On the Derived Category of Strongly Homotopy Associative Algebras

Thomas Wilskow Thorbjørnsen

September 12, 2022

Abstract

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Acknowledgements

Thank the people in your life who has made this journey easier :D

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Chapter 1

Bar and Cobar Construction

In Stasheffs papers [1] and [2], a strongly homotopy associative algebra, or A_∞-algebra, over a field is a graded vector space together with homogenous linear maps $m_n : A^{\otimes n} \to A$ of degree $2-n$ satisfying some homotopical relations. This will be made precise later. We will regard m_2 to be a multiplication of A, but it is however not a priori associative. We choose m_3 to be a homotopy of m_2's associator. In this manner, we know that the homotopy of A is an associative algebra. The maps m_n corresponds uniquely to a map $m^c : BA \to \overline{A}[1]$, which extends to a coderivation $m^c : BA \to BA$ of the bar construction of A. By this relation we could instead define an A_∞-algebra to be a coalgebra on the form BA, and we will prefer to do so in this thesis.

In order to understand the bar construction, we will first study it on associative algebras. Given a differential graded coassociative coalgebra C and a differential graded associative algebra A, we say that a homogenous linear transformation $\alpha : C \to A$ is twisting if it satisfies the Maurer-Cartan equation;

$$\partial\alpha + \alpha \star \alpha = 0.$$

Let $\mathsf{Tw}(C, A)$ be the set of twisting morphisms. By considering it as a functor $\mathsf{Tw} : \mathsf{coAlg}_{\mathbb{K}}^{op} \times \mathsf{Alg}_{\mathbb{K}} \to Ab$ we want to show that it is represented in both arguments. Moreover, these representations give rise to an adjoint pair of functors, called the bar and cobar construction.

$$\begin{tikzcd} \mathsf{Alg}^{\bullet}_{\mathbb{K},+} \arrow[r, bend left, "B"] \arrow[r, phantom, "\top"] & \mathsf{Coalg}^{\bullet}_{\mathbb{K},conil} \arrow[l, bend left, "\Omega"] \end{tikzcd}$$

The bar and cobar construction will be the basis for our discussion of A_∞-algebras. As the bar construction can be used to define A_∞-algebras, we may easily dualize this to define A_∞-coalgebras in terms of the cobar construction. This chapter will follow the notions and progression presented in Loday and Vallete [3] to develop the theory for the bar-cobar adjunction.

1.1 Preliminaries

1.1.1 Algebras

This section presents a review of associative algebras over a field $\mathbb{K}$. The main goal of this section is to establish some conventions before we look at how they are dualized to coalgebras. This collection of algebras, together with ring homomorphisms, form the category $\mathsf{Alg}_{\mathbb{K}}$ of associative algebras over $\mathbb{K}$.

Definition 1.1.1 ($\mathbb{K}$-Algebra)**.** Let $\mathbb{K}$ be a field with unit 1. A $\mathbb{K}$-algebra A, or an algebra A over $\mathbb{K}$, is a vector space with structure morphisms called multiplication and unit,

$$(\cdot_A) : A \otimes_{\mathbb{K}} A \to A$$
$$1_A : \mathbb{K} \to A,$$

satisfying the associativity and identity laws.

$$\text{(associativity)} \quad (a \cdot_A b) \cdot_A c = a \cdot_A (b \cdot_A c)$$
$$\text{(unitality)} \quad 1_A(1) \cdot_A a = a = a \cdot_A 1_A(1)$$

Whenever A does not posess a unit morphism, we will call A a non-unital algebra. In this case only the associativity law must hold.

By abuse of notation we will confuse the unit of $\mathbb{K}$ by the unit of A. Since 1_A is a ring homomorphism, this should not be a cause of confusion. However, when we want to use the unit as a morphism, we will stick to the 1_A notation. When there are no confusion, we will exchange the symbol $(\cdot_A)$ with words in A. $(\cdot_A)$ is then concatenation of variables.

Definition 1.1.2 (Algebra homomorphisms)**.** Let A and B be algebras. Then $f : A \to B$ is an algebra homomorphism if

1. f is $\mathbb{K}$-linear
2. $f(ab) = f(a)f(b)$
3. $f \circ 1_A = 1_B$

Whenever A and B are non-unital, we must drop the condition that f preserves units.

Definition 1.1.3 (Category of algebras)**.** We let $\mathsf{Alg}_{\mathbb{K}}$ denote the category of $\mathbb{K}$-algebras. Its objects consists of every algebra A, and the morphisms are algebra homomorphisms. The sets of morphisms between A and B are denoted as $\mathsf{Alg}_{\mathbb{K}}(A, B)$.

Let $\widehat{\mathsf{Alg}}_{\mathbb{K}}$ denote the category of non-unital algebras. It's objects consists of every non-unital algebra A, and the morphisms are non-unital algebra homomorphisms. The sets of morphisms between A and B are denoted as $\widehat{\mathsf{Alg}}_{\mathbb{K}}(A, B)$.

The cagtegory $(\mathsf{Mod}_{\mathbb{K}}, \otimes_{\mathbb{K}}, \mathbb{Z})$ is symmetric monoidal. Observe that given any algebra A in $\mathsf{Mod}_{\mathbb{K}}$, the triple $(A, (\cdot_A), 1_A)$ is a monoid. This gives us an isomorphism of categories, namely that $\mathsf{Alg}_{\mathbb{K}}$ is the category of monoids in $\mathsf{Mod}_{\mathbb{K}}$. Therefore, the algebra axioms are the following commutative diagrams:

$$\begin{tikzcd}
A \otimes_{\mathbb{K}} A \otimes_{\mathbb{K}} A \arrow[r, "(\cdot_A)\otimes id_{\mathbb{K}}"] \arrow[d, "id_{\mathbb{K}}\otimes(\cdot_A)"] & A \otimes_{\mathbb{K}} A \arrow[d, "(\cdot_A)"] \\
A \otimes_{\mathbb{K}} A \arrow[r, "(\cdot_A)"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes_{\mathbb{K}} \mathbb{K} \arrow[r, "id_A\otimes 1_A"] \arrow[dr, "\simeq"'] & A \otimes_{\mathbb{K}} A \arrow[d, "(\cdot_A)"] & \mathbb{K} \otimes_{\mathbb{K}} A \arrow[l, "1_A\otimes id_A"'] \arrow[dl, "\simeq"] \\
& A &
\end{tikzcd}$$

In any symmetric monoidal category $\mathcal{C}$ we may reformulate these defintions by using the monoidal structure more heavily. In section 3 we will introduce electronic circuits, which are inspired by some of the proofs found in [3]. These conventions will give us a graphical calculus of morphisms in $\mathcal{C}$.

We supply some examples of algebras one may encounter in nature.

Example 1.1.4. Let $\mathbb{K}$ be any field. The field is trivially an algebra over itself.

Example 1.1.5. The complex numbers $\mathbb{C}$ is an algebra over $\mathbb{R}$. This is true since it is a vector space over $\mathbb{R}$, and complex multiplication respects scalar multiplication.

Example 1.1.6. Let $\mathbb{K}$ be any field. The ring of n-dimensional matrices $M_n(\mathbb{K})$ is an algebra over $\mathbb{K}$. The multiplication is taken to be the matrix multiplication and the unit is the n-dimensional identity matrix.

Augmented algebras will be central to our discussion. An algebra A is augmented if it is equipped with an algebra homomorphism which splits the algebra into an ideal component and a unit component. We make this precise with the following definition

Definition 1.1.7 (Augmented algebras)**.** Let A be an algebra. It is called augmented if there is an algebra homomorphism $\varepsilon_A : A \to \mathbb{K}$.

Given this algebra homomorphism we know that it has to preserve the unit. Thus the kernel $\mathsf{Ker}\varepsilon_A \subseteq A$ is almost A, but without its unit. In the module category $Mod_{\mathbb{K}}$ the morphism ε_A is automatically a split-epimorphism, where the splitting is given by the unit 1_A. Thus as a module we have $A \simeq \overline{A} \oplus \mathbb{K}$, where $\overline{A} = \mathsf{Ker}\varepsilon_A$. $\overline{A}$ is called the augmentation ideal, or the reduced algebra of A.

A morphism $f : A \to B$ of augmented algebras is an algebra homomorphism, but with the added condition that it must preserve the augmentation; i.e. $\varepsilon_B \circ f = \varepsilon_A$. The collection of all augmented algebras over $\mathbb{K}$ together with the morphisms defines the category of augmented algebras over $\mathbb{K}$, $\mathsf{Alg}_{\mathbb{K},+}$.

Given an augmented algebra A, taking kernels of ε_A gives a functor $\overline{} : \mathsf{Alg}_{\mathbb{K},+} \to \widehat{\mathsf{Alg}}_{\mathbb{K}}$. This functor is well defined on morphisms of augmented algebras, as each morphism is required to preserve the splitting. This functor has a quasi-inverse, given by the free augmentation $\_^+ : \widehat{\mathsf{Alg}}_{\mathbb{K}} \to \mathsf{Alg}_{\mathbb{K},+}$. Given a non-unital algebra A, the free augmentation is defined as $A^+ = A \oplus \mathbb{K}$, where the multiplication is given by:

$$(a, k)(a', k') = (aa' + ak' + a'k, kk').$$

The unit is given by the element $(0, 1)$. We summarize this in the statement below.

Proposition 1.1.8. *The functors $\overline{\_}$ and $\_^+$ are quasi-inverse to each other.*

Proof. We show that the free augmentation functor is fully-faithfull and essentially surjective.

Let A and B be non-unital $\mathbb{K}$-algebras, and let $f, g : A \to B$ morphisms in $\widehat{\mathsf{Alg}}_{\mathbb{K}}$. Suppose that $f^+ = g^+$, then $f = \overline{f^+} = \overline{g^+} = g$. Now suppose that $h : A^+ \to B^+$, then $h = \overline{h}^+$.

Suppose that $A \in \mathsf{Alg}_{\mathbb{K},+}$. We want to show that $A \simeq \overline{A}^+$. As $\mathbb{K}$-modules, $A = \overline{A}^+$, so we propose that $id_A : A \to \overline{A}^+$ induces an isomorphism. To see that id_A is an algebra homomorphism is to see that the multiplication in A decomposes as $(a_1 + k)(a_2 + l) = (a_1 a_2 + a_1 l + k a_2) + kl$, where $a_1, a_2 \in \overline{A}$ and $k, l \in \mathbb{K}$. The second condition is equivalent to the existence of ε_A. id_A also preserves the augmentation as $\overline{A} \simeq \overline{\overline{A}^+}$. ☐

There are many augmented algebras to encounter in nature. We will note some examples.

Example 1.1.9 (Group algebra). Pick any group G and any field $\mathbb{K}$. The group ring $K[G]$ is an augmented algebra where the augmentation $\varepsilon_{\mathbb{K}[G]} : \mathbb{K}[G] \to \mathbb{K}$ is given as

$$\varepsilon_{\mathbb{K}[G]}(\sum_{g \in G} k_g g) = \sum_{g \in G} k_g.$$

Among our most important example of algebras is the tensor algebra. This is also the free algebra over $\mathbb{K}$.

Example 1.1.10 (Tensor algebra). Let V be a $\mathbb{K}$-module. We define the tensor algebra $T(V)$ of V as the module

$$T(V) = \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \cdots.$$

The tensor algebra is then the algebra consisting of words in V. Given two words $v^1..v^i$ and $w^1...w^j$ in $T(V)$ we define the multiplication by the concatenation operation,

$$\begin{aligned} \nabla_{T(V)} : T(V) \otimes_{\mathbb{K}} T(V) &\to T(V), \\ (v^1...v^i) \otimes (w^1...w^j) &\mapsto v^1...v^i w^1...w^j. \end{aligned}$$

The unit is given by including $\mathbb{K}$ into $T(V)$,

$$\begin{aligned} \upsilon_{T(V)} : \mathbb{K} &\to T(V), \\ 1 &\mapsto 1. \end{aligned}$$

Observe that the tensor algebra is augmented. The projection from $T(V)$ into $\mathbb{K}$ is an algebra homomorphism, so we may split the tensor algebra into its unit and its augmentation ideal $T(V) \simeq \mathbb{K} \oplus \overline{T}(V)$. We call $\overline{T}(V)$ the reduced tensor algebra.

Proposition 1.1.11 (Tensor algebras are free)**.** *The tensor algebras are the free algebras over the category of $\mathbb{K}$-modules, i.e. for any $\mathbb{K}$-module V there is a natural isomorphism Hom$_{\mathbb{K}}(V, A) \simeq$ Alg$_{\mathbb{K}}(T(V), A)$.*

The reduced tensor algebra is the free non-unital algebra over the category of $\mathbb{K}$-modules, i.e. for any $\mathbb{K}$-module V there is a natural isomorphism Hom$_{\mathbb{K}}(V, A) \simeq \widehat{\text{Alg}}_{\mathbb{K}}(\overline{T}(V), A)$.

Proof. This proposition should be evident from the description of an algebra homomorphism from a tensor algebra. If $f : T(V) \to A$ is an algebra homomorphism, then f must satisfy the following conditions:

- Unitality: $f(1) = 1$
- Homomorphism property: Given $v, w \in V$, then $f(vw) = f(v) \cdot_A f(w)$

By induction, we see that f is completely determined by where it sends the elements of V. Thus restriction by the inclusion of V into $T(V)$ induces a bijection. □

Just as for rings, every algebra A has a module category of its own. The definition is very much like that for rings.

Definition 1.1.12 (Modules)**.** Let A be an algebra over $\mathbb{K}$. A $\mathbb{K}$-module M is said to be a left (right) A-module if there exists a structure morphism $\mu_M : A \otimes_{\mathbb{K}} M \to M$ ($\mu_M : M \otimes_{\mathbb{K}} A \to M$) called multiplication. We require that μ_M is associative and preserves the unit of A; i.e. we have the commutative diagrams in $Mod_{\mathbb{K}}$,

$$\begin{tikzcd} A \otimes_{\mathbb{K}} A \otimes_{\mathbb{K}} M \arrow[r, "A\otimes\mu_M"] \arrow[d, "\cdot_A \otimes M"] & A \otimes_{\mathbb{K}} M \arrow[d, "\mu_M"] \\ A \otimes_{\mathbb{K}} M \arrow[r, "\mu_M"] & M \end{tikzcd} \qquad \begin{tikzcd} \mathbb{K} \otimes_{\mathbb{K}} M \arrow[r, "1_A\otimes M"] \arrow[dr, "\simeq"] & A \otimes_{\mathbb{K}} M \arrow[d, "\mu_M"] \\ & M \end{tikzcd}$$

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tum???

Definition 1.1.13 (A-linear homomorphisms)**.** Let M, N be two left A-modules. A morphism $f : M \to N$ is called A-linear if it is $\mathbb{K}$-linear and for any a in A $f(am) = af(m)$.

The category of left A-modules is denoted as Mod_A, where the morphisms Hom$_A(_, _)$ are A-linear. Likewise, the category of right A-modules is denoted as Mod^A. There is a free functor from the category of $\mathbb{K}$-modules to left A-modules.

Proposition 1.1.14. *Let M be a $\mathbb{K}$-module. The module $A \otimes_{\mathbb{K}} M$ is a left A-module. Moreover, it is the free left module over $\mathbb{K}$-modules, i.e. there is a natural isomorphism Hom$_{\mathbb{K}}(M, N) \simeq$ Hom$_A(A \otimes_{\mathbb{K}} M, N)$.*

Proof. We define natural transformations in each direction and then show that they are inverses.

We define morphisms ϕ and ψ as

$$\begin{aligned} \phi : \mathsf{Hom}_A(A \otimes_{\mathbb{K}} M, N) &\to \mathsf{Hom}_{\mathbb{K}}(M, N) \\ f &\mapsto f \circ (1_A \otimes M), \\ \psi : \mathsf{Hom}_{\mathbb{K}}(M, N) &\to \mathsf{Hom}_A(A \otimes_{\mathbb{K}} M, N) \\ g &\mapsto \mu_N \circ (A \otimes g). \end{aligned}$$

Pick an $f \in \mathsf{Hom}_A(A \otimes_{\mathbb{K}} M, N)$, then

$$\psi \circ \phi(f) = \mu_N \circ (A \otimes \phi(f)) = \mu_N \circ (A \otimes f(1_A \otimes M)) = f(A \otimes M) = f.$$

Pick a $g \in \mathsf{Hom}_{\mathbb{K}}(M, N)$, then

$$\phi \circ \psi(g) = \phi(\mu_N \circ (A \otimes g)) = \mu_N \circ (1_A \otimes g) = g.$$

□

Corollary 1.1.14.1. *A as a left A-module is the free left A-module over $\mathbb{K}$; i.e. for any left A-module M, $M \simeq$ Hom$_{\mathbb{K}}(\mathbb{K}, M) \simeq$ Hom$_A(A, M)$*

1.1.2 Coalgebras

In this section we want to dualize the definitions given in the last section. As we will see, some strange artifacts from the structure of algebras appears in the dualization process. We would like to define a coalgebra as an algebra, but where all of the diagrams are reversed. For many purposes this dualization is good, but as we will observe, some finiteness conditions are necessary. The category of "ill-behaved" coalgebras will be denoted coAlg$_{\mathbb{K}}$, and we will look at this first.

Definition 1.1.15 ($\mathbb{K}$-Coalgebra)**.** Let $\mathbb{K}$ be a field. A coalgebra C over $\mathbb{K}$ is a $\mathbb{K}$-module with structure morphisms called comultiplication and counit,

$$\begin{aligned} (\Delta_C) : C &\to C \otimes_{\mathbb{K}} C \\ \varepsilon_C : C &\to \mathbb{K}, \end{aligned}$$

satisfying the coassociativity and coidentity laws.

$$\begin{aligned} &\text{(coassociativity)} \quad (\Delta_C \otimes id_C) \circ \Delta_C(c) = (id_C \otimes \Delta_C) \circ \Delta_C(c) \\ &\text{(counitality)} \quad (id_C \otimes \varepsilon_C) \circ \Delta_C(c) = c = (\varepsilon_C \otimes id_C) \circ \Delta_C(c) \end{aligned}$$

In the same way as for algebras, we say that a coalgebra is non-counital if it is without a counit.

Unlike algebras, coalgebras does not admit a single intuitive method for writing repeated application of the comulitplication. For instance, given an element $c \in C$, we may apply the comultiplication twice on c in two different ways:

$$\Delta^2_{C,(1)}(c) = (\Delta_C \otimes C)\Delta_C(c),$$
$$\Delta^2_{C,(2)}(c) = (C \otimes \Delta_C)\Delta_C(c).$$

One should immediately note that $\Delta^2_{C,(1)}(c) = \Delta^2_{C,(2)}(c)$ is coassociativity. Hence there is in fact a unique way to do repeated application of Δ_C on c. We denote the n-fold repeated application of Δ_C by Δ^n_C. Since the element $\Delta^n_C(c)$ represents a finite sum in $C^{\otimes n}$, one may use Sweedlers notation [3], as it is well-defined,

$$\Delta^n_C(c) = \sum c_{(1)} \otimes ... \otimes c_{(n)}.$$

Definition 1.1.16 (Coalgebra homomorphism)**.** Let C and D be coalgebras. Then $f : C \to D$ is a coalgebra morphism if

1. f is $\mathbb{K}$-linear
2. $(f \otimes f) \circ \Delta_C(c) = \Delta_D(f(c))$
3. $\varepsilon_D \circ f = \varepsilon_C$

Whenever C and D are non-counital, we only require 1. and 2. for a homomorphism of non-counital coalgebras.

Definition 1.1.17 (Category of coalgebras)**.** Let $\mathsf{coAlg}_{\mathbb{K}}$ denote the category of coalgebras. Its objects consists of coalgebras C, and the morphisms are coalgebra homomorphisms. The set of morphisms between C and D are denoted as $\mathsf{coAlg}_{\mathbb{K}}(C, D)$.

Let $\widehat{\mathsf{coAlg}}_{\mathbb{K}}$ denote the category of non-counital algebras. Its objects consists of non-counital algebras C, and the morphisms are non-uconital coalgebra homomorphisms. The set of morphisms between C and D are denoted as $\widehat{\mathsf{coAlg}}_{\mathbb{K}}(C, D)$.

At a first glance, coalgebras may seem weird and unnatural, but they appear many places in nature.

Example 1.1.18 ($\mathbb{K}$ as a coalgebra)*.* The field $\mathbb{K}$ can be given a coalgebra structure over itself. Since $\{1\}$ is a basis for $\mathbb{K}$ we define the structure morphisms as

$$\Delta_{\mathbb{K}}(1) = 1 \otimes 1$$
$$\varepsilon(1) = 1.$$

One may check that these morphisms are indeed coassociative and counital. Thus we may regard our field as either an algebra or a coalgebra over itself.

Example 1.1.19 ($\mathbb{K}[G]$ as a coalgebra)*.* The group algebra has a natural coalgebra structure. We may take duplication of group elements as the comultiplication, i.e.

$$\Delta_{\mathbb{K}[G]}(kg) = kg \otimes g.$$

Coincidentally we have already defined the counit, this is the augmentation $\varepsilon_{\mathbb{K}[G]}$ for the group algebra $\mathbb{K}[G]$. Recall that this was

$$\varepsilon_C(\sum k_g g) = \sum k_g.$$

One may see that these morphisms satisfy coassociativity and counitality.

Example 1.1.20 (The linear dual coalgebra). Let M be any finite dimensional $\mathbb{K}$-module. There is a natural isomorphism $\xi : M^* \otimes_{\mathbb{K}} M^* \to (M \otimes_{\mathbb{K}} M)^*$, given on elementary tensors as

$$\xi(f \otimes g)(m \otimes n) = f(m)g(n).$$

Let A be a finite dimensional algebra, then its linear dual A^* is a coalgebra. The linear dual of the multiplication $(\cdot_A)$ is defined as

$$(\cdot_A)^* : A^* \to (A \otimes_{\mathbb{K}} A)^*.$$

We define the comulitplication of A^* as $\xi^{-1}(\cdot_A)^*$.

The counit of A^* is the morphism 1_A^*.

Before we state our main example, we will first introduce its essential structure.

Definition 1.1.21 (Coaugmented coalgebras)**.** Let C be a coalgebra. C is coagumented if there is a coalgebra homomorphism $\eta_C : \mathbb{K} \to C$.

Similar to augmented algebras, each coaugmented coalgebra splits in the category $Mod_{\mathbb{K}}$. We first notice that given a coalgebra homomorphism f, the cokernel Cok$f$ is also a coalgebra. This is some induced quotient structure. Given a coaugmentation $\eta_C : \mathbb{K} \to C$, we call Cok$\eta_C = \overline{C}$ for the coaugmentation quotient or reduced coalgebra of C. Then we obtain the splitting $C \simeq \overline{C} \oplus \mathbb{K}$. The reduced coproduct, denoted $\overline{\Delta}_C$ may explicitly be given as

$$\overline{\Delta}_C(c) = \Delta_C(c) - 1 \otimes c - c \otimes 1.$$

Example 1.1.22 (Tensor Coalgebras). Let V be a $\mathbb{K}$-module. We define the tensor coalgebra $T^c(V)$ of V as the module

$$T^c(V) = \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \cdots.$$

Given a string $v^1 ... v^i$ in $T(V)$ we define the comultiplication by the deconcatenation operation,

$$\begin{aligned}\Delta_{T^c(V)} &: T^c(V) \to T^c(V) \otimes_{\mathbb{K}} T^c(V) \\ v^1 ... v^i &\mapsto 1 \otimes (v^1 ... v^i) + (\sum_{j=1}^{i-1} (v^1 ... v^j) \otimes (v^{j+1} ... v^i)) + (v^1 ... v^i) \otimes 1.\end{aligned}$$

The counit is given by projecting $T^c(V)$ onto $\mathbb{K}$,

$$\begin{aligned}\varepsilon_{T^c(V)} &: T^c(V) \to \mathbb{K} \\ 1 &\mapsto 1 \\ v^1 ... v^i &\mapsto .\end{aligned}$$

We observe that the tensor coalgebra is coaugmented. Its coaugmentation is given by the inclusion of $\mathbb{K}$ into $T^c(V)$. We can split $T^c(V) \simeq \mathbb{K} \oplus \overline{T}^c(V)$, where $\overline{T}^c(V)$ denotes the reduced tensor coalgebra.

Cofreeness does not come for free for the tensor coalgebra. Our problem is a mismatch in the behavior of algebras and coalgebras. The problem arises when we try to do evaluation. Suppose that A is an algebra and that we have n elements of A, i.e. an element of $A^{\otimes n}$. On this element we may apply the multiplication of A a maximum of n-times, we cannot multiply an element with nothing non-trivially. However, given an element in a coalgebra C, we may use the comultiplication on this element n times, $n+1$ times and so on ad infinitum. In the coalgebra we may comultiply an element possibly an infinite amount of times. This is a property which is sometimes ill-behaved with our dualization of algebras to coalgebras.

However, when we dualized the tensor algebra to the tensor coalgebra we did not lose this property, that an element may only be comultiplied a finite number of times. This is due to the composition of $T^c(V)$ as a direct sum of $V^{\otimes n}$, i.e. any element is a finite sum of finite tensors.

This extra assumption we need for coalgebras will be called conilpotent. Let $C \simeq \mathbb{K} \oplus \overline{C}$ be a coaugmented coalgebra, we define the coradical filtration of C as a filtration $Fr_0 C \subseteq Fr_1 C \subseteq \ldots \subseteq Fr_r C \subseteq \ldots$ by the submodules:

$$\begin{aligned} Fr_0 C &= \mathbb{K} \\ Fr_r C &= \mathbb{K} \oplus \{c \in \overline{C} \mid \forall n \geqslant r, \overline{\Delta}_C(c) = 0\}. \end{aligned}$$

Definition 1.1.23 (Conilpotent coalgebras)**.** Let C be a coaugmented coalgebra. We say that C is conilpotent if its coradical filtration is exhaustive, i.e.

$$\varinjlim_r Fr_r C \simeq C.$$

The full subcategory of conilpotent coalgebras will be denoted as $\mathsf{coAlg}_{\mathbb{K},conil}$.

Proposition 1.1.24 (Conilpotent tensor coalgebra)**.** *Let V be a $\mathbb{K}$-module. The tensor coalgebra $T^c(V)$ is conilpotent.*

Proof. Let $v \in V$, then $\Delta_{T^c(V)}(v) = 1 \otimes v + v \otimes 1$ and $\overline{\Delta}_{T^c(V)}(v) = 0$. We then observe the following:

$$\begin{aligned} Fr_0 T^c(V) &= \mathbb{K}, \\ Fr_1 T^c(V) &= \mathbb{K} \oplus V, \\ Fr_r T^c(V) &= \bigoplus_{i \leqslant r} V^{\otimes i}. \end{aligned}$$

This shows that the coradical filtration is exhaustive. ☐

Proposition 1.1.25 (Cofree tensor coalgebra)**.** *The tensor coalgebra is the cofree conilpotent coalgebra over the category of $\mathbb{K}$-modules, i.e. for any $\mathbb{K}$-module V and any conilpotent coalgebra C there is a natural isomorphism $\mathrm{Hom}_{\mathbb{K}}(\overline{C}, V) \simeq \mathsf{coAlg}_{\mathbb{K},conil}(C, T^c(V))$.*

Proof. This proposition should be evident from the description of a coalgebra homomorphism into the a tensor coalgebra. If $g : C \to T^c(V)$ is a coalgebra homomorphism, then g must satisfy the following conditions:

1. (Coaugmentation) $g(1) = 1$,
2. (Counitality) Given $c \in \overline{C}$ then $\varepsilon_{T^c(V)} \circ g(c) = 0$,
3. (Homomorphism property) Given $c \in C$ then $\Delta_{T^c(V)}(g(c)) = (g \otimes g) \circ \Delta_C(c)$.

We will construct the maps for the isomorphism explicitly. If $g : C \to T^c(V)$ is a coalgebra homomorphism, then composing with projection gives a map $\pi \circ g : C \to V$. Note that $\pi \circ g(1) = 0$, so this is essentially a map $\pi \circ g : \overline{C} \to V$. For the other direction, let $\overline{g} : \overline{C} \to V$. We will then define g as

$$g = id_{\mathbb{K}} \oplus \sum_{i=1}^{\infty} (\otimes^i \overline{g}) \overline{\Delta}_C^{i-1}.$$

Observe that g is well defined, since convergence of the sum follows from conilpotency of C. One may then check that g is a coalgebra homomorphism, which yields the result. $\square$

Essential to our dualization are comodules. We quickly give a definition of them.

Definition 1.1.26 (Comodules)**.** Let C be a coalgebra. A $\mathbb{K}$-module M is said to ba left (right) C-comodule if there exist a structure morphism $\omega_M : M \to C \otimes_{\mathbb{K}} M$ ($\omega_M : M \to M \otimes_{\mathbb{K}} C$) called comultiplication. We require that ω_M is coassociative with respect to the comultiplication of C and preserves the counit of C; i.e. we have the following commutative diagrams in $Mod_{\mathbb{K}}$,

$$\begin{tikzcd}
C \otimes_{\mathbb{K}} C \otimes_{\mathbb{K}} M & C \otimes_{\mathbb{K}} M \arrow[l, "C\otimes\omega_M"'] \\
C \otimes_{\mathbb{K}} M \arrow[u, "\Delta_C \otimes M"] & M \arrow[l, "\omega_M"] \arrow[u, "\omega_M"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{K} \otimes_{\mathbb{K}} M & C \otimes_{\mathbb{K}} M \arrow[l, "\varepsilon_C\otimes M"'] \\
 & M \arrow[ul, "\simeq"] \arrow[u, "\omega_M"']
\end{tikzcd}$$

Definition 1.1.27 (C-colinear homomorphism)**.** Let M, N be two left C-comodules. A morphism $g : M \to N$ is called C-colinear if it is $\mathbb{K}$-linear and for any m in M, $\omega_N(g(m)) = (id_C \otimes g)\omega_M(m)$. In Sweedlers notation this looks like

$$\sum g(c)_{(1)} \otimes g(c)_{(2)} = \sum c_{(1)} \otimes g(c_{(2)}).$$

The category of left C-comodules is denoted as CoMod$_C$, where the morphisms Hom$_C(_,_)$ are C-colinear. We would also like to restrict our attention to those C-comodules which are conilpotent, i.e. those comodules which have an exhaustive coradical filtration. The coradical filtration is defined analogously, as we only care for the $\mathbb{K}$-module structure. Notice that for conilpotent coalgebras this requirement is automatic. Likewise, the category of right C-comodules is denoted as CoMod^C.

Proposition 1.1.28. *Let M be a $\mathbb{K}$-module. The module $C \otimes_{\mathbb{K}} M$ is a left C-comodule. Moreover, it is the cofree left comodule over $\mathbb{K}$-modules, i.e. there is an isomorphism Hom$_{\mathbb{K}}(N, M) \simeq$ Hom$_C(N, C \otimes_{\mathbb{K}} M)$.*

Proof. This is dual to proposition 1.1.14. We only give the isomorphism, its validity is clear.

$$\begin{aligned} \phi' : \mathrm{Hom}_C(N, C \otimes_{\mathbb{K}} M) &\to \mathrm{Hom}_{\mathbb{K}}(N, M) \\ f &\mapsto (\varepsilon_C \otimes M) \circ f, \\ \psi' : \mathrm{Hom}_{\mathbb{K}}(N, M) &\to \mathrm{Hom}_C(N, C \otimes_{\mathbb{K}} M) \\ g &\mapsto (C \otimes g) \circ \omega_N. \end{aligned}$$

□

Corollary 1.1.28.1. *C as a left C-comodule is the cofree C-comodule over $\mathbb{K}$; i.e. for any left C-comodule N, $N^* \simeq$ Hom$_{\mathbb{K}}(N, \mathbb{K}) \simeq$ Hom$_C(N, C)$.*

1.1.3 Electronic Circuits

Calculations involving both algebras and coalgebras tends to become convoluted and unmanagable. Since we want to study the interplay between algebras and coalgebras using other tools to write down equations can be handy. We will develop the graphical calculus briefly mentioned in [3]. This graphical calculus will consists of string diagrams, refered to as electronic circuits, which describes function compositions of tensors. Since we only care about the interplay of tensors, we may develop this graphical calculus in any closed symmetric monoidal category. Why do we want to introduce this abstraction? A closed symmetric monoidal category is a good category to model functions, or morphisms, which may take several variables in its argument. Moreover, in the next section we are going to switch categories. Thus the same notions may be applied and the differences are only seen in the implementations.

The definition of a closed symmetric monoidal category may be found in the appendix . We recall Skriv appen
that it is a category $\mathcal{C}$ together with a bifunctor $\_ \otimes \_ : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ usually called tensor, and a unit object $Z \in \mathcal{C}$. We are also given four natural isomorphisms relating the functors and the unit to what they are meant to be:

Associator	$\alpha : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$.
Right unit	$\rho : A \otimes Z \to A$.
Left unit	$\lambda : Z \otimes A \to A$.
Braiding/Symmetry	$\beta : A \otimes B \to B \otimes A$.

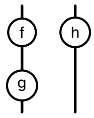
These natural isomorphisms are supposed to satisfy some laws as well. See the appendix for the full definition.

In an electronic circuit we want to rewrite equations, possibly involving tensors, into string diagrams. To illustrate with some simple examples, let $f : A \to B$, $g : B \to C$ and $h : D \to E$. We may consider the composition

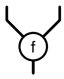
$$(g \otimes E) \circ (f \otimes h) : A \otimes D \to C \otimes E.$$

An electronic circuit is written from top to bottom and is composed of levels. This means that the first morphisms which we apply will be at the top, descending downwards with each function composition. We write each argument in the composition as its own string. Thus this example above will look like the circuit below. Notice how f and h are at the same level, indicating that they should be thought of as $f \otimes h$. Thus an $\otimes$ indicates a change of string, while a $\circ$ indicates a change of level.

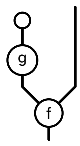
Beware that when many tensors are used, we should really remember how each string is tensored. We may call adding tensors for horizontal composition and composition of morphism for vertical composition. Both of these have a choice in how we associate them, but both have unique choices up to isomorphism given by the associator.



The true power of electronic circuit comes to light when we consider morphisms that in some sense "creates" or "destroys" strings. For example a morphism of 2 variables in some sense "destroys" a string by applying them to each other. Consider now a morphism $f : A \otimes B \to C$, we represent this morphism in an electronic circuit by using a converging fork. Likewise, "creation" of strings is seen as a diverging fork.

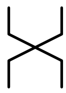


We may write the unit object Z without any strings in a circuit. By the right and left unit, an object is isomorphic to itself tensored with the unit. In this manner, whenever we have a morphism entering or exiting the unit Z we start a new string by using a sink or a source. For example, consider f as before and a morphism $g : Z \to A$, then we may write $f \circ (g \otimes B)$ as the circuit below. Notice that again this is only well-defined up to isomorphism by the right and left unit.



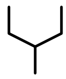
The final operation we have is braiding. Whenever we apply the braiding morphism on the tensors, we may simply denote this as interchanging the strings. For example, $\beta_{A,B} : A \otimes B \to B \otimes A$

is the circuit below. Notice that by naturality of β, we may move a braiding along the circuit. In this manner, if we have two braids, they may undo each other. A braid may also be moved to either end, and in this manner they may be ignored during calculations.

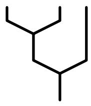
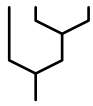


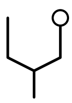
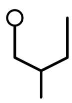
With the language of electronic circuits, we may now write down the axioms of an algebra or coalgebra by circuits. The axioms states the existence of morphisms, as well as commutativity of some diagrams. We may do this electronically. The structure maps of algebras and coalgebras will be given special notation, since we will use these often.

For convenience we will let $\mathcal{C} = \mathsf{Mod}_{\mathbb{K}}$. This is a closed symmetric monoidal category, with $\otimes_{\mathbb{K}}$ as $\otimes$. Recall that an algebra is a $\mathbb{K}$-module A together with maps $(\cdot_A) : A \otimes A \to A$ and $1_A : \mathbb{K} \to A$. These morphisms will be denoted electronically as the diagrams below.

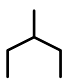
$(\cdot_A)$ =  1_A = 

We write the electronic laws for an algebra as how one would write equations. Associativity and unitality then becomes as follows.

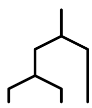
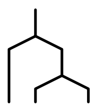
Associativity  $=$ 

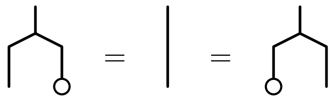
Unitality  $=$ $=$ 

Dually, given a coalgebra C we make the same notation, but everything will be turned upside down. The maps $\Delta_C : C \to C \otimes C$ and $\varepsilon_C : C \to \mathbb{K}$ will be denoted as the following electronic circuits.

Δ_C =  ε_C = 

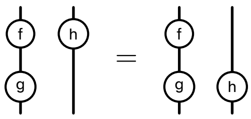
The electronic laws for C becomes the following diagrams.

Coassociativity  $=$ 

Counitality 

This notation will be adopted for our algebras and coalgebras when convenient. This will compactify the notation for comultiplication and make tensors more readable. Moreover, the intuiton for algebras is easier to dualize to coalgebras with this notation. One simply has to work out an algebra statement and then turn the diagram upside down to make it into a coalgebra statement.

