On the Derived Category of Strongly Homotopy Associative Algebras

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Abstract

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Sammendrag

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Acknowledgements

Thank the people in your life who has made this journey easier :D

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Chapter 1

Bar and Cobar Construction

In Stasheffs papers [1] and [2], a strongly homotopy associative algebra, or A_∞-algebra, over a field is a graded vector space together with homogenous linear maps $m_n : A^{\otimes n} \to A$ of degree $2-n$ satisfying some homotopical relations. This will be made precise later. We may regard m_2 to be a multiplication of A, it is however not a priori associative. The associator of m_2 is taken to be the homotopical relation of m_3. Thus, we know that the homotopy of A is an associative algebra. The maps m_n corresponds uniquely to a map $m^c : BA \to \bar{A}[1]$, which extends to a coderivation $m^c : BA \to BA$ of the bar construction of A. So we could instead define an A_∞-algebra to be a coalgebra on the form BA.

In order to understand the bar construction we will first study it on associative algebras. Given a differential graded coassociative coalgebra C and a differential graded associative algebra A, we say that a homogenous linear transformation $\alpha : C \to A$ is twisting if it satisfies the Maurer-Cartan equation:

$$\partial\alpha + \alpha \star \alpha = 0.$$

Let $Tw(C, A)$ be the set of twisting morphisms, then considering it as a functor $Tw : CoAlg_{\mathbb{K}}^{op} \times Alg_{\mathbb{K}} \to Ab$ we want to show that it is represented in both arguments. Moreover, these representations give rise to an adjoint pair of functors, called the bar and cobar construction.

$$\begin{tikzcd} Alg^{\bullet}_{\mathbb{K},+} \arrow[r, bend left, "B"] \arrow[r, phantom, "\top"] & Coalg^{\bullet}_{\mathbb{K},conil} \arrow[l, bend left, "\Omega"] \end{tikzcd}$$

The bar and cobar construction will be the basis for our discussion of A_∞-algebras. As the bar construction can be used to define A_∞-algebras, we may easily dualize this to define A_∞-coalgebras in terms of the cobar construction. This chapter will follow the notations and progression presented in Loday and Vallete [3] to develop the theory for the bar-cobar adjunction.

1.1 Prelimaries

1.1.1 Algebras

This section is a review of associative algebras. We will define unital associative algebras and possibly non-unital associative algebras, which we will call algebras and non-unital algebras respectively. The collection of algebras together with homomorphisms between them form the category $Alg_{\mathbb{K}}$ of algebras. Other types of algebras such as augmented and tensor algebras will be defined as well.

Definition 1.1.1 (Algebra)**.** Let $\mathbb{K}$ be a field with unit 1. An algebra A over $\mathbb{K}$ is a vector space with structure morphisms called multiplication and unit,

$$(\nabla_A) : A \otimes_{\mathbb{K}} A \to A$$
$$\upsilon_A : \mathbb{K} \to A,$$

satisfying the associativity and identity laws.

$$\text{(associativity)} \quad (a\nabla_A b)\nabla_A c = a\nabla_A(b\nabla_A c)$$
$$\text{(unitality)} \quad \upsilon_A(1)\nabla_A a = a = a\nabla_A \upsilon_A(1)$$

Whenever A does not posess a unit morphism, we will call A a non-unital algebra. Only the associativity law must hold.

Definition 1.1.2 (Algebra homomorphisms)**.** Let A and B be algebras. Then $f : A \to B$ is an algebra homomorphism if

1. f is $\mathbb{K}$-linear
2. $f(ab) = f(a)f(b)$
3. $f \circ \upsilon_A = \upsilon_B$

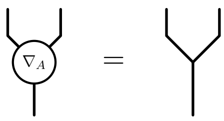
Whenever A and B are non-unital, we only require 1 and 2 for a homomorphism of non-unital algebras.

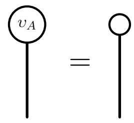
Definition 1.1.3 (Category of algebras)**.**
- Let $Alg_{\mathbb{K}}$ denote the category of algebras. It's objects consists of every algebra A, and the morphisms are algebra homomorphisms. The sets of morphisms between A and B are denoted as $Alg_{\mathbb{K}}(A, B)$.
- Let $\widehat{Alg}_{\mathbb{K}}$ denote the category of non-unital algebras. It's objects consists of every non-unital algebra A, and the morphisms are non-unital algebra homomorphisms. The sets of morphisms between A and B are denoted as $\widehat{Alg}_{\mathbb{K}}(A, B)$.

Observe that for an algebra A, the triple $(A, \nabla_A, \upsilon_A)$ is a monoid in $mod_{\mathbb{K}}$. Thus, we may say that an algebra is a triple where the following diagrams commute.

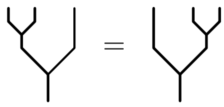
$$
\begin{tikzcd}
A \otimes_{\mathbb{K}} A \otimes_{\mathbb{K}} A \arrow[r, "(\nabla_A)\otimes id_{\mathbb{K}}"] \arrow[d, "id_{\mathbb{K}}\otimes(\nabla_A)"] & A \otimes_{\mathbb{K}} A \arrow[d, "(\nabla_A)"] \\
A \otimes_{\mathbb{K}} A \arrow[r, "(\nabla_A)"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes_{\mathbb{K}} \mathbb{K} \arrow[r, "id_A\otimes v_A"] \arrow[dr, "\simeq"'] & A \otimes_{\mathbb{K}} A \arrow[d, "(\nabla_A)"] & \mathbb{K} \otimes_{\mathbb{K}} A \arrow[l, "v_A\otimes id_A"'] \arrow[dl, "\simeq"] \\
& A &
\end{tikzcd}
$$

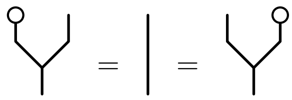
The final method we will use to represent an algebra are electric circuits. An electric circuit is a diagram read from top to bottom, where each column represent a different vector space in a tensor. Morphisms in such diagrams are figures, conjunctions, twistings and etc. E.g. The multiplication operator may be represented as a converging fork, and the unit as a source.

(Multiplication) 

(Unit) 

Using these operations we can now reformulate the algebra laws. These are the electric laws for an algebra:

(Associativity) 

(unitality) 

Definition 1.1.4 (Augmented algebras)**.** Let A be an algebra. It is called augmented if there is an algebra homomorphism $\varepsilon : A \to \mathbb{K}$.

If A is an augmented algebra, then it decomposes into $\mathbb{K} \oplus Ker\varepsilon$ as a module. The splitting is given by unitality of the morphism $\varepsilon : A \to \mathbb{K}$, as we know that $\varepsilon(v_A) = id_{\mathbb{K}}$. The kernel of ε is called the augmentation ideal or redecued algebra and we will denote it as $\bar{A}$. Taking kernels gives an equivalence of categories between augmented algebras and non-unital algebras, with unitization as the quasi-inverse. The category of augmented algebras is denoted as $Alg_{\mathbb{K},+}$ or $Alg_{\mathbb{K},+}$.

Definition 1.1.5 (Tensor algebra)**.** Let V be a $\mathbb{K}$-module. We define the tensor algebra $T(V)$ of V as the module

$$T(V) = \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus ...$$

Given two strings $v^1..v^i$ and $w^1...w^j$ in $T(V)$ we define the multiplication by the concatenation operation.

$$
\begin{aligned}
\nabla_{T(V)} : T(V) \otimes_{\mathbb{K}} T(V) &\to T(V) \\
(v^1...v^i) \otimes (w^1...w^j) &\mapsto v^1...v^i w^1...w^j
\end{aligned}
$$

The unit is given by including $\mathbb{K}$ into $T(V)$.

$$\begin{aligned} \upsilon_{T(V)} : \mathbb{K} &\to T(V) \\ 1 &\mapsto 1 \end{aligned}$$

Observe that the tensor algebra is augmented. The projection from $T(V)$ into $\mathbb{K}$ is an algebra homomorphism, so we may split the tensor algebra into its unit and its augmentation ideal $T(V) \simeq \mathbb{K} \oplus \bar{T}(V)$. We call $\bar{T}(V)$ the reduced tensor algebra.

Proposition 1.1.6 (Tensor algebra is free)**.** *The tensor algebra is the free algebra over the category of $\mathbb{K}$-modules, i.e. for any $\mathbb{K}$-module V there is a natural isomorphism $Hom_{\mathbb{K}}(V, A) \simeq Alg_{\mathbb{K}}(T(V), A)$.*

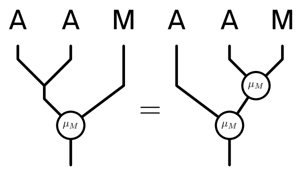
The reduced tensor algebra is the fre non-unital algebra over the category of $\mathbb{K}$-modules, i.e. for any $\mathbb{K}$-module V there is a natural isomorphism $Hom_{\mathbb{K}}(V, A) \simeq \widehat{Alg}_{\mathbb{K}}(\bar{T}(V), A)$.

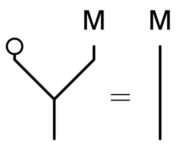
Proof. This proposition should be evident from the description of an algebra homomorphism from a tensor algebra. If $f : T(V) \to A$ is an algebra homomorphism, then f must satisfy the following conditions:

- Unitality: $f(1) = 1$
- Homomorphism property: Given $v, w \in V$, then $f(vw) = f(v)\nabla_A f(w)$

By induction, we see that f is completely determined by where it sends the elements of V. Thus restriction by the inclusion of V into $T(V)$ induces a bijection. □

Definition 1.1.7 (Modules)**.** Let A be an algebra. A $\mathbb{K}$-module M is said to be a left (right) A-module if there exists a structure morphism $\mu_M : A \otimes_{\mathbb{K}} M \to A$ ($\mu_M : M \otimes_{\mathbb{K}} A \to A$) called multiplication. We require that μ_M is associative with respect to the multiplication, and it preserves the unit of A. In other words, the electric laws below are satisfied.

(Associativity)


(unitality)


Definition 1.1.8 (A-linear homomorphisms)**.** Let M, N be two left A-modules. A morphism $f : M \to N$ is called A-linear if it is $\mathbb{K}$-linear and for any a in A, i.e. $f(am) = af(m)$.

The category of left A-modules is denoted as Mod_A, where the morphisms $Hom_A(\_,\_)$ are A-linear. Likewise, the category of right A-modules is denoted as Mod^A. One may check that there is a free functor from $Mod_{\mathbb{K}}$ to Mod_A.

Proposition 1.1.9. *Let M be a $\mathbb{K}$-module. The module $A \otimes_{\mathbb{K}} M$ is a left A-module. Moreover, it is the free left module over $\mathbb{K}$-modules, i.e. there is an isomorphism $Hom_{\mathbb{K}}(M,N) \simeq Hom_A(A \otimes_{\mathbb{K}} M, N)$.*

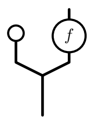
Proof. Define the isomorphism ϕ as

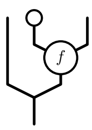
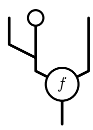
$$\begin{aligned} \phi : Hom_{\mathbb{K}}(M,N) &\to Hom_A(A \otimes_{\mathbb{K}} M, N) \\ \phi(f) &= \mu_N \circ (A \otimes f). \end{aligned}$$

Its inverse σ is then

$$\begin{aligned} \sigma : Hom_A(A \otimes_{\mathbb{K}} M, N) &\to Hom_{\mathbb{K}}(M,N) \\ \sigma(f) &= f \circ (\upsilon_A \otimes M). \end{aligned}$$

We check that this does indeed define an isomorphism by electric diagrams.

$\sigma \circ \phi(f) =$  $= f$

$\phi \circ \sigma(f) =$  $=$  $= f$

$\square$

1.1.2 Coalgebras

This section aims to dualize the definitions from last section. To this end we will define counital coassociative coalgebras and non-counital coassociative coalgebras, which will be called coalgebras and non-counital coalgebras respectively. The collection of coalgebras together with coalgebra homomorphisms is the category $CoAlg_{\mathbb{K}}$. Due to some ill-behavior, this dualization is only a true dualization under some finiteness conditions for the coalgebras. Thus we will see that the proper dual concept will be thath of conilpotent coalgebras.

Definition 1.1.10 (Coalgebra)**.** Let $\mathbb{K}$ be a field. A coalgebra C over $\mathbb{K}$ is a $\mathbb{K}$-module with structure morphisms called comultiplication and counit,

$$(\Delta_C) : C \to C \otimes_{\mathbb{K}} C$$
$$\varepsilon_C : C \to \mathbb{K},$$

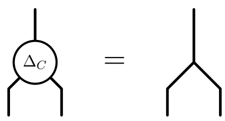
satisfying the coassociativity and coidentity laws.

$$\text{(coassociativity)} \quad (\Delta_C \otimes id_C) \circ \Delta_C(c) = (id_C \otimes \Delta_C) \circ \Delta_C(c)$$
$$\text{(counitality)} \quad (id_C \otimes \varepsilon_C) \circ \Delta_C(c) = c = (\varepsilon_C \otimes id_C) \circ \Delta_C(c)$$

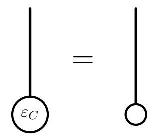
We define repeated application of comultiplication as $\Delta_C^n = (\Delta_C \otimes id_C \otimes ...) \circ \Delta_C^{n-1}$. Notice that the choice of where we put comultiplication in the tensor does not matter, as coassociativity require all of the choices to be equal.

We may dualize the electric circuits of an algebra to coalgebras. In this manner our structure morphisms would be upside down relative to the algebra morphisms. Thus comultiplication becomes a diverging fork and counit becomes a sink.

(Comultiplication)

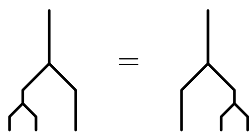


(Counit)

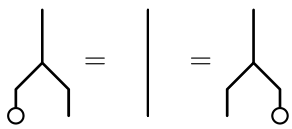


We then obtain the electric laws for a coalgebra by flipping the circuits around.

(Coassociativity)



(Counitality)



Definition 1.1.11 (Coalgebra homomorphism)**.** Let C and D be coalgebras. Then $f : C \to D$ is a coalgebra morphism if

1. f is $\mathbb{K}$-linear
2. $(f \otimes f) \circ \Delta_C(c) = \Delta_D(f(c))$
3. $\varepsilon_D(f) = \varepsilon_C$

Whenever C and D are non-counital, we only require 1 and 2 for a homomorphism of non-counital coalgebras.

Definition 1.1.12 (Category of Coalgebras)**.**
- Let $CoAlg_{\mathbb{K}}$ denote the category of coalgebras. It's objects consists of every coalgebra C, and the morphisms are coalgebra homomorphisms. The sets of morphisms between C and D are denoted as $CoAlg_{\mathbb{K}}(C, D)$.
- Let $\widehat{CoAlg}_{\mathbb{K}}$ denote the category of non-unital algebras. It's objects consists of every non-unital algebra C, and the morphisms are non-unital algebra homomorphisms. The sets of morphisms between C and D are denoted as $\widehat{CoAlg}_{\mathbb{K}}(C, D)$.

Example 1.1.13 (The coalgebra $\mathbb{K}$). The field $\mathbb{K}$ can be given a coalgebra structure over itself. Since $\{1\}$ is a basis for $\mathbb{K}$ we define the structure morphisms as

$$\Delta_{\mathbb{K}}(1) = 1 \otimes 1$$
$$\varepsilon(1) = 1.$$

One may check that these morphisms are indeed coassociative and counital. Thus we may regard our field as either an algebra or coalgebra over itself.

Definition 1.1.14 (Coaugmented coalgebras)**.** Let C be a coalgebra. C is coagumented if there is a coalgebra homomorphism $\upsilon : \mathbb{K} \to C$.

If C is a coaugmented coalgebra, then it splits as $C \simeq \mathbb{K} \oplus Cok\upsilon$. The splitting is given by counitality of υ, as $\varepsilon_C(\upsilon) = id_{\mathbb{K}}$. We call the cokernel $Cok\upsilon = \bar{C}$ for the coaugmentation quotient or reduced coalgebra, and its reduced coproduct may be explicitly given as

$$\bar{\Delta}_C(c) = \Delta_C(c) - 1 \otimes c - c \otimes 1.$$

Definition 1.1.15 (Tensor Coalgebras)**.** Let V be a $\mathbb{K}$-module. We define the tensor coalgebra $T^c(V)$ of V as the module

$$T^c(V) = \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus ...$$

Given a string $v^1...v^i$ in $T(V)$ we define the comultiplication by the deconcatenation operation.

$$\begin{aligned} \Delta_{T^c(V)} : T^c(V) &\to T^c(V) \otimes_{\mathbb{K}} T^c(V) \\ v^1...v^i &\mapsto 1 \otimes (v^1...v^i) + (\sum_{j=1}^{n-1} (v^1...v^j) \otimes (v^{j+1}...v^i)) + (v^1...v^i) \otimes 1 \end{aligned}$$

The counit is given by projecting $T^c(V)$ onto $\mathbb{K}$.

$$\begin{aligned} \varepsilon_{T^c(V)} : T^c(V) &\to \mathbb{K} \\ 1 &\mapsto 1 \\ v^1...v^i &\mapsto 0 \end{aligned}$$

Notice that the tensor coalgebra is coaugmented. Its coaugmentation is given by the inclusion of $\mathbb{K}$ into $T^c(V)$. We may split $T^c(V) \simeq \mathbb{K} \oplus \bar{T}^c(V)$, where $\bar{T}^c(V)$ is the reduced tensor coalgebra.

In order to get cofreeness for the tensor coalgebra we need some finiteness conditions. This is one of the properties which is ill-behaved when we are dualizing the tensor algebra. The extra assumption which we will need is to assume that the coalgebras are conilpotent. Let $C \simeq \mathbb{K} \oplus \bar{C}$ be a coaugmented coalgebra, we define the coradical filtration of C as a filtration $Fr_0C \subseteq Fr_1C \subseteq ... \subseteq Fr_rC \subseteq ...$ by the submodules:

$$\begin{aligned} &Fr_0C = \mathbb{K} \\ &Fr_rC = \mathbb{K} \oplus \{c \in \bar{C} \mid \forall n \geqslant r\bar{\Delta}_C(c) = 0\}. \end{aligned}$$

Definition 1.1.16 (Conilpotent coalgebras)**.** Let C be a coaugmented coalgebra. We say that C is conilpotent if its coradical filtration is exhaustive, i.e. $\varinjlim_r Fr_rC \simeq C$. The subcategory of conilpotent coalgebras will be denoted as $CoAlg_{\mathbb{K},conil}$.

Proposition 1.1.17 (Conilpotent tensor coalgebra)**.** *Let V be a $\mathbb{K}$-module. The tensor coalgebra $T^c(V)$ is conilpotent.*

Proof. Let $v \in V$, then $\Delta_{T^c(V)}(v) = 1 \otimes v + v \otimes 1$ and $\bar{\Delta}_{T^c(V)}(v) = 0$. We then observe the following:

$$\begin{aligned} Fr_0T^c(V) &= \mathbb{K} \\ Fr_1T^c(V) &= \mathbb{K} \oplus V \\ Fr_rT^c(V) &= \bigoplus_{i \leqslant r} V^{\otimes i} \end{aligned}$$

This shows that the coradical filtration is exhaustive. □

Proposition 1.1.18 (Cofree tensor coalgebra)**.** *The tensor coalgebra is the cofree conilpotent coalgebra over the category of $\mathbb{K}$-modules, i.e. for any $\mathbb{K}$-module V and any conilpotent coalgebra C there is a natural isomorphism $Hom_{\mathbb{K}}(\bar{C}, V) \simeq CoAlg_{\mathbb{K},conil}(C, T^c(V))$.*

Proof. This proposition should be evident from the description of a coalgebra homomorphism into the a tensor coalgebra. If $g : C \to T^c(V)$ is a coalgebra homomorphism, then g must satisfy the following conditions:

1. (Coaugmentation) $g(1) = 1$
2. (Counitality) Given $c \in \bar{C}$ then $\varepsilon_{T^c(V)} \circ g(c) = 0$
3. (Homomorphism property) Given $c \in C$ then $\Delta_{T^c(V)}(g(c)) = (g \otimes g) \circ \Delta_C(c)$

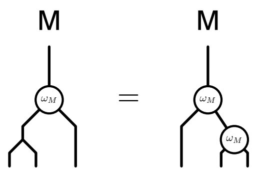
We will construct the maps for the isomorphism explicitly. If $g : C \to T^c(V)$ is a coalgebra homomorphism, then composing with projection gives a map $\pi \circ g : C \to V$. Note that $\pi \circ g(1) = 0$, so this is essentially a map $\pi \circ g : \bar{C} \to V$. For the other direction, let $\bar{g} : \bar{C} \to V$. We will then define g as

$$g = id_{\mathbb{K}} \oplus \sum_{i=1}^{\infty} (\otimes^i \bar{g}) \bar{\Delta}_C^{i-1}.$$

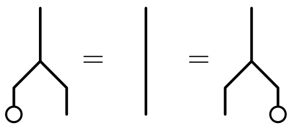
Observe that g is well defined, since convergence of the sum follows from conilpotency of C. One may then check that g is a coalgebra homomorphism, which yields the result. □

Definition 1.1.19 (Comodules)**.** Let C be a coalgebra. A $\mathbb{K}$-module M is said to ba left (right) C-comodule if there exist a structure morphism $\omega_M : M \to C \otimes_{\mathbb{K}} M$ ($\omega_M : M \to M \otimes_{\mathbb{K}} C$) called comultiplication. We require that ω_M is coassociative with respect to the comultiplication of C and preserves the counit of C, i.e. the electric laws are satisfied.

(Coassociativity)



(Counitality)



Definition 1.1.20 (C-colinear homomorphism)**.** Let M, N be two left C-comodules. A morphism $g : M \to N$ is called C-colinear if it is $\mathbb{K}$-linear and for any m in M, $\omega_N(g(m)) = (id_C \otimes g)\omega_M(m)$.

The category of left C-comodules is denoted as $CoMod_C$, where the morphisms $Hom_C(\_,\_)$ are C-colinear. We would also like to restrict our attention to those C-comodules which are conilpotent, i.e. those comodules which have an exhaustive coradical filtration. Notice that for conilpotent coalgebras this requirement is automatic. Likewise, the category of right C-comodules is denoted as $CoMod^C$.

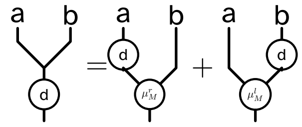
Proposition 1.1.21. *Let M be a $\mathbb{K}$-module. The module $C \otimes_{\mathbb{K}} M$ is a left C-comodule. Moreover, it is the cofree left comodule over $\mathbb{K}$-modules, i.e. there is an isomorphism $Hom_{\mathbb{K}}(N, M) \simeq Hom_C(N, C \otimes_{\mathbb{K}} M)$.*

Proof. This is dual to 1.1.9. □

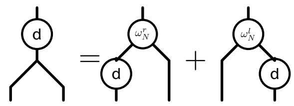
1.1.3 Derivations and DG-Algebras

In this section we will look at differential graded objects and convolution products. We will define derivations and coderivations to obtain differential graded algebras and coalgebras. Moreover we will see that the set of homogenous homomorphisms between differential graded objects is itself differential graded. Moreover, whenever we look at morphisms between dg coalgebras and dg algebras, we can give this object the convolution operator, making the set a dg algebra.

Definition 1.1.22 (Derivations and Coderivations)**.** Let M be an A-bimodule. A $\mathbb{K}$-linear morphism $d : A \to M$ is called a derivation if $d(ab) = d(a)b + ad(b)$, i.e. electrically:



Let N be a C-bicomodule. A $\mathbb{K}$-linear morphism $d : N \to C$ is called a coderivation if $\Delta_C \circ d = (d \otimes id_C) \circ \omega_N^r + (id_C \otimes d) \circ \omega_N^l$, i.e. electrically:



Proposition 1.1.23. *Let V be a $\mathbb{K}$-module and M be a $T(V)$-bimodule. A $\mathbb{K}$-linear morphism $f : V \to M$ uniquely determines a derivation $d_f : T(V) \to M$, i.e. there is an isomorphism $Hom_{\mathbb{K}}(V, M) \simeq Der(T(V), M)$.*

Let N be a $T^c(V)$-cobimodule. A $\mathbb{K}$-linear morphism $g : M \to V$ uniquely determines a coderivation $d_g^c : N \to T^c(V)$, i.e. there is an isomorphism $Hom_{\mathbb{K}}(N, V) \simeq Coder(N, T^c(V))$.

Proof. Let $a_1 \otimes ... \otimes a_n$ be an elementary tensor of $T(V)$. We define $d_f(a_1 \otimes ... \otimes a_n) = \sum_{i=1}^n a_1 ... f(a_i) ... a_n$ and $d_f(1) = 0$. Notice that d_f is by definition a derivation.

Restriction to V gives the natural isomorphism. Let $i : V \to T(V)$, then $i^* d_f = f$. Let $d : T(V) \to M$ be a derivation, then $d_{i^*d} = d$. Suppose that $g : M \to N$ is a morphism between $T(V)$-bimodules, then naturality follows from bilinearity.

In the dual case, d_g^c is a bit tricky to define. Let $\omega_N^l : N \to N \otimes T^c(V)$ and $\omega_N^r : N \to T^c(V) \otimes N$ denote the coactions on N. Since $T^c(V)$ is conilpotent we get the same kind of finiteness restrictions on N. We define the reduced coactions as $\bar{\omega}_N^l = \omega_N^l - \_ \otimes 1$ and $\bar{\omega}_N^r = \omega_N^r - 1 \otimes \_$, this is well-defined by coassociativity. Observe that for any $n \in N$ there are $k, k' > 0$ such that $\bar{\omega}_N^{l^k}(n) = 0$ and $\bar{\omega}_N^{r^{k'}}(n) = 0$.

Let $n_{(k)}^{(i)}$ denote the extension of n by k coactions at position i, i.e. $n_{(k)}^{(i)} = \bar{\omega}_N^{r^i} \bar{\omega}_N^{l^{k-i}}(n)$. The extension of n by k coactions is then the sum over every position i, $n_{(k)} = \sum_{i=0}^k n_{(k)}^{(i)}$. Observe that $n_{(0)} = n$. The grade of n may be thought of as the smallest k such that $n_{(k)}$ is zero. This grading gives us the coradical filtration of N, and it is exhaustive by the finiteness restrictions given above. So every element of N may be given a finite grade.

If $g : N \to V$ is a linear map, we may think of it as a map sending every element of N to an element of $T^c(V)$ of grade 1. To get a map which sends element of grade k to grade k, we must extend the morphism. Let $\pi : T^c(V) \to V$ be the linear projection and define $g_{(k)}^{(i)} = \pi \otimes ... \otimes \pi \circ g \otimes \pi$ as a morphism which is g at the i-th argument, but the projection otherwise.

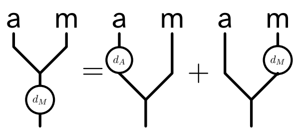
d_g^c is then defined as the sum over each coaction and coordinate.

$$d_g^c(n) = \sum_{k=0}^{\infty} \sum_{i=0}^{k} g_{(k)}^{(i)}(n_{(k)}^{(i)})$$

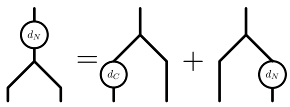
Upon closer inspection we may observe that this is the dual construction of the derivation morphism. It is well-defined as the sum is finite by the finiteness restrictions. The map is a coderivation by duality, and the natural isomorphism is given by composition with the projection map π. ☐

Definition 1.1.24 (Differential algebra)**.** Let A be an algebra. We say that A is a differential algebra if it is equipped with at least one derivation $d : A \to A$. Dually, a coalgebra C is called differential if it is equipped with at least one coderivation $d : C \to C$.

Definition 1.1.25 (A-derivation)**.** Let (A, d_A) be a differential algebra and M a left A-module. A $\mathbb{K}$-linear morphism $d_M : M \to M$ is called an A-derivation if $d_M(am) = d_A(a)m + ad_M(m)$, or electrically:



Dually, given a differential coalgebra (C, d_C) and N a left C-comodule, a $\mathbb{K}$-linear morphism $d_N : N \to N$ is a coderivation if $\omega_N \circ d_N = (d_C \otimes id_N + id_C \otimes d_N) \circ \omega_N$, or electrically:



Proposition 1.1.26. *Let A be a differential algebra and M a $\mathbb{K}$-module. A $\mathbb{K}$-linear morphism $f : M \to A \otimes_{\mathbb{K}} M$ uniquely determines a derivation $d_f : A \otimes M \to A \otimes M$, i.e. there is an isomorphism $Hom_{\mathbb{K}}(M, A \otimes_{\mathbb{K}} M) \simeq Der(A \otimes_{\mathbb{K}} M)$. Moreover, d_f is given as $(\nabla_A \otimes id_M) \circ (id_A \otimes f) + d_A \otimes id_M$.*

Dually, if C is a differential coalgebra and N is a $\mathbb{K}$-module, then a $\mathbb{K}$-linear morphism $g : C \otimes N \to N$ uniquely determines a coderivation $d_g : C \otimes_{\mathbb{K}} N \to C \otimes_{\mathbb{K}} N$. There is an isomorphism $Hom_{\mathbb{K}}(C \otimes_{\mathbb{K}} N, N) \simeq Coder(C \otimes_{\mathbb{K}} N)$, and d_g is given as $(id_C \otimes g) \circ (\Delta_C \otimes id_N) + d_C \otimes id_N$.

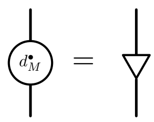
Proof. We must check that the given d in facts defines an isomorphism. This is checked by calculation. ☐

Recall that a module M^* is $\mathbb{Z}$ graded if it decomposes as a sum $M^* = \bigoplus_{z:\mathbb{Z}} M^z$. Let M^*, N^* be graded modules and $f : M^* \to N^*$ is a homogenous $\mathbb{K}$-linear morphism of degree n if it preserves the grading, that is $f(M^i) \subseteq N^{n+i}$. We denote the degree of f as $|f|$. The category of graded modules will be denoted as $GrMod_{\mathbb{K}}$ or $Mod^*_{\mathbb{K}}$. Generally $\mathcal{C}^*$ is the category of graded objects whenever it makes sense, and the graded $\mathbb{K}$-module of morphisms between two graded objects is denoted as $Hom^*_{\mathbb{K}}(M^*, N^*)$.

