Strongly Homotopy Associative Quasi-isomorphisms

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Abstract

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Sammendrag

Fyll inn sammendraget

Acknowledgements

Thank the people in your life who has made this journey easier :D

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Chapter 1

Bar and Cobar Construction

In Stasheffs papers [1] and [2], a strongly homotopy associative algebra, or A_∞-algebra, over a field is a graded vector space together with homogenous linear maps $m_n : A^{\otimes n} \to A$ of degree $n-2$ satisfying some homotopical relations. This will be made precise later. We may regard m_2 to be a multiplication of A, it is however not a priori associative. The associator of m_2 is taken to be the homotopical relation of m_3. Thus, we know that the homotopy of A is an associative algebra. The maps m_n corresponds uniquely to a map $m^c : BA \to \bar{A}[1]$, which extends to a coderivation $m^c : BA \to BA$ of the bar construction of A. So we could instead define an A_∞-algebra to be a coalgebra on the form BA.

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In order to understand the bar construction we will first study it on associative algebras. Given a differential graded coassociative coalgebra C and a differential graded associative algebra A, we say that a homogenous linear transformation $\alpha : C \to A$ is twisting if it satisfies the Maurer-Cartan equation:

$$\partial\alpha + \alpha \star \alpha = 0.$$

Let $Tw(C, A)$ be the set of twisting morphisms, then considering it as a functor $Tw : CoAlg_{\mathbb{K}}^{op} \times Alg_{\mathbb{K}} \to Ab$ we want to show that it is represented in both arguments. Moreover, these representations give rise to an adjoint pair of functors, called the bar and cobar construction.

$$\begin{tikzcd} \begin{matrix} Aug^{\bullet} \\ Alg_{\mathbb{K}} \end{matrix} \arrow[r, bend left, "B"] \arrow[r, phantom, "\top"] & \begin{matrix} Conil^{\bullet} \\ CoAlg_{\mathbb{K}} \end{matrix} \arrow[l, bend left, "\Omega"] \end{tikzcd}$$

The bar and cobar construction will be the basis for our discussion of A_∞-algebras. As the bar construction can be used to define A_∞-algebras, we may easily dualize this to define A_∞-coalgebras in terms of the cobar construction. This chapter will follow the notations and progression presented in Loday and Vallete [3] to develop the theory for the bar-cobar adjunction.

1.1 Algebras

This section is a review of associative algebras. We will define unital associative algebras and possibly non-unital associative algebras, which we will call algebras and non-unital algebras respectively. The collection of algebras together with homomorphisms between them form the category $Alg_{\mathbb{K}}$ of algebras. Other types of algebras such as augmented and tensor algebras will be defined as well.

Definition 1.1.1 (Algebra)**.** Let $\mathbb{K}$ be a field with unit 1. An algebra A over $\mathbb{K}$ is a vector space with structure morphisms called multiplication and unit,

$$(\nabla_A) : A \otimes_{\mathbb{K}} A \to A$$
$$\upsilon_A : \mathbb{K} \to A,$$

satisfying the associativity and identity laws.

$$\text{(associativity)} \quad (a\nabla_A b)\nabla_A c = a\nabla_A(b\nabla_A c)$$
$$\text{(unitality)} \quad \upsilon_A(1)\nabla_A a = a = a\nabla_A \upsilon_A(1)$$

Whenever A does not posess a unit morphism, we will call A a non-unital algebra. Only the associativity law must hold.

Definition 1.1.2 (Algebra homomorphisms)**.** Let A and B be algebras. Then $f : A \to B$ is an algebra homomorphism if

1. f is $\mathbb{K}$-linear
2. $f(ab) = f(a)f(b)$
3. $f \circ \upsilon_A = \upsilon_B$

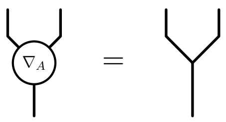
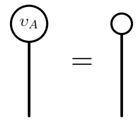
Whenever A and B are non-unital, we only require 1 and 2 for a homomorphism of non-unital algebras.

Definition 1.1.3 (Category of algebras)**.**
- Let $Alg_{\mathbb{K}}$ denote the category of algebras. It's objects consists of every algebra A, and the morphisms are algebra homomorphisms. The sets of morphisms between A and B are denoted as $Alg_{\mathbb{K}}(A, B)$.
- Let $nAlg_{\mathbb{K}}$ denote the category of non-unital algebras. It's objects consists of every non-unital algebra A, and the morphisms are non-unital algebra homomorphisms. The sets of morphisms between A and B are denoted as $nAlg_{\mathbb{K}}(A, B)$.

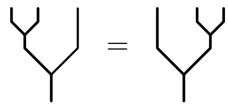
Observe that for an algebra A, the triple $(A, \nabla_A, \upsilon_A)$ is a monoid in $mod_{\mathbb{K}}$. Thus, we may say that an algebra is a triple where the following diagrams commute.

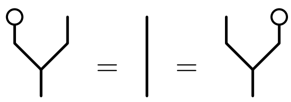
$$\begin{tikzcd}
A \otimes_{\mathbb{K}} A \otimes_{\mathbb{K}} A \arrow[r, "(\nabla_A)\otimes id_{\mathbb{K}}"] \arrow[d, "id_{\mathbb{K}}\otimes(\nabla_A)"] & A \otimes_{\mathbb{K}} A \arrow[d, "(\nabla_A)"] \\
A \otimes_{\mathbb{K}} A \arrow[r, "(\nabla_A)"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes_{\mathbb{K}} \mathbb{K} \arrow[r, "id_A\otimes\upsilon_A"] \arrow[dr, "\simeq"'] & A \otimes_{\mathbb{K}} A \arrow[d, "(\nabla_A)"] & \mathbb{K} \otimes_{\mathbb{K}} A \arrow[l, "\upsilon_A\otimes id_A"'] \arrow[dl, "\simeq"] \\
& A &
\end{tikzcd}$$

The final method we will use to represent an algebra are electric circuits. An electric circuit is a diagram read from top to bottom, where each column represent a different vector space in a tensor. Morphisms in such diagrams are figures, conjunctions, twistings and etc. E.g. The multiplication operator may be represented as a converging fork, and the unit as a source.

(Multiplication)  (Unit) 

Using these operations we can now reformulate the algebra laws. These are the electric laws for an algebra:

(Associativity) 

(unitality) 

Definition 1.1.4 (Augmented algebras)**.** Let A be an algebra. It is called augmented if there is an algebra homomorphism $\varepsilon : A \to \mathbb{K}$.

If A is an augmented algebra, then it decomposes into $\mathbb{K} \oplus Ker\varepsilon$ as a module. The splitting is given by unitality of the morphism $\varepsilon : A \to \mathbb{K}$, as we know that $\varepsilon(\upsilon_A) = id_{\mathbb{K}}$. The kernel of ε is called the augmentation ideal or redecued algebra and we will denote it as $\bar{A}$. Taking kernels gives an equivalence of categories between augmented algebras and non-unital algebras, with unitization as the quasi-inverse. The category of augmented algebras is denoted as $AugAlg_{\mathbb{K}}$ or $\overset{Aug}{Alg}_{\mathbb{K}}$.

Definition 1.1.5 (Tensor algebra)**.** Let V be a $\mathbb{K}$-module. We define the tensor algebra $T(V)$ of V as the module

$$T(V) = \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus ...$$

Given two strings $v^1..v^i$ and $w^1...w^j$ in $T(V)$ we define the multiplication by the concatenation operation.

$$\begin{aligned} \nabla_{T(V)} : T(V) \otimes_{\mathbb{K}} T(V) &\to T(V) \\ (v^1...v^i) \otimes (w^1...w^j) &\mapsto v^1...v^i w^1...w^j \end{aligned}$$

The unit is given by including $\mathbb{K}$ into $T(V)$.

$$\begin{aligned} \upsilon_{T(V)} : \mathbb{K} &\to T(V) \\ 1 &\mapsto 1 \end{aligned}$$

Observe that the tensor algebra is augmented. The projection from $T(V)$ into $\mathbb{K}$ is an algebra homomorphism, so we may split the tensor algebra into its unit and its augmentation ideal $T(V) \simeq \mathbb{K} \oplus T(\bar{V})$. We call $T(\bar{V})$ the reduced tensor algebra.

Proposition 1.1.6 (Tensor algebra is free)**.** *The tensor algebra is the free algebra over the category of $\mathbb{K}$-modules, i.e. for any $\mathbb{K}$-module V there is a natural isomorphism $Hom_{\mathbb{K}}(V, A) \simeq Alg_{\mathbb{K}}(T(V), A)$.*

The reduced tensor algebra is the fre non-unital algebra over the category of $\mathbb{K}$-modules, i.e. for any $\mathbb{K}$-module V there is a natural isomorphism $Hom_{\mathbb{K}}(V, A) \simeq nAlg_{\mathbb{K}}(T(\bar{V}), A)$.

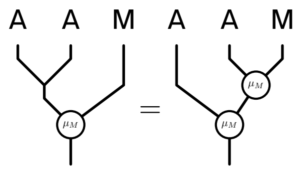
Proof. This proposition should be evident from the description of an algebra homomorphism from a tensor algebra. If $f : T(V) \to A$ is an algebra homomorphism, then f must satisfy the following conditions:

- (Unitality) $\quad f(1) = 1$
- (Homomorphism property) $\quad$ Given $v, w \in V$, then $f(vw) = f(v)\nabla_A f(w)$

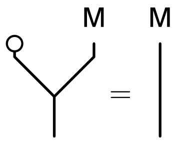
By induction, we see that f is completely determined by where it sends the elements of V. Thus restriction by the inclusion of V into $T(V)$ induces a bijection. ☐

Definition 1.1.7 (Modules)**.** Let A be an algebra. A $\mathbb{K}$-module M is said to be a left (right) A-module if there exists a structure morphism $\mu_M : A \otimes_{\mathbb{K}} M \to A$ ($\mu_M : M \otimes_{\mathbb{K}} A \to A$) called multiplication. We require that μ_M is associative with respect to the multiplication and preserves the unit of A, i.e. the electric laws are satisfied.

(Associativity)



(unitality)



Definition 1.1.8 (A-linear homomorphisms)**.** Let M, N be two left A-modules. A morphism $f : M \to N$ is called A-linear if it is $\mathbb{K}$-linear and for any a in A, $f(am) = af(m)$.

The category of left A-modules is denoted as Mod_A, where the morphisms $Hom_A(\_, \_)$ are A-linear. Likewise, the category of right A-modules is denoted as Mod^A.

Proposition 1.1.9. *Let M be a $\mathbb{K}$-module. The module $A \otimes_{\mathbb{K}} M$ is a left A-module. Moreover, it is the free left module over $\mathbb{K}$-modules, i.e. there is an isomorphism $Hom_{\mathbb{K}}(M, N) \simeq Hom_A(A \otimes_{\mathbb{K}} M, N)$.*

1.2 Coalgebras

This section aims to dualize the definitions from last section. To this end we will define counital coassociative coalgebras and non-counital coassociative coalgebras, which will be called coalgebras and non-counital coalgebras respectively. The collection of coalgebras together with coalgebra homomorphisms is the category $CoAlg_{\mathbb{K}}$. Due to some ill-behavior, this dualization is only a true dualization under some finiteness conditions for the algebras. Thus we will see that the proper dual concept will be of conilpotent coalgebras. We will see that the cofree coalgebra is conilpotent.

Definition 1.2.1 (Coalgebra)**.** Let $\mathbb{K}$ be a field. A coalgebra C over $\mathbb{K}$ is a $\mathbb{K}$-module with structure morphisms called comultiplication and counit,

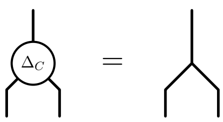
$$(\Delta_C) : C \to C \otimes_{\mathbb{K}} C$$
$$\varepsilon_C : C \to \mathbb{K},$$

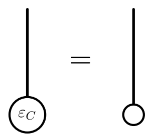
satisfying the coassociativity and coidentity laws.

$$\text{(coassociativity)} \quad (\Delta_C \otimes id_C) \circ \Delta_C(c) = (id_C \otimes \Delta_C) \circ \Delta_C(c)$$
$$\text{(counitality)} \quad (id_C \otimes \varepsilon_C) \circ \Delta_C(c) = c = (\varepsilon_C \otimes id_C) \circ \Delta_C(c)$$

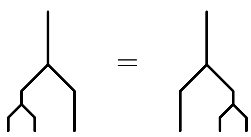
We define repeated application of comultiplication as $\Delta_C^n = (\Delta_C \otimes id_C \otimes ...) \circ \Delta_C^{n-1}$. Notice that the choice of where we put comultiplication in the tensor does not matter, as coassociativity require all of the choices to be equal.

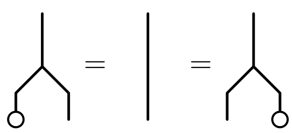
We may dualize the electric circuits of an algebra to coalgebras. In this manner our structure morphisms would be upside down relative to the algebra morphisms. Thus comultiplication becomes a diverging fork and counit is a sink.

(Comultiplication) 

(Counit) 

We then obtain the electric laws for a coalgebra by flipping the circuits around.

(Coassociativity) 

(Counitality) 

Definition 1.2.2 (Coalgebra homomorphism)**.** Let C and D be coalgebras. Then $f : C \to D$ is a coalgebra morphism if

1. f is $\mathbb{K}$-linear
2. $(f \otimes f) \circ \Delta_C(c) = \Delta_D(f(c))$
3. $\varepsilon_D(f) = \varepsilon_C$

Whenever C and D are non-counital, we only require 1 and 2 for a homomorphism of non-counital coalgebras.

Definition 1.2.3 (Category of Coalgebras)**.**
- Let $CoAlg_{\mathbb{K}}$ denote the category of coalgebras. It's objects consists of every coalgebra C, and the morphisms are coalgebra homomorphisms. The sets of morphisms between C and D are denoted as $CoAlg_{\mathbb{K}}(C, D)$.
- Let $nCoAlg_{\mathbb{K}}$ denote the category of non-unital algebras. It's objects consists of every non-unital algebra C, and the morphisms are non-unital algebra homomorphisms. The sets of morphisms between C and D are denoted as $nCoAlg_{\mathbb{K}}(C, D)$.

Example 1.2.4 (The coalgebra $\mathbb{K}$). The field $\mathbb{K}$ can be given a coalgebra structure over itself. Since $\{1\}$ is a basis for $\mathbb{K}$ we define the structure morphisms as

$$\Delta_{\mathbb{K}}(1) = 1 \otimes 1$$
$$\varepsilon(1) = 1.$$

One may check that these morphisms are indeed coassociative and counital. Thus we may regard our field as either an algebra or coalgebra over itself.

Definition 1.2.5 (Coaugmented coalgebras)**.** Let C be a coalgebra. C is coagumented if there is a coalgebra homomorphism $v : \mathbb{K} \to C$.

If C is a coaugmented coalgebra, then it splits as $C \simeq \mathbb{K} \oplus Cokv$. The splitting is given by counitality of v, as $\varepsilon_C(v) = id_{\mathbb{K}}$. We call the cokernel $Cokv = \bar{C}$ for the coaugmentation quotient or reduced coalgebra, and its reduced coproduct may be explicitly given as

$$\bar{\Delta}_C(c) = \Delta_C(c) - 1 \otimes c - c \otimes 1.$$

Definition 1.2.6 (Tensor Coalgebras)**.** Let V be a $\mathbb{K}$-module. We define the tensor coalgebra $T^c(V)$ of V as the module

$$T^c(V) = \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus ...$$

Given a string $v^1...v^i$ in $T(V)$ we define the comultiplication by the deconcatenation operation.

$$\begin{aligned} \Delta_{T^c(V)} &: T^c(V) \to T^c(V) \otimes_{\mathbb{K}} T^c(V) \\ &v^1...v^i \mapsto 1 \otimes (v^1...v^i) + (\sum_{j=1}^{n-1} (v^1...v^j) \otimes (v^{j+1}...v^i)) + (v^1...v^i) \otimes 1 \end{aligned}$$

The counit is given by projecting $T^c(V)$ onto $\mathbb{K}$.

$$\begin{aligned} \varepsilon_{T^c(V)} : T^c(V) &\to \mathbb{K} \\ 1 &\mapsto 1 \\ v^1 ... v^i &\mapsto 0 \end{aligned}$$

Notice that the tensor coalgebra is coaugmented. Its coaugmentation is given by the inclusion of $\mathbb{K}$ into $T^c(V)$. We may split $T^c(V) \simeq \mathbb{K} \oplus \bar{T}^c(V)$, where $\bar{T}^c(V)$ is the reduced tensor coalgebra.

In order to get cofreeness for the tensor coalgebra we need some finiteness conditions. This is one of the properties which is ill-behaved when we are dualizing the tensor algebra. The extra assumption which we will need is to assume that the coalgebras are conilpotent. Let $C \simeq \mathbb{K} \oplus \bar{C}$ be a coaugmented coalgebra, we define the coradical filtration of C as a filtration $Fr_0 C \subseteq Fr_1 C \subseteq ... \subseteq Fr_r C \subseteq ...$ by the submodules:

$$\begin{aligned} &Fr_0 C = \mathbb{K} \\ &Fr_r C = \mathbb{K} \oplus \{c \in \bar{C} \mid \forall n \geq r \bar{\Delta}_C(c) = 0\}. \end{aligned}$$

Definition 1.2.7 (Conilpotent coalgebras)**.** Let C be a coaugmented coalgebra. We say that C is conilpotent if its coradical filtration is exhaustive, i.e. $\varinjlim_r Fr_r C \simeq C$. The subcategory of conilpotent coalgebras will be denoted as $ConilCoAlg_{\mathbb{K}}$ or ${}^{Conil}_{CoAlg_{\mathbb{K}}}$.

Proposition 1.2.8 (Conilpotent tensor coalgebra)**.** *Let V be a $\mathbb{K}$-module. The tensor coalgebra $T^c(V)$ is conilpotent.*

Proof. Let $v \in V$, then $\Delta_{T^c(V)}(v) = 1 \otimes v + v \otimes 1$ and $\bar{\Delta}_{T^c(V)}(v) = 0$. We then observe the following:

$$\begin{aligned} Fr_0 T^c(V) &= \mathbb{K} \\ Fr_1 T^c(V) &= \mathbb{K} \oplus V \\ Fr_r T^c(V) &= \bigoplus_{i \leq r} V^{\otimes i} \end{aligned}$$

This shows that the coradical filtration is exhaustive. □

Proposition 1.2.9 (Cofree tensor coalgebra)**.** *The tensor coalgebra is the cofree conilpotent coalgebra over the category of $\mathbb{K}$-modules, i.e. for any $\mathbb{K}$-module V and any conilpotent coalgebra C there is a natural isomorphism $Hom_{\mathbb{K}}(\bar{C}, V) \simeq {}^{Conil}_{CoAlg_{\mathbb{K}}}(C, T^c(V))$.*

Proof. This proposition should be evident from the description of a coalgebra homomorphism into the a tensor coalgebra. If $g : C \to T^c(V)$ is a coalgebra homomorphism, then g must satisfy the following conditions:

1. (Coaugmentation) $g(1) = 1$

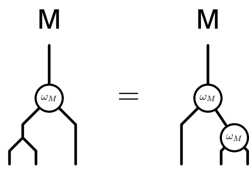
2. (Counitality) Given $c \in \bar{C}$ then $\varepsilon_{T^c(V)} \circ g(c) = 0$
3. (Homomorphism property) Given $c \in C$ then $\Delta_{T^c(V)}(g(c)) = (g \otimes g) \circ \Delta_C(c)$

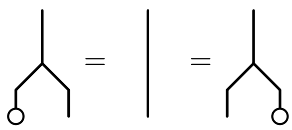
We will construct the maps for the isomorphism explicitly. If $g : C \to T^c(V)$ is a coalgebra homomorphism, then composing with projection gives a map $\pi \circ g : C \to V$. Note that $\pi \circ g(1) = 0$, so this is essentially a map $\pi \circ g : \bar{C} \to V$. For the other direction, let $\bar{g} : \bar{C} \to V$. We will then define g as

$$g = id_{\mathbb{K}} \oplus \sum_{i=1}^{\infty} (\otimes^i \bar{g}) \bar{\Delta}_C^{i-1}.$$

Observe that g is well defined, since convergence of the sum follows from conilpotency of C. One may then check that g is a coalgebra homomorphism, which yields the result. □

Definition 1.2.10 (Comodules)**.** Let C be a coalgebra. A $\mathbb{K}$-module M is said to ba left (right) C-comodule if there exist a structure morphism $\omega_M : M \to C \otimes_{\mathbb{K}} M$ ($\omega_M : M \to M \otimes_{\mathbb{K}} C$) called comultiplication. We require that ω_M is coassociative with respect to the comultiplication of C and preserves the counit of C, i.e. the electric laws are satisfied.

(Coassociativity) 

(Counitality) 

Definition 1.2.11 (C-colinear homomorphism)**.** Let M, N be two left C-comodules. A morphism $g : M \to N$ is called C-colinear if it is $\mathbb{K}$-linear and for any m in M, $\omega_N(g(m)) = (id_C \otimes g)\omega_M(m)$.

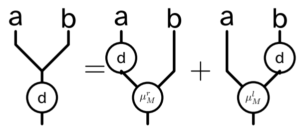
The category of left C-comodules is denoted as $CoMod_C$, where the morphisms $CoHom_C(\_,\_)$ are C-colinear. Likewise, the category of right C-comodules is denoted as $CoMod^C$.

