Lambda-Calculus (II)

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Plan

- local confluency
- Church Rosser theorem
- Redexes and residuals
- Finite developments theorem
- Standardization theorem

Confluency

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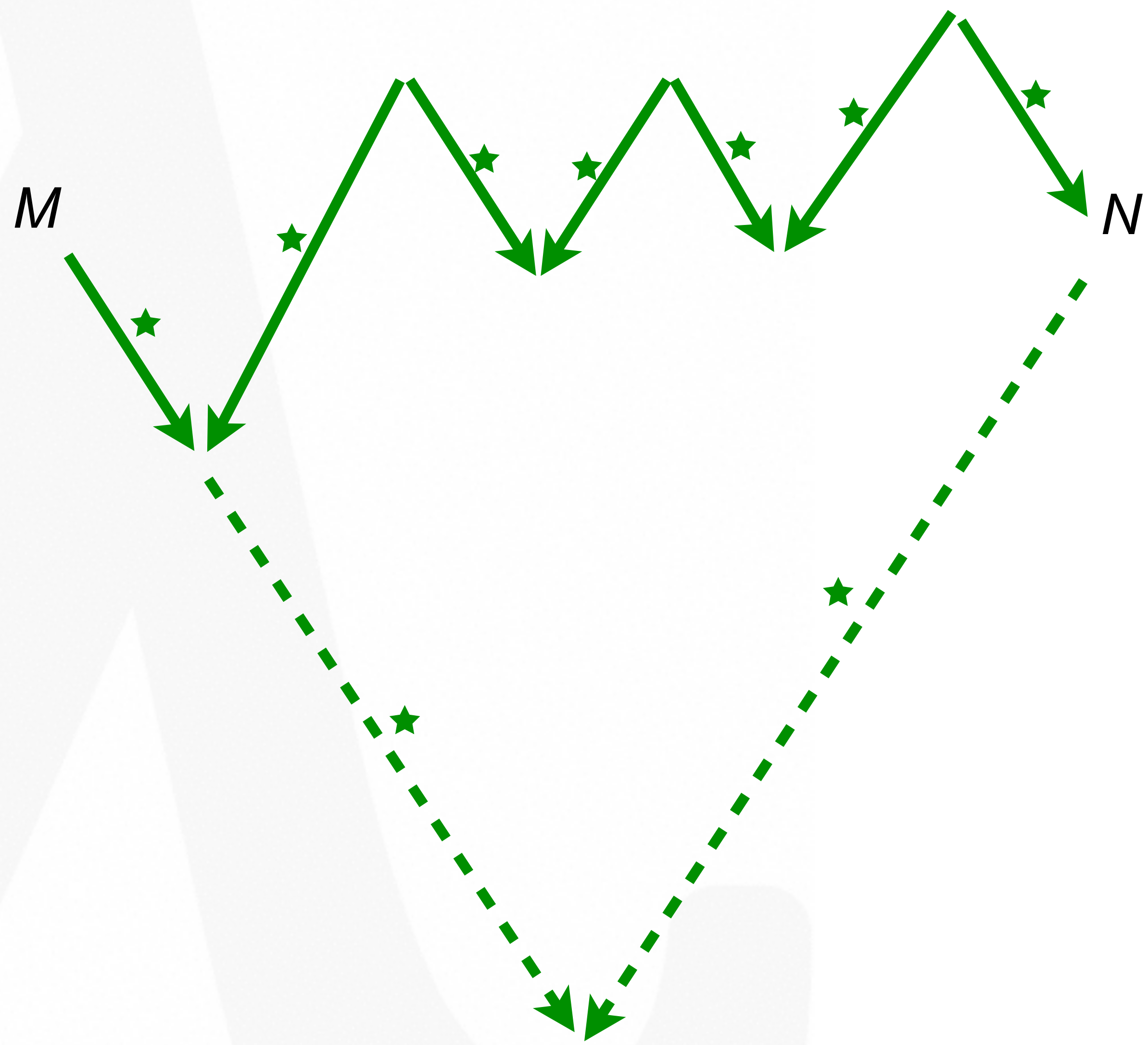
Consistency

Question: Can we get $M \xrightarrow{*} 2$ and $M \xrightarrow{*} 3$??

Consequence: $2 =_\beta 3$!!

Confluency

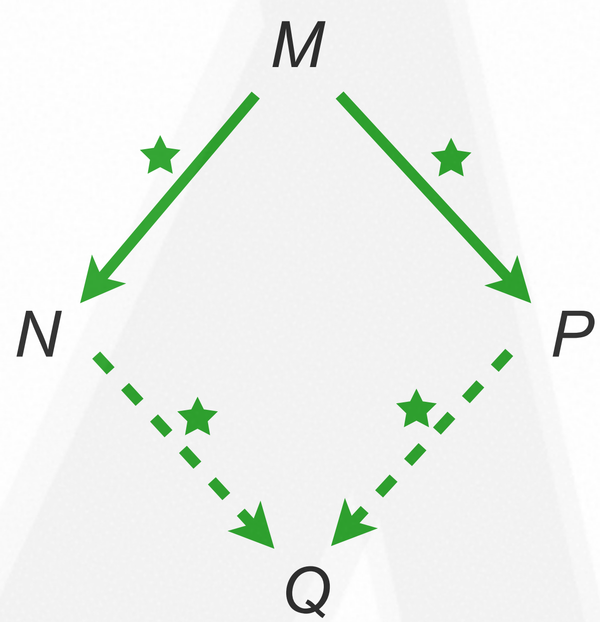
Question: If $M =_\beta N$, then $M \xrightarrow{\star} P$ and $N \xrightarrow{\star} P$ for some P ??

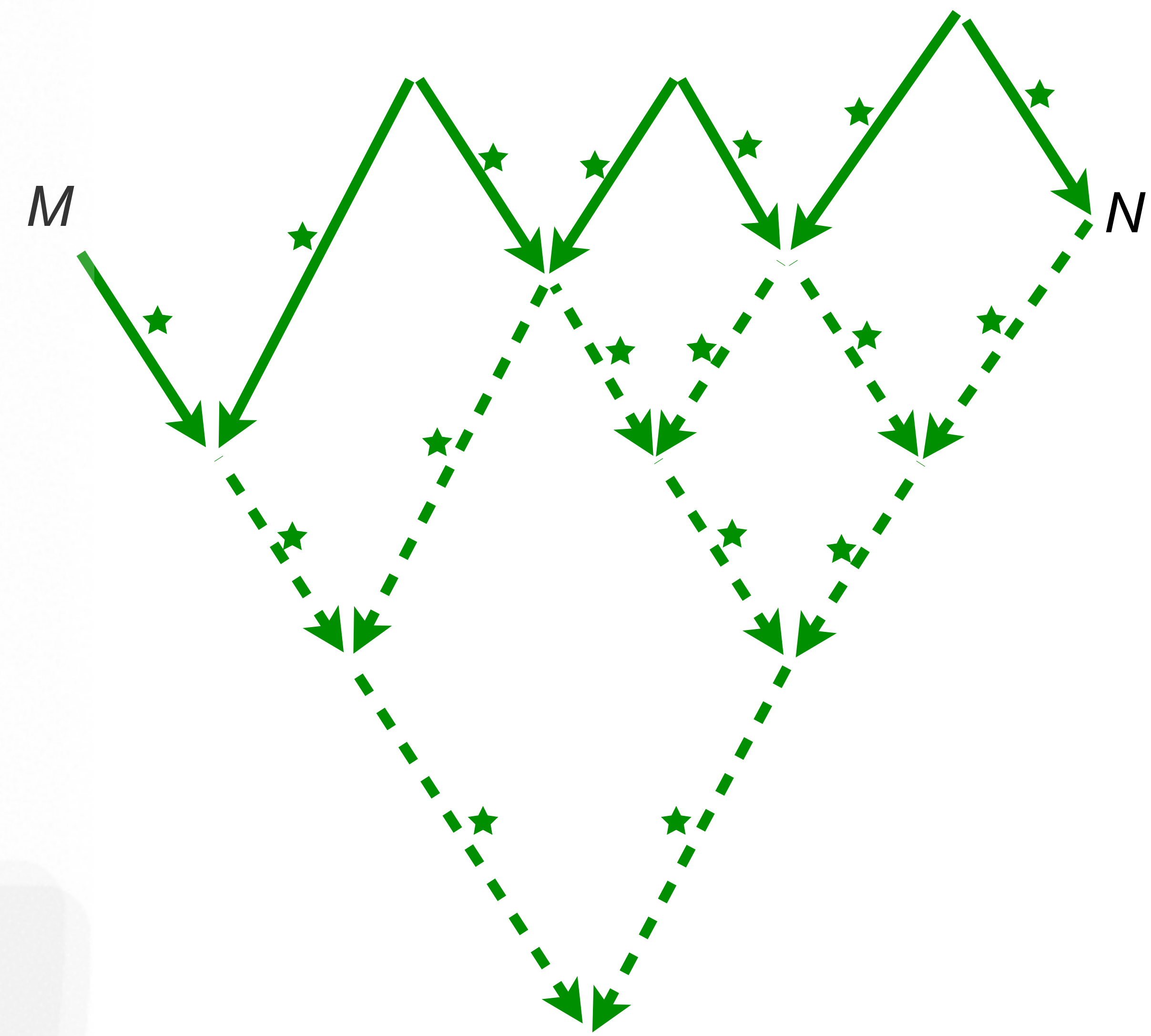


Then impossible to get $2 =_\beta 3$

Confluency

Question: If $M \xrightarrow{\star} N$ and $M \xrightarrow{\star} P$, then $N \xrightarrow{\star} Q$ and $P \xrightarrow{\star} Q$ for some Q ?





Corollary: [unicity of normal forms]

If $M \xrightarrow{\star} N$ in normal form and $M \xrightarrow{\star} N'$ in normal form, then $N = N'$.