Previously we talked about braiding and how that relates to interchanging strings. In the same manner that we have a horizontal and vertical associator, we also have a vertical and horizontal braiding. The braiding we first encountered is the horizontal braiding. The vertical braiding refers then to function composition of tensors. This manifests in electronic circuits as sliding a morphism along a string. Whenever the given braiding of $\mathcal{C}$ is nice enough we can get away by ignoring it whenever we move a morphism along a string. For instance, look at the category of $\mathbb{K}$-modules where we may define the braiding on elementary tensors as $\beta(a \otimes b) = b \otimes a$. In this case the braiding is agnostic to how we move our morphisms along a string. This means that we have the following equality of circuits.

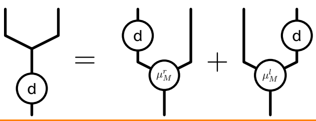


However, in nature we may encounter a braiding which is not nice enough. Here we should take a step back to figure out how we can move morphisms along strings, before we continue to use this graphical calculus of function composition. We will meet such a braiding soon.

1.1.4 Derivations and DG-Algebras

This section aims to define differential graded algebras and their modules. Given an algebra A, we may define a derivation as a map satisfying the Leibniz rule. In the dual case for a coalgebra we may define a coderivation as a map satisfying the Zinbiel rule, but for brevity we will refer to these maps as derivations. Once we get a grasp on how to make derivations we introduce graded algebras and modules, to equip these with derivations. This will give us the categories of differential graded algebras and chain complexes. Throughout this section we will also develop electronic circuits for these notions.

Definition 1.1.29 (Derivations and Coderivations)**.** Let M be an A-bimodule. A $\mathbb{K}$-linear morphism $d : A \to M$ is called a derivation if $d(ab) = d(a)b + ad(b)$, i.e. electronically,



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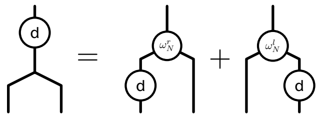
Let N be a C-bicomodule. A $\mathbb{K}$-linear morphism $d : N \to C$ is called a coderivation if $\Delta_C \circ d = (d \otimes id_C) \circ \omega_N^r + (id_C \otimes d) \circ \omega_N^l$, i.e. electronically,



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We remark that this translation between equations and electronic circuits is not at the same level of generalization. Due to this, the electronic circuit description has more advantages as it allows us to think with elements when we are really only dealing with morphisms. This will later be abused, to derive results which is e.g. independent of the given braiding on the category.

A useful fact about derivations is that they will always map the identity to 0. We obtain this from Leibniz rule as one would get $d(1) = 2d(1)$, and thus $d(1) = 0$.

Proposition 1.1.30. *Let V be a $\mathbb{K}$-module and M be a $T(V)$-bimodule. A $\mathbb{K}$-linear morphism $f : V \to M$ uniquely determines a derivation $d_f : T(V) \to M$, i.e. there is an isomorphism $Hom_{\mathbb{K}}(V, M) \simeq Der(T(V), M)$.*

Let N be a $T^c(V)$-bicomodule. A $\mathbb{K}$-linear morphism $g : M \to V$ uniquely determines a coderivation $d_g^c : N \to T^c(V)$, i.e. there is an isomorphism $Hom_{\mathbb{K}}(N, V) \simeq Coder(N, T^c(V))$.

Proof. Let $a_1 \otimes ... \otimes a_n$ be an elementary tensor of $T(V)$. We define a map $d_f : T(V) \to M$ as

$$d_f(a_1 \otimes ... \otimes a_n) = \sum_{i=1}^{n} a_1 ... f(a_i) ... a_n$$
$$d_f(1) = 0.$$

d_f is a derivation by defintion.

Restriction to V gives the natural isomorphism. Let $i : V \to T(V)$, then $i^* d_f = f$. Let $d : T(V) \to M$ be a derivation, then $d_{i^*d} = d$. Suppose that $g : M \to N$ is a morphism between $T(V)$-bimodules, then naturality follows from bilinearity.

In the dual case, $d_g^c : N \to T^c(V)$ is a bit tricky to define. Let $\omega_N^l : N \to N \otimes T^c(V)$ and $\omega_N^r : N \to T^c(V) \otimes N$ denote the coactions on N. Since $T^c(V)$ is conilpotent, we get the same kind of finiteness restrictions on N. Define the reduced coactions as $\overline{\omega}_N^l = \omega_N^l - \_ \otimes 1$ and $\overline{\omega}_N^r = \omega_N^r - 1 \otimes \_$, this is well-defined by coassociativity. Observe that for any $n \in N$ there are k and $k' > 0$ such that $\overline{\omega}_N^{l^k}(n) = 0$ and $\overline{\omega}_N^{r^{k'}}(n) = 0$.

Let $n_{(k)}^{(i)}$ denote the extension of n by k coactions at position i, i.e.

$$n_{(k)}^{(i)} = \overline{\omega}_N^{r^i} \overline{\omega}_N^{l^{k-i}}(n).$$

The extension of n by k coactions is then the sum over every position i,

$$n_{(k)} = \sum_{i=0}^{k} n_{(k)}^{(i)}.$$

Observe that $n_{(0)} = n$. The grade of n may be thought of as the smallest k such that $n_{(k)}$ is zero. This grading gives us the coradical filtration of N, and it is exhaustive by the finiteness restrictions given above. With this notion, every element of N may be given a finite grade.

If $g : N \to V$ is a linear map, we may think of it as a map sending every element of N to an element of $T^c(V)$ of grade 1. To get a map which sends element of grade k to grade k, we must extend the morphism. Let $\pi : T^c(V) \to V$ be the linear projection and define $g_{(k)}^{(i)} = \pi \otimes ... \otimes g \otimes \pi$ as a morphism which of k tensors which is g at the i-th argument, but the projection otherwise. d_g^c is then defined as the sum over each coaction and coordinate,

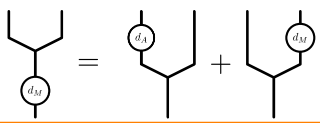
$$d_g^c(n) = \sum_{k=0}^{\infty} \sum_{i=0}^{k} g_{(k)}^{(i)}(n_{(k)}^{(i)}).$$

Upon closer inspection we may observe that this is the dual construction of the derivation morphism. It is well-defined as the sum is finite by the finiteness restrictions. The map is a coderivation by duality, and the natural isomorphism is given by composition with the projection map π. □

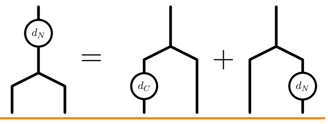
We now wish to get into the territory of derivations as we know them from homological algebra. This is obtained by choosing an algebra and equipping it with a derivation. This will give us a universal derivation to work with when we want to define a differential on a graded module.

Definition 1.1.31 (Differential algebra)**.** Let A be an algebra. We say that A is a differential algebra if it is equipped with a derivation $d : A \to A$. Dually, a coalgebra C is called a differential coalgebra if it is equipped with at least one coderivation $d : C \to C$.

Definition 1.1.32 (A-derivation)**.** Let (A, d_A) be a differential algebra and M a left A-module. A $\mathbb{K}$-linear morphism $d_M : M \to M$ is called an A-derivation if $d_M(am) = d_A(a)m + ad_M(m)$, or electronically,



Dually, given a differential coalgebra (C, d_C) and N a left C-comodule, a $\mathbb{K}$-linear morphism $d_N : N \to N$ is a coderivation if $\omega_N \circ d_N = (d_C \otimes id_N + id_C \otimes d_N) \circ \omega_N$, or electronically,



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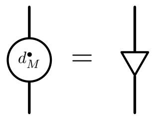
When there is no ambiguity, we will start to adopt writing the differential in electronic circuits as a triangle,



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Proposition 1.1.33. *Let A be a differential algebra and M a $\mathbb{K}$-module. A $\mathbb{K}$-linear morphism $f : M \to A \otimes_{\mathbb{K}} M$ uniquely determines a derivation $d_f : A \otimes M \to A \otimes M$, i.e. there is an isomorphism Hom$_{\mathbb{K}}(M, A \otimes_{\mathbb{K}} M) \simeq$ Der$(A \otimes_{\mathbb{K}} M)$. Moreover, d_f is given as $((\cdot_A) \otimes id_M) \circ (id_A \otimes f) + d_A \otimes id_M$.*

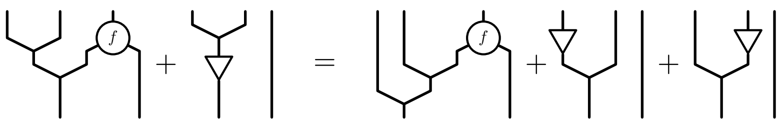
Dually, if C is a differential coalgebra and N is a $\mathbb{K}$-module, then a $\mathbb{K}$-linear morphism $g : C \otimes N \to N$ uniquely determines a coderivation $d_g : C \otimes_{\mathbb{K}} N \to C \otimes_{\mathbb{K}} N$. There is an isomorphism Hom$_{\mathbb{K}}(C \otimes_{\mathbb{K}} N, N) \simeq$ Coder$(C \otimes_{\mathbb{K}} N)$, and d_g is given as $(id_C \otimes g) \circ (\Delta_C \otimes id_N) + d_C \otimes id_N$.

Proof. This is only proved in the case of an algebra. The case of a coalgebra is dual.

We first got to prove that the morphism $d\_$: Hom$_{\mathbb{K}}(M, A \otimes_{\mathbb{K}} M) \to$ Der$(A \otimes_{\mathbb{K}} M)$ is well-defined. This means to check that for any morphism $f : M \to A \otimes_{\mathbb{K}} M$ the morphism d_f satisfies the Leibniz rule.

Assume that we have elements $a, b \in A$ and $m \in M$. Then $d_f(ab \otimes m) = d_f(a(b \otimes m))$. We now make an abuse of notation to write an equality between an element and a circuit. Recall that this really means that we have to think of a, b and m as generalized elements,

$d_f(ab \otimes m) \;=$

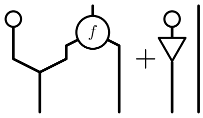


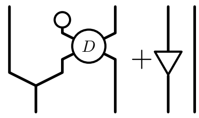
$$= d_A(a)b \otimes m + a d_f(b \otimes m).$$

Next we show that $d\_$ has an inverse. This is given by "restriction to M", also known as

$$(1_A \otimes M)^* : \text{Hom}_{\mathbb{K}}(A \otimes_{\mathbb{K}} M, N) \to \text{Hom}_{\mathbb{K}}(M, N).$$

Let $f : M \to A \otimes_{\mathbb{K}} M$ be a linear map and $D : A \otimes_{\mathbb{K}} M \to A \otimes_{\mathbb{K}} M$ be a derivation, then a quick calculation verifies that $d\_$ is inverse to restriction.

$d_f \circ (1_A \otimes M) =$  $= f$

$d_{D \circ (1_A \otimes M)} =$  $= D$

Notice that in the last equation we use the Leibniz rule to get the equality to D.

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We are now going to slowly work our way towards the context of homological algebra. We say that a $\mathbb{K}$-module M^* admits a $\mathbb{Z}$-grading if it decomposes into either summands or factors

$$M^* = \bigoplus_{z:\mathbb{Z}} M^z \text{ or } M^* = \prod_{z:\mathbb{Z}} M^z.$$

An element of $m \in M$ is said to be homogenous if it is properly contained in a single summand, i.e. $m \in M^n$. m is then said to have degree n. We say that a morphism of graded modules $f : M^* \to N^*$ is homogenous of degree n if it preserves the grading, i.e. $f(M^i) \subseteq N^{n+i}$. The degree of a homogenous element m or morphism f is denoted as $|m|$ or $|f|$.

There is a distinction between the ordinary and self-enriched category of graded modules. We are going to work with the self-enriched category, and its hom-objects is the graded module of homogenous morphism. We denote a factor in the grading as $\mathrm{Hom}^w_{\mathbb{K}}(M^*, N^*) = \{f : M^* \to N^* \mid f \text{ is homogenous and } |f| = w\}$, so the graded hom is

$$\mathrm{Hom}^*_{\mathbb{K}} = \prod_{w \in \mathbb{Z}} \mathrm{Hom}^w_{\mathbb{K}}.$$

The ordinary hom-objects are the hom-objects of the underlying modules, $\mathrm{Hom}_{\mathbb{K}}(M, N)$. This category is denoted as $\mathrm{Mod}^*_{\mathbb{K}}$. In general and whenever it makes sense, we denote $\mathcal{C}^*$ as the category of $\mathbb{Z}$-graded objects from $\mathcal{C}$.

The category $\mathrm{Mod}^*_{\mathbb{K}}$ is a closed symmetric monoidal category. The tensor is given by the following formula, using the ordinary tensor of $\mathrm{Mod}_{\mathbb{K}}$,

$$M^* \otimes N^* = \bigoplus_{n \in \mathbb{Z}} \bigoplus_{p \in \mathbb{Z}} M^p \otimes_{\mathbb{K}} N^q, \text{ where } q = n - p.$$

The associator of $\mathrm{Mod}_{\mathbb{K}}$ may be lifted to this tensor. The unit is the module $\mathbb{K}^* \simeq \mathbb{K}$ concentrated in degree 0. The right and left unit transformation may also be lifted from $\mathbb{K}$.

The category $\mathrm{Mod}^*_{\mathbb{K}}$ is closed. This means that our graded tensor fixed in one variable is left adjoint to the graded hom. We may obtain the graded hom as a right adjoint for the other variable as well by using the braiding which we will define later. Showing closedness is a routine calculation by using the tensor-hom adjunction from $\mathsf{Mod}_{\mathbb{K}}$.

$$
\begin{aligned}
\mathsf{Hom}^*_{\mathbb{K}}(A^* \otimes B^*, C^*) &= \prod_{w\in\mathbb{Z}}\prod_{n\in\mathbb{Z}} \mathsf{Hom}^w_{\mathbb{K}}(\bigoplus_{p\in\mathbb{Z}} A^p \otimes_{\mathbb{K}} B^{n-p}, C^n) \\
&= \prod_{w\in\mathbb{Z}}\prod_{n\in\mathbb{Z}}\prod_{p\in\mathbb{Z}} \mathsf{Hom}_{\mathbb{K}}(A^p \otimes B^{n-(p+w)}, C^n) \simeq \prod_{w\in\mathbb{Z}}\prod_{n\in\mathbb{Z}}\prod_{p\in\mathbb{Z}} \mathsf{Hom}_{\mathbb{K}}(A^p, \mathsf{Hom}_{\mathbb{K}}(B^{n-(p+w), C^n})) \\
&\simeq \prod_{w\in\mathbb{Z}}\prod_{p\in\mathbb{Z}} \mathsf{Hom}_{\mathbb{K}}(A^p, \prod_{n\in\mathbb{Z}} \mathsf{Hom}_{\mathbb{K}}(B^{n-(p+w), C^n})) = \prod_{w\in\mathbb{Z}}\prod_{p\in\mathbb{Z}} \mathsf{Hom}_{\mathbb{K}}(A^p, \mathsf{Hom}^{p+w}_{\mathbb{K}}(B^*, C^*)) \\
&= \prod_{w\in\mathbb{Z}} \mathsf{Hom}^w_{\mathbb{K}}(A^*, \mathsf{Hom}^*_{\mathbb{K}}(B^*, C^*)) = \mathsf{Hom}^*_{\mathbb{K}}(A^*, \mathsf{Hom}^*_{\mathbb{K}}(B^*, C^*)).
\end{aligned}
$$

We give a braiding on homogenous elementary tensors as

$$\beta(a \otimes b) = (-1)^{|a||b|} b \otimes a.$$

It is immediate that $\beta_{A,B}$ is inverse to $\beta_{B,A}$. Observe that this category also admits a braiding where we don't introduce a sign. This does however not work when we want to add differentials to our graded modules, so this is why we stick with this sign. This braiding is also commonly known as the Koszul sign convention.

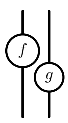
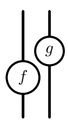
Since $\mathsf{Mod}^*_{\mathbb{K}}$ is a closed symmetric monoidal category it admits electronic circuits. Thus the previous results which we have proved by electronic circuits also apply to this category, as the proof is identical in this language. One should note that the specific implementation may be different as vertical braiding may work different. We will now study the vertical braiding in more detail. By definition [4] the application of two homogenous morphisms $f : A \to A'$ and $g : B \to B'$ on elements $a \in A$ and $b \in B$ on tensors is

$$(f \otimes g)(a \otimes b) = (-1)^{|g||a|} f(a) \otimes g(b).$$

Viewing a and b as generalized elements again, we get Koszul sign rule on morphisms, i.e. given homogenous composable morphisms f, f', g, g' we get that

$$(f' \otimes g') \circ (f \otimes g) = (-1)^{|g'||f|}(f' \circ f) \otimes (g' \circ g).$$

Electronically we may represent this as a 2-string circuit where a morphism on the left wants to downwards pass a morphism on its right,

 $= (-1)^{|g||f|}$ 

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A good way of thinking about moving components in a circuit is that whenever we move a component downwards it has to pass over every component to the left of it on its current level, and every component to the right of it on the level below. Thus, in a 2-string component we may say that if we move a component downwards to-, or completely past another component to its right, a sign is introduced. If we move a component upwards completely past another component to its left, we introduce a sign. In an n-string circuit it gets more complicated as the component may have to move past components on both the left and right side.

Unlike the other electronic equations in which we may substitute parts of an electronic circuit with other equal parts this does not however work a priori in this context. This is because how we defined levels. If we are given a 3-string circuit, then the formula changes, this is because we want to manipulate every element on a level simultanously. In the case of moving a left-most component downwards pass many components, we may regard them as a component on a single string. We will use this to prove a interchange of components on an n-string circuit formula.

Proposition 1.1.34. *Let $n \geqslant 1$ and suppose that we have $a_i \in A_i \to B_i$ and $b_i : B_i \to C_i$ for any $0 < i \leqslant n$. Then we get that*

$$(b_i \circ a_i) \otimes \cdots \otimes (b_n \circ a_n) = (-1)^s (b_1 \otimes \cdots \otimes b_n) \circ (a_1 \otimes \cdots \otimes a_n),$$

$$\textit{where } s = \sum_{i=1}^{n} |b_i| (\sum_{1 \leqslant j < i} |a_j|).$$

Proof. We prove this by induction. If $n = 1$ this is true. $s = 0$ since the sum is empty, so $b_1 \circ a_1 = (-1)^s b_1 \circ a_1$.

Assume that the conclusion holds for $n-1$, and that we have a_i and b_i as in the hypothesis. Let $s' = \sum_{i=1}^{n-1} |b_i| (\sum_{1 \leqslant j < i} |a_j|)$, then

$$s = s' + |b_n| (\sum_{i=1}^{n-1} |a_i|).$$

The conclusion follows from this calculation.

$$\begin{aligned}(b_1 \circ a_1) \otimes \cdots \otimes (b_n \circ a_n) &= (-1)^{s'} ((b_1 \otimes \cdots \otimes b_{n-1}) \circ (a_1 \otimes \cdots \otimes a_{n-1})) \otimes (b_n \circ a_n) \\ &= (-1)^{s' + |b_n| (\sum_{i=1}^{n-1} |a_i|)} (b_1 \otimes \cdots \otimes b_n) \circ (a_1 \otimes \cdots \otimes a_n).\end{aligned}$$

□

A final remark on this braiding is that it affects any scenario where we compose functions and they move past each other. Since function composition factors through this tensor, moving functions around is actually a braiding. An important example of this is the pre-composition functor. If f and g are homogenous and composable, then

$$f^*(g) = (-1)^{|f||g|} g \circ f.$$

The graphical calculus we have developed will be the same for any symmetric monoidal category where the braiding is similar. What this means will soon be evident when we add extra structure to the objects of $\mathrm{Mod}_{\mathbb{K}}^*$.

A graded $\mathbb{K}$-module $M^\bullet$ is called a cochain complex if it comes equipped with a differential $d_M : M^\bullet \to M^\bullet$. By a differential we mean a homogenous morphism of degree 1 such that $d_M^2 = 0$. Be cautious of bad notation, as d_M^2 might mean $d_M^2 = d_M \circ d_M$ and $d_M^2 : M^2 \to M^3$.

Given a cochain complex $M^\bullet$, we know by definition that the image of the differential lies inside the kernel of the differential. We denote this at the i'th coordinate as $B^i M \subseteq Z^i M$. $B^* M$ is the graded submodule of images, also called boundaries. $Z^* M$ is the graded submodule of kernels, also called cycles. The graded cohomology module $H^* M$ is defined as the quotient ${Z^*M}/{B^*M}$. A cochain complex is said to be exact if $H^* M \simeq 0$.

Cochain complexes are plentiful in nature.

Example 1.1.35 ($\mathbb{K}$ as a cochain complex). Let $\mathbb{K}^\bullet = (\mathbb{K}, 0)$ be the graded $\mathbb{K}$-module concentrated in degree 0 together with a 0 differential. This is trivially a cochain complex.

Example 1.1.36 (Trivial cochain complexes). Let M^* be a graded $\mathbb{K}$-module. Let $M^\bullet = (M^*, 0)$ be the same graded module with the 0 differential, this is also a cochain complex.

Example 1.1.37. We can create a chain complex as the following diagram.

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{K} \xrightarrow{id_{\mathbb{K}}} \mathbb{K} \longrightarrow 0 \longrightarrow \cdots$$

Example 1.1.38 (Cone of a chain map). Suppose that $f : A^\bullet \to B^\bullet$ is a homogenous morphism of degree 0 such that $f \circ d_A = d_B \circ f$. There is a cochain complex associated to f, yielding a short-exact sequence of cochain complexes. We define $\mathrm{cone}(f)$ at each degree by

$$\mathrm{cone}(f)^n = A^{n+1} \oplus B^n,$$
$$d^n_{\mathrm{cone}(f)} = \begin{pmatrix} d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{pmatrix}.$$

This gives us a short exact sequence,

$$B^\bullet \rightarrowtail \mathrm{cone}(f) \twoheadrightarrow A^\bullet[1].$$ punktum?

Example 1.1.39 (Normalized cochain complex). Let $A : \Delta^{op} \to \mathrm{Mod}_{\mathbb{K}}$ be a simplicial $\mathbb{K}$-module.
We define a collection of diagrams J^n as $J^0 = A_0$, and every other as No sources?

$$\begin{tikzcd} J^n = A_n \arrow[r, bend left, "0"] \arrow[r, "d_1"] \arrow[r, phantom, "\vdots"] \arrow[r, "d_n"'] & A_{n-1} \end{tikzcd}$$

A's normalized cochain complex is the complex given as

$$NA^{-n} = \varprojlim J^n.$$

In a complete pointed category, such as $\mathsf{Mod}_{\mathbb{K}}$, the limit is the same as the intersection of every kernel:

$$\varprojlim J^n = \bigcap_{i=1}^{n} \mathsf{Ker} d_i.$$

The differential of NA is given as d_0. Since we have turned the complex around, this is a morphism of degree 1. By taking the limit we force $d_0^2 = 0$ as well.

Example 1.1.40 (Associated cochain complex). Let $A : \Delta^{op} \to \mathsf{Mod}_{\mathbb{K}}$ be a simplical $\mathbb{K}$-module. We define a differential as

$$d = \sum_{i=0}^{n} (-1)^i d_i.$$

Let $\mathsf{C}A$ be the complex given in each degree as

$$\mathsf{C}A^{-n} = A_n.$$

d defines a differential on $\mathsf{C}A$ of degree 1.

Example 1.1.41 (Singular chain complex with $\mathbb{K}$-coefficents). Let M be a topological space. There is a simplical set defined as $\mathsf{Sing}(M) = \mathsf{Top}(\Delta-, M) : \Delta^{op} \to \mathsf{Set}$. Here $\Delta^{[n]}$ in Top refers to the topological standard n-simplex. We get a simplical $\mathbb{K}$-module by creating the free one, $\mathbb{K}\mathsf{Sing}(M)$. By the above example defines a chain complex in $\mathsf{Mod}_{\mathbb{K}}$.

We make a distinction for some cochain complexes which is of particular interest.

Definition 1.1.42 (Almost free cochain complexes)**.** Suppose that $M^\bullet$ is a cochain complex. We say that $M^\bullet$ is almost free if the underlying graded module M^* is free, i.e. a tensor algebra.

Likewise, we say that $M^\bullet$ is almost cofree if M^* is cofree, i.e. a tensor coalgebra.

The category of cochain complexes will be denoted as $\mathsf{Mod}_{\mathbb{K}}^\bullet$. Note that this category is built upon $\mathsf{Mod}_{\mathbb{K}}^*$, and we inherit the braiding β. It can be shown that this is the only braiding which will make the tensors of this category well-defined. For this category we would like to entertain different collections of morphisms. This is because the morphisms which respects all of the structure and the morphisms which makes this category self-enriched are different. We will usually denote both of these categories as $\mathsf{Mod}_{\mathbb{K}}^\bullet$, but when we want to emphesize the structure preserving maps we will rather denote this as $\mathsf{Ch}(\mathbb{K})$.

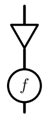
When $A^\bullet$ and $B^\bullet$ are cochain complexes the graded $\mathbb{K}$-module $\mathsf{Hom}_{\mathbb{K}}^*(A^\bullet, B^\bullet)$ admits a derivative. Let $f : A^\bullet \to B^\bullet$ be any homogenous morphism, then the derivative-, or boundary of f is given by

$$\partial f = [d, f] = (d_{B*} + d_A^*)(f) = d_B \circ f - (-1)^{|f|} f \circ d_A.$$

We see that $|\partial| = |d_{B*} + d_A^*| = 1$, and

$$\partial^2 f = (d_{B*} + d_A^*)(d_B \circ f - (-1)^{|f|} f \circ d_A) = d_B^2 f + (-1)^{|f|} d_B f d_A - (-1)^{|f|} d_B f d_A - f d_A^2 = 0.$$

Thus, $\text{Hom}^\bullet_{\mathbb{K}}(A^\bullet, B^\bullet) = (\text{Hom}^*_{\mathbb{K}}(A^\bullet, B^\bullet), \partial)$ is a cochain complex. We endow $Mod^\bullet_{\mathbb{K}}$ with these hom-objects. In an electronic diagram we write ∂f as a sum of circuits,

$\partial f =$  $+(-1)^{|f|}$ 

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Notice how this construction of $\text{Hom}^\bullet_{\mathbb{K}}$ is the same as the (product) total complex of an anti-commutative double complex. An anti-commutative double complex is a graded module of cochain complexes, together with a differential between the cochain complexes. These different differentials are supposed to anti-commute. An anti-commutative double complex may be illustrated in this manner.

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r, "d_C^h"] & C^{-1,1} \arrow[r, "d_C^h"] \arrow[u, "d_C^v"] & C^{0,1} \arrow[r, "d_C^h"] \arrow[u, "d_C^v"] & C^{1,1} \arrow[r, "d_C^h"] \arrow[u, "d_C^v"] & \cdots \\
\cdots \arrow[r, "d_C^h"] & C^{-1,0} \arrow[r, "d_C^h"] \arrow[u, "d_C^v"] & C^{0,0} \arrow[r, "d_C^h"] \arrow[u, "d_C^v"] & C^{1,0} \arrow[r, "d_C^h"] \arrow[u, "d_C^v"] & \cdots \\
\cdots \arrow[r, "d_C^h"] & C^{-1,-1} \arrow[r, "d_C^h"] \arrow[u, "d_C^v"] & C^{0,-1} \arrow[r, "d_C^h"] \arrow[u, "d_C^v"] & C^{1,-1} \arrow[r, "d_C^h"] \arrow[u, "d_C^v"] & \cdots \\
 & \vdots \arrow[u, "d_C^v"] & \vdots \arrow[u, "d_C^v"] & \vdots \arrow[u, "d_C^v"] &
\end{tikzcd}$$

Another way of thinking of an anti-commutative double complex $C^{\bullet,\bullet}$ is that it is a bigraded $\mathbb{K}$-module with a vertical and horizontal differential such that $d_C^v \circ d_C^h = -d_C^h \circ d_C^v$.

Definition 1.1.43. Let $C^{\bullet,\bullet}$ be an anti-commutative double complex. We define the sum and product total complex. The differential at each $C^{p,q}$ is defined as $d_{\text{Tot}C} = d_C^v + d_C^h$, and

$$\text{Tot}^{\oplus}(C^{\bullet,\bullet}) = \bigoplus_{n\in\mathbb{Z}} \bigoplus_{p+q=n} C^{p,q},$$
$$\text{Tot}^{\prod}(C^{\bullet,\bullet}) = \prod_{n\in\mathbb{Z}} \prod_{p+q=n} C^{p,q}.$$

If $C^{\bullet,\bullet}$ is bounded, then $\text{Tot}^{\oplus}(C^{\bullet,\bullet}) \simeq \text{Tot}^{\prod}(C^{\bullet,\bullet})$.

If we let $\mathsf{Hom}_{\mathbb{K}}(A^\bullet, B^\bullet)^{\bullet,\bullet} = (\prod_{p,q\in\mathbb{Z}} \mathsf{Hom}_{\mathbb{K}}(A^p, B^q), d_A^*, d_{B*})$, then it is clear that

$$\mathsf{Hom}^\bullet_{\mathbb{K}}(A^\bullet, B^\bullet) = \mathsf{Tot}^{\prod}(\mathsf{Hom}_{\mathbb{K}}(A^\bullet, B^\bullet)^{\bullet,\bullet}).$$

From this we are able to deduce that $\mathsf{Mod}^\bullet_{\mathbb{K}}$ is a closed symmetric monoidal category. The tensor may be deduced from the data of $\mathsf{Hom}^\bullet_{\mathbb{K}}$. Define an anti-commutative double complex $(A^\bullet \otimes_{\mathbb{K}} B^\bullet)^{\bullet,\bullet} = (\bigoplus_{n\in\mathbb{Z}} \bigoplus_{p+q=n} A^p \otimes B^q, d_A \otimes B, A \otimes d_B)$, then the tensor is defined as

$$A^\bullet \otimes B^\bullet = \mathsf{Tot}^{\oplus}((A^\bullet \otimes B^\bullet)^{\bullet,\bullet}).$$

It is clear that this tensor is left adjoint to $\mathsf{Hom}^\bullet_{\mathbb{K}}$. All of the structure morphisms for a closed symmetric monoidal category are inherited from $\mathsf{Mod}^*_{\mathbb{K}}$. This also means that $\mathsf{Mod}^\bullet_{\mathbb{K}}$ employs the same electronic circuits as $\mathsf{Mod}^*_{\mathbb{K}}$.

The category of cochain complexes with chain maps $\mathsf{Ch}(\mathbb{K})$ is defined to have its hom-objects as $Z^0\mathsf{Hom}^\bullet_{\mathbb{K}}(A^\bullet, B^\bullet)$. By abuse of notation we may write $\mathsf{Ch}(\mathbb{K}) = Z^0\mathsf{Mod}^\bullet_{\mathbb{K}}$. Notice that this condition means that the derivative of any morphism $f : A^\bullet \to B^\bullet$ in $\mathsf{Ch}(\mathbb{K})$ is 0; i.e. that $\partial f = 0$, or $f \circ d_A = d_B \circ f$. We will call these morphisms for chain maps.

The homotopy category $\mathsf{K}(\mathbb{K})$ is defined to be the quotient category of $\mathsf{Ch}(\mathbb{K})$ at null-homotopic chain maps. Observe that $\mathsf{K}(\mathbb{K}) = H^0\mathsf{Mod}^\bullet_{\mathbb{K}}$. This is definitional, because the chain maps $f, g : A^\bullet \to B^\bullet$ are homotopic if there is a homogenous morphism $h : A^\bullet \to B^\bullet$ of degree -1 such that $\partial h = f - g$.

A chain map $f : A^\bullet \to B^\bullet$ induces homogenous morphisms of degree 0.

$$B^*f : B^*A \to B^*B$$
$$Z^*f : Z^*A \to Z^*B$$
$$H^*f : H^*A \to H^*B$$

We say that f is a quasi-isomorphism if H^*f is an isomorphism. This is equivalent to say that $\mathsf{cone}(f)$ is exact.

A cochain complex $N^\bullet$ is said to be contractible if id_N is null-homotopic. Then it follows for any other cochain complexes $M^\bullet$ that $H^0\mathsf{Hom}^\bullet_{\mathbb{K}}(M^\bullet, N^\bullet) \simeq 0$.

We define the shift functor $\_[n] : \mathsf{Mod}^\bullet_{\mathbb{K}} \to \mathsf{Mod}^\bullet_{\mathbb{K}}$ is defined on cochains $M^\bullet$ as

$$(M^\bullet, d_M)[n] = (M^\bullet[n], (-1)^n d_M).$$

With this definition, shifting is naturally isomorphic to tensoring. That is if $\mathbb{K}^\bullet[n]$ denotes the field concentrated in dimension $-n$, then

$$\mathbb{K}^\bullet[n] \otimes_{\mathbb{K}} M^\bullet \simeq M^\bullet[n] \simeq M^\bullet \otimes_{\mathbb{K}} \mathbb{K}^\bullet[n].$$

By writing out the total tensor product, one may see how the differential gets its sign. We usually call $\_[1]$ shifting, desuspension or looping; and $\_[-1]$ for inverse-shifting, suspension or delooping.

We are now ready to talk about algebras in $Mod_{\mathbb{K}}^{\bullet}$.

Definition 1.1.44 (Differential graded algebra)**.** $(A^{\bullet}, d_A)$ is a differential graded algebra if:

- $A^{\bullet}$ is a differential algebra in Mod$_{\mathbb{K}}^{\bullet}$.
- The structure morphisms $(\cdot_A)$ and 1_A are chain maps.
- The derivation and differential coincides.

Example 1.1.45 (The unit). $\mathbb{K}^{\bullet} = (\mathbb{K}, 0)$ is a differential graded algebra in the trivial way. It is concentrated in degree 0 and the differential is the trivial derivation.

Example 1.1.46 (De Rham complex). Given a manifold M, the exterior algebra ΩM is a differential graded algebra. See Tu [5] for a thouough explanation.

Given a differential graded, or dg-algebra $A^{\bullet}$, we may form the category of left $A^{\bullet}$-modules, Mod$_A$.

Definition 1.1.47. $M^{\bullet}$ is a left $A^{\bullet}$-module if

- $M^{\bullet}$ is a cochain complex,
- there is a chain map $\mu_M : A^{\bullet} \otimes_{\mathbb{K}} M^{\bullet}$ satisfying associativity and unitality,
- d_M is an $A^{\bullet}$-derivation.

The hom-objects are defined analogously. We use Hom$_{A^{\bullet}}^{\bullet}$ to denote the $\mathbb{K}$-linear cochain complex.

With this definition the categories Mod$_{\mathbb{K}^{\bullet}}$ and Mod$_{\mathbb{K}}^{\bullet}$ is the same category. This is because a chain complex already satisfy the first two bullet points by definition. Being a $\mathbb{K}^{\bullet}$-derivation is trivial, so every map satisfies this.

We also have the dual definition to obtain dg-coalgebras and their comodules.

Definition 1.1.48. $C^{\bullet}$ is a differential graded coalgebra if

- $C^{\bullet}$ is a differential coalgebra in Mod$_{\mathbb{K}}^{\bullet}$,
- the structure morphisms Δ_C and ε_C are chain maps,
- the coderivation and differential coincides

Definition 1.1.49. $N^{\bullet}$ is a left $C^{\bullet}$-comodule if

- $N^{\bullet}$ is a cochain complex,
- there is a chain map $\omega_C : N^{\bullet} \to C^{\bullet} \otimes_{\mathbb{K}} N^{\bullet}$ satisfying coassociativity and counitalit,
- d_N is a $C^{\bullet}$-coderivation.

By these definitions we may extend proposition 1.1.33 to the category of cochain complexes.

Corollary 1.1.49.1. *Let $A^\bullet$ be a differential graded algebra and $M^\bullet$ a cochain complex. A homogenous $\mathbb{K}$-linear morphism $f : M \to A \otimes_{\mathbb{K}} M$ uniquely determines a derivation $d_f : A \otimes M \to A \otimes M$ of same degree, i.e. there is an isomorphism Hom$^*_{\mathbb{K}}(M^\bullet, A^\bullet \otimes_{\mathbb{K}} M^\bullet) \simeq$ Der$^*(A^\bullet \otimes_{\mathbb{K}} M^\bullet)$. Moreover, d_f is given as $(\nabla_{A^\bullet} \otimes id_M) \circ (id_A \otimes f) + d_{A\otimes M}$.*

Dually, if $C^\bullet$ is a differential graded coalgebra and $N^\bullet$ is a cochain complex, then a homogenous $\mathbb{K}$-linear morphism $g : C^\bullet \otimes N^\bullet \to N^\bullet$ uniquely determines a coderivation $d_g : C^\bullet \otimes_{\mathbb{K}} N^\bullet \to C^\bullet \otimes_{\mathbb{K}} N^\bullet$. There is an isomorphism Hom$^*_{\mathbb{K}}(C^\bullet \otimes_{\mathbb{K}} N^\bullet, N^\bullet) \simeq$ Coder$^*(C^\bullet \otimes_{\mathbb{K}} N^\bullet)$, and d_g is given as $(id_C \otimes g) \circ (\Delta_{C^\bullet} \otimes id_N) + d_{C\otimes N}$.

Proof. The same electronic circuits as in the proof of proposition 1.1.33 suffice to prove this statement. ☐

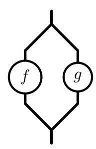
It is notable that this statement carries an additional two duals. That is we have the same result when we consider right modules as well. The same proof applies in these cases.