Proposition 1.2.12. *Let M be a $\mathbb{K}$-module. The module $C \otimes_{\mathbb{K}} M$ is a left C-comodule. Moreover, it is the cofree left comodule over $\mathbb{K}$-modules, i.e. there is an isomorphism $Hom_{\mathbb{K}}(N, M) \simeq CoHom_C(N, C \otimes_{\mathbb{K}} M)$.*

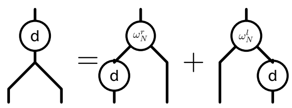
1.3 Derivations and DG-Algebras

In this section we will look at differential graded objects and convolution products. We will define derivations and coderivations to obtain differential graded algebras and coalgebras. Moreover we will see that the set of homogenous homomorphisms between differential graded objects is itself differential graded. Moreover, whenever we look at morphisms between dg coalgebras and dg algebras, we can give this object the convolution operator, making the set a dg algebra.

Definition 1.3.1 (Derivations and Coderivations)**.** Let M be an A-bimodule. A $\mathbb{K}$-linear morphism $d : A \to M$ is called a derivation if $d(ab) = d(a)b + ad(b)$, i.e. electrically:



Let N be a C-bicomodule. A $\mathbb{K}$-linear morphism $d : N \to C$ is called a coderivation if $\Delta_C \circ d = (d \otimes id_C) \circ \omega^r_N + (id_C \otimes d) \circ \omega^l_N$, i.e. electrically:



Proposition 1.3.2. *Let V be a $\mathbb{K}$-module and M be a $T(V)$-bimodule. A $\mathbb{K}$-linear morphism $f : V \to M$ uniquely determines a derivation $d_f : T(V) \to M$, i.e. there is an isomorphism $Hom_{\mathbb{K}}(V, M) \simeq Der(T(V), M)$.*

Let N be a $T^c(V)$-cobimodule. A $\mathbb{K}$-linear morphism $g : M \to V$ uniquely determines a coderivation $d^c_g : N \to T^c(V)$, i.e. there is an isomorphism $Hom_{\mathbb{K}}(N, V) \simeq Coder(N, T^c(V))$.

Proof. Let $a_1 \otimes ... \otimes a_n$ be an elementary tensor of $T(V)$. We define $d_f(a_1 \otimes ... \otimes a_n) = \sum_{i=1}^n a_1 ... f(a_i) ... a_n$ and $d_f(1) = 0$. Notice that d_f is by definition a derivation.

Restriction to V gives the natural isomorphism. Let $i : V \to T(V)$, then $i^* d_f = f$. Let $d : T(V) \to M$ be a derivation, then $d_{i^*d} = d$. Suppose that $g : M \to N$ is a morphism between $T(V)$-bimodules, then naturality follows from bi-linearity.

In the dual case, d^c_g is a bit tricky to define. Let $\omega^l_N : N \to N \otimes T^c(V)$ and $\omega^r_N : N \to T^c(V) \otimes N$ denote the coactions on N. Since $T^c(V)$ is conilpotent we get the same kind of finiteness restrictions on N. We define the reduced coactions as $\bar{\omega}^l_N = \omega^l_N - \_ \otimes 1$ and $\bar{\omega}^r_N = \omega^r_N - 1 \otimes \_$, this is well-defined by coassociativity. Observe that for any $n \in N$ there are $k, k' > 0$ such that $\bar{\omega}^{l^k}_N(n) = 0$ and $\bar{\omega}^{r^{k'}}_N(n) = 0$.

Let $n^{(i)}_{(k)}$ denote the extension of n by k coactions at position i, i.e. $n^{(i)}_{(k)} = \bar{\omega}^{r^i}_N \bar{\omega}^{l^{k-i}}_N(n)$. The extension of n by k coactions is then the sum over every position i, $n_{(k)} = \sum_{i=0}^k n^{(i)}_{(k)}$. Observe that $n_{(0)} = n$. The grade of n may be thought of as the smallest k such that $n_{(k)}$ is zero. This grading gives us the coradical filtration of N, and it is exhaustive by the finiteness restrictions given above. So every element of N may be given a finite grade.

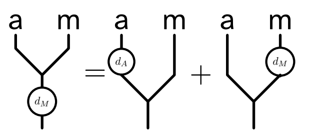
If $g : N \to V$ is a linear map, we may think of it as a map sending every element of N to an element of $T^c(V)$ of grade 1. To get a map which sends element of grade k to grade k, we must extend the morphism. Let $\pi : T^c(V) \to V$ be the linear projection and define $g^{(i)}_{(k)} = \pi \otimes ... \otimes \pi \circ g \otimes \pi$ as a morphism which is g at the i-th argument, but the projection otherwise. d^c_g is then defined as the sum over each coaction and coordinate.

$$d^c_g(n) = \sum_{k=0}^{\infty} \sum_{i=0}^{k} g^{(i)}_{(k)}(n^{(i)}_{(k)})$$

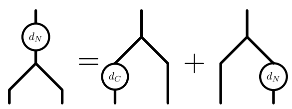
Upon closer inspection we may observe that this is the dual construction of the derivation morphism. It is well-defined as the sum is finite by the finiteness restrictions. The map is a coderivation by duality, and the natural isomorphism is given by composition with the projection map π. $\square$

Definition 1.3.3 (Differential algebra)**.** Let A be an algebra. We say that A is a differential algebra if it is equipped with at least one derivation $d : A \to A$. Dually, a coalgebra C is called differential if it is equipped with at least one coderivation $d : C \to C$.

Definition 1.3.4 (A-derivation)**.** Let (A, d_A) be a differential algebra and M a left A-module. A $\mathbb{K}$-linear morphism $d_M : M \to M$ is called an A-derivation if $d_M(am) = d_A(a)m + ad_M(m)$, or electrically:



Dually, given a differential coalgebra (C, d_C) and N a left C-comodule, a $\mathbb{K}$-linear morphism $d_N : N \to N$ is a coderivation if $\omega_N \circ d_N = (d_C \otimes id_N + id_C \otimes d_N) \circ \omega_N$, or electrically:



Proposition 1.3.5. *Let A be a differential algebra and M a $\mathbb{K}$-module. A $\mathbb{K}$-linear morphism $f : M \to A \otimes_{\mathbb{K}} M$ uniquely determines a derivation $d_f : A \otimes M \to A \otimes M$, i.e. there is an isomorphism $Hom_{\mathbb{K}}(M, A \otimes_{\mathbb{K}} M) \simeq Der(A \otimes_{\mathbb{K}} M)$. Moreover, d_f is given as $(\nabla_A \otimes id_M) \circ (id_A \otimes f) + d_A \otimes id_M$.*

Dually, if C is a differential coalgebra and N is a $\mathbb{K}$-module, then a $\mathbb{K}$-linear morphism $g : C \otimes N \to N$ uniquely determines a coderivation $d_g : C \otimes_{\mathbb{K}} N \to C \otimes_{\mathbb{K}} N$. There is an isomorphism $Hom_{\mathbb{K}}(C \otimes_{\mathbb{K}} N, N) \simeq Coder(C \otimes_{\mathbb{K}} N)$, and d_g is given as $(id_C \otimes g) \circ (\Delta_C \otimes id_N) + d_C \otimes id_N$.

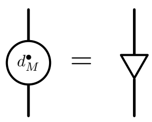
Proof. ... $\square$

Recall that a module M^* is $\mathbb{Z}$ graded if it decomposes as a sum $M^* = \bigoplus_{z:\mathbb{Z}} M^z$. Let M^*, N^* be graded modules and $f : M^* \to N^*$ is a homogenous $\mathbb{K}$-linear morphism of degree n if it preserves the grading, that is $f(M^i) \subseteq N^{n+i}$. We denote the degree of f as $|f|$. The category of graded modules will be denoted as $GrMod_{\mathbb{K}}$ or $Mod^*_{\mathbb{K}}$. Generally $\mathcal{C}^*$ is the category of graded objects whenever it makes sense, and the graded $\mathbb{K}$-module of morphisms between two graded objects is denoted as $Hom^*_{\mathbb{K}}(M^*, N^*)$.

$M^\bullet$ is called a chain complex if it comes equipped with a homogenous morphism of degree 1, like $d^\bullet_M : M^\bullet \to M^\bullet$, such that ${d^\bullet_M}^2 = 0$. This morphism is called differential. A chain morphism $f : M^\bullet \to N^\bullet$ is a homogenous $\mathbb{K}$-linear morphism of degree 0, such that $f \circ d^\bullet_M = d^\bullet_N \circ f$. The category of chain complexes will be denoted as $ChMod_{\mathbb{K}}$ or $Mod^\bullet_{\mathbb{K}}$. Generally $\mathcal{C}^\bullet$ is the category of chain complexes whenever it makes sense, and the $\mathbb{K}$-module of morphisms between two chain complexes is denoted as $Hom^\bullet_{\mathbb{K}}(M^\bullet, N^\bullet)$.

The functor $\_[n] : Mod^\bullet_{\mathbb{K}} \to Mod^\bullet_{\mathbb{K}}$ shifts the degree on each object by adding n to each grade, it is called the shift functor. Let $\otimes$ denote the total tensor product in $Mod^\bullet_{\mathbb{K}}$. There is an isomorphism between the identity shift functor and total tensor of the stalk of $\mathbb{K}$, $\_[0] \simeq \bar{\mathbb{K}} \otimes \_$. In the same manner, shifting n-fold becomes isomorphic to tensoring with the shifted stalk of $\mathbb{K}$, $\_[n] \simeq \bar{\mathbb{K}}[n] \otimes \_$. For our purposes we will let $(A^\bullet, d^\bullet_A)[n] = (A^{\bullet+n}, -d^{\bullet+n}_A)$. The koszul sign rule gives us a switching map for the tensor product. Thus, if $f^* : A^\bullet \to B^\bullet$ is a morphism of degree k, then $f^*[n] = (-1)^{k \cdot n} f^{*+n}$.

In electric diagrams we will write triangles for the differential if there are no ambiguity.



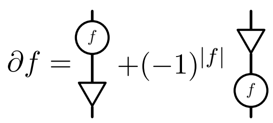
Proposition 1.3.6. *Let $M^\bullet$ and $N^\bullet$ be two chain complexes. The graded module of morphisms $Hom^*_{\mathbb{K}}(M^\bullet, N^\bullet)$ is a chain complex, given by the differential $\partial(f) = d^\bullet_N \circ f - (-1)^{|f|} f \circ d^\bullet_M$.*

Proof. We observe that $\partial : Hom^*_{\mathbb{K}}(M^\bullet, N^\bullet) \to Hom^*_{\mathbb{K}}(M^\bullet, N^\bullet)$ is a morphism of degree 1. It remains to check that $\partial^2 = 0$. Pick any homogenous morphism $f : M^\bullet \to N^\bullet$.

$$\begin{aligned} \partial^2(f) = \partial(d^\bullet_N \circ f - (-1)^{|f|} f \circ d^\bullet_M) &= \partial(d^\bullet_N \circ f) - (-1)^{|f|} \partial(f \circ d^\bullet_M) \\ &= -(-1)^{|d^\bullet_N \circ f|} d^\bullet_N \circ f \circ d^\bullet_M - (-1)^{|f|} d^\bullet_N \circ f \circ d^\bullet_M = 0 \end{aligned}$$

□

In an electric diagram we write ∂f as a sum of circuits.



Observe that $f : M^\bullet \to N^\bullet$ of degree 0 is a chain morphism if and only if $\partial(f) = 0$. We then observe that $Hom^\bullet_{\mathbb{K}}(M^\bullet, N^\bullet) \simeq Z^0 Hom^*_{\mathbb{K}}(M^\bullet)$.

To complete the definitions of graded modules and chain complexes to algebras we would like the structure morphisms to respect the given structure. E.g. if a and b are homogenous elements, we would like that the degree of ab is the sum of its parts, i.e. $|ab| = |a| + |b|$. Since multiplication by identity doesn't do anything, we want that the identity lives in the 0'th degree, and so forth.

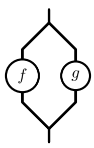
Definition 1.3.7 (Graded algebra)**.** Let A^* be a graded $\mathbb{K}$-module. We say that A^* is a graded algebra if A^* is an algebra such that ∇_A and υ_A are homogenous and of degree 0. Dually, C^* is a graded coalgebra if Δ_C and ε_C are homogenous and of degree 0.

Definition 1.3.8 (Differential graded algebra)**.** Let $A^\bullet$ be a chain complex over $\mathbb{K}$. We say that $A^\bullet$ is a differential graded algebra, or dg algebra, if it is a graded algebra and the differential is a graded derivation, i.e. $d_A(ab) = d_A(a)b + (-1)^{|a|} a d_A(b)$.

Dually, $C^\bullet$ is a differential graded coalgebra if $C^\bullet$ is a graded coalgebra and the differential is a graded coderivation.

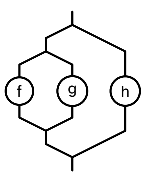
1.4 Convolution Algebras

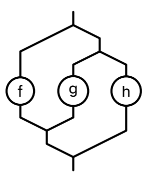
Let C be a coalgebra and A an algebra, then if $f, g : C \to A$ are $\mathbb{K}$-linear morphism we may define $f \star g = \nabla_A(f \otimes g)\Delta_C$. We call the operation $\star$ for convolution.

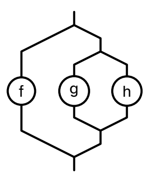
$f \star g =$ 

Proposition 1.4.1 (Convolution algebra)**.** *The $\mathbb{K}$-module $Hom_{\mathbb{K}}(C, A)$ is an associative algebra when equipped with convolution $\star : Hom_{\mathbb{K}}(C, A) \to Hom_{\mathbb{K}}(C, A)$. The unit is given by $1 \mapsto \upsilon_A \circ \varepsilon_C$.*

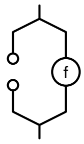
Proof. This proposition follows from (co)associativity and (co)unitality of (C) A.

$(f \star g) \star h \;=$ 

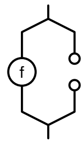
$=$ 

$=$ 

$= \; f \star (g \star h)$

$(\upsilon_A \circ \varepsilon_C) \star f =$ 

$=$ 

$=$ 

$= f \star (\upsilon_A \circ \varepsilon_C)$

□

If A is an algebra and C is a coalgebra, then they may be given the structure of a differential algebra by attaching the 0 morphism to each algebra as the (co)derivation. In this case proposition 1.3.5 says that a morphism $f : M \to A \otimes_{\mathbb{K}} M$ determines the derivation given as $d_f = (\nabla_A \otimes id_M) \circ (id_A \otimes f)$. Dually, a morphism $g : C \otimes_{\mathbb{K}} M \to M$ determines the coderivation $d_g = (id_C \otimes g) \circ (\Delta_C \otimes id_N)$.

If $\alpha : C \to A$ is a $\mathbb{K}$-linear morphism, then there are two ways to extend α to obtain a (co)derivation. Precomposing with Cs comultiplication gives us a morphism from C to the free A-module $A \otimes_{\mathbb{K}} C$.

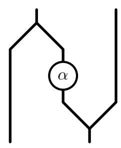
$$(\alpha \otimes id_C) \circ \Delta_C : C \to A \otimes_{\mathbb{K}} C$$

Postcomposing with As multiplication gives us a morphism from to the cofree C-comodule $C \otimes_{\mathbb{K}} A$ to A.

$$\nabla_A \circ (\alpha \otimes id_A) : C \otimes_{\mathbb{K}} A \to A$$

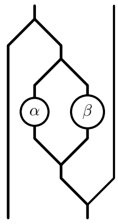
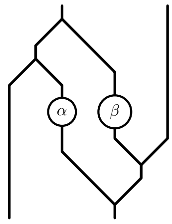
Notice that when applying proposition 1.3.5 to both morphisms yields the same map, and it is thus both a derivation and a coderivation.

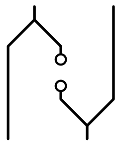
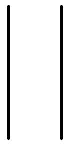
$$d_\alpha = (\nabla_A \otimes id_C) \circ (id_A \otimes \alpha \otimes id_C) \circ (id_A \otimes \Delta_C)$$

$d_\alpha \quad =$ 

Proposition 1.4.2. *$d_{(\_)} : Hom_{\mathbb{K}}(C, A) \to End(C \otimes_{\mathbb{K}} A)$ is a morphism of algebras. Moreover, if $\alpha \star \alpha = 0$, then $d_\alpha^2 = 0$.*

Proof. The proof quickly follows from (co)associativity and (co)unitality.

$d_{\alpha\star\beta}$ =  =  = $d_\alpha \circ d_\beta$

$d_{\upsilon_A \circ \varepsilon_C}$ =  =  = $id_{C\otimes_{\mathbb{K}} A}$

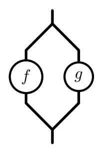
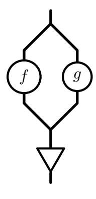
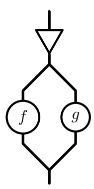
$\square$

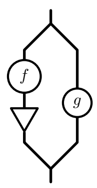
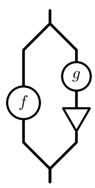
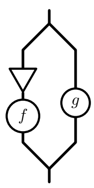
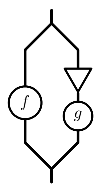
Suppose that C and A are differential graded (co)algebras. We want to expect that the differential ∂ makes $Hom^*_{\mathbb{K}}(C, A)$ into a dg-algebra.

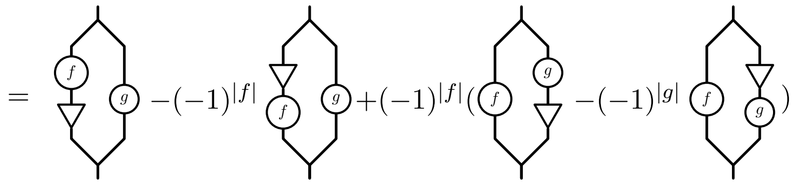
Proposition 1.4.3. *The convolution algebra* $(Hom^*_{\mathbb{K}}(C, A), \star)$ *is a dg-algebra with differential* ∂*.*

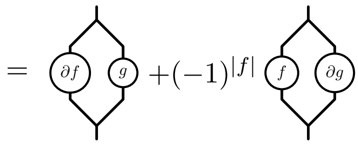
Proof. We know that $(Hom^*_{\mathbb{K}}(C, A), \star)$ is a convolution algebra and that $(Hom^*_{\mathbb{K}}(C, A), \partial)$ is a chain complex. It remains to verify that the differential is compatible with the multiplication, i.e. $\partial(f \star g) = \partial f \star g + (-1)^{|f|} f \star \partial g$.

Let $f, g \in Hom^*_{\mathbb{K}}(C, A)$ be two homogenous morphisms. The key property to arrive at the result is that the differential in a dg-(co)algebra is a (co)derivation. We denote the degree of $f \star g$ as $|f \star g| = |f| + |g| = d$

$\partial(f \star g) =\partial$  =  $-(-1)^d$ 

=  $+(-1)^{|f|}$  $-(-1)^d((-1)^{|g|}$  + )





$$= \partial(f) \star g + (-1)^{|f|} f \star \partial(g)$$

□

1.5 Twisting Morphisms

In this section we will define twisting morphisms from coalgebras to algebras. They are of importance as the bifunctor $Tw(C, A)$ is represented in both arguments. To understand the elements of Tw we start this section be reviewing the Maurer-Cartan equation.

Suppose that C is a dg-coalgebra and A is a dg-algebra. We say that a morphism $\alpha \in Hom^*_{\mathbb{K}}(C, A)$ is twisting if it is of degree -1 and satisfies the Maurer-Cartan equation:

$$\partial\alpha + \alpha \star \alpha = 0.$$

We say that α is an element of $Tw(C, A) \subset Hom^{-1}_{\mathbb{K}}(C, A) \subset Hom^*_{\mathbb{K}}(C, A)$. In light of proposition 1.4.2, every morphism between coalgebras and algebras extend to a unique (co)derivation on the tensor product $C \otimes_{\mathbb{K}} A$. Let d^r_α denote this unique morphism. In the case of dg-coalgebras and dg-algebras we perturbate the total differential on the tensor with d^r_α, as in proposition 1.3.5. We call this derivation for the perturbated derivative.

$$d^\bullet_\alpha = d^\bullet_{C \otimes_{\mathbb{K}} A} + d^r_\alpha = d^\bullet_C \otimes id_A + id_C \otimes d^\bullet_A + d^r_\alpha$$

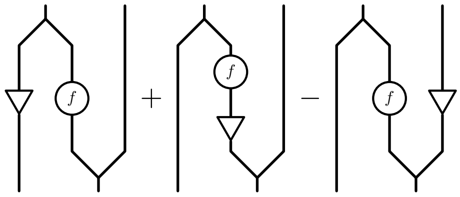
Proposition 1.5.1. *Suppose that C is a dg-coalgebra and A is a dg-algebra, and $\alpha \in Hom^*_{\mathbb{K}}(C, A)$. The perturbated derivation satisfies the following relation.*

$${d^\bullet_\alpha}^2 = d^r_{\partial\alpha + \alpha\star\alpha}$$

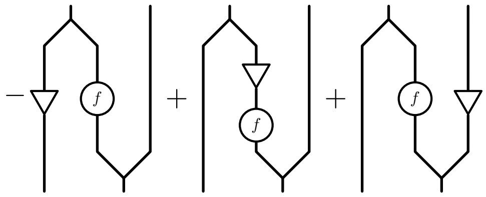
Moreover, a morphism is twisting if and only if the perturbated derivative is a differential.

