

# Sha-Categories & Quasi-isomorphisms

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## 0.1 Localization

We will briefly explain the localization of a category at a set. We will look at local objects and localizations by adjoint functors. This section is based on [Krause21].

### 0.1.1 Definition and Local Objects

The localization of a category at some set of morphisms is a universal category where these morphisms have been inverted. In this manner, we will think of the localization as adding formal inverses to the morphisms in this set to the category.

**Definition 0.1.1.** Let  $\mathcal{C}$  be a category and  $S \subseteq \text{Mor}(\mathcal{C})$  a set of morphisms in  $\mathcal{C}$ . We define the localization of  $\mathcal{C}$  at  $S$  as a category  $\mathcal{C}[S^{-1}]$  together with a functor  $Q : \mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$  satisfying the properties:

1. For any morphism  $s \in S$ ,  $Qs$  is an invertible morphism.
2. Any functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  satisfying  $Fs$  is invertible if  $s \in S$  factors through  $Q$ . Thus there exists a unique functor  $\bar{F} : \mathcal{C}[S^{-1}] \rightarrow \mathcal{D}$  and a natural isomorphism  $\alpha : F \Rightarrow \bar{F} \circ Q$  relating the functors.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ & \searrow Q \quad \nearrow \bar{F} & \\ & \mathcal{C}[S^{-1}] & \end{array}$$

Informally, the category  $\mathcal{C}[S^{-1}]$  may be constructed as the category with the same objects as  $\mathcal{C}$ . The morphisms are constructed with paths of morphisms in  $\mathcal{C}$ , where we also allow formal inverse paths  $s^{-1}$  for any morphism in  $S$ . The constant paths would be the new identity morphisms, while every path from an object  $X$  to  $Y$  represents a morphism.

*Remark 0.1.2.* Beware whenever localizing a locally small category. Adding more morphisms to this category may make some set of morphisms between objects big.

By the universality of the functor  $Q$  we have the following lemma.

**Lemma 0.1.3.** Let  $\mathcal{D}$  be a category, then the pre-composition functor is fully faithful.

$$\begin{aligned} \_ \circ Q : \text{Fun}(\mathcal{C}[S^{-1}], \mathcal{D}) &\rightarrow \text{Fun}(\mathcal{C}, \mathcal{D}) \\ F &\mapsto F \circ Q \end{aligned}$$

Moreover, we may identify  $\text{Fun}(\mathcal{C}[S^{-1}], \mathcal{D})$  as the full subcategory of functors in  $\text{Fun}(\mathcal{C}, \mathcal{D})$  which sends every morphism in  $S$  to an isomorphism.

We proceed to define  $S$ -local objects, and informally, these are all the objects which does not change after the localization process.

**Definition 0.1.4.** An object  $Y \in \mathcal{C}$  is called  $S$ -local ( $S$ -closed or  $S$ -orthogonal) if for any  $s \in S$ , the map  $\mathcal{C}(s, Y)$  is a bijection. Define  $S \perp$  to be the full subcategory of  $S$ -local objects.

The following lemma gives an equivalent definition for "localness".

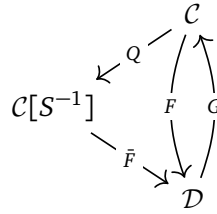
**Lemma 0.1.5.** *An object  $Y \in \mathcal{C}$  is  $S$ -local if and only if the canonical map  $q$  is a natural isomorphism.*

$$q_{X,Y} : \mathcal{C}(X, Y) \rightarrow \mathcal{C}[S^{-1}](X, Y)$$

### 0.1.2 Localizing at Adjoint Functors

Given two functors  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$  such that they form an adjoint pair  $F \dashv G$ , we want to see how they react with localizations.

**Proposition 0.1.6.** *Let  $F \dashv G$  be an adjoint pair as above, and define  $S = \{s \in \text{Mor } \mathcal{C} \mid Fs \text{ is invertible}\}$ . Draw out the following diagram, with the relations  $F = \bar{F} \circ Q$ .*

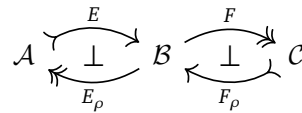


The following are equivalent:

1.  $G$  is fully faithful.
2. The counit  $\varepsilon : FG \Rightarrow \text{Id}_{\mathcal{D}}$  is invertible for any object  $X \in \mathcal{D}$ .
3. The functor  $F$  induces an equivalence  $\bar{F} : \mathcal{C}[S^{-1}] \rightarrow \mathcal{D}$ .

In the light of the last proposition, we may define "short exact sequences" of pre-additive categories.

**Definition 0.1.7.** Suppose there is a diagram of additive functors like below.



It is called a localization sequence whenever the following holds:

1.  $(E, E_\rho)$  and  $(F, F_\rho)$  are adjoint pairs.
2.  $E$  and  $F_\rho$  are fully faithful.
3.  $\text{Im } E \simeq \text{Ker } F$ .