1.2 Cobar-Bar Adjunction

1.2.1 Convolution Algebras

Given a coalgebra C and an algebra A, we obtain a special product on the hom-object Hom$_{\mathbb{K}}(C, A)$ by twisting the comulitplication and multiplication together. The convolution algebra forms the backbone of our proof of the cobar-bar adjunction.

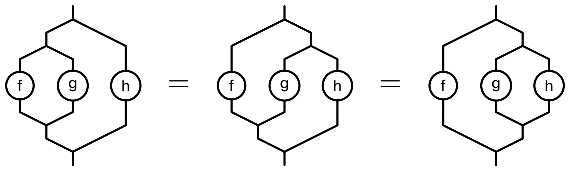
Let C be a coalgebra and A an algebra, then if $f, g : C \to A$ is a $\mathbb{K}$-linear morphism we may define $f \star g = (\cdot_A)(f \otimes g)\Delta_C$. This operation is called $\star$ convolution.

$f \star g =$ 

Proposition 1.2.1 (Convolution algebra)**.** *The $\mathbb{K}$-module Hom$_{\mathbb{K}}(C, A)$ is an associative algebra when equipped with convolution $\star$: Hom$_{\mathbb{K}}(C, A) \to$ Hom$_{\mathbb{K}}(C, A)$. The unit is given by $1 \mapsto v_A \circ \varepsilon_C$.*

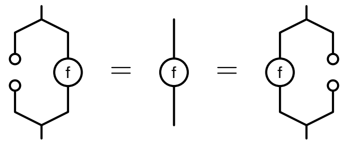
Proof. This proposition follows from (co)associativity and (co)unitality of (C) A.

$(f \star g) \star h =$



$= f \star (g \star h)$

$(\upsilon_A \circ \varepsilon_C) \star f =$



$= f \star (\upsilon_A \circ \varepsilon_C)$

□

This proof does not rely on braiding and lifts to any closed symmetric monoidal category.

If we are given an algebra A, then it may be considered as a differential algebra with the the trivial derivation. That is, $(A, 0)$ is a differential algebra. For such structures, the set of A-derivations is precisely the set of A-linear morphisms. Dually, every coalgebra C may be thought of as a differential coalgebra.

We may apply a trivialization of proposition 1.1.33 to A and C considered as differential (co)algebra. When we look at the module $C \otimes_{\mathbb{K}} A$, it is both free over A on the right and cofree over C on the left. Consider a morphism $\alpha : C \to A$, then there are two ways to extend α to obtain a (co)derivation. Precomposing with C's comultiplication gives us a morphism from C to the free A-module $C \otimes_{\mathbb{K}} A$,

$$(id_C \otimes \alpha) \circ \Delta_C : C \to C \otimes_{\mathbb{K}} A.$$

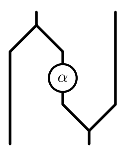
Postcomposing with the multiplication of A gives us a morphism from to the cofree C-comodule $C \otimes_{\mathbb{K}} A$ to A,

$$(\cdot_A) \circ (\alpha \otimes id_A) : C \otimes_{\mathbb{K}} A \to A.$$

When we apply proposition 1.1.33 to both morphisms, it yields the same map. Therefore it is both a derivation and a coderivation, as

$$d^r_\alpha = (id_C \otimes (\cdot_A)) \circ (id_C \otimes \alpha \otimes id_A) \circ (\Delta_C \otimes id_A)$$

$d^r_\alpha =$



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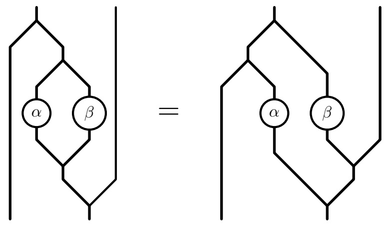
One of our main tools will be this coderivation. In the ungraded case it determines a ring morphism with little to no complications.

Proposition 1.2.2. *d^r_- : Hom$_{\mathbb{K}}(C, A) \to$ End$(C \otimes_{\mathbb{K}} A)$ is a morphism of algebras. Moreover, if $\alpha \star \alpha = 0$, then $(d^r_\alpha)^2 = 0$.*

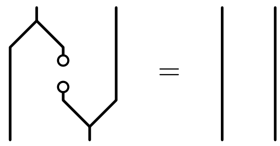
Proof. The proof quickly follows from (co)associativity and (co)unitality.

$d^r_{\alpha\star\beta} =$



$= d^r_\alpha \circ d^r_\beta$

$d^r_{\upsilon_A \circ \varepsilon_C} =$



$= id_{C\otimes_{\mathbb{K}}A}$

□

This proof relies on braiding, so we will run into some problems when we try to lift this proposition to the graded case. We may observe that the above has no problem lifting, this is because the β has no morphisms of odd degrees to the right or over itself. However, the dual will introduce some signs when lifted.

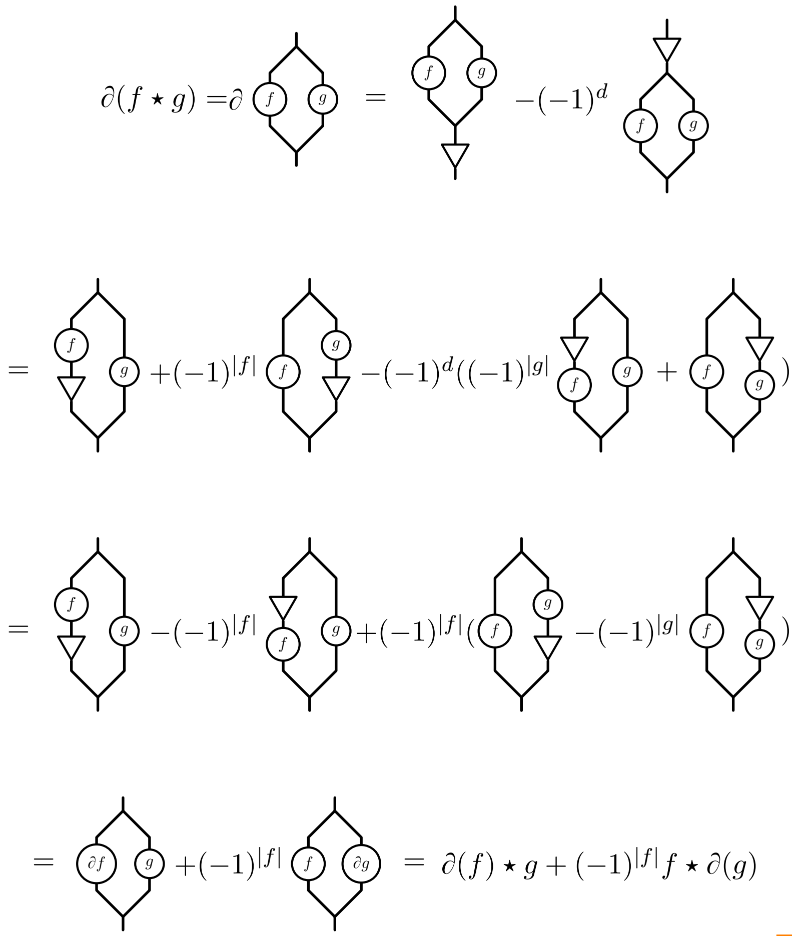
Corollary 1.2.2.1. *Suppose that C and A are differential graded co/algebras. d^r_- : Hom$^*_{\mathbb{K}}(C, A) \to$ End$^*(C \otimes_{\mathbb{K}} A)$ extends to a homogenous ring morphism of degree 0.*

Suppose that C and A are differential graded (co)algebras. We want to expect that the differential ∂ makes (Hom$^*_{\mathbb{K}}(C, A), \star)$ into a dg-algebra.

Proposition 1.2.3. *The convolution algebra (Hom$^*_{\mathbb{K}}(C, A), \star)$ is a dg-algebra with differential ∂.*

Proof. We know that (Hom$^*_{\mathbb{K}}(C, A), \star)$ is a convolution algebra and that (Hom$^*_{\mathbb{K}}(C, A), \partial)$ is a chain complex. It remains to verify that the differential is compatible with the multiplication, i.e. $\partial(f \star g) = \partial f \star g + (-1)^{|f|} f \star \partial g$.

Let $f, g \in$ Hom$^*_{\mathbb{K}}(C, A)$ be two homogenous morphisms. The key property to arrive at the result is that the differential in a dg-(co)algebra is a (co)derivation. We denote the degree of $f \star g$ as $|f \star g| = |f| + |g| = d$. Then



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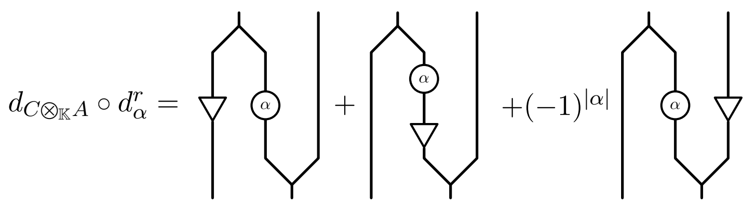
□

Proposition 1.2.4. *The morphism d^r_{-} : Hom$^\bullet_{\mathbb{K}}(C, A) \to$ End$^\bullet(C \otimes_{\mathbb{K}} A)$ is a chain map.*

Proof. We already know from corollary 1.2.2.1 that d^r_{-} is a homogenous ring map. It remains to see that it commutes with the differentials. That is, $\partial d^r_\alpha = d^r_{\partial\alpha}$. We write out each summand in ∂d^r_α,



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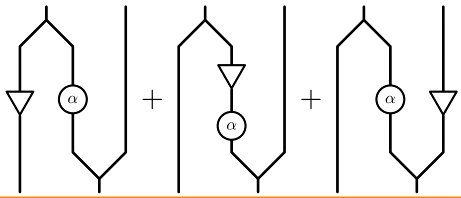
$$d^r_\alpha \circ d_{C\otimes_{\mathbb{K}}A} = (-1)^{|\alpha|}$$



When α is of even degree, $\partial d^r_\alpha = d_{C\otimes_{\mathbb{K}}A} \circ d^r_\alpha - d^r_\alpha \circ d_{C\otimes_{\mathbb{K}}A}$. The outer summands cancel and we are left with

$$\partial d^r_\alpha = d_{d_A\alpha - \alpha d_C} = d_{\partial\alpha}.$$

When α is of odd degree, $\partial d^r_\alpha = d_{C\otimes_{\mathbb{K}}A} \circ d^r_\alpha + d^r_\alpha \circ d_{C\otimes_{\mathbb{K}}A}$. The outer summands cancel and we are left with

$$\partial d^r_\alpha = d_{d_A\alpha + \alpha d_C} = d_{\partial\alpha}.$$

□

1.2.2 Twisting Morphisms

In this section we will define twisting morphisms from coalgebras to algebras. They are of importance as the bifunctor Tw(C, A) is represented in both arguments. To understand the elements of Tw we start this section be reviewing the Maurer-Cartan equation.

Suppose that C is a coaugmented dg-coalgebra and A is an augmented dg-algebra. We say that a morphism $\alpha \in \mathrm{Hom}^*_{\mathbb{K}}(C, A)$ is twisting if it is of degree 1, is 0 on the coaugmentation of C, is 0 on the augmentation of A and satisfies the Maurer-Cartan equation:

$$\partial\alpha + \alpha \star \alpha = 0.$$

We say that α is an element of Tw$(C, A) \subset \mathrm{Hom}^1_{\mathbb{K}}(C, A) \subset \mathrm{Hom}^*_{\mathbb{K}}(C, A)$. Notice that these requirements means that $\mathrm{Im}\alpha|_{\overline{C}} \subseteq \overline{A}$. In light of proposition 1.2.2, every morphism between (coalgebras) algebras extend to a unique (co)derivation on the tensor product $C \otimes_{\mathbb{K}} A$. Let d^r_α denote this unique morphism. In the case of dg-coalgebras and dg-algebras we perturbate the total differential on the tensor with d^r_α, as in proposition 1.1.33. We call this derivation for the perturbated derivative,

$$d_\alpha = d_{C\otimes_{\mathbb{K}}A} + d^r_\alpha = d_C \otimes id_A + id_C \otimes d_A + d^r_\alpha.$$

Proposition 1.2.5. *Suppose that C is a dg-coalgebra and A is a dg-algebra, and $\alpha \in \mathrm{Hom}^{-1}_{\mathbb{K}}(C, A)$. The perturbated derivation satisfies the following relation.*

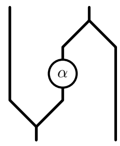
$${d_\alpha}^2 = d^r_{\partial\alpha + \alpha\star\alpha}$$

Moreover, a morphism satisfies the Maurar-Cartan equation if and only if its associated perturbated derivative is a differential.

Proof. ${d_\alpha}^2 = d_{C\otimes_{\mathbb{K}}A} \circ d^r_\alpha + d^r_\alpha \circ d_{C\otimes_{\mathbb{K}}A} + {d^r_\alpha}^2$. By proposition 1.2.4 the result is immediate. □

Corollary 1.2.5.1. *If $\alpha : C \to A$ is a twisting morphism, then $(C \otimes_{\mathbb{K}} A, d^\bullet_\alpha)$ is a chain complex which is also a left C-comodule and a right A-module. This is called the right twisted tensor product and is denoted as $C \otimes_\alpha A$.*

Normally $A \otimes C$ and $C \otimes A$ are isomorphic as modules. In general, it is not true that $C \otimes_\alpha A$ and $A \otimes_\alpha C$ are isomorphic, since we have to choose a particular side to perform the twisting. However, if A is commutative and C is cocommutative then they are isomorphic. To illustrate we realize the unique derivation above as a right derivative. The left derivative d^l_α is then defined analogously,

$d^l_\alpha =$  .

$d^l_{\_}$: $\mathrm{Hom}^\bullet_{\mathbb{K}}(C, A) \to \mathrm{End}^\bullet(C, A)$ does no longer define a ring morphism. Note that this still commutes with the differential, the problem lies in the ring homomorphism property. Observe that we get

$$d^l_{\alpha\star\beta} = (-1)^{|\alpha||\beta|} d^l_\beta \circ d^l_\alpha.$$

We summarize this in the next proposition.

Proposition 1.2.6. *The morphism $d^l_{\_}$: $\mathrm{Hom}^\bullet_{\mathbb{K}}(C, A) \to \mathrm{End}^\bullet(C, A)$ is a skew chain map.*

Proof. This is clear by the previous discussion. □

Functoriality of the right twisted tensor at the level of chain maps does not really work. To show where it may go wrong, pick two twisting morphisms $\alpha : C \to A$ and $\beta : C' \to A'$. Given a pair of morphisms $f : C \to C'$ and $g : A \to A'$, it is not clear if $f \otimes g$ will preserve the perturbated differential, and it should not be true in general.

It is however the case that the right twisted tensor product defines a tri-functor from the category of elements to chain complexes,

$$\_ \otimes_{\_} \_ : \sum_{\mathsf{Coalg}\otimes\mathsf{Alg}} \mathsf{Tw} \to \mathsf{Mod}^A_C.$$

This means that any commutative square as below gets mapped to a morphism of its right twisted tensors. Here f is a morphism of coalgebras and g is a morphism of algebras,

$$
\begin{tikzcd}
C \arrow[r, "\alpha"] \arrow[d, "f"] & A \arrow[d, "g"] & & C \otimes_\alpha A \arrow[d, "f\otimes g"] \\
C' \arrow[r, "\alpha'"] & A' \arrow[r, phantom, "\rightsquigarrow"] & & C' \otimes_{\alpha'} A'
\end{tikzcd}
$$

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The important property to obtain this is that f and g are morphisms in their respective categories. This allows us to collapse the different compositions to the same map, up to sign.

1.2.3 Bar and Cobar Construction

The bar construction was first formalized for augmented skew-commutative dg-rings by Eilenberg and Mac Lane [6]. The bar construction then served as a tool to calculate the homology of the Eilenberg-Mac Lane spaces. This construction was later dualized by Adams [7] to obtain the cobar construction. Its first purpose was to serve as a method for constructing an injective resolution in order to calculate the Cotor resolution [8]. With time, the bar-cobar construction have been subjected to much generalization, such as a fattened tensor product on simplicially enriched, tensored and cotensored categories [9]. We will mainly follow the work of [3] to obtain the bar and cobar construction. The approach which we are going to take is also slightly inspired by MacLanes [10] canonical resolutions of comonads.

For our purposes, the bar construction of an augmented algebra is a simplicial resoulution as a cofree coalgebra structure. Given a dg-algebra, we will realize this as the total complex of its resoultion. Dually, the cobar construction of a conilpotent coalgebra is a cosimplicial resolution as a free algebra structure. We will see that these constructions defines an adjoint pair of functors.

An algebra A is a monoid in the monoidal category $(Mod_{\mathbb{K}}, \otimes_{\mathbb{K}}, \mathbb{K})$. By proposition A.1.5 we may think of A as an augmented cosimplicial object $A : \Delta_+ \to Mod_{\mathbb{K}}$. Notice that all of the cosimplical identities follow from associativity and unitality. If A is an augmented algebra, we may instead give it the structure of an augmented simplicial set. Let $d_0^0 = \varepsilon_A$ be the augmentation. We define $d_n^n = A^{\otimes n-1} \otimes \varepsilon_A$ and set $d_n^i = A^{i-1} \otimes (\cdot_A) \otimes A^{\otimes n-i-1}$. All the degeneracies are set to be the units, i.e. $s_n^i = A^{\otimes i} \otimes \upsilon_A \otimes A^{\otimes n-i-1}$. One may check that this structure defines an augmented simplical object $A : \Delta_+^{op} \to Mod_{\mathbb{K}}$. Observe that the chain complex CA is exactly the Hochschild complex of A. We depict the simplicial object as the following diagram:

$$
\begin{tikzcd}
\mathbb{K} & A \arrow[l, "\varepsilon_A"'] & A^{\otimes 2} \arrow[l, shift right, "(\cdot_A)"'] \arrow[l, shift left, "A\otimes\varepsilon_A"] & A^{\otimes 3} \arrow[l, shift right=2, "(\cdot_A)"'] \arrow[l] \arrow[l, shift left=2, "A^{\otimes 2}\otimes\varepsilon_A"] & \cdots \arrow[l, shift right=3, "(\cdot_A)"'] \arrow[l, shift right=1] \arrow[l, shift left=1] \arrow[l, shift left=3, "A^{\otimes 4}\otimes\varepsilon_A"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{K} & A \arrow[r, "s^1"] & A^{\otimes 2} \arrow[r, shift left, "s^i"] \arrow[r, shift right] & A^{\otimes 3} \arrow[r, shift left=2, "s^i"] \arrow[r] \arrow[r, shift right=2] & \cdots
\end{tikzcd}
$$

The augmentation ideal $\overline{A}$ carries a natural semi-simplical structure induced by A. By restricting each of the face maps $\overline{d}^i = d^i|_{\overline{A}} : \overline{A}^{\otimes n} \to \overline{A}^{\otimes n-1}$ we obtain the maps together with the simplical identity 1. This is the non-unital Hochschild complex of A. We may depict the semi-simplical object as the following diagram:

$$\begin{tikzcd} \mathbb{K} & \overline{A} \arrow[l, "0"'] & \overline{A}^{\otimes 2} \arrow[l, shift right, "(\cdot_A)"'] \arrow[l, shift left, "0"] & \overline{A}^{\otimes 3} \arrow[l, shift right=2, "(\cdot_A)"'] \arrow[l] \arrow[l, shift left=2, "0"] & \cdots \arrow[l, shift right=3, "(\cdot_A)"'] \arrow[l, shift right] \arrow[l, shift left] \arrow[l, shift left=3, "0"] \end{tikzcd}$$

As graded modules, the chain complex C$\overline{A}$ is isomorphic to $T^c(\overline{A})$. Here we think of the grading $T^c(\overline{A})$ as starting at 0 and going down to negative degrees. Consider instead the looped non-unital algebra $\overline{A}[1]$. Every algebra may be considered as a graded algebra concentrated in degree 0, the shift functor then recontextualize the degree the algebra is concentrated in. This object $\overline{A}[1]$ is no longer an associative algebra. In order to understand this looped multiplication we will first consider $\mathbb{K}\{\omega\}$, where $|\omega| = -1$. We define a looped multiplication $(\cdot) : \mathbb{K}\{\omega\}^{\otimes 2} \to \mathbb{K}\{\omega\}$ as

$$\omega \cdot \omega = \omega.$$

Given an algebra A, the looped multiplication of $A[1]$ is defined as the composite

$$(\cdot_{A[1]}) = ((\cdot) \otimes (\cdot_A)) \circ (\mathbb{K}\{\omega\} \otimes \beta \otimes \overline{A}).$$

As an example, suppose that ωa_1 and ωa_2 are elements of $A[1]$, then their multiplication would look like

$$(\cdot_{A[1]})(\omega a_1 \otimes \omega a_2) = (-1)^{|a_1||\omega|}((\cdot) \otimes \cdot_A)(\omega^{\otimes 2} \otimes a_1 \otimes a_2) = (-1)^{|a_1|}\omega a_1 a_2.$$

Observe that the resulting morphism $(\cdot_{A[1]})$ is of degree 1.

Proposition 1.2.7. *Suppose that A is an augmented algebra. The differential $d_{\overline{A}[1]}$ is a coderivation for the cofree coalgebra $T^c(\overline{A}[1])$. Thus $(C\overline{A}[1], d_{\overline{A}[1]})$ is a dg-coalgebra.*

Proof. By injecting $\overline{A}[1]$ into $T^c(\overline{A}[1])$, we may think of $(\cdot_{\overline{A}[1]}) : \overline{A}[1]^{\otimes 2} \to T^c(\overline{A}[1])$ as a morphism into the tensor coalgebra. By using proposition 1.1.30, $(\cdot_{\overline{A}[1]})$ extends uniquely into a coderivation:

$$d^c_{\overline{A}[1]} = \sum_{n=0}^{\infty} \sum_{i=0}^{n} (\cdot_{\overline{A}[1]})^{(n)}_{(i)} = d_{\overline{A}[1]}.$$

□

If (A, d_A) is an augmented dg-algebra, then A is a simplical object of $Mod^{\bullet}_{\mathbb{K}}$. This also has an associated complex CA by taking the alternate sum of face maps. The complex CA may be seen to be the total complex of the double complex represented below.

$$\begin{tikzcd}
\cdots \arrow[r, "0"'] & 0 \arrow[r, "0"'] & \mathbb{K} \arrow[r, "0"'] & 0 \arrow[r, "0"'] & \cdots \\
\cdots \arrow[r, "-d_A"'] & A^{-1} \arrow[r, "-d_A"'] \arrow[u, "\varepsilon_A"] & A^{0} \arrow[r, "-d_A"'] \arrow[u, "\varepsilon_A"] & A^{1} \arrow[r, "-d_A"'] \arrow[u, "\varepsilon_A"] & \cdots \\
\cdots \arrow[r, "d_{A^{\otimes 2}}"'] & (A^{\otimes 2})^{-1} \arrow[r, "d_{A^{\otimes 2}}"'] \arrow[u, shift left, "(\cdot_A)"] \arrow[u, shift right, "A\otimes\varepsilon_A"'] & (A^{\otimes 2})^{0} \arrow[r, "d_{A^{\otimes 2}}"'] \arrow[u, shift left, "(\cdot_A)"] \arrow[u, shift right, "A\otimes\varepsilon_A"'] & (A^{\otimes 2})^{1} \arrow[r, "d_{A^{\otimes 2}}"'] \arrow[u, shift left, "(\cdot_A)"] \arrow[u, shift right, "A\otimes\varepsilon_A"'] & \cdots \\
& \vdots \arrow[u] \arrow[u, shift left=2] \arrow[u, shift right=2] & \vdots x \arrow[u] \arrow[u, shift left=2] \arrow[u, shift right=2] & \vdots \arrow[u] \arrow[u, shift left=2] \arrow[u, shift right=2] &
\end{tikzcd}$$

For simplicity, we will write d_1 for the horizontal differential and d_2 for the vertical differential. CA is thus the total complex of the double complex above. Again we may instead consider the looped algebra $\overline{A}[1]$ to obtain a double complex similar as above. The following lemma states that this double complex is well-defined.

Proposition 1.2.8. *Let A an augmented dg-algebra. The bar complex BA is the total associated chain complex of the augmentation ideal $\overline{A}$. $(BA, d^{\bullet}_{BA})$ is the cofree conilpotent coalgebra equipped with $d^{\bullet}_{BA} = d_1 + d_2$ as coderivation.*

Proof. It is apparent that d_1 and d_2 are coderivations with respect to deconcatenation. Since the multiplication $(\cdot_A)$ is a chain map, we should have ${d^{\bullet}_{BA}}^2 = d_1 \circ d_2 + d_2 \circ d_1 = 0$. This will be shown for each element in $A^{\otimes 2}$, the result may then be extended to all of BA. Instead of decorating each a_i with an ω we will follow MacLanes notation and use brackets and bars, $\omega a_1 \otimes \omega a_2 = [a_1|a_2]$. This is how this coalgebra got its name as the bar construction.

$$\begin{aligned}
d_1 \circ d_2[a_1|a_2] &= (-1)^{|a_1|} d_1[a_1 a_2] = (-1)^{|a_1|} d_{A[1]}[a_1 a_2] \\
&= (-1)^{|a_1|+1}[d_A(a_1 a_2)] = (-1)^{|a_1|+1}([d_A(a_1)a_2] + (-1)^{|a_1|}[a_1 d_A(a_2)]) \\
&= (-1)^{|a_1|+1}[d_A(a_1)a_2] - [a_1 d_A(a_2)]
\end{aligned}$$

$$\begin{aligned}
d_2 \circ d_1[a_1|a_2] &= d_2 \circ (d_{A[1]} \otimes id_{A[1]} + id_{A[1]} \otimes d_{A[1]})[a_1 \otimes a_2] \\
&= -d_2 \circ ([d_A(a_1)|a_2] + (-1)^{|a_1|+1}[a_1|d_A(a_2)]) \\
&= (-1)^{|d_A(a_1)|+1}[d_A(a_1)a_2] + (-1)^{2|a_1|+2}[a_1 d_A(a_2)] \\
&= (-1)^{|a_1|}[d_A(a_1)a_2] + [a_1 d_A(a_2)] = -d_1 \circ d_2[a_1|a_2]
\end{aligned}$$

□

Remark 1.2.9. For now we don't need to show that BA is a functor. This property follows from BA being the representing object of Tw$(_, A)$.

On the other hand, a coalgebra C is a comonoid in $Mod_{\mathbb{K}}$. By the dual of proposition A.1.5 we may think of it as an augmented simplical object $C : (\Delta_+)^{op} \to Mod_{\mathbb{K}}$. Dually, all of the simplical identities follows from coassociativity and counitality. A coaugmented coalgebra C may be given an augmented cosimplicial structure in the opposite way of algebras. We then get that the coaugmentation quotient $\overline{C}$ is a semi-cosimplical object of $Mod_{\mathbb{K}}$. Observe that $\overline{C}$ has an associated chain complex like $\overline{A}$, but every arrow goes in the opposite direction.

$$\begin{tikzcd}
\mathbb{K} \arrow[r, "\upsilon_C"] & C \arrow[r, shift left, "\Delta_C"] \arrow[r, shift right, "A\otimes\upsilon_C"'] & C^{\otimes 2} \arrow[r, shift left=3, "\Delta_C"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3, "C^{\otimes 2}\otimes\upsilon_C"'] & C^{\otimes 3} \arrow[r, shift left=3, "\Delta_C"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3, "C^{\otimes 4}\otimes\upsilon_C"'] & \dots \\
\mathbb{K} & C & C^{\otimes 2} \arrow[l, "s_1"'] & C^{\otimes 3} \arrow[l, shift right, "s_i"'] \arrow[l, shift left] & \dots \arrow[l, shift right=2, "s_i"'] \arrow[l] \arrow[l, shift left=2]
\end{tikzcd}$$

The cobar construction is made from the suspended coalgebra $C[-1]$. We may also denote suspension by tensoring with a formal generator s, such that $|s| = 1$. Then we have an isomorphism $C[-1] \simeq \mathbb{K}\{s\}\otimes C$. The cobar construction is realized as the free tensor algebra $T(\overline{C}[-1])$, where the comultiplication $\Delta_{\overline{C}[-1]}$ induces a derivation $d_{\overline{C}[-1]}$ by proposition 1.1.30.

Remark 1.2.10. As we have chosen to define $(\cdot_{A[1]})(a_1 \otimes a_2) = (-1)^{|a_1|}a_1a_2$, we are forced by the linear dual to define $\Delta_{C[-1]}(c) = -(-1)^{|c_{(1)}|}c_{(1)} \otimes c_{(2)}$. Here we use the Sweedlers notation without sums to denote the comultiplication. Note that this really should be a sum of many different elementary tensors. Lastly, observe that this definition also agrees with Koszuls sign rule.

The associated cochain complex CC is the total complex of the double complex below. In the same manner as before, we want to study $C[-1]$ to obtain an analogous result to the bar construction.

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\dots \arrow[r, "d_{\overline{C}^{\otimes 2}}"] & (\overline{C}^{\otimes 2})^{-1} \arrow[u, shift left] \arrow[u, shift right, "\Delta_C\otimes\overline{C}"] \arrow[r, "d_{\overline{C}^{\otimes 2}}"] & (\overline{C}^{\otimes 2})^{0} \arrow[u, shift left] \arrow[u, shift right, "\Delta_C\otimes\overline{C}"] \arrow[r, "d_{\overline{C}^{\otimes 2}}"] & (\overline{C}^{\otimes 2})^{1} \arrow[u, shift left] \arrow[u, shift right, "\Delta_C\otimes\overline{C}"] \arrow[r, "d_{\overline{C}^{\otimes 2}}"] & \dots \\
\dots \arrow[r, "d_{\overline{C}}"] & \overline{C}^{-1} \arrow[u, "\Delta_C"] \arrow[r, "d_{\overline{C}}"] & \overline{C}^{0} \arrow[u, "\Delta_C"] \arrow[r, "d_{\overline{C}}"] & \overline{C}^{1} \arrow[u, "\Delta_C"] \arrow[r, "d_{\overline{C}}"] & \dots \\
\dots \arrow[r] & 0 \arrow[u] \arrow[r] & \mathbb{K} \arrow[u, "0"] \arrow[r] & 0 \arrow[u] \arrow[r] & \dots
\end{tikzcd}$$

Proposition 1.2.11. *Let C be a coaugmented dg-coalgebra. The cobar complex ΩC is the total associated chain complex of the suspended coaugmentation quotient $\overline{C}[-1]$. $(\Omega C, d_{\Omega C})$ is the free algebra equipped with the differential $d_{\Omega C} = d_1 + d_2$ as derivation.*

Proof. This proof is similar to the one given for the bar construction. □

Given a string of elements in the cobar $sc_1 \otimes \cdots$, we write it by using brackets and bars instead,

$$sc_1 \otimes sc_2 \otimes \cdots \otimes sc_n = \langle c_1 | c_2 | \cdots | c_n \rangle.$$

The bar and cobar construction defines an adjoint pair of functors. We want to show that for any conilpotent dg-coalgebra C, the object ΩC represents a functor in the category of augmented algebras. By Yoneda's lemma, Ω does truly define a functor.

Theorem 1.2.12. *Let C be a conilpotent dg-coalgebra and A an augmented dg-algebra. The functor Tw(C, A) is represented in both arguments, i.e.*

$$\textit{Alg}^{\bullet}_{\mathbb{K},+}(\Omega C, A) \simeq \textit{Tw}(C, A) \simeq \textit{coAlg}^{\bullet}_{\mathbb{K},Conil}(C, BA).$$

Proof. We will show that ΩC represents the set of twisting morphisms in the first argument. Showing that BA represents the second argument uses every dual proposition. Thus, it is necessary that C is conilpotent, in order to dualize the arguments.

Suppose that $f : \Omega C \to A$ is an augmented dg-algebra homomorphism. f is then a morphism of degree 0. By freeness, f is uniquely determined by a morphism $f \mid_{\overline{C}[-1]} : \overline{C}[-1] \to \overline{A}$ of degree 0, which corresponds to a morphism $f' : C \to A$ of degree 1 which is 0 on the augmentation and coaugmentation.

Since f is a morphism of chain complexes it commutes with the differential, i.e.

$$\begin{aligned} & f \circ d_{\Omega C} = d_A \circ f \\ \Leftrightarrow \quad & f \circ (d_1 + d_2) = d_A \circ f \end{aligned}$$

By 1.1.11, to establish these conditions it is enough to consider the summand where $d_1 = -d_C$ and $d_2 = \overline{\Delta}_{C[-1]}$. Then the right hand side becomes $-f' \circ d_C - (-1)^{|f|}(\cdot_A)(f' \otimes f')\Delta_C$. This is equivalent to say that $-f' \circ d_C - f' \star f' = d_A \circ f'$. Thus f' is a twisting morphism as desired.

Since every step to establish that f' is a twisting morphism was a logic equivalence, we arrive at the desired conclusion. □

For our convenience, we will give these isomorphisms some names. Whenever $\tau : C \to A$ is a twisting morphism, the induced morphism of algebras is denoted $f_\tau : \Omega C \to A$ and the induced morphism of coalgebras is denoted $g_\tau : C \to BA$.

Remark 1.2.13. We could have defined a twisting morphism from any coalgebra C to algebra A. In this case we could have defined a twisting morphism as a morphism of degree 1 which satifies the Cartan-Maurer equation. This definition of twisting morphism is however not represented by the cobar and bar construction on augmented algebras. The subclass of twisting morphisms which also (co)restricts to twisting morphisms on its coaugmentation quotient and augmentation ideal would be represented in this manner. This is simply because our definition requires that a twisting morphism on (co)augmented (co)algebras restricts to twisting morphisms in this new sense.

The cobar-bar adjunction consists of a composition with the augmentation ideal (quotient) and then the (co)free tensor (co)algebra. By reversing these operations we obtain another adjunction which is more or less the same adjunction. By abuse of language, we will call these functors for the bar and cobar construction as well, and they establish an adjoint pair between dg-algebras and reduced conilpotent dg-coalgebras. In other words, given a dg-algebra A and a reduced conilpotent dg-coalgebra C, $BA = \overline{T}^c(A[1])$ and $\Omega C = \overline{T}(C[-1])$.

$$\begin{tikzcd} \mathsf{Alg}_{\mathbb{K}}^{\bullet} \arrow[r, bend right, "B"'] \arrow[r, phantom, "\perp"] & \mathsf{coAlg}_{\mathbb{K},conil,-}^{\bullet} \arrow[l, bend right, "\Omega"'] \end{tikzcd}$$

Associated to this adjunction, we obtain universal elements, together with universal properties. Let A be an augmented dg-algebra, then the identity of the coalgebras $id_{BA} : BA \to BA$, the counit $\varepsilon_A : \Omega BA \to A$ and a twisting morphism $\pi_A : BA \to A$ are equivalent by the adjunction and representation. Dually, the identity of algebras $id_{\Omega C} : \Omega C \to \Omega C$, the unit $\eta_C : C \to B\Omega C$ and the twisting morphism $\iota_C : C \to \Omega C$ are equivalent. The morphisms π_A and ι_C are called the universal elements. We summarize their universal property in the following corollary.

Corollary 1.2.13.1. *Let A be an augmented dg-algebra, and C a conilpotent dg-coalgebra. Any twisting morphism $\alpha : C \to A$ factors uniquely through either π_A or ι_C.*

$$\begin{tikzcd} & \Omega C \arrow[dr, dashed, "g_\alpha"] & \\ C \arrow[ur, "\iota_C"] \arrow[rr, "\alpha"] \arrow[dr, dashed, "f_\alpha"'] & & A \\ & BA \arrow[ur, "\pi_A"'] & \end{tikzcd}$$

Moreover, the morphism f_α is a morphism of dg-coalgebras, and g_α is a morphism of dg-algebras.

Definition 1.2.14 (Augmented Bar-Cobar construction)**.** Let A be an augmented dg-algebra. The (right) augmented bar construction is the right twisted tensor product $BA \otimes_{\pi_A} A$, where π_A is the universal twisting morphism.

Let C be a conilpotent dg-coalgebra. The (right) augmented cobar construction is the right twisted tensor product $C \otimes_{\iota_C} \Omega C$, where ι_C is the universal twisting morphism.

Remark 1.2.15. We could have defined the augmented bar-cobar construction as the left twisted tensor product. There is really no preference of handedness. Whenever we wish to be precise which handedness we will use it will be specified, e.g. the left augmented bar construction of A.

Proposition 1.2.16. *The augmentation ideal and quotient of the augmented bar and cobar construction are acyclic, i.e.* $BA\overline{\otimes}_{\pi_A}A$ *(*$A\overline{\otimes}_{\pi_A}BA$*) and* $C\overline{\otimes}_{\iota_C}\Omega C$ *(*$\Omega C\overline{\otimes}_{\iota_C}C$*) are acyclic.*

Proof. We will postpone this proof until chapter 3. This is a part of the fundamental theorem of twisting morphisms, and will not be relevant until then. □

1.3 Strongly Homotopy Associative Algebras and Coalgebras

1.3.1 SHA-Algebras

We have seen from corollary 1.2.7 that any dg-algebra A defines a dg-coalgebra $T^c(A[1])$, the bar construction, with a coderivation m^c of degree -1. Does this however work in reverse? I.e. if A is a vector space such that the coalgebra $T^c(A[1])$ together with a coderivation m^c is a dg-coalgebra, is then A an algebra? The answer to this is no, but it leads to the definition of a strongly homotopy associative algebra.