$M^\bullet$ is called a cochain complex if it comes equipped with a homogenous morphism of degree 1, like $d^\bullet_M : M^\bullet \to M^\bullet$, such that ${d^\bullet_M}^2 = 0$. This morphism is called differential. A chain morphism $f : M^\bullet \to N^\bullet$ is a homogenous $\mathbb{K}$-linear morphism of degree 0, such that $f \circ d^\bullet_M = d^\bullet_N \circ f$. The category of chain complexes will be denoted as $ChMod_{\mathbb{K}}$ or $Mod^\bullet_{\mathbb{K}}$. Generally $\mathcal{C}^\bullet$ is the category of chain complexes whenever it makes sense, and the $\mathbb{K}$-module of morphisms between two chain complexes is denoted as $Hom^\bullet_{\mathbb{K}}(M^\bullet, N^\bullet)$.

The functor $\_[n] : Mod^\bullet_{\mathbb{K}} \to Mod^\bullet_{\mathbb{K}}$ shifts the degree on each object by adding n to each grade, it is called the shift functor. Let $\otimes$ denote the total tensor product in $Mod^\bullet_{\mathbb{K}}$. There is an isomorphism between the identity shift functor and total tensor of the stalk of $\mathbb{K}$, $\_[0] \simeq \bar{\mathbb{K}} \otimes \_$. In the same manner, shifting n-fold becomes isomorphic to tensoring with the shifted stalk of $\mathbb{K}$, $\_[n] \simeq \bar{\mathbb{K}}[n] \otimes \_$. For our purposes we will let $(A^\bullet, d^\bullet_A)[n] = (A^{\bullet+n}, -d^{\bullet+n}_A)$. The koszul sign rule gives us a switching map for the tensor product. Thus, if $f^* : A^\bullet \to B^\bullet$ is a morphism of degree k, then $f^*[n] = (-1)^{k \cdot n} f^{*+n}$.

In electric diagrams we will write triangles for the differential if there are no ambiguity.



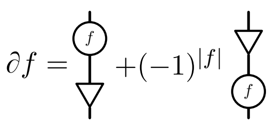
Proposition 1.1.27. *Let $M^\bullet$ and $N^\bullet$ be two chain complexes. The graded module of morphisms $Hom^*_{\mathbb{K}}(M^\bullet, N^\bullet)$ is a chain complex, given by the differential $\partial(f) = d^\bullet_N \circ f - (-1)^{|f|} f \circ d^\bullet_M$.*

Proof. We observe that $\partial : Hom^*_{\mathbb{K}}(M^\bullet, N^\bullet) \to Hom^*_{\mathbb{K}}(M^\bullet, N^\bullet)$ is a morphism of degree 1. It remains to check that $\partial^2 = 0$. Pick any homogenous morphism $f : M^\bullet \to N^\bullet$.

$$\begin{aligned} \partial^2(f) = \partial(d^\bullet_N \circ f - (-1)^{|f|} f \circ d^\bullet_M) &= \partial(d^\bullet_N \circ f) - (-1)^{|f|} \partial(f \circ d^\bullet_M) \\ &= -(-1)^{|d^\bullet_N \circ f|} d^\bullet_N \circ f \circ d^\bullet_M - (-1)^{|f|} d^\bullet_N \circ f \circ d^\bullet_M = 0 \end{aligned}$$

□

In an electric diagram we write ∂f as a sum of circuits.



Observe that $f : M^\bullet \to N^\bullet$ of degree 0 is a chain morphism if and only if $\partial(f) = 0$. We then observe that $Hom^\bullet_{\mathbb{K}}(M^\bullet, N^\bullet) \simeq Z^0 Hom^*_{\mathbb{K}}(M^\bullet)$. Moreover, if f is a boundary, i.e. there is some h such that $f = \partial h$, then f null-homotopic. Thus the 0'th homology $H^0 Hom^*_{\mathbb{K}}(M^\bullet, N^\bullet)$ is the set of every chain map which is not null-homotopic up to null homotopy. We obtain this lemma immediate from this discussion.

Lemma 1.1.28. *Suppose that $N^\bullet$ is a contractible chain complex. Then for any chain complex $M^\bullet$, the 0'th homology of $Hom^*_{\mathbb{K}}(M^\bullet, N^\bullet)$ is $H^0 Hom^*_{\mathbb{K}}(M^\bullet, N^\bullet) \simeq 0$*

To complete the definitions of graded modules and chain complexes to algebras we would like the structure morphisms to respect the given structure. E.g. if a and b are homogenous elements, we would like that the degree of ab is the sum of its parts, i.e. $|ab| = |a| + |b|$. Since multiplication by identity doesn't do anything, we want that the identity lives in the 0'th degree, and so forth.

Definition 1.1.29 (Graded algebra)**.** Let A^* be a graded $\mathbb{K}$-module. We say that A^* is a graded algebra if A^* is an algebra such that ∇_A and υ_A are homogenous and of degree 0. Dually, C^* is a graded coalgebra if Δ_C and ε_C are homogenous and of degree 0.

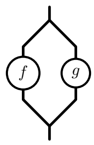
Definition 1.1.30 (Differential graded algebra)**.** Let $A^\bullet$ be a chain complex over $\mathbb{K}$. We say that $A^\bullet$ is a differential graded algebra, or dg algebra, if it is a graded algebra and the differential is a graded derivation, i.e. $d_A(ab) = d_A(a)b + (-1)^{|a|} a d_A(b)$.

Dually, $C^\bullet$ is a differential graded coalgebra if $C^\bullet$ is a graded coalgebra and the differential is a graded coderivation.

1.2 Cobar-Bar Adjunction

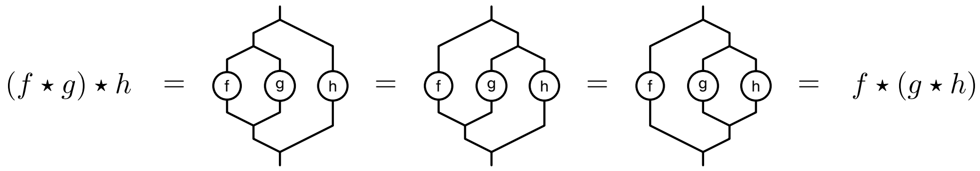
1.2.1 Convolution Algebras

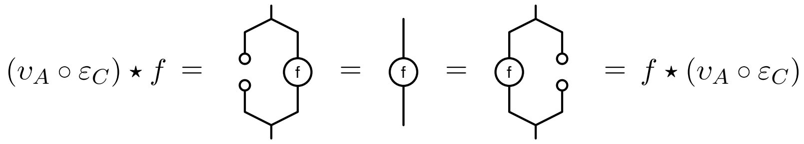
Let C be a coalgebra and A an algebra, then if $f, g : C \to A$ are $\mathbb{K}$-linear morphism we may define $f \star g = \nabla_A (f \otimes g) \Delta_C$. We call the operation $\star$ for convolution.

$f \star g =$ 

Proposition 1.2.1 (Convolution algebra)**.** *The $\mathbb{K}$-module $Hom_{\mathbb{K}}(C, A)$ is an associative algebra when equipped with convolution $\star : Hom_{\mathbb{K}}(C, A) \to Hom_{\mathbb{K}}(C, A)$. The unit is given by $1 \mapsto \upsilon_A \circ \varepsilon_C$.*

Proof. This proposition follows from (co)associativity and (co)unitality of (C) A.





□

If A is an algebra and C is a coalgebra, then they may be given the structure of a differential algebra by attaching the 0 morphism to each algebra as the (co)derivation. In this case proposition 1.1.26 says that a morphism $f : M \to A \otimes_{\mathbb{K}} M$ determines the derivation given as $d_f = (\nabla_A \otimes id_M) \circ (id_A \otimes f)$. Dually, a morphism $g : C \otimes_{\mathbb{K}} M \to M$ determines the coderivation $d_g = (id_C \otimes g) \circ (\Delta_C \otimes id_N)$.

If $\alpha : C \to A$ is a $\mathbb{K}$-linear morphism, then there are two ways to extend α to obtain a (co)derivation. Precomposing with Cs comultiplication gives us a morphism from C to the free A-module $A \otimes_{\mathbb{K}} C$.

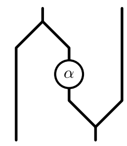
$$(\alpha \otimes id_C) \circ \Delta_C : C \to A \otimes_{\mathbb{K}} C$$

Postcomposing with the multiplication of A gives us a morphism from to the cofree C-comodule $C \otimes_{\mathbb{K}} A$ to A.

$$\nabla_A \circ (\alpha \otimes id_A) : C \otimes_{\mathbb{K}} A \to A$$

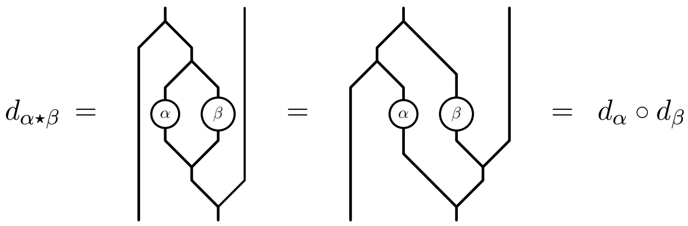
Notice that when applying proposition 1.1.26 to both morphisms yields the same map, and it is thus both a derivation and a coderivation.

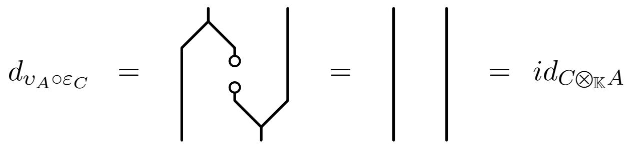
$$d_\alpha = (\nabla_A \otimes id_C) \circ (id_A \otimes \alpha \otimes id_C) \circ (id_A \otimes \Delta_C)$$

$d_\alpha \quad =$ 

Proposition 1.2.2. *$d_{(\_)} : Hom_{\mathbb{K}}(C, A) \to End(C \otimes_{\mathbb{K}} A)$ is a morphism of algebras. Moreover, if $\alpha \star \alpha = 0$, then $d_\alpha^2 = 0$.*

Proof. The proof quickly follows from (co)associativity and (co)unitality.





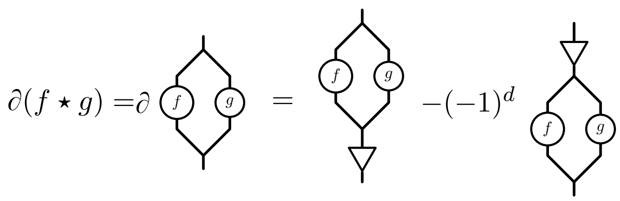
□

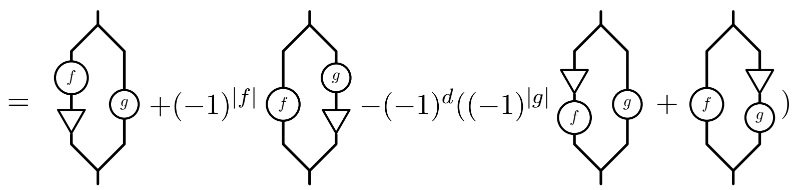
Suppose that C and A are differential graded (co)algebras. We want to expect that the differential ∂ makes $Hom^*_{\mathbb{K}}(C, A)$ into a dg-algebra.

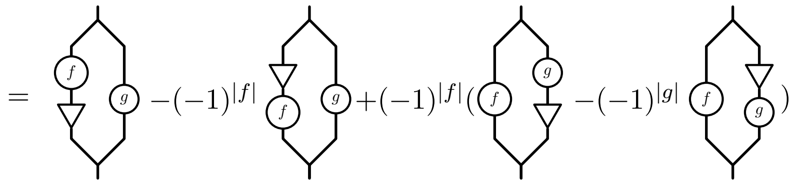
Proposition 1.2.3. *The convolution algebra* $(Hom^*_{\mathbb{K}}(C, A), \star)$ *is a dg-algebra with differential* ∂.

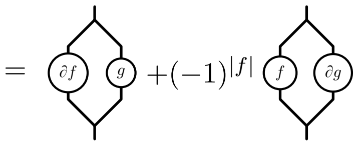
Proof. We know that $(Hom^*_{\mathbb{K}}(C, A), \star)$ is a convolution algebra and that $(Hom^*_{\mathbb{K}}(C, A), \partial)$ is a chain complex. It remains to verify that the differential is compatible with the multiplication, i.e. $\partial(f \star g) = \partial f \star g + (-1)^{|f|} f \star \partial g$.

Let $f, g \in Hom^*_{\mathbb{K}}(C, A)$ be two homogenous morphisms. The key property to arrive at the result is that the differential in a dg-(co)algebra is a (co)derivation. We denote the degree of $f \star g$ as $|f \star g| = |f| + |g| = d$









$$= \partial(f) \star g + (-1)^{|f|} f \star \partial(g)$$

□

1.2.2 Twisting Morphisms

In this section we will define twisting morphisms from coalgebras to algebras. They are of importance as the bifunctor $Tw(C, A)$ is represented in both arguments. To understand the elements of Tw we start this section be reviewing the Maurer-Cartan equation.

Suppose that C is a dg-coalgebra and A is a dg-algebra. We say that a morphism $\alpha \in Hom^*_{\mathbb{K}}(C, A)$ is twisting if it is of degree 1 and satisfies the Maurer-Cartan equation:

$$\partial\alpha + \alpha \star \alpha = 0.$$

We say that α is an element of $Tw(C, A) \subset Hom^1_{\mathbb{K}}(C, A) \subset Hom^*_{\mathbb{K}}(C, A)$. In light of proposition 1.2.2, every morphism between coalgebras and algebras extend to a unique (co)derivation on the tensor product $C \otimes_{\mathbb{K}} A$. Let d^r_α denote this unique morphism. In the case of dg-coalgebras and dg-algebras we perturbate the total differential on the tensor with d^r_α, as in proposition 1.1.26. We call this derivation for the perturbated derivative.

$$d^\bullet_\alpha = d^\bullet_{C\otimes_{\mathbb{K}}A} + d^r_\alpha = d^\bullet_C \otimes id_A + id_C \otimes d^\bullet_A + d^r_\alpha$$

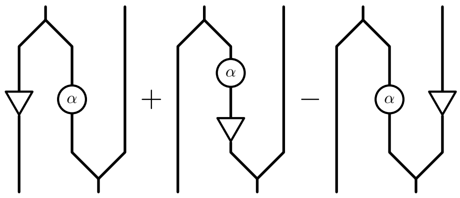
Proposition 1.2.4. *Suppose that C is a dg-coalgebra and A is a dg-algebra, and $\alpha \in Hom^{-1}_{\mathbb{K}}(C, A)$. The perturbated derivation satisfies the following relation.*

$${d^\bullet_\alpha}^2 = d^r_{\partial\alpha+\alpha\star\alpha}$$

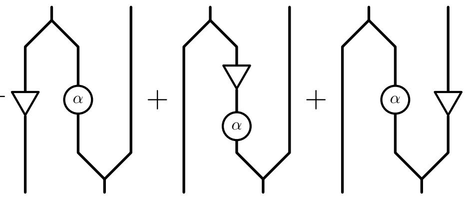
Moreover, a morphism is twisting if and only if the perturbated derivative is a differential.

Proof. ${d^\bullet_\alpha}^2 = d^\bullet_{C\otimes_{\mathbb{K}}A} \circ d^r_\alpha + d^r_\alpha \circ d^\bullet_{C\otimes_{\mathbb{K}}A} + {d^r_\alpha}^2$. By proposition 1.2.2 $d^r_?$ is an algebra homomorphism from the convolution algebra to the endomorphism algebra, thus ${d^r_\alpha}^2 = d^r_{\alpha\star\alpha}$.

$$d^{\bullet}_{C\otimes_{\mathbb{K}}A} \circ d^{r}_{\alpha} =$$



$$d^{r}_{\alpha} \circ d^{\bullet}_{C\otimes_{\mathbb{K}}A} = -$$



By summing the above terms we get

$$d^{\bullet}_{C\otimes_{\mathbb{K}}A} \circ d^{r}_{\alpha} + d^{r}_{\alpha} \circ d^{\bullet}_{C\otimes_{\mathbb{K}}A} = d^{r}_{d^{\bullet}_{A}\circ\alpha+\alpha\circ d^{\bullet}_{C}} = d^{r}_{\partial\alpha},$$

to obtain the result.

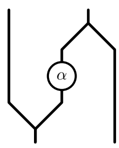
$${d^{\bullet}_{\alpha}}^{2} = d^{\bullet}_{C\otimes_{\mathbb{K}}A} \circ d^{r}_{\alpha} + d^{r}_{\alpha} \circ d^{\bullet}_{C\otimes_{\mathbb{K}_{\precsim\approx}}A} + d{\alpha^{r}}^{2} = d^{r}_{\partial\alpha} + d^{r}_{\alpha\star\alpha} = d_{\partial\alpha+\alpha\star\alpha}$$

□

Corollary 1.2.4.1. *If $\alpha : C \to A$ is a twisting morphism, then $(C \otimes_{\mathbb{K}} A, d^{\bullet}_{\alpha})$ is a chain complex. It is called the right twisted tensor product and is denoted as $C \otimes_{\alpha} A$.*

Normally $A \otimes C$ and $C \otimes A$ are isomorphic as modules. In general, it is not true that $C \otimes_{\alpha} A$ and $A \otimes_{\alpha} C$ are isomorphic, since we choose a particular side to perform the twisting. However, if A is commutative and C is cocommutative then they are isomorphic. To illustrate we realize the unique derivation above as a right derivative. The left derivative d^{l}_{α} is then defined analogously.

$$d^{l}_{\alpha} =$$



Remark 1.2.5. Functoriality of $\otimes_{\alpha}$ is obtained from the category of elements. I propose that there is an equivalence of categories, that is:

$$\int_{(C,A)} Tw(C, A) \simeq \text{right twisted tensors.}$$

1.2.3 Bar and Cobar Construction

The bar construction was first formalized for augmented skew-commutative dg-rings by Eilenberg and Mac Lane [4]. The bar construction then served as a tool to calculate the homology of the Eilenberg-Mac Lane spaces. This construction was later dualized by Adams [5] to obtain the cobar construction. It's first purpose was to serve as a method for constructing an injective resolution in order to calculate the cotor resolution [6]. With time, the bar-cobar construction have been subjected to much generalization, such as a fattened tensor product on simplicially enriched, tensored and cotensored categories [7]. We will mainly follow the work of [3] to obtain the bar and cobar construction. The approach which we are going to take is also slightly inspired by MacLanes[8] canonical resolutions of comonads.

For our purposes, the bar construction of an augmented algebra is a simplicial resoulution with the cofree coalgebra structure. For a dg-algebra, we will realize this resoultion as the total complex of its resoultion. Dually, the cobar construction of a conilpotent coalgebra is a cosimplicial resolution with the free algebra structure. We will see that these constructions defines an adjoint pair of functors.

Definition 1.2.6. The simplex category Δ consists of ordered sets $[n] = \{0, ..., n\}$ for any $n \in \mathbb{N}$. A morphism in Δ is a monotone function between the sets.

Δ_+ is the augmented simplex category, where we add the object $[-1] = \varnothing$. Δ^+ is the non-full subcategory of Δ, where all morphisms are injective functions.

The simplex category comes equipped with coface and codegeneracy morphisms. The coface maps are the injective morphisms $\delta_i : [n] \to [n+1]$, and the codegeneracy maps are the surjective morphisms $\sigma_i : [n] \to [n-1]$.

$$\delta_i(k) = \begin{cases} k, \text{ if } k<i \\ k+1, \text{ otherwise} \end{cases} \qquad \sigma_i(k) = \begin{cases} k, \text{ if } k\leqslant i \\ k-1, \text{ otherwise} \end{cases}$$

Every morphism in Δ may be realized as a composition of coface and codegeneracy maps, see [8]. Furthermore, these maps are characterized by some identites, called the cosimplicial identites.

1. $\delta_j\delta_i = \delta_i\delta_{j-1}$, if $i < j$
2. $\sigma_j\delta_i = \delta_i\sigma_{j-1}$, if $i < j$
3. $\sigma_j\delta_i = id$, if $i = j$ or $i = j+1$
4. $\sigma_j\delta_i = \delta_{i-1}\sigma_j$, if $i > j+1$
5. $\sigma_j\sigma_i = \sigma_i\sigma_{j+1}$, if $i \leqslant j$

We may arrange the arrows of the augmented simplex category in the following way:

$$\begin{tikzcd} {[-1]} \arrow[r] & {[0]} \arrow[r, shift left, "\delta_i"] \arrow[r, shift right] & {[1]} \arrow[r, shift left=2, "\delta_i"] \arrow[r] \arrow[r, shift right=2] & {[2]} \arrow[r, shift left=3, "\delta_i"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \cdots \end{tikzcd}$$

$$[-1] \qquad [0] \xleftarrow{\sigma_1} [1] \xleftarrow{\sigma_i} [2] \xleftarrow{\sigma_i} \dots$$

Let $\mathcal{C}$ be a category. A simplicial object in $\mathcal{C}$ is a functor $S : \Delta^{op} \to \mathcal{C}$. It may be viewed as a collection of objects $\{S_n\}_{n\in\mathbb{N}}$ together with face maps $d^i : S_n \to S_{n-1}$ and degeneracy maps $s^i : S_n \to S_{n+1}$ satisfying the simplicial identities. An augmented simplicial object is a functor $S : \Delta_+^{op} \to \mathcal{C}$. The restricted functor $\bar{S} : \Delta^{op} \to \mathcal{C}$ is the augmentation ideal of S. An augmented semi-simplicial object is a functor $S : (\Delta_+^+)^{op} \to \mathcal{C}$. Dually, a cosimplicial object is a functor $S : \Delta \to \mathcal{C}$, it may be regarded as a sequence of objects with coface and codegeneracy maps satisfying the cosimplicial identities.

Let $\mathcal{A}$ be an abelian category. To each semi-simplical object $M : (\Delta^+)^{op} \to \mathcal{A}$ there is an associated chain complex $M^\bullet$. Let $M^{-i} = M_i$ with differential $d_M^{-n} = \sum_{i=1}^{n+1}(-1)^{i-1}d^i$. This differential is well-defined by simplicial identity 1.

$$\dots \longrightarrow M_2 \xrightarrow{d^1-d^2+d^3} M_1 \xrightarrow{d^1-d^2} M_0 \xrightarrow{0} 0 \longrightarrow \dots$$

As face maps and degeneracy maps have the same identites, but flipped around, we could also have defined a chain complex by using the degeneracies instead.

The augmented simplex category has a universal monoid. Let $+ : \Delta_+ \times \Delta_+ \to \Delta_+$ be a functor acting on objects and morphisms as:

$$[m] + [n] = [m+n+1]$$
$$(f+g)(k) = \begin{cases} f(k), \text{ if } k\leqslant m \\ g(k)+m, \text{ otherwise} \end{cases}$$

Notice that $[-1] + \_ \simeq Id_\Delta$, so $(\Delta, +, [-1])$ is a monoidal category. Since $[0]$ is terminal in Δ it becomes a monoid with $\delta_{-1} : [-1] \to [0]$ as unit and $\sigma_0 : [1] \to [0]$ as multiplication. Associativity and unitality is satisfied by uniqueness of morphisms $f : [n] \to [0]$.

Proposition 1.2.7. *Let $(\mathcal{C}, \otimes, Z)$ be a monoidal category. If (C, η, μ) is a monoid in $\mathcal{C}$, then there is a strong monoidal functor $: \Delta_+ \to \mathcal{C}$, such that $F[0] \simeq C$, $F\delta_{-1} \simeq \eta$ and $F\sigma_0 \simeq \mu$.*

Proof. This is proved in Mac Lanes book [8]. □

An algebra A is a monoid in the monoidal category $(Mod_{\mathbb{K}}, \otimes_{\mathbb{K}}, \mathbb{K})$. By proposition 1.2.7 we may think of A as an augmented cosimplicial object $A : \Delta_+ \to Mod_{\mathbb{K}}$. Notice that all of the cosimplical identities follow from associativity and unitality. If A is an augmented algebra, we may instead give it the structure of an augmented simplicial set. Let $d_0^0 = \varepsilon_A$ be the augmentation. We define $d_n^n = A^{\otimes n-1} \otimes \varepsilon_A$ and set $d_n^i = A^{i-1} \otimes \nabla_A \otimes A^{\otimes n-i-1}$. All the degeneracies are set to be the units, i.e. $s_n^i = A^{\otimes i} \otimes \upsilon_A \otimes A^{\otimes n-i-1}$. One may check that this structure defines an augmented simplical object $A : \Delta_+^{op} \to Mod_{\mathbb{K}}$. Observe that the associated chain complex $A^\bullet$ is exactly the Hochschild complex of A. We depict the simplicial object as the following diagram:

$$\mathbb{K} \xleftarrow{\varepsilon_A} A \underset{A\otimes\varepsilon_A}{\overset{\nabla_A}{\leftleftarrows}} A^{\otimes 2} \underset{A^{\otimes 2}\otimes\varepsilon_A}{\overset{\nabla_A}{\Lleftarrow}} A^{\otimes 3} \underset{A^{\otimes 4}\otimes\varepsilon_A}{\overset{\nabla_A}{\Lleftarrow}} \cdots$$

$$\mathbb{K} \qquad A \xrightarrow{s^1} A^{\otimes 2} \overset{s^i}{\rightrightarrows} A^{\otimes 3} \overset{s^i}{\Rrightarrow} \cdots$$

The augmentation ideal $\bar{A}$ carries a natural semi-simplical structure induced by A. By restricting each of the face maps $\bar{d^i} = d^i|_{\bar{A}} : \bar{A}^{\otimes n} \to \bar{A}^{\otimes n-1}$ we obtain the maps together with the simplical identity 1. This is the non-unital Hochschild complex of A. We may depict the semi-simplical object as the following diagram:

$$\mathbb{K} \xleftarrow{0} \bar{A} \underset{0}{\overset{\nabla_A}{\leftleftarrows}} \bar{A}^{\otimes 2} \underset{0}{\overset{\nabla_A}{\Lleftarrow}} \bar{A}^{\otimes 3} \underset{0}{\overset{\nabla_A}{\Lleftarrow}} \cdots$$

As graded modules, the chain complex $\bar{A}^\bullet$ is isomorphic to $T^c(\bar{A})$. Here we think of the grading $T^c(\bar{A})$ as starting at 0 and going down to negative degrees. Consider instead the looped, or shifted, non-unital algebra $\bar{A}[1]$. Every algebra may be considered as a graded algebra concentrated in degree 0, the shift functor then recontextualize the degree the algebra is concentrated in. Since the tensor product respects shifting, we see that shifting may be regarded as a tensor.

$$M[n] \simeq \mathbb{K} \otimes M[n] \simeq \mathbb{K}[n] \otimes M \simeq \mathbb{K}\{\sigma\} \otimes M$$

Here σ is a formal generator of $\mathbb{K}$ such that $|\sigma| = -n$. If we let ω be a formal generator s.t. $|\omega| = -1$, then $\bar{A}[1] \simeq \mathbb{K}\{\omega\} \otimes \bar{A}$. Every element of $\bar{A}[1]$ may then be written on the form ωa, where $a \in \bar{A}$. Using MacLanes and Eilenbergs notation [4], we would write $[a]$ instead.

The looped algebra $\bar{A}[1]$ is no longer associative. For $\mathbb{K}\{\omega\}$ we define a map $(\cdot)$.

$$\begin{aligned} (\cdot) : \mathbb{K}\{\omega\}^{\otimes 2} &\to \mathbb{K}\{\omega\} \\ \omega \cdot \omega &= \omega \end{aligned}$$

The looped multiplication may then be defined as the composite $\nabla_{\bar{A}[1]} = ((\cdot) \otimes \nabla_A) \circ (\mathbb{K}\{\omega\} \otimes \beta \otimes \bar{A})$. Multiplying elements like ωa_1 and ωa_2 would then look like:

$$\nabla_{\bar{A}[1]}(\omega a_1 \otimes \omega a_2) = (-1)^{|a_1||\omega|}((\cdot) \otimes \nabla_{\bar{A}})(\omega^{\otimes 2} \otimes a_1 \otimes a_2) = (-1)^{|a_1|}\omega a_1 a_2$$

It is now also evident that the new map $\nabla_{\bar{A}[1]}$ is no longer a morphism of degree 0, but of degree 1.