Confluency

Goal: If $M \xrightarrow{*} N$ and $M \xrightarrow{*} P$, there is Q such that $N \xrightarrow{*} Q$ and $P \xrightarrow{*} Q$

$$\begin{tikzcd} & M \arrow[dl, "*"'] \arrow[dr, "*"] & \\ N \arrow[dr, dashed, "*"'] & & P \arrow[dl, dashed, "*"] \\ & Q & \end{tikzcd}$$

How to prove confluency ?

Local confluency

- **Theorem 1:** If $M \longrightarrow N$ and $M \longrightarrow P$ there is Q such that $N \xrightarrow{*} Q$ and $P \xrightarrow{*} Q$

$$\begin{tikzcd}
 & M \arrow[dl] \arrow[dr] & \\
N \arrow[dr, dashed, "*"'] & & P \arrow[dl, dashed, "*"] \\
 & Q &
\end{tikzcd}$$

- Example:

$$\begin{tikzcd}
(\lambda x.xx)(Iz) \arrow[r] \arrow[dr] & (\lambda x.xx)z \arrow[dr] & \\
 & Iz(Iz) \arrow[r, "*"] & zz
\end{tikzcd}$$

where $I = \lambda x.x$

- **Lemma 1:** $M \longrightarrow N$ implies $P\{x := M\} \xrightarrow{*} P\{x := N\}$

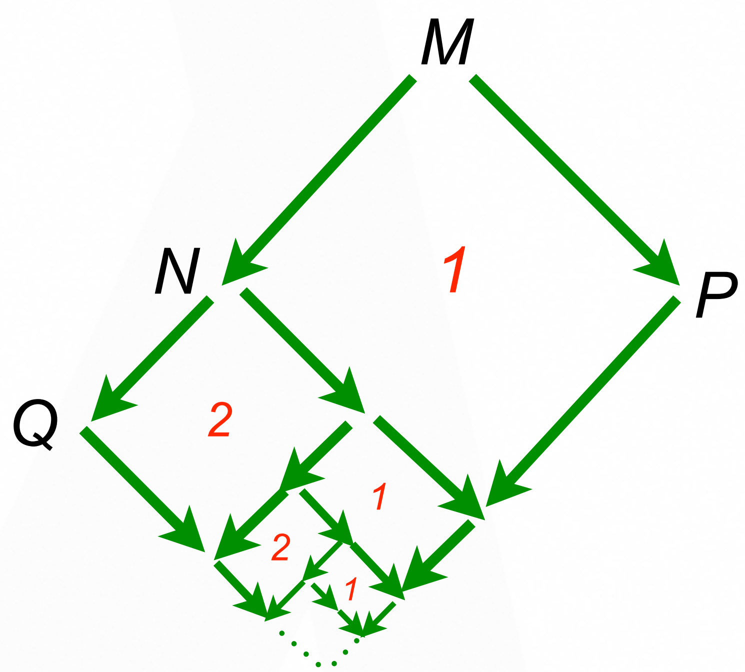
- **Lemma 2:** $M \longrightarrow N$ implies $M\{x := P\} \longrightarrow N\{x := P\}$

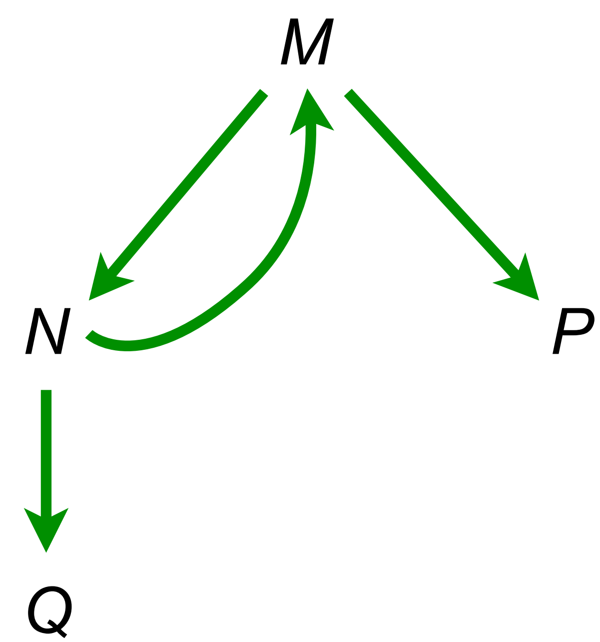
- **Substitution lemma:** $M\{x := N\}\{y := P\} = M\{y := P\}\{x := N\{y := P\}\}$

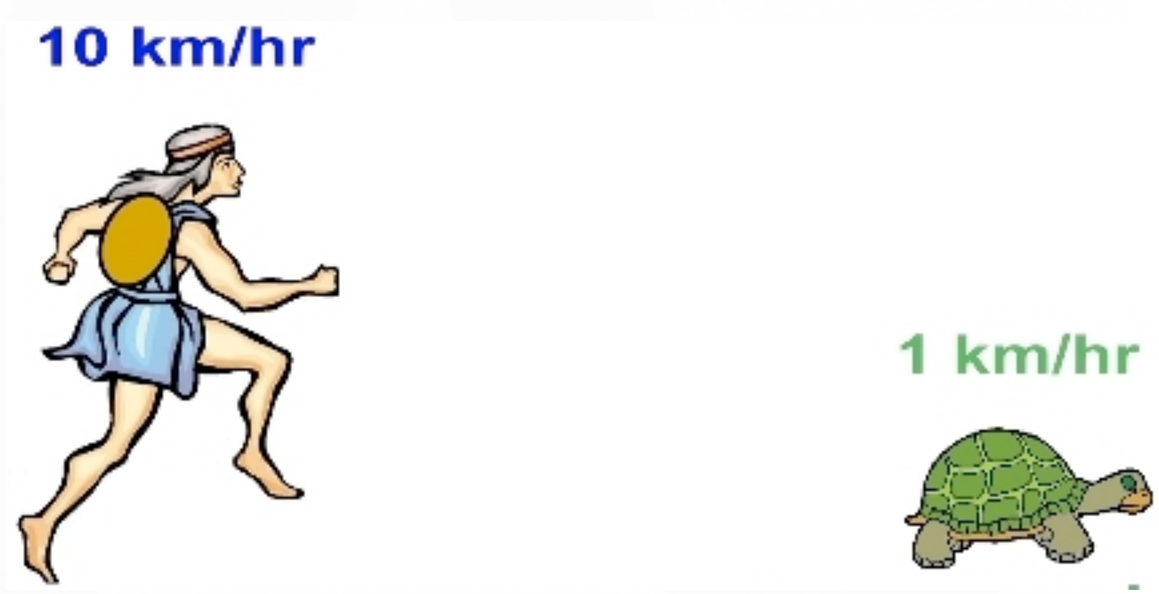
  when x not free in P

Confluency

- **Fact:** local confluency does not imply confluency







Confluency

We define $\twoheadrightarrow\!\!\!\!\!\!/\!/$ such that $\rightarrow \subset \twoheadrightarrow\!\!\!\!\!\!/\!/ \subset \xrightarrow{*}$

- **Definition [parallel reduction]:**

[Var Axiom] $x \twoheadrightarrow\!\!\!\!\!\!/\!/ \; x$ [Const Axiom] $c \twoheadrightarrow\!\!\!\!\!\!/\!/ \; c$

[App Rule] $\dfrac{M \twoheadrightarrow\!\!\!\!\!\!/\!/ \; M' \quad N \twoheadrightarrow\!\!\!\!\!\!/\!/ \; N'}{MN \twoheadrightarrow\!\!\!\!\!\!/\!/ \; M'N'}$ [Abs Rule] $\dfrac{M \twoheadrightarrow\!\!\!\!\!\!/\!/ \; M'}{\lambda x.M \twoheadrightarrow\!\!\!\!\!\!/\!/ \; \lambda x.M'}$

[//Beta Rule] $\dfrac{M \twoheadrightarrow\!\!\!\!\!\!/\!/ \; M' \quad N \twoheadrightarrow\!\!\!\!\!\!/\!/ \; N'}{(\lambda x.M)N \twoheadrightarrow\!\!\!\!\!\!/\!/ \; M'\{x := N'\}}$

- Example:

$$\dfrac{\dfrac{x \twoheadrightarrow\!\!\!\!\!\!/\!/ \; x \quad z \twoheadrightarrow\!\!\!\!\!\!/\!/ \; z}{Iz \twoheadrightarrow\!\!\!\!\!\!/\!/ \; z} \quad \dfrac{x \twoheadrightarrow\!\!\!\!\!\!/\!/ \; x \quad z \twoheadrightarrow\!\!\!\!\!\!/\!/ \; z}{Iz \twoheadrightarrow\!\!\!\!\!\!/\!/ \; z}}{Iz(Iz) \twoheadrightarrow\!\!\!\!\!\!/\!/ \; zz}$$