Definition 1.3.1. An A_∞-algebra is a graded vector space A together with a differential $m : \overline{T}^c(A[1]) \to \overline{T}^c(A[1])$ that is a coderivation of degree 1.

The differential m induces structure morphisms on $A[1]$. By proposition 1.1.30, there is a natural bijection $\mathrm{Hom}_{\mathbb{K}}(\overline{T}^c(A[1]), A[1]) \simeq \mathrm{Coder}(\overline{T}^c(A[1]), \overline{T}^c(A[1]))$ given by the projection onto $A[1]$. Thus $m : \overline{T}^c(A[1]) \to \overline{T}^c(A[1])$ corresponds to maps $\widetilde{m}_n : A[1]^{\otimes n} \to A[1]$ of degree 1 for any $n \geqslant 1$. We define maps $m_n : A^{\otimes n} \to A$ by the composite $s\widetilde{m}_n\omega^{\otimes n}$. Since $\omega^{\otimes n}$ is of degree $-n$, $\widetilde{m}_n$ and s is of degree 1, we get that m_n is of degree $2-n$.

$$\begin{tikzcd}
A^{\otimes n} \arrow[r, "m_n"] \arrow[d, "\omega^{\otimes n}"', "\simeq"] & A \\
A[1]^{\otimes n} \arrow[r, "\widetilde{m}_n"] & A[1] \arrow[u, "s", "\simeq"']
\end{tikzcd}$$

Proposition 1.3.2. *An* A_∞*-algebra is equivalent to a graded vector space* A *together with homogenous morphisms* $m_n : A^{\otimes n} \to A$ *of degree* $2-n$*. Moreover, the morphism must satisfy the following relations for any* $n \geqslant 1$*:*

$$(\mathit{rel}_n) \qquad \sum_{p+q+r=n} (-1)^{pq+r} m_{p+1+r} \circ (id^{\otimes p} \otimes m_q \otimes id^{\otimes r}) = 0$$

Remark 1.3.3. We make a more convenient notation for (rel_n), called partial composition $\circ_i$,

$$m_{p+1+r} \circ_{p+1} m_q = m_k \circ (id^{\otimes p} \otimes m_q \otimes id^{\otimes r})$$

$$(\mathrm{rel}_n) \qquad \sum_{p+q+r=n} (-1)^{pq+r} m_{p+1+r} \circ_{p+1} m_q = 0.$$

Before starting with the proof, we will need a lemma for checking whether a coderivation $m : T^c(A) \to T^c(A)$ is a differential.

Lemma 1.3.4. *Let $m : T^c(A) \to T^c(A)$ be a coderivation, and denote $m_n = m|_{A^{\otimes n}}$. m is a differential if and only if the following relations are satisfied,*

$$\sum_{p+q+r=n} m_{p+1+r} \circ_{p+1} m_q = 0.$$

Proof. By proposition 1.1.30 we may write $m = \sum_{n=0}^{\infty} \sum_{i=0}^{n} m_{(n)}^{(i)}$. By using partial composition, we rewrite its n'th component as,

$$m_n = \sum_{q=1}^{n} \sum_{p=1}^{n} id^{\otimes(n-q)} \circ_p m_q = \sum_{p+q+r=n} id^{\otimes(p+1+r)} \circ_{p+1} m_q.$$

For m^2 we denote it's n'th component as m_n^2. Let $\pi : T^c(A) \to A$ denote the projection onto A. Observe the following:

$$m_n^2 = m \circ m_n = m \circ \sum_{p+q+r=n} id^{\otimes(p+1+r)} \circ_{p+1} m_q = \sum_{p+q+r=n} m \circ_{p+1} m_q,$$

$$\pi m_n^2 = \pi \sum_{p+q+r=n} m \circ_{p+1} m_q = \sum_{p+q+r=n} m_{p+1+r} \circ_{p+1} m_q.$$

By 1.1.33, every coderivation is uniquely determined by π, we get that $m^2 = 0$ if and only if

$$\sum_{p+q+r=n} m_{p+1+r} \circ_{p+1} m_q = 0.$$

□

Proof of proposition 1.3.2. Let (A, m) be an A_∞-algebra. We denote the n'th component of m as $\widetilde{m}_n$. The n'th components thus define maps $m_n : A^{\otimes n} \to A$ as $m_n = s\widetilde{m}_n \omega^{\otimes n}$.

By the above lemma we know that the n'th component of m^2 is,

$$\begin{aligned}&\sum_{p+q+r=n} \widetilde{m}_{p+1+r} \circ_{p+1} \widetilde{m}_q \\ &\quad = \sum_{p+q+r=n} \omega m_{p+1+r} s^{\otimes(p+1+r)} \circ_{p+1} \omega m_q s^{\otimes q} = \sum_{p+q+r=n} (-1)^{pq+r} \omega m_{p+1+r} \circ_{p+1} m_q s^{\otimes n}.\end{aligned}$$

The last equation is given by applying proposition 1.1.34 twice. In other words, we want to find a parity $p = p_1 + p_2$ which determines the sign above. To get p_1 we start with moving the s on the left,

$$s^{\otimes p+1+r} \circ (id^{\otimes p} \otimes \omega m_q s^{\otimes q} \otimes id^{\otimes r}) = (-1)^{p_1}(s^{\otimes q} \otimes m_q s^{\otimes q} \otimes s^{\otimes r}).$$

By proposition 1.1.34,

$$p_1 = \sum_{i=1}^{n} \sum_{1 \leqslant j < i} (\text{if } j = p+1 \text{ then } 1 \text{ otherwise } 0) = r.$$

In the next step we separate the s on the right,

$$(id^{\otimes p} \otimes m_q \otimes id^{\otimes r}) \circ s^{\otimes n} = (-1)^{p_2}(s^{\otimes q} \otimes m_q s^{\otimes q} \otimes s^{\otimes r}).$$

We calculate p_2 to be,

$$p_2 = (2-q) \sum_{1 \leqslant j < p+1} 1 = 2p - qp.$$

Thus the parity of p is $p = 2p - qp + r = pq + r$ modulo 2.

Since suspension and loop are isomorphisms, we get that $m^2 = 0$ if and only if (rel_n) are 0 for every $n \geqslant 1$, i.e.

$$\sum_{p+q+r=n} (-1)^{pq+r} m_{p+1+r} \circ_{p+1} m_q = 0.$$

$\square$

Given an A_∞ algebra A, we may either think of it as a differential tensor coalgebra $\overline{T}^c(A[1])$ with differential $m : \overline{T}^c(A[1]) \to \overline{T}^c(A[1])$, or as a graded vector space with morphisms $m_n : A^{\otimes n} \to A$ satisfying (rel_n). We will calculate (rel_n) for $n = 1, 2, 3$:

(rel_1) $\quad m_1 \circ m_1 = 0$

(rel_2) $\quad m_1 \circ m_2 - m_2 \circ_1 m_1 - m_2 \circ_2 m_1 = 0$

(rel_3) $\quad m_1 \circ m_3 - m_2 \circ_1 m_2 + m_2 \circ_2 m_2 + m_3 \circ_1 m_1 + m_3 \circ_2 m_1 + m_3 \circ_3 m_1 = 0$

We see that (rel_1) states that m_1 should be a differential. Thus we may think of (A, m_1) as a chain complex. Furthermore, (rel_2) says that $m_2 : (A^{\otimes 2}, m_1 \otimes id_A + id_A \otimes m_1) \to (A, m_1)$ is a morphism of chain complexes. Lastly, (rel_3) gives us a homotopy for the associator of m_2, namely m_3. Thus we may regard (A, m_1, m_2) as an algebra which is associative up to the homotopy m_3. Regarding A as a chain complex instead we obtain our final definition of an A_∞-algebra.

Proposition 1.3.5. *Suppose that (A, d) is a chain complex, and that there exists morphisms $m_n : A^{\otimes n} \to A$ of degree $2-n$ for any $n \geqslant 2$. A is an A_∞-algebra if and only it satisfies the following relations:*

$$(\textit{rel'}_n) \qquad \partial(m_n) = -\sum_{\substack{n=p+q+r\\k=p+1+r\\k>1,q>1}} (-1)^{pq+r} m_k \circ_{p+1} m_q$$

We define the homotopy of an A_∞-algebra to be the homology of the chain complex (A, m_1). Since $\partial(m_3) = m_2 \circ_1 m_2 - m_2 \circ_2 m_2$, we get that m_2 is associative in homology. Thus for any A_∞-algebra A, the homotopy HA is an associative algebra. The operadic homology of A is defined as the homology of $(T^c(A[1]), m)$, which is the non-unital augmented Hochschild homology of A.

Example 1.3.6. Suppose that V is a cochain complex with differential d. Then V is an A_∞-algebra with trivial multiplication. In other words $m^1 = d$ and $m^i = 0$ for any $i > 1$.

Example 1.3.7. Suppose that A is a dg-algebra. Then A is an A_∞-algebra where $m^1 = d$, $m^2 = (\cdot)$ and $m^i = 0$ for any $i > 2$.

Example 1.3.8. Let A be a connected weight-graded algebra over $\mathbb{K}$. We may then think of $\mathbb{K}$ as an A-module with trivial action, i.e. as the quotient ${}^{A}/_{A^{(i)}}{}_{(i\geqslant 1)}$, then $Ext^*_A(\mathbb{K}, \mathbb{K})$ is an A_∞-algebra. This was shown by Lu, Palmari, Wu and Zhang [11].

Next we want to understand the category of A_∞-algebras. A morphism between A_∞-algebras is called an ∞-morphism. We define such an ∞-morphism $f : A \rightsquigarrow B$ between A_∞-algebras as associated dg-coalgebra homomorphism $Bf : (\overline{T}^c(A[1]), m^A) \to (\overline{T}^c(B[1]), m^B)$. Here Bf is purely formal, we will make sense of this soon.

Proposition 1.3.9. *Let A, B be two A_∞-algebras. A collection of morphisms $f_n : A^{\otimes n} \to B$ of degree $1-n$ for any $n \geqslant 1$ defines an ∞-morphism $f : A \rightsquigarrow B$ if and only if f_1 is a morphism of chain complexes and for any $n \geqslant 2$ the following relations are satisfied:*

$$(\textit{rel}_n) \qquad \partial(f_n) = \sum_{\substack{p+1+r=k\\p+q+r=n}} (-1)^{pq+r} f_k \circ_{p+1} m^A_q - \sum_{\substack{k\geqslant 2\\i_1+\ldots+i_k=n}} (-1)^e m^B_k \circ (f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_k}),$$

where e is given as

$$e = \sum_{l=1}^{k} (1 - i_l) \sum_{1\leqslant m<l} i_m.$$

Proof. Establishing the shape of this equation is immediate by the universal property of cofree coalgebras. The parity e is obtained by factorizing out the s on the right.

$$(f_{i_1} \otimes \cdots \otimes f_{i_k}) \circ s^{\otimes n} = (-1)^e (f_{i_1} s^{\otimes i_1} \otimes \cdots \otimes f_{i_k} s^{\otimes i_k}).$$

By proposition 1.1.34, we arrive at the conclusion,

$$e = \sum_{l=1}^{k} |f_{i_l}| \sum_{1 \leqslant m < l} |s^{\otimes i_m}| = \sum_{l=1}^{k} (1 - i_l) \sum_{1 \leqslant m < l} i_m$$

□

Since the composition of two dg-coalgebra homomorphisms is again a dg-coalgebra homomorphism, we get that the composition of two ∞-morphisms is again an ∞-morphism. More explicitly if $f : A \rightsquigarrow B$ and $g : B \rightsquigarrow C$ are two ∞-morphisms, then their composition is defined as

$$(fg)_n = \sum_r \sum_{i_1 + \ldots + i_r = n} (-1)^e g_r (f_{i_1} \otimes \ldots \otimes f_{i_r}).$$

Here e denotes the same parity as above.

Definition 1.3.10. An ∞-morphism $f : A \rightsquigarrow B$ is called strict if $f_n = 0$ for any $n \geqslant 2$.

Definition 1.3.11. Alg_∞ denotes the category of A_∞-algebras, and the morphisms in this category are the ∞-morphisms.

Observe that we may extend the bar construction to $B : \mathsf{Alg}_\infty \to \mathsf{CoAlg}^{\bullet}_{\mathbb{K},conil}$ to a fully fatihful functor. This may be done explicitly by using proposition 1.1.30. The subcategory of the essential image is the full subcategory of every almost cofree dg-coalgebra. Notice that the bar construction on the category of dg-algebras is a non-full injection into the category of A_∞-algebras. This inclusion gives us a recontextualization of a dg-algebra as an A_∞-algebra.

A quasi-isomorphism between A_∞-algebras is called an ∞-quasi-isomorphism. Given an ∞-morphism $f : A \rightsquigarrow B$, then we say that it is an ∞-quasi-isomorphism if f_1 is a quasi-isomorphism. If we wanted to be more stringent with this definition, we would want an ∞-quasi-isomorphism to be an ∞-morphism which is a quasi-isomorphism of dg-coalgebras. We will later see that these definitions are equivalent.

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A homotopy between two A_∞-algebras is a homotopy between the dg-coalgebras they define. We may however trace this definition back along the quasi-inverse of the bar construction to get a new definition in terms of many morphisms. Let $f, g : A \rightsquigarrow B$ be two ∞-morphisms, we say that $f - g$ is null-homotopic if there is a collection of morphisms $h_n : A^{\otimes n} \to B$ of degree n such that the following relations are satisfied for any $n \geqslant 1$:

$$f_n - g_n = \sum (-1)^s m^B_{r+1+t} \circ (f_{i_1} \otimes \ldots \otimes f_{i_r} \otimes h_k \otimes g_{j_1} \otimes \ldots \otimes g_{j_t}) + \sum (-1)^{j+kl} h_i \circ_{j+1} m^A_k.$$

Where s is some constant depending on t, r and k. More details may be found in [12].

As in the same case for algebras, there is also a notion of unital A_∞-algebras and augmented A_∞-algebras. For this discussion it is important to note that the field $\mathbb{K}$ is also an A_∞-algebra. This algebra will serve as the initial algebra, in the same way as it does for ordinary algebras.

Definition 1.3.12. A strictly unital A_∞-algebra is an A_∞-algebra A together with a unit morphism $\upsilon_A : \mathbb{K} \to A$ of degree 0 such that the following are satisfied:

- $m_1 \circ \upsilon_A = 0$.
- $m_2(id_A \otimes \upsilon_A) = id_A = m_2(\upsilon_A \otimes id_A)$.
- $m_i \circ_k \upsilon_A = 0$ for any $i \geqslant 3$ and $1 \leqslant k < i$.

A strictly unital ∞-morphism $f : A \rightsquigarrow B$ between strictly unital A_∞-algebras is a morphism which preserves the unit. This means that $f_1\upsilon_A = \upsilon_B$ and $f_i \circ_k \upsilon_A = 0$ for any $i \geqslant 2$ and $1 \leqslant k < i$. The collection of strictly unital A_∞-algebras and strictly unital ∞-morphisms form a non-full subcategory of A_∞-algbras. A strict ∞-morphism which is unital at the level of chain complexes is automatically stricly unital. Strict unital will then mean strict and strictly unital. Note that $\mathbb{K}$ is strictly unital where the unit is $id_{\mathbb{K}}$.

Definition 1.3.13. An augmented A_∞-algebra is a strictly unital A_∞-algebra A together with a strict unital morphism $\varepsilon_A : A \to \mathbb{K}$. The ∞-morphism ε_A is called the augmentation of A.

The collection of augmented A_∞-algebras and strictly unital morphism is the category of augmented A_∞-algebras, denoted as $\mathsf{Alg}_{\infty,+}$. As in the same way for algebras, there is an equivalence of categories $\mathsf{Alg}_\infty \simeq \mathsf{Alg}_{\infty,+}$. The augmentation ideal, or the reduced A_∞-algebra is the kernel of the augmentation ε_A. A priori, it does not make sense to talk about this limit as we do not know if it exists. However, we will see in section 2.3.3 that such morphisms does in fact have a kernel. This defines a functor, $\overline{\_} : \mathsf{Alg}_{\infty,+} \to \mathsf{Alg}_\infty$, where $Ker\varepsilon_A = \overline{A}$. The quasi-inverse to this functor is given by free augmentations. Given an A_∞-algebra A, we may construct the A_∞-algebra $A \oplus \mathbb{K}$. The structure morphisms are given by m_i^A, but there is now a unit $\upsilon_{A\oplus\mathbb{K}}$. Thus we get that $m_1(1) = 0$, $m_2(a \otimes 1) = a$ and $m_i \circ_k 1 = 0$ in the same manner. We obtain a functor $\_^+ : \mathsf{Alg}_\infty \to \mathsf{Alg}_{\infty,+}$, where $A \oplus \mathbb{K} = A^+$.

1.3.2 A_∞-Coalgebras

Dual to A_∞-algebras we got conilpotent A_∞-coalgebras. Here we instead ask ourselves if the cobar construction has some converse, i.e. if C is a graded vector space such that $T(C[-1])$ together with a derivation m is a dg-algebra, is then C a coalgebra? Again, the answer to this is no, but we do obtain a definition for conilpotent A_∞-coalgebras.

Definition 1.3.14. A graded vector space C is called a conilpotent A_∞-coalgebra if it is a dg-algebra of the form $(\overline{T}(C[-1]), d)$ where d is a derivation of degree 1.

Remark 1.3.15. For the rest of this thesis, an A_∞-coalgebra should be understood as a conilpotent A_∞-coalgebra unless otherwise specified.

Corollary 1.3.15.1. *C is an A_∞-coalgebra with differential d then there is a chain complex (C, d^1), where d^1 is of degree 1, and together with morphisms $d^n : C \to C^{\otimes n}$ such that d*

uniquely determines each d^i for any $i > 0$. Conversely, if the morphisms d^i satisfy (rel)$_n$, then they uniquely determine a d such that C is an A_∞-coalgebra,

$$(\textit{rel}_n) \text{ is } \qquad \sum_{p+q+r=n} (-1)^{pq+r} d^{p+1+q} \circ^{op}_{p+1} d^q = 0$$

A morphism of A_∞-coalgebras would be defined in the same manner, but opposite. An ∞-morphism $f : C \rightsquigarrow D$ is then either a morphism $\widetilde{f} : (T(C[-1]), m^C) \to (T(D[-1]), m^D)$ of dg-algebras; or equivalently it is a collection of morphisms $f_n : C \to D^{\otimes n}$ of degree $1-n$ such that f_1 is a morphism of chain complexes, and for any $n \geqslant 2$ the following relations are satisfied:

$$(\mathrm{rel}_n) \qquad \partial(f_n) = \sum_{\substack{p+1+r=k \\ p+q+r=n}} (-1)^{pq+r} f_k \circ^{op}_{p+1} m^D_q - \sum_{\substack{k\geqslant 2 \\ i_1+\ldots+i_k=n}} (-1)^e m^C_k \circ^{op} (f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_k}),$$

where e is given as

$$e = \sum_{l=1}^{k} (1-i_l) \sum_{1\leqslant m<l} i_m.$$

We denote coAlg$_\infty$ as the category of A_∞-coalgebras. In the same manner, the cobar construction extends to this category and gives us an identification of A_∞-coalgebras and a subcategory of dg-algebras. This subcategory consists of every dg-algebra that is isomorphic, as an algebra, to a free tensor algebra. Lastly, every dg-coalgebra is an A_∞-coalgebra by letting every morphism $m^i = 0$ where $i > 2$. This gives a non-full inclusion.

Chapter 2

Homotopy Theory of Algebras

Quillen envisioned a more general approach to homotopy theory, which he dubbed homotopical algebra. A homotopy theory was first enclosed by the structure of a model category, and now we mostly consider closed model categories. Many of the results from classical homotopy theory was recovered in this new setting of model categories. The theorem which we are most concerned about is Whiteheads theorem:

Theorem 2.0.1 (Whiteheads Theorem)**.** *Let X and Y be two CW-complexes. If $f : X \to Y$ is a weak equivalence, then it is also a homotopy equivalence. I.e. there exists a morphism $g : Y \to X$ such that $gf \sim id_X$ and $fg \sim id_Y$.*

If we endow a Quillen model category onto the category Top, we get that a space X is bifibrant if and only if it is a CW-complex. The natural generalization is then to not ask of X to be a CW-complex, but a bifibrant object.

Theorem 2.0.2 (Generalized Whiteheads Theorem)**.** *Let $\mathcal{C}$ be a model category. Suppose that X and Y are bifibrant objects of $\mathcal{C}$, and that there is a weak-equivalence $f : X \to Y$. Then f is also a homotopy equivalence, i.e. there exists a morphism $g : Y \to X$ such that $gf \sim id_X$ and $fg \sim id_Y$.*

The category of differential graded algebras employs such a model category, and here we let the weak-equivalences be quasi-isomorphisms. On the other hand the category of differential graded coalgebras has a model structure where the weak equivalences are the maps which are sent to quasi-isomorphism by the cobar construction. Moreover, in this case the bar and cobar construction defines a Quillen equivalence between these model structures. As we will see, a dg-coalgebra will be bifibrant exactly when it is an A_∞-algebra. Thus, by Whiteheads theorem, quasi-isomorphisms lift to homotopy equivalences. In this case the derived category of A_∞-algebras is equivalent to the homotopy category of A_∞-algebras.

We will conclude this section by looking at the category of algebras as a subcategory of A_∞-algebras. The derived category may then be expressed as the homotopy category A_∞-algebras, restricted to algebras.

2.1 Model categories

In this section we will define Quillens model category. As one may see is that in practice there are a plethora of semantically different definitions of model categories, however they are all made to be equivalent. The difference comes down to preference. In this thesis we will use the definitions as they are developed in Mark Hoveys book [13]. We will then go on to prove the basic results known about model categories, its associated homotopy category and Quillen functors between model categories.

Before we state the definition of a model category we need some preliminary definitions. For this section, let $\mathcal{C}$ be a category.

Definition 2.1.1 (Retract)**.** A morphism $f : A \to B$ in $\mathcal{C}$ is a retract of a morphism $g : C \to D$ if it fits in a commutative diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, "id_A"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "id_B"'] & D \arrow[r] & B \end{tikzcd}$$

Definition 2.1.2 (Functorial factorization)**.** A pair of functors $\alpha, \beta : \mathcal{C}^{\to} \to \mathcal{C}^{\to}$ is called a functorial factorization if for any morphism $f \in$ Mor$(\mathcal{C})$, $f = \beta(f) \circ \alpha(f)$. We will denote the morphisms in the factorization as f_α and f_β. The functorial factorization may be depicted by the following commutative diagram:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "f_\alpha"'] & & B \\ & C \arrow[ur, "f_\beta"'] & \end{tikzcd}$$

Definition 2.1.3 (Lifting properties)**.** Suppose that the morphisms $i : A \to B$ and $p : C \to D$ fits inside a commutative square. i is said to have the left lifting property with resptect to p, or p has the right lifting property with respect to i, if there is an $h : B \to C$ such that the two triangles commute.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & C \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed, "h"] & D \end{tikzcd}$$

Remark 2.1.4. We will call the left lifting property for LLP and the right lifting property for RLP.

Definition 2.1.5 (Wide subcategory)**.** A subcategory $\mathcal{W} \subset \mathcal{C}$ is called wide if $\mathcal{W}$ has every object $\mathcal{C}$ has. In particular, $\mathcal{W}$ is a subcategory having every identity morphism.

2.1.1 Model categories

Definition 2.1.6 (Model category)**.** Let $\mathcal{C}$ be a category with all finite limits and colimits. $\mathcal{C}$ admits a model structure if there are three wide subcategories each defining a class of morphisms:

- Ac $\subset$ Mor$(\mathcal{C})$ are called weak equivalences
- Cof $\subset$ Mor$(\mathcal{C})$ are called cofibrations
- Fib $\subset$ Mor$(\mathcal{C})$ are called fibrations

In addition we call morphisms in Cof $\cap$ Ac for acyclic cofibrations and Fib $\cap$ Ac for acyclic fibrations. Moreover, $\mathcal{C}$ has two functorial factorizations (α, β) and (γ, δ). The following axioms should be satisfied:

MC1 The class of weak equivalences satisfy the 2-out-of-3 property, i.e. if f and g are composable morphisms such that 2 out of f, g and gf are weak equivalences, then so is the third.

MC2 The three classes Ac, Cof and Fib are retraction closed, i.e. if f is a retraction of g, and g is either a weak-equivalence, cofibration or fibration, then so is f.

MC3 The class of cofibrations have the left lifting property with respect to acyclic fibrations, and fibrations have the right lifting property with respect to acyclic cofibrations.

MC4 Given any morphism f, f_α is a cofibration, f_β is an acyclic fibration, f_γ is an acyclic cofibration and f_δ is a fibration.

Remark 2.1.7. The class Ac has every isomorphism. This is because every isomorphism is a retract of some identity morphism.

Remark 2.1.8. The type of category which has been introduced above was first called a closed model category by Quillen [14]. In his sense, a model category does not require to have either finite limits or finite colimits. In our case, we will explicitly state whenever a model category is non-closed, i.e. does not have every finite limit or colimit.

A model category $\mathcal{C}$ is now defined to be a category equipped with a particular model structure. Notice that a category may admit several model structures. We will postpone examples until sufficient theory have been developed. For more topological examples, we refer to Dwayer-Spalinski [15] and Hovey [13].

An interesting and a not so non-trivial property of model categories is that giving all three classes Ac, Cof and Fib is redundant. Given the class of weak equivalences and either cofibrations or fibrations, the model structure is determined. Thus the classes of fibrations are determined by acyclic cofibrations and cofibrations are determined by fibrations. The next two results will show this.

Lemma 2.1.9 (The retract argument)**.** *Let $\mathcal{C}$ be a category. Suppose there is a factorization $f = pi$ and that f has LLP with respect to p, then f is a retract of i. Dually, if f har RLP to i, then it is a retract of p.*

Proof. We assume that $f : A \to C$ has LLP with respect to $p : B \to C$. Then we may find a lift $r : C \to B$, which realize f as a retract of i.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[d, "p"] \\ C \arrow[r, equal] \arrow[ur, dashed, "r"] & C \end{tikzcd} \Longrightarrow \begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"] & A \arrow[r, equal] \arrow[d, "i"] & A \arrow[d, "f"] \\ C \arrow[r, "r"] & B \arrow[r, "p"] & C \end{tikzcd}$$

□

Proposition 2.1.10. *Let $\mathcal{C}$ be a model category. A morphism f is a cofibration (acyclic cofibration) if and only if f has LLP with respect acyclic fibrations (fibrations). Dually, f is a fibration (acyclic fibration) if and only if it has RLP with respect to acyclic cofibrations (cofibrations).*

Proof. Assume that f is a cofibration. By MC3, we know that f has LLP with respect to acyclic fibrations. Assume instead that f has LLP with respect to ever acyclic fibration. By MC4 we factor $f = f_\alpha \circ f_\beta$, where f_α is a cofibration and f_β is an acyclic fibration. Since we assume f to have LLP with respect to f_β, by lemma 2.1.9 we know that f is a retract of f_α. Thus by MC2, we know that f is a cofibration. □

Corollary 2.1.10.1. *Let $\mathcal{C}$ be a model category. (Acyclic) Cofibrations are stable under pushouts, i.e. if f is an (acyclic) cofibration, then f' is an (acyclic) cofibration.*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & C \arrow[d, "f'"] \\ B \arrow[r] \arrow[ur, phantom, "\lrcorner"] & D \end{tikzcd}$$

Dually, fibrations are stable under pullbacks.

Proof. This is clear by the universal property of pushouts. □

Since we assume that every model category $\mathcal{C}$ is admits finite limits and colimits, we know that it has both an initial and a terminal object. We let $\varnothing$ denote the initial object and $*$ denote the terminal object.

Definition 2.1.11 (Cofibrant, fibrant and bifibrant objects)**.** Let $\mathcal{C}$ be a model category. An object X is called cofibrant if the unique morphism $\varnothing \to X$ is a cofibration. Dually, X is called fibrant if the unique morphism $X \to *$ is fibrant. If X is both cofibrant and fibrant, we call it bifibrant.

There is no reason for every object to be either cofibrant or fibrant. However, we may see that every object is weakly equivalent to an object which is either fibrant or cofibrant. We will make it precise what it means for two objects to be weakly equivalent later.

Construction 2.1.12. Let X be an object of a model category $\mathcal{C}$. The morphism $i : \varnothing \to X$ has a functorial factorization $i = i_\beta \circ i_\alpha$, where $i_\alpha : \varnothing \to QX$ is a cofibration and $i_\beta : QX \to X$ is an acyclic fibration. By definition QX is cofibrant and weakly equivalent to X.

$Q : \mathcal{C} \to \mathcal{C}$ defines a functor called the cofibrant replacement. To see this we first look at the slice category $\varnothing/\mathcal{C}$. The objects are morphisms $f : \varnothing \to X$ for any object X in $\mathcal{C}$, while morphisms are commutative triangles. We first observe that $\varnothing/\mathcal{C} \subset \mathcal{C}^{\to}$ is a subcategory of the arrow category. Thus (α, β) may be interpreted as functors on the slice category to the arrow category. Moreover, since every arrow $f : \varnothing \to X$ is unique, we observe that this category is equivalent to $\mathcal{C}$. Thus (α, β) may be interpreted as functors on $\mathcal{C}$ into arrows. We define Q as the composition $Q = \text{tar} \circ \alpha$.

Dually, we get a fibrant replacement R by dualizing the above argument.

We collect the following properties

Lemma 2.1.13. *The cofibrant replacement Q and fibrant replacement R preserves weak equivalences.*

Proof. Clear from the 2-out-of-3 property. □

Lemma 2.1.14 (Ken Brown's lemma)**.** *Let $\mathcal{C}$ be a model category and $\mathcal{D}$ be a category with weak equivalences satisfying the 2-out-of-3 property. If $F : \mathcal{C} \to \mathcal{D}$ is a functor sending acyclic cofibrations between cofibrant objects to weak equivalences, then F takes all weak equivalences between cofibrant objects to weak equivalences. Dually, if F takes all acyclic fibrations between fibrant objects to weak equivalences, then F takes all weak equivalences between fibrant objects to weak equivalences.*

Proof. Suppose that A and B are cofibrant objects and that $f : A \to B$ is a weak equivalence. Using the universal property of the coproduct we define the map $(f, id_B) = p : A \coprod B \to B$. p has a functorial factorization into a cofibration and acyclic fibration, $p = p_\beta \circ p_\alpha$. We recollect the maps in the following pushout diagram:

$$
\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & B \arrow[d, "i_2"] \arrow[ddrr, bend left, "id_B"] & & \\
A \arrow[r, "i_1"] \arrow[drrr, bend right, "f"'] & A\coprod B \arrow[dr, "p_\alpha"] & & \\
& & C \arrow[dr, "p_\beta"] & \\
& & & D
\end{tikzcd}
$$

By lemma 2.1.10.1 both i_1 and i_2 are cofibrations. Since f, id_B and p_β are weak equivalences, so are $p_\alpha \circ i_1$ and $p_\alpha \circ i_2$ by MC2. Moreover, they are acyclic cofibrations.

Assume that $F : \mathcal{C} \to \mathcal{D}$ is a functor as described above. Then by assumption, $F(p_\alpha \circ i_1)$ and $F(p_\alpha \circ i_2)$ are weak equivalences. Since a functor sends identity to identity, we also know that $F(id_B)$ is a weak equivelnce. Thus by the 2-out-of-3 property $F(p_\beta)$ is a weak equivalence, as $F(p_\beta) \circ F(p_\alpha \circ i_2) = id_{F(B)}$. Again, by 2-out-of-3 property $F(f)$ is a weak equivelnce, as $F(f) = F(p_\beta) \circ F(p_\alpha \circ i_1)$. □

2.1.2 Homotopy category

Homotopy theory at it's most abstract is the study of categories and functors up to weak equivalences. Here, a weak equivalence may be anything, but most commonly it is a weak equivalence in topological homotopy or a quasi-isomorphism in homological algebra. The biggest concern when dealing with such concepts is to make a functor well-defined up to these chosen weak-equivalences. To this end, there is a construction to amend these problems, known as derived functors. We define a homotopical category in the sense of Riehl [16].

Definition 2.1.15 (Homotopical Category)**.** Let $\mathcal{C}$ be a category. $\mathcal{C}$ is Homotopical if there is a wide subcategory constituting a class of morphisms known as weak equivalences, Ac $\subset MorC$. The weak equivalences should satisfy the 2**-out-of-**6 **property**, i.e. given three composable morphisms f, g and g, if gf and hg are weak equivalences, then so are f, g, h and hgf.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "gf"] \arrow[drr, dotted] & B \arrow[d, "g"] \arrow[dr, "hg"] & \\
& C \arrow[r, "h"] & D
\end{tikzcd}
$$

Remark 2.1.16. Notice that the 2-out-of-6 property is stronger than the 2-out-of-3 property. To see this, let either f, g or h be the identity, and then conclude with the 2-out-of-3 property.

Remark 2.1.17. The collection of weak equivalences contains every isomorphism. To see this pick an isomorphism f and f^{-1}, then the compositions are the identity on the domain and codomain, which are assumed to be in Ac.

Given such a homotopical category $\mathcal{C}$, we want to invert every weak equivalence and create the homotopy category of $\mathcal{C}$. This construction is developed in Gabriel and Zisman [17] called calculus of fractions. This method essentially tries to mimic localization for commutative rings in a category theoretic fashion. We will not give an account of existence or construction of localizations, however the reader is encouraged to read in Gabriel-Zisman.

Definition 2.1.18. Let $\mathcal{C}$ be a homotopical category. It's homotopy category Ho$\mathcal{C} = \mathcal{C}[\text{Ac}^{-1}]$, together with a localization functor $q : \mathcal{C} \to \text{Ho}\mathcal{C}$. Recall that the localization are determined by the following universal property: If $F : \mathcal{C} \to \mathcal{D}$ is a functor sending weak equivalences to isomorphisms, then it uniquely factors through the homotopy category up to a unique natural isomorphism η.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "q"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{D} \\ & \text{Ho}\mathcal{C} \arrow[ur, "F'"'] & \end{tikzcd}$$

Definition 2.1.19. Suppose that $\mathcal{C}$ is a homotopical category. Two objects of $\mathcal{C}$ are said to be weakly equivalent if they are isomorphic in Ho$\mathcal{C}$. I.e. there is some zig-zag relation between the objects, consisting only of weak equivalences.

Remark 2.1.20. A renown problem with localizations is that even if $\mathcal{C}$ is a locally small category, any localization $\mathcal{C}[S^{-1}]$ does not need to be. Thus, without a good theory of classes or higher universes, we cannot in general ensure that the localization still exists as a locally small category.

We see from the definition of the homotopy category that a functor F admits a lift F' to the homotopy category whenever weak equivalences are sent to isomorphisms. Moreover, if we have a functor F between homotopical categories which preserves weak equivalences, it then induces a functor between the homotopy categories.

Definition 2.1.21 (Homotopical functors)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ between homotopical categories is homotopical if it preserves weak equivalences. Moreover, there is a lift of functors as in the following diagram where η is a natural isomorphism.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "q_{\mathcal{C}}"] & \mathcal{D} \arrow[d, "q_{\mathcal{D}}"] \arrow[dl, Rightarrow, "\eta"] \\ \text{Ho}\mathcal{C} \arrow[r, dashed, "F'"] & \text{Ho}\mathcal{D} \end{tikzcd}$$

Derived functors come into play whenever this is not the case. These lifts are however the closest approximation which we can make functorial. The general exposition of derived functors is beyond the scope of this thesis, but an account of it may be found in [16]. As we will see, model categories are a nice environment to work with these concepts. Firstly we will amend the problem with localizations, where the homotopy category may not exists. Secondly, we will obtain a simple description of some important derived functors.