Proposition 1.2.8. *The differential $d^\bullet_{\bar{A}[1]}$ is a coderivation for the cofree coalgebra $T^c(\bar{A}[1])$. Thus $(\bar{A}[1]^\bullet, d^\bullet_{\bar{A}[1]})$ is a dg-coalgebra.*

Proof. By injecting $\bar{A}[1]$ into $T^c(\bar{A}[1])$ we may think of $\nabla_{\bar{A}[1]} : \bar{A}[1]^{\otimes 2} \to T^c(\bar{A}[1])$ as a morphism into the tensor coalgebra. By using proposition 1.1.23, $\nabla_{\bar{A}[1]}$ extends uniquely into a coderivation:

$$d^c_{\bar{A}[1]} = \sum_{n=0}^{\infty} \sum_{i=0}^{n} (\nabla_{\bar{A}[1]})^{(n)}_{(i)} = d^\bullet_{\bar{A}[1]}.$$

$\square$

If $(A, d^\bullet_A)$ is an augmented dg-algebra, then A is a simplical object of $Mod^\bullet_{\mathbb{K}}$. It has an associated chain complex. Taking the alternate sum of face maps gives us a double complex as below. We define the double complex $A^\bullet$ as the associated chain complex to A.

$$\begin{tikzcd}
\cdots \arrow[r, "0"'] & 0 \arrow[r, "0"'] & \mathbb{K} \arrow[r, "0"'] & 0 \arrow[r, "0"'] & \cdots & & A[1]^{\otimes 0} \\
\cdots \arrow[r, "d^\bullet_{A[1]}"'] & A^{-1} \arrow[r, "d^\bullet_{A[1]}"'] \arrow[u, "\varepsilon_A"] & A^0 \arrow[r, "d^\bullet_{A[1]}"'] \arrow[u, "\varepsilon_A"] & A^1 \arrow[r, "d^\bullet_{A[1]}"'] \arrow[u, "\varepsilon_A"] & \cdots & = & A[1]^{\otimes 1} \arrow[u] \\
\cdots \arrow[r, "d^\bullet_{A[1]^{\otimes 2}}"'] & (A^{\otimes 2})^{-1} \arrow[r, "d^\bullet_{A[1]^{\otimes 2}}"'] \arrow[u, shift left, "\nabla_A"] \arrow[u, shift right, "A\otimes\varepsilon_A"'] & (A^{\otimes 2})^{0} \arrow[r, "d^\bullet_{A[1]^{\otimes 2}}"'] \arrow[u, shift left, "\nabla_A"] \arrow[u, shift right, "A\otimes\varepsilon_A"'] & (A^{\otimes 2})^{1} \arrow[r, "d^\bullet_{A[1]^{\otimes 2}}"'] \arrow[u, shift left, "\nabla_A"] \arrow[u, shift right, "A\otimes\varepsilon_A"'] & \cdots & & A[1]^{\otimes 2} \arrow[u] \\
& \vdots \arrow[u, shift left=2, "\nabla_A"] \arrow[u] \arrow[u, shift right=2, "A^{\otimes 2}\otimes\varepsilon_A"'] & \vdots \arrow[u, shift left=2, "\nabla_A"] \arrow[u] \arrow[u, shift right=2, "A^{\otimes 2}\otimes\varepsilon_A"'] & \vdots \arrow[u, shift left=2, "\nabla_A"] \arrow[u] \arrow[u, shift right=2, "A^{\otimes 2}\otimes\varepsilon_A"'] & & & \vdots \arrow[u]
\end{tikzcd}$$

For simplicity we write d_1 for the horizontal differential and d_2 for the vertical differential. The total associated chain complex is the total complex for $Tot(A^\bullet)$, denoted $A^\bullet$ if there are no confusion. Whenever we go down to the total complex we should switch out the multiplication with the suspended multiplication. The following proposition shows that this is well-defined, among other things. This total complex is then isomorphic, as graded modules, to $T^c(\bar{A}[1])$

Proposition 1.2.9. *Let A an augmented dg-algebra. The bar complex BA is the total associated chain complex of the augmentation ideal $\bar{A}$. $(BA, d^\bullet_{BA})$ is the cofree conilpotent coalgebra equipped with $d^\bullet_{BA} = d_1 + d_2$ as coderivation.*

Proof. It is apparent that d_1 and d_2 are coderivations with respect to deconcatenation. Since the multiplication ∇_A is a chain map, we should have ${d^\bullet_{BA}}^2 = d_1 \circ d_2 + d_2 \circ d_1 = 0$. This will be shown for each element in $A^{\otimes 2}$, the result may then be extended to all of BA. In the following calculation we will not decorate each a with an ω, and it should be understood that these elements are looped.

$$
\begin{aligned}
d_1 \circ d_2(a_1 \otimes a_2) = (-1)^{|a_1|} d_1(a_1a_2) = (-1)^{|a_1|} d^\bullet_{A[1]}(a_1a_2) \\
= (-1)^{|a_1|+1} d^\bullet_A(a_1a_2) = (-1)^{|a_1|+1}(d^\bullet_A(a_1)a_2 + (-1)^{|a_1|}a_1 d^\bullet_A(a_2)) \\
= (-1)^{|a_1|+1} d^\bullet_A(a_1)a_2 - a_1 d^\bullet_A(a_2)
\end{aligned}
$$

$$
\begin{aligned}
d_2 \circ d_1(a_1 \otimes a_2) = d_2 \circ (d^\bullet_{A[1]} \otimes id_{A[1]} + id_{A[1]} \otimes d^\bullet_{A[1]}[1])(a_1 \otimes a_2) \\
= -d_2 \circ (d^\bullet_A(a_1) \otimes a_2 + (-1)^{|a_1|+1} a_1 \otimes d^\bullet_A(a_2)) \\
= (-1)^{|d^\bullet_A(a_1)|+1} d^\bullet_A(a_1)a_2 + (-1)^{2|a_1|+2} a_1 d^\bullet_A d^\bullet_A(a_2) \\
= (-1)^{|a_1|} d^\bullet_A(a_1)a_2 + a_1 d^\bullet_A(a_2) = -d_1 \circ d_2(a_1 \otimes a_2)
\end{aligned}
$$

□

Remark 1.2.10. For now we don't need to show that BA is a functor. This property follows from BA being the representing object of $Tw(\_, A)$.

On the other hand, a coalgebra C is a comonoid in $Mod_{\mathbb{K}}$. By the dual of proposition 1.2.7 we may think of it as an augmented simplical object $C : (\Delta_+)^{op} \to Mod_{\mathbb{K}}$. Dually, all of the simplical identities follows from coassociativity and counitality. A coaugmented coalgebra C may be given an augmented cosimplicial structure in the opposite way of algebras. We then get that the coaugmentation quotient $\bar{C}$ is a semi-cosimplical object of $Mod_{\mathbb{K}}$. Observe that $\bar{C}$ has an associated chain complex like $\bar{A}$, but every arrow goes in the opposite direction.

$$
\begin{tikzcd}
\mathbb{K} \arrow[r, "\upsilon_C"] & C \arrow[r, shift left, "\Delta_C"] \arrow[r, shift right, "A\otimes\upsilon_C"'] & C^{\otimes 2} \arrow[r, shift left=2, "\Delta_C"] \arrow[r] \arrow[r, shift right=2, "C^{\otimes 2}\otimes\upsilon_C"'] & C^{\otimes 3} \arrow[r, shift left=3, "\Delta_C"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3, "C^{\otimes 4}\otimes\upsilon_C"'] & \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{K} & C & C^{\otimes 2} \arrow[l, "s_1"'] & C^{\otimes 3} \arrow[l, shift left, "s_i"'] \arrow[l, shift right] & \cdots \arrow[l, shift left=2, "s_i"'] \arrow[l] \arrow[l, shift right=2]
\end{tikzcd}
$$

The cobar construction is made from the inverse shifted, or suspended coalgebra $C[-1]$. We may also denote suspension by tensoring with the formal generator s, where $|s| = 1$. That is we have the isomorphism $C[-1] \simeq \mathbb{K}\{s\} \otimes C$. The cobar construction is realized as the free tensor algebra $T(\bar{C}[-1])$, where the comultiplication $\Delta_{\bar{C}[-1]}$ induces a derivation $d_{\bar{C}[-1]}$ by proposition 1.1.23.

Remark 1.2.11. As we have chosen to define $\nabla_{A[1]}(a_1 \otimes a_2) = (-1)^{|a_1|}a_1a_2$, we are forced by the linear dual to define $\Delta_{C[-1]}(c) = -(-1)^{|c_{(1)}|}c_{(1)} \otimes c_{(2)}$. Here we use the Sweedlers notation without sums to denote the comultiplication. Note that this really should be a sum of many different elementary tensors Lastly, observe that this definition also agrees with Koszuls sign rule.

Proposition 1.2.12. *Let C be a coaugmented dg-coalgebra. The cobar complex ΩC is the total associated chain complex of the coaugmentation quotient $\bar{C}$. $(\Omega C, D^{\bullet}_{\Omega C})$ is the free algebra equipped with $d^{\bullet}_{\Omega C} = d_1 + d_2$ as derivation.*

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\dots \arrow[r, "d_{\bar{C}[-1]^{\otimes 2}}"] & (\bar{C}^{\otimes 2})^{-1} \arrow[u, shift left] \arrow[u, shift right, "\Delta_{C \otimes \bar{C}}"] \arrow[r, "d_{\bar{C}[-1]^{\otimes 2}}"] & (\bar{C}^{\otimes 2})^{0} \arrow[u, shift left] \arrow[u, shift right, "\Delta_{C \otimes \bar{C}}"] \arrow[r, "d_{\bar{C}[-1]^{\otimes 2}}"] & (\bar{C}^{\otimes 2})^{1} \arrow[u, shift left] \arrow[u, shift right, "\Delta_{C \otimes \bar{C}}"] \arrow[r, "d_{\bar{C}[-1]^{\otimes 2}}"] & \dots \\
\dots \arrow[r, "d_{\bar{C}[-1]}"] & \bar{C}^{-1} \arrow[u, "\Delta_C"] \arrow[r, "d_{\bar{C}[-1]}"] & \bar{C}^{0} \arrow[u, "\Delta_C"] \arrow[r, "d_{\bar{C}[-1]}"] & \bar{C}^{1} \arrow[u, "\Delta_C"] \arrow[r, "d_{\bar{C}[-1]}"] & \dots \\
\dots \arrow[r] & 0 \arrow[u] \arrow[r] & \mathbb{K} \arrow[u, "0"] \arrow[r] & 0 \arrow[u] \arrow[r] & \dots
\end{tikzcd}$$

We will now see that the bar and cobar construction defines an adjoint pair of functors. Note that since for any conilpotent dg-coalgebra C, the object ΩC represents the functor in the category of augmented algebras. By Yoneda's lemma, the data of morphisms are then defined, so Ω does truly define a functor.

Theorem 1.2.13. *Let C be a conilpotent dg-coalgebra and A an augmented dg-algebra. The functor $Tw(C, A)$ is represented in both arguments, i.e.*

$$Alg^{\bullet}_{\mathbb{K},+}(\Omega C, A) \simeq Tw(C, A) \simeq CoAlg^{\bullet}_{\mathbb{K},Conil}(C, BA).$$

Proof. We will show that ΩC represents the set of twisting morphisms in the first argument. Showing that BA represents the second argument uses every dual proposition. Thus, it is necessary that C is conilpotent, in order to dualize the arguments.

Suppose that $f : \Omega C \to A$ is an augmented dg-algebra homomorphism. f is then a morphism of degree 0. By freeness, f is uniquely determined by a morphism $f \mid_{\bar{C}[-1]}: \bar{C}[-1] \to \bar{A}$ of degree 0, which corresponds to a morphism $f' : C \to A$ of degree 1.

Since f is a morphism of chain complexes it commutes with the differential, i.e.

$$f \circ d^{\bullet}_{\Omega C} = d^{\bullet}_{A} \circ f$$
$$f \circ (d_1 + d_2) = d^{\bullet}_{A} \circ f$$

By free-forget, this is equivalent to say that $-f' \circ d^{\bullet}_{C} - f' \star f' = d^{\bullet}_{A} \circ f'$. Thus f' is a twisting morphism as desired. □

For our convenience, we will give these isomorphisms some names. Whenever $\tau : C \to A$ is a twisting morphism, the induced morphism of algebras is denoted $f_\tau : \Omega C \to A$ and the induced morphism of coalgebras is denoted $g_\tau : C \to BA$.

Remark 1.2.14. These functors consists of a composition with the augmentation ideal (quotient) and then the (co)free tensor (co)algebra. By reversing these operations we obtain another adjunction which is more or less the same adjunction. By abuse of notation we will call these functors for the bar and cobar construction as well, and they induce and adjoint pair between dg-algebras and reduced conilpotent dg-coalgebras. In other words, given a dg-algebra A and a reduced conilpotent dg-coalgebra C, $BA = \bar{T}^c(A[1])$ and $\Omega C = \bar{T}(C[-1])$.

$$\begin{tikzcd} Alg^{\bullet}_{\mathbb{K}} \arrow[r, bend right, "B"'] \arrow[r, phantom, "\perp"] & CoAlg^{\bullet}_{\mathbb{K},conil} \arrow[l, bend right, "\Omega"'] \end{tikzcd}$$

Associated to this adjunction, we obtain universal elements, together with universal properties. Let A be an augmented dg-algebra, then the identity of the coalgebras $id_{BA} : BA \to BA$, the counit $\varepsilon_A : \Omega BA \to A$ and a twisting morphism $\pi_A : BA \to A$ are equivalent by the adjunction and representation. Dually, the identity of algebras $id_{\Omega C} : \Omega C \to \Omega C$, the unit $\eta_C : C \to B\Omega C$ and the twisting morphism $\iota_C : C \to \Omega C$ are equivalent. The morphisms π_A and ι_C are called the universal elements. We summarize their universal property in the following corollary.

Corollary 1.2.14.1. *Let A be an augmented dg-algebra, and C a conilpotent dg-coalgebra. Any twisting morphism $\alpha : C \to A$ factors uniquely through either π_A or ι_C.*

$$\begin{tikzcd} & \Omega C \arrow[dr, dashed, "g_\alpha"] & \\ C \arrow[ur, "\iota_C"] \arrow[rr, "\alpha"] \arrow[dr, dashed, "f_\alpha"'] & & A \\ & BA \arrow[ur, "\pi_A"'] & \end{tikzcd}$$

Moreover, the morphism f_α is a morphism of dg-coalgebras, and g_α is a morphism of dg-algebras.

Definition 1.2.15 (Augmented Bar-Cobar construction)**.** Let A be an augmented dg-algebra. The (right) augmented bar construction is the right twisted tensor product $BA \otimes_{\pi_A} A$, where π_A is the universal twisting morphism.

Let C be a conilpotent dg-coalgebra. The (right) augmented cobar construction is the right twisted tensor product $C \otimes_{\iota_C} \Omega C$, where ι_C is the universal twisting morphism.

Remark 1.2.16. We could have defined the augmented bar-cobar construction as the left twisted tensor product. There is really no preference of handedness. Whenever we wish to be precise which handedness we will use it will be specified, e.g. the left augmented bar construction of A.

Proposition 1.2.17. *The augmentation ideal of the augmented bar (cobar) construction is acyclic, i.e. $BA\bar{\otimes}_{\pi_A}A$ ($A\bar{\otimes}_{\pi_A}BA$) and $C\bar{\otimes}_{\iota_C}\Omega C$ ($\Omega C\bar{\otimes}_{\iota_C}C$) are acyclic.*

Proof. We will postpone this proof until chapter 3. This is a part of the fundamental theorem of twisting morphisms, and will not be relevant until then. □

1.3 Strongly Homotopy Associative Algebras and Coalgebras

1.3.1 SHA-Algebras

We have seen from corollary 1.2.8 that any dg-algebra A defines a dg-coalgebra $T^c(A[1])$, the bar construction, with a coderivation m^c of degree -1. Does this however work in reverse? I.e. if A is a vector space such that the coalgebra $T^c(A[1])$ together with a coderivation m^c is a dg-coalgebra, is then A an algebra? The answer to this is no, but it leads to the definition of a strongly homotopy associative algebra.

Definition 1.3.1. An A_∞-algebra is a graded vector space A together with a differential $m : \bar{T}^c(A[1]) \to \bar{T}^c(A[1])$ that is a coderivation of degree 1.

The differential m induces structure morphisms on $A[1]$. By proposition 1.1.23 there is a natural bijection $Hom_{\mathbb{K}}(\bar{T}^c(A[1]), A[1]) \simeq Coder(\bar{T}^c(A[1]), \bar{T}^c(A[1]))$ given by the projection onto $A[1]$. Thus $m : \bar{T}^c(A[1]) \to \bar{T}^c(A[1])$ corresponds to maps $\widetilde{m}_n : A[1]^{\otimes n} \to A[1]$ of degree 1 for any $n \geqslant 1$. We define maps $m_n : A^{\otimes n} \to A$ by the composite $s\widetilde{m}_n\omega^{\otimes n}$. Since $\omega^{\otimes n}$ is of degree $-n$, $\widetilde{m}_n$ and s is of degree 1, we get that m_n is of degree $2-n$.

$$\begin{tikzcd} A^{\otimes n} \arrow[r, "m_n"] \arrow[d, "\omega^{\otimes n}"', "\simeq"] & A \\ A[1]^{\otimes n} \arrow[r, "\widetilde{m}_n"] & A[1] \arrow[u, "s", "\simeq"'] \end{tikzcd}$$

Proposition 1.3.2. *An A_∞-algebra is equivalent to a graded vector space A together with homogenous morphisms $m_n : A^{\otimes n} \to A$ of degree $2-n$. Moreover, the morphism must satisfy the following relations for any $n \geqslant 1$:*

$$(\text{rel}_n) \qquad \sum_{p+q+r=n} (-1)^{pq+r} m_{p+1+r} \circ (id^{\otimes p} \otimes m_q \otimes id^{\otimes r}) = 0$$

Remark 1.3.3. We make a more convenient notation for (rel_n), called partial composition $\circ_i$.

$$m_{p+1+r} \circ_{p+1} m_q = m_k \circ (id^{\otimes p} \otimes m_q \otimes id^{\otimes r})$$

$$(\text{rel}_n) \qquad \sum_{p+q+r=n} (-1)^{pq+r} m_{p+1+r} \circ_{p+1} m_q = 0$$

Before starting with the proof we will use a lemma for checking whether a coderivation $m : T^c(A) \to T^c(A)$ is a differential.

Lemma 1.3.4. *Let $m : T^c(A) \to T^c(A)$ be a coderivation, and denote $m_n = m|_{A^{\otimes n}}$. m is a differential if and only if the following relations are satisfied:*

$$\sum_{p+q+r=n} m_{p+1+r} \circ_{p+1} m_q = 0$$

Proof. By proposition 1.1.23 we may write $m = \sum_{n=0}^{\infty} \sum_{i=0}^{n} m_{(n)}^{(i)}$. By using partial composition, we rewrite its n'th component as:

$$m_n = \sum_{q=1}^{n} \sum_{p=1}^{n} id^{\otimes(n-q)} \circ_p m_q = \sum_{p+q+r=n} id^{\otimes(p+1+r)} \circ_{p+1} m_q$$

For m^2 we denote it's n'th component as m_n^2. Observe the following:

$$m_n^2 = m \circ m_n = m \circ \sum_{p+q+r=n} id^{\otimes(p+1+r)} \circ_{p+1} m_q = \sum_{p+q+r=n} m \circ_{p+1} m_q$$
$$\pi m_n^2 = \pi \sum_{p+q+r=n} m \circ_{p+1} m_q = \sum_{p+q+r=n} m_{p+1+r} \circ_{p+1} m_q$$

Since every coderivation are uniquely determined by π, its projection onto A we get that $m^2 = 0$ if and only if

$$\sum_{p+q+r=n} m_{p+1+r} \circ_{p+1} m_q = 0.$$

□

Proof of proposition 1.3.2. Let (A, m) be an A_∞-algebra. We denote the n'th component of m as $\widetilde{m}_n$. The n'th components thus define maps $m_n : A^{\otimes n} \to A$ as $m_n = s\widetilde{m}_n \omega^{\otimes n}$.

By the above lemma we know that the n'th component of m^2 is:

$$\sum_{p+q+r=n} \widetilde{m}_{p+1+r} \circ_{p+1} \widetilde{m}_q$$
$$= \sum_{p+q+r=n} \omega m_{p+1+r} s^{\otimes(p+1+r)} \circ_{p+1} \omega m_q s^{\otimes q} = \sum_{p+q+r=n} (-1)^{pq+r} \omega m_{p+1+r} \circ_{p+1} m_q s^{\otimes n}$$

Since suspension and desuspension are isomorphism we get that $m^2 = 0$ if and only if (rel_n) are 0 for every $n \geqslant 1$, i.e.

$$\sum_{p+q+r=n} (-1)^{pq+r} m_{p+1+r} \circ_{p+1} m_q = 0$$

□

Given an A_∞ algebra A we may either think of it as a differential tensor coalgebra $\bar{T}^c(A[1])$ with differential $m : \bar{T}^c(A[1]) \to \bar{T}^c(A[1])$ or as a graded vector space with morphisms $m_n : A^{\otimes n} \to A$ satisfying (rel$_n$). We will calculate (rel$_n$) for $1, 2, 3$:

$$
\begin{array}{ll}
(\text{rel}_1) & m_1 \circ m_1 = 0 \\
(\text{rel}_2) & m_1 \circ m_2 - m_2 \circ_1 m_1 - m_2 \circ_2 m_1 = 0 \\
(\text{rel}_3) & m_1 \circ m_3 - m_2 \circ_1 m_2 + m_2 \circ_2 m_2 + m_3 \circ_1 m_1 + m_3 \circ_2 m_1 + m_3 \circ_3 m_1 = 0
\end{array}
$$

We see that (rel$_1$) states that m_1 should be a differential, we may thus think of (A, m_1) as a chain complex. Furthermore, (rel$_2$) says that $m_2 : (A^{\otimes 2}, m_1 \otimes id_A + id_A \otimes m_1) \to (A, m_1)$ is a morphism of chain complexes. Lastly, (rel$_3$) gives us a homotopy for the associator of m_2, namely m_3. Thus we may regard (A, m_1, m_2) as an algebra which is associative up to homotopy. Regarding A as a chain complex instead we obtain our final definition of an A_∞-algebra.

Proposition 1.3.5. *Suppose that (A, d) is a chain complex, and that there exists morphisms $m_n : A^{\otimes n} \to A$ of degree $2 - n$ for any $n \geqslant 2$. A is an A_∞-algebra if and only it satisfies the following relations:*

$$
(\textit{rel'}_n) \qquad \partial(m_n) = - \sum_{\substack{n=p+q+r \\ k=p+1+r \\ k>1, q>1}} (-1)^{pq+r} m_k \circ_{p+1} m_q
$$

We define the homotopy of an A_∞-algebra to be the homology of the chain complex (A, m_1). Since $\partial(m_3) = m_2 \circ_1 m_2 - m_2 \circ_2 m_2$, we get that m_2 is associative in homology. Thus for any A_∞-algebra A, the homotopy HA is an associative algebra. The operadic homology of A is defined as the homology of $(T^c(A[1]), m)$, which is the non-unital augmented Hochschild homology of A.

Example 1.3.6. Suppose that V is a cochain complex with differential d. Then V is an A_∞-algebra with trivial multiplication. In other words $m^1 = d$ and $m^i = 0$ for any $i > 1$.

Example 1.3.7. Suppose that A is a dg-algebra. Then A is an A_∞-algebra where $m^1 = d$, $m^2 = \cdot$ and $m^i = 0$ for any $i > 2$.

Example 1.3.8. Let A be a connected weight-graded algebra over $\mathbb{K}$. We may then think of $\mathbb{K}$ as an A-module with trivial action, i.e. as the quotient ${}^{A}/_{A^{(i)}}{}_{(i \geqslant 1)}$, then $Ext^*_A(\mathbb{K}, \mathbb{K})$ is an A_∞-algebra. This was shown by Lu, Palmari, Wu and Zhang [9].

Next we want to understand the category of A_∞-algebras. A morphism between A_∞-algebras is called an ∞-morphism. We define such an ∞-morphism $f : A \rightsquigarrow B$ between A_∞-algebras as associated dg-coalgebra homomorphism $Bf : (\bar{T}^c(A[1]), m^A) \to (\bar{T}^c(B[1]), m^B)$. Here Bf is purely formal, we will make sense of this soon.

Proposition 1.3.9. *Let A, B be two A_∞-algebras. A collection of morphisms $f_n : A^{\otimes n} \to B$ of degree $1 - n$ for any $n \geqslant 1$ defines an ∞-morphism $f : A \rightsquigarrow B$ if and only if f_1 is a morphism* *of chain complexes and for any $n \geqslant 2$ the following relations are satisfied:*

$$(\textit{rel}_n) \qquad \partial(f_n) = \sum_{\substack{p+1+r=k \\ p+q+r=n}} (-1)^{pq+r} f_k \circ_{p+1} m_q^A - \sum_{\substack{k\geqslant 2 \\ i_1+\ldots+i_k=n}} (-1)^e m_k^B \circ (f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_k})$$

where e is given as: $e = (k-1)(1-i_1) + (k-2)(1-i_2) + \ldots + 2(1-i_{k-2}) + (1-i_{k-1})$

Proof. This is immediate by the universa l property of cofree coalgebras. To see how we get the exponent e we explain the first term. The factor $(1-i_1)$ is precisely the degree of f_{i_1}, and $k-1$ is exactly how many tensors which appear after f_{i_1}. We get the first term as we have to move $k-1$ ωs over f_{i_1}. The same applies to every other tensor after f_{i_1}, but for f_{i_2} there are only $k-2$ tensors which comes after, and so on. □

Since the composition of two dg-coalgebra homomorphism is again a dg-algebra homomorphism, we get that the composition of two ∞-morphisms is again an ∞-morphism. More explicitly if $f : A \rightsquigarrow B$ and $g : B \rightsquigarrow C$ are two ∞-morphisms, then their composition is defined as:

$$(fg)_n = \sum_r \sum_{i_1+\ldots+i_r=n} (-1)^e g_r(f_{i_1} \otimes \ldots \otimes f_{i_r}).$$

Definition 1.3.10. An ∞-morphism $f : A \rightsquigarrow B$ is called strict if $f_n = 0$ for any $n \geqslant 2$.

Definition 1.3.11. Alg_∞ denotes the category of A_∞-algebras, and the morphisms in this category are the ∞-morphisms.

Observe that we may extend the bar construction to $B : Alg_\infty \to CoAlg^\bullet_{\mathbb{K},conil}$ to a fully fatihful functor. This may be done explicitly by using proposition 1.1.23. The subcategory of the essential image is the full subcategory of every dg-coalgebra that is isomorphic, as a graded coalgebra, to a cofree tensor coalgebra. Notice that the bar construction on the category of dg-algebras is a non-full injection into the category of A_∞-algebras. Observe that this inclusion is the recontextualization of a dg-algebra into an A_∞-algebra.

We call a quasi-isomorphism between A_∞-algebras for an ∞-quasi-isomorphism. Given an ∞-morphism $f : A \rightsquigarrow B$, then we say that it is an ∞-quasi-isomorphism if f_1 is a quasi-isomorphism. If we wanted to be more stringent with this definition we would want an ∞-quasi-isomorphism to be an ∞-morphism which is a quasi-isomorphism of dg-coalgebras. We will see that these definitions are equivalent later.

A homotopy between two A_∞-algebras is a homotopy between the dg-coalgebras they define. We may however trace this definition back along the quasi-inverse of the bar construction to get a new definition in terms of many morphisms. Let $f, g : A \rightsquigarrow B$ be two ∞-morphisms, we say that $f - g$ is null-homotopic if there is a collection of morphisms $h_n : A^{\otimes n} \to B$ of degree n such that the following relations are satisfied for any $n \geqslant 1$:

$$f_n - g_n = \sum (-1)^s m^B_{r+1+t} \circ (f_{i_1} \otimes \ldots \otimes f_{i_r} \otimes h_k \otimes g_{j_1} \otimes \ldots \otimes g_{j_t}) + \sum (-1)^{j+kl} h_i \circ_{j+1} m_k^A$$

Where s is some constant depending on t, r and k. More details may be found in [10].

As in the same case for algebras, there is also a notion of unital A_∞-algebras and augmented A_∞-algebras. For this discussion it is important to note that the field $\mathbb{K}$ is also an A_∞-algebra. This algebra will serve as the initial algebra, in the same way as it did for ordinary algebras.

Definition 1.3.12. A strictly unital A_∞-algebra is an A_∞-algebra A together with a unit morphism $\upsilon_A : \mathbb{K} \to A$ of degree 0 such that the following are satisfied:

- $m_1 \circ \upsilon_A = 0$.
- $m_2(id_A \otimes \upsilon_A) = id_A = m_2(\upsilon_A \otimes id_A)$.
- $m_i \circ_k \upsilon_A = 0$ for any $i \geqslant 3$ and $1 \leqslant k < i$.

A strictly unital ∞-morphism $f : A \rightsquigarrow B$ between strictly unital A_∞-algebras is a morphism which preserves the unit. This means that $f_1\upsilon_A = \upsilon_B$ and $f_i \circ_k \upsilon_A = 0$ for any $i \geqslant 2$ and $1 \leqslant k < i$. The collection of strictly unital A_∞-algebras and strictly unital ∞-morphisms form a non-full subcategory of A_∞-algbras. A strict ∞-morphism which is unital at the level of chain complexes is automatically stricly unital. Strict unital will then mean strict and strictly unital. Note that $\mathbb{K}$ is strictly unital where the unit is $id_{\mathbb{K}}$.

Definition 1.3.13. An augmented A_∞-algebra is a strictly unital A_∞-algebra A together with a strict unital morphism $\varepsilon_A : A \to \mathbb{K}$. The ∞-morphism ε_A is called the augmentation of A.

The collection of augmented A_∞-algebras and strictly unital morphism is the category of augmented A_∞-algebras, denoted as $Alg_{\infty,+}$. As in the same way for algebras, there is an equivalence of categories $Alg_\infty \simeq Alg_{\infty,+}$. The augmentation ideal, or the reduced A_∞-algebra is the kernel of the augmentation ε_A. A priori, it does not make sense to talk about this limit as we do not know if it exists. However, we will see in section 2.3.3 that such morphisms does in fact have a kernel. This defines a functor, $\bar{\_} : Alg_{\infty,+} \to Alg_\infty$, where $Ker\varepsilon_A = \bar{A}$. The quasi-inverse to this functor is free augmentations. Given an A_∞-algebra A, we may construct the A_∞-algebra $A \oplus \mathbb{K}$. The structure morphisms are given by m_i^A, but there is now a unit $\upsilon_{A\oplus\mathbb{K}}$. Thus we get that $m_1(1) = 0$, $m_2(a \otimes 1) = a$ and $m_i \circ_k 1 = 0$ in the same manner. We obtain a functor $\_^+ : Alg_\infty \to Alg_{\infty,+}$, where $A \oplus \mathbb{K} = A^+$.

1.3.2 A_∞-Coalgebras

Dual to A_∞-algebras we got conilpotent A_∞-coalgebras. Here we instead ask ourselves if the cobar construction has some converse. I.e. if C is a graded vector space such that $T(C[-1])$ together with a derivation m is a dg-algebra, is then C a coalgebra? Again, the answer to this is no, but we do obtain a definition for conilpotent A_∞-coalgebras.

Definition 1.3.14. A graded vector space C is called a conilpotent A_∞-coalgebra if it is a dg-algebra of the form $(\bar{T}(C[-1]), d)$ where d is a derivation of degree 11.

Remark 1.3.15. For the rest of this thesis, an A_∞-coalgebra should be understood as a conilpotent A_∞-coalgebra unless otherwise specified.