Proof. ${d^\bullet_\alpha}^2 = d^\bullet_{C\otimes_{\mathbb{K}}A} \circ d^r_\alpha + d^r_\alpha \circ d^\bullet_{C\otimes_{\mathbb{K}}A} + {d^r_\alpha}^2$. By proposition 1.4.2 $d^r_?$ is an algebra homomorphism from the convolution algebra to the endomorphism algebra, thus ${d^r_\alpha}^2 = d^r_{\alpha\star\alpha}$.

$$d^{\bullet}_{C\otimes_{\mathbb{K}}A} \circ d^r_\alpha =$$



$$d^r_\alpha \circ d^{\bullet}_{C\otimes_{\mathbb{K}}A} = -$$



By summing the above terms we get

$$d^{\bullet}_{C\otimes_{\mathbb{K}}A} \circ d^r_\alpha + d^r_\alpha \circ d^{\bullet}_{C\otimes_{\mathbb{K}}A} = d^r_{d^{\bullet}_C\circ\alpha+\alpha\circ d^{\bullet}_A} = d^r_{\partial\alpha},$$

to obtain the result.

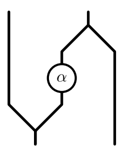
$$d^{\bullet}_{\alpha}{}^2 = d^{\bullet}_{C\otimes_{\mathbb{K}}A} \circ d^r_\alpha + d^r_\alpha \circ d^{\bullet}_{C\otimes_{\mathbb{K}_{\precsim}}A} + d\alpha^{r2} = d^r_{\partial\alpha} + d^r_{\alpha\star\alpha} = d_{\partial\alpha+\alpha\star\alpha}$$

□

Corollary 1.5.1.1. *If $\alpha : C \to A$ is a twisting morphism, then $(C \otimes_{\mathbb{K}} A, d^{\bullet}_\alpha)$ is a chain complex. It is called the right twisted tensor product and is denoted as $C \otimes_\alpha A$.*

Normally $A \otimes C$ and $C \otimes A$ are isomorphic as modules. In general, it is not true that $C \otimes_\alpha A$ and $A \otimes \alpha C$ are isomorphic, since we choose a particular side to perform the twisting. However, if A is commutative and C is cocommutative then they are isomorphic. To illustrate we realize the unique derivation above as a right derivative. The left derivative d^l_α is then defined analogously.

$$d^l_\alpha =$$



Remark 1.5.2. Functoriality of $\otimes_\alpha$ is obtained from the category of elements. I propose that there is an equivalence of categories, that is:

$$\int_{(C,A)} Tw(C,A) \simeq \text{right twisted tensors.}$$

1.6 Bar and Cobar Construction

The bar construction was first formalized for augmented skew-commutative dg-rings by Eilenberg and Mac Lane [4]. The bar construction then served as a tool to calculate the homology of the Eilenberg-Mac Lane spaces. This construction was later dualized by Adams [5] to obtain the cobar construction. It's first purpose was to serve as a method for constructing an injective resolution in order to calculate the cotor resolution [6]. With time, the bar-cobar construction have been subjected to much generalization, such as a fattened tensor product on simplicially enriched, tensored and cotensored categories [7]. We will mainly follow the work of [3] to obtain the bar and cobar construction. The approach which we are going to take is also slightly inspired by MacLanes[8] canonical resolutions of comonads.

For our purposes, the bar construction of an augmented algebra is a simplicial resoulution with the cofree coalgebra structure. For a dg-algebra, we will realize this resoultion as the total complex of its resoultion. Dually, the cobar construction of a conilpotent coalgebra is a cosimplicial resolution with the free algebra structure. We will see that these constructions defines an adjoint pair of functors.

Definition 1.6.1. The simplex category Δ consists of ordered sets $[0] = \emptyset$ and $[n] = \{1, ..., n\}$ for any $n \in \mathbb{N}$. A morphism is a monotone function between the sets.

Δ^+ is the full subcategory of Δ where $n > 0$. Δ_+ is the wide subcategory of Δ with only injective functions.

The simplex category comes equipped with coface and codegeneracy morphisms. The coface maps are the injective morphisms $\delta_i : [n] \to [n+1]$, and the codegeneracy maps are the surjective morphisms $\sigma_i : [n] \to [n-1]$.

$$\delta_i(k) = \begin{cases} k, \text{ if } k<i \\ k+1, \text{ otherwise} \end{cases} \qquad \sigma_i(k) = \begin{cases} k, \text{ if } k\leq i \\ k-1, \text{ otherwise} \end{cases}$$

Every morphism in Δ may be realized as a composition of coface and codegeneracy maps, see [8]. Furthermore, these maps are characterized by some identites, called the cosimplicial identites.

1. $\delta_j\delta_i = \delta_i\delta_{j-1}$, if $i < j$
2. $\sigma_j\delta_i = \delta_i\sigma_{j-1}$, if $i < j$
3. $\sigma_j\delta_i = id$, if $i = j$ or $i = j + 1$
4. $\sigma_j\delta_i = \delta_{i-1}\sigma_j$, if $i > j + 1$
5. $\sigma_j\sigma_i = \sigma_i\sigma_{j+1}$, if $i \leq j$

We may arrange the arrows of the simplex category in the following way:

$$
\begin{tikzcd}
{[0]} \arrow[r] & {[1]} \arrow[r, shift left, "\delta_i"] \arrow[r, shift right] & {[2]} \arrow[r, shift left=2, "\delta_i"] \arrow[r] \arrow[r, shift right=2] & {[3]} \arrow[r, shift left=3, "\delta_i"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \dots
\end{tikzcd}
$$

$$
\begin{tikzcd}
{[0]} & {[1]} & {[2]} \arrow[l, "\sigma_1"'] & {[3]} \arrow[l, shift left, "\sigma_i"'] \arrow[l, shift right] & \dots \arrow[l, shift left=2, "\sigma_i"'] \arrow[l] \arrow[l, shift right=2]
\end{tikzcd}
$$

Let $\mathcal{C}$ be a category. A simplicial object in $\mathcal{C}$ is a functor $S : (\Delta^+)^{op} \to \mathcal{C}$. It may be viewed as a collection of objects $\{S_n\}_{n\in\mathbb{N}^+}$ together with face maps $d^i : S_n \to S_{n-1}$ and degeneracy maps $s^i : S_n \to S_{n+1}$ satisfying the simplicial identities. An augmented simplicial object is a functor $S : \Delta^{op} \to \mathcal{C}$. The restricted functor $S^+ : (\Delta^+)^{op} \to \mathcal{C}$ is the augmentation ideal of S. An augmented semi-simplicial object is a functor $S : (\Delta_+)^{op} \to \mathcal{C}$. Dually, a cosimplicial object is a functor $S : \Delta^+ \to \mathcal{C}$, it may be regarded as a sequence of objects with coface and codegeneracy maps satisfying the cosimplicial identities.

Let $\mathcal{A}$ be an abelian category. To each semi-simplical object $M : (\Delta_+^+)^{op} \to \mathcal{A}$ there is an associated chain complex $M^\bullet$. Let $M^\bullet = \bigoplus_{i=1}^{\infty} M[i]$ with differential $d_M^n = \sum_{i=1}^{n}(-1)^{i-1}d^i$. This differential is well-defined by simplicial identity 1.

$$
\dots \longrightarrow M_3 \xrightarrow{d^1-d^2+d^3} M_2 \xrightarrow{d^1-d^2} M_1 \xrightarrow{0} 0 \longrightarrow \dots
$$

As face maps and degeneracy maps have the same identites, but flipped around, we could also have defined a chain complex by using the degeneracies instead.

The simplex category has a universal monoid. Let $+ : \Delta \to \Delta$ be a functor acting on objects and morphisms as:

$$
[m] + [n] = [m+n]
$$
$$
(f+g)(k) = \begin{cases} f(k), \text{ if } k \leq m \\ g(k)+m, \text{ otherwise} \end{cases}
$$

Notice that $[0] + \_ \simeq Id_\Delta$, so $(\Delta, +, [0])$ is a monoidal category. Since $[1]$ is terminal in Δ it becomes a monoid with $\delta_0 : [0] \to [1]$ as unit and $\sigma_1 : [2] \to [1]$ as multiplication. Associativity and unitality is satisfied by uniqueness of morphisms $f : [n] \to [1]$.

Proposition 1.6.2. *Let $(\mathcal{C}, \otimes, Z)$ be a monoidal category. If (C, η, μ) is a monoid in $\mathcal{C}$, then there is a strong monoidal functor : $\Delta \to \mathcal{C}$, such that $F[1] \simeq C$, $F\delta_0 \simeq \eta$ and $F\sigma_1 \simeq \mu$.*

Proof. This is proved in Mac Lanes book [8]. □

An algebra A is a monoid in the monoidal category $(Mod_{\mathbb{K}}, \otimes_{\mathbb{K}}, \mathbb{K})$. By proposition 1.6.2 we may think of A as an augmented cosimplicial object $A : \Delta \to Mod_{\mathbb{K}}$. Notice that all of the cosimplical identities follow from associativity and unitality. If A is an augmented algebra, we may instead give it the structure of an augmented simplicial set. Let $d_1^1 = \varepsilon_A$ be the augmentation. We define

$d_n^n = A^{\otimes n-1} \otimes \varepsilon_A$ and set $d_n^i = A^{i-1} \otimes \nabla_A \otimes A^{\otimes n-i-1}$. All the degeneracies are set to be the units, i.e. $s_n^i = A^{\otimes i} \otimes \upsilon_A \otimes A^{\otimes n-i-1}$. One may check that this structure defines a simplical object $A : \Delta^{op} \to Mod_{\mathbb{K}}$. Observe that the associated chain complex $A^\bullet$ is exactly the Hochschild complex of A. We depict the simplicial object as the following diagram:

$$\mathbb{K} \xleftarrow{\varepsilon_A} A \underset{A\otimes\varepsilon_A}{\overset{\nabla_A}{\leftleftarrows}} A^{\otimes 2} \underset{A^{\otimes 2}\otimes\varepsilon_A}{\overset{\nabla_A}{\Lleftarrow}} A^{\otimes 3} \underset{A^{\otimes 4}\otimes\varepsilon_A}{\overset{\nabla_A}{\Lleftarrow}} \dots$$

$$\mathbb{K} \qquad A \xrightarrow{s^1} A^{\otimes 2} \overset{s^i}{\rightrightarrows} A^{\otimes 3} \overset{s^i}{\Rrightarrow} \dots$$

The augmentation ideal $\bar{A}$ carries a natural semi-simplical structure induced by A. By restricting each of the face maps $\bar{d^i} = d^i|_{\bar{A}} : \bar{A}^{\otimes n} \to \bar{A}^{\otimes n-1}$ we obtain the maps together with the simplical identity 1. This is the non-unital Hochschild complex of A. We may depict the semi-simplical object as the following diagram:

$$\mathbb{K} \xleftarrow{0} \bar{A} \underset{0}{\overset{\nabla_A}{\leftleftarrows}} \bar{A}^{\otimes 2} \underset{0}{\overset{\nabla_A}{\Lleftarrow}} \bar{A}^{\otimes 3} \underset{0}{\overset{\nabla_A}{\Lleftarrow}} \dots$$

Notice that as graded modules, the chain complex $\bar{A}^\bullet$ is isomorphic to $T^c(\bar{A})$. We will now instead consider the suspended non-unital algebra $\bar{A}[1]$. Every algebra may be considered as a graded algebra concentrated in degree 0, the shift functor then recontextualize the degree the algebra is concentrated in. With Koszul sign rule, we may define the suspended multiplication as $\nabla_{A[1]}(a_1 \otimes a_2) = (-1)^{|a_1|} a_1 a_2$. Notice that $\nabla_{A[1]}$ is a morphism of degree -1. Repeating Koszul sign rule, we may se that associativity does not longer hold, as multiplying the multiplication on the right first introduces a sign, contrary to first multiplying on the left side.

Proposition 1.6.3. *The suspended augmentation ideal $\bar{A}[1]$ is a semi-simplical set with face maps:*

$$\bar{d^i} = (-1)^{i-1} d^i = (-1)^{i-1} (\nabla_{A[1]})^{(n-1)}_{(i-1)}.$$

Corollary 1.6.3.1. *The differential $d^\bullet_{\bar{A}[1]}$ is a coderivation for the cofree coalgebra $T^c(\bar{A}[1])$. Thus $(\bar{A}[1]^\bullet, d^\bullet_{\bar{A}[1]})$ is a dg-coalgebra.*

Proof. The differential is given by the alternating sum of face maps.

$$d^n_{\bar{A}[1]} = \sum_{i=1}^{n} (-1)^{i-1} \bar{d^i} = \sum_{i=1}^{n} (-1)^{2(i-1)} d^i = \sum_{i=1}^{n} (\nabla_{A[1]})^{(n-1)}_{(i-1)}$$

By injecting $\bar{A}[1]$ into $T^c(\bar{A}[1])$ we may think of $\nabla_{\bar{A}[1]} : \bar{A}[1]^{\otimes 2} \to T^c(\bar{A}[1])$ as a morphism into the tensor coalgebra. By using proposition 1.3.2, $\nabla_{\bar{A}[1]}$ extends uniquely into a coderivation:

$$d^c_{\bar{A}[1]} = \sum_{n=0}^{\infty} \sum_{i=0}^{n} (\nabla_{\bar{A}[1]})^{(n)}_{(i)} = d^\bullet_{\bar{A}[1]}.$$

□

If $(A, d^\bullet_A)$ is an augmented dg-algebra, then A is a simplical object of $Mod^\bullet_{\mathbb{K}}$. It has an associated chain complex. Taking the alternate sum of face maps gives us a double complex as below. We define the double complex $A^\bullet$ as the associated chain complex to A.

$$\begin{tikzcd}
 & \vdots \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2, "\nabla_A"'] \arrow[d, phantom, shift left=6, "A^{\otimes 2}\otimes\varepsilon_A"] & \vdots \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2, "\nabla_A"'] \arrow[d, phantom, shift left=6, "A^{\otimes 2}\otimes\varepsilon_A"] & \vdots \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2, "\nabla_A"'] \arrow[d, phantom, shift left=6, "A^{\otimes 2}\otimes\varepsilon_A"] & \\
\cdots \arrow[r, "d^\bullet_{A\otimes 2}"'] & (A^{\otimes 2})^1 \arrow[r, "d^\bullet_{A\otimes 2}"'] \arrow[d, shift right, "\nabla_A"'] \arrow[d, shift left, "A\otimes\varepsilon_A"] & (A^{\otimes 2})^0 \arrow[r, "d^\bullet_{A\otimes 2}"'] \arrow[d, shift right, "\nabla_A"'] \arrow[d, shift left, "A\otimes\varepsilon_A"] & (A^{\otimes 2})^{-1} \arrow[r, "d^\bullet_{A\otimes 2}"'] \arrow[d, shift right, "\nabla_A"'] \arrow[d, shift left, "A\otimes\varepsilon_A"] & \cdots \\
\cdots \arrow[r, "d^\bullet_A"'] & A^1 \arrow[r, "d^\bullet_A"'] \arrow[d, "\varepsilon_A"] & A^0 \arrow[r, "d^\bullet_A"'] \arrow[d, "\varepsilon_A"] & A^{-1} \arrow[r, "d^\bullet_A"'] \arrow[d, "\varepsilon_A"] & \cdots \\
\cdots \arrow[r, "0"'] & 0 \arrow[r, "0"'] & \mathbb{K} \arrow[r, "0"'] & 0 \arrow[r, "0"'] & \cdots
\end{tikzcd}$$

For simplicity we write d_1 for the horizontal differential and d_2 for the vertical differential. The total associated chain complex is the total complex for $Tot(A^\bullet)$, denoted $A^\bullet$ if there are no confusion. In the case of the suspended algebra, the signs mess up commutativity of the squares, thus we change the sign of the horizontal differential to $(-1)^n$. We may also define the differential of the total complex simply as the sum of d_1 and d_2.

Proposition 1.6.4. *Let A an augmented dg-algebra. The bar complex BA is the total associated chain complex $\bar{A}[1]^\bullet$ of the suspended augmentation ideal $\bar{A}$. $(BA, d^\bullet_{BA})$ is the cofree conilpotent coalgebra equipped with $d^\bullet_{BA} = d_1 + d_2$ as coderivation.*

Proof. It is apparent that d_1 and d_2 are coderivations with respect to deconcatenation. Since the multiplication ∇_A is a chain map ${d^\bullet_{BA}}^2 = d_1 \circ d_2 + d_2 \circ d_1 = 0$. We will show this for each element in $A^{\otimes 2}$, then the result may be extended to all of BA.

$$\begin{aligned}
d_1 \circ d_2(a_1 \otimes a_2) &= (-1)^{|a_1|} d_1(a_1 a_2) = (-1)^{|a_1|} d^\bullet_A[1](a_1 a_2) \\
&= (-1)^{|a_1|+1} d^\bullet_A(a_1 a_2) = (-1)^{|a_1|+1} (d^\bullet_A(a_1) a_2 + (-1)^{|a_1|} a_1 d^\bullet_A(a_2)) \\
&= (-1)^{|a_1|+1} d^\bullet_A(a_1) a_2 - a_1 d^\bullet_A(a_2)
\end{aligned}$$

$$\begin{aligned}
d_2 \circ d_1(a_1 \otimes a_2) &= d_2 \circ (d_A^\bullet[1] \otimes id_{A[1]} + id_{A[1]} \otimes d_A^\bullet[1])(a_1 \otimes a_2) \\
&= -d_2 \circ (d_A^\bullet(a_1) \otimes a_2 + (-1)^{|a_1|+1} a_1 \otimes d_A^\bullet(a_2)) \\
&= (-1)^{|d_A^\bullet(a_1)|+1} d_A^\bullet(a_1)a_2 + (-1)^{2|a_1|+2} a_1 d_A^\bullet d_A^\bullet(a_2) \\
&= (-1)^{|a_1|} d_A^\bullet(a_1)a_2 + a_1 d_A^\bullet(a_2) = -d_1 \circ d_2(a_1 \otimes a_2)
\end{aligned}$$

□

Remark 1.6.5. For now we don't need to show that BA is a functor. This property follows from BA being the representing object of $Tw(\_, A)$.

On the other hand, a coalgebra C is a comonoid in $Mod_{\mathbb{K}}$. By the dual of proposition 1.6.2 we may think of it as a simplical object $C : (\Delta)^{op} \to Mod_{\mathbb{K}}$. Dually, all of the simplical identities follows from coassociativity and counitality. A coaugmented coalgebra C may be given a cosimplicial structure in the opposite way of algebras. We then get that the coaugmentation quotient $\bar{C}$ is a semi-cosimplical object of $Mod_{\mathbb{K}}$. Observe that $\bar{C}$ has an associated chain complex like $\bar{A}$, but every arrow goes in the opposite direction.

$$\begin{tikzcd}
\mathbb{K} \arrow[r, "\upsilon_C"] & C \arrow[r, shift left, "\Delta_C"] \arrow[r, shift right, "A\otimes\upsilon_C"'] & C^{\otimes 2} \arrow[r, shift left=2, "\Delta_C"] \arrow[r] \arrow[r, shift right=2, "C^{\otimes 2}\otimes\upsilon_C"'] & C^{\otimes 3} \arrow[r, shift left=3, "\Delta_C"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3, "C^{\otimes 4}\otimes\upsilon_C"'] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{K} & C & C^{\otimes 2} \arrow[l, "s_1"'] & C^{\otimes 3} \arrow[l, shift right, "s_i"'] \arrow[l, shift left] & \cdots \arrow[l, shift right=2, "s_i"'] \arrow[l] \arrow[l, shift left=2]
\end{tikzcd}$$

The cobar construction is made from the inverse shifted, or desuspended coalgebra $C[-1]$. We realize it as the free tensor algebra $T(\bar{C}[-1])$, where the comultiplication $\Delta_{\bar{C}[-1]}$ induces a derivation $d_{\bar{C}[-1]}$ by proposition 1.3.2.

Remark 1.6.6. As we have chosen to define $\nabla_{A[1]}(a_1 \otimes a_2) = (-1)^{|a_1|} a_1 a_2$, we are forced by the linear dual to define $\Delta_{C[-1]}(c) = -(-1)^{|c_{(1)}|} c_{(1)} \otimes c_{(2)}$.

Proposition 1.6.7. *Let C be a coaugmented dg-coalgebra. The cobar complex ΩC is the total associated chain complex $\bar{C}[-1]^\bullet$ of the desuspended coaugmentation quotient $\bar{C}$. $(\Omega C, D_{\Omega C}^\bullet)$ is the free algebra equipped with $d_{\Omega C}^\bullet = d_1 + d_2$ as derivation.*

We will now see that the bar and cobar construction defines an adjoint pair of functors. Note that since for any conilpotent dg-coalgebra C, the object ΩC represents the functor in the category of augmented algebras. By Yoneda's lemma, the data of morphisms are then defined, so Ω does truly define a functor.

Theorem 1.6.8. *Let C be a conilpotent dg-coalgebra and A an augmented dg-algebra. The functor $Tw(C, A)$ is represented in both arguments, i.e.*

$$\tfrac{Aug}{Alg}^\bullet(\Omega C, A) \simeq Tw(C, A) \simeq \tfrac{Conil}{CoAlg}^\bullet(C, BA).$$

Proof. We will show that ΩC represents the set of twisting morphisms in the first argument. Showing that BA represents the second argument uses every dual proposition. Thus, it is necessary that C is conilpotent, in order to dualize the arguments.

Suppose that $f : \Omega C \to A$ is an augmented dg-algebra homomorphism. f is then a morphism of degree 0. By freeness, f is uniquely determined by a morphism $f \mid_{\bar{C}[-1]} : \bar{C} \to \bar{A}$ of degree 0, which corresponds to a morphism $f' : C \to A$ of degree -1.