Proposition 2.1.22. *Any model category $\mathcal{C}$ is a homotopical category.*

Proof. To show that a model category is homotopical it suffices to show that Ac satisfies the 2-out-of-6 property. Assume there are 3 composable morphisms f, g, h such that $gf, hg \in$ Ac. By the 2-out-of-3 property for Ac it is enough to show that at least one of f, g, h, fgh is a weak equivalence to deduce that every other morphism is a weak equivalence.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "gf"'] \arrow[drr, dotted] & B \arrow[d, "g"] \arrow[dr, "hg"] & \\
& C \arrow[r, "h"] & D
\end{tikzcd}$$

To be able to use the model structure, we will first show that we may assume f, g to be cofibrant and g, h to be fibrant. We know by MC4 that f, g, gf may be factored into a cofibration composed with an acyclic fibration, e.g. $f = f_\beta f_\alpha$. Since gf is a weak equivalence, so is $(gf)_\alpha$ by the 2-out-of-3 property.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "f_\alpha"'] & & B \\
& B' \arrow[ur, "f_\beta"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
B \arrow[rr, "g"] \arrow[dr, "g_\alpha"'] & & C \\
& C' \arrow[ur, "g_\beta"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[rr, "gf"] \arrow[dr, "(gf)_\alpha"'] & & C \\
& C'' \arrow[ur, "(gf)_\beta"'] &
\end{tikzcd}$$

Notice that the "cofibrant approximation" of the map from A to C either goes through C' or C''. We conjoin these by taking the pullback. Since acyclic fibrations are stable over pullbacks, we get a pullback square where every morphism is an acyclic fibration. Thus the map $A \to \widetilde{C}$ is a weak equivalence by 2-out-of-3.

$$\begin{tikzcd}
A \arrow[dr, dashed] \arrow[drr, bend left, "g_\alpha"] \arrow[ddr, bend right, "(gf)_\alpha"'] & & \\
& \widetilde{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & C' \arrow[d, "g_\beta"] \\
& C'' \arrow[r, "(gf)_\beta"] & C
\end{tikzcd}$$

To replace f with f_α we must lift the composition into our "new" C, which is $\widetilde{C}$. This is done by using MC3, as f_α is a cofibration and the pullback square above consists entirely of acyclic fibrations.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f_\alpha"] & \widetilde{C} \arrow[d] \\
B' \arrow[r] \arrow[ur, dashed] & C
\end{tikzcd}$$

To summarize we have the following diagram, where every squiggly arrow is a weak equivalence.

$$\begin{tikzcd}
A \arrow[r, "f_\alpha"] \arrow[dr, rightsquigarrow, "t"] & B' \arrow[r, rightsquigarrow] \arrow[d, "s"] & B \arrow[d] \arrow[dr, rightsquigarrow] & \\
& \widetilde{C} \arrow[r, rightsquigarrow] & C \arrow[r] & D
\end{tikzcd}$$

We now wish to promote the arrow $s : B' \to \widetilde{C}$ into a cofibration. We do this by factoring both s and t with $MC4$. Notice that s_β, t_β and t_α are weak equivalences.

$$\begin{tikzcd}
B' \arrow[rr, "s"] \arrow[dr, "s_\alpha"'] & & \widetilde{C} & A \arrow[rr, "t"] \arrow[dr, "t_\alpha"'] & & \widetilde{C} \\
& \widetilde{C}' \arrow[ur, "s_\beta"'] & & & \widetilde{C}'' \arrow[ur, "t_\beta"'] &
\end{tikzcd}$$

To obtain our final factorization we use RLP of s_β on t_α.

$$\begin{tikzcd}
& B' \arrow[d, "s_\alpha"] \\
A \arrow[ur, "f_\alpha"] \arrow[r] \arrow[d, "t_\alpha"] & \widetilde{C}' \arrow[d, "s_\beta"] \\
\widetilde{C}'' \arrow[r, "t_\beta"] \arrow[ur, dashed] & \widetilde{C}
\end{tikzcd}$$

We have now obtained a factorization of gf which are two cofibrations followed by an acyclic fibration, in such a manner that it is compatible with the original composition. The dual to this claim is that we may also factor hg into two fibrations which is preceeded by an acyclic cofibration. In other words, we may assume without loss of generality that f and g are cofibrations, and that g and h are fibrations.

In this case we will show that g is an isomorphism. Consider the diagram below with lifts i and j, these exists since we assume gf and hg to be weak equivalences.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "gf"] & B \arrow[r, "id_B"] \arrow[d, "g"] & B \arrow[d, "hg"] \\
C \arrow[r, "id_C"] \arrow[ur, dashed, "i"] & C \arrow[r, "h"] \arrow[ur, dashed, "j"] & D
\end{tikzcd}$$

Since the diagram is commutative, we get that $i = j$, and that g is both split mono and split epi, with i as its splitting. □

Since every model category is homotopical, it also has an associated homotopy category Ho$\mathcal{C}$. Let $\mathcal{C}_c$, $\mathcal{C}_f$ and $\mathcal{C}_{cf}$ denote the full subcategories consisting of cofibrant, fibrant and bifibrant objects respectively.

Proposition 2.1.23. *Let $\mathcal{C}$ be a model category. The following categories are equivalent:*

- *Ho$\mathcal{C}$*
- *Ho$\mathcal{C}_c$*
- *Ho$\mathcal{C}_f$*
- *Ho$\mathcal{C}_{cf}$*

Proof. We show that Ho$\mathcal{C} \simeq$ Ho$\mathcal{C}_c$. The inclusion $i : \mathcal{C}_c \to \mathcal{C}$ clearly preserves weak equivalences, thus i is homotopical and admits a lift. Moreover, since the cofibrant replacement is also homotopical, it also has a lift.

$$\begin{tikzcd}
\mathcal{C}_c \arrow[r, "i"] \arrow[d] & \mathcal{C} \arrow[d] \\
\mathrm{Ho}\mathcal{C}_c \arrow[r, dashed, bend right, "\mathrm{Ho}\ i"'] & \mathcal{C} \arrow[l, dashed, bend right, "\mathrm{Ho}\ Q"']
\end{tikzcd}$$

It is clear that Ho Q is the quasi-inverse of Ho i. □

As of now we still don't see how model categories will fix the size issues. To do this we will develop the notion of homotopy equivalence, $\sim$. On the subcategory of bifibrant objects $\mathcal{C}_{cf}$, this homotopy equivalence will be a congruence relation. This, together with the fact that there is an equivalence of categories Ho$\mathcal{C}_{cf} \simeq {}^{\mathcal{C}_{cf}}/_{\sim}$, is enough to solve the size issues.

Definition 2.1.24 (Cylinder and path objects)**.** Let $\mathcal{C}$ be a model category. Given an object X, a cylinder object $X \wedge I$ is a factorization of the fold map $i : X \coprod X \to X$, such that p_0 is a cofibration and that p_1 is a weak equivalence.

$$\begin{tikzcd} X\coprod X \arrow[rr, "i"] \arrow[dr, "p_0"'] & & X \\ & X \wedge I \arrow[ur, "p_1"'] & \end{tikzcd}$$

Dually, a path object X^I is a factorization of the diagonal map $i : X \to X \prod X$, such that p_0 is a weak equivalence and that p_1 is a fibration.

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "p_0"'] & & X\prod X \\ & X^I \arrow[ur, "p_1"'] & \end{tikzcd}$$

Remark 2.1.25. Even though we have written $X \wedge I$ suggestively to be a functor, it is not. There may be many choices for a cylinder object. However, by using the functorial factorization from MC4, we get a canonical choice of a cylinder object, as it factors every map into a cofibration and an acyclic fibration. By letting the cylinder object be this object, we do indeed obtain a functor.

Proposition 2.1.26. *Let $\mathcal{C}$ be a model category and X an object of $\mathcal{C}$. Given two cylinder objects $X \wedge I$ and $X \wedge I'$, then they are weakly equivalent.*

Proof. It is enough to show that there is a weak equivalence from any cylinder object into one specified cylinder object. This is in fact true for the functorial cylinder object $X \wedge I$, as the final morphism p_1 is an acyclic fibration, which enables a lift which is a weak equivalence by the 2-out-of-3 property.

$$\begin{tikzcd} X\coprod X \arrow[r, "p_0"] \arrow[d, "p_0'"] & X \wedge I \arrow[d, "p_1"] \\ X \wedge I' \arrow[r, "p_1'"] \arrow[ur, dashed] & X \end{tikzcd}$$

□

Definition 2.1.27 (Homotopy equivalence)**.** Let $f, g : X \to Y$. A left homotopy between f and g is a morphism $H : X \wedge I \to Y$ such that $Hi_0 = f$ and $Hi_1 = g$. f and g are left homotopic if a left homotopy exists, and it is denoted $f \overset{l}{\sim} g$.

$$
\begin{tikzcd}
X \arrow[d] \arrow[dr, dotted, "i_0"] \arrow[drr, bend left, "f"] & & \\
X \coprod X \arrow[r, "p_0"] & X \wedge I \arrow[r, "H"] & Y \\
X \arrow[u] \arrow[ur, dotted, "i_1"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}
$$

A right homotopy between f and g is a morphism $H : X \to Y^I$ such that $i_0 H = f$ and $i_1 H = g$. f and g are right homotopic if a right homotopy exists, and it is denoted $f \overset{r}{\sim} g$.

$$
\begin{tikzcd}
& & Y \\
X \arrow[r, "H"] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & Y^I \arrow[r, "p_1"] \arrow[ur, dotted, "i_0"] \arrow[dr, dotted, "i_1"] & Y \prod Y \arrow[u] \arrow[d] \\
& & Y
\end{tikzcd}
$$

f and g are said to be homotopic if they are both left and right homotopic, it is denoted $f \sim g$. f is said to be a homotopy equivalence, if it has a homotopy inverse $h : Y \to X$, such that $hf \sim id_X$ and $fh \sim id_Y$.

It is important to know that this is not a priori an equivalence relation. This is amended by taking both fibrant and cofibrant replacements. We see this in the following proposition.

Proposition 2.1.28. *Let $\mathcal{C}$ be a model category, and $f, g : X \to Y$ be morphisms. We have the following:*

1. *If $f \overset{l}{\sim}$ and $h : Y \to Z$, then $hf \overset{l}{\sim} hg$.*
2. *If Y is fibrant, $f \overset{l}{\sim} g$ and $h : W \to X$, then $fh \overset{l}{\sim} gh$.*
3. *If X is cofibrant, then left homotopy is an equivalence relation on $\mathcal{C}(X, Y)$.*
4. *If X is cofibrant and $f \overset{l}{\sim} g$, then $f \overset{r}{\sim} g$.*

Proof. $(1.)$ Assume that $f \overset{l}{\sim} g$ and $h : Y \to Z$. Let $H : X \wedge I \to Y$ denote the left homotopy between f and g. The left homotopy between hf and hg is given as hH.

$(2.)$ Assume that Y is fibrant, $f \overset{l}{\sim} g$ and that $h : W \to X$. Let $H : X \wedge I \to Y$ be a left homotopy. We construct a new cylinder object for the homotopy. Factor $p_1 : X \wedge I \to X$ as $q_1 \circ q_0$ where $q_0 : X \wedge I \to X \wedge I'$ is an acyclic cofibration and $q_1 : X \wedge I' \to X$ is a fibration. By the 2-out-of-3

property q_1 is an acyclic fibration, as p_1 and q_0 are weak equivalences. $X \wedge I'$ is a cylinder object as $q_0 \circ p_0$ is a cofibration and q_1 is a weak equivalence. Since we assume Y to be fibrant we lift the left homotopy $H : X \wedge I \to Y$ to the left homotopy $H' : X \wedge I' \to Y$ with the following diagram:

$$\begin{tikzcd} X \wedge I \arrow[r, "H"] \arrow[d, "q_0"] & Y \arrow[d] \\ X \wedge I' \arrow[r] \arrow[ur, dashed, "H'"] & * \end{tikzcd}$$

We can find the appropriate homotopy needed with lift given by the following diagram:

$$\begin{tikzcd} W \coprod W \arrow[r, "q_0 p_0 (h \coprod h)"] \arrow[d, "p_0'"] & X \wedge I' \arrow[d, "q_1"] \\ W \wedge I \arrow[r, "hp_1'"] \arrow[ur, dashed, "k"] & X \end{tikzcd}$$

Then the morphism $H'k$ is the desired left homotopy witnessing $fh \overset{l}{\sim} gh$.

$(3.)$ Assume that X is cofibrant. First observe that a left homotopy is reflexive and symmetric. We must show that in this case it is also transitative. Thus, assume that $f, g, h : X \to Y$ and that $H : X \wedge I \to Y$ is a left homotopy witnessing $f \overset{l}{\sim} g$ and that $H' : X \wedge I' \to Y$ is a left homotopy witnessing $g \overset{l}{\sim} h$. We first observe that $i_0 : X \to X \wedge I$ is a weak equivalence, as $id_X = p_1 i_0$ where id_X and p_1 are weak equivalences. Since X is assumed to be cofibrant, we see that $X \coprod X$ is cofibrant by the following pushout:

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X \arrow[d, "inr"] \\ X \arrow[r, "inl"] \arrow[ur, phantom, "\lrcorner", very near end] & X \coprod X \end{tikzcd}$$

Moreover, both inl and inr are cofibrations. This shows that i_0 is a cofibration as $i_0 = p_0 \circ inr$ is a composition of two cofibrations. i_0 is thus an acyclic cofibration. We define an almost cylinder object C by the pushout of i_1 and i_0'. We define the maps t and H'' by using the universal property in the following manner:

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[d, "i_0'"] & X \wedge I \arrow[d] \arrow[ddr, bend left, "p_1"] & \\
X \wedge I' \arrow[r] \arrow[drr, bend right, "p_1'"'] & C \arrow[dr, dashed, "t"] & \\
& & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[d, "i_0'"] & X \wedge I \arrow[d] \arrow[ddr, bend left, "H"] & \\
X \wedge I' \arrow[r] \arrow[drr, bend right, "H'"'] & C \arrow[dr, dashed, "H''"] & \\
& & X
\end{tikzcd}$$

Observe that there is a factorization of the fold map $X \coprod X \xrightarrow{s} C \xrightarrow{t} X$. However, s may not be a cofibration, so we replace C with the cylinder object $X \wedge I''$ such that we have the factorization $X \coprod X \xrightarrow{s_\alpha} X \wedge I'' \xrightarrow{ts_\beta} X$. The morphism Hs_β is then our required homotopy for $f \overset{l}{\sim} g$.

$(4.)$ Suppose that X is cofibrant and that $H : X \wedge I \to Y$ is a left homotopy for $f \overset{l}{\sim} g$. Pick a path object for Y, such that we have the factorization $Y \xrightarrow{q_0} Y^I \xrightarrow{q_1} Y \prod Y$ where q_0 is a weak equivalence and q_1 is a fibration. Again, as X is cofibrant we get that i_0 is an acyclic cofibration, so we have the following lift of the homotopy:

$$\begin{tikzcd}
X \arrow[r, "q_0 f"] \arrow[d, "i_0"] & Y^I \arrow[d, "q_1"] \\
X \wedge I \arrow[r, "(fp_1, H)"] \arrow[ur, dashed, "J"] & Y \prod Y
\end{tikzcd}$$

The right homotopy is given by injecting away from f, i.e. $H' = Ji_1$. □

Corollary 2.1.28.1. *We collect the dual results of the above proposition, and thus have the following.*

1. *If $f \overset{r}{\sim}$ and $h : W \to X$, then $fh \overset{r}{\sim} gh$.*
2. *If X is cofibrant, $f \overset{r}{\sim} g$ and $h : Y \to Z$, then $hf \overset{r}{\sim} hg$.*
3. *If Y is fibrant, then left homotopy is an equivalence relation on $\mathcal{C}(X, Y)$.*
4. *If Y is fibrant and $f \overset{r}{\sim} g$, then $f \overset{l}{\sim} g$.*

Corollary 2.1.28.2. *Homotopy is a congruence relation on $\mathcal{C}_{cf}$. In this manner, the category $\mathcal{C}_{cf}/\sim$ is well-defined, exists and inverts every homotopy equivalence.*

Lemma 2.1.29 (Weird Whitehead)**.** *Let $\mathcal{C}$ be a model category. Suppose that C is cofibrant and $h : X \to Y$ is an acyclic fibration or a weak equivalence between fibrant objects, then h induces an isomorphism:*

$$\mathcal{C}(C, X)/\overset{l}{\sim} \xrightarrow[\simeq]{h_*} \mathcal{C}(C, Y)/\overset{l}{\sim}$$

Dually, if X is fibrant and $h : C \to D$ is an acyclic cofibration or a weak equivalence between cofibrant objects, then h induces an isomorphism:

$$\mathcal{C}(D,X)/\overset{r}{\sim} \xrightarrow{\overset{h^*}{\simeq}} \mathcal{C}(C,X)/\overset{r}{\sim}$$

Proof. We assume $\mathcal{C}$ to be cofibrant and $h : X \to Y$ to ba an acyclic fibration. We first prove that h is surjective. Let $f : C \to Y$. By RLP of h there is a morphism $f' : C \to X$ such that $f = hf'$.

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & X \arrow[d, "h"] \\
C \arrow[r, "f"'] \arrow[ur, dashed, "f'"] & Y
\end{tikzcd}$$

To show injectivity we assume $f, g : C \to X$ such that $hf \overset{l}{\sim} hg$, in particular there is a left homotopy $H : C \wedge I \to Y$. Remember that since C is cofibrant, the map p_0 is a cofibration. We find a left homotopy $H : C \wedge I \to X$ witnessing $f \overset{l}{\sim} g$ by the following lift.

$$\begin{tikzcd}
C \coprod C \arrow[r, "f+g"] \arrow[d, "p_0"] & X \arrow[d, "h"] \\
C \wedge I \arrow[r, "H"'] \arrow[ur, dashed, "H'"] & Y
\end{tikzcd}$$

Moreover, if we assume both X and Y to be fibrant, the functor $\mathcal{C}(C,\_)/\overset{l}{\sim}$ sends acyclic fibrations to isomorphisms, i.e. to weak equivalences. By Ken Brown's lemma, lemma 2.1.14, the afformentioned functor sends weak equivalences between fibrant objects to isomorphisms. □

Theorem 2.1.30 (Generalized Whiteheads theorem)**.** *Let $\mathcal{C}$ be a model category. Suppose that $f : X \to Y$ is a morphism of bifibrant objects, then f is a weak equivalence if and only if f is a homotopy equivalence.*

Proof. Suppose first that f is a weak equivalence. Pick a bifibrant object A, then by lemma 2.1.29 $f_* : \mathcal{C}(A,X)/\sim \to \mathcal{C}(A,Y)/\sim$ is an isomorphism. Letting $A = Y$ we know that there is a morphism $g : Y \to X$, such that $f_*g = fg \sim id_Y$. Furthermore, by proposition 2.1.28, since X is bifibrant composing on the right preserves homotopy equivalence, e.g. $fgf \sim f$. By letting $A = X$, we get that $f_*gf = fgf \sim f = f_*id_X$, thus $gf \sim id_X$.

For the opposite direction, assume that f is a homotopy equivalence. We factor f into an acyclic cofibration f_γ and a fibration f_δ, i.e. $X \xrightarrow{f_\gamma} Z \xrightarrow{f_\delta} Y$. Observe that Z is bifibrant as X and Y is, in particular, f_γ is a weak equivalence of bifibrant objects, so it is a homotopy equivalence.

It is enough to show that f_δ is a weak equivalence. Let g be the homotopy inverse of f, and $H : Y \wedge I \to Y$ is a left homotopy witnessing $fg \sim id_Y$. Since Y is bifibrant, the following square has a lift.

$$\begin{tikzcd} Y \arrow[r, "f_\gamma g"] \arrow[d, "i_0"'] & Z \arrow[d, "f_\delta"] \\ Y \wedge I \arrow[r, "H"'] \arrow[ur, dashed, "H'"] & Y \end{tikzcd}$$

Let $h = H'i_1$, then by definition we know that $f_\delta H' i_1 = id_Y$. Moreover, H is a left homotopy witnessing $f_\gamma g \sim h$. Let $g' : Z \to X$ be the homotopy inverse of f_γ. We have the following relations $f_\delta \sim f_\delta f_\gamma g' \sim fg'$, and $hf_\delta \sim (f_\gamma g)(fg') \sim f_\gamma g' \sim id_Z$. Let $H'' : Z \wedge I \to Z$ be a left homotopy witnessing this homotopy. Since Z is bifibrant, i_0 and i_1 are weak equivalences. By the 2-out-of-3 property H'' and hf_δ are weak equivalences. Since $f_\delta h = id_Y$, it follows that f_δ is a retract of $f_\delta h$, and is thus a weak equivalence. □

Corollary 2.1.30.1. *The category ${}^{\mathcal{C}_{cf}}/_{\sim}$ satisfy the universal property of the localization of $\mathcal{C}_{cf}$ by the weak equivalences. I.e. there is a categorical equivalence Ho$\mathcal{C}_{cf} \simeq {}^{\mathcal{C}_{cf}}/_{\sim}$.*

Proof. By generalized Whiteheads theorem, theorem 2.1.30 weak equivalences and homotopy equivalences coincide. The corollary follows steadily from both the universal property of the localization category and the quotient category. □

We collect the results from above in the following theorem.

Theorem 2.1.31 (Fundamental theorem of model categories)**.** *Let $\mathcal{C}$ be a model category and denote $q : \mathcal{C} \to$ Ho$\mathcal{C}$ the localization functor. Let X and Y be objects of $\mathcal{C}$.*

1. *There is an equivalence of categories Ho$\mathcal{C} \simeq {}^{\mathcal{C}_{cf}}/_{\sim}$.*
2. *There are natural isomorphisms ${}^{\mathcal{C}_{cf}}/_{\sim}(QRX, QRY) \simeq$ Ho$\mathcal{C}(X, Y) \simeq {}^{\mathcal{C}_{cf}}/_{\sim}(RQX, RQY)$. Additionally, Ho$\mathcal{C}(X, Y) \simeq {}^{\mathcal{C}_{cf}}/_{\sim}(QX, RY)$.*
3. *The localization q identifies left or right homotopic morphisms.*
4. *A morphism $f : X \to Y$ is a weak equivalence if and only if qf is an isomorphism.*

Proof. This is clear by the results above. □

2.1.3 Quillen adjoints

We now want to study morphisms, or certain functors, between model categories. Like in the case of homotopical functors we want these morphisms to induce a functor between the homotopy categories. However, we also want them to respect the cofibration and fibration structure, not just weak equivalences. In this way we will instead look towards derived functors to be able to define this extension to the homotopy category. We recall the definition of a total (left/right) derived functor. In the case of model categories, we get a simple description for some of these derived functors which are of special interest.

Definition 2.1.32 (Total derived functors)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be homotopical categories, and $F : \mathcal{C} \to \mathcal{D}$ a functor. Whenever it exists, a total left derived functor of F, is a functor $\mathbb{L}F : \text{Ho}\mathcal{C} \to \text{Ho}\mathcal{D}$ with a natural transformation $\varepsilon : \mathbb{L}F \circ q \Rightarrow q \circ F$ satisfying the universal property: If $G : \text{Ho}\mathcal{C} \to \text{Ho}\mathcal{D}$ is a functor and there is a natural transformation $\alpha : G \circ q \Rightarrow q \circ F$, then it factors uniquely up to unique isomorphism through ε.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "q"] & \mathcal{D} \arrow[d, "q"] \\ \text{Ho}\mathcal{C} \arrow[r, dashed, "\mathbb{L}F"'] \arrow[ur, phantom, "\overset{\varepsilon}{\Nearrow}"] & \text{Ho}\mathcal{D} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "q"] & \mathcal{D} \arrow[d, "q"] \\ \text{Ho}\mathcal{C} \arrow[r, bend left, "\mathbb{L}F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\exists!\Uparrow"] \arrow[ur, phantom, "\overset{\varepsilon}{\Nearrow}"] & \text{Ho}\mathcal{D} \end{tikzcd}$$

Dually, whenever it exists, a total right derived functor of F, is a functor $\mathbb{R}F : \text{Ho}\mathcal{C} \to \text{Ho}\mathcal{D}$ with a natural transformation $\eta : q \circ F \Rightarrow \mathbb{R}F \circ q$ having the opposite universal property.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "q"] & \mathcal{D} \arrow[d, "q"] \\ \text{Ho}\mathcal{C} \arrow[r, dashed, "\mathbb{R}F"'] \arrow[ur, phantom, "\overset{\eta}{\Swarrow}"] & \text{Ho}\mathcal{D} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "q"] & \mathcal{D} \arrow[d, "q"] \\ \text{Ho}\mathcal{C} \arrow[r, bend left, "\mathbb{R}F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\exists!\Downarrow"] \arrow[ur, phantom, "\overset{\eta}{\Swarrow}"] & \text{Ho}\mathcal{D} \end{tikzcd}$$

Definition 2.1.33 (Deformation)**.** A left (right) deformation on a homotopical category $\mathcal{C}$ is an endofunctor Q together with a natural weak equivalence $q : Q \Rightarrow Id_{\mathcal{C}}$ ($q : Id_{\mathcal{C}} \Rightarrow Q$).

A left (right) deformation on a functor $F : \mathcal{C} \to \mathcal{D}$ between homotopical categories, is a left (right) deformation Q on $\mathcal{C}$ such that weak equivalences in the image of Q is preserved by F.

Remark 2.1.34 (Cofibrant and fibrant replacement)*.* If $\mathcal{C}$ is a model category, then we have a left and a right deformation. The cofibrant replacement Q defines a left deformation, and the fibrant replacement defines a right deformation. Notice that this is only due to the fact that the factorization system is functorial.

Proposition 2.1.35. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between homotopical categories. If F has a left deformation Q, then the total left derived functor $\mathbb{L}F$ exists. Moreover, the functor FQ is homotopical, and $\mathbb{L}F$ is the unique extension of FQ.*

Proof. Since we already have a candidate for the derived functor, the proof must just check that it has the universal property. A proof may be found in Reihl [16] under proposition 6.4.11. □

Remark 2.1.36. There is a somewhat weaker statement by Dwayer and Spalinski [15]. If we instead ask for functors F which have the cofibrant replacement Q (fibrant replacement R) as a left (right) deformation we may make this proof more explicit. This is theorem 9.3.

Equipped with the above proposition and remark, it makes sense to define Quillen functors as left and right Quillen functors. A left Quillen functor should be left deformable by the cofibrant replacement. Moreover, for the composition of two left Quillen functors to make sense, we also need weak equivalences between cofibrant objects to be mapped to weak equivalences between cofibrant objects. We make the following definition.

Definition 2.1.37 (Quillen adjunction)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be model categories.

1. A left Quillen functor is a functor $F : \mathcal{C} \to \mathcal{D}$ such that it preserves cofibrations and acyclic cofibrations.
2. A right Quillen functor is a fucntor $F : \mathcal{C} \to \mathcal{D}$ such that it preserves fibrations and acyclic fibrations.
3. Suppose that (F, U) is an adjunction where $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to U. (F, U) is called a Quillen adjunction if F is a left Quillen functor and U is a right Quillen functor.

Remark 2.1.38. By Ken Browns lemma, lemma 2.1.14, we see that a left Quillen functor F is left deformable to the cofibrant replacement functor Q. Thus the total left derived functor exists and is given by $\mathbb{L}F = \text{Ho}FQ$.

In order to eliminate the choice of left or right derivedness, we will think of a morphism of model categories as a Quillen adjunction. The direction of the arrow can be chosen to be along either the left or right adjoints, we make the convention of following the left adjoint functors. We summarize the following properties.

Lemma 2.1.39. *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories, and suppose there is an adjunction $F : \mathcal{C} \rightleftharpoons \mathcal{D} : U$. The following are equivalent:*

1. *(F, U) is a Quillen adjunction.*
2. *F is a left Quillen functor.*
3. *U is a right Quillen functor.*

Proof. This follows from naturality of the adjunction. I.e. any square in $\mathcal{C}$, with the right side from $\mathcal{D}$ is commutative if and only if any square in $\mathcal{D}$ with the left side from $\mathcal{C}$ is commutative. Now, f has LLP with respect to Ug if and only if Ff has LLP with respect to g.

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "f"'] & UX \arrow[d, "Ug"] & FA \arrow[r, "k^T"] \arrow[d, "Ff"'] & X \arrow[d, "g"] \\ B \arrow[r, "l"] \arrow[ur, dotted, "h"] & UY \arrow[r, phantom, "\rightsquigarrow"] & FB \arrow[r, "l^T"] \arrow[ur, dotted, "h^T"] & Y \end{tikzcd}$$

□

Proposition 2.1.40. *Suppose that $(F, U) : \mathcal{C} \to \mathcal{D}$ is a Quillen adjunction. The functors $\mathbb{L}F$: Ho$\mathcal{C} \to$ Ho$\mathcal{D}$ and $\mathbb{R}U$: Ho$\mathcal{D} \to$ Ho$\mathcal{C}$ forms an adjoint pair.*

Proof. We must show that Ho$\mathcal{D}(\mathbb{L}FX, Y) \simeq$ Ho$\mathcal{D}(X, \mathbb{R}UY)$. By using the fundamental theorem of model categories, theorem 2.1.31, we have the following isomorphisms: Ho$\mathcal{D}(\mathbb{L}FX, Y) \simeq$ $\mathcal{C}(FQX, RY)/\sim$ and Ho$\mathcal{D}(X, \mathbb{R}UY) \simeq \mathcal{D}(QX, URY)/\sim$. In other words, if we assume X to be cofibrant, and Y to be fibrant, we must show that the adjunction preserves homotopy equivalences.

We show it for one direction. Suppose that the morphisms $f, g : FA \to B$ are homotopic, witnessed by a right homotopy $H : FA \to B^I$. Since we assume U to preserve products, fibrations and weak equivalences between fibrant objects, $U(B^I)$ is a path object for UB. Thus the transpose $H^T : A \to U(B^I)$ is the desired homotopy witnessing $f^T \sim g^T$ $\square$

Definition 2.1.41 (Quillen equivalence)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be model categories, and $(F, U) : \mathcal{C} \to \mathcal{D}$ be a Quillen adjunction. (F, U) is called a Quillen equivalence if for any cofibrant X in $\mathcal{C}$, fibrant Y in $\mathcal{D}$ and any morphism $f : FX \to Y$ is a weak equivalence if and only if its transpose $f^T : X \to UY$ is a weak equivalence.

Proposition 2.1.42. *Suppose that $(F, U) : \mathcal{C} \to \mathcal{D}$ is a Quillen adjunction. The following are equivalent:*

1. *(F, U) is a Quillen equivalence.*
2. *Let $\eta : Id_{\mathcal{C}} \Rightarrow UF$ denote the unit, and $\varepsilon : FU \Rightarrow Id_{\mathcal{D}}$ denote the counit. The composite $Ur_F\eta : Id_{\mathcal{C}_c} \Rightarrow URF|_{\mathcal{C}_c}$, and $\varepsilon_{FQU}Fq_U : FQU|_{\mathcal{D}_f} \Rightarrow Id_{\mathcal{D}_f}$ are natural weak equivalences.*
3. *The derived adjunction $(\mathbb{L}F, \mathbb{R}U)$ is an equivalence of categories.*

Proof. Firstly observe that $2. \implies 3.$ by definition. Secondly observe that equivalences both preserves and reflects isomorphisms, from this we get $3. \implies 1.$. We now show $1. \implies 2.$. Pick X in $\mathcal{C}$ such that X is cofibrant. Since (F, U) is assumed to be a Quillen adjunction we know that FX is still cofibrant. The fibrant replacement $r_{FX} : FX \to RFX$ gives us a weak equivalence. Furthermore, since (F, U) is assumed to be a Quillen equivalence, its transpose $r_{FX}^T : X \to URFX$ is a weak equivalence. Unwinding the definition of the transpose we get that $r_{FX}^T = Ur_{rFX}\eta_X$.

$\square$

We have the following refinement.

Corollary 2.1.42.1. *Suppose that $(F, U) : \mathcal{C} \to \mathcal{D}$ is a Quillen adjunction. The following are equivalent:*

1. *(F, U) is a Quillen equivalence.*
2. *F reflects weak equivalences between cofibrant objects, and $\varepsilon_{FQU}F_{qU} : FQU|_{\mathcal{D}_f} \Rightarrow Id_{\mathcal{D}_f}$ is a natural weak equivalence.*
3. *U reflects weak equivalences between fibrant objects, and $U_{rF}\eta : Id_{\mathcal{C}_c} \Rightarrow URF|\mathcal{C}_c$ is a natural weak equivalence.*

Proof. We start by showing $1. \implies 2.$ and $3.$. We already know that the derived unit and counit are isomorphism in homotopy, so we only need to show that F (U) reflects weak equivalences between cofibrant (fibrant) objects. Suppose that $Ff : FX \to FY$ is a weak equivalence between cofibrant objects. Since F preserves weak equivalences between cofibrant objects, we get that FQf is a weak equivalence, or that $\mathbb{L}Ff$ is an isomorphism. By assumption, $\mathbb{L}F$ is an equivalence of categories, so f is a weak equivalence as needed.

We will show $2. \implies 1.$, the case $3. \implies 1.$ is dual. We assume that the counit map is an isomorphism in homotopy. By assumption, the derived unit $\mathbb{L}\eta$ is split-mono on the image of $\mathbb{L}F$. Moreover, the derived counit $\mathbb{R}\varepsilon$ is assumed to be an isomorphism, in particular the derived unit $\mathbb{L}F\mathbb{L}\eta$ is an isomorphism. Unpacking this, we have a morphism, call it $\eta'_X : FQX \to FQURFQX$, which is a weak equivalence. Since F and Q reflects weak equivalences, we get that $\eta_X : X \to URFQX$ is a weak equivalence. □

2.2 Model structures on Algebraic Categories

In order to understand ∞-quasi-isomorphism of strongly homotopy associative algebras we will study different homotopy theories of various categories. Munkholm [18] successfully showed that the derived category of augmented algebras is equivalent to the derived category of augmented algebras equipped with ∞-morphisms. Well, to be more precise, he showed that certain sub-categories of augmented algebras had this property. Lefevre-Hasagawas phd thesis [12] builds upon this identification, but with help of further devolpment within the field. We will follow the approach of Lefevre-Hasagawa, by comparing the model structure for algebras and coalgebras,

2.2.1 DG-Algebras as a Model Category

Bousefield and Gugenheim [19] proved that the category of commutative dg-algebras had a model structure whenever the base field was a field of characteristic 0. In a joint project, Jardine's paper from 1997 [20] shows that this construction may be extended to dg-algebras over any commutative ring. On the other hand, Munkholm expanded on the ideas from Bousfield and Gugenheim to get an identification of derived categories. Also, Hinichs paper from 1997 [21] details another method to obtain the model category which we want. We will follow the approach of Hinich, as it will be usefull later on. Notice that where Hinich use theory of algebraic operads to show that the category of algebras is a model category, we will however give a more explicit formulation.

Let $\mathbb{K}$ be a field, and $\mathcal{C}$ be a category such that there is an adjunction $F : \mathcal{C} \rightleftharpoons Ch(\mathbb{K}) : \#$, where F is left adjoint to #. Furthermore, suppose that $\mathcal{C}$ satsifies the 2 conditions:

(H0) $\mathcal{C}$ admits finite limits and every small colimit. The functor # commutes with filtered colimits.
(H1) Let M be the complex below, concentrated in 0 and 1.

$$\dots \longrightarrow 0 \longrightarrow \mathbb{K} \xrightarrow{id} \mathbb{K} \longrightarrow 0 \longrightarrow \dots$$

For any $d \in \mathbb{Z}$ and for any $A \in \mathcal{C}$ the injection $A \to A \coprod F(M[d])$ induces a quasi-isomorphism $A^\# \to (A \coprod F(M[d]))^\#$.

With this adjunction in mind, we define weak equivalences, fibrations and cofibrations as follows: Let $f \in \mathcal{C}$ be a morphism

- $f \in$ Ac if $f^\#$ is a quasi-isomorphism.
- $f \in$ Fib if $f^\#$ is surjective on each component.
- $f \in$ Cof if f has LLP to acyclic fibrations.

Theorem 2.2.1. *The category $\mathcal{C}$ equipped with the weak equivalences, fibrations and cofibrations as defined above is a model category.*

Before we show this theorem we need to understand the cofibrations better. Let $A \in \mathcal{C}$, $M \in Ch(\mathbb{K})$ and $\alpha : M \to A^\#$ a morphism in $Ch(\mathbb{K})$. We define a functor

$$h_{A,\alpha}(B) = \{(f,t) \mid f \in \mathcal{C}(A,B), t \in \mathrm{Hom}_{\mathbb{K}}^{-1}(M, B^\#)\ s.t.\ \partial t = f^\# \circ \alpha\}.$$

Note that t is not a morphism of chain maps. This is a homogenous morphism of degree -1. The differential then promotes this morphism to a chain map, and t is thus a homotopy for the comoposite $f^\# \circ \alpha$.

This functor is represented by an object of $\mathcal{C}$. We define this representing object $A\langle M, \alpha\rangle$ as the pushout:

$$\begin{tikzcd}
F(A^\#) \arrow[r, "\varepsilon_A"] \arrow[d] & A \arrow[d, "a"] \\
F(\mathrm{cone}(\alpha)) \arrow[r, "e"] \arrow[ur, phantom, "\lrcorner"] & A\langle M, \alpha\rangle
\end{tikzcd}$$

Let $i : M[1] \to \mathrm{cone}(\alpha)$ be a homogenous morphism which is the injection when considered as graded modules. Notice that we have a pair of morphisms $(a, e^T i) \in h_{A,\alpha}(A\langle M, \alpha\rangle)$.

Proposition 2.2.2. *The functor $h_{A,\alpha}$ is represented by $A\langle M, \alpha\rangle$, i.e. $h_{A,\alpha} \simeq \mathcal{C}(A\langle M, \alpha\rangle, \_)$ is a natural isomorphism. Moreover, the pair $(a, e^T i)$ is the universal element of the functor $h_{A,\alpha}$, i.e. the natural isomorphism is induced by this element under Yoneda's lemma.*

Proof. Let $(f,t) \in h_{A,\alpha}(B)$ for some $B \in \mathcal{C}$. The condition that $\partial t = f^\# \alpha$ is equivalent to say that $f^\#$ extends to a morphism $f' : \mathrm{cone}(\alpha) \to B^\#$ along t, i.e. $f' = \begin{pmatrix} f^\# & t \end{pmatrix}$. This concludes the isomorphism part, as being an element (f,t) is equivalent to the existence of the diagram below, where $\widetilde{f}$ is uniquely determined.

$$\begin{tikzcd}
F(A^{\#}) \arrow[r, "\varepsilon_A"] \arrow[d] & A \arrow[d, "a"] \arrow[ddr, bend left, "f"] & \\
F(\mathsf{cone}(\alpha)) \arrow[r, "e"] \arrow[drr, bend right, "f'"] & A\langle M, \alpha \rangle \arrow[dr, dashed, "\tilde{f}"] & \\
& & B
\end{tikzcd}$$

To obtain naturality, we use the adjunction to observe that the element $(a, e^T i)$ is in fact universal. $\square$

We are now in a position to explicitly find some important cofibrations. We collect these morphisms into the ”standard” cofibrations.