Corollary 1.3.15.1. *C is an A_∞-coalgebra with differential d then there is a chain complex (C, d^1), where d^1 is of degree 1, and together with morphisms $d^n : C \to C^{\otimes n}$ such that d uniquely determines each d^i for any $i > 0$. Conversely, if the morphisms d^i satisfy (rel)$_n$, then they uniquely determine a d such that C is an A_∞-coalgebra.*

$$(\textit{rel}_n) \qquad \sum_{p+q+r=n} (-1)^{pq+r} d^{p+1+q} \circ^{op}_{p+1} d^q = 0$$

A morphism of A_∞-coalgebras would be defined in the same manner, but opposite. So an ∞-comorphism $f : C \rightsquigarrow D$ is either a morphism $\widetilde{f} : (T(C[-1]), m^C) \to (T(D[-1]), m^D)$ of dg-algebras. Equivalently such an ∞-comorphism is a collection of morphisms $f_n : C \to D^{\otimes n}$ of degree $1-n$ such that f_1 is a morphism of chain complexes and for any $n \geqslant 2$ the following relations are satisfied:

$$(\mathrm{rel}_n) \qquad \partial(f_n) = \sum_{\substack{p+1+r=k \\ p+q+r=n}} (-1)^{pq+r} f_k \circ^{op}_{p+1} m^D_q - \sum_{\substack{k \geqslant 2 \\ i_1+\ldots+i_k=n}} (-1)^e m^C_k \circ^{op} (f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_k})$$

where e is given as: $e = (k-1)(1-i_1) + (k-2)(1-i_2) + \ldots + 2(1-i_{k-2}) + (1-i_{k-1})$

We denote $Coalg_\infty$ as the category of A_∞-coalgebras. In the same manner, the cobar construction extends to this category and gives us an identification of A_∞-coalgebras and a subcategory of dg-algebras. This subcategory consists of every dg-algebra that is isomorphic, as an algebra, to a free tensor algebra. Lastly, every dg-coalgebra is an A_∞-coalgebra by letting every morphism $m^i = 0$ where $i > 2$. This gives a non full inclusion.

Chapter 2

Homotopy Theory of Algebras

Quillen envisioned a more general approach to homotopy theory, which he dubbed homotopical algebra. A homotopy theory was first enclosed by the structure of a model category, and now we mostly consider closed model category. Many of the results from classical homotopy theory was recovered in this new setting of model categories. The theorem which we are most concerned about is Whiteheads theorem:

Theorem 2.0.1 (Whiteheads Theorem)**.** *Let X and Y be two CW-complexes. If $f : X \to Y$ is a weak equivalence, then it is also a homotopy equivalence. I.e. there exists a morphism $g : Y \to X$ such that $gf \sim id_X$ and $fg \sim id_Y$.*

If we employ Quillens model category onto the category Top, we get that a space X is bifibrant if and only if it is a CW-complex. The natural generalization is then to not ask of X to be a CW-complex, but a bifibrant object.

Theorem 2.0.2 (Generalized Whiteheads Theorem)**.** *Let $\mathcal{C}$ be a model category. Suppose that X and Y are bifibrant objects of $\mathcal{C}$, and that there is a weak-equivalence $f : X \to Y$. Then f is also a homotopy equivalence, i.e. there exists a morphism $g : Y \to X$ such that $gf \sim id_X$ and $fg \sim id_Y$.*

The category of differential graded algebras employs such a model category, and here we let the weak-equivalences be quasi-isomorphisms. On the other hand the category of differential graded coalgebras has a model structure where the weak equivalences are the maps which are sent to quasi.-isomorphism by the cobar construction. Moreover, in this case the bar and cobar construction defines a Quillen equivalence between these model structures. As we will see, a dg-coalgebra will be bifibrant exactly when it is an A_∞-algebra. Thus, by Whiteheads theorem, quasi-isomorphisms lift to homotopy equivalences. In this case the derived category of A_∞-algebras is equivalent to the homotopy category of A_∞-algebras.

We will conclude this section by looking at the category of algebras as a subcategory of A_∞-algebras. The derived category may then be expressed as the homotopy category A_∞-algebras, restricted to algebras.

2.1 Model categories

In this section we will define Quillens model category. As one may see is that in practice there are a plethora of semantically different definitions of model categories, however they are all made to be equivalent. The difference comes down to preference. In this thesis we will use the definitions as they are developed in Mark Hoveys book [11]. We will then go on to prove the basic results known about model categories, its associated homotopy category and Quillen functors between model categories.

Before we state the definition of a model category we need some preliminary definitions. For this section, let $\mathcal{C}$ be a category.

Definition 2.1.1 (Retract)**.** A morphism $f : A \to B$ in $\mathcal{C}$ is a retract of a morphism $g : c \to D$ if it fits in a commutative diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, "id_A"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "id_B"'] & D \arrow[r] & B \end{tikzcd}$$

Definition 2.1.2 (Functorial factorization)**.** A pair of functors $\alpha, \beta : \mathcal{C}^{\to} \to \mathcal{C}^{\to}$ is called a functorial factorization if for any morphism $f = \beta(f) \circ \alpha(f)$. We will denote the morphisms in the factorization as f_α and f_β. The functorial factorization may be depicted by the following commutative diagram:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "f_\alpha"'] & & B \\ & C \arrow[ur, "f_\beta"'] & \end{tikzcd}$$

Definition 2.1.3 (Lifting properties)**.** Suppose that the morphisms $i : A \to B$ and $p : C \to D$ fits inside a commutative square. i is said to have the left lifting property with resptect to p, or p has the right lifting property with respect to i, if there is an $h : B \to C$ such that the two triangles commute.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & C \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed, "h"] & D \end{tikzcd}$$

Remark 2.1.4. We will call the left lifting property for LLP and the right lifting property for RLP.

2.1.1 Model categories

Definition 2.1.5 (Model category)**.** Let $\mathcal{C}$ be a category with all finite limits and colimits. $\mathcal{C}$ admits a model structure if there are three wide subcategories each defining a class of morphisms:

- $Ac \subset Mor(\mathcal{C})$ are called weak equivalences
- $Cof \subset Mor(\mathcal{C})$ are called cofibrations
- $Fib \subset Mor(\mathcal{C})$ are called fibrations

In addition we call morphisms in $Cof \cap Ac$ for acyclic cofibrations and $Fib \cap Ac$ for acyclic fibrations. Moreover, $\mathcal{C}$ has two functorial factorizations (α, β) and (γ, δ). The following axioms should be satisfied:

MC1 The class of weak equivalences satisfy the 2-out-of-3 property, i.e. if f and g are composable morphisms such that 2 out of f, g and gf are weak equivalences, then so is the third.

MC2 The three classes Ac, Cof and Fib are retraction closed, i.e. if f is a retraction of g, and g is either a weak-equivalence, cofibration or fibration, then so is f.

MC3 The class of cofibrations have the left lifting property with respect to acyclic fibrations, and fibrations have the right lifting property with respect to acyclic cofibrations.

MC4 Given any morphism f, f_α is a cofibration, f_β is an acyclic fibration, f_γ is an acyclic cofibration and f_δ is a fibration.

Remark 2.1.6. The class Ac has every isomorphism. This is because every isomorphism is a retract of some identity morphism.

Remark 2.1.7. The type of category which has been introduced above was first called a closed model category by Quillen [12]. In his sense, a model category does not require to have either finite limits or finite colimits. In our case, we will explicitly state whenever a model category is non-closed, i.e. does not have every limit or colimit.

A model category $\mathcal{C}$ is now defined to be a category equipped with a particular model structure. Notice that a category may admit several model structures. We will postpone examples until sufficient theory have been developed. For more topological examples, we refer to Dwayer-Spalinski [13] and Hovey [11].

An interesting and a not so non-trivial property of model categories is that giving all three classes Ac, Cof and Fib is redundant. Given the class of weak equivalences and either cofibrations or fibrations, the model structure is determined. Thus the classes of fibrations are determined by acyclic cofibrations and cofibrations are determined by fibrations. The next two results will show this.

Lemma 2.1.8 (The retract argument)**.** *Let $\mathcal{C}$ be a category. Suppose there is a factorization $f = pi$ and that f has LLP with respect to p, then f is a retract of i. Dually, if f har RLP to i, then it is a retract of p.*

Proof. We assume that $f : A \to C$ has LLP with respect to $p : B \to C$. Then we may find a lift $r : C \to B$, which realize f as a retract of i.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "p"] \\ C \arrow[r, equal] \arrow[ur, dashed, "r"] & C \end{tikzcd} \implies \begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[r, equal] \arrow[d, "i"] & A \arrow[d, "f"] \\ C \arrow[r, "r"] & B \arrow[r, "p"] & C \end{tikzcd}$$

□

Proposition 2.1.9. *Let $\mathcal{C}$ be a model category. A morphism f is a cofibration (acyclic cofibration) if and only if f has LLP with respect acyclic fibrations (fibrations). Dually, f is a fibration (acyclic fibration) if and only if it has RLP with respect to acyclic cofibrations (cofibrations).*

Proof. Assume that f is a cofibration. By MC3, we know that f has LLP with respect to acyclic fibrations. Assume instead that f has LLP with respect to ever acyclic fibration. By MC4 we factor $f = f_\alpha \circ f_\beta$, where f_α is a cofibration and f_β is an acyclic fibration. Since we assume f to have LLP with respect to f_β, by lemma 2.1.8 we know that f is a retract of f_α. Thus by MC2, we know that f is a cofibration. □

Corollary 2.1.9.1. *Let $\mathcal{C}$ be a model category. (Acyclic) Cofibrations are stable under pushouts, i.e. if f is an (acyclic) cofibration, then f' is an (acyclic) cofibration.*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[d, "f'"] \\ B \arrow[r] & D \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

Dually, fibrations are stable under pullbacks.

Proof. This is clear by the universal property of pushouts. □

Since we assume that every model category $\mathcal{C}$ is admits finite limits and colimits, we know that it has both an initial and a terminal object. We let $\varnothing$ denote the initial object and $*$ denote the terminal object.

Definition 2.1.10 (Cofibrant, fibrant and bifibrant objects)**.** Let $\mathcal{C}$ be a model category. An object X is called cofibrant if the unique morphism $\varnothing \to X$ is a cofibration. Dually, X is called fibrant if the unique morphism $X \to *$ is fibrant. If X is both cofibrant and fibrant, we call it bifibrant.

There is no reason for every object to be either cofibrant or fibrant. However, we may see that every object is weakly equivalent to an object which is either fibrant or cofibrant. We will make it precise what it means for two objects to be weakly equivalent later.

Construction 2.1.11. Let X be an object of a model category $\mathcal{C}$. The morphism $i : \varnothing \to X$ has a functorial factorization $i = i_\beta \circ i_\alpha$, where $i_\alpha : \varnothing \to QX$ is a cofibration and $i_\beta : QX \to X$ is an acyclic fibration. By definition QX is cofibrant and weakly equivalent to X.

$Q : \mathcal{C} \to \mathcal{C}$ defines a functor called the cofibrant replacement. To see this we first look at the slice category $\varnothing/c$. The objects are morphisms $f : \varnothing \to X$ for any object X in $\mathcal{C}$, while morphisms are commutative triangles. We first observe that $\varnothing/c \subset \mathcal{C}^{\to}$ is a subcategory of the arrow category. Thus (α, β) may be interpreted as functors on the slice category to the arrow category. Moreover, since every arrow $f : \varnothing \to X$ is unique, we observe that this category is equivalent to $\mathcal{C}$. Thus (α, β) may be interpreted as functors on $\mathcal{C}$ into arrows. We define Q as the composition $Q = tar \circ \alpha$.

Dually, we get a fibrant replacement R by dualizing the above argument.

We collect the following properties

Lemma 2.1.12. *The cofibrant replacement Q and fibrant replacement R preserves weak equivalences.*

Proof. Clear from the 2-out-of-3 property. □

Lemma 2.1.13 (Ken Brown's lemma)**.** *Let $\mathcal{C}$ be a model category and $\mathcal{D}$ be a category with weak equivalences satisfying the 2-out-of-3 property. If $F : \mathcal{C} \to \mathcal{D}$ is a functor sending acyclic cofibrations between cofibrant objects to weak equivalences, then F takes all weak equivalences between cofibrant objects to weak equivalences. Dually, if F takes all acyclic fibrations between fibrant objects to weak equivalences, then F takes all weak equivalences between fibrant objects to weak equivalences.*

Proof. Suppose that A and B are cofibrant objects and that $f : A \to B$ is a weak equivalence. Using the universal property of the coproduct we define the map $(f, id_B) = p : A \coprod B \to B$. p has a functorial factorization into a cofibration and acyclic fibration, $p = p_\beta \circ p_\alpha$. We recollect the maps in the following pushout diagram:

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & B \arrow[d, "i_2"] \arrow[ddrr, bend left, "id_B"] & & \\
A \arrow[r, "i_1"] \arrow[drrr, bend right, "f"'] & A\coprod B \arrow[dr, "p_\alpha"] \arrow[ul, phantom, "\lrcorner", very near start] & & \\
& & C \arrow[dr, "p_\beta"] & \\
& & & D
\end{tikzcd}$$

By lemma 2.1.9.1 both i_1 and i_2 are cofibrations. Since f, id_B and p_β are weak equivalences, so are $p_\alpha \circ i_1$ and $p_\alpha \circ i_2$ by MC2. Moreover, they are acyclic cofibrations.

Assume that $F : \mathcal{C} \to \mathcal{D}$ is a functor as described above. Then by assumption, $F(p_\alpha \circ i_1)$ and $F(p_\alpha \circ i_2)$ are weak equivalences. Since a functor sends identity to identity, we also know that $F(id_B)$ is a weak equivelnce. Thus by the 2-out-of-3 property $F(p_\beta)$ is a weak equivalence, as $F(p_\beta) \circ F(p_\alpha \circ i_2) = id_{F(B)}$. Again, by 2-out-of-3 property $F(f)$ is a weak equivelnce, as $F(f) = F(p_\beta) \circ F(p_\alpha \circ i_1)$. □

2.1.2 Homotopy category

Homotopy theory at it's most abstract is the study of categories and functors up to weak equivalences. Here, a weak equivalence may be anything, but most commonly it is a weak equivalence in topological homotopy or a quasi-isomorphism in homological algebra. The biggest concern when dealing with such concepts is to make a functor well-defined up to these chosen weak-equivalences. To this end, there is a construction to amend these problems, known as derived functors. We define a homotopical category in the sense of Riehl [14].

Definition 2.1.14 (Homotopical Category)**.** Let $\mathcal{C}$ be a category. $\mathcal{C}$ is Homotopical if there is a wide subcategory constituting a class of morphisms known as weak equivalences, $Ac \subset Mor\mathcal{C}$. The weak equivalences should satisfy the **2-out-of-6 property**, i.e. given three composable morphisms f, g and g, if gf and hg are weak equivalences, then so are f, g, h and hgf.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "gf"'] \arrow[drr, dotted] & B \arrow[d, "g"] \arrow[dr, "hg"] & \\
& C \arrow[r, "h"] & D
\end{tikzcd}$$

Remark 2.1.15. Notice that the 2-out-of-6 property is stronger than the 2-out-of-3 property. To see this, let either f, g or h be the identity, and then conclude with the 2-out-of-3 property.

Remark 2.1.16. The collection of weak equivalences contains every isomorphism. To see this pick an isomorphism f and f^{-1}, then the compositions are the identity on the domain and codomain, which are assumed to be in Ac.

Given such a homotopical category $\mathcal{C}$, we want to invert every weak equivalence and create the homotopy category of $\mathcal{C}$. This construction is developed in Gabriel and Zisman [15] called calculus of fractions. This method essentially tries to mimic localization for commutative rings in a category theoretic fashion.

Definition 2.1.17. Let $\mathcal{C}$ be a homotopical category. It's homotopy category $Ho\mathcal{C} = \mathcal{C}[Ac^{-1}]$, together with a localization functor $q : \mathcal{C} \to Ho\mathcal{C}$. Recall that the localization are determined by the following universal property: If $F : \mathcal{C} \to \mathcal{D}$ is a functor sending weak equivalences to isomorphisms, then it uniquely factors through the homotopy category up to a unique natural isomorphism η.

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "q"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{D} \\
& Ho\mathcal{C} \arrow[ur, "F'"'] &
\end{tikzcd}$$

Definition 2.1.18. Suppose that $\mathcal{C}$ is a homotopical category. Two objects of $\mathcal{C}$ are said to be weakly equivalent if they are isomorphic in $Ho\mathcal{C}$. I.e. there is some zig-zag relation between the objects, consisting only of weak equivalences.

Remark 2.1.19. A renown problem with localizations is that even if $\mathcal{C}$ is a locally small category, any localization $\mathcal{C}[S^{-1}]$ does not need to be. Thus, without a good theory of classes or higher universes, we cannot in general ensure that the localization still exists as a locally small category.

We see from the definition of the homotopy category that a functor F admits a lift F' to the homotopy category whenever weak equivalences are sent to isomorphisms. Moreover, if we have a functor F between homotopical categories which preserves weak equivalences, it then induces a functor between the homotopy categories.

Definition 2.1.20 (Homotopical functors)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ between homotopical categories is homotopical if it preserves weak equivalences. Moreover, there is a lift of functors as in the following diagram.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "q_{\mathcal{C}}"] & \mathcal{D} \arrow[d, "q_{\mathcal{D}}"] \arrow[dl, Rightarrow, "\eta"] \\
Ho\mathcal{C} \arrow[r, dashed, "F'"] & Ho\mathcal{D}
\end{tikzcd}$$

Derived functors come into play whenever this is not the case. These lifts are however the closest approximation which we can make functorial. The general exposition of derived functors is beyond the scope of this thesis, but an account of it may be found in [14]. As we will see, model categories are a nice environment to work with these concepts. Firstly we will amend the problem with localizations, where the homotopy category may not exists. Secondly, we will obtain a simple description of some important derived functors.

Proposition 2.1.21. *Any model category $\mathcal{C}$ is a homotopical category.*

Proof. To show that a model category is homotopical it suffices to show that Ac satisfies the 2-out-of-6 property. Assume there are 3 composable morphisms f, g, h such that $gf, hg \in Ac$. By the 2-out-of-3 property for Ac it is enough to show that at least one of f, g, h, fgh is a weak equivalence to deduce that every other morphism is a weak equivalence.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "gf"'] \arrow[drr, dotted] & B \arrow[d, "g"] \arrow[dr, "hg"] & \\
& C \arrow[r, "h"] & D
\end{tikzcd}$$

To be able to use the model structure, we will first show that we may assume f, g to be cofibrant and g, h to be fibrant. We know by MC4 that f, g, gf may be factored into a cofibration composed with an acyclic fibration, e.g. $f = f_\beta f_\alpha$. Since gf is a weak equivalence, so is $(gf)_\alpha$ by the 2-out-of-3 property.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "f_\alpha"'] & & B & B \arrow[rr, "g"] \arrow[dr, "g_\alpha"'] & & C & A \arrow[rr, "gf"] \arrow[dr, "(gf)_\alpha"'] & & C \\
& B' \arrow[ur, "f_\beta"'] & & & C' \arrow[ur, "g_\beta"'] & & & C'' \arrow[ur, "(gf)_\beta"'] &
\end{tikzcd}$$

Notice that the "cofibrant approximation" of the map from A to C either goes through C' or C''. We conjoin these by taking the pullback. Since acyclic fibrations are stable over pullbacks, we get a pullback square where every morphism is an acyclic fibration. Thus the map $A \to \widetilde{C}$ is a weak equivalence by 2-out-of-3.

$$\begin{tikzcd}
A \arrow[dr, dashed] \arrow[drr, bend left, "g_\alpha"] \arrow[ddr, bend right, "(gf)_\alpha"'] & & \\
& \widetilde{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C' \arrow[d, "g_\beta"] \\
& C'' \arrow[r, "(gf)_\beta"] & C
\end{tikzcd}$$

To replace f with f_α we must lift the composition into our "new" C, which is $\widetilde{C}$. This is done by using MC3, as f_α is a cofibration and the pullback square above consists entirely of acyclic fibrations.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f_\alpha"] & \widetilde{C} \arrow[d] \\
B' \arrow[r] \arrow[ur, dashed] & C
\end{tikzcd}$$

To summarize we have the following diagram, where every squiggly arrow is a weak equivalence.

$$\begin{tikzcd}
A \arrow[r, "f_\alpha"] \arrow[dr, squiggly, "t"] & B' \arrow[r, squiggly] \arrow[d, "s"] & B \arrow[d] \arrow[dr, squiggly] & \\
& \widetilde{C} \arrow[r, squiggly] & C \arrow[r] & D
\end{tikzcd}$$

We now wish to promote the arrow $s : B' \to \widetilde{C}$ into a cofibration. We do this by factoring both s and t with $MC4$. Notice that s_β, t_β and t_α are weak equivalences.

$$\begin{tikzcd}
B' \arrow[rr, "s"] \arrow[dr, "s_\alpha"'] & & \widetilde{C} & A \arrow[rr, "t"] \arrow[dr, "t_\alpha"'] & & \widetilde{C} \\
& \widetilde{C}' \arrow[ur, "s_\beta"'] & & & \widetilde{C}'' \arrow[ur, "t_\beta"'] &
\end{tikzcd}$$

To obtain our final factorization we use RLP of s_β on t_α.

$$\begin{tikzcd}
& B' \arrow[d, "s_\alpha"] \\
A \arrow[ur, "f_\alpha"] \arrow[r] \arrow[d, "t_\alpha"] & \widetilde{C}' \arrow[d, "s_\beta"] \\
\widetilde{C}'' \arrow[r, "t_\beta"] \arrow[ur, dashed] & \widetilde{C}
\end{tikzcd}$$

We have now obtained a factorization of gf which are two cofibrations followed by an acyclic fibration, in such a manner that it is compatible with the original composition. The dual to this claim is that we may also factor hg into two fibrations which is preceeded by an acyclic cofibration. In other words, we may assume without loss of generality that f and g are cofibrations, and that g and h are fibrations.

In this case we will show that g is an isomorphism. Consider the diagram below with lifts i and j, these exists since we assume gf and hg to be weak equivalences.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "gf"] & B \arrow[r, "id_B"] \arrow[d, "g"] & B \arrow[d, "hg"] \\
C \arrow[r, "id_C"] \arrow[ur, dashed, "i"] & C \arrow[r, "h"] \arrow[ur, dashed, "j"] & D
\end{tikzcd}$$

Since the diagram is commutative, we get that $i = j$, and that g is both split mono and split epi, with i as its splitting. □

Since every model category is homotopical, it also has an associated homotopy category $Ho\mathcal{C}$. Let $\mathcal{C}_c$, $\mathcal{C}_f$ and $\mathcal{C}_{cf}$ denote the full subcategories consisting of cofibrant, fibrant and bifibrant objects respectively.

Proposition 2.1.22. *Let $\mathcal{C}$ be a model category. The following categories are equivalent:*

- $Ho\mathcal{C}$
- $Ho\mathcal{C}_c$
- $Ho\mathcal{C}_f$
- $Ho\mathcal{C}_{cf}$

Proof. We show that $Ho\mathcal{C} \simeq Ho\mathcal{C}_c$. The inclusion $i : \mathcal{C}_c \to \mathcal{C}$ clearly preserves weak equivalences, thus i is homotopical and admits a lift. Moreover, since the cofibrant replacement is also homotopical, it also has a lift.

$$\begin{tikzcd}
\mathcal{C}_c \arrow[r, "i"] \arrow[d] & \mathcal{C} \arrow[d] \\
Ho\mathcal{C}_c \arrow[r, dashed, bend right, "Ho\ i"'] & \mathcal{C} \arrow[l, dashed, bend right, "Ho\ Q"']
\end{tikzcd}$$

It is clear that $Ho\ Q$ is the quasi-inverse of $Ho\ i$. □

As of now we still don't see how model categories will fix the size issues. To do this we will develop the notion of homotopy equivalence, $\sim$. On the subcategory of bifibrant objects $\mathcal{C}_{cf}$, this homotopy equivalence will be a congruence relation. This, together with the fact that there is an equivalence of categories $Ho\mathcal{C}_{cf} \simeq {}^{\mathcal{C}_{cf}}/_{\sim}$, is enough to solve the size issues.

Definition 2.1.23 (Cylinder and path objects)**.** Let $\mathcal{C}$ be a model category. Given an object X, a cylinder object $X \wedge I$ is a factorization of the fold map $i : X \coprod X \to X$, such that p_0 is a cofibration and that p_1 is a weak equivalence.

$$\begin{tikzcd} X \coprod X \arrow[rr, "i"] \arrow[dr, "p_0"'] & & X \\ & X \wedge I \arrow[ur, "p_1"'] & \end{tikzcd}$$

Dually, a path object X^I is a factorization of the diagonal map $i : X \to X \prod X$, such that p_0 is a weak equivalence and that p_1 is a fibration.

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "p_0"'] & & X \prod X \\ & X^I \arrow[ur, "p_1"'] & \end{tikzcd}$$

Remark 2.1.24. Even though we have written $X \wedge I$ suggestively to be a functor, it is not. There may be many choices for a cylinder object. However, by using the functorial factorization from MC4, we get a canonical choice of a cylinder object, as it factors every map into a cofibration and an acyclic fibration. By letting the cylinder object be this object, we do indeed obtain a functor.

Proposition 2.1.25. *Let $\mathcal{C}$ be a model category and X an object of $\mathcal{C}$. Given two cylinder objects $X \wedge I$ and $X \wedge I'$, then they are weakly equivalent.*

Proof. It is enough to show that there is a weak equivalence from any cylinder object into one specified cylinder object. This is in fact true for the functorial cylinder object $X \wedge I$, as the final morphism p_1 is an acyclic fibration, which enables a lift which is a weak equivalence by the 2-out-of-3 property.

$$\begin{tikzcd} X \coprod X \arrow[r, "p_0"] \arrow[d, "p_0'"] & X \wedge I \arrow[d, "p_1"] \\ X \wedge I' \arrow[r, "p_1'"] \arrow[ur, dashed] & X \end{tikzcd}$$

□

Definition 2.1.26 (Homotopy equivalence)**.** Let $f, g : X \to Y$. A left homotopy between f and g is a morphism $H : X \wedge I \to Y$ such that $Hi_0 = f$ and $Hi_1 = g$. f and g are left homotopic if a left homotopy exists, and it is denoted $f \overset{l}{\sim} g$.

$$\begin{tikzcd}
X \arrow[d] \arrow[dr, dotted, "i_0"] \arrow[drr, bend left, "f"] & & \\
X \coprod X \arrow[r, "p_0"] & X \wedge I \arrow[r, "H"] & Y \\
X \arrow[u] \arrow[ur, dotted, "i_1"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

A right homotopy between f and g is a morphism $H : X \to Y^I$ such that $i_0 H = f$ and $i_1 H = g$. f and g are right homotopic if a right homotopy exists, and it is denoted $f \overset{r}{\sim} g$.

$$\begin{tikzcd}
& & Y \\
X \arrow[r, "H"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & Y^I \arrow[r, "p_1"] \arrow[ur, dotted, "i_0"] \arrow[dr, dotted, "i_1"'] & Y \prod Y \arrow[u] \arrow[d] \\
& & Y
\end{tikzcd}$$

f and g are said to be homotopic if they are both left and right homotopic, it is denoted $f \sim g$. f is said to be a homotopy equivalence, if it has a homotopy inverse $h : Y \to X$, such that $hf \sim id_X$ and $fh \sim id_Y$.

It is important to know that this is not a priori an equivalence relation. This is amended by taking both fibrant and cofibrant replacements. We see this in the following proposition.

Proposition 2.1.27. *Let $\mathcal{C}$ be a model category, and $f, g : X \to Y$ be morphisms. We have the following:*

1. *If $f \overset{l}{\sim}$ and $h : Y \to Z$, then $hf \overset{l}{\sim} hg$.*
2. *If Y is fibrant, $f \overset{l}{\sim} g$ and $h : W \to X$, then $fh \overset{l}{\sim} gh$.*
3. *If X is cofibrant, then left homotopy is an equivalence relation on $\mathcal{C}(X, Y)$.*
4. *If X is cofibrant and $f \overset{l}{\sim} g$, then $f \overset{r}{\sim} g$.*

Proof. $(1.)$ Assume that $f \overset{l}{\sim} g$ and $h : Y \to Z$. Let $H : X \wedge I \to Y$ denote the left homotopy between f and g. The left homotopy between hf and hg is given as hH.

$(2.)$ Assume that Y is fibrant, $f \overset{l}{\sim} g$ and that $h : W \to X$. Let $H : X \wedge I \to Y$ be a left homotopy. We construct a new cylinder object for the homotopy. Factor $p_1 : X \wedge I \to X$ as $q_1 \circ q_0$ where $q_0 : X \wedge I \to X \wedge I'$ is an acyclic cofibration and $q_1 : X \wedge I' \to X$ is a fibration. By the 2-out-of-3

property q_1 is an acyclic fibration, as p_1 and q_0 are weak equivalences. $X \wedge I'$ is a cylinder object as $q_0 \circ p_0$ is a cofibration and q_1 is a weak equivalence. Since we assume Y to be fibrant we lift the left homotopy $H : X \wedge I \to Y$ to the left homotopy $H' : X \wedge I' \to Y$ with the following diagram:

$$\begin{tikzcd} X \wedge I \arrow[r, "H"] \arrow[d, "q_0"] & Y \arrow[d] \\ X \wedge I' \arrow[r] \arrow[ur, dashed, "H'"] & * \end{tikzcd}$$

We can find the appropriate homotopy needed with lift given by the following diagram:

$$\begin{tikzcd} W \coprod W \arrow[r, "q_0 p_0 (h \coprod h)"] \arrow[d, "p_0'"] & X \wedge I' \arrow[d, "q_1"] \\ W \wedge I \arrow[r, "h p_1'"] \arrow[ur, dashed, "k"] & X \end{tikzcd}$$

Then the morphism $H'k$ is the desired left homotopy witnessing $fh \overset{l}{\sim} gh$.