$I = \lambda x.x$

Confluency

- Goal is to prove **strongly local confluency:**

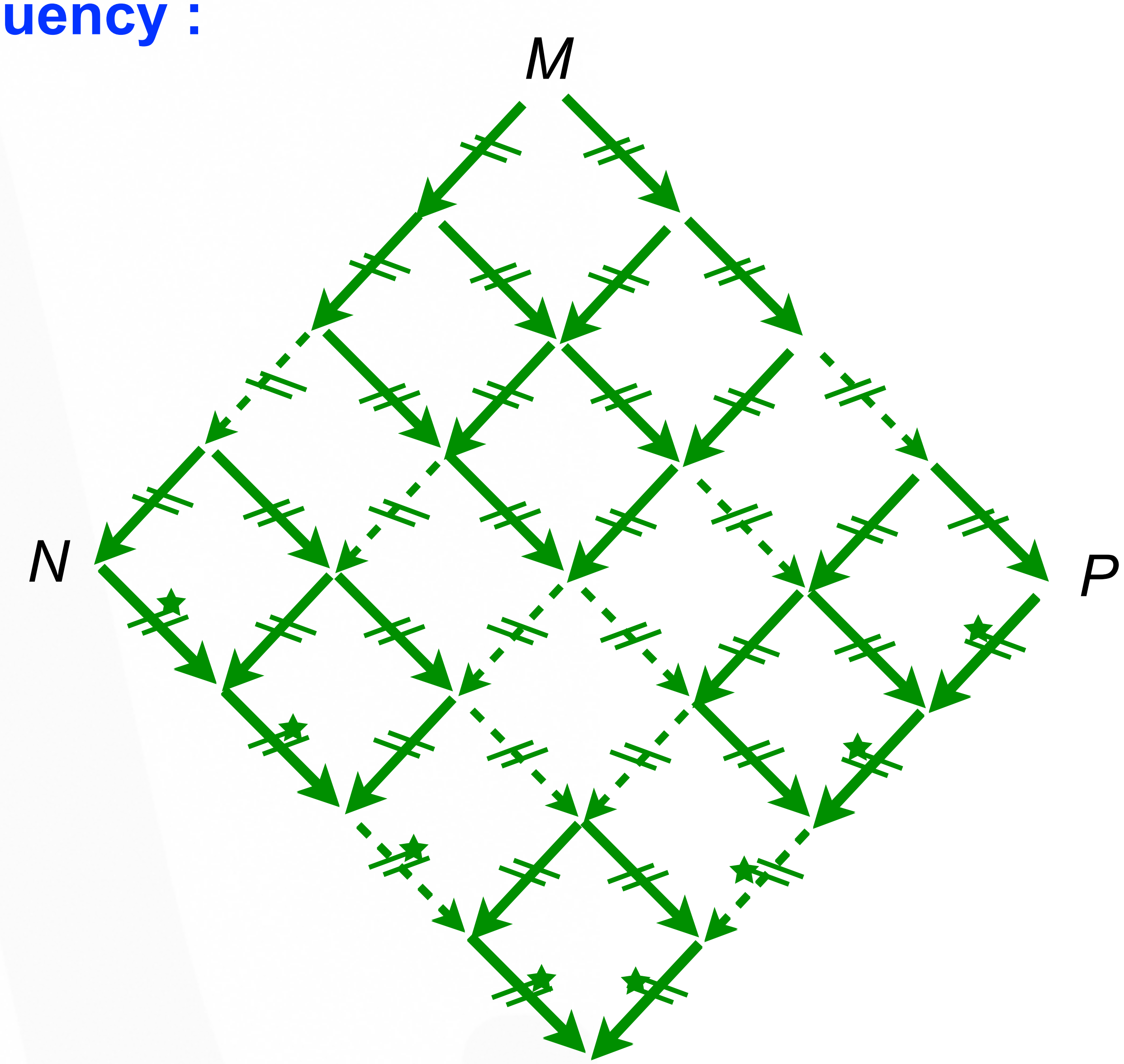
$$\begin{tikzcd}
 & M \arrow[dl, "/\!/" marking] \arrow[dr, "/\!/" marking] & \\
N \arrow[dr, dashed, "/\!/" marking] & & P \arrow[dl, dashed, "/\!/" marking] \\
 & Q &
\end{tikzcd}$$

- Example:

$$\begin{tikzcd}
(\lambda x.xx)(Iz) \arrow[r, "/\!/" marking] \arrow[dr, "/\!/" marking] & (\lambda x.xx)z \arrow[dr, "/\!/" marking] & \\
 & Iz(Iz) \arrow[r, "/\!/" marking] & zz
\end{tikzcd}$$

Confluency

- **Proof of confluency :**



Confluency

- **Lemma 4:** $M \mathrel{/\!\!/\!\!\!\!\twoheadrightarrow} N$ and $P \mathrel{/\!\!/\!\!\!\!\twoheadrightarrow} Q$ implies $M\{x := P\} \mathrel{/\!\!/\!\!\!\!\twoheadrightarrow} N\{x := Q\}$

Proof: by structural induction on M.

Case 1: $M = x \mathrel{/\!\!/\!\!\!\!\twoheadrightarrow} x = N$. Then $M\{x := P\} = P \mathrel{/\!\!/\!\!\!\!\twoheadrightarrow} Q = N\{x := Q\}$

Case 2: $M = y \mathrel{/\!\!/\!\!\!\!\twoheadrightarrow} y = N$. Then $M\{x := P\} = y \mathrel{/\!\!/\!\!\!\!\twoheadrightarrow} y = N\{x := Q\}$

Case 3: $M = \lambda y.M_1 \mathrel{/\!\!/\!\!\!\!\twoheadrightarrow} \lambda y.N_1 = N$ with $M_1 \mathrel{/\!\!/\!\!\!\!\twoheadrightarrow} N_1$. By induction $M_1\{x := P\} \mathrel{/\!\!/\!\!\!\!\twoheadrightarrow} N_1\{x := Q\}$. So $M\{x := P\} = \lambda y.M_1\{x := P\} \mathrel{/\!\!/\!\!\!\!\twoheadrightarrow} \lambda y.N_1\{x := Q\} = N$.

Case 4: $M = M_1M_2 \mathrel{/\!\!/\!\!\!\!\twoheadrightarrow} N_1N_2 = N$ with $M_1 \mathrel{/\!\!/\!\!\!\!\twoheadrightarrow} N_1$ and $M_2 \mathrel{/\!\!/\!\!\!\!\twoheadrightarrow} N_2$. By induction $M_1\{x := P\} \mathrel{/\!\!/\!\!\!\!\twoheadrightarrow} N_1\{x := Q\}$ and $M_2\{x := P\} \mathrel{/\!\!/\!\!\!\!\twoheadrightarrow} N_2\{x := Q\}$. So $M\{x := P\} = M_1\{x := P\}M_2\{x := P\} \mathrel{/\!\!/\!\!\!\!\twoheadrightarrow} N_1\{x := Q\}N_2\{x := Q\} = N\{x := Q\}$.

Case 5: $M = (\lambda y.M_1)M_2 \mathrel{/\!\!/\!\!\!\!\twoheadrightarrow} N_1\{y := N_2\} = N$ with $M_1 \mathrel{/\!\!/\!\!\!\!\twoheadrightarrow} N_1$ and $M_2 \mathrel{/\!\!/\!\!\!\!\twoheadrightarrow} N_2$. By induction $M_1\{x := P\} \mathrel{/\!\!/\!\!\!\!\twoheadrightarrow} N_1\{x := Q\}$ and $M_2\{x := P\} \mathrel{/\!\!/\!\!\!\!\twoheadrightarrow} N_2\{x := Q\}$. So $M\{x := P\} = (\lambda y.M_1\{x := P\})(M_2\{x := P\}) \mathrel{/\!\!/\!\!\!\!\twoheadrightarrow} N_1\{x := Q\}\{y := N_2\{x := Q\}\} = N_1\{y := N_2\}\{x := Q\} = N$ by **substitution lemma**, since $y \notin \mathrm{var}(Q) \subset \mathrm{var}(P)$. $\square$

Confluency

- **Lemma 5:** If $M \mathrel{/\!\!/\!\!\!\!\to} N$ and $M \mathrel{/\!\!/\!\!\!\!\to} P$, then $N \mathrel{/\!\!/\!\!\!\!\to} Q$ and $N \mathrel{/\!\!/\!\!\!\!\to} Q$ for some Q.

Proof: by structural induction on M.

Case 1: $M = x$. Then $M = x \mathrel{/\!\!/\!\!\!\!\to} x = N$ and $M = x \mathrel{/\!\!/\!\!\!\!\to} x = P$. We have too $N \mathrel{/\!\!/\!\!\!\!\to} x = Q$ and $P \mathrel{/\!\!/\!\!\!\!\to} x = Q$.

Case 2: $M = \lambda y.M_1 \mathrel{/\!\!/\!\!\!\!\to} \lambda y.N_1 = N$ with $M_1 \mathrel{/\!\!/\!\!\!\!\to} N_1$. Same for $M = \lambda y.M_1 \mathrel{/\!\!/\!\!\!\!\to} \lambda y.P_1 = P$ with $M_1 \mathrel{/\!\!/\!\!\!\!\to} P_1$. By induction $N_1 \mathrel{/\!\!/\!\!\!\!\to} Q_1$ and $P_1 \mathrel{/\!\!/\!\!\!\!\to} Q_1$ for some Q_1. So $N = \lambda y.N_1 \mathrel{/\!\!/\!\!\!\!\to} \lambda y.Q_1 = Q$ and $P = \lambda y.P_1 \mathrel{/\!\!/\!\!\!\!\to} \lambda y.Q_1 = Q$.

Case 3: $M = M_1 M_2 \mathrel{/\!\!/\!\!\!\!\to} N_1 N_2 = N$ and $M = M_1 M_2 \mathrel{/\!\!/\!\!\!\!\to} P_1 P_2 = P$ with $M_i \mathrel{/\!\!/\!\!\!\!\to} N_i$, $M_i \mathrel{/\!\!/\!\!\!\!\to} P_i$ $(1 \leq i \leq 2)$. By induction $N_i \mathrel{/\!\!/\!\!\!\!\to} Q_i$ and $P_i \mathrel{/\!\!/\!\!\!\!\to} Q_i$ for some Q_i. So $N \mathrel{/\!\!/\!\!\!\!\to} Q_1 Q_2 = Q$ and $P \mathrel{/\!\!/\!\!\!\!\to} Q_1 Q_2 = Q$.

Case 4: $M = (\lambda x.M_1)M_2 \mathrel{/\!\!/\!\!\!\!\to} N_1\{x := N_2\} = N$ and $M = (\lambda x.M_1)M_2 \mathrel{/\!\!/\!\!\!\!\to} P' P_2 = P$ with $M_i \mathrel{/\!\!/\!\!\!\!\to} N_i$ $(1 \leq i \leq 2)$ and $\lambda x.M_1 \mathrel{/\!\!/\!\!\!\!\to} P'$, $M_2 \mathrel{/\!\!/\!\!\!\!\to} P_2$. Therefore $P' = \lambda x.P_1$ with $M_1 \mathrel{/\!\!/\!\!\!\!\to} P_1$. By induction $N_i \mathrel{/\!\!/\!\!\!\!\to} Q_i$ and $P_i \mathrel{/\!\!/\!\!\!\!\to} Q_i$ for some Q_i.
So $N \mathrel{/\!\!/\!\!\!\!\to} Q_1\{x := Q_2\} = Q$ by **lemma 4**. And $P \mathrel{/\!\!/\!\!\!\!\to} Q_1\{x := Q_2\} = Q$ by definition.