Definition 2.2.3. Let $f : A \to B$ be a morphism in $\mathcal{C}$. Suppose that f factors as a transfinite composition of morphisms on the form $A_i \to A_i\langle M, \alpha \rangle$, i.e. f factors into the diagram below, where $A_{i+1} = A_i\langle M, \alpha \rangle$.

$$A \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \dots \longrightarrow B$$

- If every such M is a complex consisting of free $\mathbb{K}$-modules and has a 0-differential, we call f a standard cofibration.
- If every such M is a contractible complex and $\alpha = 0$, we call f a standard acyclic cofibration.

Proposition 2.2.4. *Every standard cofibration is a cofibration, and every standard acyclic cofibration is an acyclic cofibration. We will see that these morphisms in some sense generate every (acyclic) cofibrations.*

Proof. First observe that every standard cofibration may be made iteratively from the chain complexes $\mathbb{K}[n]$, and likewise, every standard acyclic cofibration may be made iteratively from M as in $H1$.

We first prove that if $M \simeq \mathbb{K}[n]$, and $\alpha : M \to A^{\#}$ is any map, then the map $A \to A\langle M, \alpha \rangle$ is a cofibration. This amounts to show that it has LLP to every acyclic fibration. Suppose that $h : B \to C$ is an acyclic fibration and that there is a commutative square as below.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[d, "h"] \\
A\langle M, \alpha \rangle \arrow[r, "g"] & C
\end{tikzcd}$$

By the universal property of $h_{A,\alpha}$ 2.2.2 it suffices to find a pair (f', t') such that $f : A \to B$, $t' : M \to B^{\#}$ is homogenous of degree -1, $\partial t = f^{\#}\alpha$ and that h induces a morphism $h : (f', t') \to g$. We see that we are forced to choose $f' = f$ as $hf = ga$. By the existence of g, there exists a $t : M \to C^{\#}$ such that $\partial t = g^{\#}a^{\#}\alpha = h^{\#}f^{\#}\alpha$. Since h is an acyclic fibration $h^{\#}$ is a surjective quasi-isomorphism. Since $M \simeq \mathbb{K}[n]$, the morphism t is really an element of $(C^{\#})^{n-1}$. By surjectivity of $h^{\#}$ there is an element u of $(B^{\#})^{n-1}$ such that $h^{\#}(u) = t$. Moreover, the difference $h^{\#}(\partial u - f^{\#}\alpha) = 0$, so $\partial u - f^{\#}\alpha$ factors through the kernel Ker$h^{\#}$, which is assumed to be acyclic. This element is furthermore a cycle, so by acyclicity there is another element u' such that $\partial u' = \partial u - f^{\#}\alpha$. We may now see that $(f, u - u')$ is our desired factorization.

Secondly we prove that if M is in as $H1$ and $\alpha = 0$, then the map $A \to A\langle M, \alpha\rangle$ is an acyclic cofibration. This amounts to show that it has LLP to every fibration.

Suppose that $h : B \to C$ is a fibration and that there is a commutative square as below.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle h} \\ A\langle M, \alpha\rangle & \xrightarrow{g} & C \end{array}$$

We will again use 2.2.2, so it suffices to find a t' such that $\partial t' = f^{\#}\alpha = 0$. By g, let $t : M \to C^{\#}$ such that $\partial t = g^{\#}a^{\#}\alpha = h^{\#}f^{\#}\alpha = 0$. Since $h^{\#}$ is surjective t admits a lift $t' : M \to B^{\#}$ such that $t' = h^{\#}t$. The result follows as $\partial t' = \partial h^{\#}t = h^{\#}\partial t = 0$.

□

In light of the above proposition we would like to make some more convenient notation. If $M \simeq \mathbb{K}[n]$ and $\alpha : M \to Z^n(A^{\#})$, s.t. $\alpha(1) = a$, we write $A\langle M, \alpha\rangle$ as $A\langle T; dT = a\rangle$ instead. Hinich calls this "adding a variable to kill a cycle". If M is the acyclic complex as below and $\alpha = 0$, we write $A\langle T, S; dT = S\rangle$. This could be thought of "adding a variable and a cycle to kill itself".

$$\dots \longrightarrow 0 \longrightarrow \mathbb{K} \xrightarrow{id} \mathbb{K} \longrightarrow 0 \longrightarrow \dots$$

proof of theorem. **MC1** and **MC2** are satisfied. By definition we also have the first part of **MC3**. We start by checking **MC4**.

Let $f : A \to B$ be a morphism in $\mathcal{C}$. Given any $b \in B^{\#}$, let $C_b = A\langle T_b, S_b; dT_b = S_b\rangle$. We define $g_b : C_b \to B$ by the conditions that it acts on A as f, $g_b^{\#}(T_b) = b$ and $g_b^{\#}(S_b) = db$. Iterating this construction for every $b \in B$, we obtain an object C, such that the injection $A \to C$ is an acyclic standard cofibration, and the map $g : C \to B$ is a fibration. This gives us a factorization $f = f_\delta \circ f_\gamma$, where f_γ is the injection and $f_\delta = g$.

To obtain the other factorization we want to make a standard cofibration. We already know that the map $A \to C$ is a standard cofibration, so let $C_0 = C$. From here on, we will make each C_i inductively, such that $\varinjlim C_i$ has the factorization property which we desire. Notice that from C_0 there is a morphism $g_0 : C_0 \to B$, which is surjective and surjective on every kernel. This morphism may fail to be a quasi-isomorphism, so it is not an acyclic fibration.

To construct C_1 we assign to every pair of elements (c, b), such that $c \in ZC_0^\#$ and $g_0^\#(c) = db$, a variable to kill a cycle. If (c, b) is such a pair, then we add a variable T such that $dT = c$ and $g_1^\#(T) = b$. C_1 is then the complex where each cycle c has been killed by adding a variable T. Now if we suppose that we have constructed C_i, then C_{i+1} is constructed in the same manner, by adding a variable to kill each cycle which is a boundary in the image.

When adding a variable, we have also updated the morphism g_i by letting $g_{i+1}^\#(T) = b$. Thus in each step we have also made a new morphism g_{i+1}. If g denote the morphism at the colimit, then it is clear that this is still a fibration and it has also become a quasi-isomorphism. This is because every cycle which failed to be in the homology of B have been killed.

It remains to check the last part of **CM3**. Suppose that $f : A \to B$ is an acyclic cofibration. By **CM4**, we know that it factors as $f = f_\delta \circ f_\gamma$, where f_δ is an acyclic fibration and f_γ is a standard acyclic fibration. We thus obtain that f is a retract of f_γ by the commutative diagram below.

$$\begin{tikzcd}
A \arrow[r, "f_\gamma"] \arrow[d, "f"] & C \arrow[d, "f_\delta"] \\
B \arrow[r, equal] \arrow[ur, dashed] & B
\end{tikzcd}$$

□

The following corollary will concretize what it means that the standard cofibrations generate every cofibration. This corollary is really a step used within in the proof.

Corollary 2.2.4.1. *Any (acyclic) cofibration is a retract of a standard (acyclic) cofibration.*

We may immedietly apply this theorem to some familiar examples.

Corollary 2.2.4.2. *Let A be a dg-algebra over the field $\mathbb{K}$. The category Mod$_A$ of left modules is a model category.*

sketch of proof. We establish the adjunction by letting $F = A \otimes_{\mathbb{K}} \_$. H0 is satisfied as this category is bicomplete, and filtered colimits may be thought of as unions of sets. Moreover, since mod_A is an Abelian category, the forgetful functor # commutes with coproducts, or direct sums, which makes H1 trivially satisfied. □

Corollary 2.2.4.3. *The categories Alg$_{\mathbb{K}}^\bullet$ (Alg$_{\mathbb{K},+}^\bullet$) are model categories.*

Proof. We establish the adjunction by letting $F = T(M)$, the tensor algebra of a cochain complex. For the same reasons as above, H0 is trivially satisfied.

Given two algebras A and B, their coproduct should be analogous to the free group. Following [22], we define

$$A * B = T(A \otimes_{\mathbb{K}} B)/I$$

where I is an ideal representing the relations

$$(a \otimes b) \otimes (1 \otimes b') = a \otimes (bb')$$
$$(a \otimes 1) \otimes (a' \otimes b) = (aa') \otimes b$$
$$1 \otimes 1 = 1.$$

This looks like the formal words definition of groups, except that we allow each element coming from the field $\mathbb{K}$ commute. We see that this construction does indeed define the coproduct by having maps $f : A \to T$ and $g : B \to T$ to then define a map $f * g$ which acts as $f \otimes g$ everywhere.

Given a cochain complex $N^\bullet$, we may consider the free dg-algebra $T(N^\bullet)$. In this case the coproduct $A * T(N^\bullet)$ has an easier description. We define a complex

$$A[N^\bullet] = A \oplus (A \otimes N^\bullet \otimes A) \oplus (A \otimes N^\bullet \otimes A \otimes N^\bullet \otimes A) \oplus \cdots.$$

The differential on $A[N^\bullet]$ is the differential induced by the tensor product. We define a multiplication on $A[N^\bullet]$ by the following formula

$$(a_1 \otimes \cdots \otimes a_i) \cdot (a'_1 \otimes \cdots \otimes a'_j) = a_1 \otimes \cdots \otimes a_i a'_1 \otimes \cdots a'_j.$$

Let $i : A \to A[N^\bullet]$ denote the inclusion, and $\iota : T(N^\bullet) \to A[N^\bullet]$ is defined by interspersing the $N^\bullet$ tensors with 1s. I.e. $\iota(n_1 \otimes \cdots \otimes n_j) = 1 \otimes n_1 \otimes 1 \otimes \cdots \otimes 1 \otimes n_j \otimes 1$.

To define a map $f : A[N^\bullet] \to T$ it is enough by the ring homomorphism property to define a map $g : A \to T$ and a map $h : T(N^\bullet) \to T$. We observe that this choice of g and h is unique for any f, establishing the universal property. I.e. $A[N^\bullet] \simeq A * T(N^\bullet)$.

To see that the map $i^\# : A^\# \to A[M^\bullet]^\#$ is a quasi-isomorphism, it is enough to see that contractible complexes are stable under tensoring. Given a contractible complex $C^\bullet$, then there is a homotopy $h : C^\bullet \to C^\bullet$ such that $\partial h = id_C$. Observe that $id_N \otimes h : N^\bullet \otimes C^\bullet \to N^\bullet \otimes C^\bullet$ is a homotopy witnessing $id_{N^\bullet \otimes C^\bullet} \sim 0$. Since M is contractible, we know that the homology of the inclusion is $H^* i = id_{H^* A}$. This shows H1. ☐

We summarize the last result:

The category of augmented dg-algebras $\mathsf{Alg}^\bullet_{\mathbb{K},+}$ is a model category. Let $f : X \to Y$ be a homomorphism of augmented algebras.

- $f \in$ Ac if $f^{\#}$ is a quasi-isomorphism.
- $f \in$ Fib if $f^{\#}$ is an epimorphism (surjective onto every component).
- $f \in$ Cof if f has LLP with respect to to every acyclic fibration.

The category of augmented dg-algebras has an initial and a terminal object. The initial object is the stalk $\mathbb{K}$ and the terminal object is the 0-ring. We see that every object is fibrant, as 0 is preserved by the forgetful functor and every map into 0 is surjective. Every dg-algebra which is isomorphic to a tensor algebra when considered as a graded algebra is cofibrant.

wrong

2.2.2 A Model Structure on DG-Coalgebras

We now want to equip the category of dg-coalgebras with a suitable model structure. This model structure should be suitable in the sense that it give rise to the same homotopy theory as dg-algebras. The bar-cobar construction will be crucial in this construction, as it is in fact a Quillen adjunction. To this end we will follow the setup as presented by Lefevre-Hasegawa [12]. His method is a modification of Hinichs paper [23]. Let $f : C \to D$, the category of dg-coalgerbas will be equipped with the three following classes of morphisms:

- $f \in$ Ac if Ωf is a quasi-isomorphism.
- $f \in$ Fib if f has RLP with respect to every acyclic cofibration.
- $f \in$ Cof if $f^{\#}$ is a monomorphism (injective in every component).

To see that these classes of morphisms does define a model structure it is crucial to figure out what these weak equivalences are. This can only be done by calculating homologies, since f is a weak equivalence if and only if $H^*\text{cone}(\Omega f) \simeq 0$. Using spectral sequences to calculate these homologies is not crucial, but it gives us a method to formalize the basic idea of our most important tool.

Definition 2.2.5. A filtered chain map $f : M \to N$ of filtered complexes M and N is a filtered quasi-isomorphism if $\text{gr}f : \text{gr}M \to \text{gr}N$ is a quasi-isomorphism of the associated graded complexes.

Lemma 2.2.6. *Let $f : C \to C'$ be a graded quasi-isomorphism between conilpotent dg-coalgebras, then $\Omega f : \Omega C \to \Omega C'$ is a quasi-isomorphism.*

Proof. We do this by considering a spectral sequence. Endow C with a grading (as a vector space) induced by the coradical filtration, i.e. $c \in C$ has degree $|c| = n$ if n is the smallest number such that $\overline{\Delta}^n c = 0$. We define a filtration on ΩC by

$$F_p\Omega C = \{\langle c_1|\cdots|c_n\rangle \mid |c_1| + ... + |c_n| \leqslant p\}$$

Since C is a dg-coalgebra, the coradical filtration respects the differential. In other words, $F_p\Omega C$ is still a chain complex, which is a subcomplex of ΩC. This filtration is clearly bounded below and exhaustive. Thus by the classical convergence theorem of spectral sequences, theorem 5.5.1 [24], the spectral sequence converges to the homology $E\Omega C \Rightarrow H^*\Omega C$.

By definition, the 0'th page is defined as

$$E^0_{p,q}\Omega C = {}^{(F_p\Omega C)_{p+q}}\!/\!_{(F_{p-1}\Omega C)_{p+q}}.$$

Furthermore, notice that at this page we have the following isomorphism $E^0_{p,q}\Omega C \simeq (\Omega\text{gr}C)^{(p)}_{p+q}$, where $(\Omega\text{gr}C)^{(p)} = \{\langle c_1|\cdots|c_n\rangle \mid |c_1| + ... + |c_n| = p\}$.

Evaluating f at the 0'th page would look like $E^0\Omega f \simeq \Omega\text{gr}f$. By the mapping lemma, exercise 5.2.3 [24], it is enough to check that $\Omega\text{gr}f$ is a quasi-isomorphism to see that Ωf is a quasi-isomorphism. We show that $\Omega\text{gr}f$ is a quasi-isomorphism by inspecting every $E^0_{p,\bullet}\Omega C$.

We define a filtration G_k on $E^0_{p,\bullet}\Omega C$ as

$$G_k = \{\langle c_1|\cdots|c_n\rangle \mid n \geqslant -k\}.$$

We see that $G_0 = E^0_{p,\bullet}\Omega C$ by definition and $G_{-p-1} \simeq 0$ on the coaugmentation quotient $\overline{C}$. By the classical convergence theorem of spectral sequences, this defines a spectral sequence such that $EG \Rightarrow H^*E^0_{p,\bullet}\Omega C$.

To see that $\Omega\text{gr}f$ is a quasi-isomorphism, it is now enough to see that E^0Gf is a quasi-isomorphism for any p. Notice that $E^0_{l,\bullet}G \subseteq (\text{gr}C[-1])^{\otimes l}$ where the total grading is p. Since f is a filtered quasi-isomorphisms and by the Kunneth-formula, theorem 3.6.3 [24], it follows that E^0Gf is a quasi-isomorphism. □

This proof will serve as a template for how we want to approach many of the proofs we will encounter. The lemma says that to show that f is a weak equivalence it is sufficient to show that f is a filtered quasi-isomorphism. However, to show that f is a filtered quasi-isomorphism we first need a good filtering, and once we have a filtering we look at its spectral sequence. If we still have a problem, we look at complexes within a page on a spectral sequence, the mapping lemma says that it is enough to verify that a morphism becomes a quasi-isomorphism on any page to see that it is a quasi-isomorphism. We will call this technique for an iterated spectral sequence argument.

For completeness we include the following statement.

Lemma 2.2.7. *Let $f : A \to A'$ be a quasi-isomorphism between dg-algebras, then $Bf : BA \to BA'$ is both a filtered and unfiltered quasi-isomorphism.*

Proof. Notice that the homology of BA may calculated from the double complex used to define BA. In fact, at the 0'th page we have $E^0_{p,\bullet}f \simeq f^{\otimes p}$. It follows that f is a quasi-isomorphism on the 0'th page from the Kunneth formula, theorem 3.6.3 [24]. □

Let A (C) be a filtered dg-algebra (coalgebra). Given an element $a \in A$ ($c \in C$) we say that its filtered degree f-deg(a) (f-deg(c)) is the number such that $a \in F_{\text{f-deg}(a)}A$ ($c \in F_{\text{f-deg}(c)}C$) but not $a \in F_{\text{f-deg}(a)+1}A$ ($c \in F_{\text{f-deg}(c)+1}C$). There is then an associated filtration on the bar (cobar) construction of this complex, defined as

$$F_pBA = \{[a_1 \mid \cdots \mid a_n] \mid \sum \text{f-deg}(a_i) \leqslant p\}$$
$$(F_p\Omega C = \{\langle c_1 \mid \cdots \mid c_n\rangle \mid \sum \text{f-deg}(c_i) \leqslant p\}).$$

Proposition 2.2.8. *Let A be an augmented dg-algebra and C a conilpotent dg-coalgebra. The counit $\varepsilon_A : \Omega BA \to A$ a quasi-isomorphism. The unit $\eta_C : C \to B\Omega C$ is a filtered quasi-isomorphism, moreover $B\eta_C$ is a quasi-isomorphism.*

The following proof is due to [12], but with corrections given by [25]. Some minor modifications are given to the proof as it resembles a proof given earlier.

Proof. We will show that the counit is quasi-invertible. The proof technique for quasi-invertability of the unit is analogous.

Define the following filtration for A.

$$F_0A = \mathbb{K}$$
$$F_1A = A$$
$$F_pA = F_1A$$

We clearly see that this filtration endows A with the structure of a filtered dg-algebra. For ΩBA we will use the induced filtration from the coradical filtration of BA.

The counit acts on ΩBA as tensorwise projection, followed by multiplication in A. This morphism respects the filtration, so it is a filtered morphism. Notice that both of these filtrations are bounded below, and exhaustive, so the classical convergence theorem of spectral sequences apply.

Let $E_r\Omega BA$ and E_rA be the spectral sequences given by these filtrations. We have that $E_1^p\Omega BA \simeq \text{gr}_p\Omega BA$ and $E_1^pA \simeq \text{gr}_pA$. For $p = 1$ both complexes are isomorphic to the same complex, $\overline{A}$. Moreover, $E_1^1\varepsilon_A = id_{\overline{A}}$. Whenever $p \neq 1$ we get that $E_1^pA \simeq 0$, so it remains to show that $E_1^p\Omega BA \simeq \text{gr}_p\Omega BA$ is contractible for any $p \geqslant 2$.

The differential of ΩBA is generated by three actions: the differential on A, the multiplication on A and the comultiplication on BA. With the induced filtration on ΩBA, we see that the multiplication on A is the only action which maps $F_p\Omega BA \to F_{p-1}\Omega BA$. Thus this action is 0 in the associated graded and the spectral sequence.

There is a homotopy of the identity given as $r : \text{gr}_i\Omega BA \to \text{gr}_i\Omega BA$, which is 0 unless if there is an element on the form $\langle [a] \mid [\cdots] \mid [\cdots]\rangle$. In this case h is

$$r\langle [a] \mid [\cdots] \mid \cdots\rangle = (-1)^{|a|+1}\langle [a \mid [\cdots] \mid \cdots\rangle$$

We will show that this is a homotopy by induction on i.

Let $i = 2$. Then there are two cases we must handle, either an element is on the form $\langle [a_1] \mid [a_2] \rangle$ or $\langle [a_1 \mid a_2] \rangle$. We consider the latter case first. If we apply r to this element we are returned 0.

$$(r \circ d_{\Omega BA} + d_{\Omega BA} \circ r)\langle [a_1 \mid a_2] \rangle = r(-1)^{|a_1|+1}\langle [a_1] \mid [a_2] \rangle = \langle [a_1 \mid a_2] \rangle$$

Then we treat the former case

$$\begin{aligned}
&(r \circ d_{\Omega BA} + d_{\Omega BA} \circ r)\langle [a_1] \mid [a_2] \rangle \\
&\qquad = r\langle [d_A a_1] \mid [a_2] \rangle + (-1)^{|a_1|} r\langle [a_1] \mid [d_A a_2] \rangle + d_{\Omega BA}(-1)^{|a_1|+1}\langle [a_1 \mid a_2] \rangle \\
&\qquad\qquad = (-1)^{|a_1|}\langle [d_A a_1 \mid a_2] \rangle - \langle [a_1 \mid d_A a_2] \rangle + \langle [a_1] \mid [a_2] \rangle \\
&\qquad\qquad\qquad + (-1)^{|a|+1}\langle [d_A a_1 \mid a_2] \rangle + \langle [a_1 \mid d_A a_2] \rangle = \langle [a_1] \mid [a_2] \rangle.
\end{aligned}$$

We see that this is in fact a homotopy, which makes $id_{\mathrm{gr}_2 \Omega BA}$ null-homotopic.

To extend this argument by induction, we will hypothesize that the terms where the differential is applied will have opposite signs, such that they cancel. Then the result follows for any i, by observing that the tensors far enough out to the right is not affected by r.

If C is a dg-coalgebra, we use the same technique as in 2.2.6. Consider the filtration on $B\Omega C$ given as

$$F_p B\Omega C = \{[\langle sc_{1,1} \mid \cdots \mid sc_{1,n_1} \rangle \mid \cdots \mid \langle sc_{m,1} \mid \cdots \mid sc_{m,n_m} \rangle] \mid |c_{1,1}| + \cdots + |c_{m,n_m}| \leqslant p\}.$$

This filtration is both bounded below and exhaustive, so the classical convergence theorem says that the associated spectral sequence converges. If we denote this sequence as EF, then $EF \implies H^* B\Omega C$. Let EC denote the spectral sequence associated to C. Since C is conilpotent $EC \implies H^* C$. The unit $\eta_C : C \to B\Omega C$ is now a map acting on EC^0 as the identity, sending each element in $EC^0_{p,q}$ to itself in $EF^0_{p,q}$.

On each row $EF^0_{p,\bullet}$ we make another filtration called G.

$$G_k EF^0_{p,\bullet} = \{[\langle \ldots \rangle_1 \mid \ldots \mid \langle \ldots \rangle_n] \mid n \geqslant -k\}$$

Similarly as in 2.2.6, this filtration is bounded below and exhaustive, so we may again apply the classical convergence theorem to obtain a spectral sequence $E_p G$ such that $E_p G \implies H^* EF^0_{p,\bullet} \simeq EF^1_{p,\bullet}$. Since the unit acts as the identity on EC^0, it descends to a morphism $\mathrm{gr}_p C \to E_p G^0_{k,\bullet}$ which is the identity when $k = -1$ and 0 otherwise. Notice that every string of length $\geqslant 2$ is not hit by this morphism. However, by employing r as above we may show that these summands are contractible.

To show that this map is a quasi-isomorphism it is sufficient to show that $E_p G^0_{k,\bullet}$ is contractible for $k \neq 1$. This looks like the same situation as for algebras. We use the same homotopy r as in the first part to get a null-homotopy of the identity. □

Lemma 2.2.9. *Let $f : C \to D$ be a morphism of dg-coalgebras, then:*

- *if f is a cofibration, then Ωf is a standard cofibration.*
- *if f is a weak equivalence, then Ωf is as well.*

Almost dually, let $f : A \to B$ be a morphism of dg-algebras, then:

- *if f is a fibration, then Bf is a fibration.*
- *if f is a weak equivalence, then Bf is as well.*

Proof. First suppose that $f : C \to D$ is a cofibration. We define a filtration on D as the sum of the image of f and the coradical filtration on D: $D_i = Imf + Fr_i D$. f being a cofibration ensures us that $D_0 \simeq C$. Since D is conilpotent we know that $D \simeq \varinjlim D_i$, and that Ω commutes with colimits, there is a sequence of algebras $\Omega C \to \Omega D_1 \to \ldots \to \Omega D$. It is enough to show that each morphism $\Omega D_i \to \Omega D_{i+1}$ is a standard cofibration. The quotient coalgebra ${}^{D_{i+1}}/_{D_i}$ only has a trivial comultiplication, thus every element is primitive. This means that as a cochain complex D_{i+1} is constructed from D_i by attaching possibly very many copies of $\mathbb{K}$. We treat the case when there is only one such $\mathbb{K}$, here $D_{i+1} \simeq D_i \oplus \mathbb{K}\{x\}$ where $dx = y$ for some $y \in D_i$. We observe that this is exactly the condition for that the morphism $\Omega D_i \to \Omega D_{i+1}$ is a standard cofibration.

If f is a weak equivalence, then Ωf is a quasi-isomorphism by definition.

By lemma 2.1.39, or adjointness more specifically, B preserving fibrations is a consequence of Ω preserving cofibrations.

It remains to show that if $f : A \to B$ is a quasi-isomorphism, then Bf is a weak equivalence. Now, Bf is a weak equivalence if and only if ΩBf is a quasi-isomorphism. By 2.2.8, the counit $A \to \Omega BA$ is a quasi-isomorphism, so Bf is a weak equivalence by 2-out-of-3 property.

$$\begin{tikzcd}
A \arrow[r, "f"] & B \\
\Omega BA \arrow[u, "\varepsilon_A"] \arrow[r, "\Omega Bf"] & \Omega BB \arrow[u, "\varepsilon_B"']
\end{tikzcd}$$

□

We will need one more technical lemma.

Lemma 2.2.10. *Let A be a dg-algebra, D a dg-coalgebra and $p : A \to \Omega D$ a fibration of algebras. The projection morphism $\pi : BA \prod_{B\Omega D} D \to BA$ is an acyclic cofibration.*

$$\begin{tikzcd}
BA\prod_{B\Omega D} D \arrow[r] \arrow[d, "\pi"] & D \arrow[d, "\eta_D"] \\
BA \arrow[r, "Bp"] & B\Omega D
\end{tikzcd}$$

Proof. π being a cofibration is immediate by corollary 2.1.10.1. To see that π is a quasi-isomorphism it is enough to understand that it is a quasi-isomorphism as chain complexes. This is checked by Lefevre-Hasegawa [12]. Keller has pointed out a slight mistake in this proof which he has proposed a fix to [26]. □

Theorem 2.2.11. *The category coAlg$^{\bullet}_{\mathbb{K},conil}$ is a model category with the classes Ac, Fib and Cof as defined above.*

Proof. The axioms **MC1** and **MC2** are immediet. Also, fibrations having RLP with respect to acyclic cofibrations is by definition.

We show **MC4** first. Let $f : C \to D$ be a morphism of coalgebras. There is a factorization $\Omega f = pi$ of morphisms between algebras, where i is a cofibration, p is a fibration and at least one of i and p are quasi-isomorphisms. Applying bar we get a factorization $B\Omega f = BiBp$, where Bp is a fibration and at least one of Bi and Bp are weak equivalences.

$$\begin{tikzcd}
\Omega C \arrow[rr, "\Omega f"] \arrow[dr, "i"'] & & \Omega D & \rightsquigarrow & B\Omega C \arrow[rr, "B\Omega F"] \arrow[dr, "Bi"'] & & B\Omega D \\
& A \arrow[ur, "p"'] & & & & BA \arrow[ur, "Bp"'] &
\end{tikzcd}$$

We construct a pullback with Bp and η_D. By 2.2.10 the morphism π is an acyclic cofibration. We collect our morphisms in a big diagram. The dashed arrow exists since the rightmost square is a pullback.

$$\begin{tikzcd}
& BA\prod_{B\Omega D} D \arrow[dd] \arrow[dr, "q"] \arrow[d, phantom, "\ulcorner"] & \\
C \arrow[ur, dashed, "j"] \arrow[rr, "f"] \arrow[ddd, "\eta_C"] & & D \arrow[ddd, "\eta_D"] \\
& BA \arrow[ddr, "Bp"] & \\
& & \\
B\Omega C \arrow[uur, "Bi"] \arrow[rr, "B\Omega f"] & & B\Omega D
\end{tikzcd}$$

First notice that q is a fibration, since fibrations are stable under pullbacks. j is a cofibration, or a monomorphism, as the composition $Bi\eta_C$ is a monomorphism. Thus it remains to see that if Bi (Bp) is a weak equivalence, then j (q) is as well. This is evident from the 2-out-of-3 property, as η is a natural weak equivalence, π is a weak equivalence and Bi (Bp) is a weak equivalence.

We now show **CM3**. Suppose that there is a square as below, where i is a cofibration and t is an acyclic cofibration.

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "i"] & C \arrow[d, "t"] \\
F \arrow[r] & D
\end{tikzcd}$$

We can factor t as $t = qj$ by **CM4**. Notice that t is a retract of q, i.e. there is a commutative diagram as below.

$$\begin{tikzcd}
C \arrow[r, equal] \arrow[d, "j"] & C \arrow[d, "t"] \\
BA\prod_{B\Omega A} D \arrow[r, "q"] \arrow[ur] & D
\end{tikzcd}$$

So in order to find a lift to C, we may instead find a lift to $BA\prod_{B\Omega D} D$. Since p is an acyclic fibration by construction and Ωi is a cofibration by 2.2.9, there is a lift $h : \Omega E \to A$ of algebras. We obtain our desired lift from the bar-cobar adjunction and the universal property of the pullback.

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "i"] & BA\prod_{B\Omega D} D \arrow[r, "\pi"] \arrow[d, "q"] \arrow[dr, phantom, "\ulcorner", very near start] & BA \arrow[d, "Bp"] & {} \arrow[r, leftrightsquigarrow] & {} & \Omega E \arrow[r] \arrow[d, "\Omega i"] & A \arrow[d, "p"] \\
F \arrow[r] \arrow[ur, dashed] \arrow[urr, dotted, "h^T"] & D \arrow[r, "\eta_D"] & B\Omega D & & & \Omega F \arrow[r] \arrow[ur, dotted, "h"] & \Omega D
\end{tikzcd}$$

□

We restate the corollary of the adjunction.

Corollary 2.2.11.1. *The bar-cobar construction* B : $Alg^{\bullet}_{\mathbb{K},+} \rightleftharpoons$ *coAlg*$^{\bullet}_{\mathbb{K},conil}$: Ω *as a Quillen equivalence.*

Proof. We first observe that (B, Ω) is a Quillen adjunction by lemma 2.2.9. Moreover, since the unit and counit are weak equivalences by proposition 2.2.8, it follows by either proposition 2.1.42 or its corollary 2.1.42.1 that (B, Ω) is a Quillen equivalence. □

2.2.3 Homotopy theory of A_∞-algebras

This section aims to finalize the discussion of the homotopy theory of A_∞-algebras. We will look at the homotopy invertability of every strongly homotopy associative quasi-isomorphism, and the relation to associative algebras. This discussion will end with mentioning different results which gives a clearer description of fibrations, cofibrations and homotopy equivalences. This section follows Lefevre-Hasegawa [12]. Before we get to the main theorem, we start by discussing a non-closed model structure on the category of Alg_∞.

Let $f : A \rightsquigarrow B$ be a morphism between A_∞-algebras, the category of A_∞-algebras will be equipped with the three following classes of morphisms:

- $f \in \mathrm{Ac}$ if f is an ∞-quasi-isomorphism, i.e. f_1 is a quasi-isomorphism.
- $f \in \mathrm{Fib}$ if f_1 is an epimorphism.
- $f \in \mathrm{Cof}$ if f_1 is a monomorphism.

This category does not make a model category in the sense of a closed model category, as we are lacking many finite limits. It does however come quite close to be such a category.

Theorem 2.2.12. *The category Alg_∞ equipped with the three classes as defined above satisfies:*

*a The axioms **MC1** through **MC4**.*
b Given a diagram as below, where p is a fibration, then its limit exists.

$$\begin{tikzcd} & A \arrow[d, "p"] \\ B \arrow[r] & C \end{tikzcd}$$

We will first prove one lemma which technique we will reuse. A seconde lemma used to simplify the theorem will be stated without proof.

Lemma 2.2.13. *let A be an A_∞-algebra, and K a contractible complex considered as an A_∞-algebra. If $g : (A, m_1^A) \to (K, m_1^K)$ is a cochain map, then it extends to an ∞-morphism $f : A \rightsquigarrow K$.*

Proof. We construct each f_i inductively. The case $i = 1$ is degenerate as we have assumed $f_1 = g$.

Assume that we have already constructed f_1 through f_n. We observe that the sum below is a cycle of $\mathrm{Hom}^*_{\mathbb{K}}(A, K)$.

$$\sum_{\substack{p+1+r=k \\ p+q+r=n}} (-1)^{pq+r} f_k \circ_{p+1} m_q^A - \sum_{\substack{k \geqslant 2 \\ i_1+\ldots+i_k=n}} (-1)^e m_k^B \circ (f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_k})$$

Thus since K is contractible, $\mathrm{Hom}^*_{\mathbb{K}}(A, K)$ is acyclic and there exists some morphism f_{n+1} such that $\partial(fn+1)$ is the sum above. This says that this extension does in fact satisfy (rel_{n+1}). □

Lemma 2.2.14. *Let $j : A \rightsquigarrow D$ be a cofibration of A_∞-algberas, then there is an isomorphism $k : D \rightsquigarrow D'$ such that the composition $k \circ j : A \rightsquigarrow D'$ is a strict morphism of A_∞-algebras.*

Dually, if $j : A \rightsquigarrow D$ is a fibration, then there is an isomorphism $l : A' \rightsquigarrow A$ such that the composition $j \circ l : A' \rightsquigarrow D$ is a strict morphism of A_∞-algebras.

Proof. A proof is given as lemma 1.3.3.3 in [12]. □

proof of 2.2.12. We start by showing (b). Suppose that we have a diagram of A_∞-algebras, such that g_1 is an epimorphism.

$$\begin{tikzcd} & A \arrow[d, two heads, "g"] \\ A' \arrow[r, "f"] & A'' \end{tikzcd}$$

First notice that as dg-coalgebras, this pullback exists and defines a new dg-coalgebra $BA \prod_{BA''} BA'$.

Since g_1 is an epimorphism, $A[1]$ as a graded vector space splits into $A''[1] \oplus K$, where $K = Kerg_1$. The pullback is then naturally identified with $BA \prod_{BA''} BA' \simeq \overline{T}^c(K) \prod \overline{T}^c(A'[1])$ as graded vector spaces. Since the cofree coalgebra is right adjoint to forget, it commutes with products and we get, $\overline{T}^c(A'[1]) \prod \overline{T}^c(K) \simeq \overline{T}^c(A'[1] \oplus K)$. Thus the pullback is isomorphic to a cofree coalgebra as a graded coalgebra, i.e. it is an A_∞-algebra.

We now prove (a). MC1 and MC2 are immediate, so we will not prove them.

We start by proving MC3. Suppose that there is a square of A_∞-algebras as below, where j is a cofibration and q is a fibration.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"] & B \arrow[d, "q"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

By lemma 2.2.14, we may assume assume that both j and q are strict morphisms. We now assume that q is an ∞-quasi-isomorphism, the proof will be analogous if j is an ∞-quasi-isomorphism instead.

Our goal is to construct a lifting in this diagram inductively. Having a lift means finding an ∞-morphism $a : C \rightsquigarrow B$, such that the following hold for any $n \geqslant 1$:

- a satisfy (rel_n).
- $a_n \circ j_1 = f_n$.
- $q_1 \circ a_n = g_n$.

We start by showing there is such an a_1. Consider the diagram below of chain complexes over $\mathbb{K}$.

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, tail, "j_1"'] & B \arrow[d, two heads, "q_1"] \\ C \arrow[r, "g_1"'] \arrow[ur, dashed, "a_1"] & D \end{tikzcd}$$

The lift exists since the category Ch($\mathbb{K}$) is a model category. Here j_1 is a cofibration, while q_1 is an acyclic fibration, so the lift a_1 exists.

We now wish to extend this. Suppose that we have been able to create morphisms a_1 up to a_n, all satisfying the above points. A naive solution to make a_{n+1} is $b = f_{n+1}r^{\otimes n+1} + sg_{n+1} - sq_1 f_{n+1}r^{\otimes n+1}$. Notice that this satisfy the two last points by definition. We will augment b to get an a_{n+1} which also satisfies (rel_{n+1}).