$(3.)$ Assume that X is cofibrant. First observe that a left homotopy is reflexive and symmetric. We must show that in this case it is also transitative. Thus, assume that $f, g, h : X \to Y$ and that $H : X \wedge I \to Y$ is a left homotopy witnessing $f \overset{l}{\sim} g$ and that $H' : X \wedge I' \to Y$ is a left homotopy witnessing $g \overset{l}{\sim} h$. We first observe that $i_0 : X \to X \wedge I$ is a weak equivalence, as $id_X = p_1 i_0$ where id_X and p_1 are weak equivalences. Since X is assumed to be cofibrant, we see that $X \coprod X$ is cofibrant by the following pushout:

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X \arrow[d, "inr"] \\ X \arrow[r, "inl"] & X \coprod X \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

Moreover, both inl and inr are cofibrations. This shows that i_0 is a cofibration as $i_0 = p_0 \circ inr$ is a composition of two cofibrations. i_0 is thus an acyclic cofibration. We define an almost cylinder object C by the pushout of i_1 and i_0'. We define the maps t and H'' by using the universal property in the following manner:

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[d, "i_0'"] & X \wedge I \arrow[d] \arrow[ddr, bend left, "p_1"] & \\ X \wedge I' \arrow[r] \arrow[drr, bend right, "p_1'"] & C \arrow[dr, dashed, "t"] & \\ & & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "i_1"] \arrow[d, "i_0'"] & X \wedge I \arrow[d] \arrow[ddr, bend left, "H"] & \\ X \wedge I' \arrow[r] \arrow[drr, bend right, "H'"] & C \arrow[dr, dashed, "H''"] & \\ & & X \end{tikzcd}$$

Observe that there is a factorization of the fold map $X \coprod X \xrightarrow{s} C \xrightarrow{t} X$. However, s may not be a cofibration, so we replace C with the cylinder object $X \wedge I''$ such that we have the factorization $X \coprod X \xrightarrow{s_\alpha} X \wedge I'' \xrightarrow{ts_\beta} X$. The morphism Hs_β is then our required homotopy for $f \overset{l}{\sim} g$.

$(4.)$ Suppose that X is cofibrant and that $H : X \wedge I \to Y$ is a left homotopy for $f \overset{l}{\sim} g$. Pick a path object for Y, such that we have the factorization $Y \xrightarrow{q_0} Y^I \xrightarrow{q_1} Y \prod Y$ where q_0 is a weak equivalence and q_1 is a fibration. Again, as X is cofibrant we get that i_0 is an acyclic cofibration, so we have the following lift of the homotopy:

$$\begin{tikzcd} X \arrow[r, "q_0 f"] \arrow[d, "i_0"] & Y^I \arrow[d, "q_1"] \\ X \wedge I \arrow[r, "(fp_1, H)"] \arrow[ur, dashed, "J"] & Y \prod Y \end{tikzcd}$$

The right homotopy is given by injecting away from f, i.e. $H' = Ji_1$. □

Corollary 2.1.27.1. *We collect the dual results of the above proposition, and thus have the following.*

1. *If $f \overset{r}{\sim}$ and $h : W \to X$, then $fh \overset{r}{\sim} gh$.*
2. *If X is cofibrant, $f \overset{r}{\sim} g$ and $h : Y \to Z$, then $hf \overset{r}{\sim} hg$.*
3. *If Y is fibrant, then left homotopy is an equivalence relation on $\mathcal{C}(X, Y)$.*
4. *If Y is fibrant and $f \overset{r}{\sim} g$, then $f \overset{l}{\sim} g$.*

Corollary 2.1.27.2. *Homotopy is a congruence relation on $\mathcal{C}_{cf}$. In this manner, the category $\mathcal{C}_{cf}/\sim$ is well-defined, exists and inverts every homotopy equivalence.*

Lemma 2.1.28 (Weird Whitehead)**.** *Let $\mathcal{C}$ be a model category. Suppose that C is cofibrant and $h : X \to Y$ is an acyclic fibration or a weak equivalence between fibrant objects, then h induces an isomorphism:*

$$\mathcal{C}(C, X)/\overset{l}{\sim} \xrightarrow[\simeq]{h_*} \mathcal{C}(C, Y)/\overset{l}{\sim}$$

Dually, if X is fibrant and $h : C \to D$ is an acyclic cofibration or a weak equivalence between cofibrant objects, then h induces an isomorphism:

$$\mathcal{C}(D,X)/\overset{r}{\sim} \xrightarrow[\simeq]{h^*} \mathcal{C}(C,X)/\overset{r}{\sim}$$

Proof. We assume $\mathcal{C}$ to be cofibrant and $h : X \to Y$ to ba an acyclic fibration. We first prove that h is surjective. Let $f : C \to Y$. By RLP of h there is a morphism $f' : C \to X$ such that $f = hf'$.

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & X \arrow[d, "h"] \\
C \arrow[r, "f"'] \arrow[ur, dashed, "f'"] & Y
\end{tikzcd}$$

To show injectivity we assume $f, g : C \to X$ such that $hf \overset{l}{\sim} hg$, in particular there is a left homotopy $H : C \wedge I \to Y$. Remember that since C is cofibrant, the map p_0 is a cofibration. We find a left homotopy $H : C \wedge I \to X$ witnessing $f \overset{l}{\sim} g$ by the following lift.

$$\begin{tikzcd}
C \coprod C \arrow[r, "f+g"] \arrow[d, "p_0"] & X \arrow[d, "h"] \\
C \wedge I \arrow[r, "H"'] \arrow[ur, dashed, "H'"] & Y
\end{tikzcd}$$

Moreover, if we assume both X and Y to be fibrant, the functor $\mathcal{C}(C,\_)/\overset{l}{\sim}$ sends acyclic fibrations to isomorphisms, i.e. to weak equivalences. By Ken Brown's lemma, lemma 2.1.13, the afformentioned functor sends weak equivalences between fibrant objects to isomorphisms. $\square$

Theorem 2.1.29 (Generalized Whiteheads theorem)**.** *Let $\mathcal{C}$ be a model category. Suppose that $f : X \to Y$ is a morphism of bifibrant objects, then f is a weak equivalence if and only if f is a homotopy equivalence.*

Proof. Suppose first that f is a weak equivalence. Pick a bifibrant object A, then by lemma 2.1.28 $f_* : \mathcal{C}(A,X)/\sim \to \mathcal{C}(A,Y)/\sim$ is an isomorphism. Letting $A = Y$ we know that there is a morphism $g : Y \to X$, such that $f_* g = fg \sim id_Y$. Furthermore, by proposition 2.1.27, since X is bifibrant composing on the right preserves homotopy equivalence, e.g. $fgf \sim f$. By letting $A = X$, we get that $f_* gf = fgf \sim f = f_* id_X$, thus $gf \sim id_X$.

For the opposite direction, assume that f is a homotopy equivalence. We factor f into an acyclic cofibration f_γ and a fibration f_δ, i.e. $X \xrightarrow{f_\gamma} Z \xrightarrow{f_\delta} Y$. Observe that Z is bifibrant as X and Y is, in particular, f_γ is a weak equivalence of bifibrant objects, so it is a homotopy equivalence.

It is enough to show that f_δ is a weak equivalence. Let g be the homotopy inverse of f, and $H : Y \wedge I \to Y$ is a left homotopy witnessing $fg \sim id_Y$. Since Y is bifibrant, the following square has a lift.

$$\begin{tikzcd} Y \arrow[r, "f_\gamma g"] \arrow[d, "i_0"'] & Z \arrow[d, "f_\delta"] \\ Y \wedge I \arrow[r, "H"] \arrow[ur, dashed, "H'"] & Y \end{tikzcd}$$

Let $h = H'i_1$, then by definition we know that $f_\delta H' i_1 = id_Y$. Moreover, H is a left homotopy witnessing $f_\gamma g \sim h$. Let $g' : Z \to X$ be the homotopy inverse of f_γ. We have the following relations $f_\delta \sim f_\delta f_\gamma g' \sim fg'$, and $hf_\delta \sim (f_\gamma g)(fg') \sim f_\gamma g' \sim id_Z$. Let $H'' : Z \wedge I \to Z$ be a left homotopy witnessing this homotopy. Since Z is bifibrant, i_0 and i_1 are weak equivalences. By the 2-out-of-3 property H'' and hf_δ are weak equivalences. Since $f_\delta h = id_Y$, it follows that f_δ is a retract of $f_\delta h$, and is thus a weak equivalence. □

Corollary 2.1.29.1. *The category ${}^{\mathcal{C}_{cf}}/_{\sim}$ satisfy the universal property of the localization of $\mathcal{C}_{cf}$ by the weak equivalences. I.e. there is a categorical equivalence $Ho\mathcal{C}_{cf} \simeq {}^{\mathcal{C}_{cf}}/_{\sim}$.*

Proof. By generalized Whiteheads theorem, theorem 2.1.29 weak equivalences and homotopy equivalences coincide. The corollary follows steadily from both the universal property of the localization category and the quotient category. □

We collect the results from above in the following theorem.

Theorem 2.1.30 (Fundamental theorem of model categories)**.** *Let $\mathcal{C}$ be a model category and denote $q : \mathcal{C} \to Ho\mathcal{C}$ the localization functor. Let X and Y be objects of $\mathcal{C}$.*

1. *There is an equivalence of categories $Ho\mathcal{C} \simeq {}^{\mathcal{C}_{cf}}/_{\sim}$.*
2. *There are natural isomorphisms ${}^{\mathcal{C}_{cf}}/_{\sim}(QRX, QRY) \simeq Ho\mathcal{C}(X, Y) \simeq {}^{\mathcal{C}_{cf}}/_{\sim}(RQX, RQY)$. Additionally, $Ho\mathcal{C}(X, Y) \simeq {}^{\mathcal{C}_{cf}}/_{\sim}(QX, RY)$.*
3. *The localization q identifies left or right homotopic morphisms.*
4. *A morphism $f : X \to Y$ is a weak equivalence if and only if qf is an isomorphism.*

Proof. This is clear by the results above. □

2.1.3 Quillen adjoints

We now want to study morphisms, or certain functors, between model categories. Like in the case of homotopical functors we want these morphisms to induce a functor between the homotopy categories. However, we also want them to respect the cofibration and fibration structure, not just weak equivalences. In this way we will instead look towards derived functors to be able to define this extension to the homotopy category. We recall the definition of a total (left/right) derived functor. In the case of model categories, we get a simple description for some of these derived functors which are of special interest.

Definition 2.1.31 (Total derived functors)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be homotopical categories, and $F : \mathcal{C} \to \mathcal{D}$ a functor. Whenever it exists, a total left derived functor of F, is a functor $\mathbb{L}F : Ho\mathcal{C} \to Ho\mathcal{D}$ with a natural transformation $\varepsilon : \mathbb{L}F \circ q \Rightarrow q \circ F$ satisfying the universal property: If $G : Ho\mathcal{C} \to Ho\mathcal{D}$ is a functor and there is a natural transformation $\alpha : G \circ q \Rightarrow q \circ F$, then it factors uniquely up to unique isomorphism through ε.

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "q"] & \mathcal{D} \arrow[d, "q"] \\
Ho\mathcal{C} \arrow[r, dashed, "\mathbb{L}F"'] \arrow[ur, phantom, "\overset{\varepsilon}{\Nearrow}"] & Ho\mathcal{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "q"] & \mathcal{D} \arrow[d, "q"] \\
Ho\mathcal{C} \arrow[r, bend left, dashed, "\mathbb{L}F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\exists!\Uparrow"] \arrow[ur, phantom, "\overset{\varepsilon}{\Nearrow}"] & Ho\mathcal{D}
\end{tikzcd}
$$

Dually, whenever it exists, a total right derived functor of F, is a functor $\mathbb{R}F : Ho\mathcal{C} \to Ho\mathcal{D}$ with a natural transformation $\eta : q \circ F \Rightarrow \mathbb{R}F \circ q$ having the opposite universal property.

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "q"] & \mathcal{D} \arrow[d, "q"] \\
Ho\mathcal{C} \arrow[r, dashed, "\mathbb{R}F"'] \arrow[ur, phantom, "\underset{\eta}{\Swarrow}"] & Ho\mathcal{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "q"] & \mathcal{D} \arrow[d, "q"] \\
Ho\mathcal{C} \arrow[r, bend left, dashed, "\mathbb{R}F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\exists!\Downarrow"] \arrow[ur, phantom, "\overset{\eta}{\Swarrow}"] & Ho\mathcal{D}
\end{tikzcd}
$$

Definition 2.1.32 (Deformation)**.** A left (right) deformation on a homotopical category $\mathcal{C}$ is an endofunctor Q together with a natural weak equivalence $q : Q \Rightarrow Id_{\mathcal{C}}$ ($q : Id_{\mathcal{C}} \Rightarrow Q$).

A left (right) deformation on a functor $F : \mathcal{C} \to \mathcal{D}$ between homotopical categories, is a left (right) deformation Q on $\mathcal{C}$ such that weak equivalences in the image of Q is preserved by F.

Remark 2.1.33 (Cofibrant and fibrant replacement). If $\mathcal{C}$ is a model category, then we have a left and a right deformation. The cofibrant replacement Q defines a left deformation, and the fibrant replacement defines a right deformation. Notice that this is only due to the fact that the factorization system is functorial.

Proposition 2.1.34. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between homotopical categories. If F has a left deformation Q, then the total left derived functor $\mathbb{L}F$ exists. Moreover, the functor FQ is homotopical, and $\mathbb{L}F$ is the unique extension of FQ.*

Proof. Since we already have a candidate for the derived functor, the proof must just check that it has the universal property. A proof may be found in Reihl [14] under proposition 6.4.11. $\square$

Remark 2.1.35. There is a somewhat weaker statement by Dwayer and Spalinski [13]. If we instead ask for functors F which have the cofibrant replacement Q (fibrant replacement R) as a left (right) deformation we may make this proof more explicit. This is theorem 9.3.

Equipped with the above proposition and remark, it makes sense to define Quillen functors as left and right Quillen functors. A left Quillen functor should be left deformable by the cofibrant replacement. Moreover, for the composition of two left Quillen functors to make sense, we also need weak equivalences between cofibrant objects to be mapped to weak equivalences between cofibrant objects. We make the following definition.

Definition 2.1.36 (Quillen adjunction)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be model categories.

1. A left Quillen functor is a functor $F : \mathcal{C} \to \mathcal{D}$ such that it preserves cofibrations and acyclic cofibrations.
2. A right Quillen functor is a fucntor $F : \mathcal{C} \to \mathcal{D}$ such that it preserves fibrations and acyclic fibrations.
3. Suppose that (F, U) is an adjunction where $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to U. (F, U) is called a Quillen adjunction if F is a left Quillen functor and U is a right Quillen functor.

Remark 2.1.37. By Ken Browns lemma, lemma 2.1.13, we see that a left Quillen functor F is left deformable to the cofibrant replacement functor Q. Thus the total left derived functor exists and is given by $\mathbb{L}F = HoFQ$.

In order to eliminate the choice of left or right derivedness, we will think of a morphism of model categories as a Quillen adjunction. The direction of the arrow can be chosen to be along either the left or right adjoints, we make the convention of following the left adjoint functors. We summarize the following properties.

Lemma 2.1.38. *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories, and suppose there is an adjunction $F : \mathcal{C} \rightleftharpoons \mathcal{D} : U$. The following are equivalent:*

1. *(F, U) is a Quillen adjunction.*
2. *F is a left Quillen functor.*
3. *U is a right Quillen functor.*

Proof. This follows from naturality of the adjunction. I.e. any square in $\mathcal{C}$, with the right side from $\mathcal{D}$ is commutative if and only if any square in $\mathcal{D}$ with the left side from $\mathcal{C}$ is commutative. Now, f has LLP with respect to Ug if and only if Ff has LLP with respect to g.

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "f"'] & UX \arrow[d, "Ug"] \\ B \arrow[r, "l"] \arrow[ur, dotted, "h"] & UY \end{tikzcd} \rightsquigarrow \begin{tikzcd} FA \arrow[r, "k^T"] \arrow[d, "Ff"'] & X \arrow[d, "g"] \\ FB \arrow[r, "l^T"] \arrow[ur, dotted, "h^T"] & Y \end{tikzcd}$$

$\square$

Proposition 2.1.39. *Suppose that $(F, U) : \mathcal{C} \to \mathcal{D}$ is a Quillen adjunction. The functors $\mathbb{L}F : Ho\mathcal{C} \to Ho\mathcal{D}$ and $\mathbb{R}U : Ho\mathcal{D} \to Ho\mathcal{C}$ forms an adjoint pair.*

Proof. We must show that $Ho\mathcal{D}(\mathbb{L}FX, Y) \simeq Ho\mathcal{D}(X, \mathbb{R}UY)$. By using the fundamental theorem of model categories, theorem 2.1.30, we have the following isomorphisms: $Ho\mathcal{D}(\mathbb{L}FX, Y) \simeq {}^{\mathcal{C}(FQX, RY)}/\sim$ and $Ho\mathcal{D}(X, \mathbb{R}UY) \simeq {}^{\mathcal{D}(QX, URY)}/\sim$. In other words, if we assume X to be cofibrant, and Y to be fibrant, we must show that the adjunction preserves homotopy equivalences.

We show it for one direction. Suppose that the morphisms $f, g : FA \to B$ are homotopic, witnessed by a right homotopy $H : FA \to B^I$. Since we assume U to preserve products, fibrations and weak equivalences between fibrant objects, $U(B^I)$ is a path object for UB. Thus the transpose $H^T : A \to U(B^I)$ is the desired homotopy witnessing $f^T \sim g^T$ □

Definition 2.1.40 (Quillen equivalence)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be model categories, and $(F, U) : \mathcal{C} \to \mathcal{D}$ be a Quillen adjunction. (F, U) is called a Quillen equivalence if for any cofibrant X in $\mathcal{C}$, fibrant Y in $\mathcal{D}$ and any morphism $f : FX \to Y$ is a weak equivalence if and only if its transpose $f^T : X \to UY$ is a weak equivalence.

Proposition 2.1.41. *Suppose that $(F, U) : \mathcal{C} \to \mathcal{D}$ is a Quillen adjunction. The following are equivalent:*

1. *(F, U) is a Quillen equivalence.*
2. *Let $\eta : Id_{\mathcal{C}} \Rightarrow UF$ denote the unit, and $\varepsilon : FU \Rightarrow Id_{\mathcal{D}}$ denote the counit. The composite $Ur_F\eta : Id_{\mathcal{C}_c} \Rightarrow URF|_{\mathcal{C}_c}$, and $\varepsilon_{FQU}Fq_U : FQU|_{\mathcal{D}_f} \Rightarrow Id_{\mathcal{D}_f}$ are natural weak equivalences.*
3. *The derived adjunction $(\mathbb{L}F, \mathbb{R}U)$ is an equivalence of categories.*

Proof. Firstly observe that $2. \implies 3.$ by definition. Secondly observe that equivalences both preserves and reflects isomorphisms, from this we get $3. \implies 1.$. We now show $1. \implies 2.$. Pick X in $\mathcal{C}$ such that X is cofibrant. Since (F, U) is assumed to be a Quillen adjunction we know that FX is still cofibrant. The fibrant replacement $r_{FX} : FX \to RFX$ gives us a weak equivalence. Furthermore, since (F, U) is assumed to be a Quillen equivalence, its transpose $r_{FX}^T : X \to URFX$ is a weak equivalence. Unwinding the definition of the transpose we get that $r_{FX}^T = Ur_{rFX}\eta_X$.

□

We have the following refinement.

Corollary 2.1.41.1. *Suppose that $(F, U) : \mathcal{C} \to \mathcal{D}$ is a Quillen adjunction. The following are equivalent:*

1. *(F, U) is a Quillen equivalence.*
2. *F reflects weak equivalences between cofibrant objects, and $\varepsilon_{FQU}F_{qU} : FQU|_{\mathcal{D}_f} \Rightarrow Id_{\mathcal{D}_f}$ is a natural weak equivalence.*
3. *U reflects weak equivalences between fibrant objects, and $U_{rF}\eta : Id_{\mathcal{C}_c} \Rightarrow URF|\mathcal{C}_c$ is a natural weak equivalence.*

Proof. We start by showing $1. \implies 2.$ and $3.$. We already know that the derived unit and counit are isomorphism in homotopy, so we only need to show that F (U) reflects weak equivalences between cofibrant (fibrant) objects. Suppose that $Ff : FX \to FY$ is a weak equivalence between cofibrant objects. Since F preserves weak equivalences between cofibrant objects, we get that FQf is a weak equivalence, or that $\mathbb{L}Ff$ is an isomorphism. By assumption, $\mathbb{L}F$ is an equivalence of categories, so f is a weak equivalence as needed.

We will show $2. \implies 1.$, the case $3. \implies 1.$ is dual. We assume that the counit map is an isomorphism in homotopy. By assumption, the derived unit $\mathbb{L}\eta$ is split-mono on the image of $\mathbb{L}F$. Moreover, the derived counit $\mathbb{R}\varepsilon$ is assumed to be an isomorphism, in particular the derived unit $\mathbb{L}F\mathbb{L}\eta$ is an isomorphism. Unpacking this, we have a morphism, call it $\eta'_X : FQX \to FQURFQX$, which is a weak equivalence. Since F and Q reflects weak equivalences, we get that $\eta_X : X \to URFQX$ is a weak equivalence. □

2.2 Model structures on Algebraic Categories

In order to understand ∞-quasi-isomorphism of strongly homotopy associative algebras we will study different homotopy theories of various categories. Munkholm [16] successfully showed that the derived category of augmented algebras is equivalent to the derived category of augmented algebras equipped with ∞-morphisms. Well, to be more precise, he showed that certain subcategories of augmented algebras had this property. Lefevre-Hasagawas phd thesis [10] builds upon this identification, but with help of further devolpment within the field. We will follow the approach of Lefevre-Hasagawa, by comparing the model structure for algebras and coalgebras,

2.2.1 DG-Algebras as a Model Category

Bousefield and Gugenheim [17] proved that the category of commutative dg-algebras had a model structure whenever the base field was a field of characteristic 0. In a joint project, Jardines paper from 1997 [18] shows that this construction may be extended to dg-algebras over any commutative ring. On the other hand, Munkholm expanded on the ideas from Bousfield and Gugenheim to get an identification of derived categories. Also, Hinichs paper from 1997 [19] details another method to obtain the model category which we want. We will follow the approach of Hinich, as it will be usefull later on. Notice that where Hinich use theory of algebraic operads to show that the category of algebras is a model category, we will however give a more explicit formulation.

Let $\mathbb{K}$ be a field, and $\mathcal{C}$ be a category such that there is an adjunction $F : \mathcal{C} \rightleftharpoons Ch(\mathbb{K}) : \#$, where F is left adjoint to #. Furthermore, suppose that $\mathcal{C}$ satsifies the 2 conditions:

(H0) $\mathcal{C}$ admits finite limits and every small colimit. The functor # commutes with filtered colimits.
(H1) Let M be the complex below, concentrated in 0 and 1.

$$\ldots \longrightarrow 0 \longrightarrow \mathbb{K} \xrightarrow{id} \mathbb{K} \longrightarrow 0 \longrightarrow \ldots$$

For any $d \in \mathbb{Z}$ and for any $A \in \mathcal{C}$ the injection $A \to A \coprod F(M[d])$ induces a quasi-isomorphism $A^\# \to (A \coprod F(M[d]))^\#$.

With this adjunction in mind, we define weak equivalences, fibrations and cofibrations as follows: Let $f \in \mathcal{C}$ be a morphism

- $f \in Ac$ if $f^\#$ is a quasi-isomorphism.
- $f \in Fib$ if $f^\#$ is surjective on each component.
- $f \in Cof$ if f has LLP to acyclic fibrations.

Theorem 2.2.1. *The category $\mathcal{C}$ equipped with the weak equivalences, fibrations and cofibrations as defined above is a model category.*

Before we show this theorem we need to understand the cofibrations better. Let $A \in \mathcal{C}$, $M \in Ch(\mathbb{K})$ and $\alpha : M \to A^\#$ a morphism in $Ch(\mathbb{K})$. We define a functor

$$h_{A,\alpha}(B) = \{(f,t) \mid f \in \mathcal{C}(A,B), t \in Hom_{\mathbb{K}}^{-1}(M, B^\#) \ s.t. \ \partial t = f^\# \circ \alpha\}.$$

Note that t is not a morphism of chain maps. This is a homogenous morphism of degree -1. The differential then promotes this morphism to a chain map, and t is thus a homotopy for the comoposite $f^\# \circ \alpha$.

This functor is represented by an object of $\mathcal{C}$. We define this representing object $A\langle M, \alpha\rangle$ as the pushout:

$$\begin{tikzcd}
F(A^\#) \arrow[r, "\varepsilon_A"] \arrow[d] & A \arrow[d, "a"] \\
F(cone(\alpha)) \arrow[r, "e"] & A\langle M, \alpha\rangle \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

Let $i : M[1] \to cone(\alpha)$ be a homogenous morphism which is the injection when considered as graded modules. Notice that we have a pair of morphisms $(a, e^T i) \in h_{A,\alpha}(A\langle M, \alpha\rangle)$.

Proposition 2.2.2. *The functor $h_{A,\alpha}$ is represented by $A\langle M, \alpha\rangle$, i.e. $h_{A,\alpha} \simeq \mathcal{C}(A\langle M, \alpha\rangle, \_)$ is a natural isomorphism. Moreover, the pair $(a, e^T i)$ is the universal element of the functor $h_{A,\alpha}$, i.e. the natural isomorphism is induced by this element under Yoneda's lemma.*

Proof. Let $(f,t) \in h_{A,\alpha}(B)$ for some $B \in \mathcal{C}$. The condition that $\partial t = f^\# \alpha$ is equivalent to say that $f^\#$ extends to a morphism $f' : cone(\alpha) \to B^\#$ along t, i.e. $f' = \begin{pmatrix} f^\# & t \end{pmatrix}$. This concludes the isomorphism part, as being an element (f,t) is equivalent to the existence of the diagram below, where $\widetilde{f}$ is uniquely determined.

$$\begin{tikzcd}
F(A^{\#}) \arrow[r, "\varepsilon_A"] \arrow[d] & A \arrow[d, "a"] \arrow[ddr, bend left, "f"] & \\
F(cone(\alpha)) \arrow[r, "e"] \arrow[drr, bend right, "f'"] & A\langle M, \alpha \rangle \arrow[dr, dashed, "\widetilde{f}"] & \\
& & B
\end{tikzcd}$$

To obtain naturality, we use the adjunction to observe that the element $(a, e^T i)$ is in fact universal. $\square$

We are now in a position to explicitly find some important cofibrations. We collect these morphisms into the "standard" cofibrations.

Definition 2.2.3. Let $f : A \to B$ be a morphism in $\mathcal{C}$. Suppose that f factors as a transfinite composition of morphisms on the form $A_i \to A_i\langle M, \alpha \rangle$, i.e. f factors into the diagram below, where $A_{i+1} = A_i\langle M, \alpha \rangle$.

$$A \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \dots \longrightarrow B$$

- If every such M is a complex consisting of free $\mathbb{K}$-modules and has a 0-differential, we call f a standard cofibration.
- If every such M is a contractible complex (and $\alpha = 0$), we call f a standard acyclic cofibration.

Proposition 2.2.4. *Every standard cofibration is a cofibration, and every standard acyclic cofibration has LLP with respect to fibrations. Moreover, if $\alpha = 0$, then every standard acyclic cofibration is also a weak equivalence. We will see that these morphisms in some sense generate every (acyclic) cofibration.*

Proof. First observe that every standard cofibration may be made iteratively from the chain complexes $\mathbb{K}[n]$, and likewise, every standard acyclic cofibration may be made iteratively from M as in $H1$.

We first prove that if $M \simeq \mathbb{K}[n]$, and $\alpha : M \to A^{\#}$ is any map, then the map $A \to A\langle M, \alpha \rangle$ is a cofibration. This amounts to show that it has LLP to every acyclic fibration. Suppose that $h : B \to C$ is an acyclic fibration and that there is a commutative square as below.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[d, "h"] \\
A\langle M, \alpha \rangle \arrow[r, "g"] & C
\end{tikzcd}$$

By the universal property of $h_{A,\alpha}$ 2.2.2 it suffices to find a pair (f', t') such that $f : A \to B$, $t' : M \to B^{\#}$ is homogenous of degree -1, $\partial t = f^{\#}\alpha$ and that h induces a morphism $h : (f', t') \to g$. We see that we are forced to choose $f' = f$ as $hf = ga$. We know there exists a $t : M \to C^{\#}$ such that $\partial t = g^{\#}a^{\#}\alpha = h^{\#}f^{\#}\alpha$. Since h is an acyclic fibration $h^{\#}$ is a surjective quasi-isomorphism. Since $M \simeq \mathbb{K}[n]$, the morphism t is really an element of $C^{\#^{n-1}}$. By surjectivity of $h^{\#}$ there is an element u of $B^{\#^{n-1}}$ such that $h^{\#}(u) = t$. Moreover, the difference $h^{\#}(\partial u - f^{\#}\alpha) = 0$, so $\partial u - f^{\#}\alpha$ factors through the kernel $Kerh^{\#}$, which is acyclic. This element is furthermore a cycle, so by acyclicity there is another element u' such that $\partial u' = \partial u - f^{\#}\alpha$. We may now see that $(f, u - u')$ is our desired factorization.

Secondly we prove that if M is in as $H1$ and $\alpha = 0$, then the map $A \to A\langle M, \alpha\rangle$ is an acyclic cofibration.This amounts to show that it has LLP to every fibration. Suppose that $h : B \to C$ is a fibration and that there is a commutative square as below.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle h} \\ A\langle M, \alpha\rangle & \xrightarrow{g} & C \end{array}$$

We will again use 2.2.2, so it suffices to find a (t') such that $\partial t' = f^{\#}\alpha$. Let $t : M \to C^{\#}$ such that $\partial t = h^{\#}f^{\#}\alpha$.

Finally we will assume that $\alpha = 0$. In this case the cone is a direct sum $cone(\alpha) = A^{\#} \oplus M$. Since F is left adjoint to #, we know that $Fcone(\alpha) \simeq F(A^{\#}) \coprod FM$. By $H1$ we get that the map $A \to A\langle M, \alpha\rangle \simeq A \coprod FM$ is a weak equivalence.

□

In light of the above proposition we would like to make some more convenient notation. If $M \simeq \mathbb{K}[n]$ and $\alpha : M \to Z^n(A^{\#})$, s.t. $\alpha(1) = a$, we write $A\langle M, \alpha\rangle$ as $A\langle T; dT = a\rangle$ instead. Hinich calls this "adding a variable to kill a cycle". If M is the acyclic complex as below and $\alpha = 0$, we write $A\langle T, S; dT = S\rangle$. This could be thought of "adding a variable and a cycle to kill itself".

$$... \longrightarrow 0 \longrightarrow \mathbb{K} \xrightarrow{id} \mathbb{K} \longrightarrow 0 \longrightarrow ...$$

proof of theorem. **MC1** and **MC2** are satisfied. By definition we also have the first part of **MC3**. We start by checking **MC4**.

