Set} \to \text{Ab}$, where we define

$$\mathbb{Z}_{\_} : \text{Set} \to \text{Ab},$$
$$M \mapsto \mathbb{Z}M.$$

The binary operation on the group is given by formal linear combinations. This adjoint pair is monadic.

Example A.1.9 (Mod^R is monadic over Mod$^{\mathbb{K}}$)**.** The adjoint pair of functors $\_ \otimes_{\mathbb{K}} R \dashv \text{forget} : \text{Mod}^{\mathbb{K}} \to \text{Mod}^R$ is monadic.

Example A.1.1Ø (Alg$_{\mathbb{K},+}$ is monadic over Mod$^{\mathbb{K}}$)**.** The adjoint pair $T_{\_} \dashv \text{forget} : \text{Mod}^{\mathbb{K}} \to \text{Alg}_{\mathbb{K},+}$, where T is the tensor algebra, is monadic.

One very good property about categories of algebras, is that their small limits is very well-behaved. These are exactly the same limits as in $\mathcal{C}$. We have the following result:

Theorem A.1.11 ([Theorem 5.6.5 [10], p. 181]). *A monadic functor* $G : \mathcal{D} \to \mathcal{C}$

- *creates any limits which $\mathcal{C}$ has,*
- *and creates any colimits $\mathcal{C}$ has and which are preserved by the monad T and its square $T \circ T$.*

A.2   Comonads and Categories of Coalgebras

In this section we will dualize the definitions and results from the last section. One could think of the dual themselves, but we do this for clearity.

Definition A.2.1 (Comonad). Let $\mathcal{C}$ be a category. We say that an endofunctor $W : \mathcal{C} \to \mathcal{C}$ together with

- a comulitplication $\nu : W \Rightarrow W \circ W$
- and a counit $\varepsilon : W \Rightarrow \mathrm{Id}_{\mathcal{C}}$

is a comonad, if the following diagrams commute

$$
\begin{array}{ccc}
W \circ W \circ W & \xleftarrow{W(\nu)} & W \circ W \\
{\scriptstyle \nu_W} \uparrow & & \uparrow {\scriptstyle \nu} \\
W \circ W & \xleftarrow{\quad \nu \quad} & W
\end{array}
\qquad
\begin{array}{ccccc}
W \circ \mathrm{Id}_{\mathcal{C}} & \xleftarrow{W(\varepsilon)} & W \circ W & \xrightarrow{\varepsilon_W} & \mathrm{Id}_{\mathcal{C}} \circ W \\
& \nwarrow & \uparrow {\scriptstyle \nu} & \nearrow & \\
& & W & &
\end{array}
$$

Lemma A.2.2 (Comonads from adjunctions). *Given an adjunction $F \dashv G : \mathcal{C} \to \mathcal{D}$ with*

- *unit $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$*
- *and a counit $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$,*

there is an associated comonad (W, ν, ε). Let $W = FG$, together with

- *a comulitplication given by the unit $\nu = F(\eta_G) : W \Rightarrow W \circ W$*
- *and the counit $\varepsilon : W \Rightarrow \mathrm{Id}_{\mathcal{D}}$*

is a comonad on $\mathcal{D}$.

Given any comonad $(W : \mathcal{D} \to \mathcal{D}, \nu, \varepsilon)$, we say that M is a W-coalgebra if there exists a morphism $w : M \to W(M)$ such that the following diagrams commute

$$W \circ W(M) \xleftarrow{W(w)} W(M) \qquad M \xleftarrow{\varepsilon_M} W(M)$$

$$\nu_M \uparrow \qquad\qquad w \uparrow \qquad\qquad\qquad w \uparrow$$

$$W(M) \xleftarrow{\quad w \quad} M \qquad\qquad\qquad M$$

Given two W-coalgebras M and N we say that a morphism $f : M \to N$ is a W-coalgebra morphism if the following diagram commutes

$$W(M) \xrightarrow{W(f)} W(N)$$

$$w \uparrow \qquad\qquad u \uparrow$$

$$M \xrightarrow{\quad f \quad} N$$

Definition A.2.3 (Category of coalgebras)**.** The category of coalgebras $\mathcal{C}_W$ is the category having

- objects M as W-coalgebras
- and morphisms $f : M \to N$ as W-coalgebra morphisms.

