# Sha-Categories & Quasi-isomorphisms

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## Abstract

*Fill inn abstract*

## Sammendrag

*Fyll inn sammendraget*

## Acknowledgements

*Thank the people in your life who has made this journey easier :D*

# Contents

# Chapter 1

# Preliminaries

## 1.1 Localization

We will briefly explain the localization of a category at a set. We will look at local objects and localizations by adjoint functors. This section is based on [1].

### 1.1.1 Definition and Local Objects

The localization of a category at some set of morphisms is a universal category where these morphisms have been inverted. In this manner, we will think of the localization as adding formal inverses to the morphisms in this set to the category.

**Definition 1.1.1.** Let $\mathcal{C}$ be a category and $S \subseteq Mor(\mathcal{C})$ a set of morphisms in $\mathcal{C}$. We define the localization of $\mathcal{C}$ at $S$ as a category $\mathcal{C}[S^{-1}]$ together with a functor $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ satisfying the properties:

1. For any morphism $s \in S$, $Qs$ is an invertible morphism.
2. Any functor $F : \mathcal{C} \to \mathcal{D}$ satisfying $Fs$ is invertible if $s \in S$ factors through $Q$. Thus there exists a unique functor $\bar{F} : \mathcal{C}[S]^{-1} \to \mathcal{D}$ and a natural isomorphism $\alpha : F \implies \bar{F} \circ Q$ relating the functors.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "Q"'] & & \mathcal{D} \\ & \mathcal{C}[S^{-1}] \arrow[ur, dashed, "\bar{F}"'] & \end{tikzcd}$$

Informally, the category $\mathcal{C}[S^{-1}]$ may be constructed as the category with the same objects as $\mathcal{C}$. The morphisms are constructed with paths of morphisms in $\mathcal{C}$, where we also allow formal inverse paths $s^{-1}$ for any morphism in $S$. The constant paths would be the new identity morphisms, while every path from an object $X$ to $Y$ represents a morphism.

*Remark* 1.1.2. Beware whenever localizing a locally small category. Adding more morphisms to this category may make some set of morphisms between objects big.

By the universality of the functor $Q$ we have the following lemma.

**Lemma 1.1.3.** *Let $\mathcal{D}$ be a category, then the pre-composition functor is fully faithful.*

$$\begin{aligned} \_ \circ Q : Fun(\mathcal{C}[S^{-1}], \mathcal{D}) &\to Fun(\mathcal{C}, \mathcal{D}) \\ F &\mapsto F \circ Q \end{aligned}$$

*Moreover, we may identify $Fun(\mathcal{C}[S^{-1}], \mathcal{D})$ as the full subcategory of functors in $Fun(\mathcal{C}, \mathcal{D})$ which sends every morphism in $S$ to an isomorphism.*

We proceed to define $S$-local objects, and informally, these are all the objects which does not change after the localization process.

**Definition 1.1.4.** An object $Y \in \mathcal{C}$ is called $S$-local ($S$-closed or $S$-orthogonal) if for any $s \in S$, the map $\mathcal{C}(s, Y)$ is a bijection. Define $S \perp$ to be the full subcategory of $S$-local objects.

The following lemma gives an equivalent definition for "localness".

**Lemma 1.1.5.** *An object $Y \in \mathcal{C}$ is $S$-local if and only if the canonical map $q$ is a natural isomorphism.*

$$q_{X,Y} : \mathcal{C}(X, Y) \to \mathcal{C}[S^{-1}](X, Y)$$

### 1.1.2 Localizing at Adjoint Functors

Given two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that they form an adjoint pair $F \dashv G$, we want to see how they react with localizations.

**Proposition 1.1.6.** *Let $F \dashv G$ be an adjoint pair as above, and define $S =$\{$s \in Mor\mathcal{C} \mid Fs$ is invertible\}. Draw out the following diagram, with the relations $F = \bar{F} \circ Q$.*

$$\begin{tikzcd} & \mathcal{C} \arrow[dl, "Q"'] \arrow[dd, bend right, "F"'] \\ \mathcal{C}[S^{-1}] \arrow[dr, "\bar{F}"'] & \\ & \mathcal{D} \arrow[uu, bend right, "G"'] \end{tikzcd}$$

*The following are equivalent:*

1. *$G$ is fully faithful.*
2. *The counit $\varepsilon : FG \implies Id_{\mathcal{D}}$ is invertible for any object $X \in \mathcal{D}$.*
3. *The functor $F$ induces an equivalence $\bar{F} : \mathcal{C}[S^{-1}] \to \mathcal{D}$.*

In the light of the last proposition, we may define "short exact sequences" of pre-additive categories.

**Definition 1.1.7.** Suppose there is a diagram of additive functors like below.

$$\begin{tikzcd} \mathcal{A} \arrow[r, tail, bend left, "E"] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, two heads, bend left, "E_\rho"] \arrow[r, two heads, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, tail, bend left, "F_\rho"] \end{tikzcd}$$

It is called a localization sequence whenever the following holds:

1. $(E, E_\rho)$ and $(F, F_\rho)$ are adjoint pairs.
2. $E$ and $F_\rho$ are fully faithful.
3. $ImE \simeq KerF$.

## 1.2 Triangulated Categories

### 1.2.1 Homotopy colimits

### 1.2.2 A generalization for algebraic triangulated categories

# Bibliography


[1] H. Krause, *Homological Theory of Representations -Draft Version of a Book Project-*. Cambridge University Press, Aug. 2021. [Online]. Available: `https://www.math.uni-bielefeld.de/~hkrause/HomTheRep.pdf`.