Let $f : A \to B$ be a morphism in $\mathcal{C}$. Given any $b \in B^{\#}$, let $C_b = A\langle T_b, S_b; dT_b = S_b\rangle$. We define $g_b : C_b \to B$ by the conditions that it acts on A as f, $g_b^{\#}(T_b) = b$ and $g_b^{\#}(S_b) = db$. Iterating this construction for every $b \in B$, we obtain an object C, such that the injection $A \to C$ is an acyclic standard cofibration, and the map $g : C \to B$ is a fibration. This gives us a factorization $f = f_\delta \circ f_\gamma$, where f_γ is the injection and $f_\delta = g$.

To obtain the other factorization we want to make a standard cofibration. We already know that the map $A \to C$ is a standard cofibration, so let $C_0 = C$. From here on, we will make each C_i inductively, such that $\varinjlim C_i$ has the factorization property which we desire. Notice that from C_0 there is a morphism $g_0 : C_0 \to B$, which is surjective and surjective on every kernel. This morphism may fail to be a quasi-isomorphism, so it is not an acyclic fibration.

To construct C_1 we assign to every pair of elements (c, b), such that $c \in ZC_0^\#$ and $g_0^\#(c) = db$, a variable to kill a cycle. If (c, b) is such a pair, then we add a variable T such that $dT = c$ and $g_1^\#(T) = b$. C_1 is then the complex where each cycle c has been killed by adding a variable T. Now if we suppose that we have constructed C_i, then C_{i+1} is constructed in the same manner, by adding a variable to kill each cycle which is a boundary in the image.

When adding a variable, we have also updated the morphism g_i by letting $g_{i+1}^\#(T) = b$. Thus in each step we have also made a new morphism g_{i+1}. If g denote the morphism at the colimit, then it is clear that this is still a fibration and it has also become a quasi-isomorphism. This is because every cycle which failed to be in the homology of B have been killed.

It remains to check the last part of **CM3**. Suppose that $f : A \to B$ is an acyclic cofibration. By **CM4**, we know that it factors as $f = f_\delta \circ f_\gamma$, where f_δ is an acyclic fibration and f_γ is a standard acyclic fibration. We thus obtain that f is a retract of f_γ by the commutative diagram below.

$$\begin{tikzcd} A \arrow[r, "f_\gamma"] \arrow[d, "f"] & C \arrow[d, "f_\delta"] \\ B \arrow[r, equal] \arrow[ur, dashed] & B \end{tikzcd}$$

□

The following corollary will concretize what it means that the standard cofibrations generate every cofibration. This corollary is really a step used within in the proof.

Corollary 2.2.4.1. *Any (acyclic) cofibration is a retract of a standard (acyclic) cofibration.*

We may immedietly apply this theorem to some familiar examples.

Corollary 2.2.4.2. *Let A be a dg-algebra over the field $\mathbb{K}$. The category mod_A of left modules is a model category.*

sketch of proof. We establish the adjunction by letting $F = A \otimes_{\mathbb{K}} \_$. H0 is satisfied as this category is bicomplete, and filtered colimits may be thought of as unions of sets. Moreover, since mod_A is an Abelian category, the forgetful functor # commutes with coproducts, or direct sums, which makes H1 trivially satisfied. □

Corollary 2.2.4.3. *The categories $Alg^\bullet_{\mathbb{K}}$ ($AugAlg^\bullet_{\mathbb{K}}$) are model categories.*

Proof. We establish the adjunction by letting $F = \bar{T}(T)$, the reduced tensor algebra of a cochain complex. For the same reasons as above, H0 is trivially satisfied. To show H1 we must find the coproduct and what $\bar{T}(M)$ is. ... □

We summarize the last result:

The category of augmented dg-algebras $AugAlg^\bullet_{\mathbb{K}}$ is a model category. Let $f : X \to Y$ be a homomorphism of augmented algebras.

- $f \in Ac$ if $f^\#$ is a quasi-isomorphism.
- $f \in Fib$ if $f^\#$ is an epimorphism (surjective onto every component).
- $f \in Cof$ if f has LLP wrt. to every acyclic fibration.

The category of augmented dg-algebras has an initial and a terminal object. The initial object is the stalk $\bar{\mathbb{K}}$ and the terminal object is the 0-ring. We see that every object is fibrant, as 0 is preserved by the forgetful functor and every map into 0 is surjective. Every dg-algebra which is isomorphic to a tensor algebra when considered as a graded algebra is cofibrant. This is wron

2.2.2 A Model Structure on DG-Coalgebras

We now want to equip the category of dg-coalgebras with a suitable model structure. This model structure should be suitable in the sense that it give rise to the same homotopy theory of dg-algebras. The bar-cobar construction will be crucial in this construction, as it is in fact a Quillen adjunction. To this end we will follow the setup as presented by Lefevre-Hasegawa [10]. His method is a modification of Hinichs paper [20] which describes a model structure on dg-coalgebras, but in relation to dg-lie algebras.

Let $f : C \to D$, the category of dg-coalgerbas will be equipped with the three following classes of morphisms:

- $f \in Ac$ if Ωf is a quasi-isomorphism.
- $f \in Fib$ if f has RLP wrt. to every acyclic cofibration.
- $f \in Cof$ if $f^\#$ is a monomorphism (injective in every component).

Recall that a filtration on a chain complex M is a sequence of inclusions $M_0 \subseteq M_1 \subseteq \ldots$. The filtration is called exhaustive if $\varinjlim M_i \simeq M$, and admissable if $Fr_0 M \simeq \mathbb{K}$ as well. Since each inclusion respect the differential, we may find the cokernel of each inclusion. The associated graded grM is then the graded complex given by $gr_0 M = M_0$ and $gr_i M = M_i / M_{i-1}$. A dg-(co)algebra is filtered if the filtration respects the (co)multiplication. If $f : M \to N$ is a morphism of filtered cochain complexes, then it defines a morphism $grf : grM \to grN$ on the associated graded. We call f a graded quasi-isomorphism if grf is a quasi-isomorphism.

Let C be a conilpotent dg-coalgebra. The coradical filtration $Fr_0C \subseteq Fr_1C \subseteq ...$ is an exhaustive filtration by assumption and $Fr_0C \simeq \mathbb{K}$, so it is admissable. Observe that $\Delta Fr_{i+1}C \subseteq Fr_iC \otimes Fr_iC$. To obtain quasi-isomorphisms of chain complexes we do not need the extra structure of the coalgebra. If not specified otherwise, whenever C is assumed to be a conilpotent dg-coalgebra then grC should be the associated graded of the coradical filtration.

Lemma 2.2.5. *Let $f : C \to C'$ be a graded quasi-isomorphism between conilpotent dg-coalgebras, then $\Omega f : \Omega C \to \Omega C'$ is a quasi-isomorphism.*

Proof. We do this by considering a spectral sequence. Endow C with a grading (as a vector space) induced by the coradical filtration, i.e. $c \in C$ has degree $|c| = n$ if n is the smallest number such that $\bar{\Delta}^n c = 0$. We define a filtration on ΩC by

$$F_p\Omega C = \{sc_1 \otimes ... \otimes sc_n \mid |c_1| + ... + |c_n| \leqslant p\}$$

Since C is a dg-coalgebra, the coradical filtration respects the differential. In other words, $F_p\Omega C$ is still a chain complex, which is a subcomplex of ΩC. This filtration is clearly bounded below and exhaustive. Thus by the classical convergence theorem of spectral sequences, theorem 5.5.1 [21], the spectral sequence converges to the homology $E\Omega C \Rightarrow H^*\Omega C$.

By definition, the 0'th page is defined as $E^0_{p,q}\Omega C = {}^{(F_p\Omega C)_{p+q}}/{}_{(F_{p-1}\Omega C)_{p+q}}$. Furthermore, notice that at this page we have the following isomorphism $E^0_{p,q}\Omega C \simeq (\Omega grC)^{(p)}_{p+q}$, where $(\Omega grC)^{(p)} = \{sc_1 \otimes ... \otimes sc_n \mid |c_1| + ... + |c_n| = p\}$.

Evaluating f at the 0'th page would the look like $E^0\Omega f \simeq \Omega grf$. By the mapping lemma, exercise 5.2.3 [21], it is enough to check that Ωgrf is a quasi-isomorphism, to see that Ωf is a quasi-isomorphism. We show that Ωgrf is a quasi-isomorphism by inspecting every $E^0_{p,\bullet}\Omega C$.

We define a filtration G_k on $E^0_{p,\bullet}\Omega C$ as

$$G_k = \{sc_1 \otimes ... \otimes sc_n \mid n \geqslant -k\}.$$

We see that $G_0 = E^0_{p,\bullet}\Omega C$ by definition and $G_{-p-1} \simeq 0$ on the coaugmentation quotient $\bar{C}$. Again, by the classical convergence theorem of spectral sequences, this defines a spectral sequence such that $EG \Rightarrow H^*E^0_{p,\bullet}\Omega C$.

To see that Ωgrf is a quasi-isomorphism, it is now enough to see that E^0Gf is a quasi-isomorphism for any p. Notice that $E^0_{l,\bullet}G \subseteq (grC[-1])^{\otimes l}$ where the total grading is p. Since f is a graded quasi-isomorphisms and by the Kunneth-formual, theorem 3.6.3 [21], it follows that E^0Gf is a quasi-isomorphism. □

For completeness we include the following statement.

Lemma 2.2.6. *Let $f : A \to A'$ be a quasi-isomorphism between dg-algebras, then $Bf : BA \to BA'$ is both a filtered and unfiltered quasi-isomorphism.*

Proof. Notice that the homology of BA may calculated from the double complex used to define BA. In fact, at the 0'th page we have $E^0_{p,\bullet} f \simeq f^{\otimes p}$. It follows that f is a quasi-isomorphism on the 0'th page from the Kunneth formula, theorem 3.6.3 [21]. □

Let A (C) be a filtered dg-algebra (coalgebra). Given an element $a \in A$ ($c \in C$) we say that its filtered degree $fdeg(a)$ ($fdeg(c)$) is the number such that $a \in F_{fdeg(a)}A$ ($c \in F_{fdeg(c)}C$) but not $a \in F_{fdeg(a)+1}A$ ($c \in F_{fdeg(c)+1}C$). There is then an associated filtration on the bar (cobar) construction of this complex, defined as $F_pBA = \{\omega a_1 \otimes ... \otimes \omega a_n \mid \sum fdeg(a_i) \leqslant p\}$ ($F_p\Omega C = \{sc_1 \otimes ...sc_n \mid \sum fdeg(c_i) \leqslant p\}$).

Proposition 2.2.7. *Let A be an augmented dg-algebra and C a conilpotent dg-coalgebra. The counit $\varepsilon_A : \Omega BA \to A$ a quasi-isomorphism. The unit $\eta_C : C \to B\Omega C$ is a filtered quasi-isomorphism, moreover $B\eta_C$ is a quasi-isomorphism.*

The following proof is due to [10], but with corrections given by [22]. Some minor modifications are given to the proof as it resembles a proof given earlier.

Proof. We will show that the counit is quasi-invertible. The proof technique for quasi-invertability of the unit is analogous.

Define the following filtration for A.

$$\begin{aligned} F_0A &= \mathbb{K} \\ F_1A &= A \\ F_pA &= F_1A \end{aligned}$$

We clearly see that this filtration endows A with the structure of a filtered dg-algebra. For ΩBA we will use the induced filtration from the coradical filtration of BA.

The counit acts on ΩBA as tensorwise projection, following with multiplication in A. This morphism respects the filtration, so it is a filtered morphism. Notice that both of these filtrations are bounded below, and exhaustive, so the classical convergence theorem of spectral sequences apply.

Let $E_r\Omega BA$ and E_rA be the spectral sequences given by the filtration. We have that $E_1^p\Omega BA \simeq gr_p\Omega BA$ and $E_1^pA \simeq gr_pA$. For $p = 1$ both complexes are isomorphic to the same complex, $\bar{A}$. Moreover, $E_1^1\varepsilon_A = id_{\bar{A}}$. Whenever $p \neq 1$ we get that $E_1^pA \simeq 0$, so it remains to show that $E_1^p\Omega BA \simeq gr_p\Omega BA$ is contractible for any $p \geqslant 2$.

The differential of ΩBA is generated by three actions: the differential on A, the multiplication on A and the comultiplication on BA. With the induced filtration on ΩBA, we see that the multiplication on A is the only action which maps $F_p\Omega BA \to F_{p-1}\Omega BA$. Thus this action is 0 in the associated graded and the spectral sequence.

There is a homotopy of the identity given as $h : gr_i\Omega BA \to gr_i\Omega BA$, which is 0 unless if there is an element on the form $s^{-1}(sa) \otimes s^{-1}(...) \otimes ...$. In this case h is

$$r(s(\omega a) \otimes s(...) \otimes ...) = (-1)^{|a|+1} s(\omega a \otimes ...) \otimes ...$$

We will show that this is a homotopy by induction on i.

Let $i = 2$. Then there are two cases we must handle, either an element is on the form $s(\omega a_1) \otimes s(\omega a_2)$ or $s(\omega a_1 \otimes \omega a_2)$. We consider the latter case first. If we apply r to this element we are returned 0.

$$(r \circ d_{\Omega BA} + d_{\Omega BA} \circ r)(s(\omega a_1 \otimes \omega a_2)) = r((-1)^{|a_1|+1} s(\omega a_1) \otimes s(\omega a_2)) = s(\omega a_1 \otimes \omega a_2)$$

The we treat the former case

$$\begin{aligned}
&(r \circ d_{\Omega BA} + d_{\Omega BA} \circ r)(s(\omega a_1) \otimes s(\omega a_2)) \\
&\quad = r(s(\omega d_A a_1) \otimes s(\omega a) + (-1)^{|a_1|} s(\omega a_1) \otimes s(\omega d_A a_2)) + d_{\Omega BA}((-1)^{|a_1|+1} s(\omega a_1 \otimes \omega a_2)) \\
&\qquad = (-1)^{|a_1|} s(\omega d_A a_1 \otimes \omega a_2) - s(\omega a_1 \otimes \omega d_A a_2) + s(\omega a_1) \otimes s(\omega a_2) \\
&\qquad\qquad + (-1)^{|a|+1} s(\omega d_A a_1 \otimes \omega a_2) + s(\omega a_1 \otimes \omega d_A a_2) = s(\omega a_1) \otimes s(\omega a_2)
\end{aligned}$$

We see that this is in fact a homotopy, which makes $id_{gr_2\Omega BA}$ a null-homotopy.

To extend this argument by induction, we will hypothesize that the terms where the differential is applied will have opposite signs, such that they cancel. Then the result follows for any i, by observing that the tensors far enough out to the right is not affected by r.

If C is a dg-coalgebra, we use the same technique as in 2.2.5. Consider the filtration on $B\Omega C$ given as

$$F_p B\Omega C = \{\omega(sc_{1,1} \otimes ... \otimes sc_{1,n_1}) \otimes ... \otimes \omega(sc_{m,1} \otimes ... \otimes sc_{m,n_m}) \mid |c_{1,1}| + ... + |c_{m,n_m}| \leqslant p\}.$$

This filtration is both bounded below and exhaustive, so the classical convergence theorem says that the associated spectral sequence converges. If we denote this sequence as EF, then $EF \implies H^* B\Omega C$. Let EC denote the spectral sequence associated to C. Since C is conilpotent $EC \implies H^* C$. The unit $\eta_C : C \to B\Omega C$ is now a map acting on EC^0 as the identity, sending each element in $EC^0_{p,q}$ to itself in $EF^0_{p,q}$.

On each row $EF^0_{p,\bullet}$ we make another filtration called G.

$$G_k EF^0_{p,\bullet} = \{\omega(...)_1 \otimes ... \otimes \omega(...)_n \mid n \geqslant -k\}$$

Similarly as in 2.2.5, this filtration is bounded below and exhaustive, so we may again apply the classical convergence theorem to obtain a spectral sequence E_pG such that $E_pG \implies$

$H^*EF^0_{p,\bullet} \simeq EF^1_{p,\bullet}$. Since the unit acts as the identity on EC^0, it descends to a morphism $gr_pC \to E_pG^0_{k,\bullet}$ which is the identity when $k = -1$ and 0 otherwise. Notice that every string of length $\geqslant 2$ is not hit by this morphism. However, by employing r as above we may show that these summands are contractible.

To show that this map is a quasi-isomorphism it is sufficient to show that $E_pG^0_{k,\bullet}$ is contractible for $k \neq 1$. This looks like the same situation as for algebras. We use the same homotopy r as in the first part to get a null-homotopy of the identity. □

Lemma 2.2.8. *Let $f : C \to D$ be a morphism of dg-coalgebras, then:*

- *if f is a cofibration, then Ωf is a standard cofibration.*
- *if f is a weak equivalence, then Ωf is as well.*

Almost dually, let $f : A \to B$ be a morphism of dg-algebras, then:

- *if f is a fibration, then Bf is a fibration.*
- *if f is a weak equivalence, then Bf is as well.*

Proof. First suppose that $f : C \to D$ is a cofibration. We define a filtration on D as the sum of the image of f and the coradical filtration on D: $D_i = Imf + Fr_iD$. f being a cofibration ensures us that $D_0 \simeq C$. Since D is conilpotent we know that $D \simeq \varinjlim D_i$, and that Ω commutes with colimits, there is a sequence of algebras $\Omega C \to \Omega D_1 \to ... \to \Omega D$. It is enough to show that each morphism $\Omega D_i \to \Omega D_{i+1}$ is a standard cofibration. The quotient coalgebra ${}^{D_{i+1}}/{}_{D_i}$ only has a trivial comultiplication, thus every element is primitive. This means that as a cochain complex D_{i+1} is constructed from D_i by attaching possibly very many copies of $\mathbb{K}$. We treat the case when there is only one such $\mathbb{K}$, here $D_{i+1} \simeq D_i \oplus \mathbb{K}\{x\}$ where $dx = y$ for some $y \in D_i$. We observe that this is exactly the condition for that the morphism $\Omega D_i \to \Omega D_{i+1}$ is a standard cofibration.

If f is a weak equivalence, then Ωf is a quasi-isomorphism by definition.

By lemma 2.1.38, or adjointness more specifically, B preserving fibrations is a consequence of Ω preserving cofibrations.

It remains to show that if $f : A \to B$ is a quasi-isomorphism, then Bf is a weak equivalence. Now, Bf is a weak equivalence if and only if ΩBf is a quasi-isomorphism. By 2.2.7, the counit $A \to \Omega BA$ is a quasi-isomorphism, so Bf is a weak equivalence by 2-out-of-3 property.

$$
\begin{tikzcd}
A \arrow[r, "f"] & B \\
\Omega BA \arrow[u, "\varepsilon_A"] \arrow[r, "\Omega Bf"] & \Omega BB \arrow[u, "\varepsilon_B"]
\end{tikzcd}
$$

□

We will need one more technical lemma.

Lemma 2.2.9. *Let A be a dg-algebra, D a dg-coalgebra and $p : A \to \Omega D$ a fibration of algebras. The projection morphism $\pi : BA \prod_{B\Omega D} D \to BA$ is an acyclic cofibration.*

$$\begin{tikzcd}
BA \prod_{B\Omega D} D \arrow[r] \arrow[d, "\pi"] & D \arrow[d, "\eta_D"] \\
BA \arrow[r, "Bp"] & B\Omega D
\end{tikzcd}$$

Proof. π being a cofibration is immediate by corollary 2.1.9.1. To see that π is a quasi-isomorphism it is enough to understand that it is a quasi-isomorphism as chain complexes. This is checked by Lefevre-Hasegawa [10]. Keller has pointed out a slight mistake in this proof which he has proposed a fix to [23]. □

Theorem 2.2.10. *The category $ConilCoalg^{\bullet}_{\mathbb{K}}$ is a model category with the classes Ac, Fib and Cof as defined above.*

Proof. The axioms **MC1** and **MC2** are immediet. Also, fibrations having RLP wrt. acyclic cofibrations is by definition.

We show **MC4** first. Let $f : C \to D$ be a morphism of coalgebras. There is a factorization $\Omega f = pi$ of morphisms between algebras, where i is a cofibration, p is a fibration and at least one of i and p are quasi-isomorphisms. Applying bar we get a factorization $B\Omega f = BiBp$, where Bp is a fibration and at least one of Bi and Bp are weak equivalences.

$$\begin{tikzcd}
\Omega C \arrow[rr, "\Omega f"] \arrow[dr, "i"'] & & \Omega D \\
& A \arrow[ur, "p"'] &
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
B\Omega C \arrow[rr, "B\Omega F"] \arrow[dr, "Bi"'] & & B\Omega D \\
& BA \arrow[ur, "Bp"'] &
\end{tikzcd}$$

We construct a pullback with Bp and η_D. By 2.2.9 the morphism π is an acyclic cofibration. We collect our morphisms in a big diagram. The dashed arrow exists since the rightmost square is a pullback.

$$\begin{tikzcd}
 & BA\prod_{B\Omega D} D \arrow[dr, "q"] \arrow[dd] \arrow[dr, phantom, very near start, "\ulcorner"] & \\
C \arrow[ur, dashed, "j"] \arrow[rr, "f"] \arrow[dd, "\eta_C"] & & D \arrow[dd, "\eta_D"] \\
 & BA \arrow[dr, "Bp"] & \\
B\Omega C \arrow[ur, "Bi"] \arrow[rr, "B\Omega f"] & & B\Omega D
\end{tikzcd}$$

First notice that q is a fibration, since fibrations are stable under pullbacks. j is a cofibration, or a monomorphism, as the composition $Bi\eta_C$ is a monomorphism. Thus it remains to see that if Bi (Bp) is a weak equivalence, then j (q) is as well. This is evident from the 2-out-of-3 property, as η is a natural weak equivalence, π is a weak equivalence and Bi (Bp) is a weak equivalence.

We now show **CM3**. Suppose that there is a square as below, where i is a cofibration and t is an acyclic cofibration.

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "i"] & C \arrow[d, "t"] \\
F \arrow[r] & D
\end{tikzcd}$$

We can factor t as $t = qj$ by **CM4**. Notice that t is a retract of q, i.e. there is a commutative diagram as below.

$$\begin{tikzcd}
C \arrow[r, equal] \arrow[d, "j"] & C \arrow[d, "t"] \\
BA\prod_{B\Omega A} D \arrow[ur] \arrow[r, "q"] & D
\end{tikzcd}$$

So in order to find a lift to C, we may instead find a lift to $BA\prod_{B\Omega D} D$. Since p is an acyclic fibration by construction and Ωi is a cofibration by 2.2.8, there is a lift $h : \Omega E \to A$ of algebras. We obtain our desired lift from the bar-cobar adjunction and the universal property of the pullback.

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "i"] & BA\prod_{B\Omega D} D \arrow[r, "\pi"] \arrow[d, "q"] \arrow[dr, phantom, very near start, "\ulcorner"] & BA \arrow[d, "Bp"] \\
F \arrow[ur, dashed] \arrow[urr, dotted, "h^T"] \arrow[r] & D \arrow[r, "\eta_D"] & B\Omega D
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
\Omega E \arrow[r] \arrow[d, "\Omega i"] & A \arrow[d, "p"] \\
\Omega F \arrow[ur, dotted, "h"] \arrow[r] & \Omega D
\end{tikzcd}$$

$\square$

We restate the corollary of the adjunction.

Corollary 2.2.10.1. *The bar-cobar construction $B : AugAlg_{\mathbb{K}}^{\bullet} \rightleftharpoons ConilCoalg_{\mathbb{K}}^{\bullet} : \Omega$ as a Quillen equivalence.*

Proof. We first observe that (B, Ω) is a Quillen adjunction by lemma 2.2.8. Moreover, since the unit and counit are weak equivalences by proposition 2.2.7, it follows by either proposition 2.1.41 or its corollary 2.1.41.1 that (B, Ω) is a Quillen equivalence. □

2.2.3 Homotopy theory of A_∞-algebras

This section aims to finalize the discussion of the homotopy theory of A_∞-algebras. We will look at the homotopy invertability of every strongly homotopy associative quasi-isomorphism, and the relation to associative algebras. This discussion will end with mentioning different results which gives a clearer description of fibrations, cofibrations and homotopy equivalences. This section follows Lefevre-Hasegawa [10]. Before we get to the main theorem, we start by discussing a non-closed model structure on the category of Alg_∞.

Let $f : A \rightsquigarrow B$ be a morphism between A_∞-algebras, the category of A_∞-algebras will be equipped with the three following classes of morphisms:

- $f \in Ac$ if f is an ∞-quasi-isomorphism, i.e. f_1 is a quasi-isomorphism.
- $f \in Fib$ if f_1 is an epimorphism.
- $f \in Cof$ if f_1 is a monomorphism.

This category does not make a model category in the sense of a closed model category, as we are lacking many finite limits. It does however come quite close to be such a category.

Theorem 2.2.11. *The category Alg_∞ equipped with the three classes as defined above satisfies:*

*a The axioms **MC1** through **MC4**.*
b Given a diagram as below, where p is a fibration, then its limit exists.

$$\begin{tikzcd} & A \arrow[d, "p"] \\ B \arrow[r] & C \end{tikzcd}$$

We will first prove one lemma which technique we will reuse. A seconde lemma used to simplify the theorem will be stated without proof.

Lemma 2.2.12. *let A be an A_∞-algebra, and K a contractible complex considered as an A_∞-algebra. If $g : (A, m_1^A) \to (K, m_1^K)$ is a cochain map, then it extends to an ∞-morphism $f : A \rightsquigarrow K$.*

Proof. We construct each f_i inductively. The case $i = 1$ is degenerate as we have assumed $f_1 = g$.

Assume that we have already constructed f_1 through f_n. We observe that the sum below is a cycle of $Hom^*_{\mathbb{K}}(A, K)$.

$$\sum_{\substack{p+1+r=k\\ p+q+r=n}} (-1)^{pq+r} f_k \circ_{p+1} m_q^A - \sum_{\substack{k\geqslant 2\\ i_1+\ldots+i_k=n}} (-1)^e m_k^B \circ (f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_k})$$

Thus since K is contractible, $Hom^*_{\mathbb{K}}(A, K)$ is acyclic and there exists some morphism f_{n+1} such that $\partial(fn + 1)$ is the sum above. This says that this extension does in fact satisfy (rel_{n+1}). □

Lemma 2.2.13. *Let $j : A \rightsquigarrow D$ be a cofibration of A_∞-algberas, then there is an isomorphism $k : D \rightsquigarrow D'$ such that the composition $k \circ j : A \rightsquigarrow D'$ is a strict morphism of A_∞-algebras.*

Dually, if $j : A \rightsquigarrow D$ is a fibration, then there is an isomorphism $l : A' \rightsquigarrow A$ such that the composition $j \circ l : A' \rightsquigarrow D$ is a strict morphism of A_∞-algebras.

Proof. A proof is given as lemma 1.3.3.3 in [10]. □

proof of 2.2.11. We start by showing b. Suppose that we have a diagram of A_∞-algebras, such that g_1 is an epimorphism.

$$\begin{tikzcd} & A \arrow[d, twoheadrightarrow, "g"] \\ A' \arrow[r, "f"] & A'' \end{tikzcd}$$

First notice that as dg-coalgebras, this pullback exists and defines a new dg-coalgebra $BA \prod_{BA''} BA'$.

Since g_1 is an epimorphism, $A[1]$ as a graded vector space splits into $A''[1] \oplus K$, where $K = Kerg_1$. The pullback is then naturally identified with $BA \prod_{BA''} BA' \simeq \bar{T}^c(K) \prod \bar{T}^c(A'[1])$ as graded vector spaces. Since the cofree coalgebra is right adjoint to forget, it commutes with products and we get, $\bar{T}^c(A'[1]) \prod \bar{T}^c(K) \simeq \bar{T}^c(A'[1] \oplus K)$. Thus the pullback is isomorphic to a cofree coalgebra as a graded coalgebra, i.e. it is an A_∞-algebra.

We now prove a. MC1 and MC2 are immediate, so we will not prove them.

We start by proving MC3. Suppose that there is a square of A_∞-algebras as below, where j is a cofibration and q is a fibration.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"] & B \arrow[d, "q"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

By lemma 2.2.13, we may assume assume that both j and q are strict morphisms. We now assume that q is an ∞-quasi-isomorphism, the proof will be analogous if j is an ∞-quasi-isomorphism instead.

Our goal is to construct a lifting in this diagram inductively. Having a lift means finding an ∞-morphism $a : C \rightsquigarrow B$, such that the following hold for any $n \geqslant 1$:

- a satisfy (rel_n).
- $a_n \circ j_1 = f_n$.
- $q_1 \circ a_n = g_n$.

We start by showing there is such an a_1. Consider the diagram below of chain complexes over $\mathbb{K}$.

$$
\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, tail, "j_1"'] & B \arrow[d, two heads, "q_1"] \\
C \arrow[r, "g_1"'] \arrow[ur, dashed, "a_1"] & D
\end{tikzcd}
$$

The lift exists since the category $Ch\mathbb{K}$ is a model category. Here j_1 is a cofibration, while q_1 is an acyclic fibration, so the lift a_1 exists.

We now wish to extend this. Suppose that we have been able to create morphisms a_1 up to a_n, all satisfying the above points. A naive solution to make a_{n+1} is $b = f_{n+1}r^{\otimes n+1} + sg_{n+1} - sq_1f_{n+1}r^{\otimes n+1}$. Notice that this satisfy the two last points by definition. We will augment b to get an a_{n+1} which also satisfies (rel_{n+1}).