Since f is a morphism of chain complexes it commutes with the differential, i.e.

$$f \circ d^\bullet_{\Omega C} = d^\bullet_A \circ f$$
$$f \circ (d_1 + d_2) = d^\bullet_A \circ f$$

This is equivalent to say that $-f' \circ d^\bullet_C - f' \star f' = d^\bullet_A \circ f'$. Thus f' is a twisting morphism. □

Associated to this adjunction, we obtain universal elements, together with universal properties. Let A be an augmented dg-algebra, then the identity of the coalgebras $id_{BA} : BA \to BA$, the counit $\varepsilon_A : \Omega BA \to A$ and a twisting morphism $\pi_A : BA \to A$ are equivalent by the adjunction and representation. Dually, the identity of algebras $id_{\Omega C} : \Omega C \to \Omega C$, the unit $\eta_C : C \to B\Omega C$ and the twisting morphism $\iota_C : C \to \Omega C$ are equivalent. The morphisms π_A and ι_C are called the universal elements. We summarize their universal property in the following corollary.

Corollary 1.6.8.1. *Let A be an augmented dg-algebra, and C a conilpotent dg-coalgebra. The any twisting morphism $\alpha : C \to A$ factors uniquely through either π_A or ι_C.*

$$\begin{tikzcd}
 & \Omega C \arrow[dr, dashed, "g_\alpha"] & \\
C \arrow[ur, "\iota_C"] \arrow[rr, "\alpha"] \arrow[dr, dashed, "f_\alpha"'] & & A \\
 & BA \arrow[ur, "\pi_A"'] &
\end{tikzcd}$$

Moreover, the morphism f_α is a morphism of dg-coalgebras, and g_α is a morphism of dg-algebras.

Definition 1.6.9 (Augmented Bar-Cobar construction)**.** Let A be an augmented dg-algebra. The (right) augmented bar construction is the right twisted tensor product $BA \otimes_{\pi_A} A$, where π_A is the universal twisting morphism.

Let C be a conilpotent dg-coalgebra. The (right) augmented cobar construction is the right twisted tensor product $C \otimes_{\iota_C} \Omega C$, where ι_C is the universal twisting morphism.

Remark 1.6.10. We could have defined the augmented bar-cobar construction as the left twisted tensor product. There is really no preference of handedness. Whenever we wish to be precise which handedness we will use it will be specified, e.g. the left augmented bar construction of A.

Proposition 1.6.11. *The augmentation ideal of the augmented bar (cobar) construction is acyclic, i.e.* $BA\bar{\otimes}_{\pi_A}A$ *(*$A\bar{\otimes}_{\pi_A}BA$*) and* $C\bar{\otimes}_{\iota_C}\Omega C$ *(*$\Omega C\bar{\otimes}_{\iota_C}C$*) are acyclic.*

Proof. calculate kernel of the augmentation map, and find a homotopy of the identity. □

1.7 Comparison Lemma

In this section we wish to deduce the fundamental theorem of twisting morphisms. In order to this we will need a result by Henri Cartan [9], called the comparison lemma. This section follows Loday [3], and takes inspiration from Lefevre-Hasagawa [10].

In order to state the comparison lemma we need to understand what it means for a dg-algebra to be connected. We define weight, which is a second grading for the objects.

Definition 1.7.1 (Weight graded cochain complexes)**.** Let $N = \{0, 1, ...\}$ be an indexing set, which is possibly finite. A cochain complex M is weight graded if it splits as a direct sum $M \simeq \bigoplus_{n\in N} M_{(n)}$ indexed over N. Let $m \in M$, then m has weight n if $m \in M_{(n)}$, has homological degree n' if $m \in M^{n'}$ and (total) degree $|m| = n + n'$.

A dg-(co)algebra is weight graded if the weight on the cochain complex respects the (co)multiplication. A weight graded dg-(co)algebra will be called a wdg-(co)algebra.

Definition 1.7.2 (Connected cochain complexes)**.** A weight graded cochain complex M is called connected if $M_(0) \simeq \mathbb{K}$ and is concentrated in homological degree 0.

Lemma 1.7.3 (Comparison Lemma)**.** *Let* $g : A \to A'$ *be a morphism of connected wdg-algebras, and* $f : C \to C'$ *be a morphism of connected wdg-coalgebras. Suppose there are twisting morphisms* $\alpha : C \to A$ *and* $\alpha' : C' \to A'$ *such that* f *and* g *becomes a morphism of twisted tensors. If two out of* f*,* g *and* $f \otimes g$ *are quasi-isomorphisms, then so is the third.*

Proof. A proof may be found in either Cartans paper [9] or Loday and Valletes book [3]. □

Theorem 1.7.4 (Fundamental Theorem of Twisting Morphisms)**.** *Let* A *be a connected wdg-algebra,* C *be a connected wdg-coalgebra and* $\alpha : C \to A$ *a twisting morphism. The following are equivalent:*

1. *The augmentation ideal of the right twisted tensor product* $C\bar{\otimes}_\alpha A$ *is acyclic*
2. *The augmentation ideal of the left twisted tensor product* $A\bar{\otimes}_\alpha C$ *is acyclic*
3. *The morphism* $f_\alpha : C \to BA$ *is a quasi-isomorphism*
4. *The morphism* $g_\alpha : \Omega C \to A$ *is a quasi-isomorphism*

Proof. In order to do this proof we must first observe that the bar and cobar construction preserve the weight grading, and therefore the connectedness.

We prove 1. $\iff$ 3., the other equivalences are analogous. By corollary 1.6.8.1, the morphism $id_C \otimes g_\alpha : C \otimes_{\iota_C} \Omega C \to C \otimes_\alpha A$ is a morphism of twisting tensor products. Since id_C is a quasi-isomorphism, we get by 1.7.3 that $id_C \otimes g_\alpha$ is a quasi-isomorphism if and only if g_α is a quasi-isomorphism if and only if $\bar{g}_\alpha$ is a quasi-isomorphism. By 1.6.11 $C\bar{\otimes}_{\iota_C}\Omega C$ is acyclic, so $\bar{g}_\alpha$ is a quasi-isomorphism if and only if $C\bar{\otimes}_\alpha A$ is acyclic. □

Corollary 1.7.4.1. *Let A be a connected wdg-algebra, C be a connected wdg-coalgebra. The counit $\varepsilon_A : \Omega BA \to A$, and the unit $\eta_C : C \to B\Omega C$ are quasi-isomorphisms.*

We now know some cases where the unit and the counit are quasi-isomorphisms. However, we would like to promote this result to every (conilpotent) augmented dg-(co)algebra. To this end we will find suitable filtrations to realize the associated graded as a connected wdg-(co)algebra and then get isomorphisms on the associated graded. The problem would then be to lift such quasi-isomorphisms back to our original objects.

Recall that a filtration on a chain complex M is a sequence of inclusions $M_0 \subseteq M_1 \subseteq ....$ The filtration is called exhaustive if $\varinjlim M_i \simeq M$, and admissable if $Fr_0M \simeq \mathbb{K}$ as well. Since each inclusion respect the differential, we may find the cokernel of each inclusion. The associated graded grM is then the graded complex given by $gr_0M = M_0$ and $gr_iM = M_i/M_{i-1}$. A dg-(co)algebra is filtered if the filtration respects the (co)multiplication. If $f : M \to N$ is a morphism of filtered cochain complexes, then it defines a morphism $grf : grM \to grN$ on the associated graded. We call f a graded quasi-isomorphism if grf is a quasi-isomorphism.

Let C be a conilpotent dg-coalgebra. The coradical filtration $Fr_0C \subseteq Fr_1C \subseteq ...$ is an exhaustive filtration by assumption and $Fr_0C \simeq \mathbb{K}$, so it is admissable. Moreover, since $\Delta Fr_{i+1}C \subseteq Fr_iC \otimes Fr_iC$ we may define the comultiplication of the associated graded to be trivial, and this we will do. To obtain quasi-isomorphisms of chain complexes we do not need the extra structure of the coalgebra. If not specified otherwise, whenever C is assumed to be a conilpotent dg-coalgebra then grC should be the associated graded of the coradical filtration.

Lemma 1.7.5. *Let $f : C \to C'$ be a graded quasi-isomorphism between conilpotent dg-coalgebras, then $\Omega f : \Omega C \to \Omega C'$ is a quasi-isomorphism.*

Proof. We do this by considering a spectral sequence. Endow C with a grading (as a vector space) induced by the coradical filtration, i.e. $c \in C$ has degree $|c| = n$ if n is the smallest number such that $\bar{\Delta}^n c = 0$. We define a filtration on ΩC by

$$F_p\Omega C = \{c_1[-1] \otimes ... \otimes c_n[-1] \mid |c_1| + ... + |c_n| \leq p\}$$

Since C is a dg-coalgebra, the coradical filtration respects the differential. In other words, $F_p\Omega C$ is still a chain complex, which is a subcomplex of ΩC. This filtration is clearly bounded below and exhaustive. Thus by the classical convergence theorem of spectral sequences, theorem 5.5.1 [11], the spectral sequence converges to the homology $E\Omega C \Rightarrow H^*\Omega C$.

By definition, the 0'th page is defined as $E^0_{p,q}\Omega C = {}^{(F_p\Omega C)_{p+q}}\big/_{(F_{p-1}\Omega grC)_{p+q}}$. Furthermore, notice that at this page we have the following isomorphism $E^0_{p,q}\Omega C \simeq (\Omega grC)^{(p)}_{p+q}$, where $(\Omega grC)^{(p)} = \{c_1[-1] \otimes ... \otimes c_n[-1] \mid |c_1| + ... + |c_n| = p\}$.

Evaluating f at the 0'th page would the look like $E^0\Omega f \simeq \Omega grf$. By the mapping lemma, exercise 5.2.3 [11], it is enough to check that Ωgrf is a quasi-isomorphism, to see that Ωf is a quasi-isomorphism. We show that Ωgrf is a quasi-isomorphism by inspecting every $E^0_{p,\bullet}\Omega C$.

We define a filtration G_k on $E^0_{p,\bullet}\Omega C$ as

$$G_k = \{c_1[-1] \otimes ... \otimes c_n[-1] \mid n \geq -k\}.$$

We see that $G_0 = E^0_{p,\bullet}\Omega C$ by definition and $G_{-p-1} \simeq 0$ on the augmentation ideal $\bar{C}$. Again, by the classical convergence theorem of spectral sequences, this defines a spectral sequence such that $EG \Rightarrow H^*E^0_{p,\bullet}\Omega C$.

To see that Ωgrf is a quasi-isomorphism, it is now enough to see that E^0Gf is a quasi-isomorphism for any p. Notice that $E^0_{l,\bullet}G \subseteq (grC[-1])^{\otimes l}$ where the total grading is p. Since f is a graded quasi-isomorphisms and by the Kunneth-formual, theorem 3.6.3 [11], it follows that E^Gf is a quasi-isomorphism. □

For completeness we include the following statement.

Lemma 1.7.6. *Let $f : A \to A'$ be a quasi-isomorphism between dg-algebras, then $Bf : BA \to BA'$ is a quasi-isomorphism.*

Proof. Notice that the homology of BA may calculated from the double complex used to define BA. In fact, at the 0'th page we have $E^0_{p,\bullet}f \simeq f^{\otimes p}$. It follows that f is a quasi-isomorphism on the 0'th page from the Kunnet formula, theorem 3.6.3 [11]. □

Proposition 1.7.7. *Let A be an augmented dg-algebra and C a conilpotent dg-coalgebra. The counit $\varepsilon_A : \Omega BA \to A$ a quasi-isomorphism. The unit $\eta_C : C \to B\Omega C$ is a filtered quasi-isomorphism, moreover $B\eta_C$ is a quasi-isomorphism.*

Proof. We pick the filtration $\mathbb{K} \subseteq A \subseteq A \subseteq ...$ for A. The associated graded grA is A but with almost trivial multiplication $grA \simeq \mathbb{K} \oplus \bar{A}$. Given a dg-coalgebra M, we may find a filtration on ΩM as $\Omega M_i = \bigoplus_{k \leq i} \Omega M^k$. Notice that the counit $\varepsilon_A : \Omega BA \to A$ becomes a filtered morphism together with these filtrations. thus $gr\varepsilon_A : gr\Omega BA \to grA$ is a morphism of augmented dg-algebras. By 1.7.4 $gr\varepsilon_A$ is a quasi-isomorphism if and only if the augmentation ideal of $gr\Omega BA \otimes grA$ is acyclic.

Considered as cochain complexes, $A \simeq grA$ and $gr\Omega BA \simeq \Omega BA$, so there is a quasi-isomorphism $gr\varepsilon_A : \Omega BA \to A$.

The other direction is analogously, but we use the coradical filtration for C to see that $gr\eta_C : grC \to grB\Omega C$ is a quasi-isomorphism. □

1.8 Strongly Homotopy Associative Algebras and Coalgebras

We have seen from corollary 1.6.3.1 that any dg-algebra A defines a dg-coalgebra $T^c(A[1])$, the bar construction, with a coderivation m^c of degree -1. Does this however work in reverse? I.e. if A is a vector space such that the coalgebra $T^c(A[1])$ together with a coderivation m^c is a dg-coalgebra, is then A an algebra? The answer to this is no, but it leads to the definition of a strongly homotopy associative algebra.

Definition 1.8.1. An A_∞-algebra is a graded vector space A together with a differential $m : \bar{T}^c(A[1]) \to \bar{T}^c(A[1])$ that is a coderivation of degree -1.

The differential m induces structure morphisms on $A[1]$. By proposition 1.3.2 there is a natural bijection $Hom_{\mathbb{K}}(\bar{T}^c(A[1]), A[1]) \simeq Coder(\bar{T}^c(A[1]), \bar{T}^c(A[1]))$ given by the projection onto $A[1]$. Thus $m : \bar{T}^c(A[1]) \to \bar{T}^c(A[1])$ corresponds to maps $\widetilde{m}_n : A[1]^{\otimes n} \to A[1]$ of degree -1 for any $n \geq 1$. We define maps $m_n : A^{\otimes n} \to A$ by the composite $s^{-1}\widetilde{m}_n s^{\otimes n}$. Since $s^{\otimes n}$ is of degree n, $\widetilde{m}_n$ and s^{-1} is of degree -1, we get that m_n is of degree $n-2$.

$$\begin{tikzcd} A^{\otimes n} \arrow[r, "m_n"] \arrow[d, "s^{\otimes n}"', "\simeq"] & A \\ A[1]^{\otimes n} \arrow[r, "\widetilde{m}_n"] & A[1] \arrow[u, "s^{-1}"', "\simeq"] \end{tikzcd}$$

Proposition 1.8.2. *An A_∞-algebra is equivalent to a graded vector space A together with homogenous morphisms $m_n : A^{\otimes n} \to A$ of degree $n-2$. Moreover, the morphism must satisfy the following relations for any $n \geq 1$:*

$$(\textit{rel}_n) \qquad \sum_{p+q+r=n} (-1)^{p+qr} m_{p+1+r} \circ (id^{\otimes p} \otimes m_q \otimes id^{\otimes r}) = 0$$

Remark 1.8.3. We make a more convenient notation for (rel_n), called partial composition $\circ_i$.

$$m_{p+1+r} \circ_{p+1} m_q = m_k \circ (id^{\otimes p} \otimes m_q \otimes id^{\otimes r})$$

$$(\mathrm{rel}_n) \qquad \sum_{p+q+r=n} (-1)^{p+qr} m_{p+1+r} \circ_{p+1} m_q = 0$$

Before starting with the proof we will use a lemma for checking whether a coderivation $m : T^c(A) \to T^c(A)$ is a differential.

Lemma 1.8.4. *Let $m : T^c(A) \to T^c(A)$ be a coderivation, and denote $m_n = m|_{A^{\otimes n}}$. m is a differential if and only if the following relations are satisfied:*

$$\sum_{p+q+r=n} m_{p+1+r} \circ_{p+1} m_q = 0$$

Proof. By proposition 1.3.2 we may write $m = \sum_{n=0}^{\infty} \sum_{i=0}^{n} m_{(n)}^{(i)}$. By using partial composition, we rewrite its n'th component as:

$$m_n = \sum_{q=1}^{n} \sum_{p=1}^{n} id^{\otimes(n-q)} \circ_p m_q = \sum_{p+q+r=n} id^{\otimes(p+1+r)} \circ_{p+1} m_q$$

For m^2 we denote it's n'th component as m_n^2. Observe the following:

$$m_n^2 = m \circ m_n = m \circ \sum_{p+q+r=n} id^{\otimes(p+1+r)} \circ_{p+1} m_q = \sum_{p+q+r=n} m \circ_{p+1} m_q$$
$$\pi m_n^2 = \pi \sum_{p+q+r=n} m \circ_{p+1} m_q = \sum_{p+q+r=n} m_{p+1+r} \circ_{p+1} m_q$$

Since every coderivation are uniquely determined by π, its projection onto A we get that $m^2 = 0$ if and only if

$$\sum_{p+q+r=n} m_{p+1+r} \circ_{p+1} m_q = 0.$$

□

Proof of proposition 1.8.2. Let (A, m) be an A_∞-algebra. We denote the n'th component of m as $\widetilde{m}_n$. The n'th components thus define maps $m_n : A^{\otimes n} \to A$ as $m_n = s^{-1}\widetilde{m}_n s^{\otimes n}$.

By the above lemma we know that the n'th component of m^2 is:

$$\sum_{p+q+r=n} \widetilde{m}_{p+1+r} \circ_{p+1} \widetilde{m}_q$$
$$= \sum_{p+q+r=n} sm_{p+1+r}s^{-\otimes(p+1+r)} \circ_{p+1} sm_q s^{-\otimes q} = \sum_{p+q+r=n} (-1)^{pq+r} sm_{p+1+r} \circ_{p+1} m_q s^{-\otimes n}$$

Since suspension and desuspension are isomorphism we get that $m^2 = 0$ if and only if (rel_n) are 0 for every $n \geq 1$, i.e.

$$\sum_{p+q+r=n} (-1)^{p+qr} m_{p+1+r} \circ_{p+1} m_q = 0$$

□

Given an A_∞ algebra A we may either think of it as a differential tensor coalgebra $\bar{T}^c(A[1])$ with differential $m : \bar{T}^c(A[1]) \to \bar{T}^c(A[1])$ or as a graded vector space with morphisms $m_n : A^{\otimes n} \to A$ satisfying (rel_n). We will calculate (rel_n) for $1, 2, 3$:

$$(\mathrm{rel}_1) \qquad m_1 \circ m_1 = 0$$
$$(\mathrm{rel}_2) \qquad m_1 \circ m_2 - m_2 \circ_1 m_1 - m_2 \circ_2 m_1 = 0$$
$$(\mathrm{rel}_3) \qquad m_1 \circ m_3 + m_2 \circ_1 m_2 - m_2 \circ_2 m_2 + m_3 \circ_1 m_1 + m_3 \circ_2 m_1 + m_3 \circ_3 m_1 = 0$$

We see that (rel_1) states that m_1 should be a differential, we may thus think of (A, m_1) as a chain complex. Furthermore, (rel_2) says that $m_2 : (A^{\otimes 2}, m_1 \otimes id_A + id_A \otimes m_1) \to (A, m_1)$ is a morphism of chain complexes. Lastly, (rel_3) gives us a homotopy for the associator of m_2, namely m_3. Thus we may regard (A, m_1, m_2) as an algebra which is associative up to homotopy. Regarding A as a chain complex instead we obtain our final definition of an A_∞-algebra.

Proposition 1.8.5. *Suppose that (A, d) is a chain complex, and that there exists morphisms $m_n : A^{\otimes n} \to A$ for any $n \geq 2$. A is an A_∞-algebra if and only it satisfies the following relations:*

$$(\textit{rel'}_n) \qquad \partial(m_n) = -\sum_{\substack{n=p+q+r\\ k=p+1+r\\ k>1, q>1}} (-1)^{p+qr} m_k \circ_p +1 m_q$$

We define the homotopy of an A_∞-algebra to be the homology of the chain complex (A, m_1). Since $\partial(m_3) = m_2 \circ_1 m_2 - m_2 \circ_2 m_2$, we get that m_2 is associative in homology. Thus for any A_∞-algebra A, the homotopy HA is an associative algebra. The operadic homology of A is defined as the homology of $(T^c(A[1]), m)$, which is the non-unital augmented Hochschild homology of A.

Example 1.8.6. Suppose that V is a cochain complex with differential d. Then V is an A_∞-algebra with trivial multiplication. In other words $m^1 = d$ and $m^i = 0$ for any $i > 1$.

Example 1.8.7. Suppose that A is a dg-algebra. Then A is an A_∞-algebra where $m^1 = d$, $m^2 = \cdot$ and $m^i = 0$ for any $i > 2$.

Example 1.8.8. Eksemplet jeg fikk fra Torgeir.