There is a cofree functor from $\mathcal{D}$ to W-coalgebras

$$F_W : \mathcal{D} \to \mathcal{D}_W,$$
$$M \mapsto (W(M), \nu_M).$$

By forgetting the W-coalgebra structure we obtain a forgetful functor

$$U_W : \mathcal{D}_W \to \mathcal{D},$$
$$(M, w) \mapsto M.$$

Lemma A.2.4 (Adjunctions from comonads)**.** *Given any comonad $(W, \nu, \varepsilon) : \mathcal{D} \to \mathcal{D}$, the the pair of functors U_W and F_W defines an adjunction*

$$U_W \dashv F_W : \mathcal{D}_W \to \mathcal{D}.$$

In the category of coalgebras $\mathcal{D}_W$ every object may be cogenerated from cofree W-coalgebras.

Definition A.2.5 (Cofree W-coalgebras)**.** (M, w) is a cofree W-coalgebra if there is an object $N \in \mathcal{D}$ and an isomorphism $(M, w) \simeq F_W(N)$.

Proposition A.2.6 (Cofree resolutions)**.** *Given any W-coalgebra M, then*

$$(M, m) \xrightarrow{\quad w \quad} (W(M), \nu_M) \overset{W(w)}{\underset{\nu_M}{\rightrightarrows}} (W \circ W(M), \nu_{W(M)})$$

is a limit diagram in $\mathcal{D}_W$.

Definition A.2.7 (Comonadicity)**.** Suppose that there is an adjunction $F \dashv G : \mathcal{C} \to \mathcal{D}$ such that $W = FG$. We say that the adjunction, or the $F : \mathcal{C} \to \mathcal{D}$, is comonadic if there exists an equivalence of categories $K : \mathcal{D}_W \to \mathcal{C}$ such that there are natural isomorphisms $F \circ K \simeq F_W$ and $K \circ U_W \simeq G$.

As we would except, we have the comonadic categories.

Example A.2.8 (coMod^C is comonadic over Mod$^{\mathbb{K}}$)*.* The adjoint pair of functors forget $\dashv$ _ $\otimes_{\mathbb{K}} C :$ coMod$^C \to$ Mod$^{\mathbb{K}}$ is comonadic.

Example A.2.9 (coAlg$_{\mathbb{K},\text{conil}}$ is comonadic over Mod$^{\mathbb{K}}$)*.* The adjoint pair of functors forget $\dashv T^c$ _ : coAlg$_{\mathbb{K},\text{conil}} \to$ Mod$^{\mathbb{K}}$.

Theorem A.2.10. *A comonadic functor $F : \mathcal{C} \to \mathcal{D}$*

- *creates any colimits which $\mathcal{D}$ has*
- *and creates and limits $\mathcal{D}$ has and which are preserved by the comonad W and its square $W \circ W$.*

A.3 Canonical Resolutions

As described by MacLane [11, p. 180]: "Monads and their duals, the comonads, play via Δ a central role in homological algebra,...". We will here look at a method to construct resolutions associated to comonads.

Let (W, ν, ε) be a comonad over an abelian category $\mathcal{D}$, then this is a comonoid in the category of endofunctors $(\text{End}\mathcal{D}, \circ, \text{Id}_{\mathcal{D}})$. By Proposition B.1.5, there is then a strong monoidal functor, which we denote by $W^{\circ?}$, $W^{\circ?} : \Delta_+^{\text{op}} \to \text{End}\mathcal{D}$. Using the standard representation of simplicial objects we see that the face and degeneracy maps are given as

$$\text{Id}_{\mathcal{D}} \qquad W \xrightarrow{\ \nu\ } W^{\circ 2} \underset{\nu_W}{\overset{W(\nu)}{\rightrightarrows}} W^{\circ 3} \rightrightarrows \cdots$$

$$\text{Id}_{\mathcal{D}} \xleftarrow{\ \varepsilon\ } W \underset{\varepsilon_W}{\overset{W(\varepsilon)}{\leftleftarrows}} W^{\circ 2} \Lleftarrow W^{\circ 3} \Lleftarrow \cdots$$

Let M be an object of $\mathcal{D}$. Evaluating $W^{\circ?}$ at M gives us a functor $W^{\circ?}(M) : \Delta_+^{\text{op}} \to \mathcal{D}$. This may be made into a cochain complex by Example 1.1.50,

$$\cdots \longrightarrow W^{\circ 3}(M) \longrightarrow W^{\circ 2}(M) \longrightarrow W(M) \xrightarrow{\ \varepsilon_M\ } M \longrightarrow 0 \longrightarrow \cdots.$$

Definition A.3.1 (Canonical W-resoultion)**.** The cochain complex as defined above is the canonical W-resolution at M.