Case 5: symmetric.

Confluency

Proof: ....

Case 6: $M = (\lambda x.M_1)M_2 \not\twoheadrightarrow N_1\{x := N_2\} = N$ and $M = (\lambda x.M_1)M_2 \not\twoheadrightarrow P_1\{x := P_2\} = P$ with $M_i \not\twoheadrightarrow N_i, M_i \not\twoheadrightarrow P_i$ $(1 \leq i \leq 2)$. By induction $N_i \not\twoheadrightarrow Q_i$ and $P_i \not\twoheadrightarrow Q_i$ for some Q_i.
So $N \not\twoheadrightarrow Q_1\{x := Q_2\} = Q$ and $P \not\twoheadrightarrow Q_1\{x := Q_2\} = Q$ by **lemma 4**. $\square$

- **Lemma 6:** If $M \longrightarrow N$, then $M \not\twoheadrightarrow N$.
- **Lemma 7:** If $M \not\twoheadrightarrow N$, then $M \xrightarrow{*} N$.

Proofs: obvious.

- **Theorem 2 [Church-Rosser]:**

  If $M \xrightarrow{*} N$ and $M \xrightarrow{*} P$, then $N \xrightarrow{*} Q$ and $P \xrightarrow{*} Q$ for some Q.

Confluency

- previous axiomatic method is due to **Martin-Löf**
- Martin-Löf’s method models inside-out parallel reductions
- there are other proofs with explicit redexes

$$\downarrow$$

- Curry’s finite developments

Finite developments

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Residuals of redexes

- tracking redexes while contracting others
- examples:

$$\Delta(\underbrace{Ia}) \longrightarrow \underbrace{Ia}(\underbrace{Ia}) \qquad \Delta = \lambda x.\, xx \quad I = \lambda x.x \quad K = \lambda xy.x$$

$$\underbrace{Ia}(\Delta(Ib)) \longrightarrow \underbrace{Ia}(Ib(Ib))$$

$$\underbrace{I(\Delta(Ia))} \longrightarrow \underbrace{I(Ia(Ia))}$$

$$\underbrace{\Delta(Ia)} \longrightarrow Ia(Ia))$$

$$Ia(\Delta(\underbrace{Ib})) \longrightarrow Ia(\underbrace{Ib}(\underbrace{Ib}))$$

$$\underbrace{\Delta\Delta} \longrightarrow \Delta\Delta$$

$$(\lambda x.Ia)(\underbrace{Ib}) \longrightarrow Ia$$

Residuals of redexes

- when R is redex in M and $M \xrightarrow{S} N$

  the set R/S of **residuals** of R in N is defined by inspecting relative positions of R and S in M:

1- R and S disjoint, $M = \cdots \underbrace{R}_{} \cdots S \cdots \xrightarrow{S} \cdots \underbrace{R}_{} \cdots S' \cdots = N$

2- S in $R = (\lambda x.A)B$

2a- S in A, $M = \cdots \underline{(\lambda x. \cdots S \cdots)B} \cdots \xrightarrow{S} \cdots \underline{(\lambda x. \cdots S' \cdots)B} \cdots = N$

2b- S in B, $M = \cdots \underline{(\lambda x.A)(\cdots S \cdots)} \cdots \xrightarrow{S} \cdots \underline{(\lambda x.A)(\cdots S' \cdots)} \cdots = N$

3- R in $S = (\lambda y.C)D$

3a- R in C, $M = \cdots (\lambda y. \cdots \underbrace{R}_{} \cdots)D \cdots \xrightarrow{S} \cdots\cdots \underbrace{R\{y := D\}}_{} \cdots\cdots = N$

3b- R in D, $M = \cdots (\lambda y.C)(\cdots \underbrace{R}_{} \cdots) \cdots \xrightarrow{S} \cdots (\cdots \underbrace{R}_{} \cdots) \cdots (\cdots \underbrace{R}_{} \cdots) \cdots = N$

4- R is S, no residuals of R.

Residuals of redexes

- when ρ is a reduction from M to N, i.e. $\rho : M \xrightarrow{*} N$
  the set of residuals of R by ρ is defined by **transitivity** on the length of ρ
  and is written R/ρ

- notice that we can have $S \in R/\rho$ and $R \neq S$
  residuals may **not** be syntacticly **equal** (see previous 3rd example)

- residuals **depend on reductions**. Two reductions between same terms may produce two distinct sets of residuals.

- a redex is residual of a **single** redex (the inverse of the residual relation is a function): $R \in S/\rho$ and $R \in T/\rho$ implies $S = T$

Exercices

- Find redex R and reductions ρ and σ between M and N such that residuals of R by ρ and σ differ. Hint: consider $M = I(Ix)$
- Show that residuals of nested redexes keep nested.
- Show that residuals of disjoint redexes may be nested.
- Show that residuals of a redex may be nested after several reduction steps.

Created redexes

- A redex is **created by reduction** ρ if it is not a residual by ρ of a redex in initial term. Thus R is created by ρ when $\rho : M \xrightarrow{\star} N$ and $\nexists S,\ R \in S/\rho$

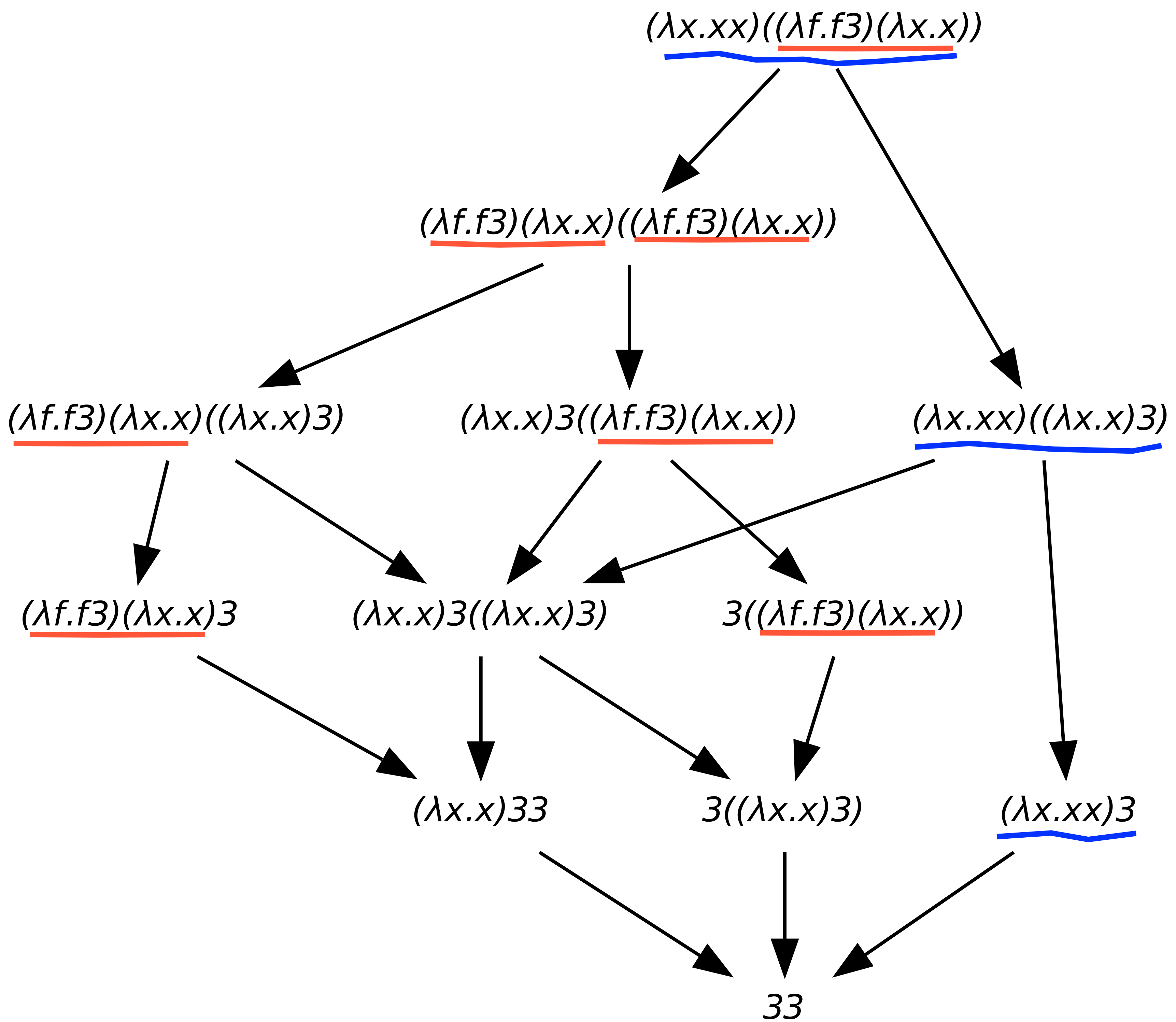
$(\lambda x.xa)I \longrightarrow \underbrace{Ia}$