For our own convenience, let $-c(f_1, ..., f_n)$ denote the right hand side of (rel_{n+1}) formula. Since both j and q are strict ∞-morphisms we get the following identites:

$$
\begin{aligned}
(\partial b + c(a_1, ..., a_n)) \circ j_1 &= \partial(b \circ j_1) + c(a_1 \circ j_1, ..., a_n \circ j_1) = \partial f_{n+1} + c(f_1, ..., f_n) = 0 \\
q_1 \circ (\partial b + c(a_1, ..., a_n)) &= \partial(q_1 \circ b) + c(q_1 \circ a_1, ..., q_1 \circ a_n) = \partial g_{n+1} + c(g_1, ..., g_n) = 0
\end{aligned}
$$

We thus obtain that the cycle $\partial b + c(a_1, ..., a_n)$ factors thorugh the cokernel of j and the kernel of q. Let us say that it factors like the diagram below:

$$
C \xrightarrow{p} Cokj_1 \xrightarrow{c'} Kerq_1 \xrightarrow{i} D
$$

Now, c' is a morphism between two A_∞-algebras. Since q is assumed to be an ∞-quasi-isomorphism, it follows that $Kerq$ is a acyclic. Since c' is a cycle, it necesserilly have to be in the image of the differential. Let h be a morphism such that $\partial h = c'$, and define $a_{n+1} = b - i \circ h \circ p$. One may check that this morphism satisfies all three properties.

We will now show MC4. Since the two properties have a similar proof, we will only show one direction. Let $f : A \rightsquigarrow B$ be an ∞-morphism. Let $C = cone(id_(B[-1]))$. The complex C may be considered as an A_∞-algebra. Let $j : A \rightsquigarrow A \prod C$ be the morphism induced by id_A and $0 : A \to C$. The canonical projection $q_1 : A \oplus C \to B$ gives a lift of the following diagram.

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "j_1"] & B \arrow[d] \\ A \oplus C \arrow[r] \arrow[ur, "q_1"] & 0 \end{tikzcd}$$

Since we have a morphism of chain complexes, lodged between an acyclic cofibration and a fibration we use the same technique as above to construruce an ∞-morphism $q : A \prod C \to B$. q is a fibration by construction. The morphism f may be factored as $f = q \circ j$, where j is an acyclic cofibration and q is a fibration. $\square$

With this model structure we are finally able to characterize the fibrant and cofibrant conilpotent dg-coalgebras.

Proposition 2.2.14. *Let C be a conilpotent dg-coalgebra. Then C is cofibrant, and C is fibrant if and only if there is a cochain complex V, such that $C \simeq T^c(V)$ as complexes.*

Proof. To see that C is cofibrant is the same as to verify that the map $\mathbb{K} \to C$ is a monomorphism, but this is clear.

We start by assuming that C is fibrant. Then there is a lift in the square below, making the unit split-mono.

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d, "\eta_C"] & C \arrow[d, "\varepsilon_C"] \\ B\Omega C \arrow[r, "\varepsilon_{B\Omega C}"] \arrow[ur, dashed, "r"] & \mathbb{K} \end{tikzcd}$$

Consider the morphism $p_1^C : C \to Fr_1 C$ which is defined as $p_1^C = Fr_1 r \circ p_1 \circ \eta_C$, where $p_1 : B\Omega C \to Fr_1 B\Omega C$ is the canonical projection on the filtration induced by the coradical filtration on C. Clearly, r makes p_1 into a universal arrow in the category of conilpotent coalgebras, so $C \simeq T^c(Fr_1 C)$.

Now, assume that C is isomorphic to $T^c(V)$ as coalgebras for some cochain complex V. Note that, by definition, C is an A_∞-algebra. By definition, we have a commutative square of A_∞-algebras. Since every A_∞-algebra is bifibrant, we know that this diagram has a lift, exhibiting C as a retract of $B\Omega C$.

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d] & C \arrow[d] \\ B\Omega C \arrow[r] \arrow[ur, dashed] & \mathbb{K} \end{tikzcd}$$

We know that ΩC is fibrant, since the map $\Omega C \to 0$ is epi. By lemma 2.2.8, we know that the bar construction preserves fibrations, so $B\Omega C$ is fibrant. Thus C is fibrant as well. □

The model structure of A_∞-algebras is compatible with the model structure of conilpotent dg-coalgebras in the following sense. If $f : A \rightsquigarrow B$ is an ∞-morphism, we denote its dg-coalgebra counterpart as $Bf : BA \to BB$. Remember that the bar construction is extended such that it is an equivalence of categories on its image. We use this to realize Alg_∞ as a subcategory of $ConilCoalg_\mathbb{K}$ to essentially obtain 2 different model structure on this category. The following proposition tells us that these structures do not differ.

Proposition 2.2.15. *Let $f : A \rightsquigarrow B$ be an ∞-morphism. Then we have the following:*

- *f is an ∞-quasi-isomorphism if and only if Bf is a weak equivalence.*
- *f_1 is an epimorphism if and only if Bf is a fibration.*
- *f_1 is a monomorphism if and only if Bf is a monomorphism.*

Proof. This is proposition 1.3.3.5 in [10]. □

2.3 The Homotopy Category of Alg_∞

With the results we have establish, we are now ready to talk about homotopies in Alg_∞.

Theorem 2.3.1. *In the category Alg_∞ we have the following:*

- *Homotopy equivalence is an equivalence relation.*
- *A morphism is an ∞-quasi-isomorphism if and only if it is a homotopy equivalence.*
- *Let $dash \subseteq Alg_\infty$ be the full subcategory consisting of dg-algebras considered as A_∞-algebras. $dash$ has an induced homotopy equivalence from Alg_∞, and the inclusion $Alg \to dash$ induces an equivalence in homotopy $Alg[Qis^{-1}] \simeq {}^{dash}/{\sim}$.*

Proof. The first point is obsreved from corollary 2.1.27.2, and the second point is Whiteheads theorem, theorem 2.1.29.

To see the final point, observe that the inclusion functor is given by the bar construction B. By corollary 2.2.10.1, we know that the bar construction induces an equivalence on the homotopy categories, i.e. $HoAlg \simeq HoCoalg$. Moreover, we know that by theorem 2.1.30 that $HoCoalg \simeq$

${}^{Alg_\infty}\!/\!\sim$. Notice that the image of B is $dash$, so in homotopy, we get that the image ${}^{dash}\!/\!\sim$ is equivalent to the essentiall image $HoAlg_\infty$. □

Homotopy equivalence in the algebraic sense and the model categorical sense differ in how they are defined. In homological algebra, two morphisms are called homotopic if their difference is a boundary of some homotopy, i.e. $f - g = \partial h$. In the model categorical sense, a homotopy is a morphism through either a cylinder or a path object. The following proposition tells us how they differ in the category of conilpotent dg-coalgebras.

Proposition 2.3.2. *Let C and D be two conilpotent dg-coalgebras, where $f, g : C \to D$ are two morphisms. Then:*

- *If $f - g$ is null homotopic, then they are left homotopic.*
- *If D is fibrant, then $f - g$ is null homotopic if and only if f and g are left homotopic.*

This tells us that in general, these concepts are usually not the same. However, we know that the subcategory of bifibrant objects in this category is exactly the subcategory Alg_∞. Thus for A_∞-algebras homological homotopy is the same as model categorical homotopy. In this sense we obtain that the homotopy category in the homological sense is equivalent to the derived category.

Chapter 3

Derived Categories of Strongly Homotopy Associative Algebras

In this chapter we wish to study the derived categories of A_∞-algebras. At the heart of homological algebra is the derived category of algebras, so it is only natural to ask how this category looks like in the A_∞ case. In the last chapter we studied the relationship between the category of algebras and coalgebras to understand how quasi-isomorphisms between A_∞-algebras worked. In this chapter we will instead study the relationship between module and comodule categories in order to understand how quasi-isomorphisms between A_∞-modules will work. At the heart of this discussion are twisting morphisms $\alpha : C \to A$, which allows us to study the relationship between Mod^A and $CoMod^C$.

From twisting morphisms we will obtain functors $L_\alpha : CoMod^C \to Mod^A$ and $R_\alpha : Mod^A \to CoMod^C$ which create an adjoint pair of functors. Whenever the twisting morphism α is acyclic, this will in fact become a Quillen Equivalence.

We wish to reuse all of the methods we have gained and acquired thorughout this thesis. This chapter will mostly be reformulation and recontextualization of previous definitions, concepts and techniques.

3.1 Twisting Morphisms

Twisting morphisms were already introduced in chapter 1. There, they were used mostly to be represented by the bar and cobar construction. Now we want twisting morphisms and twisting tensors to play a bigger role. In order to define the functors L_α and R_α, these constructions will be crucial.

3.1.1 Twisted Tensor Products

Let A be an augmented dg-algebra, C a conilpotent dg-coalgebra and $\alpha : C \to A$ a twisting morphism. The right (left) twisted tensor product is the complex $C \otimes_\alpha A$ ($A \otimes_\alpha C$) together with the differential $d^\bullet_\alpha = d^\bullet_{C\otimes A} + d^r_\alpha$. The perturbation is defined as

$$d^r_\alpha = (\nabla_A \otimes id_C) \circ (id_A \otimes \alpha \otimes id_C) \circ (id_A \otimes \Delta_C).$$

If M is a right A-module and N is a left C-comodule then the tensor product $M \otimes_{\mathbb{K}} N$ exists and is a $\mathbb{K}$-module with differential $d_{M\otimes N}$. We may define a perturbation to this differential as

$$d^r_\alpha = (\mu_M \otimes id_N) \circ (id_M \otimes \alpha \otimes id_N) \circ (id_M \otimes \nu_N).$$

By using the same line of thought as proposition 1.2.4, there is a twisted tensor product $M \otimes_\alpha N$ with differential $d^\bullet_\alpha = d_{M\otimes N} + d^r_\alpha$.

Remark 3.1.1. Koszuls sign rule forces us to define the differential of the left twisted tensor product as $d^\bullet_\alpha = d_{N\otimes M} - d^l_\alpha$.

Definition 3.1.2. Suppose that $M \in Mod^A$ ($M \in Mod_A$) and $N \in CoMod_C$ ($N \in CoMod^C$), then the left (right) twisted tensor product is the $\mathbb{K}$-module $M \otimes_\alpha N$ ($N \otimes_\alpha M$).

In this setting right handedness and left handedness for the twisted tensor product is more clear in this setting, as we only have an action or coaction from one of the chosen sides. Trying to force the other handedness on the twisted tensors would just be ill-defined.

Definition 3.1.3. Let A be an augmented dg-algebra and C a conilpotent dg-coalgebgra, such that there is a twisting morphism $\alpha : C \to A$. Given a linear map $f : N \to M$ between a right C-comodule N and a right A-module M we say that it is an α right twisted linear homomorphism, or just an α twisted morphism, if it satisfies the following equation:

$$\partial f - f \star \alpha = 0$$

This definition gives us a functor $Tw_\alpha : CoMod^C \times Mod^A \to Ab$ which is the collection of right twisting linear homomorphisms between a comodule and module.

Suppose that $\alpha : C \to A$ is a twisting morphism. We define the functor $L_\alpha = \otimes_\alpha A : CoMod^C \to Mod^A$ as an arbitrary right twisted tensor product with A. This functor does indeed hit Mod^A by using the free right A-module structure on A. Likewise, we define a functor $R_\alpha = \otimes_\alpha C : Mod^A \to CoMod^C$ as an arbitrary left twisted tensor product with C. This does also hit right C-comodules by using the free right C-comodule structure on C.

Proposition 3.1.4. *Suppose that $\alpha : C \to A$ is a twisting morphism. The functor L_α and R_α form an adjoint pair of categories.*

$$\begin{tikzcd} CoMod^C \arrow[r, bend left, "L_\alpha"] \arrow[r, phantom, "\perp"] & Mod^A \arrow[l, bend left, "R_\alpha"] \end{tikzcd}$$

Proof. This proof breaks down to showing $CoMod^C(N, L_\alpha(M)) \simeq Tw_\alpha(N, M) \simeq Mod^A(R_\alpha(N), M)$. This is a rutine calculation, much like the proof for 1.2.13. □

Let A be a dg-algebra, and M a right A-module. Recall that by the Cobar-Bar adjunction 1.2.13 there exists a universal twisting morphism $\pi_A : BA \to A$. We define the bar-construction of M as $B_A M = R_{\pi_A} M = M \otimes_{\pi_A} BA$. Likewise, given a conilpotent dg-coalgebra C and N a right C-comodule we define the cobar-construction as $\Omega_C N = L_{\iota_C} N = N \otimes_{\iota_C} \Omega C$. In these cases we obtain adjunctions $\Omega_{BA} \dashv B_A$ and $\Omega_C \dashv B_{\Omega C}$.

Let A and B be two algebras and $f : A \to B$ is an algebra morphism. Then f induces a functor between the module categories by restriction: $f^* : Mod^B \to Mod^A$. Since A and B considered as algebroids are small, and the category of abelian groups is cocomplete, so the left Kan extension (induction) along this functor exists.

$$\begin{tikzcd} Mod^B \arrow[r, "f^*"'] & Mod^A \arrow[l, bend right, "f_!"'] \end{tikzcd}$$

Dually, if C and D are two coalgebras and $g : C \to D$ is an coalgebra morphism. Then g induces a functor between the module categories by composing: $g* : CoMod^C \to CoMod^D$. Since C and D considered as coalgebroids are small, and the category of abelian groups is complete, so the right Kan extension (co-induction) along this functor exists.

$$\begin{tikzcd} CoMod^C \arrow[r, "g*"] & CoMod^D \arrow[l, bend left, "g^!"] \end{tikzcd}$$

Lemma 3.1.5. *Let $\tau : C \to A$ be a twisting morphism. The adjunction (L_τ, R_τ) factors as $(f_{\tau}!, f_\tau^*) \circ (L_{\iota_C}, R_{\iota_C})$ or $(L_{\pi_A}, R_{\pi_A}) \circ (g_{\tau *}, g_\tau^!)$.*

Proof. This follows from corollary 1.2.14.1, that is $\tau = f_\tau \circ \iota_C = \pi_A \circ g_\tau$. □

Definition 3.1.6. A twisting morphism $f : C \to A$ is called acyclic if the counit of the adjunction $L_\alpha \dashv R_\alpha$ is a pointwise quasi-isomorphism.

Lemma 3.1.7. *Let A be an augmented dg-algebra and C a conilpotent dg-coalgebra. The universal twisting morphisms π_A and ι_C are acyclic.*

Proof. We start with π_A. Recall that π_A is constructed as the twisting morphism corresponding to id_{BA}. This morphism is thus given as the projection onto the first dimension of BA, that is:

$$\pi_A sa = a$$
$$\pi_A(sa \otimes ...) = 0$$

We say that π_A is acyclic if the counit $\varepsilon : L_{\pi_A} R_{\pi_A} \Rightarrow Id_{Mod^A}$ at each object M is a quasi-isomorphism.

For each M in Mod^A, $L_{\pi_A} R_{\pi_A} M = M \otimes_{\pi_A} BA \otimes_{\pi_A} A$. We may split up the differential into two summands, d_v and d_h. d_v is the ordinary differential on the tensor product, while $d_h = (-d^l_{\pi_A} \otimes A) + M \otimes d_2 \otimes A + d^r_{\pi_A}$. Since $(d_v + d_h)^2 = 0$ and $d_v^2 = 0$ we can observe that $d_v d_h = -d_h d_v$ and $d_h^2 = 0$. This is evident as d_v changes the homological degree while d_h does not, so in order for both of the first equations to hold, the last two must hold as well. We almost obtain a double complex.

$$\begin{tikzcd}
... \arrow[r, "d_h"] & M \otimes BA_{(i)} \otimes A \arrow[loop above, "d_v"] \arrow[r, "d_h"] & ... \arrow[r, "d_h"] & M \otimes BA_{(1)} \otimes A \arrow[loop above, "d_v"] \arrow[r, "d_h"] & M \otimes A \arrow[loop above, "d_v"] \arrow[r, "0"] & 0
\end{tikzcd}$$

It is clear that the total complex of this "double complex" is in fact $L_{\pi_A} R_{\pi_A} M$. Moreover, the counit induces an augmentation to this complex resolution of M, denoted as $cone(\varepsilon_M)$.

$$\begin{tikzcd}
... \arrow[r, "d_h"] & M \otimes BA_{(i)} \otimes A \arrow[loop above, "d_{M \otimes BA_{(i)} \otimes A}"] \arrow[r, "d_h"] & ... \arrow[r, "d_h"] & M \otimes BA_{(1)} \otimes A \arrow[loop above, "d_{M \otimes BA_{(1)} \otimes A}"] \arrow[r, "d_h"] & M \otimes A \arrow[loop above, "d_{M \otimes A}"] \arrow[r, "\varepsilon_M"] & M \arrow[loop above, "d_M"] \arrow[r, "0"] & 0
\end{tikzcd}$$

To see that this is in fact a resolution we define a morphism $h : cone(\varepsilon_M) \to cone(\varepsilon_M)$ of degree -1. It works by the following formula:

$$h(m \otimes (sa_1 \otimes ... \otimes sa_n) \otimes a) = m \otimes (sa_1 \otimes ... \otimes sa_n \otimes sa) \otimes 1$$

It is clear that $id_{cone(\varepsilon_M)} = d_h h - h d_h$ and $d_v h = h d_v$. Thus to see that the cone is acyclic we let $c \in cone(\varepsilon_M)$ be a cycle, that is $(d_v + d_h)(c) = 0$. Our goal is to show that $h(c)$ is a preimage of c along $d_v + d_h$.

$$(d_v + d_h) \circ h(c) = d_v \circ h(c) + d_h \circ h(c) = h \circ d_v(c) + c + h \circ d_h(c) = h \circ (d_v + d_h)(c) + c = c$$

To treat the case of ι_C we will use a double spectral sequence argument. This proof does not use any new techniques and looks like things we have already done before. A full proof may be found in Lefevre-Hasegawa [10].z

□

3.1.2 Model Structure on Module Categories

Let A be an augmented dg-algebra, then we know that Mod^A is a model category. By corollary 2.2.4.2 we have a model structure where the fibrations, cofibrations and weak equivalences are given as follows:

- $f \in Ac$ is a weak equivalence if f is a quasi-isomorphism.
- $f \in Fib$ is a fibration if $f^{\#}$ is an epimorphism.
- $f \in Cof$ is a cofibration if it has LLP to acyclic fibrations.

Every object in this category is fibrant as the morphism $0 : M \to 0$ is always an epimorphism.

3.1.3 Model Structure on Comodule Categories

Unless stated otherwise, for this section we fix A to be an augmented dg-algebra, C as a conilpotent dg-coalgebra and $\tau : C \to A$ as an acyclic twisting morphism. We endow $CoMod^C_{conil}$ with three classes of morphisms:

- $f \in Ac$ is a weak equivalence if $L_\tau f$ is a quasi-isomorphism.
- $f \in Fib$ is a cofibration if $f^{\#}$ is a monomorphism.
- $f \in Cof$ is a fibration if it har RLP to acyclic cofibrations.

Theorem 3.1.8. *The category $CoMod^C_{conil}$ with the three classes as above forms a model category. Every object is cofibrant, and those objects which is a direct summand of $R_\tau M$ for some $M \in Mod^A$ are fibrant. The adjoint pair (L_τ, R_τ) is a Quilllen equivalence.*

We will call this model structure for the canonical model structure on $CoMod^C_{conil}$.

To be able to prove this we will need some lemmata. This proof is essentially the same as the case for dg-coalgebras. The main difference is to show independence of the choice of a twisting morphism τ. To this end we must establish the relationship between graded quasi-isomorphisms and weak equivalences, as well as a technical lemma.

Recall that given a coaugmented coalgebra C we have a filtration called the coradical filtration, defined as $Fr_iC = Ker(\bar{\Delta}_C)^i$. If N is a right C-comodule we may define the coradical filtration of N as $Fr_iN = Ker(\bar{\omega}^i_N)$. This filtration is admissable, meaning it is exhaustive and $Fr_0N = 0$.

Lemma 3.1.9. *Let C be a conilpotent dg-coalgebra, M and N be right C-comodules. Then any graded quasi-isomorphism $f : M \to N$ is a weak equivalence.*

Proof. This proof is identical to 2.2.5. □

Lemma 3.1.10. *Let M and N be two objects of Mod^A. The functor R_τ sends a quasi-isomorphism $f : M \to N$ to a weak equivalence $R_\tau f : R_\tau M \to r_\tau N$.*

The unit of the adjunction $\eta : Id \to R_\tau L_\tau$ is a pointwise weak equivalence.

Proof. $R_\tau f$ is a weak equivalence if $L_\tau R_\tau f$ is a quasi-isomorphism. By naturality of the counit we have the following commutative diagram.

$$\begin{tikzcd} M \arrow[d, "f"] & L_\tau R_\tau M \arrow[l, "\varepsilon_M"'] \arrow[d, "L_\tau R_\tau f"] \\ N & L_\tau R_\tau N \arrow[l, "\varepsilon_N"'] \end{tikzcd}$$

By assumption we know that all three of f, ε_M and ε_N are quasi-isomorphisms. It follows by the 2-out-of-3 property that $L_\tau R_\tau f$ is a quasi-isomorphism as well.

To show that $\eta : Id \to L_\tau R_\tau$ is a pointwise weak-equivalence, we must show that $L\eta$ is a pointwise quasi-isomorphism. Since L_τ is left adjoint to R_τ we know that η is split on the image of L_τ, i.e.

$$\varepsilon_{L_\tau} \circ L_\tau \eta = id_{L_\tau}$$

Since we know that the natural isomorphisms ε and id are pointwise quasi-isomorphisms, we get by the 2-out-of-3 property that $L\eta$ is a pointwise quasi-isomorphism as well. □

Lemma 3.1.11. *The functor L_τ preserves cofibrations and sends weak-equivalences to quasi-isomorphisms.*

Proof. This proof is essentially the same as 2.2.8. □

With the above lemmata we have now established that the adjunction (L_τ, R_τ) forms a Quillen equivalence if $CoMod^C$ is a model category.

The next lemma is a technical lemma which we need. There will not be given a proof for it, but this is lemma 2.2.2.9 in [10].

Lemma 3.1.12. *Let M be a right A-module and N a right C-comodule. Let $p : M \to L_\tau N$ be a fibration of modules. The projection $j : R_\tau M \prod_{R_\tau L_\tau N} N \to R_\tau M$ is an acyclic cofibration of comodules.*

Proof. This proof is omitted. □

Proof of 3.1.8. With the above lemmata established, this proof is identical to the proof of 2.2.10. □

3.1.4 Triangulation of Homotopy Categories

In this section we will show that the homotopy categories are triangulated. If we look at the category Mod^A we will observe that the category $HoMod^A$ is our beloved derived category $\mathcal{D}(A)$. It is not quite the same for the cateogry $CoMod^C$. Here we want $HoCoMod^C$ to be equivalent to the derived category of a ring, so we will see that the derived category is a further localization of $HoCoMod^C$.

Furthermore, by employing the theory of triangulated categories we will show that the model structure on $CoMod^C$ is independent on the choice of acyclic twisting morphism. This breaks down to show that every acyclic twisting morphism induce an equivalence between derived categories, as done by Bernhard Keller in [24].

Mod^A is a an abelian category, where we employ the maximal exact structure $\mathcal{E}'$ consisting of short exact sequences in Mod^A. This translates to short exact sequences which is short exact in each degree. However, this category also has an exact structure $\mathcal{E}$ which makes Mod^A into a Frobenius category, which we will now describe.

Let $f : M \to N$, be a chain map from M to N. Then $\mathcal{E}$ contains a conflation on the form:

$$N \rightarrowtail cone(f) \twoheadrightarrow M[1]$$

We define $\mathcal{E}$ to be the smallest exact structure on Mod^A which contains every conflation arising from a chain map f. Observe that these conflations are exactly the short exact sequences of Mod^A such that they are split when regarded as graded modules, i.e. forgetting the differential. Thus the smallest such $\mathcal{E}$ is exactly the collection of every conflation arising from a chain map f.

Recall that an object M is projective (injective) if the represented functor $Mod^A(M, \_)$ ($Mod^A(\_, M)$) is exact. For the category $(Mod^A, \mathcal{E})$

Proposition 3.1.13. *Let M be an object of Mod^A. The following are equivalent:*

- *M is projective*
- *M is injective*
- *M is contractible*

Proof. This is a well known statement from literature. See Krause [25], Happel [26], Buehler [27] or Thorbjørnsen [28] for an account of this result. □

To see that $(Mod^A, \mathcal{E})$ has both enough projectives and injectives we consider the following conflation:

$$M \rightarrowtail cone(id_M) \twoheadrightarrow M[1]$$

It is known that the complex $cone(id_M)$ is contractible for any complex M (and it is also universal with this property). In this way by letting M vary we can find an inflation or deflation from the identity cone from or to any complex. This concludes that $(Mod^A, \mathcal{E})$ is a Frobenius category.

Let $\underline{Mod^A}$ denote the injectively stable module category. Let $I(M, N)$ denote the set of chain maps from M to N which factors through an injective object. We define the injectively stable category as the quotient of abelian groups $\underline{Mod^A}(M, N) = {}^{Mod^A(M,N)}/_{I(M,N)}$.

Theorem 3.1.14. *Suppose that $(\mathcal{C}, \mathcal{E})$ is a Frobenius category, then the injectively stable category $\underline{\mathcal{C}}$ is triangulated. The additive auto-equivalence is given by cozyzygy, and the standard triangles is the image of the conflations into the quotient.*

Proof. This is also well known in literature. An account for it may also be found in Krause [25], Happel [26], Buehler [27] or Thorbjørnsen [28]. □

We thus obtain a triangulated category $\underline{Mod^A}$ associated to the Frobenius pair $(Mod^A, \mathcal{E})$. This category is commonly denoted as $K(A)$, and we will do this as well. Notice that with the structure given by $\mathcal{E}$, the cozyzygy is defined by the shift functor $[1]$. Every standard triangle is also on the form:

$$M \xrightarrow{f} N \longrightarrow cone(f) \longrightarrow M[1]$$

To define the derived category $D(A)$ of A we will consider the localization of $K(A)$ at the quasi-isomorphisms, $D(A) = K(A)[Qiso^{-1}]$. To see that the derived category is still triangulated we may realize it as a Verdier quotient of $K(A)$.

Proposition 3.1.15. *The derived category of A is equivalent to the Verdier quotient ${}^{K(A)}/_{Ac}$, where Ac denotes the image of acyclic objects in $K(A)$.*

Proof. A proof may be found in Buehler [27] or Thorbjørnsen [28]. □

There is another way of telling the story of the derived category $D(A)$. That is to directly localize it at the quasi-isomorphisms. We may directly see that $D(A) \simeq Mod^A[Qiso^{-1}]$ which we know is $HoMod^A$ by definition. This gives us our first important identification.

Theorem 3.1.16. *The homotopy category of Mod^A is triangulated, and moreover it is the derived category $D(A)$.*

Proof. Follows from discussion above. □

The triangulated construction for the category $hoCoMod^C$ closely resembles that of $HoMod^A$. We start by studying the Frobenius pair $(CoMod^C, \mathcal{E})$, where $\mathcal{E}$ is the same exact structure. Notice that this exact structure only takes the underlying category of chain complexes into account, so this follows from the above description.

We define the injectively stable category $\underline{CoMod^C} = K(C)$ in the same manner. The standard triangles and the additive auto-equivalence stays the same.

At this point things start to differ. The definition for the homotopy category $HoCoMod^C$ is $CoMod^C[Ac^{-1}]$, here Ac denotes the class of weak equivalences in $CoMod^C$. By abuse of notation we also let $Ac \subset K(C)$ be the collection of objects which are cones of weak equivalences. This subcategory can be characterized by being the preimage of acyclic objects $Ac \subset K(A)$ along $L_\tau : CoMod^C \to Mod^A$. This identification suffices to show that $Ac \subset K(C)$ is a triangulated subcategory. In this manner $HoCoMod^C$ is the category ${}^{K(C)}/_{Ac}$, which is a triangulated category.

Remark 3.1.17. We may show that $Ac \subset K(C)$ is a subcategory of acyclic objects. In this manner we get that $D(C) \simeq HoCoMod^C[Qiso^{-1}]$. This is done in Lefevre-Hasegawa [10] as proposition 1.3.5.1 or lemma 2.2.2.11. This follows from the fact that we have an equivalence of categories $CoMod^C[fQiso^{-1}] \simeq HoCoMod^C$, here $fQiso$ means the collection of filtered quasi-isomorphisms. Since every filtered quasi-isomorphism is in fact a quasi-isomorphism by a spectral sequence argument we get the inclusion of triangulated subcategories $\langle cone(fQiso)\rangle \subseteq \langle cone(Qiso)\rangle \subseteq K(C)$.

Let $\tau : C \to A$ and $\upsilon : C \to A'$ be two acyclic twisting morphisms. These defines independently two different model structures on $CoMod^C$ by the adjunctions (L_τ, R_τ) and (L_υ, R_υ). By lemma 3.1.5 we have the identification $(L_\tau, R_\tau) = (f_{\tau!}, f_\tau^*)(L_{\iota_C}, R_{\iota_C}) = (f_{\tau!}L_{\iota_C}, R_{\iota_C} f_\tau^*)$, and a likewise for υ. In order to show that τ and υ defines equivalent module structures on $CoMod^C$ it is enough that both define the same structure as ι_C. By symmetry it is enough to assume that $\upsilon = \iota_C$. From lemma 3.1.7 we know that ι_C is acyclic, so this technique is well-founded.

Since we already know that (L_τ, R_τ) and $(L_{\iota_C}, r_{\iota_C})$ are Quillen equivalences it remains to show that $(F_{\tau!}, f_\tau^*)$ is a Quillen equivalence. This is shown if f_τ^* is a right Quillen functor, and that it induces a triangle equivalence between $D(A)$ and $D(\Omega C)$.

We know that f_τ^* preserves fibrations (epimorphisms). This is because on morphisms this functor acts as the identity, it only changes the ring action, so epimorphisms stay epimorphisms. It remains to see that quasi-isomorphisms are preserved. We will show this by identifying the derived categories. This follows the methods given by Keller in [24]. We will however simplify this discussion by restricting our attention solely to dg-algebras.