A morphism between A_∞-algebras is called an ∞-morphism. Suppose that A and B are two A_∞-algebras, an ∞-morphism $f : A \rightsquigarrow B$ is a dg-coalgebra homomorphism $\widetilde{f} : (\bar{T}^c(A[1]), m^A) \to (\bar{T}^c(B[1]), m^B)$. By proposition 1.3.2, $\widetilde{f}$ is uniquely determined by homogenous morphisms $f_n : A^{\otimes n} \to B$ of degree $n-1$ for any $n \geq 1$. f_1 is required to be a morphism of the chain complexes $f_1 : (A, m_1^A) \to (B, m_1^B)$. For any $n \geq 2$ f should satisfy the relations:

$$(\text{rel}_n) \qquad \partial(f_n) = \sum_{\substack{p+1+r=k\\ p+q+r=n}} (-1)^{pq+r} f_k \circ_{p+1} m_q^A - \sum_{\substack{k\geq 2\\ i_1+\ldots+i_k=n}} (-1)^e m_k^B \circ (f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_k})$$

where e is given as: $e = (k-1)(i_1-1) + (k-2)(i_2-1) + \ldots + 2(i_{k-2}-1) + (i_{k-1}-1)$

Since the composition of two dg-coalgebra homomorphism is again a dg-algebra homomorphism, we get that the composition of two ∞-morphisms is again an ∞-morphism. More explicitly if $f : A \rightsquigarrow B$ and $g : B \rightsquigarrow C$ are two ∞-morphisms, then their composition is defined as:

$$(fg)_n = \sum_r \sum_{i_1+\ldots+i_r=n} (-1)^e g_r(f_{i_1} \otimes \ldots \otimes f_{i_r}).$$

An ∞-quasi-isomorphism is an ∞-morphism f such that f_1 is a quasi-isomorphism.

Let Alg_∞ denote the category of A_∞-algebras. The morphisms in this category are the ∞-morphisms. Observe that we may extend the bar construction $B : Alg_\infty \to ConilCoalg^\bullet$ to a fully fatihful functor, identifying Alg_∞ as a subcategory of $ConilCoalg^\bullet$. This subcategory is namely the full subcategory of every dg-coalgebra that is isomorphic, as a coalgebra, to a cofree tensor coalgebra. Also, the category of dg-algebras is a non-full subcategory of A_∞-algebras. This inclusion is really the recontextualization of a dg-algebra into an A_∞-algebra.

Dual to A_∞-algebras we got conilpotent A_∞-coalgebras. Here we instead ask ourselves if the cobar construction has some converse. I.e. if C is a graded vector space such that $T(C[-1])$ together with a derivation m is a dg-algebra, is then C a coalgebra? Again, the answer to this is no, but we do obtain a definition for conilpotent A_∞-coalgebras.

Definition 1.8.9. A graded vector space C is called a conilpotent A_∞-coalgebra if it is a dg-algebra of the form $(\bar{T}(C[-1]), d)$ where d is a derivation of degree -1.

Remark 1.8.10. For the rest of this thesis, an A_∞-coalgebra should be understood as a conilpotent A_∞-coalgebra unless otherwise specified.

Corollary 1.8.10.1. *C is an A_∞-coalgebra with differential d then there is a chain complex (C, d^1), where d^1 is of degree 1, and together with morphisms $d^n : C \to C^{\otimes n}$ such that d uniquely determines each d^i for any $i > 0$. Conversely, if the morphisms d^i satisfy (rel)$_n$, then they uniquely determine a d such that C is an A_∞-coalgebra.*

$$(\textit{rel}_n) \qquad \sum_{p+q+r=n} (-1)^{pq+r} d^{p+1+q} \circ^{op}_{p+1} d^q = 0$$

A morphism of A_∞-coalgebras would be defined in the same manner, but opposite. So an ∞-comorphism $f : C \rightsquigarrow D$ is either a morphism $\widetilde{f} : (T(C[-1]), m^C) \to (T(D[-1]), m^D)$ of dg-algebras. Equivalently such an ∞-comorphism is a collection of morphisms $f_n : C \to D^{\otimes n}$ of degree $n-2$ such that f_1 is a morphism of chain complexes and for any $n \geq 2$ the following relations are satisfied:

$$(\text{rel}_n) \qquad \partial(f_n) = \sum_{\substack{p+1+r=k \\ p+q+r=n}} (-1)^{pq+r} f_k \circ^{op}_{p+1} m^D_q - \sum_{\substack{k\geq 2 \\ i_1+\ldots+i_k=n}} (-1)^e m^C_k \circ^{op} (f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_k})$$

where e is given as: $e = (k-1)(i_1-1) + (k-2)(i_2-1) + \ldots + 2(i_{k-2}-1) + (i_{k-1}-1)$

We denote $Coalg_\infty$ as the category of A_∞-coalgebras. In the same manner, the cobar construction extends to this category and gives us an identification of A_∞-coalgebras and a subcategory of dg-algebras. This subcategory consists of every dg-algebra that is isomorphic, as an algebra, to a free tensor algebra. Lastly, every dg-coalgebra is an A_∞-coalgebra by letting every morphism $m^i = 0$ where $i > 2$. This gives a non full inclusion.

Chapter 2

Homotopy Theory of Algebras

Quillen envisioned a more general approach to homotopy theory, which he dubbed homotopical algebra. A homotopy theory was then enclosed by the structure of a model category, then a closed model category. Many of the results from classical homotopy theory was then recovered in this new setting of model categories. The theorem which we are concerned about is Whiteheads theorem:

Theorem 2.0.1 (Whiteheads Theorem)**.** *Let X and Y be two CW-complexes. If $f : X \to Y$ is a weak equivalence, then it is also a homotopy equivalence. I.e. there exists a morphism $g : Y \to X$ such that $gf \sim id_X$ and $fg \sim id_Y$.*

If we employ Quillens model category onto the category Top, we get that a space X is bifibrant if and only if it is a CW-complex. The natural generalization is then to not ask of X to be a CW-complex, but a bifibrant object.

Theorem 2.0.2 (Generalized Whiteheads Theorem)**.** *Let $\mathcal{C}$ be a model category. Suppose that X and Y are bifibrant objects of $\mathcal{C}$, and that there is a weak-equivalence $f : X \to Y$. Then f is also a homotopy equivalence, i.e. there exists a morphism $g : Y \to X$ such that $gf \sim id_X$ and $fg \sim id_Y$.*

The category of differential graded (co)algebras employs such a model category. Here we let the weak-equivalences be quasi-isomorphisms. Moreover, in this case the bar and cobar construction is a Quillen equivalence between the model structures. As we will see, a dg-coalgebra will be bifibrant exactly when it is an A_∞-algebra. Thus, by Whiteheads theorem, quasi-isomorphisms lift to homotopy equivalences. In this case the derived category of A_∞-algebras is equivalent to the homotopy category of A_∞-algebras.

We will conclude this section by looking at the category of algebras as a subcategory of A_∞-algebras. The derived category may then be expressed as the homotopy category A_∞-algebras, restricted to algebras.

2.1 Model categories

In this section we will define Quillens model category. As one may see is that in practice there are a plethora of semantically different definitions of model categories, however they are all made to be equivalent. The difference comes down to preference. In this thesis we will use the definitions as they are developed in Mark Hoveys book [12]. We will then go on to prove the basic results known about model categories, its associated homotopy category and Quillen functors between model categories.

Before we state the definition of a model category we need some preliminary definitions. For this section, let $\mathcal{C}$ be a category.

Definition 2.1.1 (Retract)**.** A morphism $f : A \to B$ in $\mathcal{C}$ is a retract of a morphism $g : c \to D$ if it fits in a commutative diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, "id_A"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\
B \arrow[r] \arrow[rr, bend right, "id_B"'] & D \arrow[r] & B
\end{tikzcd}$$

Definition 2.1.2 (Functorial factorization)**.** A pair of functors $\alpha, \beta : \mathcal{C}^{\to} \to \mathcal{C}^{\to}$ is called a functorial factorization if for any morphism $f = \beta(f) \circ \alpha(f)$. We will denote the morphisms in the factorization as f_α and f_β. The functorial factorization may be depicted by the following commutative diagram:

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "f_\alpha"'] & & B \\
& C \arrow[ur, "f_\beta"'] &
\end{tikzcd}$$

Definition 2.1.3 (Lifting properties)**.** Suppose that the morphisms $i : A \to B$ and $p : C \to D$ fits inside a commutative square. i is said to have the left lifting property with resptect to p, or p has the right lifting property with respect to i, if there is an $h : B \to C$ such that the two triangles commute.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"] & C \arrow[d, "p"] \\
B \arrow[r] \arrow[ur, dashed, "h"] & D
\end{tikzcd}$$

Remark 2.1.4. We will call the left lifting property for LLP and the right lifting property for RLP.

2.1.1 Model categories

Definition 2.1.5 (Model category)**.** Let $\mathcal{C}$ be a category with all finite limits and colimits. $\mathcal{C}$ admits a model structure if there are three wide subcategories each defining a class of morphisms:

- $Ac \subset Mor(\mathcal{C})$ are called weak equivalences
- $Cof \subset Mor(\mathcal{C})$ are called cofibrations
- $Fib \subset Mor(\mathcal{C})$ are called fibrations

In addition we call morphisms in $Cof \cap Ac$ for acyclic cofibrations and $Fib \cap Ac$ for acyclic fibrations. Moreover, $\mathcal{C}$ has two functorial factorizations (α, β) and (γ, δ). The following axioms should be satisfied:

MC1 The class of weak equivalences satisfy the 2-out-of-3 property, i.e. if f and g are composable morphisms such that 2 out of f, g and gf are weak equivalences, then so is the third.

MC2 The three classes Ac, Cof and Fib are retraction closed, i.e. if f is a retraction of g, and g is either a weak-equivalence, cofibration or fibration, then so is f.

MC3 The class of cofibrations have the left lifting property with respect to acyclic fibrations, and fibrations have the right lifting property with respect to acyclic cofibrations.

MC4 Given any morphism f, f_α is a cofibration, f_β is an acyclic fibration, f_γ is an acyclic cofibration and f_δ is a fibration.

Remark 2.1.6. The class Ac has every isomorphism. This is because every isomorphism is a retract of some identity morphism.

Remark 2.1.7. The type of category which has been introduced above was first called a closed model category by Quillen [13]. In his sense, a model category does not require to have either finite limits or finite colimits. In our case, we will explicitly state whenever a model category is non-closed, i.e. does not have every limit or colimit.

A model category $\mathcal{C}$ is now defined to be a category equipped with a particular model structure. Notice that a category may admit several model structures. We will postpone examples until sufficient theory have been developed. For more topological examples, we refer to Dwayer-Spalinski [14] and Hovey [12].

An interesting and a not so non-trivial property of model categories is that giving all three classes Ac, Cof and Fib is redundant. Given the class of weak equivalences and either cofibrations or fibrations, the model structure is determined. Thus the classes of fibrations are determined by acyclic cofibrations and cofibrations are determined by fibrations. The next two results will show this.

Lemma 2.1.8 (The retract argument)**.** *Let $\mathcal{C}$ be a category. Suppose there is a factorization $f = pi$ and that f has LLP with respect to p, then f is a retract of i. Dually, if f har RLP to i, then it is a retract of p.*

Proof. We assume that $f : A \to C$ has LLP with respect to $p : B \to C$. Then we may find a lift $r : C \to B$, which realize f as a retract of i.

$$
\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[d, "p"] \\
C \arrow[r, equal] \arrow[ur, dashed, "r"] & C
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "f"] & A \arrow[r, equal] \arrow[d, "i"] & A \arrow[d, "f"] \\
C \arrow[r, "r"] & B \arrow[r, "p"] & C
\end{tikzcd}
$$

□

Proposition 2.1.9. *Let $\mathcal{C}$ be a model category. A morphism f is a cofibration (acyclic cofibration) if and only if f has LLP with respect acyclic fibrations (fibrations). Dually, f is a fibration (acyclic fibration) if and only if it has RLP with respect to acyclic cofibrations (cofibrations).*

Proof. Assume that f is a cofibration. By MC3, we know that f has LLP with respect to acyclic fibrations. Assume instead that f has LLP with respect to ever acyclic fibration. By MC4 we factor $f = f_\alpha \circ f_\beta$, where f_α is a cofibration and f_β is an acyclic fibration. Since we assume f to have LLP with respect to f_β, by lemma 2.1.8 we know that f is a retract of f_α. Thus by MC2, we know that f is a cofibration. □

Corollary 2.1.9.1. *Let $\mathcal{C}$ be a model category. (Acyclic) Cofibrations are stable under pushouts, i.e. if f is an (acyclic) cofibration, then f' is an (acyclic) cofibration.*

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & C \arrow[d, "f'"] \\
B \arrow[r] & D \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}
$$

Dually, fibrations are stable under pullbacks.

Proof. This is clear by the universal property of pushouts. □

Since we assume that every model category $\mathcal{C}$ is admits finite limits and colimits, we know that it has both an initial and a terminal object. We let $\emptyset$ denote the initial object and $*$ denote the terminal object.

Definition 2.1.10 (Cofibrant, fibrant and bifibrant objects)**.** Let $\mathcal{C}$ be a model category. An object X is called cofibrant if the unique morphism $\emptyset \to X$ is a cofibration. Dually, X is called fibrant if the unique morphism $X \to *$ is fibrant. If X is both cofibrant and fibrant, we call it bifibrant.

There is no reason for every object to be either cofibrant or fibrant. However, we may see that every object is weakly equivalent to an object which is either fibrant or cofibrant. We will make it precise what it means for two objects to be weakly equivalent later.

Construction 2.1.11. Let X be an object of a model category $\mathcal{C}$. The morphism $i : \emptyset \to X$ has a functorial factorization $i = i_\beta \circ i_\alpha$, where $i_\alpha : \emptyset \to QX$ is a cofibration and $i_\beta : QX \to X$ is an acyclic fibration. By definition QX is cofibrant and weakly equivalent to X.

$Q : \mathcal{C} \to \mathcal{C}$ defines a functor called the cofibrant replacement. To see this we first look at the slice category ${}^{\emptyset}/_{\mathcal{C}}$. The objects are morphisms $f : \emptyset \to X$ for any object X in $\mathcal{C}$, while morphisms are commutative triangles. We first observe that ${}^{\emptyset}/_{\mathcal{C}} \subset \mathcal{C}^{\to}$ is a subcategory of the arrow category. Thus (α, β) may be interpreted as functors on the slice category to the arrow category. Moreover, since every arrow $f : \emptyset \to X$ is unique, we observe that this category is equivalent to $\mathcal{C}$. Thus (α, β) may be interpreted as functors on $\mathcal{C}$ into arrows. We define Q as the composition $Q = tar \circ \alpha$.

Dually, we get a fibrant replacement R by dualizing the above argument.

We collect the following properties

Lemma 2.1.12. *The cofibrant replacement Q and fibrant replacement R preserves weak equivalences.*

Proof. Clear from the 2-out-of-3 property. □

Lemma 2.1.13 (Ken Brown's lemma)**.** *Let $\mathcal{C}$ be a model category and $\mathcal{D}$ be a category with weak equivalences satisfying the 2-out-of-3 property. If $F : \mathcal{C} \to \mathcal{D}$ is a functor sending acyclic cofibrations between cofibrant objects to weak equivalences, then F takes all weak equivalences between cofibrant objects to weak equivalences. Dually, if F takes all acyclic fibrations between fibrant objects to weak equivalences, then F takes all weak equivalences between fibrant objects to weak equivalences.*

Proof. Suppose that A and B are cofibrant objects and that $f : A \to B$ is a weak equivalence. Using the universal property of the coproduct we define the map $(f, id_B) = p : A \coprod B \to B$. p has a functorial factorization into a cofibration and acyclic fibration, $p = p_\beta \circ p_\alpha$. We recollect the maps in the following pushout diagram:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & B \arrow[d, "i_2"] \arrow[dddrr, bend left, "id_B"] & & \\
A \arrow[r, "i_1"] \arrow[rrrdd, bend right, "f"'] \arrow[ur, phantom, "\lrcorner"] & A \coprod B \arrow[dr, "p_\alpha"] & & \\
& & C \arrow[dr, "p_\beta"] & \\
& & & D
\end{tikzcd}$$

By lemma 2.1.9.1 both i_1 and i_2 are cofibrations. Since f, id_B and p_β are weak equivalences, so are $p_\alpha \circ i_1$ and $p_\alpha \circ i_2$ by MC2. Moreover, they are acyclic cofibrations.

Assume that $F : \mathcal{C} \to \mathcal{D}$ is a functor as described above. Then by assumption, $F(p_\alpha \circ i_1)$ and $F(p_\alpha \circ i_2)$ are weak equivalences. Since a functor sends identity to identity, we also know that $F(id_B)$ is a weak equivelnce. Thus by the 2-out-of-3 property $F(p_\beta)$ is a weak equivalence, as $F(p_\beta) \circ F(p_\alpha \circ i_2) = id_{F(B)}$. Again, by 2-out-of-3 property $F(f)$ is a weak equivelnce, as $F(f) = F(p_\beta) \circ F(p_\alpha \circ i_1)$. □

2.1.2 Homotopy category

Homotopy theory at it's most abstract is the study of categories and functors up to weak equivalences. Here, a weak equivalence may be anything, but most commonly it is a weak equivalence in topological homotopy or a quasi-isomorphism in homological algebra. The biggest concern when dealing with such concepts is to make a functor well-defined up to these chosen weak-equivalences. To this end, there is a construction to amend these problems, known as derived functors. We define a homotopical category in the sense of Riehl [15].

Definition 2.1.14 (Homotopical Category)**.** Let $\mathcal{C}$ be a category. $\mathcal{C}$ is Homotopical if there is a wide subcategory constituting a class of morphisms known as weak equivalences, $Ac \subset Mor\mathcal{C}$. The weak equivalences should satisfy the 2**-out-of-**6 **property**, i.e. given three composable morphisms f, g and g, if gf and hg are weak equivalences, then so are f, g, h and hgf.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "gf"'] \arrow[drr, dotted] & B \arrow[d, "g"] \arrow[dr, "hg"] & \\
& C \arrow[r, "h"] & D
\end{tikzcd}
$$

Remark 2.1.15. Notice that the 2-out-of-6 property is stronger than the 2-out-of-3 property. To see this, let either f, g or h be the identity, and then conclude with the 2-out-of-3 property.

Remark 2.1.16. The collection of weak equivalences contains every isomorphism. To see this pick an isomorphism f and f^{-1}, then the compositions are the identity on the domain and codomain, which are assumed to be in Ac.

Given such a homotopical category $\mathcal{C}$, we want to invert every weak equivalence and create the homotopy category of $\mathcal{C}$. This construction is developed in Gabriel and Zisman [16] called calculus of fractions. This method essentially tries to mimic localization for commutative rings in a category theoretic fashion.

Definition 2.1.17. Let $\mathcal{C}$ be a homotopical category. It's homotopy category $Ho\mathcal{C} = \mathcal{C}[Ac^{-1}]$, together with a localization functor $q : \mathcal{C} \to Ho\mathcal{C}$. Recall that the localization are determined by the following universal property: If $F : \mathcal{C} \to \mathcal{D}$ is a functor sending weak equivalences to isomorphisms, then it uniquely factors through the homotopy category up to a unique natural isomorphism η.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "q"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{D} \\ & Ho\mathcal{C} \arrow[ur, "F'"'] & \end{tikzcd}$$

Definition 2.1.18. Suppose that $\mathcal{C}$ is a homotopical category. Two objects of $\mathcal{C}$ are said to be weakly equivalent if they are isomorphic in $Ho\mathcal{C}$. I.e. there is some zig-zag relation between the objects, consisting only of weak equivalences.

Remark 2.1.19. A renown problem with localizations is that even if $\mathcal{C}$ is a locally small category, any localization $\mathcal{C}[S^{-1}]$ does not need to be. Thus, without a good theory of classes or higher universes, we cannot in general ensure that the localization still exists as a locally small category.

We see from the definition of the homotopy category that a functor F admits a lift F' to the homotopy category whenever weak equivalences are sent to isomorphisms. Moreover, if we have a functor F between homotopical categories which preserves weak equivalences, it then induces a functor between the homotopy categories.

Definition 2.1.20 (Homotopical functors)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ between homotopical categories is homotopical if it preserves weak equivalences. Moreover, there is a lift of functors as in the following diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "q_{\mathcal{C}}"] & \mathcal{D} \arrow[d, "q_{\mathcal{D}}"] \arrow[dl, Rightarrow, "\eta"] \\ Ho\mathcal{C} \arrow[r, dashed, "F'"] & Ho\mathcal{D} \end{tikzcd}$$

Derived functors come into play whenever this is not the case. These lifts are however the closest approximation which we can make functorial. The general exposition of derived functors is beyond the scope of this thesis, but an account of it may be found in [15]. As we will see, model categories are a nice environment to work with these concepts. Firstly we will amend the problem with localizations, where the homotopy category may not exists. Secondly, we will obtain a simple description of some important derived functors.