$IIa \longrightarrow \underbrace{Ia}$

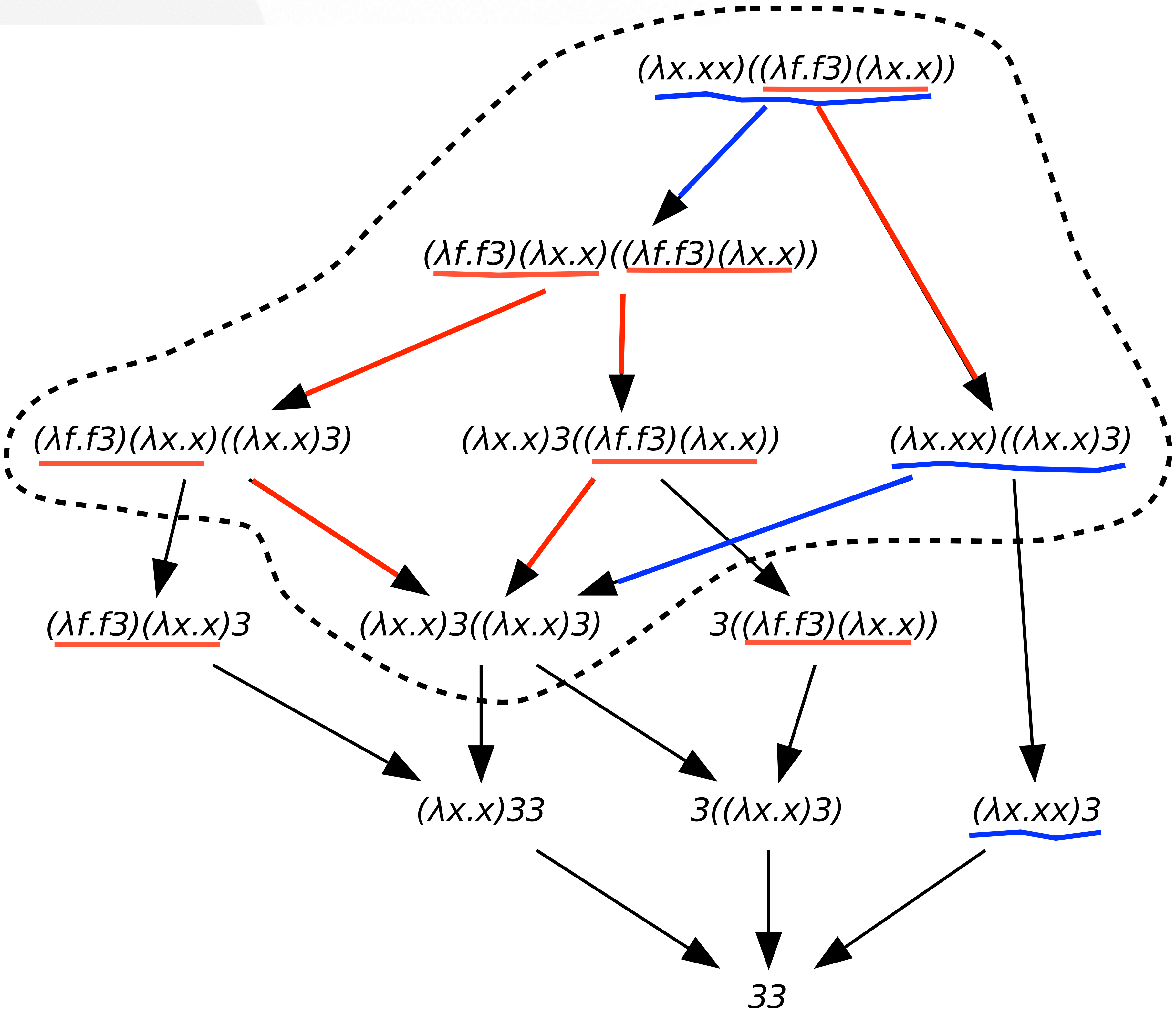
$(\lambda xy.xy)ab \longrightarrow \underbrace{(\lambda y.ay)b}$

$\Delta\Delta \longrightarrow \underbrace{\Delta\Delta}$

Residuals of redexes



Relative reductions



Finite developments

- Let $\mathcal{F}$ be a set of redexes in M. A reduction **relative to** $\mathcal{F}$ only contracts residuals of $\mathcal{F}$.
- When there are no more residuals of $\mathcal{F}$ to contract, we say the relative reduction is a **development of** $\mathcal{F}$.

- **Theorem 3 [finite developments] (Curry)** Let $\mathcal{F}$ be a set of redexes in M. Then:
  - relative reductions **cannot be infinite**; they all end in a development of $\mathcal{F}$
  - all developments end on a **same** term N
  - let R be a redex in M. Then **residuals** of R by finite developments of $\mathcal{F}$ are the same.

Finite developments

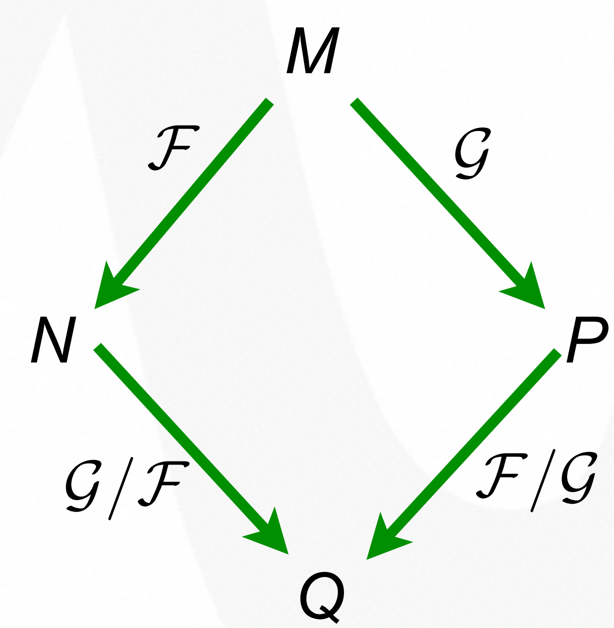
- Therefore we can define (without ambiguity) a new **parallel step** reduction:

$$\rho : M \xrightarrow{\mathcal{F}} N$$

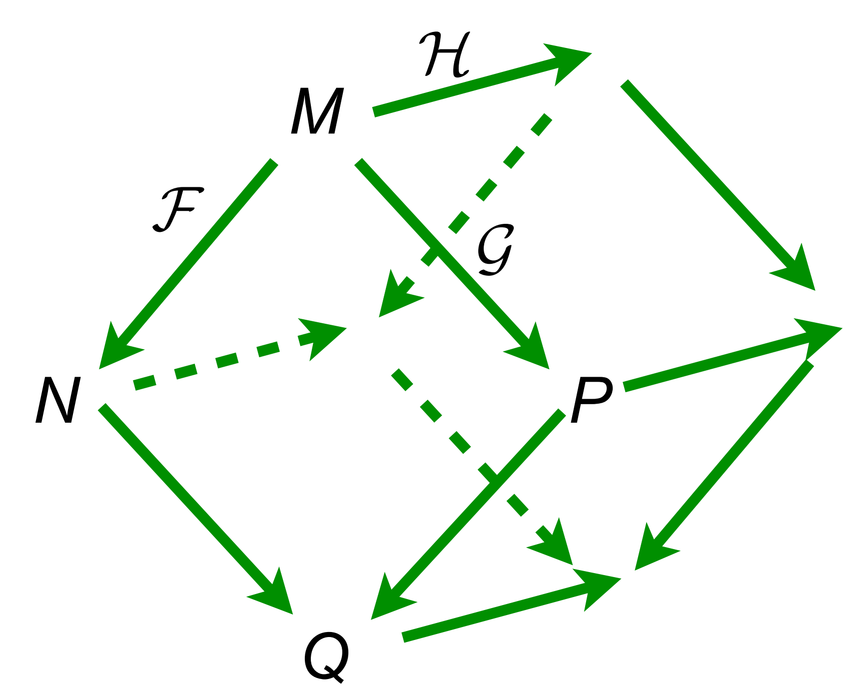
and when R is a redex in M, we can write $R/\mathcal{F}$ for its residuals in N

- **Two corollaries**:

Lemma of **Parallel Moves**



Cube Lemma



Labeled calculus

- Finite developments will be shown with a labeled calculus.
- **Lambda calculus with labeled redexes**

M, N, P	$::=$	$x, y, z, \ldots$	(variables)
	$\mid$	$(\lambda x.M)$	(M as function of x)
	$\mid$	$(M\ N)$	(M applied to N)
	$\mid$	$c, d, \ldots$	(*constants*)
	$\mid$	$(\lambda x.M)^r\, N$	(*labeled redexes)*

- **$\mathcal{F}$-labeled reduction**

$$(\lambda x.M)^r N \longrightarrow M\{x := N\} \qquad \text{when } r \in \mathcal{F}$$

- **Labeled substitution**

. . . as before

$$((\lambda x.M)^r N)\{y := P\} = ((\lambda x.M)\{y := P\})^r (N\{y := P\})$$

Labeled calculus

- **Theorem** For any $\mathcal{F}$, the labeled calculus is **confluent**.

- **Theorem** For any $\mathcal{F}$, the labeled calculus is **strongly normalizable** (no infinite labeled reductions).

- **Lemma** For any $\mathcal{F}$-reduction $\rho : M \xrightarrow{*} N$, a labeled redex in N has label r if and only if it is **residual** by ρ of a redex with label r in M.

$\downarrow$

- **Theorem 3 [finite developments] (Curry)**

Labeled calculus

- Proof of confluency is again with Martin-Löf's axiomatic method.
- Proof of residual property is by simple inspection of a reduction step.
- Proof of termination is slightly more complex with following lemmas:

- **Notation** $M \xrightarrow[\text{int}]{\star} N$ if M reduces to N without contracting a toplevel redex.

- **Lemma 1** **[Barendregt-like]** $M\{x := N\} \xrightarrow[\text{int}]{\star} (\lambda y.P)^r Q$ implies

  $M = (\lambda y.A)^r B$ with $A\{x := N\} \xrightarrow{\star} P$, $B\{x := N\} \xrightarrow{\star} Q$

  or

  $M = x$ and $N \xrightarrow{\star} (\lambda y.P)^r Q$

- **Lemma 2** $M, N \in \mathcal{SN}$ (strongly normalizing) implies $M\{x := N\} \in \mathcal{SN}$

- **Theorem** $M \in \mathcal{SN}$ for all M.

Labeled calculus proofs

- **Lemma 1** **[Barendregt-like]** $M\{x := N\} \xrightarrow[\text{int}]{*} (\lambda y.P)^r Q$ implies

  $M = (\lambda y.A)^r B$ with $A\{x := N\} \xrightarrow{*} P$, $B\{x := N\} \xrightarrow{*} Q$

  or

  $M = x$ and $N \xrightarrow{*} (\lambda y.P)^r Q$

Proof Let P^* be $P\{x := N\}$ for any P.