Let A be a dg-algebra. We denote A as a free A-module as $\hat{A} = Hom_A(\_, A)$. $\hat{A}$ is free in the enriched sense, i.e. $Hom_A^*(\hat{A}, M) \simeq M$. Recall that P is projective if it is a direct summand of Utdyp dette
$\hat{A}^n$ for some $n \in \mathbb{N}$. Given a right bounded complex M, we know how to construct a projective resolution $p : pM \to M$. Associated to this resoultion there is a triangle in $K(A)$ consisting of the complexes M, pM and aM, where aM is an acyclic complex.

$$M \xrightarrow{p} pM \longrightarrow aM \longrightarrow M[1]$$

In this sense we obtain an identification $M \simeq pM$ in $D(A)^-$. By following Kellers construction we are able to weaken this identification to all of $D(A)$ by weakening the projective resolution. In Kellers paper he calls these complexes of property (P). We will however refer to them as homotopy projective complexes, since these complexes are built up from projective complexes in a manner respecting homotopy colimits.

Definition 3.1.18. Let P be a complex of Mod^A. We say that P is homotopy projective if there exists a complex P', a homotopy equivalence $P \simeq P'$ and a filtration of P'.

$$0 = F_0 \subseteq F_1 \subseteq ... \subseteq F_n \subseteq ... \subseteq P'$$

The filtration should satisfy these properties:

F1 P' is the colimit of the filtration.
F2 Each inclusion $i_n : F_n \subseteq F_{n+1}$ is split as graded modules.
F3 The quotient ${}^{F_{n+1}}/_{F_n}$ is projective.

Remark 3.1.19. The properties $F1$ and $F2$ may be reformulated to that P should be the homotopy colimit of the filtration. Thus there is a canonical triangle in $K(A)$:

$$\bigoplus F_n \xrightarrow{\Phi} \bigoplus F_n \longrightarrow P \longrightarrow \bigoplus F_p[1]$$

Φ is given as the unique morphism which acts as the identity and the inclusion on each summand of $\bigoplus F_p$:

$$\Phi_n = \begin{pmatrix} id_{F_n} \\ -i_n \end{pmatrix}$$

In the definition of a homotopy projective complex we have required that each quotient is strictly projective. If this was true, then these objects would be ill-behaved in the homotopy category. We can weaken this assumption to (F3') the quotient ${}^{F_{n+1}}/_{F_n}$ is homotopy equivalent to a projective complex.

Lemma 3.1.20. *If P is the colimit of a filtration admitting (F2) and (F3'), then P is homotopy projective.*

Proof. Let $\{F_n\}$ denote the filtration on P. To show that P is homotopy projective is to find a homotopy equivalence P' such that P' is the homotopy colimit of a filtration admitting (F3).

Suppose that ${}^{F_{n+1}}/_{F_n} \simeq Q_{n+1}$, where each Q_{n+1} is projective. We wish to inductively define a filtration $\{F'_n\}$ which has (F2) and (F3) and a pointwise homotopy equivalence of filtrations $f : \{F_n\} \to \{F'_n\}$. The object P' is the the (homotopy) colimit of the new filtration.

Define $F_0' = Q_0$, and let $f_0 : F_0 \to F_0'$ be the projection onto Q_0. By assumption f_0 is a homotopy equivalence and we have a commutative square where the vertical arrows are homotopy equivalences. Moreover, each horizontal arrow splits as a graded arrow.

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[d, "0"] & F_0 \arrow[d, "f_0"] \\
0 \arrow[r, "0"] & Q_0
\end{tikzcd}$$

Suppose that we have been able to constructed this filtration up to F_p'. By using our known homotopy equivalences there is an isomorphism of Ext groups:

$$Ext_A({}^{F_p}\!/_{F_{p-1}}, F_{p-1}) \simeq Ext_A(Q_p, F_{p-1}')$$

Given the triangle consisting of F_{p-1}, F_p and ${}^{F_p}\!/_{F_{p-1}}$ there is an assosiated triangle with the morphisms as follows:

$$\begin{tikzcd}
F_{p-1} \arrow[r] \arrow[d, "f_{p-1}"] & F_p \arrow[r] \arrow[d, dashed] & {}^{F_p}\!/_{F_{p-1}}1111 \arrow[r] \arrow[d, "\sim"] & F_{p-1}[1] \arrow[d, "f_{p-1}[1]"] \\
F_{p-1}' \arrow[r] & F_p' \arrow[r] & Q_p \arrow[r] & F_{p-1}'
\end{tikzcd}$$

By the morphism axiom there is a morphism $f_p : F_p \to F_p'$ which is also a homotopy equivalence by the 2-out-of-3 property.

This defines a filtration $\{F_p'\}$, with (F3) and P' as its homotopy colimit. To see that P is homotopy equivalent to P' we use the maps f_p constructed to obtain a homotopy equivalence by the morphism axiom and the 2-out-of-3 property.

$$\begin{tikzcd}
\bigoplus F_p \arrow[r, "\Phi"] \arrow[d, "\oplus f_p"] & \bigoplus F_p \arrow[r] \arrow[d, "\oplus f_p"] & P \arrow[r] \arrow[d, dashed, "\sim"] & \bigoplus F_p[1] \arrow[d, "\oplus f_p[1]"] \\
\bigoplus F_p' \arrow[r, "\Phi'"] & \bigoplus F_p' \arrow[r] & P' \arrow[r] & \bigoplus F_p'[1]
\end{tikzcd}$$

□

The projective complexes are the complexes which are generated by the free module $\hat{A}$ in the sense that they are all in the smallest thick triangulated subcategory of $K(A)$ containing $\hat{A}$. By definition, we may see that the homotopy projective complexes are the complexes in the smallest thick triangulated subcategory of $K(A)$ which is closed under well-ordered homotopy colimits and contains $K(A)$. By devissage we may extend fully-fatihfullness of functors on the set $\{\hat{A}\}$ to the class of homotopy projective objects.

Lemma 3.1.21 (Devissage)**.** *Let $F : \mathcal{T} \to \mathcal{U}$ be a triangulated functor between triangulated categories. Suppose $S \subseteq \mathcal{T}$ is a class of objects closed under shift, and denote $\hat{S}$ for the smallest thick triangulated subcategory (closed under well-ordered homotopy colimits). If $F|_S$ is fully faithful, then $F|_{\hat{S}}$ is fully faithful as well.*

Proof. This is straightforward by using the Yoneda embeddings and 5-lemma. More details may be found in [25]. To get closed under homotopy colimits we also need that F commutes with infinite direct sums, and that the set $\{S\}$ only contains small objects. □

Lemma 3.1.22. *Supppose we have F and S as above. If $F|_S = 0$, then it is 0 on all of $\hat{S}$.*

Proof. The same argument as above, except we have to squeeze out zeros from exact sequences. □

The final ingredient to construct a homotopy projective resolution for our complexes is the acyclic assembly lemma [21].

Lemma 3.1.23 (acyclic assembly)**.** *Suppose that C is a double complex of R-modules. Then $Tot^{\oplus}C$ is acyclic if either:*

- *C is a lower half-plane complex with exact rows.*
- *C is a left half-plane complex with exact columns.*

Proof. This is proposition 2.7.3 in [21]. We omit the proof as the next proof is in some sense very similar. □

Corollary 3.1.23.1. *Suppose that C is a double complex of R-modules such that every column is exact and that the kernels along the rows give rise to exact columns, then $Tot^{\oplus}C$ is acyclic.*

Proof. We want to realize the images along the rows as the coimage along the horizontal differential. Write $Z^n(C)$ for the n-th horizontal kernel and $B^n(C)$ for the n-th horizontal image. We have a short exact sequence of complexes:

$$Z^n(C)^* \longrightarrow C^{n,*} \longrightarrow B^n(C)^*$$

Given that $C^{n,*}$ is acyclic we get that $Z^n(C)^*$ is acyclic if and only if $B^n(C)^*$ is acyclic.

Assuming that all of these three constructions are acyclic we make a filtration on C. Let $F_nC^{p,*} = C$ if $p \in [-n, n-1]$, $F_nC^{n,*} = Z^nC$ and $F_nC^{p,*} = 0$ otherwise.

This filtration is bounded below and exhaustive as colimits commute with colimits.

$$Tot^{\oplus}C = Tot^{\oplus} \varinjlim F_nC \simeq \varinjlim Tot^{\oplus}F_nC$$

We should be a bit careful here as the total complex is not really a coproduct, but since coproducts and cokernels are calculated pointwise we obtain the commutativity.

Now we apply the classical convergence theorem to the filtration to obtain a converging spectral sequence $EF_2C \implies H^*(Tot^{\oplus}C)$. But since we assume each column to be exact in the filtration, the second page is 0, so $H^*(Tot^{\oplus}C) \simeq 0$ as desired. □

Theorem 3.1.24. *Suppose that P is homotopy projective, N is acyclic. Then $K(A)(P,N) \simeq 0$.*

Given any module M, there is a homotopy projective object pM and an acyclic object aM giving rise to a triangle in $K(A)$.

$$pM \longrightarrow M \longrightarrow aM \longrightarrow pM[1]$$

Proof. We assume that $P \simeq \hat{A}$. By a devissage argument we may extend the isomorphism to all homotopy projective P.

$$K(A)(\hat{A},N) \simeq H^0Hom_A^*(\hat{A},N) \simeq H^0N \simeq 0$$

We want to construct two complexes pM and aM by taking total complexes. We show that aM is acyclic by using 3.1.23.1. To use it we will construct an exact sequence of complexes satisfying the assumptions. Described by MacLane [29] there is an exact structure $\mathcal{E}$ on Mod^R such that the collections on conflations are the short exact sequences such that the kernel functor is exact.

$$L \overset{f}{\rightarrowtail} M \overset{g}{\twoheadrightarrow} N$$

$$Z^*L \overset{Z^*f}{\rightarrowtail} Z^*M \overset{Z^*g}{\twoheadrightarrow} Z^*N$$

Since limits commute with limits, the kernel functor preserves any limit. Thus the kernel is left exact, and its only obstruction for exactness is to preserve cokernels. We may thus characterize the conflations by inflations and deflations, which are monomorphisms and epimorphisms which are preserved by the kernel functor. Mac Lane calls these deflations for proper epimorphisms instead.

Mac Lane also shows that there are enough $\mathcal{E}$-projectives with this exact structure. We want to construct $\mathcal{E}$-projectives be on the form of homotopy projective complexes. $\hat{A}[-n]$ is $\mathcal{E}$-projective by the following isomorphism.

$$Hom^\bullet_A(\hat{A}[-n], M) \simeq Z^0 Hom^*_A(\hat{A}, M[n]) \simeq M^n$$

Define the trivialization $trivM$ of M be the underlying graded module M endowed with a trivial differential. This trivial differential is in some sense the inclusion of graded modules into chain complexes. Thus we have the following isomorphism on hom-sets:

$$Hom^\bullet_A(trivM, trivN) \simeq Hom^*_A(M, N)$$

$triv$ is then well-defined as a functor as every morphism between chain complexes uniquely defines a morphism between their trivializations. By using the isomorphisms from Keller [24] 2.2. we get that:

$$\begin{aligned} Hom^\bullet_A(cone(id_{triv\hat{A}}), M) &\simeq Hom^\bullet_A(cone(id_{triv\hat{A}[-1]})[1], M) \\ &\simeq Hom^*_A(triv\hat{A}, trivM[-1])^0 \simeq Hom^*_A(\hat{A}, M)^{-1} \simeq M^{-1} \end{aligned}$$

This shows that if P is homotopy projective, then P and $cone(id_{trivP})$ are $\mathcal{E}$-projective. To see that there are enough $\mathcal{E}$-projectives pick an arbitrary module M. Since we know that there are enough projectives, let P be a projective such that there is an epimorphism $p : P \to M$. We don't know if this morphism is a deflation, so pick another projective Q such that there is an epimorphism $q : Q \to Z^*M$. Since Z^*M has a trivial differential we know that $d_Q q = 0$. Thus this morphism extends to $q' = \begin{bmatrix} q & 0 \end{bmatrix} : cone(id_{trivQ}) \to M$ such that Z^*q' is an epimorphism. The morphism $\begin{bmatrix} p & q' \end{bmatrix} : P \oplus cone(id_{trivQ}) \to M$ is thus a deflation. $P' = P \oplus cone(id_{trivQ})$ shows that we have enough projectives. Moreover every $cone(id_{trivQ})$ has homotopy type 0, so $P' \simeq P$ in $K(A)$.

Since we have enough $\mathcal{E}$-projective, we may construct a $\mathcal{E}$-projective resolution $P'^{*,*}$ of M in the standard way. See Keller [30] for details. Such resolutions are then double complexes, and the augmented resolution below is $\mathcal{E}$-acyclic.

$$\ldots \longrightarrow P'_1 \longrightarrow P'_0 \twoheadrightarrow M \xrightarrow{0} 0$$

Having an $\mathcal{E}$-acyclic resolution means that each row is exact, and taking kernels along the columns preserve exactness of the rows.

Denote the augmentation of $P'^{*,*}$ by $m : P'^{,*} \to M$. We define the complexes $pM = Tot^{\oplus}(P'^{*,*})$ and $aM = Tot^{\oplus}(cone(m))$.

pM carries a natural filtration $F_n pM$ from the double complex structure. Let $F_n pM$ be the truncated complex:

$$\dots \longrightarrow 0 \longrightarrow P'^{n,*} \longrightarrow \dots \longrightarrow P'^{1,*} \longrightarrow P'^{0,*} \longrightarrow 0 \longrightarrow \dots$$

The filtration $F_n pM$ satisfies F1 and F2 by construction. The quotients $F_{n+1}pM/f_n pM \simeq P'_n$ which is homotopy equivalent to a projective. By lemma 3.1.20 pM is homotopy projective.

The complex $cone(m)$ satisfies the conditions for 3.1.23.1, thus aM is acyclic. Thus we have a triangle in $K(A)$ as desired. □

Corollary 3.1.24.1. *Let M be an erbitrary module. If P is homotopy projective, then $K(A)(P, M) \simeq K(A)(P, pM)$. If N is acyclic, then $K(A)(M, N) \simeq (aM, N)$.*

a and p are well-defined functors which commutes with infinite direct sums.

Corollary 3.1.24.2. *Let $\{\hat{A}\}$ denote the smallest thick triangulated subcategory of $D(A)$ which is closed under homotopy colimits. Then $D(A) \simeq \{\hat{A}\}$.*

Corollary 3.1.24.3. *Suppose that $f : A \to B$ is a dg-ring homomorphism and a quasi-isomorphism between dg-algebras, then $D(A) \simeq D(B)$.*

Proof. f endows B with both a left and right A-module structure. We will think of $\hat{B}$ as a left A-module and right B module. There is then a natural hom-tensor adjunction between the differential graded enriched categories.

$$\begin{tikzcd} Diff(B) \arrow[r, bend left, "H"] \arrow[r, phantom, "\top"] & Diff(A) \arrow[l, bend left, "T"] \end{tikzcd}$$

We define $T = \_\otimes_A \hat{B}$ and $H = Diff(B)(B, \_)$. We see that $Diff(B)(T\hat{A}, M) \simeq Diff(A)(\hat{A}, HM) \simeq HM \simeq Diff(B)(\hat{B}, M)$. Thus $T\hat{A} \simeq \hat{B}$, and the morphism $T : Diff(A)(\hat{A}, \hat{A}) \to Diff(B)(\hat{B}, \hat{B})$ is given by f. Since we assume f to be a quasi-isomorphism, it follows that $\mathbb{L}T : D(A) \to D(B)$ is fully faithful on the set $\{\hat{A}\}$.

By devissage the functor $\mathbb{L}T$ is fully-faithful on all of $D(A)$, since $D(A)$ is generated by $\hat{A}$. Since T hits all of $D(B)$s generators, $\mathbb{L}T$ is essentially surjective as well. □

Remark 3.1.25. We have ignored smallness conditions for objects. This technique does not always work, since it depends on some unstated isomorphisms which we have used. We have these since the objects $\hat{A}$ and $\hat{B}$ are small. This detail is given more care in Keller [24].

With this result we are able to show that $HoMod^A$ and $HoMod^{\Omega C}$ are equivalent. Since we assumed the morphism $\tau : C \to A$ to be acyclic, we would expect the morphism $f^*_\tau : \Omega C \to A$ to be a quasi-isomorphism. If this is the case, we know that $D(\Omega C) \simeq D(A)$.

3.1.5 The Fundamental Theorem of Twisting Morphisms

In this section we aim to finish what we started the previous section. We will prove a characterization for the acyclic twisting morphisms.

Theorem 3.1.26 (Fundamental Theorem of Twisting Morphisms)**.** *Let $\tau : C \to A$ be a twisting morphism between augmented objects. The following are equivalent:*

1. *τ is acyclic, i.e. the natural transformation $\varepsilon : L_\tau R_\tau \implies Id_{Mod^A}$ is a pointwise quasi-isomorphism.*
2. *The unit transformation $\eta : Id_{CoMod^C} \implies R_\tau L_\tau$ is a pointwise weak equivalence.*
3. *The counit at A is a quasi-isomorphism, i.e. $\varepsilon_A : L_\tau R_\tau A \to A$ is a quasi-isomorphism.*
4. *The unit at $\mathbb{K}$ is a weak-equivalence, i.e. the algebra unit υ_A and coaugmentation υ_C assembles into a weak-equivalence: $\upsilon_A \otimes \upsilon_C : \mathbb{K} \to A \otimes_\tau C$.*
5. *The morphism of algebras $f_\tau : \Omega C \to A$ is a quasi-isomorphism.*
6. *The morphism of coalgebras $g_\tau : C \to BA$ is a weak-equivalence.*

Proof. Notice that 1. is equivalent to 2. since $\mathbb{L}L$ and $\mathbb{R}R$ are quasi-inverse. 3. is a special case of 1. and 4. is a special case of 2. Observe that 5. and 6. are equivalent since the cobar-bar-adjunction is a Quillen equivalence, 2.2.10.1.

We show 3. $\implies$ 1. Let $\mathcal{T} \subseteq D(A)$ be the full subcategory consisting of objects M where ε_M is a quasi-isomorphism. This subcategory is by assumption non-empty and contains A. By the 5-lemma, making triangles (and smallness of A), this subcategory contains the smallest thick triangulated subcategory closed under homotopy colimits which contains A. We know this to be all of $D(A)$.

To show 4. implies 5. we consider the twisting morphism ι_C. Since ι_C is acyclic we know that the counit at A is a quasi-isomorphism.

$$L_{\iota_C} R_{\iota_C} f_\tau^* A \to f_\tau^* A$$

By assumption the unit morphism $\eta_{\mathbb{K}} : \mathbb{K} \to A \otimes_\tau C$ is a weak equivalence, so the morphism $L_{\iota_C}\eta_{\mathbb{K}} : \Omega C \to L_{\iota_C} R_\tau A = L_{\iota_C} R_{\iota_C} f_\tau^* A$ is a quasi-isomorphism. Let ε' denote the counit of $L_{\iota_C} \dashv R_{\iota_C}$, then we see that $f_\tau = \varepsilon'_A \circ L_{\iota_C}\eta_{\mathbb{K}}$, so f_τ is a quasi-isomorphism by the 2-out-of-3 property.

It remains to show that 5. implies 1. Let the counit of $f_{\tau *} \dashv f_\tau^*$ be denoted as $\tilde{\varepsilon}$. Since f_τ is a quasi-isomorphism, f_τ^* descends to an equivalence between the derived categories 3.1.24.3. Thus $\tilde{\varepsilon} : f_{\tau !} f_\tau^* \implies Id$ is a pointwise quasi-isomorphism. Observe that the counit factors as

$$\varepsilon = \tilde{\varepsilon} \circ f_{\tau !}\varepsilon'_{f_\tau^*}$$

By the 2-out-of-3 property it follows that ε is a quasi-isomorphism. ☐

Corollary 3.1.26.1. *There is one canonical model structure on $CoMod^C$ defined by the acyclic twisting morphisms $\tau : C \to A$, for any algebra A. I.e. each acyclic twisting morphism defines the same model structure for $CoMod^C$.*

Proof. Apply the fundamental theorem of twisting morphisms to the discussion of the last section. □

3.2 Polydules

3.2.1 The Bar Construction

In section 1.3 we saw that we could extend the domain of the bar construction to obtain an equivalence of categories. This converse led us to the definition of an A_∞-algebra, as well as recognizing them as almost free dg-coalgebras. By employing the adjunction $(L_\tau, R_\tau) : CoMod^C \rightleftharpoons Mod^A$ we will do something similar for modules.

Let A be an augmented algebra. The bar construction of A gives us a universal adjunction $(L_{\pi_A}, R_{\pi_A}) : CoMod^{BA} \rightleftharpoons Mod^A$. We will call $R_{\pi_A}(\_[1]) = \_[1] \otimes_{\pi_A} BA$ for B_A, the bar construction on Mod^A. In this manner every A-module M gives rise to an almost free BA-comodule $B_A M$, but does the converse of this construction works?

Let us first look at what B_A does to a module M. $B_A M$ is the dg-comodule which as a graded comodule is the free comodule $M[1] \otimes BA$. The differential of $B_A M$ is given by the A-module structure of M. That is, every elementary element m' of $B_A M$ is an element of M together with a finite string of elements of A.

$$m' = \omega m \otimes (\omega a_1 \otimes ... \otimes \omega a_n)$$

The differential acts on m' by using the differential of $d_{M[1] \otimes BA}$ and multiplication from the right.

$$d_{B_A M}(m') = d_{M[1] \otimes BA}(m') + (-1)^{|m|+|a|} \omega(m \cdot a_1) \otimes (\omega a_2 \otimes ... \otimes \omega a_n)$$

By using delooping, we see that in turn that $d_{B_A M}$ defines an A-module structure for M. We may decompose $B_A M$ as:

$$B_A M = M[1] \oplus M[1] \otimes \bar{A} \oplus M[1] \otimes \bar{A}^{\otimes 2} \oplus ...$$

Let $\pi_M : R_{\pi_A} M \to M$ be the map which kills anything not on the form m. We denote $(d_{B_A M})_i$ by $d_{B_A M} \circ \iota_i$, where $\iota_i : M[-1] \otimes \bar{A}^{\otimes i-1} \hookrightarrow B_A M$. Proposition 1.1.26 tells us that we may recover the structure of M from the differential $d_{B_A M}$. This is done by conjugating the components of $d_{B_A M}$ with desuspension and applying projections appropriately. We recover the maps as:

1. The differential of M is $d_M = s \circ \pi_{M[1]} \circ (d_{B_A M})_1 \omega$

2. The right multiplication from A is $\mu_M = s \circ \pi_{M[-1]} \circ (d_{B_AM})_2 \circ \omega^{\otimes 2}$
3. For $i \geqslant 3$ we have $0 = s \circ \pi_{M[1]} \circ (d_{B_AM})_i \circ \omega^{\otimes i}$

Now, let $\widetilde{N}$ be an almost free BA-comodule. That is, $\widetilde{N} = N[1] \otimes BA$ as a graded comodule. We would now wish for that N carries an A-module structure. Unfortunately we are not that lucky, however, this defines a notion of A_∞-module to the algebra A. If we try to recover the same structure we obtain the following structure morphisms for N:

$$\begin{aligned} \text{A differential of degree } 1\text{: } & m_1 = d_N = s \circ \pi_N (d_{\widetilde{N}})_1 \circ \omega \\ \text{A 2-ary operation of degree } 0\text{: } & m_2 = s \circ \pi_N (d_{\widetilde{N}})_2 \circ \omega^{\otimes 2} \\ \text{A 3-ary operation of degree } -1\text{: } & m_3 = s \circ \pi_N (d_{\widetilde{N}})_3 \circ \omega^{\otimes 3} \\ \text{A 4-ary operation of degree } -2\text{: } & \dots \end{aligned}$$

Let $\widetilde{m}_i$ be the looped versions of the m_is. Then the sum $\sum \widetilde{m}_i : \widetilde{N} \to N$ extends to d_{B_AM}. Since $d^2_{B_AM} = 0$ we get the relations (rel_n) defined in section 1.3 imposed on the morphisms m_i.

To summarize the datum of $M[1] \otimes BA$ as a BA-comodule is equivalent to a chain complex M having maps

$$m_i : M \otimes \bar{A}^{\otimes i-1} \to M$$

of degree $2 - i$ for any $i \geqslant 2$. The maps should satisfy the relations:

$$(rel_n) \qquad \partial(m_n) = - \sum_{\substack{n=p+q+r \\ k=p+1+r \\ k>1, q>1}} (-1)^{pq+r} m_k \circ_{p+1} m_q$$

This gives M the structure of an A-module, where associativity is only well-defined up to strong homotopy. In other words, m_3 is a homotopy for the associator for m_2, m_4 is a homotopy for m_3s associator and so on. Following Lefevre-Hasegawa [10], we call the chain-complex M an A-polydule, given it has maps m_i as above.

We have defined the objects of a category Mod^A_∞. Our goal is to have that the converse bar construction defines an equivalence of categories, i.e. B_A extends to a functor $B_A : Mod^A_\infty \to CoMod^{BA}$ is fully-faithful. This makes sense as every A-module M is a non-full A-polydule by letting $m_1 = d_M$, $m_2 = \mu_M$ and $m_i = 0$ for any $i \geqslant 3$.

Since we say that B_A is fully-faithful any ∞- morphisms between polydules is determined by a morphism of almost free BA-comodules. Let $f : M \rightsquigarrow N$ be an ∞-morphism, then $B_A f : B_A M \to B_A N$ is the associated morphism of dg-comodules. By cofreeness, proposition 1.1.21, we see that Bf is uniquely determined by morphisms $f_i : M \otimes \bar{A}^{\otimes i-1} \to N$ of degree $1-i$. Since we know that $\partial Bf = 0$, we obtain the relations:

$$(rel_n) \qquad \sum_{p+q+r=n} (-1)^{pq+r} f_{p+1+r} \circ_{p+1} m^M_q = \sum_{p+q=n} m^N_{p+1} \circ_1 f_q$$

If $f : M \rightsquigarrow N$ and $g : N \rightsquigarrow P$, then their composition is:

$$(gf)_n = \sum_{p+q=n} g_{p+1} \circ_1 f_q$$

This defines the morphisms of Mod^A_∞. By definition $B_A : Mod^A_\infty \to CoMod^{BA}$ is fully-faithful. An ∞-morphism f is called strict if $f_i = 0$ for any $i \geqslant 2$. We denote the restriction of Mod^A_∞ to strict ∞-morphisms as $Mod^A_{\infty,strict}$. Observe that we obtain an equivalence of categories $Mod^A \simeq Mod^A_{\infty,strict}$.

We will give some examples of A-polydules given an augmented algebra A.

3.2.2 Polydules of SHA-algebras

In the last section we developed the notion of a polydule to an augmented/unital algebra. By applying the converse of the bar construction, we are able to extend this notion to A_∞-algebras.

Suppose that A is an A_∞-algebra. By the bar construction, BA is an almost free coalgebra. In the same manner, we may consider the almost free dg-coalgebras of $CoMod^{BA}$. This is again the collections of comodules of the form $M' = M[1] \otimes BA$. Since there is no obstruction to the above arguments the differential $d_{M'}$ is determined by a collection of morphisms $m^M_n : M \otimes A^{\otimes n-1} \to M$ satisfying (rel_n). Moreover, an ∞-morphism $f : M \rightsquigarrow N$ is a collection of morphisms $f_n : M \otimes A^{\otimes n-1} \to N$ satisfying (rel_n).

Definition 3.2.1. Let A be an A_∞-algebra. The category Mod^A_∞ has A-polydules as objects and ∞-morphisms as morphisms.

The quasi-isomorphisms in Mod^A_∞ are the ∞-morphisms f such that f_1 is a quasi-isomorphism.

Remark 3.2.2. The isomorphisms of Mod^A_∞ are the ∞-morphisms f where f_1 is an isomorphism.

We say that an ∞-morphism is strict if $f_i = 0$ for any $i \geqslant 2$. The category $Mod^A_{\infty,strict}$ is the non-full subcategory of Mod^A_∞ restricted to strict ∞-morphisms.

We may also lift homotopies between almost free BA-comodules and A-polydules. A homotopy $B_A h : B_A M \to B_A M$ is a morphism of degree -1. Thus the collection $h_n : M \otimes A^{\otimes n-1} \to N$ has morphisms of degree $-i$. Moreover, $h : M \rightsquigarrow N$ defines a homotopy of $f, g : M \rightsquigarrow N$ if we have

$$f_n - g_n = \sum_{p+q} (-1)^p m^N_{p+1} \circ_1 h_q - \sum_{p+q+r=n} (-1)^{pq+r} h_{p+1} \circ_{p+1} m^M_q$$

Suppose now that A is instead a strictly unital A_∞-algebra (1.3.12). We may define strictly unital

A-polydules as an A-polydule M such that

$$m_2^M \circ (id_M \otimes \upsilon_A) = id_M$$
$$\forall i \geqslant 3 \quad m_i^M \circ (id_M \otimes ... \otimes \upsilon_A \otimes ... \otimes id_A) = 0$$

An ∞-morphism $f : M \rightsquigarrow N$ is strictly unital if

$$\forall i \geqslant 2 \quad f_i(id_M \otimes ... \otimes \upsilon_A \otimes ... \otimes id_A) = 0$$

This definition also extends to homotopies. We may then define the categories of strictly unital polydules $iMod_\infty^A$ and $iMod_{\infty,strict}^A$.

Given an augmented A_∞-algebra A (1.3.13) we obtain an equivalence of categories. Recall that the categories Alg_∞ and $Alg_{\infty,+}$ were equivalent by taking the kernel of augmentation and applying the free augmentation as its quasi-inverse. In the same manner, given a strictly unital A-polydule M, then it defines a strictly unital $\bar{A}$-polydule $\bar{M}$ by restricting the structure maps to $\bar{A}^{\otimes n}$. This defines an equivalence of categories.

$$\begin{tikzcd} iMod_\infty^A \arrow[r, bend left, "\bar{-}"] & Mod_\infty^{\bar{A}} \arrow[l, bend left, "-^+"] \end{tikzcd}$$

We may call its quasi-inverse for the free strict unitization. This takes an $\bar{A}$-polydule M and turns it into a strictly unital A-polydule by defining the structure morphism as 0 on the unit.

We will for now restrict our attention to augmented A_∞-algebras. The reason for this is that if A is an arbitrary A_∞-algebra, then studying Mod_∞^A would be the same as studying $iMod_\infty^{A^+}$. We extend the bar construction along this equivalence. to a fully faithful functor $B_A : iMod_\infty^A \to CoMod^{BA}$.

3.2.3 Universal Enveloping Algebra

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