These canonical resolution is more of a recipe to see how a comonad on an abelian category induces a resolution.

Example A.3.2 (Free resolution)*.* Let R be a $\mathbb{K}$-algebra. Then there is an adjunction $\_ \otimes_{\mathbb{K}} R \dashv$ forget $: \mathsf{Mod}^{\mathbb{K}} \to \mathsf{Mod}^R$. The comonad $\_ \otimes_{\mathbb{K}} R : \mathsf{Mod}^R \to \mathsf{Mod}^R$ induces free R-resolutions on every right R-module M.

$$\cdots \longrightarrow M \otimes_{\mathbb{K}} R^{\otimes 3} \longrightarrow M \otimes_{\mathbb{K}} R^{\otimes 2} \longrightarrow M \otimes_{\mathbb{K}} R \longrightarrow M \longrightarrow 0 \longrightarrow \cdots$$

Appendix B

Simplicial Objects

B.1 The Simplex Category

The simplex category is in some sense the categoryfication of the standard topological simplices, Δ^n. This category carries the necessary data in order to define concepts such as homology or homotopy. This section will give a brief review of this category.

Definition B.1.1 (The simplex category)**.** The simplex category Δ consists of ordered sets $[n] = \{0, ..., n\}$ for any $n \in \mathbb{N}$. A morphism $f \in \Delta([m], [n])$ is a monotone function, i.e.

$$a \leqslant b \in [m] \implies f(a) \leqslant f(b) \in [n].$$

Definition B.1.2 (The augmented simplex category)**.** Δ_+ is called the augmented simplex category, where we add an initial object $[-1] = \varnothing$.

Definition B.1.3 (The reduced simplex category)**.** $\Delta_i nj$ is called the reduced simplex category. The morphisms consists only of the injective morphisms in Δ.

Inspired from the topological simplices, the simplex category has coface and codegeneracy morphisms. The coface maps are the injective morphisms $\delta_i : [n] \to [n+1]$, while the codegeneracy maps are the surjective morphisms $\sigma_i : [n] \to [n-1]$.

$$\delta_i(k) = \begin{cases} k, \text{ if } k<i \\ k+1, \text{ otherwise} \end{cases} \qquad \sigma_i(k) = \begin{cases} k, \text{ if } k \leqslant i \\ k-1, \text{ otherwise} \end{cases}$$

Proposition B.1.4. *Every morphism in Δ factors into coface and codegeneracy maps.*

Proof. Prop ?? in [11]. □

This result tells us that understanding how these morphisms work in tandem will be very important in understanding the simplex category. Luckily, there are five identites which characterize these maps. These are called the cosimplical identites.

1. $\delta_j \delta_i = \delta_i \delta_{j-1}$, if $i < j$
2. $\sigma_j \delta_i = \delta_i \sigma_{j-1}$, if $i < j$
3. $\sigma_j \delta_i = id$, if $i = j$ or $i = j+1$
4. $\sigma_j \delta_i = \delta_{i-1} \sigma_j$, if $i > j+1$
5. $\sigma_j \sigma_i = \sigma_i \sigma_{j+1}$, if $i \leqslant j$

If we want a more visual description of the simplex category, we may think of them in this manner. An inductive tower with a greatly increasing amount of morphisms.

$$[-1] \longrightarrow [0] \overset{\delta_i}{\Longrightarrow} [1] \overset{\delta_i}{\Longrightarrow} [2] \overset{\delta_i}{\Longrightarrow} \dots$$

$$[-1] \qquad [0] \overset{\sigma_0}{\longleftarrow} [1] \overset{\sigma_i}{\Longleftarrow} [2] \overset{\sigma_i}{\Longleftarrow} \dots$$

The augmented simplex category has a universal monoid. Let $+ : \Delta_+ \times \Delta_+ \to \Delta_+$ be the functor acting on objects and morphisms as:

$$[m] + [n] = [m + n + 1]$$
$$(f + g)(k) = \begin{cases} f(k), \text{ if } k \leqslant m \\ g(k)+m, \text{ otherwise} \end{cases}$$

$(\Delta_+, +, [-1])$ becomes a monoidal category. Unitality is satisfied as $[-1] + [m] = [1 + m - 1] = [m] = [m] + [-1]$. Associativity follows from associativity of addition. Since addition acts on morphisms by juxtaposition we get that the maps $id_{[}0] : [0] \to [0]$, $\delta_0 : [-1] \to [0]$ and $\sigma_0 : [1] \to [0]$ allows us to express any morphism in Δ by summing them.