Case 1: $M = x$. Then $M^* = N$ and $N \xrightarrow{*} (\lambda y.P)^r Q$.

Case 2: $M = y$. Then $M^* = y$. Impossible.

Case 2: $M = \lambda y.M_1$. Again impossible.

Case 3: $M = M_1 M_2$ or $M = (\lambda y.M_1)^s M_2$ with $s \neq r$. These cases are also impossible.

Case 4: $M = (\lambda y.M_1)^r M_2$. Then $M_1^* \xrightarrow{*} P$ and $M_2^* \xrightarrow{*} Q$.

Labeled calculus proofs

- **Lemma 2** $M, N \in \mathcal{SN}$ (strongly normalizing) implies $M\{x := N\} \in \mathcal{SN}$

Proof: by induction on $\langle \mathrm{depth}(M), ||M|| \rangle$. Let P^* be $P\{x := N\}$ for any P.

Case 1: $M = x$. Then $M^* = N \in \mathcal{SN}$. If $M = y$. Then $M^* = y \in \mathcal{SN}$.

Case 2: $M = \lambda y.M_1$. Then $M^* = \lambda y.M_1^*$ and by induction $M_1^* \in \mathcal{SN}$.

Case 3: $M = M_1 M_2$ and never $M^* \xrightarrow{\star} (\lambda y.A)^r B$. Same argument on M_1 and M_2.

Case 4: $M = M_1 M_2$ and $M^* \xrightarrow{\star} (\lambda y.A)^r B$. We can always consider first time when this toplevel redex appears. Hence we have $M^* \xrightarrow[\text{int}]{\star} (\lambda y.A)^r B$. By lemma 1, we have two cases:

Case 4.1: $M = (\lambda y.M_3)^r M_2$ with $M_3^* \xrightarrow{\star} A$ and $M_2^* \xrightarrow{\star} B$. Then $M^* = (\lambda y.M_3^*)^r M_2^*$. As $M_3 \in \mathcal{SN}$ and $M_2 \in \mathcal{SN}$, the internal reductions from M^* terminate by induction. If $r \notin \mathcal{F}$, there are no extra reductions. If $r \in \mathcal{F}$, we can have $M_3^* \xrightarrow{\star} A$, $M_2^* \xrightarrow{\star} B$ and $(\lambda y.A)^r B \longrightarrow A\{y := B\}$. But $M \longrightarrow M_3\{y := M_2\}$ and $(M_3\{y := M_2\})^* \xrightarrow{\star} A\{y := B\}$. As $\mathrm{depth}(A\{y := B\} \leq \mathrm{depth}(M_3\{y := M_2)\} < \mathrm{depth}(M)$, we get $A\{y := B\} \in \mathcal{SN}$ by induction.

Case 4.2: $M = x$. Impossible.

Labeled calculus proofs

- **Theorem** $M \in \mathcal{SN}$ for all M.

  Proof: by induction on $||M||$.

  Case 1: $M = x$. Obvious.

  Case 2: $M = \lambda x.M_1$. Obvious since $M_1 \in \mathcal{SN}$ by induction.

  Case 3: $M = M_1 M_2$ and $M_1 \neq (\lambda x.A)^r$. Then all reductions are internal to M_1 and M_2. Therefore $M \in \mathcal{SN}$ by induction on M_1 and M_2.

  Case 4: $M = (\lambda x.M_1)^r M_2$ and $r \notin \mathcal{F}$. Same argument on M_1 and M_2.

  Case 5: $M = (\lambda x.M_1)^r M_2$ and $r \in \mathcal{F}$. Then M_1 and M_2 in $\mathcal{SN}$ by induction. But we can also have $M \xrightarrow{\star} (\lambda x.A)^r B \longrightarrow A\{x := B\}$ with A and B in $\mathcal{SN}$. By Lemma 2, we know that $A\{x := B\} \in \mathcal{SN}$.

Standardization

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Standard reduction

Redex R is **to the left of** redex S if the λ of R is to the left of the λ of S.

$$M = \cdots \underbrace{(\lambda x.A)B}_{R} \cdots \underbrace{(\lambda y.C)D}_{S} \cdots$$

or

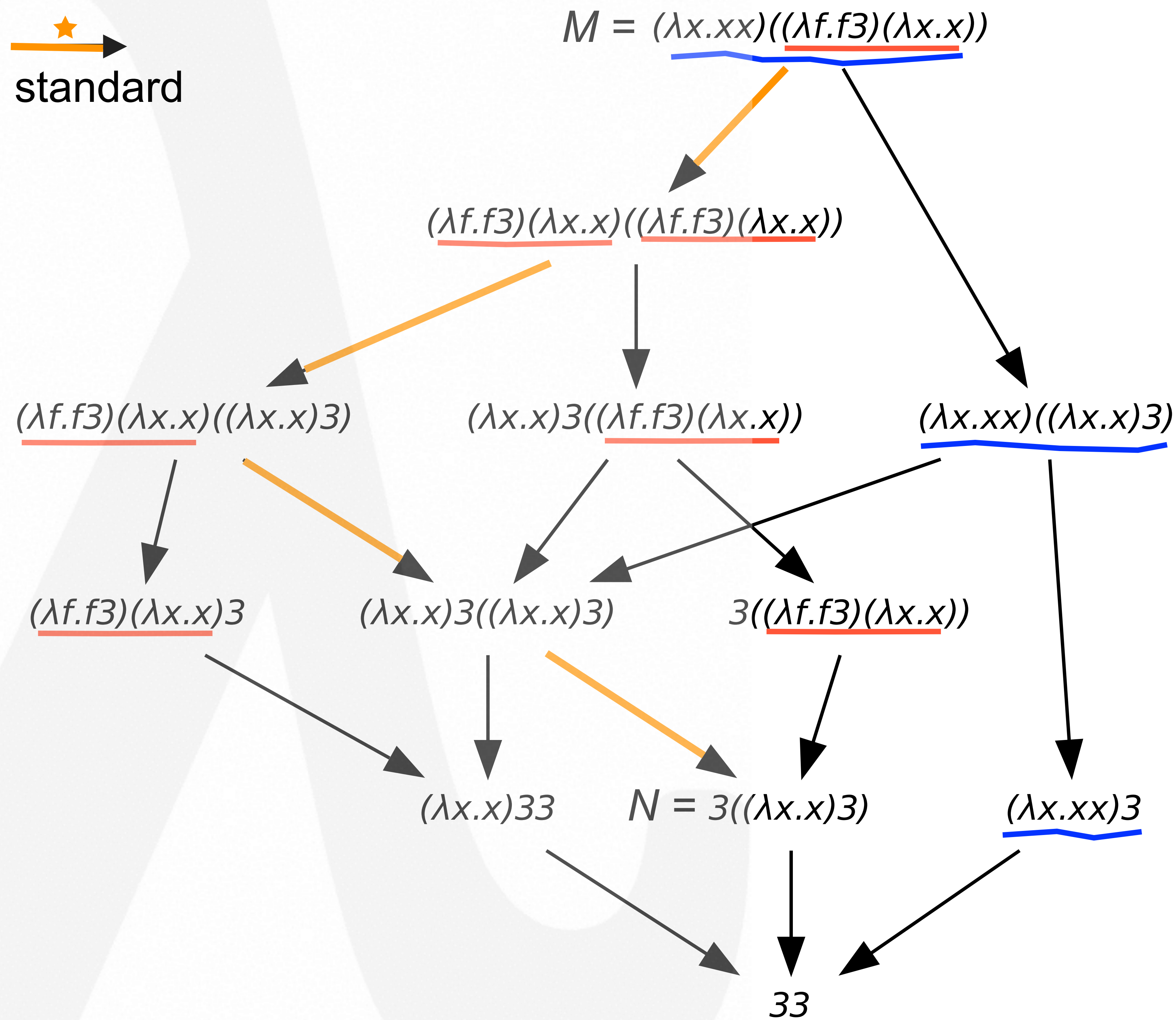
$$M = \cdots \underbrace{(\lambda x. \cdots \underbrace{(\lambda y.C)D}_{S} \cdots)B}_{R} \cdots$$

or

$$M = \cdots \underbrace{(\lambda x.A)(\cdots \underbrace{(\lambda y.C)D}_{S} \cdots)}_{R} \cdots$$

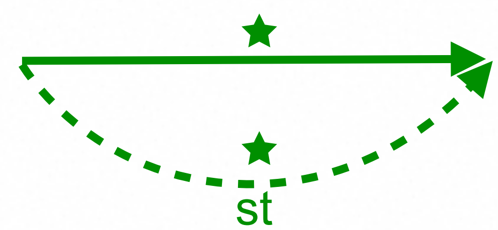
The reduction $M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N$ is standard iff for all i, j $(0 < i < j \leq n)$, redex R_j is not a residual of redex R'_j to the left of R_i in M_{i-1}.

Standard reduction



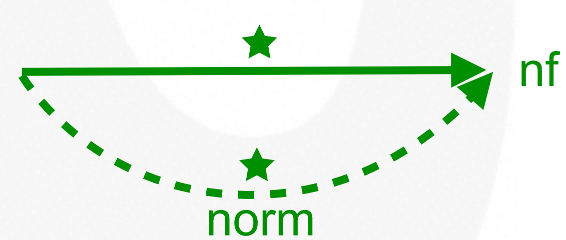
Standardization

- **Theorem [standardization] (Curry)** Any reduction can be standardized.



- The **normal reduction** (each step contracts the leftmost-outermost redex) is a standard reduction.

- **Corollary [normalization]** If *M* has a normal form, the normal reduction reaches the normal form.