For our own convenience, let $-c(f_1, ..., f_n)$ denote the right hand side of (rel_{n+1}) formula. Since both j and q are strict ∞-morphisms we get the following identites:

$$(\partial b + c(a_1, ..., a_n)) \circ j_1 = \partial(b \circ j_1) + c(a_1 \circ j_1, ..., a_n \circ j_1) = \partial f_{n+1} + c(f_1, ..., f_n) = 0$$
$$q_1 \circ (\partial b + c(a_1, ..., a_n)) = \partial(q_1 \circ b) + c(q_1 \circ a_1, ..., q_1 \circ a_n) = \partial g_{n+1} + c(g_1, ..., g_n) = 0$$

We thus obtain that the cycle $\partial b + c(a_1, ..., a_n)$ factors thorugh the cokernel of j and the kernel of q. Let us say that it factors like the diagram below:

$$C \xrightarrow{p} \text{Cok}j_1 \xrightarrow{c'} \text{Ker}q_1 \xrightarrow{i} D$$

Now, c' is a morphism between two A_∞-algebras. Since q is assumed to be an ∞-quasi-isomorphism, it follows that Kerq_1 is a acyclic. Since c' is a cycle, it necesserilly have to be in the image of the differential. Let h be a morphism such that $\partial h = c'$, and define $a_{n+1} = b - i \circ h \circ p$. One may check that this morphism satisfies all three properties.

We will now show MC4. Since the two properties have a similar proof, we will only show one direction. Let $f : A \rightsquigarrow B$ be an ∞-morphism. Let $C = \text{cone}(id_{B[-1]})$. The complex C may be considered as an A_∞-algebra. Let $j : A \rightsquigarrow A \prod C$ be the morphism induced by id_A and $0 : A \to C$. The canonical projection $q_1 : A \oplus C \to B$ gives a lift of the following diagram.

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "j_1"] & B \arrow[d] \\ A \oplus C \arrow[r] \arrow[ur, "q_1"] & 0 \end{tikzcd}$$

Since we have a morphism of chain complexes, lodged between an acyclic cofibration and a fibration we use the same technique as above to construce an ∞-morphism $q : A \prod C \to B$. q is a fibration by construction. The morphism f may be factored as $f = q \circ j$, where j is an acyclic cofibration and q is a fibration. $\square$

With this model structure we are finally able to characterize the fibrant and cofibrant conilpotent dg-coalgebras.

Proposition 2.2.15. *Let C be a conilpotent dg-coalgebra. Then C is cofibrant, and C is fibrant if and only if there is a cochain complex V, such that $C \simeq T^c(V)$ as complexes.*

Proof. To see that C is cofibrant is the same as to verify that the map $\mathbb{K} \to C$ is a monomorphism, but this is clear.

We start by assuming that C is fibrant. Then there is a lift in the square below, making the unit split-mono.

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d, "\eta_C"] & C \arrow[d, "\varepsilon_C"] \\ B\Omega C \arrow[r, "\varepsilon_{B\Omega C}"] \arrow[ur, dashed, "r"] & \mathbb{K} \end{tikzcd}$$

Consider the morphism $p_1^C : C \to Fr_1 C$ which is defined as $p_1^C = Fr_1 r \circ p_1 \circ \eta_C$, where $p_1 : B\Omega C \to Fr_1 B\Omega C$ is the canonical projection on the filtration induced by the coradical filtration on C. Clearly, r makes p_1 into a universal arrow in the category of conilpotent coalgebras, so $C \simeq T^c(Fr_1 C)$.

Now, assume that C is isomorphic to $T^c(V)$ as coalgebras for some cochain complex V. Note that, by definition, C is an A_∞-algebra. By definition, we have a commutative square of A_∞-algebras. Since every A_∞-algebra is bifibrant, we know that this diagram has a lift, exhibiting C as a retract of $B\Omega C$.

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d] & C \arrow[d] \\ B\Omega C \arrow[r] \arrow[ur, dashed] & \mathbb{K} \end{tikzcd}$$

We know that ΩC is fibrant, since the map $\Omega C \to 0$ is epi. By lemma 2.2.9, we know that the bar construction preserves fibrations, so $B\Omega C$ is fibrant. Thus C is fibrant as well. □

The model structure of A_∞-algebras is compatible with the model structure of conilpotent dg-coalgebras in the following sense. If $f : A \rightsquigarrow B$ is an ∞-morphism, we denote its dg-coalgebra counterpart as $Bf : BA \to BB$. Remember that the bar construction is extended such that it is an equivalence of categories on its image. We use this to realize Alg_∞ as a subcategory of $\mathsf{coAlg}_\mathbb{K}$ to essentially obtain 2 different model structure on this category. The following proposition tells us that these structures do not differ.

Proposition 2.2.16. *Let $f : A \rightsquigarrow B$ be an ∞-morphism. Then we have the following:*

- *f is an ∞-quasi-isomorphism if and only if Bf is a weak equivalence.*
- *f_1 is an epimorphism if and only if Bf is a fibration.*
- *f_1 is a monomorphism if and only if Bf is a monomorphism.*

Proof. This is proposition 1.3.3.5 in [12]. □

2.3 The Homotopy Category of Alg_∞

With the results we have establish, we are now ready to talk about homotopies in Alg_∞.

Theorem 2.3.1. *In the category Alg_∞ we have the following:*

- *Homotopy equivalence is an equivalence relation.*
- *A morphism is an ∞-quasi-isomorphism if and only if it is a homotopy equivalence.*
- *Let $dash \subseteq \mathsf{Alg}_\infty$ be the full subcategory consisting of dg-algebras considered as A_∞-algebras. dash has an induced homotopy equivalence from Alg_∞, and the inclusion Alg $\to$ dash induces an equivalence in homotopy Alg$[Qis^{-1}] \simeq {}^{\text{dash}}/\sim$.*

Proof. The first point is obsreved from corollary 2.1.28.2, and the second point is Whiteheads theorem, theorem 2.1.30.

To see the final point, observe that the inclusion functor is given by the bar construction B. By corollary 2.2.11.1, we know that the bar construction induces an equivalence on the homotopy categories, i.e. HoAlg $\simeq$ HocoAlg. Moreover, we know that by theorem 2.1.31 that HocoAlg $\simeq$ ${}^{\mathsf{Alg}_\infty}/\sim$. Notice that the image of B is dash, so in homotopy, we get that the image ${}^{\text{dash}}/\sim$ is equivalent to the essentiall image HoAlg$_\infty$. □

Homotopy equivalence in the algebraic sense and the model categorical sense differ in how they are defined. In homological algebra, two morphisms are called homotopic if their difference is a boundary of some homotopy, i.e. $f - g = \partial h$. In the model categorical sense, a homotopy is a morphism through either a cylinder or a path object. The following proposition tells us how they differ in the category of conilpotent dg-coalgebras.

Proposition 2.3.2. *Let C and D be two conilpotent dg-coalgebras, where $f, g : C \to D$ are two morphisms. Then:*

- *If $f - g$ is null homotopic, then they are left homotopic.*
- *If D is fibrant, then $f - g$ is null homotopic if and only if f and g are left homotopic.*

This tells us that in general, these concepts are usually not the same. However, we know that the subcategory of bifibrant objects in this category is exactly the subcategory Alg_∞. Thus for A_∞-algebras homological homotopy is the same as model categorical homotopy. In this sense we obtain that the homotopy category in the homological sense is equivalent to the derived category.

Chapter 3

Derived Categories of Strongly Homotopy Associative Algebras

In this chapter we wish to study the derived categories of A_∞-algebras. At the heart of homological algebra is the derived category of algebras, so it is only natural to ask how this category looks like in the A_∞ case. In the last chapter we studied the relationship between the category of algebras and coalgebras to understand how quasi-isomorphisms between A_∞-algebras worked. In this chapter we will instead study the relationship between module and comodule categories in order to understand how quasi-isomorphisms between A_∞-modules will work. At the heart of this discussion are twisting morphisms $\alpha : C \to A$, which allows us to study the relationship between Mod^A and $CoMod^C$.

From twisting morphisms we will obtain functors $L_\alpha : CoMod^C \to Mod^A$ and $R_\alpha : Mod^A \to CoMod^C$ which create an adjoint pair of functors. Whenever the twisting morphism α is acyclic, this will in fact become a Quillen Equivalence.

We wish to reuse all of the methods we have gained and acquired thorughout this thesis. This chapter will mostly be reformulation and recontextualization of previous definitions, concepts and techniques.

3.1 Twisting Morphisms

Twisting morphisms were already introduced in chapter 1. There, they were used mostly to be represented by the bar and cobar construction. Now we want twisting morphisms and twisting tensors to play a bigger role. In order to define the functors L_α and R_α, these constructions will be crucial.

3.1.1 Twisted Tensor Products

Let A be an augmented dg-algebra, C a conilpotent dg-coalgebra and $\alpha : C \to A$ a twisting morphism. The right (left) twisted tensor product is the complex $C \otimes_\alpha A$ ($A \otimes_\alpha C$) together with the differential $d_\alpha^\bullet = d_{C\otimes A}^\bullet + d_\alpha^r$. The perturbation is defined as

$$d_\alpha^r = (\nabla_A \otimes id_C) \circ (id_A \otimes \alpha \otimes id_C) \circ (id_A \otimes \Delta_C).$$

If M is a right A-module and N is a left C-comodule then the tensor product $M \otimes_{\mathbb{K}} N$ exists and is a $\mathbb{K}$-module with differential $d_{M\otimes N}$. We may define a perturbation to this differential as

$$d_\alpha^r = (\mu_M \otimes id_N) \circ (id_M \otimes \alpha \otimes id_N) \circ (id_M \otimes \nu_N).$$

By using the same line of thought as proposition 1.2.5, there is a twisted tensor product $M \otimes_\alpha N$ with differential $d_\alpha^\bullet = d_{M\otimes N} + d_\alpha^r$.

Remark 3.1.1. Koszuls sign rule forces us to define the differential of the left twisted tensor product as $d_\alpha^\bullet = d_{N\otimes M} - d_\alpha^l$.

Definition 3.1.2. Suppose that $M \in Mod^A$ ($M \in Mod_A$) and $N \in CoMod_C$ ($N \in CoMod^C$), then the left (right) twisted tensor product is the $\mathbb{K}$-module $M \otimes_\alpha N$ ($N \otimes_\alpha M$).

In this setting right handedness and left handedness for the twisted tensor product is more clear in this setting, as we only have an action or coaction from one of the chosen sides. Trying to force the other handedness on the twisted tensors would just be ill-defined.

Definition 3.1.3. Let A be an augmented dg-algebra and C a conilpotent dg-coalgebgra, such that there is a twisting morphism $\alpha : C \to A$. Given a linear map $f : N \to M$ between a right C-comodule N and a right A-module M we say that it is an α right twisted linear homomorphism, or just an α twisted morphism, if it satisfies the following equation:

$$\partial f - f \star \alpha = 0$$

This definition gives us a functor $Tw_\alpha : CoMod^C \times Mod^A \to Ab$ which is the collection of right twisting linear homomorphisms between a comodule and module.

Suppose that $\alpha : C \to A$ is a twisting morphism. We define the functor $L_\alpha = \otimes_\alpha A : CoMod^C \to Mod^A$ as an arbitrary right twisted tensor product with A. This functor does indeed hit Mod^A by using the free right A-module structure on A. Likewise, we define a functor $R_\alpha = \otimes_\alpha C : Mod^A \to CoMod^C$ as an arbitrary left twisted tensor product with C. This does also hit right C-comodules by using the free right C-comodule structure on C.

Proposition 3.1.4. *Suppose that $\alpha : C \to A$ is a twisting morphism. The functor L_α and R_α form an adjoint pair of categories.*

$$\begin{tikzcd} CoMod^C \arrow[r, bend left, "L_\alpha"] \arrow[r, phantom, "\perp"] & Mod^A \arrow[l, bend left, "R_\alpha"] \end{tikzcd}$$

Proof. This proof breaks down to showing $CoMod^C(N, L_\alpha(M)) \simeq Tw_\alpha(N, M) \simeq Mod^A(R_\alpha(N), M)$. This is a rutine calculation, much like the proof for 1.2.12. □

Let A be a dg-algebra, and M a right A-module. Recall that by the Cobar-Bar adjunction 1.2.12 there exists a universal twisting morphism $\pi_A : BA \to A$. We define the bar-construction of M as $B_A M = R_{\pi_A} M = M \otimes_{\pi_A} BA$. Likewise, given a conilpotent dg-coalgebra C and N a right C-comodule we define the cobar-construction as $\Omega_C N = L_{\iota_C} N = N \otimes_{\iota_C} \Omega C$. In these cases we obtain adjunctions $\Omega_{BA} \dashv B_A$ and $\Omega_C \dashv B_{\Omega C}$.

Let A and B be two algebras and $f : A \to B$ is an algebra morphism. Then f induces a functor between the module categories by restriction: $f^* : Mod^B \to Mod^A$. Since A and B considered as algebroids are small, and the category of abelian groups is cocomplete, so the left Kan extension (induction) along this functor exists.

$$\begin{tikzcd} Mod^B \arrow[r, "f^*"] & Mod^A \arrow[l, bend right, "f_!"'] \end{tikzcd}$$

Dually, if C and D are two coalgebras and $g : C \to D$ is an coalgebra morphism. Then g induces a functor between the module categories by composing: $g* : CoMod^C \to CoMod^D$. Since C and D considered as coalgebroids are small, and the category of abelian groups is complete, so the right Kan extension (co-induction) along this functor exists.

$$\begin{tikzcd} CoMod^C \arrow[r, "g*"] & CoMod^D \arrow[l, bend left, "g^!"] \end{tikzcd}$$

Lemma 3.1.5. *Let $\tau : C \to A$ be a twisting morphism. The adjunction (L_τ, R_τ) factors as $(f_{\tau!}, f_\tau^*) \circ (L_{\iota_C}, R_{\iota_C})$ or $(L_{\pi_A}, R_{\pi_A}) \circ (g_{\tau*}, g_\tau^!)$.*

Proof. This follows from corollary 1.2.13.1, that is $\tau = f_\tau \circ \iota_C = \pi_A \circ g_\tau$. □

Definition 3.1.6. A twisting morphism $f : C \to A$ is called acyclic if the counit of the adjunction $L_\alpha \dashv R_\alpha$ is a pointwise quasi-isomorphism.

Lemma 3.1.7. *Let A be an augmented dg-algebra and C a conilpotent dg-coalgebra. The universal twisting morphisms π_A and ι_C are acyclic.*

Proof. We start with π_A. Recall that π_A is constructed as the twisting morphism corresponding to id_{BA}. This morphism is thus given as the projection onto the first dimension of BA, that is:

$$\pi_A sa = a$$
$$\pi_A(sa \otimes ...) = 0$$

We say that π_A is acyclic if the counit $\varepsilon : L_{\pi_A} R_{\pi_A} \Rightarrow Id_{Mod^A}$ at each object M is a quasi-isomorphism.

For each M in Mod^A, $L_{\pi_A} R_{\pi_A} M = M \otimes_{\pi_A} BA \otimes_{\pi_A} A$. We may split up the differential into two summands, d_v and d_h. d_v is the ordinary differential on the tensor product, while $d_h = (-d^l_{\pi_A} \otimes A) + M \otimes d_2 \otimes A + d^r_{\pi_A}$. Since $(d_v + d_h)^2 = 0$ and $d_v^2 = 0$ we can observe that $d_v d_h = -d_h d_v$ and $d_h^2 = 0$. This is evident as d_v changes the homological degree while d_h does not, so in order for both of the first equations to hold, the last two must hold as well. We almost obtain a double complex.

$$\begin{tikzcd}
... \arrow[r, "d_h"] & M \otimes BA_{(i)} \otimes A \arrow["d_v", loop above] \arrow[r, "d_h"] & ... \arrow[r, "d_h"] & M \otimes BA_{(1)} \otimes A \arrow["d_v", loop above] \arrow[r, "d_h"] & M \otimes A \arrow["d_v", loop above] \arrow[r, "0"] & 0
\end{tikzcd}$$

It is clear that the total complex of this "double complex" is in fact $L_{\pi_A} R_{\pi_A} M$. Moreover, the counit induces an augmentation to this complex resolution of M, denoted as $cone(\varepsilon_M)$.

$$\begin{tikzcd}
... \arrow[r, "d_h"] & M \otimes BA_{(i)} \otimes A \arrow["d_{M\otimes BA_{(i)}\otimes A}", loop above] \arrow[r, "d_h"] & ... \arrow[r, "d_h"] & M \otimes BA_{(1)} \otimes A \arrow["d_{M\otimes BA_{(1)}\otimes A}", loop above] \arrow[r, "d_h"] & M \otimes A \arrow["d_{M\otimes A}", loop above] \arrow[r, "\varepsilon_M"] & M \arrow["d_M", loop above] \arrow[r, "0"] & 0
\end{tikzcd}$$

To see that this is in fact a resolution we define a morphism $h : cone(\varepsilon_M) \to cone(\varepsilon_M)$ of degree -1. It works by the following formula:

$$h(m \otimes (sa_1 \otimes ... \otimes sa_n) \otimes a) = m \otimes (sa_1 \otimes ... \otimes sa_n \otimes sa) \otimes 1$$

It is clear that $id_{cone(\varepsilon_M)} = d_h h - h d_h$ and $d_v h = h d_v$. Thus to see that the cone is acyclic we let $c \in cone(\varepsilon_M)$ be a cycle, that is $(d_v + d_h)(c) = 0$. Our goal is to show that $h(c)$ is a preimage of c along $d_v + d_h$.

$$(d_v + d_h) \circ h(c) = d_v \circ h(c) + d_h \circ h(c) = h \circ d_v(c) + c + h \circ d_h(c) = h \circ (d_v + d_h)(c) + c = c$$

proof To treat the case of ι_C we will use a double spectral sequence argument.

□

3.1.2 Model Structure on Module Categories

Let A be an augmented dg-algebra, then we know that Mod^A is a model category. By corollary 2.2.4.2 we have a model structure where the fibrations, cofibrations and weak equivalences are given as follows:

- $f \in Ac$ is a weak equivalence if f is a quasi-isomorphism.
- $f \in Fib$ is a fibration if $f^{\#}$ is an epimorphism.
- $f \in Cof$ is a cofibration if it has LLP to acyclic fibrations.

Every object in this category is fibrant as the morphism $0 : M \to 0$ is always an epimorphism.

3.1.3 Model Structure on Comodule Categories

Unless stated otherwise, for this section we fix A to be an augmented dg-algebra, C as a conilpotent dg-coalgebra and $\tau : C \to A$ as an acyclic twisting morphism. We endow $CoMod^C_{conil}$ with three classes of morphisms:

- $f \in Ac$ is a weak equivalence if $L_\tau f$ is a quasi-isomorphism.
- $f \in Cof$ is a cofibration if $f^{\#}$ is a monomorphism.
- $f \in Fib$ is a fibration if it har RLP to acyclic cofibrations.

Theorem 3.1.8. *The category $CoMod^C_{conil}$ with the three classes as above forms a model category. Every object is cofibrant, and those objects which is a direct summand of $R_\tau M$ for some $M \in Mod^A$ are fibrant. The adjoint pair (L_τ, R_τ) is a Quilllen equivalence.*

We will call this model structure for the canonical model structure on $CoMod^C_{conil}$. Under the hypthesis of this theorem, we may observe that every object of $CoMod^C_{conil}$ are cofibrant. Since every $M \in Mod^A$ is fibrant, and R_τ preserves fibrant objects we known that R_tauM is fibrant as well. Clearly every direct summand of $R_\tau M$ is fibrant. If $N \in CoMod^C_{conil}$ is fibrant, then it is a direct summand of $R_\tau L_\tau N$. This shows that the bifibrant objects of $CoMod^C_{conil}$ are exactly the thick image of R_τ.

To be able to prove this we will need some lemmata. This proof is essentially the same as the case for dg-coalgebras. The main difference is to show independence of the choice of a twisting morphism τ. To this end we must establish the relationship between graded quasi-isomorphisms and weak equivalences, as well as a technical lemma.

Recall that given a coaugmented coalgebra C we have a filtration called the coradical filtration, defined as $Fr_iC = Ker(\bar{\Delta}_C)^i$. If N is a right C-comodule we may define the coradical filtration of N as $Fr_iN = Ker(\bar{\omega}^i_N)$. This filtration is admissable, meaning it is exhaustive and $Fr_0N = 0$.

Lemma 3.1.9. *Let C be a conilpotent dg-coalgebra, M and N be right C-comodules. Then any graded quasi-isomorphism $f : M \to N$ is a weak equivalence.*

Proof. This proof is identical to 2.2.6. □

Lemma 3.1.10. *Let M and N be two objects of Mod^A. The functor R_τ sends a quasi-isomorphism $f : M \to N$ to a weak equivalence $R_\tau f : R_\tau M \to r_\tau N$.*

The unit of the adjunction $\eta : Id \to R_\tau L_\tau$ is a pointwise weak equivalence.

Proof. $R_\tau f$ is a weak equivalence if $L_\tau R_\tau f$ is a quasi-isomorphism. By naturality of the counit we have the following commutative diagram.

$$\begin{tikzcd} M \arrow[d, "f"] & L_\tau R_\tau M \arrow[l, "\varepsilon_M"'] \arrow[d, "L_\tau R_\tau f"] \\ N & L_\tau R_\tau N \arrow[l, "\varepsilon_N"] \end{tikzcd}$$

By assumption we know that all three of f, ε_M and ε_N are quasi-isomorphisms. It follows by the 2-out-of-3 property that $L_\tau R_\tau f$ is a quasi-isomorphism as well.

To show that $\eta : Id \to L_\tau R_\tau$ is a pointwise weak-equivalence, we must show that $L\eta$ is a pointwise quasi-isomorphism. Since L_τ is left adjoint to R_τ we know that η is split on the image of L_τ, i.e.

$$\varepsilon_{L_\tau} \circ L_\tau \eta = id_{L_\tau}$$

Since we know that the natural isomorphisms ε and id are pointwise quasi-isomorphisms, we get by the 2-out-of-3 property that $L\eta$ is a pointwise quasi-isomorphism as well. □

Lemma 3.1.11. *The functor L_τ preserves cofibrations and sends weak-equivalences to quasi-isomorphisms.*

Proof. This proof is essentially the same as 2.2.9. □

With the above lemmata we have now established that the adjunction (L_τ, R_τ) forms a Quillen equivalence if $CoMod^C$ is a model category.

The next lemma is a technical lemma which we need. There will not be given a proof for it, but this is lemma 2.2.2.9 in [12].

Lemma 3.1.12. *Let M be a right A-module and N a right C-comodule. Let $p : M \to L_\tau N$ be a fibration of modules. The projection $j : R_\tau M \prod_{R_\tau L_\tau N} N \to R_\tau M$ is an acyclic cofibration of comodules.*

Proof. This proof is omitted. □

Proof of 3.1.8. With the above lemmata established, this proof is identical to the proof of 2.2.11. □

3.1.4 Triangulation of Homotopy Categories

In this section we will show that the homotopy categories are triangulated. If we look at the category Mod^A we will observe that the category $HoMod^A$ is our beloved derived category $\mathcal{D}(A)$. It is not quite the same for the cateogry $CoMod^C$. Here we want $HoCoMod^C$ to be equivalent to the derived category of a ring, so we will see that the derived category is a further localization of $HoCoMod^C$.

Furthermore, by employing the theory of triangulated categories we will show that the model structure on $CoMod^C$ is independent on the choice of acyclic twisting morphism. This breaks down to show that every acyclic twisting morphism induce an equivalence between derived categories, as done by Bernhard Keller in [27].

Mod^A is a an abelian category, where we employ the maximal exact structure $\mathcal{E}'$ consisting of short exact sequences in Mod^A. This translates to short exact sequences which is short exact in each degree. However, this category also has an exact structure $\mathcal{E}$ which makes Mod^A into a Frobenius category, which we will now describe.

Let $f : M \to N$, be a chain map from M to N. Then $\mathcal{E}$ contains a conflation on the form:

$$N \rightarrowtail cone(f) \twoheadrightarrow M[1]$$

We define $\mathcal{E}$ to be the smallest exact structure on Mod^A which contains every conflation arising from a chain map f. Observe that these conflations are exactly the short exact sequences of Mod^A such that they are split when regarded as graded modules, i.e. forgetting the differential. Thus the smallest such $\mathcal{E}$ is exactly the collection of every conflation arising from a chain map f.

Recall that an object M is projective (injective) if the represented functor $Mod^A(M, \_)$ $(Mod^A(\_, M))$ is exact. For the category $(Mod^A, \mathcal{E})$

Proposition 3.1.13. *Let M be an object of Mod^A. The following are equivalent:*

- *M is projective*
- *M is injective*
- *M is contractible*

Proof. This is a well known statement from literature. See Krause [28], Happel [29], Buehler [30] or Thorbjørnsen [31] for an account of this result. ☐

To see that $(Mod^A, \mathcal{E})$ has both enough projectives and injectives we consider the following conflation:

$$M \rightarrowtail cone(id_M) \twoheadrightarrow M[1]$$

It is known that the complex $cone(id_M)$ is contractible for any complex M (and it is also universal with this property). In this way by letting M vary we can find an inflation or deflation from the identity cone from or to any complex. This concludes that $(Mod^A, \mathcal{E})$ is a Frobenius category.

Let $\underline{Mod^A}$ denote the injectively stable module category. Let $I(M, N)$ denote the set of chain maps from M to N which factors through an injective object. We define the injectively stable category as the quotient of abelian groups $\underline{Mod^A}(M, N) = {}^{Mod^A(M,N)}/_{I(M,N)}$.

Theorem 3.1.14. *Suppose that $(\mathcal{C}, \mathcal{E})$ is a Frobenius category, then the injectively stable category $\underline{\mathcal{C}}$ is triangulated. The additive auto-equivalence is given by cozyzygy, and the standard triangles is the image of the conflations into the quotient.*

Proof. This is also well known in literature. An account for it may also be found in Krause [28], Happel [29], Buehler [30] or Thorbjørnsen [31]. ☐

We thus obtain a triangulated category $\underline{Mod^A}$ associated to the Frobenius pair $(Mod^A, \mathcal{E})$. This category is commonly denoted as $K(A)$, and we will do this as well. Notice that with the structure given by $\mathcal{E}$, the cozyzygy is defined by the shift functor $[1]$. Every standard triangle is also on the form:

$$M \xrightarrow{f} N \longrightarrow cone(f) \longrightarrow M[1]$$

To define the derived category $D(A)$ of A we will consider the localization of $K(A)$ at the quasi-isomorphisms, $D(A) = K(A)[Qiso^{-1}]$. To see that the derived category is still triangulated we may realize it as a Verdier quotient of $K(A)$.

Proposition 3.1.15. *The derived category of A is equivalent to the Verdier quotient ${}^{K(A)}/_{Ac}$, where Ac denotes the image of acyclic objects in $K(A)$.*

Proof. A proof may be found in Buehler [30] or Thorbjørnsen [31]. ☐

There is another way of telling the story of the derived category $D(A)$. That is to directly localize it at the quasi-isomorphisms. We may directly see that $D(A) \simeq Mod^A[Qiso^{-1}]$ which we know is $HoMod^A$ by definition. This gives us our first important identification.

Theorem 3.1.16. *The homotopy category of Mod^A is triangulated, and moreover it is the derived category $D(A)$.*

Proof. Follows from discussion above. □

The triangulated construction for the category $hoCoMod^C$ closely resembles that of $HoMod^A$. We start by studying the Frobenius pair $(CoMod^C, \mathcal{E})$, where $\mathcal{E}$ is the same exact structure. Notice that this exact structure only takes the underlying category of chain complexes into account, so this follows from the above description.

We define the injectively stable category $\underline{CoMod^C} = K(C)$ in the same manner. The standard triangles and the additive auto-equivalence stays the same.

At this point things start to differ. The definition for the homotopy category $HoCoMod^C$ is $CoMod^C[Ac^{-1}]$, here Ac denotes the class of weak equivalences in $CoMod^C$. By abuse of notation we also let $Ac \subset K(C)$ be the collection of objects which are cones of weak equivalences. This subcategory can be characterized by being the preimage of acyclic objects $Ac \subset K(A)$ along $L_\tau : CoMod^C \to Mod^A$. This identification suffices to show that $Ac \subset K(C)$ is a triangulated subcategory. In this manner $HoCoMod^C$ is the category ${}^{K(C)}/_{Ac}$, which is a triangulated category.

Remark 3.1.17. We may show that $Ac \subset K(C)$ is a subcategory of acyclic objects. In this manner we get that $D(C) \simeq HoCoMod^C[Qiso^{-1}]$. This is done in Lefevre-Hasegawa [12] as proposition 1.3.5.1 or lemma 2.2.2.11. This follows from the fact that we have an equivalence of categories $CoMod^C[fQiso^{-1}] \simeq HoCoMod^C$, here $fQiso$ means the collection of filtered quasi-isomorphisms. Since every filtered quasi-isomorphism is in fact a quasi-isomorphism by a spectral sequence argument we get the inclusion of triangulated subcategories $\langle cone(fQiso)\rangle \subseteq \langle cone(Qiso)\rangle \subseteq K(C)$.

Let $\tau : C \to A$ and $\upsilon : C \to A'$ be two acyclic twisting morphisms. These defines independently two different model structures on $CoMod^C$ by the adjunctions (L_τ, R_τ) and (L_υ, R_υ). By lemma 3.1.5 we have the identification $(L_\tau, R_\tau) = (f_{\tau!}, f_\tau^*)(L_{\iota_C}, R_{\iota_C}) = (f_{\tau!}L_{\iota_C}, R_{\iota_C}f_\tau^*)$, and a likewise for υ. In order to show that τ and υ defines equivalent module structures on $CoMod^C$ it is enough that both define the same structure as ι_C. By symmetry it is enough to assume that $\upsilon = \iota_C$. From lemma 3.1.7 we know that ι_C is acyclic, so this technique is well-founded.

Since we already know that (L_τ, R_τ) and $(L_{\iota_C}, r_{\iota_C})$ are Quillen equivalences it remains to show that $(F_{\tau!}, f_\tau^*)$ is a Quillen equivalence. This is shown if f_τ^* is a right Quillen functor, and that it induces a triangle equivalence between $D(A)$ and $D(\Omega C)$.

We know that f_τ^* preserves fibrations (epimorphisms). This is because on morphisms this functor acts as the identity, it only changes the ring action, so epimorphisms stay epimorphisms. It remains to see that quasi-isomorphisms are preserved. We will show this by identifying the derived categories. This follows the methods given by Keller in [27]. We will however simplify this discussion by restricting our attention solely to dg-algebras.

Let A be a dg-algebra. We denote A as a free A-module as $\hat{A} = Hom_A(\_, A)$. $\hat{A}$ is free in the enriched sense, i.e. $Hom^*_A(\hat{A}, M) \simeq M$. Recall that P is projective if it is a direct summand of $\hat{A}^n$ for some $n \in \mathbb{N}$. Given a right bounded complex M, we know how to construct a projective resolution $p : pM \to M$. Associated to this resoultion there is a triangle in $K(A)$ consisting of the complexes M, pM and aM, where aM is an acyclic complex.

$$M \xrightarrow{p} pM \longrightarrow aM \longrightarrow M[1]$$

In this sense we obtain an identification $M \simeq pM$ in $D(A)^-$. By following Kellers construction we are able to weaken this identification to all of $D(A)$ by weakening the projective resolution. In Kellers paper he calls these complexes of property (P). We will however refer to them as homotopy projective complexes, since these complexes are built up from projective complexes in a manner respecting homotopy colimits.

Definition 3.1.18. Let P be a complex of Mod^A. We say that P is homotopy projective if there exists a complex P', a homotopy equivalence $P \simeq P'$ and a filtration of P'.

$$0 = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n \subseteq \ldots \subseteq P'$$

The filtration should satisfy these properties:

F1 P' is the colimit of the filtration.
F2 Each inclusion $i_n : F_n \subseteq F_{n+1}$ is split as graded modules.
F3 The quotient F_{n+1}/F_n is projective.

Remark 3.1.19. The properties $F1$ and $F2$ may be reformulated to that P should be the homotopy colimit of the filtration. Thus there is a canonical triangle in $K(A)$:

$$\bigoplus F_n \xrightarrow{\Phi} \bigoplus F_n \longrightarrow P \longrightarrow \bigoplus F_p[1]$$

Φ is given as the unique morphism which acts as the identity and the inclusion on each summand of $\bigoplus F_p$:

$$\Phi_n = \begin{pmatrix} id_{F_n} \\ -i_n \end{pmatrix}$$

In the definition of a homotopy projective complex we have required that each quotient is strictly projective. If this was true, then these objects would be ill-behaved in the homotopy category. We can weaken this assumption to (F3') the quotient F_{n+1}/F_n is homotopy equivalent to a projective complex.

Lemma 3.1.20. *If P is the colimit of a filtration admitting (F2) and (F3'), then P is homotopy projective.*

Proof. Let $\{F_n\}$ denote the filtration on P. To show that P is homotopy projective is to find a homotopy equivalence P' such that P' is the homotopy colimit of a filtration admitting (F3).

Suppose that ${}^{F_{n+1}}/_{F_n} \simeq Q_{n+1}$, where each Q_{n+1} is projective. We wish to inductively define a filtration $\{F'_n\}$ which has (F2) and (F3) and a pointwise homotopy equivalence of filtrations $f : \{F_n\} \to \{F'_n\}$. The object P' is the the (homotopy) colimit of the new filtration.

Define $F'_0 = Q_0$, and let $f_0 : F_0 \to F'_0$ be the projection onto Q_0. By assumption f_0 is a homotopy equivalence and we have a commutative square where the vertical arrows are homotopy equivalences. Moreover, each horizontal arrow splits as a graded arrow.

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[d, "0"] & F_0 \arrow[d, "f_0"] \\
0 \arrow[r, "0"] & Q_0
\end{tikzcd}$$

Suppose that we have been able to constructed this filtration up to F'_p. By using our known homotopy equivalences there is an isomorphism of Ext groups:

$$Ext_A({}^{F_p}/_{F_{p-1}}, F_{p-1}) \simeq Ext_A(Q_p, F'_{p-1})$$

Given the triangle consisting of F_{p-1}, F_p and ${}^{F_p}/_{F_{p-1}}$ there is an assosiated triangle with the morphisms as follows:

$$\begin{tikzcd}
F_{p-1} \arrow[r] \arrow[d, "f_{p-1}"] & F_p \arrow[r] \arrow[d, dashed] & {}^{F_p}/_{F_{p-1}}1111 \arrow[r] \arrow[d, "\sim"] & F_{p-1}[1] \arrow[d, "f_{p-1}[1]"] \\
F'_{p-1} \arrow[r] & F'_p \arrow[r] & Q_p \arrow[r] & F'_{p-1}
\end{tikzcd}$$

By the morphism axiom there is a morphism $f_p : F_p \to F'_p$ which is also a homotopy equivalence by the 2-out-of-3 property.

This defines a filtration $\{F'_p\}$, with (F3) and P' as its homotopy colimit. To see that P is homotopy equivalent to P' we use the maps f_p constructed to obtain a homotopy equivalence by the morphism axiom and the 2-out-of-3 property.

$$\begin{tikzcd}
\bigoplus F_p \arrow[r, "\Phi"] \arrow[d, "\oplus f_p"] & \bigoplus F_p \arrow[r] \arrow[d, "\oplus f_p"] & P \arrow[r] \arrow[d, dashed, "\sim"] & \bigoplus F_p[1] \arrow[d, "\oplus f_p[1]"] \\
\bigoplus F'_p \arrow[r, "\Phi'"] & \bigoplus F'_p \arrow[r] & P' \arrow[r] & \bigoplus F'_p[1]
\end{tikzcd}$$

□

The projective complexes are the complexes which are generated by the free module $\hat{A}$ in the sense that they are all in the smallest thick triangulated subcategory of $K(A)$ containing $\hat{A}$. By definition, we may see that the homotopy projective complexes are the complexes in the smallest thick triangulated subcategory of $K(A)$ which is closed under well-ordered homotopy colimits and contains $K(A)$. By devissage we may extend fully-fatihfullness of functors on the set $\{\hat{A}\}$ to the class of homotopy projective objects.

Lemma 3.1.21 (Devissage)**.** *Let $F : \mathcal{T} \to \mathcal{U}$ be a triangulated functor between triangulated categories. Suppose $S \subseteq \mathcal{T}$ is a class of objects closed under shift, and denote $\hat{S}$ for the smallest thick triangulated subcategory (closed under well-ordered homotopy colimits). If $F|_S$ is fully faithful, then $F|_{\hat{S}}$ is fully faithful as well.*

Proof. This is straightforward by using the Yoneda embeddings and 5-lemma. More details may be found in [28]. To get closed under homotopy colimits we also need that F commutes with infinite direct sums, and that the set $\{S\}$ only contains small objects. □

Lemma 3.1.22. *Supppose we have F and S as above. If $F|_S = 0$, then it is 0 on all of $\hat{S}$.*

Proof. The same argument as above, except we have to squeeze out zeros from exact sequences. □

The final ingredient to construct a homotopy projective resolution for our complexes is the acyclic assembly lemma [24].

Lemma 3.1.23 (acyclic assembly)**.** *Suppose that C is a double complex of R-modules. Then $Tot^{\oplus}C$ is acyclic if either:*

- *C is a lower half-plane complex with exact rows.*
- *C is a left half-plane complex with exact columns.*

Proof. This is proposition 2.7.3 in [24]. We omit the proof as the next proof is in some sense very similar. □

Corollary 3.1.23.1. *Suppose that C is a double complex of R-modules such that every column is exact and that the kernels along the rows give rise to exact columns, then $Tot^{\oplus}C$ is acyclic.*

Proof. We want to realize the images along the rows as the coimage along the horizontal differential. Write $Z^n(C)$ for the n-th horizontal kernel and $B^n(C)$ for the n-th horizontal image. We have a short exact sequence of complexes:

$$Z^n(C)^* \longrightarrow C^{n,*} \longrightarrow B^n(C)^*$$

Given that $C^{n,*}$ is acyclic we get that $Z^n(C)^*$ is acyclic if and only if $B^n(C)^*$ is acyclic.