Since the object $[0]$ is terminal, it automatically becomes a monoid in $(\Delta, +, [-1])$. The unit is the unique map $\delta_0 : [-1] \to [0]$, and the multiplication is the uniqe map $\sigma_0 : [1] \to [0]$. Associativity and unitality is automatically satisfied by uniqueness of any morphism $f : [n] \to [0]$.

Proposition B.1.5. *Let $(\mathcal{C}, \otimes, Z)$ be a monoidal category. If (C, η, μ) is a monoid in $\mathcal{C}$, then there is a strong monoidal functor $: \Delta_+ \to \mathcal{C}$, such that $F[0] \simeq C$, $F\delta_{-1} \simeq \eta$ and $F\sigma_0 \simeq \mu$.*

Proof. This is proved in Mac Lanes book [11]. □

B.2   Simplicial Objects

To exert the properties of the simplex category on another category $\mathcal{C}$, we look at functors from Δ into $\mathcal{C}$.

Definition B.2.1 (Simplical object)**.** A simplicial object in $\mathcal{C}$ is a functor $S : \Delta^{op} \to \mathcal{C}$.

Such an object may be viewed as a collection of objects $\{S_n\}_{n \in \mathbb{N}}$ together with face maps $d^i : S_n \to S_{n-1}$ and degeneracy maps $s^i : S_n \to S_{n+1}$. Additionally, these maps must satisfy the simplicial identites. This is the dual to the cosimplicial identites.

Definition B.2.2 (Augmented simplicial object)**.** An augmented simplicial object is then a functor $S : \Delta^{op}_+ \to \mathcal{C}$.

The restricted functor $\bar{S} : \Delta^{op} \to \mathcal{C}$ is called the augmentation ideal of S.

Definition B.2.3 (Semi-simplicial object)**.** A semi-simplicial object is a functor $S : \Delta_{inj} \to \mathcal{C}$.

Observe that a semi-simplicial object may be considered as a collection of objects $\{S_n\}$ such that we only have face maps satisfying the 1st simplicial identity.

Definition B.2.4 (cosimplical object)**.** A cosimplicial object is a functor $S : \Delta \to \mathcal{C}$.

Such an object may be regarded as a collection of objects together with coface and codegeneracy maps satisfying the cosimplicial identities.

Simplicial objects are studied across many different fields of mathematics.

Example B.2.5 (Simplicial sets)**.** A simplicial set S is a collection of sets together with face and degeneracy maps. This is a functor $S : \Delta^{op} \to$ Set. The category of simplicial sets is usually denoted as sSet or Set$_\Delta$.

Simplicial sets are important in ∞-category theory. Some special simplicial sets called quasicategories defines a model for $(\infty, 1)$-categories. A level up, simplicially enriched categories gives us a model for $(\infty, 2)$-categories.

Example B.2.6 (The standard topological n-simplex)**.** The topological n-simplex Δ^n is a topological space. Abstracting away the n we get a functor $\Delta- : \Delta \to$ Top. In this manner, the collection of standard n-simplicies is a cosimplicial object of Top.

Example B.2.7 (Rings)**.** Any ring R is by definition a monoid in the category of abelian groups. By the above proposition, this monoid is uniquely determined by a strong monoidal functor $R : \Delta_+ \to$ Ab. Thus any ring is a cosimplicial object of Ab.

An important result by Dold and Kan classyfies which chain complexes are simplicial objects in the category Mod$_\mathbb{K}$.

Theorem B.2.8 (Dold-Kan Correspondance)**.** *For any abelian category $\mathcal{A}$ there is an equivalence of categories*

$$Fun(\Delta^{op}, \mathcal{A}) \quad \overset{N}{\underset{\Gamma}{\perp}} \quad Ch^-(\mathcal{A}).$$

Example B.2.9 (Nerve of a chain complex). The above result states that every non-positive cochain complex may be thought of as a simplical object of $Mod_{\mathbb{K}}$.

B.3   Simplicial Homotopy Theory

Appendix C

Spectral Sequences

C.1   Filtrations

C.2   Spectral Sequence

Appendix D

Symmetric Monoidal Categories