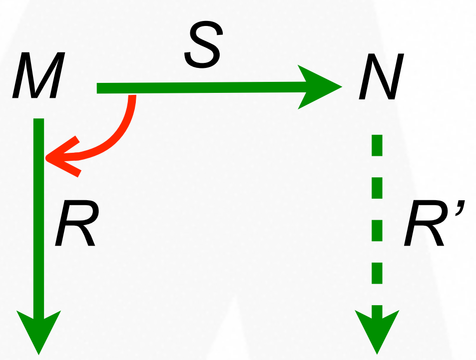
Standardization lemma

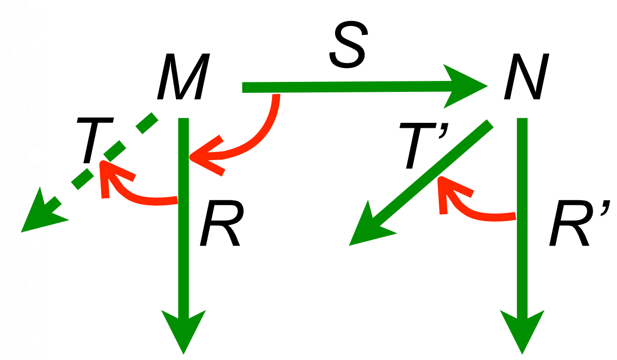
- **Notation:** write $R <_\ell S$ if redex R is to the left of redex S.

- **Lemma 1** Let R, S be redexes in M such that $R <_\ell S$. Let $M \xrightarrow{S} N$.

Then $R/S = \{R'\}$. Furthermore, if $T' <_\ell R'$, then $\exists T,\ T <_\ell R, T' \in T/S$.

[one cannot create a redex through another more-to-the-left]





- **Proof of standardization thm:** [Klop] application of the finite developments theorem and previous lemma.

Standardization axioms

- 3 axioms are sufficient to get lemma 1

- **Axiom 1 [linearity]** $S \not\leq_\ell R$ implies $\exists! R',\ R' \in R/S$

- **Axiom 2 [context-freeness]** $S \not\leq_\ell R$ and $R' \in R/S$ and $T' \in T/S$ implies $T \mathfrak{R} R$ iff $T' \mathfrak{R} R'$ where $\mathfrak{R}$ is $<_\ell$ or $>_\ell$

- **Axiom 3 [left barrier creation]** $R <_\ell S$ and $\nexists T',\ T \in T'/S$ implies $R <_\ell T$

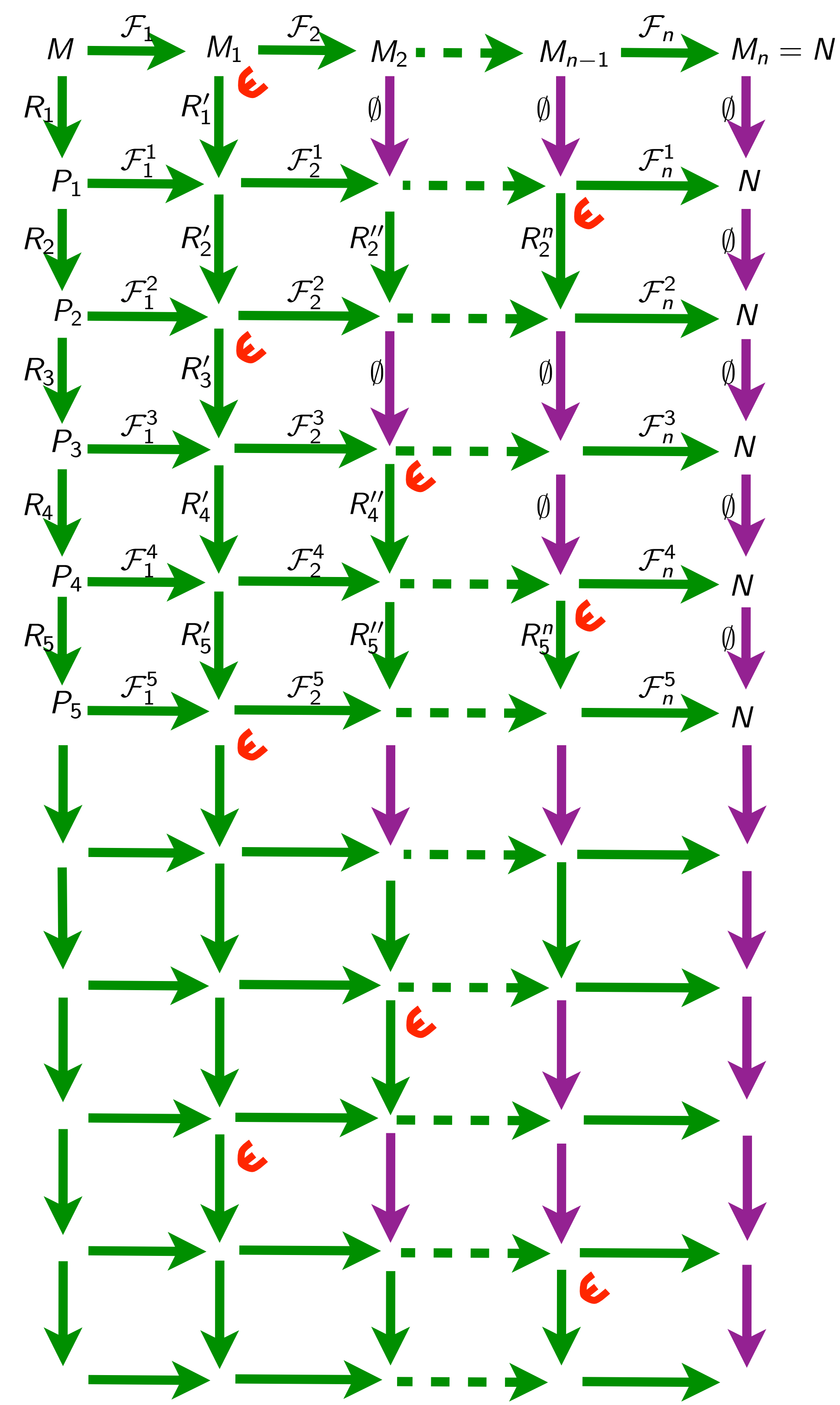
Standardization proof

- **Proof:**

  Each square is an application of the lemma of parallel moves. Let ρ_i be the horizontal reductions and σ_j the vertical ones. Each horizontal step is a parallel step, vertical steps are either elementary or empty.

  We start with reduction ρ_0 from M to N. Let R_1 be the leftmost redex in M with residual contracted in ρ_0. By lemma 1, it has a single residual R'_1 in M_1, M_2, ... until it belongs to some $\mathcal{F}_k$. Here $R'_1 \in \mathcal{F}_2$. There are no more residuals of R_1 in M_{k+1}, M_{k+2}, ...

  Let R_2 be leftmost redex in P_1 with residual contracted in ρ_1. Here the unique residual is contracted at step n. Again with R_3 leftmost with residual contracted in ρ_2. Etc.

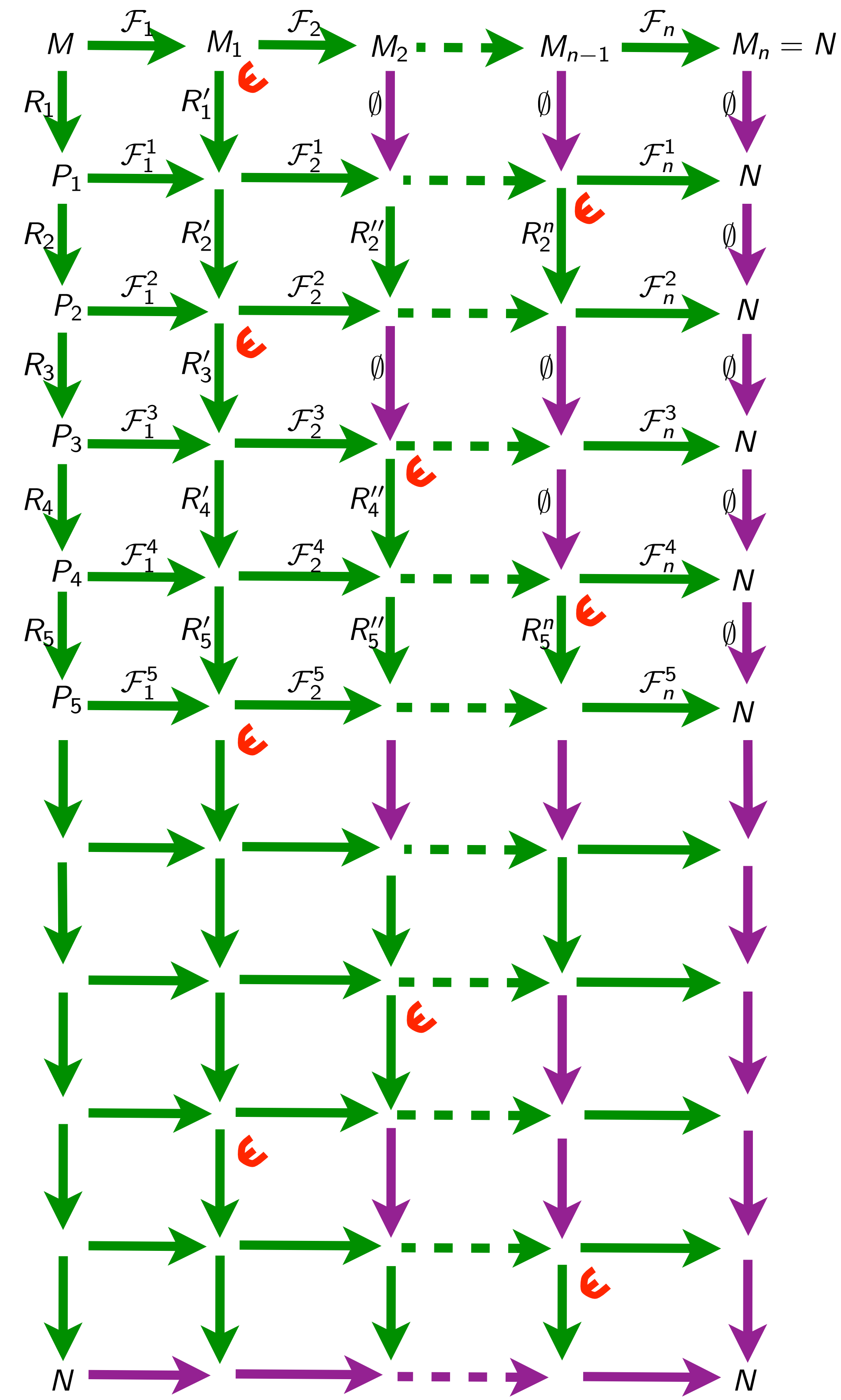


Standardization proof

- **Proof (cont'd):**

Now reduction σ_0 starting from M cannot be infinite and stops for some p. If not, there is a rightmost column σ_k with infinitely non-empty steps. After a while, this reduction is a reduction relative to a set $\mathcal{F}_i^j$, which cannot be infinite by the Finite Development theorem.