Proposition 2.1.21. *Any model category $\mathcal{C}$ is a homotopical category.*

Proof. To show that a model category is homotopical it suffices to show that Ac satisfies the 2-out-of-6 property. Assume there are 3 composable morphisms f, g, h such that $gf, hg \in Ac$. By the 2-out-of-3 property for Ac it is enough to show that at least one of f, g, h, fgh is a weak equivalence to deduce that every other morphism is a weak equivalence.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr] \arrow[drr, dotted, "gf"] & B \arrow[d, "g"] \arrow[dr, "hg"] & \\
& C \arrow[r, "h"] & D
\end{tikzcd}$$

To be able to use the model structure, we will first show that we may assume f, g to be cofibrant and g, h to be fibrant. We know by MC4 that f, g, gf may be factored into a cofibration composed with an acyclic fibration, e.g. $f = f_\beta f_\alpha$. Since gf is a weak equivalence, so is $(gf)_\alpha$ by the 2-out-of-3 property.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "f_\alpha"'] & & B \\
& B' \arrow[ur, "f_\beta"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
B \arrow[rr, "g"] \arrow[dr, "g_\alpha"'] & & C \\
& C' \arrow[ur, "g_\beta"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[rr, "gf"] \arrow[dr, "(gf)_\alpha"'] & & C \\
& C'' \arrow[ur, "(gf)_\beta"'] &
\end{tikzcd}$$

Notice that the "cofibrant approximation" of the map from A to C either goes through C' or C''. We conjoin these by taking the pullback. Since acyclic fibrations are stable over pullbacks, we get a pullback square where every morphism is an acyclic fibration. Thus the map $A \to \widetilde{C}$ is a weak equivalence by 2-out-of-3.

$$\begin{tikzcd}
A \arrow[dr, dashed] \arrow[drr, bend left, "g_\alpha"] \arrow[ddr, bend right, "(gf)_\alpha"'] & & \\
& \widetilde{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & C' \arrow[d, "g_\beta"] \\
& C'' \arrow[r, "(gf)_\beta"] & C
\end{tikzcd}$$

To replace f with f_α we must lift the composition into our "new" C, which is $\widetilde{C}$. This is done by using MC3, as f_α is a cofibration and the pullback square above consists entirely of acyclic fibrations.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f_\alpha"] & \widetilde{C} \arrow[d] \\
B' \arrow[r] \arrow[ur, dashed] & C
\end{tikzcd}$$

To summarize we have the following diagram, where every squiggly arrow is a weak equivalence.

$$\begin{tikzcd}
A \arrow[r, "f_\alpha"] \arrow[dr, squiggly, "t"] & B' \arrow[r, squiggly] \arrow[d, "s"] & B \arrow[d] \arrow[dr, squiggly] & \\
& \widetilde{C} \arrow[r, squiggly] & C \arrow[r] & D
\end{tikzcd}$$

We now wish to promote the arrow $s : B' \to \widetilde{C}$ into a cofibration. We do this by factoring both s and t with $MC4$. Notice that s_β, t_β and t_α are weak equivalences.

$$\begin{tikzcd}
B' \arrow[rr, "s"] \arrow[dr, "s_\alpha"'] & & \widetilde{C} & A \arrow[rr, "t"] \arrow[dr, "t_\alpha"'] & & \widetilde{C} \\
& \widetilde{C}' \arrow[ur, "s_\beta"'] & & & \widetilde{C}'' \arrow[ur, "t_\beta"'] &
\end{tikzcd}$$

To obtain our final factorization we use RLP of s_β on t_α.

$$\begin{tikzcd}
& B' \arrow[d, "s_\alpha"] \\
A \arrow[ur, "f_\alpha"] \arrow[r] \arrow[d, "t_\alpha"] & \widetilde{C}' \arrow[d, "s_\beta"] \\
\widetilde{C}'' \arrow[ur, dashed] \arrow[r, "t_\beta"] & \widetilde{C}
\end{tikzcd}$$

We have now obtained a factorization of gf which are two cofibrations followed by an acyclic fibration, in such a manner that it is compatible with the original composition. The dual to this claim is that we may also factor hg into two fibrations which is preceeded by an acyclic cofibration. In other words, we may assume without loss of generality that f and g are cofibrations, and that g and h are fibrations.

In this case we will show that g is an isomorphism. Consider the diagram below with lifts i and j, these exists since we assume gf and hg to be weak equivalences.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "gf"] & B \arrow[r, "id_B"] \arrow[d, "g"] & B \arrow[d, "hg"] \\
C \arrow[ur, dashed, "i"] \arrow[r, "id_C"] & C \arrow[ur, dashed, "j"] \arrow[r, "h"] & D
\end{tikzcd}$$

Since the diagram is commutative, we get that $i = j$, and that g is both split mono and split epi, with i as its splitting. □

Since every model category is homotopical, it also has an associated homotopy category $Ho\mathcal{C}$. Let $\mathcal{C}_c$, $\mathcal{C}_f$ and $\mathcal{C}_{cf}$ denote the full subcategories consisting of cofibrant, fibrant and bifibrant objects respectively.

Proposition 2.1.22. *Let $\mathcal{C}$ be a model category. The following categories are equivalent:*

- $Ho\mathcal{C}$
- $Ho\mathcal{C}_c$
- $Ho\mathcal{C}_f$
- $Ho\mathcal{C}_{cf}$

Proof. We show that $Ho\mathcal{C} \simeq Ho\mathcal{C}_c$. The inclusion $i : \mathcal{C}_c \to \mathcal{C}$ clearly preserves weak equivalences, thus i is homotopical and admits a lift. Moreover, since the cofibrant replacement is also homotopical, it also has a lift.

$$\begin{tikzcd}
\mathcal{C}_c \arrow[r, "i"] \arrow[d] & \mathcal{C} \arrow[d] \\
Ho\mathcal{C}_c \arrow[r, dashed, bend right, "Ho\ i"'] & \mathcal{C} \arrow[l, dashed, bend right, "Q"']
\end{tikzcd}$$

It is clear that Q is the quasi-inverse of i.

$\square$

As of now we still don't see how model categories will fix the size issues. To do this we will develop homotopy equivalence $\sim$. We will see that on the subcategory of bifibrant objects $\mathcal{C}_{cf}$, this homotopy equivalence will in fact be a congruence relation. Moreover, there is an equivalence of categories $Ho\mathcal{C}_{cf} \simeq {}^{\mathcal{C}_{cf}}/_{\sim}$.

Definition 2.1.23 (Cylinder and path objects)**.** Let $\mathcal{C}$ be a model category. Given an object X, a cylinder object $X \wedge I$ is a factorization of the fold map $i : X \coprod X \to X$, such that p_0 is a cofibration and that p_1 is a weak equivalence.

$$\begin{tikzcd}
X \coprod X \arrow[rr, "i"] \arrow[dr, "p_0"'] & & X \\
& X \wedge I \arrow[ur, "p_1"'] &
\end{tikzcd}$$

Dually, a path object X^I is a factorization of the diagonal map $i : X \to X \prod X$, such that p_0 is a weak equivalence and that p_1 is a fibration.

$$\begin{tikzcd}
X \arrow[rr, "i"] \arrow[dr, "p_0"'] & & X \prod X \\
& X^I \arrow[ur, "p_1"'] &
\end{tikzcd}$$

Remark 2.1.24. Even though we have written $X \wedge I$ suggestively to be a functor, it is not. There may be many choices for a cylinder object. However, by using the functorial factorization from MC4, we get a canonical choice of a cylinder object, as it factors every map into a cofibration and an acyclic fibration. By letting the cylinder object be this object, we do indeed obtain a functor.

Proposition 2.1.25. *Let $\mathcal{C}$ be a model category and X an object of $\mathcal{C}$. Given two cylinder objects $X \wedge I$ and $X \wedge I'$, then they are weakly equivalent.*

Proof. It is enough to show that there is a weak equivalence from any cylinder object into one specified cylinder object. This is in fact true for the functorial cylinder object $X \wedge I$, as the final morphism p_1 is an acyclic fibration, which enables a lift which is a weak equivalence by the 2-out-of-3 property.

$$\begin{tikzcd}
X \coprod X \arrow[r, "p_0"] \arrow[d, "p_0'"] & X \wedge I \arrow[d, "p_1"] \\
X \wedge I' \arrow[r, "p_1'"] \arrow[ur, dashed] & X
\end{tikzcd}$$

□

Definition 2.1.26 (Homotopy equivalence)**.** Let $f, g : X \to Y$. A left homotopy between f and g is a morphism $H : X \wedge I \to Y$ such that $Hi_0 = f$ and $Hi_1 = g$. f and g are left homotopic if a left homotopy exists, and it is denoted $f \overset{l}{\sim} g$.

$$\begin{tikzcd}
X \arrow[d] \arrow[dr, dotted, "i_0"] \arrow[drr, bend left, "f"] & & \\
X \coprod X \arrow[r, "p_0"] & X \wedge I \arrow[r, "H"] & Y \\
X \arrow[u] \arrow[ur, dotted, "i_1"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

A right homotopy between f and g is a morphism $H : X \to Y^I$ such that $i_0H = f$ and $i_1H = g$. f and g are right homotopic if a right homotopy exists, and it is denoted $f \overset{r}{\sim} g$.

$$\begin{tikzcd}
 & & Y \\
X \arrow[r, "H"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & Y^I \arrow[r, "p_1"] \arrow[ur, dotted, "i_0"] \arrow[dr, dotted, "i_1"'] & Y \prod Y \arrow[u] \arrow[d] \\
 & & Y
\end{tikzcd}$$

f and g are said to be homotopic if they are both left and right homotopic, it is denoted $f \sim g$. f is said to be a homotopy equivalence, if it has a homotopy inverse $h : Y \to X$, such that $hf \sim id_X$ and $fh \sim id_Y$.

It is important to know that this is not a priori an equivalence relation. This is amended by taking both fibrant and cofibrant replacements. We see this in the following proposition.

Proposition 2.1.27. *Let $\mathcal{C}$ be a model category, and $f, g : X \to Y$ be morphisms. We have the following:*

1. *If $f \stackrel{l}{\sim}$ and $h : Y \to Z$, then $hf \stackrel{l}{\sim} hg$.*
2. *If Y is fibrant, $f \stackrel{l}{\sim} g$ and $h : W \to X$, then $fh \stackrel{l}{\sim} gh$.*
3. *If X is cofibrant, then left homotopy is an equivalence relation on $\mathcal{C}(X, Y)$.*
4. *If X is cofibrant and $f \stackrel{l}{\sim} g$, then $f \stackrel{r}{\sim} g$.*

Proof. $(1.)$ Assume that $f \stackrel{l}{\sim} g$ and $h : Y \to Z$. Let $H : X \wedge I \to Y$ denote the left homotopy between f and g. The left homotopy between hf and hg is given as hH.

$(2.)$ Assume that Y is fibrant, $f \stackrel{l}{\sim} g$ and that $h : W \to X$. Let $H : X \wedge I \to Y$ be a left homotopy. We construct a new cylinder object for the homotopy. Factor $p_1 : X \wedge I \to X$ as $q_1 \circ q_0$ where $q_0 : X \wedge I \to X \wedge I'$ is an acyclic cofibration and $q_1 : X \wedge I' \to X$ is a fibration. By the 2-out-of-3 property q_1 is an acyclic fibration, as p_1 and q_0 are weak equivalences. $X \wedge I'$ is a cylinder object as $q_0 \circ p_0$ is a cofibration and q_1 is a weak equivalence. Since we assume Y to be fibrant we lift the left homotopy $H : X \wedge I \to Y$ to the left homotopy $H' : X \wedge I' \to Y$ with the following diagram:

$$\begin{tikzcd}
X \wedge I \arrow[r, "H"] \arrow[d, "q_0"] & Y \arrow[d] \\
X \wedge I' \arrow[r] \arrow[ur, dashed, "H'"] & *
\end{tikzcd}$$

We can find the appropriate homotopy needed with lift given by the following diagram:

$$\begin{tikzcd}
W \coprod W \arrow[r, "q_0 p_0(h \coprod h)"] \arrow[d, "p_0'"] & X \wedge I' \arrow[d, "q_1"] \\
W \wedge I \arrow[r, "hp_1'"] \arrow[ur, dashed, "k"] & X
\end{tikzcd}$$

Then the morphism $H'k$ is the desired left homotopy witnessing $fh \overset{l}{\sim} gh$.

$(3.)$ Assume that X is cofibrant. First observe that a left homotopy is reflexive and symmetric. We must show that in this case it is also transitative. Thus, assume that $f, g, h : X \to Y$ and that $H : X \wedge I \to Y$ is a left homotopy witnessing $f \overset{l}{\sim} g$ and that $H' : X \wedge I' \to Y$ is a left homotopy witnessing $g \overset{l}{\sim} h$. We first observe that $i_0 : X \to X \wedge I$ is a weak equivalence, as $id_X = p_1 i_0$ where id_X and p_1 are weak equivalences. Since X is assumed to be cofibrant, we see that $X \coprod X$ is cofibrant by the following pushout:

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & X \arrow[d, "inr"] \\
X \arrow[r, "inl"] \arrow[ur, phantom, "\lrcorner"] & X \coprod X
\end{tikzcd}$$

Moreover, both inl and inr are cofibrations. This shows that i_0 is a cofibration as $i_0 = p_0 \circ inr$ is a composition of two cofibrations. i_0 is thus an acyclic cofibration. We define an almost cylinder object C by the pushout of i_1 and i_0'. We define the maps t and H'' by using the universal property in the following manner:

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[d, "i_0'"] & X \wedge I \arrow[d] \arrow[ddr, bend left, "p_1"] & \\
X \wedge I' \arrow[r] \arrow[drr, bend right, "p_1'"] & C \arrow[dr, dashed, "t"] & \\
& & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[d, "i_0'"] & X \wedge I \arrow[d] \arrow[ddr, bend left, "H"] & \\
X \wedge I' \arrow[r] \arrow[drr, bend right, "H'"] & C \arrow[dr, dashed, "H''"] & \\
& & X
\end{tikzcd}$$

Observe that there is a factorization of the fold map $X \coprod X \overset{s}{\to} C \overset{t}{\to} X$. However, s may not be a cofibration, so we replace C with the cylinder object $X \wedge I''$ such that we have the factorization $X \coprod X \overset{s_\alpha}{\to} X \wedge I'' \overset{ts_\beta}{\to} X$. The morphism Hs_β is then our required homotopy for $f \overset{l}{\sim} g$.

$(4.)$ Suppose that X is cofibrant and that $H : X \wedge I \to Y$ is a left homotopy for $f \overset{l}{\sim} g$. Pick a path object for Y, such that we have the factorization $Y \overset{q_0}{\to} Y^I \overset{q_1}{\to} Y \prod Y$ where q_0 is a weak equivalence and q_1 is a fibration. Again, as X is cofibrant we get that i_0 is an acyclic cofibration, so we have the following lift of the homotopy:

$$\begin{tikzcd} X \arrow[r, "q_0 f"] \arrow[d, "i_0"] & Y^I \arrow[d, "q_1"] \\ X \wedge I \arrow[r, "(fp_1,H)"] \arrow[ur, dashed, "J"] & Y \prod Y \end{tikzcd}$$

The right homotopy is given by injecting away from f, i.e. $H' = Ji_1$. □

Corollary 2.1.27.1. *We collect the dual results of the above proposition, and thus have the following.*

1. *If* $f \overset{r}{\sim}$ *and* $h : W \to X$*, then* $fh \overset{r}{\sim} gh$.
2. *If* X *is cofibrant,* $f \overset{r}{\sim} g$ *and* $h : Y \to Z$*, then* $hf \overset{r}{\sim} hg$.
3. *If* Y *is fibrant, then left homotopy is an equivalence relation on* $\mathcal{C}(X, Y)$.
4. *If* Y *is fibrant and* $f \overset{r}{\sim} g$*, then* $f \overset{l}{\sim} g$.

Corollary 2.1.27.2. *Homotopy is a congruence relation on* $\mathcal{C}_{cf}$*. In this manner, the category* $\mathcal{C}_{cf}/\sim$ *is well-defined, exists and inverts every homotopy equivalence.*

Lemma 2.1.28 (Weird Whitehead)**.** *Let* $\mathcal{C}$ *be a model category. Suppose that* C *is cofibrant and* $h : X \to Y$ *is an acyclic fibration or a weak equivalence between fibrant objects, then* h *induces an isomorphism:*

$$\mathcal{C}(C, X)/\overset{l}{\sim} \xrightarrow[\simeq]{h_*} \mathcal{C}(C, Y)/\overset{l}{\sim}$$

Dually, if X *is fibrant and* $h : C \to D$ *is an acyclic cofibration or a weak equivalence between cofibrant objects, then* h *induces an isomorphism:*

$$\mathcal{C}(D, X)/\overset{r}{\sim} \xrightarrow[\simeq]{h^*} \mathcal{C}(C, X)/\overset{r}{\sim}$$

Proof. We assume $\mathcal{C}$ to be cofibrant and $h : X \to Y$ to ba an acyclic fibration. We first prove that h is surjective. Let $f : C \to Y$. By RLP of h there is a morphism $f' : C \to X$ such that $f = hf'$.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "h"] \\ C \arrow[r, "f"] \arrow[ur, dashed, "f'"] & Y \end{tikzcd}$$

To show injectivity we assume $f, g : C \to X$ such that $hf \overset{l}{\sim} hg$, in particular there is a left homotopy $H : C \wedge I \to Y$. Remember that since C is cofibrant, the map p_0 is a cofibration. We find a left homotopy $H : C \wedge I \to X$ witnessing $f \overset{l}{\sim} g$ by the following lift.

$$\begin{tikzcd} C\coprod C \arrow[r, "f+g"] \arrow[d, "p_0"'] & X \arrow[d, "h"] \\ C\wedge I \arrow[r, "H"'] \arrow[ur, dashed, "H'"] & Y \end{tikzcd}$$

Moreover, if we assume both X and Y to be fibrant, the functor ${}^{\mathcal{C}(C,\_)}/_{\overset{l}{\sim}}$ sends acyclic fibrations to isomorphisms, i.e. to weak equivalences. By Ken Brown's lemma, lemma 2.1.13, the afformentioned functor sends weak equivalences between fibrant objects to isomorphisms. ☐

Theorem 2.1.29 (Generalized Whiteheads theorem)**.** *Let $\mathcal{C}$ be a model category. Suppose that $f : X \to Y$ is a morphism of bifibrant objects, then f is a weak equivalence if and only if f is a homotopy equivalence.*

Proof. Suppose first that f is a weak equivalence. Pick a bifibrant object A, then by lemma 2.1.28 $f_* : {}^{\mathcal{C}(A,X)}/_{\sim} \to {}^{\mathcal{C}(A,Y)}/_{\sim}$ is an isomorphism. Letting $A = Y$ we know that there is a morphism $g : Y \to X$, such that $f_*g = fg \sim id_Y$. Furthermore, by proposition 2.1.27, since X is bifibrant composing on the right preserves homotopy equivalence, e.g. $fgf \sim f$. By letting $A = X$, we get that $f_*gf = fgf \sim f = f_*id_X$, thus $gf \sim id_X$.

For the opposite direction, assume that f is a homotopy equivalence. We factor f into an acyclic cofibration f_γ and a fibration f_δ, i.e. $X \overset{f_\gamma}{\to} Z \overset{f_\delta}{\to} Y$. Observe that Z is bifibrant as X and Y is, in particular, f_γ is a weak equivalence of bifibrant objects, so it is a homotopy equivalence.

It is enough to show that f_δ is a weak equivalence. Let g be the homotopy inverse of f, and $H : Y \wedge I \to Y$ is a left homotopy witnessing $fg \sim id_Y$. Since Y is bifibrant, the following square has a lift.

$$\begin{tikzcd} Y \arrow[r, "f_\gamma g"] \arrow[d, "i_0"'] & Z \arrow[d, "f_\delta"] \\ Y\wedge I \arrow[r, "H"'] \arrow[ur, dashed, "H'"] & Y \end{tikzcd}$$

Let $h = H'i_1$, then by definition we know that $f_\delta H' i_1 = id_Y$. Moreover, H is a left homotopy witnessing $f_\gamma g \sim h$. Let $g' : Z \to X$ be the homotopy inverse of f_γ. We have the following relations $f_\delta \sim f_\delta f_\gamma g' \sim fg'$, and $hf_\delta \sim (f_\gamma g)(fg') \sim f_\gamma g' \sim id_Z$. Let $H'' : Z \wedge I \to Z$ be a left homotopy witnessing this homotopy. Since Z is bifibrant, i_0 and i_1 are weak equivalences. By the 2-out-of-3 property H'' and hf_δ are weak equivalences. Since $f_\delta h = id_Y$, it follows that f_δ is a retract of $f_\delta h$, and is thus a weak equivalence. ☐

Corollary 2.1.29.1. *The category ${}^{\mathcal{C}_{cf}}/_{\sim}$ satisfy the universal property of the localization of $\mathcal{C}_{cf}$ by the weak equivalences. I.e. there is a categorical equivalence $Ho\mathcal{C}_{cf} \simeq {}^{\mathcal{C}_{cf}}/_{\sim}$.*

Proof. By generalized Whiteheads theorem, theorem 2.1.29 weak equivalences and homotopy equivalences coincide. The corollary follows steadily from both the universal property of the localization category and the quotient category. ☐

We collect the results from above in the following theorem.

Theorem 2.1.30 (Fundamental theorem of model categories)**.** *Let $\mathcal{C}$ be a model category and denote $q : \mathcal{C} \to Ho\mathcal{C}$ the localization functor. Let X and Y be objects of $\mathcal{C}$.*

1. *There is an equivalence of categories $Ho\mathcal{C} \simeq {}^{\mathcal{C}_{cf}}/{\sim}$.*
2. *There are natural isomorphisms ${}^{\mathcal{C}_{cf}}/{\sim}(QRX, QRY) \simeq Ho\mathcal{C}(X, Y) \simeq {}^{\mathcal{C}_{cf}}/{\sim}(RQX, RQY)$. Additionally, $Ho\mathcal{C}(X, Y) \simeq {}^{\mathcal{C}_{cf}}/{\sim}(QX, RY)$.*
3. *The localization q identifies left or right homotopic morphisms.*
4. *A morphism $f : X \to Y$ is a weak equivalence if and only if qf is an isomorphism.*

Proof. This is clear by the results above. ☐

2.1.3 Quillen adjoints

We now want to study morphisms, or certain functors, between model categories. Like in the case of homotopical functors we want these morphisms to induce a functor between the homotopy categories. However, we also want them to respect the cofibration and fibration structure, not just weak equivalences. In this way we will instead look towards derived functors to be able to define this extension to the homotopy category. We recall the definition of a total (left/right) derived functor. In the case of model categories, we get a simple description for some of these derived functors which are of special interest.