Assuming that all of these three constructions are acyclic we make a filtration on C. Let $F_nC^{p,*} = C$ if $p \in [-n, n-1]$, $F_nC^{n,*} = Z^nC$ and $F_nC^{p,*} = 0$ otherwise.

This filtration is bounded below and exhaustive as colimits commute with colimits.

$$Tot^{\oplus}C = Tot^{\oplus} \varinjlim F_nC \simeq \varinjlim Tot^{\oplus} F_nC$$

We should be a bit careful here as the total complex is not really a coproduct, but since coproducts and cokernels are calculated pointwise we obtain the commutativity.

Now we apply the classical convergence theorem to the filtration to obtain a converging spectral sequence $EF_2C \implies H^*(Tot^{\oplus}C)$. But since we assume each column to be exact in the filtration, the second page is 0, so $H^*(Tot^{\oplus}C) \simeq 0$ as desired. □

Theorem 3.1.24. *Suppose that P is homotopy projective, N is acyclic. Then $K(A)(P, N) \simeq 0$.*

Given any module M, there is a homotopy projective object pM and an acyclic object aM giving rise to a triangle in $K(A)$.

$$pM \longrightarrow M \longrightarrow aM \longrightarrow pM[1]$$

Proof. We assume that $P \simeq \hat{A}$. By a devissage argument we may extend the isomorphism to all homotopy projective P.

$$K(A)(\hat{A}, N) \simeq H^0 Hom^*_A(\hat{A}, N) \simeq H^0 N \simeq 0$$

We want to construct two complexes pM and aM by taking total complexes. We show that aM is acyclic by using 3.1.23.1. To use it we will construct an exact sequence of complexes satisfying the assumptions. Described by MacLane [32] there is an exact structure $\mathcal{E}$ on Mod^R such that the collections on conflations are the short exact sequences such that the kernel functor is exact.

$$L \overset{f}{\rightarrowtail} M \overset{g}{\twoheadrightarrow} N$$

$$Z^*L \overset{Z^*f}{\rightarrowtail} Z^*M \overset{Z^*g}{\twoheadrightarrow} Z^*N$$

Since limits commute with limits, the kernel functor preserves any limit. Thus the kernel is left exact, and its only obstruction for exactness is to preserve cokernels. We may thus characterize the conflations by inflations and deflations, which are monomorphisms and epimorphisms which are preserved by the kernel functor. Mac Lane calls these deflations for proper epimorphisms instead.

Mac Lane also shows that there are enough $\mathcal{E}$-projectives with this exact structure. We want to construct $\mathcal{E}$-projectives be on the form of homotopy projective complexes. $\hat{A}[-n]$ is $\mathcal{E}$-projective by the following isomorphism.

$$Hom_A^\bullet(\hat{A}[-n], M) \simeq Z^0 Hom_A^*(\hat{A}, M[n]) \simeq M^n$$

Define the trivialization $trivM$ of M be the underlying graded module M endowed with a trivial differential. This trivial differential is in some sense the inclusion of graded modules into chain complexes. Thus we have the following isomorphism on hom-sets:

$$Hom_A^\bullet(trivM, trivN) \simeq Hom_A^*(M, N)$$

$triv$ is then well-defined as a functor as every morphism between chain complexes uniquely defines a morphism between their trivializations. By using the isomorphisms from Keller [27] 2.2. we get that:

$$\begin{aligned} Hom_A^\bullet(cone(id_{triv\hat{A}}), M) &\simeq Hom_A^\bullet(cone(id_{triv\hat{A}[-1]})[1], M) \\ &\simeq Hom_A^*(triv\hat{A}, trivM[-1])^0 \simeq Hom_A^*(\hat{A}, M)^{-1} \simeq M^{-1} \end{aligned}$$

This shows that if P is homotopy projective, then P and $cone(id_{trivP})$ are $\mathcal{E}$-projective. To see that there are enough $\mathcal{E}$-projectives pick an arbitrary module M. Since we know that there are enough projectives, let P be a projective such that there is an epimorphism $p : P \to M$. We don't know if this morphism is a deflation, so pick another projective Q such that there is an epimorphism $q : Q \to Z^*M$. Since Z^*M has a trivial differential we know that $d_Q q = 0$. Thus this morphism extends to $q' = \begin{bmatrix} q & 0 \end{bmatrix} : cone(id_{trivQ}) \to M$ such that Z^*q' is an epimorphism. The morphism $\begin{bmatrix} p & q' \end{bmatrix} : P \oplus cone(id_{trivQ}) \to M$ is thus a deflation. $P' = P \oplus cone(id_{trivQ})$ shows that we have enough projectives. Moreover every $cone(id_{trivQ})$ has homotopy type 0, so $P' \simeq P$ in $K(A)$.

Since we have enough $\mathcal{E}$-projective, we may construct a $\mathcal{E}$-projective resolution $P'^{*,*}$ of M in the standard way. See Keller [33] for details. Such resolutions are then double complexes, and the augmented resolution below is $\mathcal{E}$-acyclic.

$$\ldots \longrightarrow P_1' \longrightarrow P_0' \longrightarrow\!\!\!\!\rightarrow M \xrightarrow{0} 0$$

Having an $\mathcal{E}$-acyclic resolution means that each row is exact, and taking kernels along the columns preserve exactness of the rows.

Denote the augmentation of $P'^{*,*}$ by $m : P'^{,*} \to M$. We define the complexes $pM = Tot^{\oplus}(P'^{*,*})$ and $aM = Tot^{\oplus}(cone(m))$.

pM carries a natural filtration F_npM from the double complex structure. Let F_npM be the truncated complex:

$$\ldots \longrightarrow 0 \longrightarrow P'^{n,*} \longrightarrow \ldots \longrightarrow P'^{1,*} \longrightarrow P'^{0,*} \longrightarrow 0 \longrightarrow \ldots$$

The filtration F_npM satisfies F1 and F2 by construction. The quotients ${}^{F_{n+1}pM}/_{f_npM} \simeq P_n'$ which is homotopy equivalent to a projective. By lemma 3.1.20 pM is homotopy projective.

The complex $cone(m)$ satisfies the conditions for 3.1.23.1, thus aM is acyclic. Thus we have a triangle in $K(A)$ as desired. □

Corollary 3.1.24.1. *Let M be an erbitrary module. If P is homotopy projective, then $K(A)(P, M) \simeq K(A)(P, pM)$. If N is acyclic, then $K(A)(M, N) \simeq (aM, N)$.*

a and p are well-defined functors which commutes with infinite direct sums.

Corollary 3.1.24.2. *Let $\{\hat{A}\}$ denote the smallest thick triangulated subcategory of $D(A)$ which is closed under homotopy colimits. Then $D(A) \simeq \{\hat{A}\}$.*

Corollary 3.1.24.3. *Suppose that $f : A \to B$ is a dg-ring homomorphism and a quasi-isomorphism between dg-algebras, then $D(A) \simeq D(B)$.*

Proof. f endows B with both a left and right A-module structure. We will think of $\hat{B}$ as a left A-module and right B module. There is then a natural hom-tensor adjunction between the differential graded enriched categories.

$$\begin{tikzcd} Diff(B) \arrow[r, bend left, "H"] \arrow[r, phantom, "\top"] & Diff(A) \arrow[l, bend left, "T"] \end{tikzcd}$$

We define $T = \_\otimes_A \hat{B}$ and $H = Diff(B)(B, \_)$. We see that $Diff(B)(T\hat{A}, M) \simeq Diff(A)(\hat{A}, HM) \simeq HM \simeq Diff(B)(\hat{B}, M)$. Thus $T\hat{A} \simeq \hat{B}$, and the morphism $T : Diff(A)(\hat{A}, \hat{A}) \to Diff(B)(\hat{B}, \hat{B})$ is given by f. Since we assume f to be a quasi-isomorphism, it follows that $\mathbb{L}T : D(A) \to D(B)$ is fully faithful on the set $\{\hat{A}\}$.

By devissage the functor $\mathbb{L}T$ is fully-faithful on all of $D(A)$, since $D(A)$ is generated by $\hat{A}$. Since T hits all of $D(B)$s generators, $\mathbb{L}T$ is essentially surjective as well. □

Remark 3.1.25. We have ignored smallness conditions for objects. This technique does not always work, since it depends on some unstated isomorphisms which we have used. We have these since the objects $\hat{A}$ and $\hat{B}$ are small. This detail is given more care in Keller [27].

With this result we are able to show that $HoMod^A$ and $HoMod^{\Omega C}$ are equivalent. Since we assumed the morphism $\tau : C \to A$ to be acyclic, we would expect the morphism $f_\tau^* : \Omega C \to A$ to be a quasi-isomorphism. If this is the case, we know that $D(\Omega C) \simeq D(A)$.

3.1.5 The Fundamental Theorem of Twisting Morphisms

In this section we aim to finish what we started the previous section. We will prove a characterization for the acyclic twisting morphisms.

Theorem 3.1.26 (Fundamental Theorem of Twisting Morphisms)**.** *Let $\tau : C \to A$ be a twisting morphism between augmented objects. The following are equivalent:*

1. *τ is acyclic, i.e. the natural transformation $\varepsilon : L_\tau R_\tau \implies Id_{Mod^A}$ is a pointwise quasi-isomorphism.*
2. *The unit transformation $\eta : Id_{CoMod^C} \implies R_\tau L_\tau$ is a pointwise weak equivalence.*
3. *The counit at A is a quasi-isomorphism, i.e. $\varepsilon_A : L_\tau R_\tau A \to A$ is a quasi-isomorphism.*
4. *The unit at $\mathbb{K}$ is a weak-equivalence, i.e. the algebra unit υ_A and coaugmentation υ_C assembles into a weak-equivalence: $\upsilon_A \otimes \upsilon_C : \mathbb{K} \to A \otimes_\tau C$.*
5. *The morphism of algebras $f_\tau : \Omega C \to A$ is a quasi-isomorphism.*
6. *The morphism of coalgebras $g_\tau : C \to BA$ is a weak-equivalence.*

Proof. Notice that 1. is equivalent to 2. since $\mathbb{L}L$ and $\mathbb{R}R$ are quasi-inverse. 3. is a special case of 1. and 4. is a special case of 2. Observe that 5. and 6. are equivalent since the cobar-bar-adjunction is a Quillen equivalence, 2.2.11.1.

We show 3. $\implies$ 1. Let $\mathcal{T} \subseteq D(A)$ be the full subcategory consisting of objects M where ε_M is a quasi-isomorphism. This subcategory is by assumption non-empty and contains A. By the 5-lemma, making triangles (and smallness of A), this subcategory contains the smallest thick triangulated subcategory closed under homotopy colimits which contains A. We know this to be all of $D(A)$.

To show 4. implies 5. we consider the twisting morphism ι_C. Since ι_C is acyclic we know that the counit at A is a quasi-isomorphism.

$$L_{\iota_C} R_{\iota_C} f_\tau^* A \to f_\tau^* A$$

By assumption the unit morphism $\eta_{\mathbb{K}} : \mathbb{K} \to A \otimes_\tau C$ is a weak equivalence, so the morphism $L_{\iota_C}\eta_{\mathbb{K}} : \Omega C \to L_{\iota_C} R_\tau A = L_{\iota_C} R_{\iota_C} f_\tau^* A$ is a quasi-isomorphism. Let ε' denote the counit of $L_{\iota_C} \dashv R_{\iota_C}$, then we see that $f_\tau = \varepsilon'_A \circ L_{\iota_C}\eta_{\mathbb{K}}$, so f_τ is a quasi-isomorphism by the 2-out-of-3 property.

It remains to show that 5. implies 1. Let the counit of $f_{\tau *} \dashv f_\tau^*$ be denoted as $\tilde{\varepsilon}$. Since f_τ is a quasi-isomorphism, f_τ^* descends to an equivalence between the derived categories 3.1.24.3. Thus $\tilde{\varepsilon} : f_{\tau !} f_\tau^* \implies Id$ is a pointwise quasi-isomorphism. Observe that the counit factors as

$$\varepsilon = \tilde{\varepsilon} \circ f_{\tau !}\varepsilon'_{f_\tau^*}$$

By the 2-out-of-3 property it follows that ε is a quasi-isomorphism. □

Corollary 3.1.26.1. *There is one canonical model structure on $CoMod^C$ defined by the acyclic twisting morphisms $\tau : C \to A$, for any algebra A. I.e. each acyclic twisting morphism defines the same model structure for $CoMod^C$.*

Proof. Apply the fundamental theorem of twisting morphisms to the discussion of the last section. □

3.2 Polydules

3.2.1 The Bar Construction

In section 1.3 we saw that we could extend the domain of the bar construction to obtain an equivalence of categories. This converse led us to the definition of an A_∞-algebra, as well as recognizing them as almost free dg-coalgebras. By employing the adjunction $(L_\tau, R_\tau) : CoMod^C \rightleftharpoons Mod^A$ we will do something similar for modules.

Let A be an augmented algebra. The bar construction of A gives us a universal adjunction $(L_{\pi_A}, R_{\pi_A}) : CoMod^{BA} \rightleftharpoons Mod^A$. We will call $R_{\pi_A}(\_[1]) = \_[1] \otimes_{\pi_A} BA$ for B_A, the bar construction on Mod^A. In this manner every A-module M gives rise to an almost free BA-comodule $B_A M$, but does the converse of this construction works?

Let us first look at what B_A does to a module M. $B_A M$ is the dg-comodule which as a graded comodule is the free comodule $M[1] \otimes BA$. The differential of $B_A M$ is given by the A-module structure of M. That is, every elementary element m' of $B_A M$ is an element of M together with a finite string of elements of A.

$$m' = \omega m \otimes (\omega a_1 \otimes ... \otimes \omega a_n)$$

The differential acts on m' by using the differential of $d_{M[1]\otimes BA}$ and multiplication from the right.

$$d_{B_A M}(m') = d_{M[1]\otimes BA}(m') + (-1)^{|m|+|a|}\omega(m \cdot a_1) \otimes (\omega a_2 \otimes ... \otimes \omega a_n)$$

By using delooping, we see that in turn that $d_{B_A M}$ defines an A-module structure for M. We may decompose $B_A M$ as:

$$B_A M = M[1] \oplus M[1] \otimes \bar{A} \oplus M[1] \otimes \bar{A}^{\otimes 2} \oplus ...$$

Let $\pi_M : R_{\pi_A} M \to M$ be the map which kills anything not on the form m. We denote $(d_{B_A M})_i$ by $d_{B_A M} \circ \iota_i$, where $\iota_i : M[-1] \otimes \bar{A}^{\otimes i-1} \hookrightarrow B_A M$. Proposition 1.1.33 tells us that we may recover the structure of M from the differential $d_{B_A M}$. This is done by conjugating the components of $d_{B_A M}$ with desuspension and applying projections appropriately. We recover the maps as:

1. The differential of M is $d_M = s \circ \pi_{M[1]} \circ (d_{B_A M})_1 \omega$
2. The right multiplication from A is $\mu_M = s \circ \pi_{M[-1]} \circ (d_{B_A M})_2 \circ \omega^{\otimes 2}$
3. For $i \geqslant 3$ we have $0 = s \circ \pi_{M[1]} \circ (d_{B_A M})_i \circ \omega^{\otimes i}$

Now, let $\widetilde{N}$ be an almost free BA-comodule. That is, $\widetilde{N} = N[1] \otimes BA$ as a graded comodule. We would now wish for that N carries an A-module structure. Unfortunately we are not that lucky, however, this defines a notion of A_∞-module to the algebra A. If we try to recover the same structure we obtain the following structure morphisms for N:

$$\begin{aligned} \text{A differential of degree } 1&: m_1 = d_N = s \circ \pi_N (d_{\widetilde{N}})_1 \circ \omega \\ \text{A 2-ary operation of degree } 0&: m_2 = s \circ \pi_N (d_{\widetilde{N}})_2 \circ \omega^{\otimes 2} \\ \text{A 3-ary operation of degree } -1&: m_3 = s \circ \pi_N (d_{\widetilde{N}})_3 \circ \omega^{\otimes 3} \\ \text{A 4-ary operation of degree } -2&: ... \end{aligned}$$

Let $\widetilde{m}_i$ be the looped versions of the m_is. Then the sum $\sum \widetilde{m}_i : \widetilde{N} \to N$ extends to $d_{B_A M}$. Since $d^2_{B_A M} = 0$ we get the relations (rel_n) defined in section 1.3 imposed on the morphisms m_i.

To summarize the datum of $M[1] \otimes BA$ as a BA-comodule is equivalent to a chain complex M having maps

$$m_i : M \otimes \bar{A}^{\otimes i-1} \to M$$

of degree $2 - i$ for any $i \geqslant 2$. The maps should satisfy the relations:

$$(rel_n) \qquad \partial(m_n) = - \sum_{\substack{n=p+q+r \\ k=p+1+r \\ k>1, q>1}} (-1)^{pq+r} m_k \circ_{p+1} m_q$$

This gives M the structure of an A-module, where associativity is only well-defined up to strong homotopy. In other words, m_3 is a homotopy for the associator for m_2, m_4 is a homotopy for m_3s associator and so on. Following Lefevre-Hasegawa [12], we call the chain-complex M an A-polydule, given it has maps m_i as above.

We have defined the objects of a category Mod^A_∞. Our goal is to have that the converse bar construction defines an equivalence of categories, i.e. B_A extends to a functor $B_A : Mod^A_\infty \to$

$CoMod^{BA}$ is fully-faithful. This makes sense as every A-module M is a non-full A-polydule by letting $m_1 = d_M$, $m_2 = \mu_M$ and $m_i = 0$ for any $i \geqslant 3$.

Since we say that B_A is fully-faithful any ∞- morphisms between polydules is determined by a morphism of almost free BA-comodules. Let $f : M \rightsquigarrow N$ be an ∞-morphism, then $B_A f : B_A M \to B_A N$ is the associated morphism of dg-comodules. By cofreeness, proposition 1.1.28, we see that Bf is uniquely determined by morphisms $f_i : M \otimes \bar{A}^{\otimes i-1} \to N$ of degree $1-i$. Since we know that $\partial Bf = 0$, we obtain the relations:

$$(rel_n) \qquad \sum_{p+q+r=n} (-1)^{pq+r} f_{p+1+r} \circ_{p+1} m_q^M = \sum_{p+q=n} m_{p+1}^N \circ_1 f_q$$

If $f : M \rightsquigarrow N$ and $g : N \rightsquigarrow P$, then their composition is:

$$(gf)_n = \sum_{p+q=n} g_{p+1} \circ_1 f_q$$

This defines the morphisms of Mod_∞^A. By definition $B_A : Mod_\infty^A \to CoMod^{BA}$ is fully-faithful. An ∞-morphism f is called strict if $f_i = 0$ for any $i \geqslant 2$. We denote the restriction of Mod_∞^A to strict ∞-morphisms as $Mod_{\infty,strict}^A$. Observe that we obtain an equivalence of categories $Mod^A \simeq Mod_{\infty,strict}^A$.

We will give some examples of A-polydules given an augmented algebra A.

3.2.2 Polydules of SHA-algebras

In the last section we developed the notion of a polydule to an augmented/unital algebra. By applying the converse of the bar construction, we are able to extend this notion to A_∞-algebras.

Suppose that A is an A_∞-algebra. By the bar construction, BA is an almost free coalgebra. In the same manner, we may consider the almost free dg-coalgebras of $CoMod^{BA}$. This is again the collections of comodules of the form $M' = M[1] \otimes BA$. Since there is no obstruction to the above arguments the differential $d_{M'}$ is determined by a collection of morphisms $m_n^M : M \otimes A^{\otimes n-1} \to M$ satisfying (rel_n). Moreover, an ∞-morphism $f : M \rightsquigarrow N$ is a collection of morphisms $f_n : M \otimes A^{\otimes n-1} \to N$ satisfying (rel_n).

Definition 3.2.1. Let A be an A_∞-algebra. The category Mod_∞^A has A-polydules as objects and ∞-morphisms as morphisms.

The quasi-isomorphisms in Mod_∞^A are the ∞-morphisms f such that f_1 is a quasi-isomorphism.

Remark 3.2.2. The isomorphisms of Mod_∞^A are the ∞-morphisms f where f_1 is an isomorphism.

We say that an ∞-morphism is strict if $f_i = 0$ for any $i \geqslant 2$. The category $Mod_{\infty,strict}^A$ is the non-full subcategory of Mod_∞^A restricted to strict ∞-morphisms.

We may also lift homotopies between almost free BA-comodules and A-polydules. A homotopy $B_A h : B_A M \to B_A M$ is a morphism of degree -1. Thus the collection $h_n : M \otimes A^{\otimes n-1} \to N$ has morphisms of degree $-i$. Moreover, $h : M \rightsquigarrow N$ defines a homotopy of $f, g : M \rightsquigarrow N$ if we have

$$f_n - g_n = \sum_{p+q} (-1)^p m^N_{p+1} \circ_1 h_q - \sum_{p+q+r=n} (-1)^{pq+r} h_{p+1} \circ_{p+1} m^M_q$$

Suppose now that A is instead a strictly unital A_∞-algebra (1.3.12). We may define strictly unital A-polydules as an A-polydule M such that

$$\begin{aligned} m^M_2 \circ (id_M \otimes v_A) &= id_M \\ \forall i \geqslant 3 \quad m^M_i \circ (id_M \otimes ... \otimes v_A \otimes ... \otimes id_A) &= 0 \end{aligned}$$

An ∞-morphism $f : M \rightsquigarrow N$ is strictly unital if

$$\forall i \geqslant 2 \quad f_i(id_M \otimes ... \otimes v_A \otimes ... \otimes id_A) = 0$$

This definition also extends to homotopies. We may then define the categories of strictly unital polydules $iMod^A_\infty$ and $iMod^A_{\infty,strict}$.

Given an augmented A_∞-algebra A (1.3.13) we obtain an equivalence of categories. Recall that the categories Alg_∞ and $Alg_{\infty,+}$ were equivalent by taking the kernel of augmentation and applying the free augmentation as its quasi-inverse. In the same manner, given a strictly unital A-polydule M, then it defines a strictly unital $\bar{A}$-polydule $\bar{M}$ by restricting the structure maps to $\bar{A}^{\otimes n}$. This defines an equivalence of categories.

$$\begin{tikzcd} iMod^A_\infty \arrow[r, bend left, "\bar{-}"] & Mod^{\bar{A}}_\infty \arrow[l, bend left, "-^+"] \end{tikzcd}$$

We may call its quasi-inverse for the free strict unitization. This takes an $\bar{A}$-polydule M and turns it into a strictly unital A-polydule by defining the structure morphism as 0 on the unit.

We will for now restrict our attention to augmented A_∞-algebras. The reason for this is that if A is an arbitrary A_∞-algebra, then studying Mod^A_∞ would be the same as studying $iMod^{A^+}_\infty$. We extend the bar construction along this equivalence to a fully faithful functor $B_A : iMod^A_\infty \to CoMod^{BA}$.

3.2.3 Universal Enveloping Algebra

Given any augmented A_∞-algebra A, there is a universal enveloping algebra UA. This algebra is universal in the sense that given any algebra A' and an ∞-morphism $B \to A$, then this factors through UA by an algebra map $B \to UA$. By the cobar-bar adjunction there is really only one way to define this algebra.

Definition 3.2.3. Let A be an A_∞-algebra. The universal enveloping algebra is the algebra defined by ΩBA.

Remark 3.2.4. In this definition we have used the extended bar construction to A_∞-algebras and the cobar construction on dg-coalgebras.

Lemma 3.2.5. *There is an isomorphism of categories* $i : Mod^{UA} \to iMod^{BA}_{\infty,strict}$ *given by delooping.*

Proof. This is immediate by the definition of a UA-module. To have a UA-module SM we must have structure maps $m_i^M : M \otimes A^{\otimes i-1} \to M$ of degree $2-i$ for any $i \geqslant 2$. Unwinding this definition and using the adjunction data establishes this isomorphism. □

3.2.4 The Derived Category $D_\infty A$ of Augmented A_∞ Algebras

In this section we wish to define the derived category of strictly unitary polydules to an augmented A_∞-algebra. If $[Qis]$ denote the class of ∞-quasi-isomorphisms, we want the derived category to be the localization at ∞-quasi-isomorphisms, e.g.

$$\mathcal{D}_\infty A = iMod^A_\infty[Qis^{-1}].$$

Like in the case of algebras, we may understand the quasi-isomorphisms better. The category $iMod^A_\infty$ is not complete, but we may in the same sense give it a model structure without limits. Within this structure we already know that every object is cofibrant, and the goal is to show that every object is fibrant as well. This will allow us to lift every ∞-quasi-isomorphism to a homotopy equivalence. With this we may see that the localization from $K_\infty A \to D_\infty A$ is given by the identity.

Within the category $iMod^A_\infty$ we define three classes of morphisms:

- $f \in Ac$ is a weak equivalence if f_1 is a quasi-isomorphism.
- $f \in Cof$ is a cofibration if f_1 is a monomorphism.
- $f \in Fib$ is a fibration if f_1 is an epimorphism.

Theorem 3.2.6. *The category* $iMod^A_\infty$ *is a model category without enough limits. Moreover, every object are bifibrant.*

Proof. This is more or less identical to the proof of 2.3.1. □

Corollary 3.2.6.1. *Homotopy equivalence defined in* $iMod^A_\infty$ *is an equivalence relation, and every* ∞*-quasi-isomorphism is a homotopy equivalence.*

If A is an ordinary associative algebra, then Mod^A fully-faithfully embeds into $iMod_\infty^A$. This inclusion defines an equivalence:

$$DA \simeq {}^{Mod^A}\!/\!_{\sim_\infty}$$

$iMod_\infty^A$ admits a "model" structure, but we want this model structure to respect the model structure of the category $CoMod_{conil}^{BA}$. In other words we want the functor $B_A : iMod_\infty^A \to CoMod_{conil}^{BA}$ to preserve and reflect the model structure of both categories.

Lemma 3.2.7. *Let M be an object of $iMod_\infty^A$. The unit $BM \to RLBM$ induces a quasi-isomorphism on the primitives.*

Proof. Fix this proof by using a double spectral sequence argument. □

Proposition 3.2.8. *Let M and M' be objects of $iMod_\infty^A$, together with an ∞-morphism $f : M \to M'$.*

- *f is an ∞-quasi-isomorphism if and only if Bf is a weak equivalence.*
- *f is a fibration if and only if Bf is a fibration.*
- *f is a cofibration if and only if Bf is a cofibration.*

Proof. Recall from theorem 3.1.8 that the morphism $\iota_{BA} : BA \to UA$ is an acyclic twisting morphism. Thus the adjoint pair $(L_{\iota_{BA}}, R_{\iota_{BA}})$ defines a Quillen equivalence.

We show only the first bullet point. The last two are identical to the proof of 2.2.16.

If f_1 is a quasi-isomorphism, then Bf is a filtered quasi-isomorphism by definition. So suppose that Bf is a weak equivalence instead. The unit transformation gives us a natural square.

$$\begin{tikzcd} BM \arrow[r] \arrow[d, "Bf"] & RLBM \arrow[d, "RLBf"] \\ BM' \arrow[r] & RLBM' \end{tikzcd}$$

In this case $R = Bi$, so this diagram is in the image of B. We pull it back by functoriality.

$$\begin{tikzcd} M \arrow[r] \arrow[d, "f"] & iLBM \arrow[d, "iLBf"] \\ M' \arrow[r] & iLBM' \end{tikzcd}$$

Since Bf is a weak equivalence, we know that $iLBf$ is an ∞-quasi-isomorphism.

By the above proposition the horizontal maps are ∞-quasi-isomorphisms as well. □

Associated to each augmented A_∞-algebra there is also a homotopy category. Since homotopy equivalence $\sim_\infty$ in $iMod_\infty^A$ defines a congruence relation we may construct the homotopy category $K_\infty A$.

Corollary 3.2.8.1. *The localization $K_\infty A \to D_\infty A$ is given by the identity. Moreover, $K_\infty A = D_\infty A$.*

Remark 3.2.9. The name homotopy category comes from homological algebra and has a priori nothing to do with the homotopy categor $Ho(iMod_\infty^A)$. However, in this particular case these naming conventions coincide.

Lemma 3.2.10. *The composition $J : Mod^{UA} \to iMod_{\infty,strict}^A \to iMod_\infty^A$ given by $J = \iota \circ i$, the equivalence then non-full inclusion is induces an equivalence of categories:*

$$DUA \simeq D_\infty A.$$

Proof. Consider the commutative square:

$$\begin{tikzcd} Mod^{UA} \arrow[r, "i"] \arrow[d, "R_{\iota_{BA}}"] & iMod_{\infty,strict}^A \arrow[d, "\iota"] \\ CoMod^{BA} & iMod_\infty^A \arrow[l, "B"] \end{tikzcd}$$

Since the three functors $R_{\iota_{BA}}$, i and B all induces equivalences on the derived categories, then ι has to as well. □

To summarize, we have established an equivalence between 4 different categories:

- $D_\infty A$, derived category of A
- $iMod_{\infty,strict}^A[Qis^{-1}]$, derived category of A with only strict morphisms
- DBA, derived category of BA as a dg-coalgebra
- DUA, derived category of universal enveloping algebra

In this sense, we may also see that in the derived category, all of the higher homotopic data of each morphism have been collapsed by the homotopy.

The triangulated structure on $D_\infty A$ may be lifted along these equivalences making them triangulated as well. Note that $R_{\iota_{BA}}$ is already triangulated, and forcing there is only one way of forcing the triangulated structure on $iMod_\infty^A$. Since $iMod_\infty^A$ isn't complete it isn't easy to obtain a description of the triangles along any ∞-morphism f. However, this problem does not appear in $iMod_{\infty,strict}^A$, so one should think of only strict morphisms instead, but in this case we are already working in the category Mod^{UA}.

3.3 The Derived Category $D_\infty A$

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Appendix A

Simplicial Objects

A.1 The Simplex Category

The simplex category is in some sense the categoryfication of the standard topological simplices, Δ^n. This category carries the necessary data in order to define concepts such as homology or homotopy. This section will give a brief review of this category.

Definition A.1.1 (The simplex category)**.** The simplex category Δ consists of ordered sets $[n] = \{0, ..., n\}$ for any $n \in \mathbb{N}$. A morphism $f \in \Delta([m], [n])$ is a monotone function, i.e.

$$a \leqslant b \in [m] \implies f(a) \leqslant f(b) \in [n].$$

Definition A.1.2 (The augmented simplex category)**.** Δ_+ is called the augmented simplex category, where we add an initial object $[-1] = \varnothing$.

Definition A.1.3 (The reduced simplex category)**.** Δ_inj is called the reduced simplex category. The morphisms consists only of the injective morphisms in Δ.

Inspired from the topological simplices, the simplex category has coface and codegeneracy morphisms. The coface maps are the injective morphisms $\delta_i : [n] \to [n+1]$, while the codegeneracy maps are the surjective morphisms $\sigma_i : [n] \to [n-1]$.

$$\delta_i(k) = \begin{cases} k, \text{ if } k<i \\ k+1, \text{ otherwise} \end{cases} \qquad \sigma_i(k) = \begin{cases} k, \text{ if } k\leqslant i \\ k-1, \text{ otherwise} \end{cases}$$

Proposition A.1.4. *Every morphism in Δ factors into coface and codegeneracy maps.*

Proof. Prop ?? in [10]. □ Find result

This result tells us that understanding how these morphisms work in tandem will be very important in understanding the simplex category. Luckily, there are five identites which characterize these maps. These are called the cosimplical identites.

1. $\delta_j \delta_i = \delta_i \delta_{j-1}$, if $i < j$
2. $\sigma_j \delta_i = \delta_i \sigma_{j-1}$, if $i < j$
3. $\sigma_j \delta_i = id$, if $i = j$ or $i = j+1$
4. $\sigma_j \delta_i = \delta_{i-1} \sigma_j$, if $i > j+1$
5. $\sigma_j \sigma_i = \sigma_i \sigma_{j+1}$, if $i \leqslant j$

If we want a more visual description of the simplex category, we may think of them in this manner. An inductive tower with a greatly increasing amount of morphisms.

$$[-1] \longrightarrow [0] \overset{\delta_i}{\rightrightarrows} [1] \overset{\delta_i}{\rightrightarrows} [2] \overset{\delta_i}{\rightrightarrows} \cdots$$

$$[-1] \qquad [0] \xleftarrow{\sigma_0} [1] \overset{\sigma_i}{\leftleftarrows} [2] \overset{\sigma_i}{\leftleftarrows} \cdots$$

The augmented simplex category has a universal monoid. Let $+ : \Delta_+ \times \Delta_+ \to \Delta_+$ be the functor acting on objects and morphisms as:

$$[m] + [n] = [m + n + 1]$$
$$(f+g)(k) = \begin{cases} f(k), \text{ if } k \leqslant m \\ g(k)+m, \text{ otherwise} \end{cases}$$

$(\Delta_+, +, [-1])$ becomes a monoidal category. Unitality is satisfied as $[-1] + [m] = [1 + m - 1] = [m] = [m] + [-1]$. Associativity follows from associativity of addition. Since addition acts on morphisms by juxtaposition we get that the maps $id_[0] : [0] \to [0]$, $\delta_0 : [-1] \to [0]$ and $\sigma_0 : [1] \to [0]$ allows us to express any morphism in Δ by summing them.

Since the object $[0]$ is terminal, it automatically becomes a monoid in $(\Delta, +, [-1])$. The unit is the unique map $\delta_0 : [-1] \to [0]$, and the multiplication is the uniqe map $\sigma_0 : [1] \to [0]$. Associativity and unitality is automatically satisfied by uniqueness of any morphism $f : [n] \to [0]$.

Proposition A.1.5. *Let $(\mathcal{C}, \otimes, Z)$ be a monoidal category. If (C, η, μ) is a monoid in $\mathcal{C}$, then there is a strong monoidal functor* $: \Delta_+ \to \mathcal{C}$*, such that $F[0] \simeq C$, $F\delta_{-1} \simeq \eta$ and $F\sigma_0 \simeq \mu$.*

Proof. This is proved in Mac Lanes book [10]. □

A.2 Simplicial Objects

To exert the properties of the simplex category on another category $\mathcal{C}$, we look at functors from Δ into $\mathcal{C}$.

Definition A.2.1 (Simplical object)**.** A simplicial object in $\mathcal{C}$ is a functor $S : \Delta^{op} \to \mathcal{C}$.

Such an object may be viewed as a collection of objects $\{S_n\}_{n\in\mathbb{N}}$ together with face maps $d^i : S_n \to S_{n-1}$ and degeneracy maps $s^i : S_n \to S_{n+1}$. Additionally, these maps must satisfy the simplicial identites. This is the dual to the cosimplical identites.

Definition A.2.2 (Augmented simplical object)**.** An augmented simplicial object is then a functor $S : \Delta_+^{op} \to \mathcal{C}$.

The restricted functor $\bar{S} : \Delta^{op} \to \mathcal{C}$ is called the augmentation ideal of S.

Definition A.2.3 (Semi-simplicial object)**.** A semi-simplicial object is a functor $S : \Delta_{inj} \to \mathcal{C}$.

Observe that a semi-simplicial object may be considered as a collection of objects $\{S_n\}$ such that we only have face maps satisfying the 1st simplicial identity.

Definition A.2.4 (cosimplical object)**.** A cosimplicial object is a functor $S : \Delta \to \mathcal{C}$.

Such an object may be regarded as a collection of objects together with coface and codegeneracy maps satisfying the cosimplicial identities.

Simplicial objects are studied across many different fields of mathematics.

Example A.2.5 (Simplicial sets). A simplicial set S is a collection of sets together with face and degeneracy maps. This is a functor $S : \Delta^{op} \to$ Set. The category of simplicial sets is usually denoted as sSet or Set$_\Delta$.

Simplicial sets are important in ∞-category theory. Some special simplicial sets called quasicategories defines a model for $(\infty, 1)$-categories. A level up, simplicially enriched categories gives us a model for $(\infty, 2)$-categories.

Example A.2.6 (The standard topological n-simplex). The topological n-simplex Δ^n is a topological space. Abstracting away the n we get a functor $\Delta - : \Delta \to$ Top. In this manner, the collection of standard n-simplicies is a cosimplical object of Top.

Example A.2.7 (Rings). Any ring R is by definition a monoid in the category of abelian groups. By the above proposition, this monoid is uniquely determined by a strong monoidal functor $R : \Delta_+ \to$ Ab. Thus any ring is a cosimplical object of Ab.

An important result by Dold and Kan classyfies which chain complexes are simplicial objects in the category Mod$_\mathbb{K}$.

Theorem A.2.8 (Dold-Kan Correspondance)**.** *For any abelian category $\mathcal{A}$ there is an equivalence of categories*

$$\begin{tikzcd} Fun(\Delta^{op}, \mathcal{A}) \arrow[r, bend right, "\Gamma"'] \arrow[r, phantom, "\perp"] & Ch^{-}(\mathcal{A}) \arrow[l, bend right, "N"'] \end{tikzcd}.$$

Example A.2.9 (Nerve of a chain complex)*.* The above result states that every non-positive cochain complex may be thought of as a simplical object of $\mathrm{Mod}_{\mathbb{K}}$.

eg dette?

A.3 Simplicial Homotopy Theory

Appendix B

Spectral Sequences

B.1 Filtrations

B.2 Spectral Sequence

Appendix C

Symmetric Monoidal Categories