Then ρ_p is an empty reduction and therefore the final term of σ_0 is N.



Standardization proof

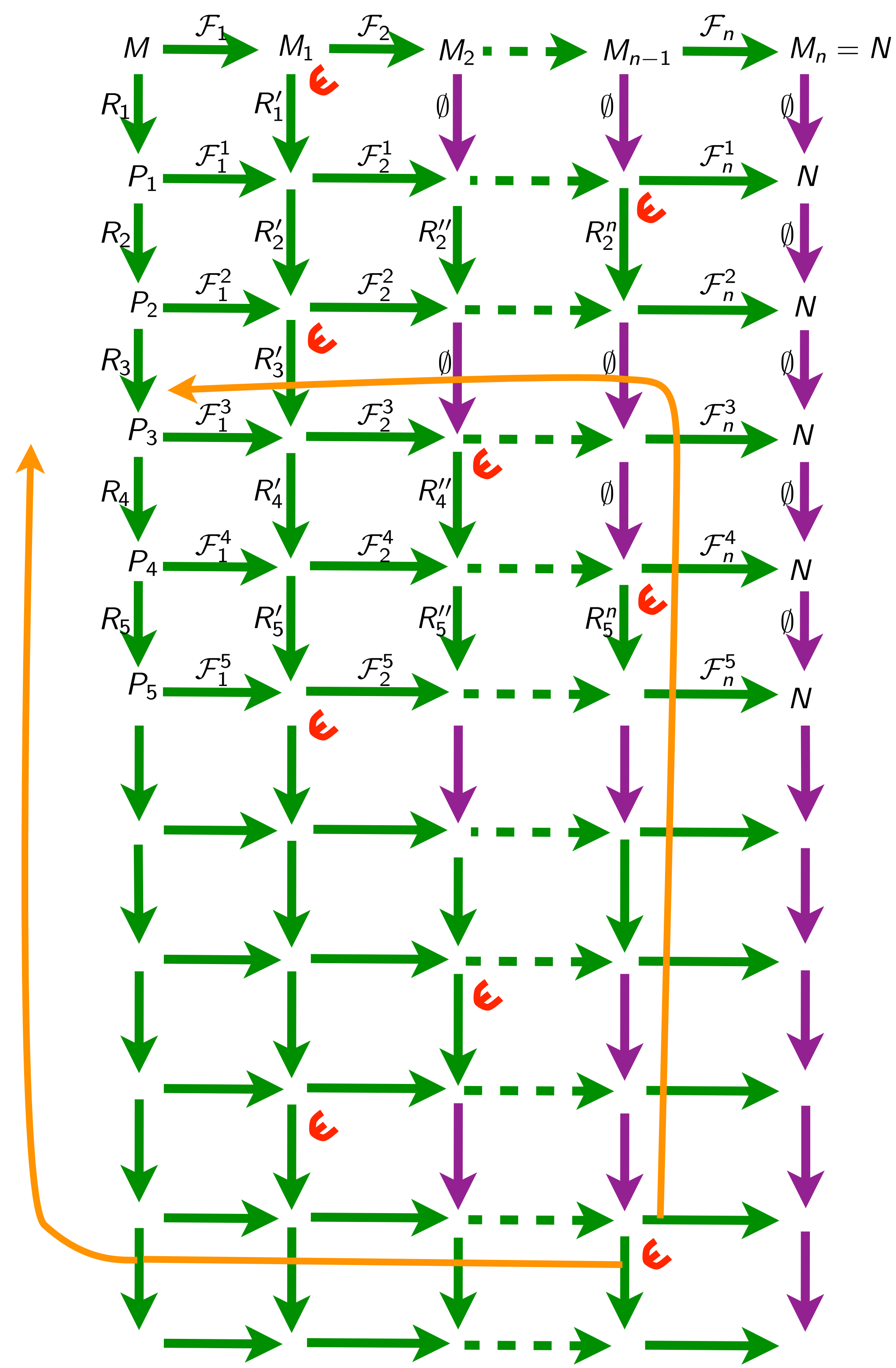
- **Proof (cont'd):**

We claim σ_0 is a standard reduction. Suppose R_k $(k > i)$ is residual of S_i to the left of R_i in P_{i-1}.

By construction R_k has residual S_k^j along ρ_{i-1} contracted at some j step. So S_k^j is residual of S_i.

By the cube lemma, it is also residual of some S_i^j along σ_{j-1}. Therefore there is S_i^j in $\mathcal{F}_i^j$ residual of S_i leftmore or outer than R_i.

Contradiction.



Redex creation

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Created redexes

- A redex is **created by reduction** ρ if it is not a residual by ρ of a redex in initial term. Thus R is created by ρ when $\rho : M \xrightarrow{*} N$ and $\nexists S,\ R \in S/\rho$

$$(\lambda x.xa)I \longrightarrow \underbrace{Ia} \qquad\qquad IIa \longrightarrow \underbrace{Ia}$$

$$(\lambda xy.xy)ab \longrightarrow \underbrace{(\lambda y.ay)b} \qquad\qquad \Delta\Delta \longrightarrow \underbrace{\Delta\Delta}$$

- By Finite Developments thm, a reduction can be infinite iff it does not stop creating new redexes.

$$\Delta\Delta \longrightarrow \underbrace{\Delta\Delta} \longrightarrow \underbrace{\Delta\Delta} \longrightarrow \underbrace{\Delta\Delta} \longrightarrow \cdots$$

- If the length of creation is bounded, there is also a generalized finite developments theorem.

Created redexes in typed calculus

- only 2 cases for creation of redexes in a reduction step

$$\underbrace{(\lambda x.\cdots xN\cdots)}_{\sigma\to\tau}\underbrace{(\lambda y.M)}_{\sigma} \rightarrow \cdots\underbrace{(\lambda y.M)}_{\sigma}N'\cdots$$

creates

$$\underbrace{(\lambda x.\underbrace{\lambda y.M}_{\tau})}_{\sigma\to\tau}NP \rightarrow \underbrace{(\lambda y.M')}_{\tau}P$$

creates

- length of creation is bounded by size of types of initial term

Other properties

Other properties

- confluency with **eta**-rules, **delta**-rules
- **generalized** finite developments theorem
- **permutation** equivalence
- redex **families**
- finite developments vs strong normalization
- completeness of reduction **strategies**
- **head** normal forms
- **Bohm trees**
- continuity theorem
- sequentiality of Bohm trees
- models of the type-free lambda-calculus
- **typed** lambda-calculi
- continuations and reduction strategies
- ...
- process calculi and lambda-calculus
- abstract reduction systems
- **explicit** substitutions
- implementation of functional languages
- lazy evaluators
- SOS
- all theory of **programming languages**
- ...
- connection to mathematical **logic**
- calculus of constructions
- ...

Homeworks

Exercices

- Show that:

  1- $M \longrightarrow_\eta N \longrightarrow P$ implies $M \longrightarrow Q \xrightarrow{*}_\eta P$ for some Q

  2- $M \xrightarrow{*}_\eta N \xrightarrow{*} P$ implies $M \xrightarrow{*} Q \xrightarrow{*}_\eta P$ for some Q

  3- $M \xrightarrow{*}_{\beta,\eta} N$ implies $M \xrightarrow{*} P \xrightarrow{*}_\eta N$ for some P

  4- $M \longrightarrow N$ and $M \longrightarrow_\eta P$ implies $N \xrightarrow{*}_\eta Q$ and $P \xrightarrow{1} Q$ for some Q

  5- $M \xrightarrow{*}_\eta N$ and $M \xrightarrow{*}_\eta P$ implies $N \xrightarrow{*}_\eta Q$ and $P \xrightarrow{*}_\eta Q$ for some Q

  6- $M \xrightarrow{*}_{\beta,\eta} N$ and $M \xrightarrow{*}_{\beta,\eta} P$ implies $N \xrightarrow{*}_{\beta,\eta} Q$ and $P \xrightarrow{*}_{\beta,\eta} Q$ for some Q

  Therefore $\xrightarrow{*}_{\beta,\eta}$ is confluent.

- Show same property for β-reduction and η-expansion $(\longrightarrow \cup \longleftarrow_\eta)^*$

Exercices

7- Show there is no M such that $M \xrightarrow{\star} Kac$ and $M \xrightarrow{\star} Kbc$ where $K = \lambda x.\lambda y.x$.

8- Find M such that $M \xrightarrow{\star} Kab$ and $M \xrightarrow{\star} Kac$.

9- (difficult) Show that $\xleftarrow{\star}$ is not confluent.

10- Show that $\Delta\Delta(II)$ has no normal form when $I = \lambda x.x$ and $\Delta = \lambda x.xx$.

11- Show that $\Delta\Delta M_1 M_2 \cdots M_n$ has no normal form for any $M_1, M_2, \ldots M_n$ $(n \geq 0)$.

12- Show there is no M whose reduction graph is exactly following:

$$\begin{tikzcd}
 & M \arrow[dl] \arrow[d] \arrow[dr] & \\
M_1 \arrow[dr] & M_2 \arrow[d] & M_3 \arrow[dl] \\
 & N &
\end{tikzcd}$$

13- Show that rightmost-outermost reduction may miss normal forms.