Definition 2.1.31 (Total derived functors)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be homotopical categories, and $F : \mathcal{C} \to \mathcal{D}$ a functor. Whenever it exists, a total left derived functor of F, is a functor $\mathbb{L}F : Ho\mathcal{C} \to Ho\mathcal{D}$ with a natural transformation $\varepsilon : \mathbb{L}F \circ q \Rightarrow q \circ F$ satisfying the universal property: If $G : Ho\mathcal{C} \to Ho\mathcal{D}$ is a functor and there is a natural transformation $\alpha : G \circ q \Rightarrow q \circ F$, then it factors uniquely up to unique isomorphism through ε.

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "q"'] & \mathcal{D} \arrow[d, "q"] \\
Ho\mathcal{C} \arrow[r, dashed, "\mathbb{L}F"'] \arrow[ur, phantom, "\nearrow\varepsilon"] & Ho\mathcal{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "q"'] & \mathcal{D} \arrow[d, "q"] \\
Ho\mathcal{C} \arrow[r, bend left, "\mathbb{L}F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\exists!\Uparrow"] \arrow[ur, phantom, "\nearrow\varepsilon"] & Ho\mathcal{D}
\end{tikzcd}
$$

Dually, whenever it exists, a total right derived functor of F, is a functor $\mathbb{R}F : Ho\mathcal{C} \to Ho\mathcal{D}$ with a natural transformation $\eta : q \circ F \Rightarrow \mathbb{R}F \circ q$ having the opposite universal property.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "q"] & \mathcal{D} \arrow[d, "q"] \arrow[dl, Rightarrow, "\eta"] \\ Ho\mathcal{C} \arrow[r, dashed, "\mathbb{R}F"] & Ho\mathcal{D} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "q"] & \mathcal{D} \arrow[d, "q"] \arrow[dl, Rightarrow, "\eta"'] \\ Ho\mathcal{C} \arrow[r, bend left, dashed, "\mathbb{R}F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\exists!\Downarrow"] & Ho\mathcal{D} \end{tikzcd}$$

Definition 2.1.32 (Deformation)**.** A left (right) deformation on a homotopical category $\mathcal{C}$ is an endofunctor Q together with a natural weak equivalence $q : Q \Rightarrow Id_{\mathcal{C}}$ ($q : Id_{\mathcal{C}} \Rightarrow Q$).

A left (right) deformation on a functor $F : \mathcal{C} \to \mathcal{D}$ between homotopical categories, is a left (right) deformation Q on $\mathcal{C}$ such that weak equivalences in the image of Q is preserved by F.

Remark 2.1.33 (Cofibrant and fibrant replacement). If $\mathcal{C}$ is a model category, then we have a left and a right deformation. The cofibrant replacement Q defines a left deformation, and the fibrant replacement defines a right deformation. Notice that this is only due to the fact that the factorization system is functorial.

Proposition 2.1.34. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between homotopical categories. If F has a left deformation Q, then the total left derived functor $\mathbb{L}F$ exists. Moreover, the functor FQ is homotopical, and $\mathbb{L}F$ is the unique extension of FQ.*

Proof. Since we already have a candidate for the derived functor, the proof must just check that it has the universal property. A proof may be found in Reihl [15] under proposition 6.4.11. □

Remark 2.1.35. There is a somewhat weaker statement by Dwayer and Spalinski [14]. If we instead ask for functors F which have the cofibrant replacement Q (fibrant replacement R) as a left (right) deformation we may make this proof more explicit. This is theorem 9.3.

Equipped with the above proposition and remark, it makes sense to define Quillen functors as left and right Quillen functors. A left Quillen functor should be left deformable by the cofibrant replacement. Moreover, for the composition of two left Quillen functors to make sense, we also need weak equivalences between cofibrant objects to be mapped to weak equivalences between cofibrant objects. We make the following definition.

Definition 2.1.36 (Quillen adjunction)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be model categories.

1. A left Quillen functor is a functor $F : \mathcal{C} \to \mathcal{D}$ such that it preserves cofibrations and acyclic cofibrations.
2. A right Quillen functor is a fucntor $F : \mathcal{C} \to \mathcal{D}$ such that it preserves fibrations and acyclic fibrations.
3. Suppose that (F, U) is an adjunction where $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to U. (F, U) is called a Quillen adjunction if F is a left Quillen functor and U is a right Quillen functor.

Remark 2.1.37. By Ken Browns lemma, lemma 2.1.13, we see that a left Quillen functor F is left deformable to the cofibrant replacement functor Q. Thus the total left derived functor exists and is given by $\mathbb{L}F = HoFQ$.

In order to eliminate the choice of left or right derivedness, we will think of a morphism of model categories as a Quillen adjunction. The direction of the arrow can be chosen to be along either the left or right adjoints, we make the convention of following the left adjoint functors. We summarize the following properties.

Lemma 2.1.38. *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories, and suppose there is an adjunction $F : \mathcal{C} \rightleftharpoons \mathcal{D} : U$. The following are equivalent:*

1. *(F, U) is a Quillen adjunction.*
2. *F is a left Quillen functor.*
3. *U is a right Quillen functor.*

Proof. This follows from naturality of the adjunction. I.e. any square in $\mathcal{C}$, with the right side from $\mathcal{D}$ is commutative if and only if any square in $\mathcal{D}$ with the left side from $\mathcal{C}$ is commutative. Now, f has LLP with respect to Ug if and only if Ff has LLP with respect to g.

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "f"'] & UX \arrow[d, "Ug"] \\ B \arrow[r, "l"] \arrow[ur, dotted, "h"] & UY \end{tikzcd} \rightsquigarrow \begin{tikzcd} FA \arrow[r, "k^T"] \arrow[d, "Ff"'] & X \arrow[d, "g"] \\ FB \arrow[r, "l^T"] \arrow[ur, dotted, "h^T"] & Y \end{tikzcd}$$

□

Proposition 2.1.39. *Suppose that $(F, U) : \mathcal{C} \to \mathcal{D}$ is a Quillen adjunction. The functors $\mathbb{L}F : Ho\mathcal{C} \to Ho\mathcal{D}$ and $\mathbb{R}U : Ho\mathcal{D} \to Ho\mathcal{C}$ forms an adjoint pair.*

Proof. We must show that $Ho\mathcal{D}(\mathbb{L}FX, Y) \simeq Ho\mathcal{D}(X, \mathbb{R}UY)$. By using the fundamental theorem of model categories, theorem 2.1.30, we have the following isomorphisms: $Ho\mathcal{D}(\mathbb{L}FX, Y) \simeq {}^{\mathcal{C}(FQX, RY)}/\sim$ and $Ho\mathcal{D}(X, \mathbb{R}UY) \simeq {}^{\mathcal{D}(QX, URY)}/\sim$. In other words, if we assume X to be cofibrant, and Y to be fibrant, we must show that the adjunction preserves homotopy equivalences.

We show it for one direction. Suppose that the morphisms $f, g : FA \to B$ are homotopic, witnessed by a right homotopy $H : FA \to B^I$. Since we assume U to preserve products, fibrations and weak equivalences between fibrant objects, $U(B^I)$ is a path object for UB. Thus the transpose $H^T : A \to U(B^I)$ is the desired homotopy witnessing $f^T \sim g^T$ □

Definition 2.1.40 (Quillen equivalence)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be model categories, and $(F, U) : \mathcal{C} \to \mathcal{D}$ be a Quillen adjunction. (F, U) is called a Quillen equivalence if for any cofibrant X in $\mathcal{C}$, fibrant Y in $\mathcal{D}$ and any morphism $f : FX \to Y$ is a weak equivalence if and only if its transpose $f^T : X \to UY$ is a weak equivalence.

Proposition 2.1.41. *Suppose that $(F, U) : \mathcal{C} \to \mathcal{D}$ is a Quillen adjunction. The following are equivalent:*

1. *(F, U) is a Quillen equivalence.*
2. *Let $\eta : Id_{\mathcal{C}} \Rightarrow UF$ denote the unit, and $\varepsilon : FU \Rightarrow Id_{\mathcal{D}}$ denote the counit. The composite $Ur_F\eta : Id_{\mathcal{C}_c} \Rightarrow URF|_{\mathcal{C}_c}$, and $\varepsilon_{FQU}Fq_U : FQU|_{\mathcal{D}_f} \Rightarrow Id_{\mathcal{D}_f}$ are natural weak equivalences.*
3. *The derived adjunction $(\mathbb{L}F, \mathbb{R}U)$ is an equivalence of categories.*

Proof. Firstly observe that $2. \implies 3.$ by definition. Secondly observe that equivalences both preserves and reflects isomorphisms, from this we get $3. \implies 1.$. We now show $1. \implies 2.$. Pick X in $\mathcal{C}$ such that X is cofibrant. Since (F, U) is assumed to be a Quillen adjunction we know that FX is still cofibrant. The fibrant replacement $r_{FX} : FX \to RFX$ gives us a weak equivalence. Furthermore, since (F, U) is assumed to be a Quillen equivalence, its transpose $r_{FX}^T : X \to URFX$ is a weak equivalence. Unwinding the definition of the transpose we get that $r_{FX}^T = Ur_{rFX}\eta_X$.

□

We have the following refinement.

Corollary 2.1.41.1. *Suppose that $(F, U) : \mathcal{C} \to \mathcal{D}$ is a Quillen adjunction. The following are equivalent:*

1. *(F, U) is a Quillen equivalence.*
2. *F reflects weak equivalences between cofibrant objects, and $\varepsilon_{FQU}F_{qU} : FQU|_{\mathcal{D}_f} \Rightarrow Id_{\mathcal{D}_f}$ is a natural weak equivalence.*
3. *U reflects weak equivalences between fibrant objects, and $U_{rF}\eta : Id_{\mathcal{C}_c} \Rightarrow URF|\mathcal{C}_c$ is a natural weak equivalence.*

Proof. We start by showing $1. \implies 2.$ and $3.$. We already know that the derived unit and counit are isomorphism in homotopy, so we only need to show that F (U) reflects weak equivalences between cofibrant (fibrant) objects. Suppose that $Ff : FX \to FY$ is a weak equivalence between cofibrant objects. Since F preserves weak equivalences between cofibrant objects, we get that FQf is a weak equivalence, or that $\mathbb{L}Ff$ is an isomorphism. By assumption, $\mathbb{L}F$ is an equivalence of categories, so f is a weak equivalence as needed.

We will show $2. \implies 1.$, the case $3. \implies 1.$ is dual. We assume that the counit map is an isomorphism in homotopy. By assumption, the derived unit $\mathbb{L}\eta$ is split-mono on the image of $\mathbb{L}F$. Moreover, the derived counit $\mathbb{R}\varepsilon$ is assumed to be an isomorphism, in particular the derived unit $\mathbb{L}F\mathbb{L}\eta$ is an isomorphism. Unpacking this, we have a morphism, call it $\eta'_X : FQX \to FQURFQX$, which is a weak equivalence. Since F and Q reflects weak equivalences, we get that $\eta_X : X \to URFQX$ is a weak equivalence. □

2.2 Model structures on Algebraic Categories

In order to understand ∞-quasi-isomorphism of strongly homotopy associative algebras we will study different homotopy theories of various categories. Munkholm [17] successfully showed that the derived category of augmented algebras is equivalent to the derived category of augmented algebras equipped with ∞-morphisms. Well, to be more precise, he showed that certain subcategories of augmented algebras had this property. Lefevre-Hasagawas phd thesis [10] builds upon this identification, but with help of further devolpment within the field. We will follow the approach of Lefevre-Hasagawa, by comparing the model structure for algebras and coalgebras,

2.2.1 DG-Algebras as a Model Category

Bousefield and Gugenheim [18] proved that the category of commutative dg-algebras had a model structure whenever the base field was a field of characteristic 0. In a joint project, Jardines paper from 1997 [19] shows that this construction may be extended to dg-algebras over any commutative ring. On the other hand, Munkholm expanded on the ideas from Bousfield and Gugenheim to get an identification of derived categories. Also, Hinichs paper from 1997 [20] details another method to obtain the model category which we want. We will follow the approach of Hinich, as it will be usefull later on. Notice that where Hinich use theory of algebraic operads to show that the category of algebras is a model category, we will however give a more explicit formulation.

Let $\mathbb{K}$ be a field, and $\mathcal{C}$ be a category such that there is an adjunction $F : \mathcal{C} \rightleftharpoons Ch(\mathbb{K}) : \#$, where F is left adjoint to #. Furthermore, suppose that $\mathcal{C}$ satsifies the 2 conditions:

(H0) $\mathcal{C}$ admits finite limits and every small colimit. The functor # commutes with filtered colimits.
(H1) Let M be the complex below, concentrated in 0 and 1.

$$\dots \longrightarrow 0 \longrightarrow \mathbb{K} \xrightarrow{id} \mathbb{K} \longrightarrow 0 \longrightarrow \dots$$

For any $d \in \mathbb{Z}$ and for any $A \in \mathcal{C}$ the injection $A \to A \coprod F(M[d])$ induces a quasi-isomorphism $A^\# \to (A \coprod F(M[d]))^\#$.

With this adjunction in mind, we define weak equivalences, fibrations and cofibrations as follows: Let $f \in \mathcal{C}$ be a morphism

- $f \in Ac$ if $f^\#$ is a quasi-isomorphism.
- $f \in Fib$ if $f^\#$ is surjective on each component.
- $f \in Cof$ if f has LLP to acyclic fibrations.

Theorem 2.2.1. *The category $\mathcal{C}$ equipped with the weak equivalences, fibrations and cofibrations as defined above is a model category.*

Before we show this theorem we need to understand the cofibrations better. Let $A \in \mathcal{C}$, $M \in Ch(\mathbb{K})$ and $\alpha : M \to A^\#$ a morphism in $Ch(\mathbb{K})$. We define a functor

$$h_{A,\alpha}(B) = \{(f,t) \mid f \in \mathcal{C}(A,B), t \in Hom_{\mathbb{K}}^{-1}(M, B^\#) \; s.t. \; \partial t = f^\# \circ \alpha\}.$$

Note that t is not a morphism of chain maps. This is a homogenous morphism of degree -1. The differential then promotes this morphism to a chain map, and t is thus a homotopy for the comoposite $f^\# \circ \alpha$.

This functor is represented by an object of $\mathcal{C}$. We define this representing object $A\langle M, \alpha\rangle$ as the pushout:

$$\begin{tikzcd} F(A^\#) \arrow[r, "\varepsilon_A"] \arrow[d] \arrow[dr, phantom, very near end, "\ulcorner"] & A \arrow[d, "a"] \\ F(cone(\alpha)) \arrow[r, "e"] & A\langle M, \alpha\rangle \end{tikzcd}$$

Let $i : M[1] \to cone(\alpha)$ be a homogenous morphism which is the injection when considered as graded modules. Notice that we have a pair of morphisms $(a, e^T i) \in h_{A,\alpha}(A\langle M, \alpha\rangle)$.

Proposition 2.2.2. *The functor $h_{A,\alpha}$ is represented by $A\langle M, \alpha\rangle$, i.e. $h_{A,\alpha} \simeq \mathcal{C}(A\langle M, \alpha\rangle, \_)$ is a natural isomorphism. Moreover, the pair $(a, e^T i)$ is the universal element of the functor $h_{A,\alpha}$, i.e. the natural isomorphism is induced by this element under Yoneda's lemma.*

Proof. Let $(f,t) \in h_{A,\alpha}(B)$ for some $B \in \mathcal{C}$. The condition that $\partial t = f^\# \alpha$ is equivalent to say that $f^\#$ extends to a morphism $f' : cone(\alpha) \to B^\#$ along t, i.e. $f' = \begin{pmatrix} f^\# & t \end{pmatrix}$. This concludes the isomorphism part, as being an element (f,t) is equivalent to the existence of the diagram below, where $\widetilde{f}$ is uniquely determined.

$$\begin{tikzcd} F(A^\#) \arrow[r, "\varepsilon_A"] \arrow[d] & A \arrow[d, "a"] \arrow[ddr, bend left, "f"] & \\ F(cone(\alpha)) \arrow[r, "e"] \arrow[drr, bend right, "f'"'] & A\langle M, \alpha\rangle \arrow[dr, dashed, "\widetilde{f}"] & \\ & & B \end{tikzcd}$$

To obtain naturality, we use the adjunction to observe that the element $(a, e^T i)$ is in fact universal. □

We are now in a position to explicitly find some important cofibrations. We collect these morphisms into the "standard" cofibrations.

Definition 2.2.3. Let $f : A \to B$ be a morphism in $\mathcal{C}$. Suppose that f factors as a transfinite composition of morphisms on the form $A_i \to A_i\langle M, \alpha\rangle$, i.e. f factors into the diagram below, where $A_{i+1} = A_i\langle M, \alpha\rangle$.

$$A \longrightarrow A_1 \longrightarrow A_2 \longrightarrow ... \longrightarrow B$$

- If every such M is a complex consisting of free $\mathbb{K}$-modules and has a 0-differential, we call f a standard cofibration.
- If every such M is a contractible complex (and $\alpha = 0$), we call f a standard acyclic cofibration.

Proposition 2.2.4. *Every standard cofibration is a cofibration, and every standard acyclic cofibration has LLP with respect to fibrations. Moreover, if $\alpha = 0$, then every standard acyclic cofibration is also a weak equivalence. We will see that these morphisms in some sense generate every (acyclic) cofibration.*

Proof. First observe that every standard cofibration may be made iteratively from the chain complexes $\mathbb{K}[n]$, and likewise, every standard acyclic cofibration may be made iteratively from M as in $H1$.

We first prove that if $M \simeq \mathbb{K}[n]$, and $\alpha : M \to A^{\#}$ is any map, then the map $A \to A\langle M, \alpha\rangle$ is a cofibration. This amounts to show that it has LLP to every acyclic fibration. Suppose that $h : B \to C$ is an acyclic fibration and that there is a commutative square as below.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle h} \\ A\langle M, \alpha\rangle & \xrightarrow{g} & C \end{array}$$

By the universal property of $h_{A,\alpha}$ 2.2.2 it suffices to find a pair (f', t') such that $f : A \to B$, $t' : M \to B^{\#}$ is homogenous of degree -1, $\partial t = f^{\#}\alpha$ and that h induces a morphism $h : (f', t') \to g$. We see that we are forced to choose $f' = f$ as $hf = ga$. We know there exists a $t : M \to C^{\#}$ such that $\partial t = g^{\#}a^{\#}\alpha = h^{\#}f^{\#}\alpha$. Since h is an acyclic fibration $h^{\#}$ is a surjective quasi-isomorphism. Since $M \simeq \mathbb{K}[n]$, the morphism t is really an element of $C^{\#^{n-1}}$. By surjectivity of $h^{\#}$ there is an element u of $B^{\#^{n-1}}$ such that $h^{\#}(u) = t$. Moreover, the difference $h^{\#}(\partial u - f^{\#}\alpha) = 0$, so $\partial u - f^{\#}\alpha$ factors through the kernel $Kerh^{\#}$, which is acyclic. This element is furthermore a cycle, so by acyclicity there is another element u' such that $\partial u' = \partial u - f^{\#}\alpha$. We may now see that $(f, u - u')$ is our desired factorization.

Secondly we prove that if M is in as $H1$ and $\alpha = 0$, then the map $A \to A\langle M, \alpha\rangle$ is an acyclic cofibration.This amounts to show that it has LLP to every fibration. Suppose that $h : B \to C$ is a fibration and that there is a commutative square as below.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle h} \\ A\langle M, \alpha\rangle & \xrightarrow{g} & C \end{array}$$

We will again use 2.2.2, so it suffices to find a (t') such that $\partial t' = f^{\#}\alpha$. Let $t : M \to C^{\#}$ such that $\partial t = h^{\#} f^{\#}\alpha$.

Finally we will assume that $\alpha = 0$. In this case the cone is a direct sum $cone(\alpha) = A^{\#} \oplus M$. Since F is left adjoint to #, we know that $Fcone(\alpha) \simeq F(A^{\#}) \coprod FM$. By $H1$ we get that the map $A \to A\langle M, \alpha\rangle \simeq A \coprod FM$ is a weak equivalence.

□

In light of the above proposition we would like to make some more convenient notation. If $M \simeq \mathbb{K}[n]$ and $\alpha : M \to Z^n(A^{\#})$, s.t. $\alpha(1) = a$, we write $A\langle M, \alpha\rangle$ as $A\langle T; dT = a\rangle$ instead. Hinich calls this "adding a variable to kill a cycle". If M is the acyclic complex as below and $\alpha = 0$, we write $A\langle T, S; dT = S\rangle$. This could be thought of "adding a variable and cycle to kill itself".

$$\ldots \longrightarrow 0 \longrightarrow \mathbb{K} \xrightarrow{id} \mathbb{K} \longrightarrow 0 \longrightarrow \ldots$$

proof of theorem. **MC1** and **MC2** are satisfied. By definition we also have the first part of **MC3**. We start by checking **MC4**.

Let $f : A \to B$ be a morphism in $\mathcal{C}$. Given any $b \in B^{\#}$, let $C_b = A\langle T_b, S_b; dT_b = S_b\rangle$. We define $g_b : C_b \to B$ by the conditions that it acts on A as f, $g_b^{\#}(T_b) = b$ and $g_b^{\#}(S_b) = db$. Iterating this construction for every $b \in B$, we obtain an object C, such that the injection $A \to C$ is an acyclic standard cofibration, and the map $g : C \to B$ is a fibration. This gives us a factorization $f = f_\delta \circ f_\gamma$, where f_γ is the injection and $f_\delta = g$.

To obtain the other factorization we start with our previous factorization. We let $C_0 = C$. From C_0 there is a morphism $g_0 : C_0 \to B$, which is surjective and surjective on kernels. This may fail to be a quasi-isomorphism. Pick a pair of elements (c, b), such that $c \in ZC_0^{\#}$ and $g_0^{\#}(c) = db$. We add a variable to C_0 to kill this cycle, i.e. let $C_1 = C_0\langle T, dT = c\rangle$. We iterate this procedure for any boundary in $B^{\#}$, to obtain C as the colimit. Then the injection $A \to C$ is a colimit of standard cofibrations and the morphism $g' : C \to B$ is an acyclic fibration.

It remains to check the last part of **CM3**. Suppose that $f : A \to B$ is an acyclic cofibration. By **CM4**, we know that it factors as $f = f_\delta \circ f_\gamma$, where f_δ is an acyclic fibration and f_γ is a standard acyclic fibration. We thus obtain that f is a retract of f_γ by the commutative diagram below.

$$\begin{tikzcd} A \arrow[r, "f_\gamma"] \arrow[d, "f"'] & C \arrow[d, "f_\delta"] \\ B \arrow[r, equal] \arrow[ur, dashed] & B \end{tikzcd}$$

□

The following corollary will concretize what it means that the standard cofibrations generate every cofibration. This corollary is really a step used within in the proof.

Corollary 2.2.4.1. *Any (acyclic) cofibration is a retract of a standard (acyclic) cofibration.*

We may immedietly apply this theorem to some familiar examples.

Corollary 2.2.4.2. *Let A be a dg-algebra over the field $\mathbb{K}$. The category mod_A of left modules is a model category.*

sketch of proof. We establish the adjunction by letting $F = A \otimes_{\mathbb{K}} \_$. H0 is satisfied as this category is bicomplete, and filtered colimits may be thought of as unions of sets. Moreover, since mod_A is an Abelian category, the forgetful functor # commutes with coproducts, or direct sums, which makes H1 trivially satisfied. □

Corollary 2.2.4.3. *The categories $Alg^\bullet_{\mathbb{K}}$ ($AugAlg^\bullet_{\mathbb{K}}$) are model categories.*

Proof. We establish the adjunction by letting $F = \bar{T}(T)$, the reduced tensor algebra of a cochain complex. For the same reasons as above, H0 is trivially satisfied. To show H1 we must find the coproduct and what $\bar{T}(M)$ is. ... □

We summarize the last result:

The category of augmented dg-algebras $AugAlg^\bullet_{\mathbb{K}}$ is a model category. Let $f : X \to Y$ be a homomorphism of augmented algebras.

- $f \in Ac$ if $f^\#$ is a quasi-isomorphism.
- $f \in Fib$ if $f^\#$ is an epimorphism (surjective onto every component).
- $f \in Cof$ if f has LLP wrt. to every acyclic fibration.

The category of augmented dg-algebras has an initial and a terminal object. The initial object is the stalk $\mathbb{K}$ and the terminal object is the 0-ring. We see that every object is fibrant, as 0 is preserved by the forgetful functor and every map into 0 is surjective. Every dg-algebra which is isomorphic to a tensor algebra when considered as a graded algebra is cofibrant.

2.2.2 A Model Structure on DG-Coalgebras

We now want to equip the category of dg-coalgebras with a suitable model structure. This model structure should be suitable in the sense that it give rise to the same homotopy theory of dg-algebras. The bar-cobar construction will be crucial in this construction, as it is in fact a Quillen adjunction. To this end we will follow the setup as presented by Lefevre-Hasegawa [10]. His method is a modification of Hinichs paper [21] which describes a model structure on dg-coalgebras, but in relation to dg-lie algebras.

Let $f : C \to D$, the category of dg-coalgerbas will be equipped with the three following classes of morphisms:

- $f \in Ac$ if Ωf is a quasi-isomorphism.
- $f \in Fib$ if f has RLP wrt. to every acyclic cofibration.
- $f \in Cof$ if $f^{\#}$ is a monomorphism (injective in every component).

Before we show that this defines a model structure on dg-coalgebras we need some preliminary results. First recall that $f : C \to D$ a morphism between dg-coalgebras is a graded quasi-isomorphism if grf is a quasi-isomorphism. Lemma 1.7.5 says that whenever such f are graded quasi-isomorphisms, f is a weak equivalence. This is quite convenient, since this gives us a method to check that a morphism is a weak equivalence without evaluating it on the cobar construction. Moreover, both the unit and the counit of the bar-cobar adjunction are weak equivalences by proposition 1.7.7. With the following lemma we see that the bar-cobar adjunction is a Quillen equivalence if the category of dg-coalgebras is a model category.

Lemma 2.2.5. *Let $f : C \to D$ be a morphism of dg-coalgebras, then:*

- *if f is a cofibration, then Ωf is a standard cofibration.*
- *if f is a weak equivalence, then Ωf is as well.*

Almost dually, let $f : A \to B$ be a morphism of dg-algebras, then:

- *if f is a fibration, then Bf is a fibration.*
- *if f is a weak equivalence, then Bf is as well.*

Proof. First suppose that $f : C \to D$ is a cofibration. We define a filtration on D as the sum of the image of f and the coradical filtration on D: $D_i = Imf + Fr_i D$. f being a cofibration ensures us that $D_0 \simeq C$. Since D is conilpotent we know that $D \simeq \varinjlim D_i$, and that Ω commutes with colimits, there is a sequence of algebras $\Omega C \to \Omega D_1 \to ... \to \Omega D$. It is enough to show that each morphism $\Omega D_i \to \Omega D_{i+1}$ is a standard cofibration. The quotient coalgebra D_{i+1}/D_i only has a trivial comultiplication, thus every element is primitive. This means that as a cochain complex D_{i+1} is constructed from D_i by attaching possibly very many copies of $\mathbb{K}$. We treat the case when there is only one such $\mathbb{K}$, here $D_{i+1} \simeq D_i \oplus \mathbb{K}\{x\}$ where $dx = y$ for some $y \in D_i$.

We observe that this is exactly the condition for that the morphism $\Omega D_i \to \Omega D_{i+1}$ is a standard cofibration.

If f is a weak equivalence, then Ωf is a quasi-isomorphism by definition.

By lemma 2.1.38, or adjointness more specifically, B preserving fibrations is a consequence of Ω preserving cofibrations.

It remains to show that if $f : A \to B$ is a quasi-isomorphism, then Bf is a weak equivalence. Now, Bf is a weak equivalence if and only if ΩBf is a quasi-isomorphism. By 1.7.7, the counit $A \to \Omega BA$ is a quasi-isomorphism, so Bf is a weak equivalence by 2-out-of-3 property.

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ \Omega BA \arrow[u, "\varepsilon_A"] \arrow[r, "\Omega Bf"] & \Omega BB \arrow[u, "\varepsilon_B"'] \end{tikzcd}$$

□

We will need one more technical lemma.

Lemma 2.2.6. *Let A be a dg-algebra, D a dg-coalgebra and $p : A \to \Omega D$ a fibration of algebras. The projection morphism $\pi : BA \prod_{B\Omega D} D \to BA$ is an acyclic cofibration.*

$$\begin{tikzcd} BA \prod_{B\Omega D} D \arrow[r] \arrow[d, "\pi"] & D \arrow[d, "\eta_D"] \\ BA \arrow[r, "Bp"] & B\Omega D \end{tikzcd}$$

Proof. π being a cofibration is immediate by corollary 2.1.9.1. To see that π is a quasi-isomorphism it is enough to understand that it is a quasi-isomorphism as chain complexes. This is checked by Lefevre-Hasegawa [10]. □

Theorem 2.2.7. *The category $ConilCoalg^\bullet_{\mathbb{K}}$ is a model category with the classes Ac, Fib and Cof as defined above.*

Proof. The axioms **MC1** and **MC2** are immediet. Also, fibrations having RLP wrt. acyclic cofibrations is by definition.

We show **MC4** first. Let $f : C \to D$ be a morphism of coalgebras. There is a factorization $\Omega f = pi$ of morphisms between algebras, where i is a cofibration, p is a fibration and at least one of i and p are quasi-isomorphisms. Applying bar we get a factorization $B\Omega f = BiBp$, where Bp is a fibration and at least one of Bi and Bp are weak equivalences.

$$\begin{tikzcd} \Omega C \arrow[rr, "\Omega f"] \arrow[dr, "i"'] & & \Omega D \\ & A \arrow[ur, "p"'] & \end{tikzcd} \rightsquigarrow \begin{tikzcd} B\Omega C \arrow[rr, "B\Omega F"] \arrow[dr, "Bi"'] & & B\Omega D \\ & BA \arrow[ur, "Bp"'] & \end{tikzcd}$$

We construct a pullback with Bp and η_D. By 2.2.6 the morphism π is an acyclic cofibration. We collect our morphisms in a big diagram. The dashed arrow exists since the rightmost square is a pullback.

$$\begin{tikzcd} & BA\prod_{B\Omega D} D \arrow[dd] \arrow[dr, "q"] \arrow[dr, phantom, "\ulcorner", very near start] & \\ C \arrow[ur, dashed, "j"] \arrow[rr, "f"] \arrow[dd, "\eta_C"] & & D \arrow[dd, "\eta_D"] \\ & BA \arrow[dr, "Bp"] & \\ B\Omega C \arrow[ur, "Bi"] \arrow[rr, "B\Omega f"] & & B\Omega D \end{tikzcd}$$

First notice that q is a fibration, since fibrations are stable under pullbacks. j is a cofibration, or a monomorphism, as the composition $Bi\eta_C$ is a monomorphism. Thus it remains to see that if Bi (Bp) is a weak equivalence, then j (q) is as well. This is evident from the 2-out-of-3 property, as η is a natural weak equivalence, π is a weak equivalence and Bi (Bp) is a weak equivalence.

We now show **CM3**. Suppose that there is a square as below, where i is a cofibration and t is an acyclic cofibration.

$$\begin{tikzcd} E \arrow[r] \arrow[d, "i"] & C \arrow[d, "t"] \\ F \arrow[r] & D \end{tikzcd}$$

We can factor t as $t = qj$ by **CM4**. Notice that t is a retract of q, i.e. there is a commutative diagram as below.

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d, "j"] & C \arrow[d, "t"] \\ BA\prod_{B\Omega A} D \arrow[r, "q"] \arrow[ur] & D \end{tikzcd}$$

So in order to find a lift to C, we may instead find a lift to $BA \prod_{B\Omega D} D$. Since p is an acyclic fibration by construction and Ωi is a cofibration by 2.2.5, there is a lift $h : \Omega E \to A$ of algebras. We obtain our desired lift from the bar-cobar adjunction and the universal property of the pullback.

$$
\begin{tikzcd}
E \arrow[r] \arrow[d, "i"] & BA \prod_{B\Omega D} D \arrow[r, "\pi"] \arrow[d, "q"] \arrow[dr, phantom, "\ulcorner", very near start] & BA \arrow[d, "Bp"] & & \Omega E \arrow[r] \arrow[d, "\Omega i"] & A \arrow[d, "p"] \\
F \arrow[r] \arrow[ur, dashed] \arrow[urr, dotted, "h^T"] & D \arrow[r, "\eta_D"] & B\Omega D \arrow[r, phantom, "\leftrightsquigarrow"] & {} & \Omega F \arrow[r] \arrow[ur, dotted, "h"] & \Omega D
\end{tikzcd}
$$

□

We restate the corollary of the adjunction.

Corollary 2.2.7.1. *The bar-cobar construction $B : AugAlg^{\bullet}_{\mathbb{K}} \rightleftharpoons ConilCoalg^{\bullet}_{\mathbb{K}} : \Omega$ as a Quillen equivalence.*

Proof. We first observe that (B, Ω) is a Quillen adjunction by lemma 2.2.5. Moreover, since the unit and counit are weak equivalences by proposition 1.7.7, it follows by either proposition 2.1.41 or its corollary 2.1.41.1 that (B, Ω) is a Quillen equivalence. □

2.2.3 Homotopy theory of A_∞-algebras

This section aims to finalize the discussion of the homotopy theory of A_∞-algebras. We will look at the homotopy invertability of every strongly homotopy associative quasi-isomorphism, and the relation to associative algebras. This discussion will end with mentioning different results which gives a clearer description of fibrations, cofibrations and homotopy equivalences. This section follows Lefevre-Hasegawa [10]. Before we get to the main theorem, we start by discussing a non-closed model structure on the category of Alg_∞.

Let $f : A \rightsquigarrow B$ be a morphism between A_∞-algebras, the category of A_∞-algebras will be equipped with the three following classes of morphisms:

- $f \in Ac$ if f is an ∞-quasi-isomorphism, i.e. f_1 is a quasi-isomorphism.
- $f \in Fib$ if f_1 is an epimorphism.
- $f \in Cof$ if f_1 is a monomorphism.

This category does not make a model category in the sense of a closed model category, as we are lacking many finite limits. It does however come quite close to be such a category.

Theorem 2.2.8. *The category Alg_∞ equipped with the three classes as defined above satisfies:*

*a The axioms **MC1** through **MC4**.*

b *Given a diagram as below, where p is a fibration, then its limit exists.*

$$\begin{tikzcd} & A \arrow[d, "p"] \\ B \arrow[r] & C \end{tikzcd}$$

Before we are able to prove this, we need some lemmata.

Lemma 2.2.9. *let A be an A_∞-algebra, and K a contractible complex considered as an A_∞-algebra. If $g : (A, m_1^A) \to (K, m_1^K)$ is a cochain map, then it extends to an ∞-morphism $f : A \rightsquigarrow K$.*

Proof. We construct each f_i inductively. The case $i = 1$ is degenerate as we have assumed $f_1 = g$.

Assume that we have already constructed f_1 through f_n. We observe that the sum below is a cycle of $Hom^*_{\mathbb{K}}(A, K)$.

$$\sum_{\substack{p+1+r=k \\ p+q+r=n}} (-1)^{pq+r} f_k \circ_{p+1} m_q^A - \sum_{\substack{k \geq 2 \\ i_1+...+i_k=n}} (-1)^e m_k^B \circ (f_{i_1} \otimes f_{i_2} \otimes ... \otimes f_{i_k})$$

Thus since K is contractible, $Hom^*_{\mathbb{K}}(A, K)$ is acyclic and there exists some morphism f_{n+1} such that $\partial(fn + 1)$ is the sum above. This says that this extension does in fact satisfy (rel_{n+1}). ☐

Lemma 2.2.10. *Let $j : A \rightsquigarrow D$ be a cofibration of A_∞-algbera s, then there is an isomorphism $k : D \rightsquigarrow D'$ such that the composition $k \circ j : A \rightsquigarrow D'$ is a strict morphism of A_∞-algebras.*

Dually, if $j : A \rightsquigarrow D$ is a fibration, then there is an isomorphism $l : A' \rightsquigarrow A$ such that the composition $j \circ l : A' \rightsquigarrow D$ is a strict morphism of A_∞-algebras.

Proof. A proof is given as lemma 1.3.3.3 in [10]. ☐

proof of 2.2.8. We start by showing b. Suppose that we have a diagram of A_∞-algebras, such that g_1 is an epimorphism.

$$\begin{tikzcd} & A \arrow[d, two heads, "g"] \\ A' \arrow[r, "f"] & A'' \end{tikzcd}$$

First notice that as dg-coalgebras, this pullback exists and defines a new dg-coalgebra $BA \prod_{BA''} BA'$.

Since g_1 is an epimorphism, $A[1]$ as a graded vector space splits into $A''[1] \oplus K$, where $K = Ker g_1$. The pullback is then naturally identified with $BA \prod_{BA''} BA' \simeq \bar{T}^c(K) \prod \bar{T}^c(A'[1])$ as graded vector spaces. Since the cofree coalgebra is right adjoint to forget, it commutes with products and we get, $\bar{T}^c(A'[1]) \prod \bar{T}^c(K) \simeq \bar{T}^c(A'[1] \oplus K)$. Thus the pullback is isomorphic to a cofree coalgebra as a graded coalgebra, i.e. it is an A_∞-algebra.

We now prove a. MC1 and MC2 are immediate, so we will not prove them.

We start by proving MC3. Suppose that there is a square of A_∞-algebras as below, where j is a cofibration and q is a fibration.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"] & B \arrow[d, "q"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

By lemma 2.2.10, we may assume assume that both j and q are strict morphisms. We now assume that q is an ∞-quasi-isomorphism, the proof will be analogous if j is an ∞-quasi-isomorphism instead.

Our goal is to construct a lifting in this diagram inductively. Having a lift means finding an ∞-morphism $a : C \rightsquigarrow B$, such that the following hold for any $n \geq 1$:

- a satisfy (rel_n).
- $a_n \circ j_1 = f_n$.
- $q_1 \circ a_n = g_n$.

We start by showing there is such an a_1. Consider the diagram below of chain complexes over $\mathbb{K}$.

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, tail, "j_1"] & B \arrow[d, two heads, "q_1"] \\ C \arrow[r, "g_1"] \arrow[ur, dashed, "a_1"] & D \end{tikzcd}$$

The lift exists since the category $Ch\mathbb{K}$ is a model category. Here j_1 is a cofibration, while q_1 is an acyclic fibration, so the lift a_1 exists.

We now wish to extend this. Suppose that we have been able to create morphisms a_1 up to a_n, all satisfying the above points. A naive solution to make a_{n+1} is $b = f_{n+1} r^{\otimes n+1} + s g_{n+1} - s q_1 f_{n+1} r^{\otimes n+1}$. Notice that this satisfy the two last points by definition. We will augment b to get an a_{n+1} which also satisfies (rel_{n+1}).

For our own convenience, let $-c(f_1, ..., f_n)$ denote the right hand side of (rel_{n+1}) formula. Since both j and q are strict ∞-morphisms we get the following identites:

$$(\partial b + c(a_1, ..., a_n)) \circ j_1 = \partial(b \circ j_1) + c(a_1 \circ j_1, ..., a_n \circ j_1) = \partial f_{n+1} + c(f_1, ..., f_n) = 0$$
$$q_1 \circ (\partial b + c(a_1, ..., a_n)) = \partial(q_1 \circ b) + c(q_1 \circ a_1, ..., q_1 \circ a_n) = \partial g_{n+1} + c(g_1, ..., g_n) = 0$$

We thus obtain that the cycle $\partial b + c(a_1, ..., a_n)$ factors thorugh the cokernel of j and the kernel of q. Let us say that it factors like the diagram below:

$$C \xrightarrow{p} Cokj \xrightarrow{c'} Kerq \xrightarrow{i} D$$

Now, c' is a morphism between two A_∞-algebras. Since q is assumed to be an ∞-quasi-isomorphism, it follows that $Kerq$ is a acyclic. By lemma 2.2.9, we get that c' may be extended to an ∞-morphism , call it h. We let $a_{n+1} = b - i \circ h \circ p$. This morphism satisfies all three properties.

I dont unde this :(

We will now show MC4. Since the two properties have a similar proof, we will only show one direction. Let $f : A \rightsquigarrow B$ be an ∞-morphism. Let $C = cone(id_(B[-1]))$. The complex C may be considered as an A_∞-algebra. Let $j : A \rightsquigarrow A \prod C$ be the morphism induced by id_A and $0 : A \to C$. The canonical projection $q_1 : A \oplus C \to B$ gives a lift of the following diagram.

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "j_1"] & B \arrow[d] \\
A \oplus C \arrow[r] \arrow[ur, "q_1"] & 0
\end{tikzcd}$$

Since we have a morphism of chain complexes, lodged between an acyclic cofibration and a fibration we use the same techniqꞏue as above to construce an ∞-morphism $q : A \prod C \to B$. q is a fibration by construction. The morphism f may be factored as $f = q \circ j$, where j is an acyclic cofibration and q is a fibration. $\square$

With this model structure we are finally able to characterize the fibrant and cofibrant conilpotent dg-coalgebras.

Proposition 2.2.11. *Let C be a conilpotent dg-coalgebra. Then C is cofibrant, and C is fibrant if and only if there is a cochain complex V, such that $C \simeq T^c(V)$ as complexes.*

Proof. To see that C is cofibrant is the same as to verify that the map $\mathbb{K} \to C$ is a monomorphism, but this is clear.

We start by assuming that C is fibrant. Then there is a lift in the square below, making the unit split-mono.

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d, "\eta_C"] & C \arrow[d, "\varepsilon_C"] \\ B\Omega C \arrow[r, "\varepsilon_{B\Omega C}"] \arrow[ur, dashed, "r"] & \mathbb{K} \end{tikzcd}$$

Consider the morphism $p_1^C : C \to Fr_1 C$ which is defined as $p_1^C = Fr_1 r \circ p_1 \circ \eta_C$, where $p_1 : B\Omega C \to Fr_1 B\Omega C$ is the canonical projection on the filtration induced by the coradical filtration on C. Clearly, r makes p_1 into a universal arrow in the category of conilpotent coalgebras, so $C \simeq T^c(Fr_1 C)$.

Now, assume that C is isomorphic to $T^c(V)$ as coalgebras for some cochain complex V. Note that, by definition, C is an A_∞-algebra. By definition, we have a commutative square of A_∞-algebras. Since every A_∞-algebra is bifibrant, we know that this diagram has a lift, exhibiting C as a retract of $B\Omega C$.

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d] & C \arrow[d] \\ B\Omega C \arrow[r] \arrow[ur, dashed] & \mathbb{K} \end{tikzcd}$$

We know that ΩC is fibrant, since the map $\Omega C \to 0$ is epi. By lemma 2.2.5, we know that the bar construction preserves fibrations, so $B\Omega C$ is fibrant. Thus C is fibrant as well. □

Before we end this section we will state some results of the homotopy theory of A_∞-algebras, relating them to homological algebra.

2.3 The Homotopy Category of Alg_∞

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