Control Principles of Complex Systems

This is a paper reading note of Control Principles of Complex Systems.

1.Introduction

- Control theory asks how to influence the behavior of a dynamical system with appropriately chosen inputs so that the system's output follows a desired trajectory or final state.
- A key notion in control theory is the feedback process: The difference between the actual and desired output is applied as feedback to the system's input, forcing the system's output to converge to the desired output.
- **Nonlinear Dynamical System**:

$$\text{State Equation:} \quad \dot{x}(t) = f(t, x(t), u(t); \Theta) \tag{1a}$$

$$\text{Output Equation:} \quad y(t) = h(t, x(t), u(t); \Theta) \tag{1b}$$

  where
  - $x(t) \in \mathbb{R}^N$ is the state vector representing the internal state of the system at time t.
  - $y(t) \in \mathbb{R}^R$ is the output vector capturing the set of experimentally measured variables.
  - $u(t) \in \mathbb{R}^M$ captures the known input signals.
  - Θ is a set of parameters collects the system's parameters.
  - $f(\cdot)$ and $h(\cdot)$ are generally nonlinear maps.
- **Linear Dynamical System**:
  - **Mathematical Formulation**

$$\text{State Equation:} \quad \dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{2a}$$

$$\text{Output Equation:} \quad y(t) = C(t)x(t) + D(t)u(t) \tag{2b}$$

    where
    - $A(t) \in \mathbb{R}^{N\times N}$ is the state or system matrix that defines the interactions between system components and their strength or nature,
    - $B(t) \in \mathbb{R}^{N\times M}$ is the input matrix,
    - $C(t) \in \mathbb{R}^{R\times N}$ is the output matrix,
    - $D(t) \in \mathbb{R}^{R\times M}$ is the feedthrough or feedforward matrix.
  - **Classification**: When $A(t), B(t), C(t)$, and $D(t)$ are time-varying, the system is classified as a linear time-varying (LTV) system. If $A(t), B(t), C(t)$, and $D(t)$ are constant matrices, the system is a **linear time-invariant (LTI) system**, which is fundamental in control theory.

    Many nonlinear systems like Eqs. (1a) and (1b) can be **linearized** around their equilibrium points, resulting in an LTI system.
  - **Simplification**: Given that $u(t)$ and $D(t)$ are typically known, the system can be simplified by defining a new output vector:

$$\tilde{y}(t) \equiv y(t) - D(t)u(t) = C(t)x(t)$$

    This allows the $D(t)u(t)$ term to be ignored, focusing on the state-dependent part of the output.
- **Major Challenges**: To extract the predictive power of Eqs.(1a) and (1b), we need

  - the accurate wiring diagram of the system,
  - a description of the nonlinear dynamics that governs the interactions between the components,
  - a precise knowledge of the system parameters.

  For most complex systems, we lack some of these prerequisites.
- **Fundamental Issues in Control of Complex Systems**:
  - **Controllability**:

    - Before deciding how to control a system, we must make sure that it is possible to control it. **Controllability**, a key notion in modern control theory, quantifies our ability to steer a dynamical system to a desired final state in finite time.
    - A system is controllable if we can drive it from any initial state to any desired final state in finite time (Kalman, 1963).
- **Observability**:
    - As a dual concept of controllability, **observability** describes the possibility of inferring the initial state of a dynamical system by monitoring its time-dependent outputs.
    - We also explore a closely related concept, **identifiability**, the ability to determine system parameters via input-output measurements.
- **Steering Complex Systems to Desired States or Trajectories**:
    - Goal: Drive systems from current states to desired final states or trajectories.
    - Methods:
        - Small perturbations to feasible parameters.
        - Compensatory perturbations exploiting the basin of attraction of the desired state.
        - Mapping control problems to combinatorial optimization on the network.
- **Controlling Collective Behavior**:
    - **Collective behavior** can result from the coordinated local activity of many interdependent components. We review a broad spectrum of methods to determine the conditions for the emergence of collective behavior and discuss pinning control as an effective control strategy.

2.Controllability of Linear Systems

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3.Controllability of Nonlinear Systems

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4.Observability

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5.Toward Desired Final States or Trajectories

- In many cases, we are interested in steering the system toward a desired **trajectory** or **attractor**, instead of an arbitrary final state.
- A **trajectory** or an orbit of a dynamical system is a collection of points (states) in the state space.
- An **attractor** is a closed subset $\mathcal{A}$ of a dynamical system's state space such that for "many" choices of initial states the system will evolve toward states in $\mathcal{A}$.
- Simple attractors correspond to fundamental geometric objects, such as points, lines, surfaces, spheres, toroids, manifolds, or their simple combinations. Fixed (or equilibrium) point and limit cycle are common simple attractors.
    - Fixed point: $x^* = f(x^*)$.
    - Equilibrium point: $\dot{x} = 0$.
- An attractor is called *strange* (奇异) if it has a fractal (分形) structure. A strange attractor often emerges in chaotic dynamics.

5.1.Controlling Chaos

- A deterministic dynamical system is said to be **chaotic** if its evolution is highly sensitive to its initial conditions. This **sensitivity** means that arbitrary small measurement errors in the initial conditions grow exponentially with time, destroying the long-term predictability of the system's future state (butterfly effect).
- It has been realized that well-designed control laws can overcome the butterfly effect, **forcing chaotic systems to follow some desired behavior**.

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5.2.Compensatory Perturbations of State Variables

- The basic insight of the approach is that each desirable state has a **"basin of attraction"** representing a region of initial conditions whose trajectories converge to it. For a system that is in an undesired state, we need to identify perturbations to the state variables that can bring the system to the basin of attraction of the desired target state. Once there, the system will evolve spontaneously to the target state.
- We use a nonlinear optimization approach to identify the perturbation δx that can bring the system to the basin of attraction of the desired target state. If the system's orbit reaches a small ball of radius κ around x^* within a certain time, we declare success and recognize δx as a compensatory perturbation.

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5.3.Small Perturbations to System Parameters

- The key step of this approach is to choose a set of experimentally adjustable parameters and determine whether small perturbations to these parameters can steer the system toward the desired attractor.
- Considering each attractor as a node, and the control paths as directed edges between them, we can construct an **attractor network**, whose properties determine the controllability of the original dynamic network.

5.4.Dynamics and Control at Feedback Vertex Sets

A Feedback Vertex Set (FVS) is a subset of nodes in the absence of which the digraph becomes acyclic.

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6.Controlling Collective Behavior

- Dynamical agents interacting through complex networks can display various collective behaviors. In this section, we mainly discuss **synchronization** and **flocking**.
- **Pinning Control**: If we aim to achieve a desired collective behavior, it is often infeasible to directly control all nodes of a large network. This difficulty is partially alleviated by the notion of **pinning control**. In pinning control, a feedback control input is applied to a small subset of nodes called **pinned nodes**, which propagates to the rest of the network through the edges.

  > Conceptually, pinning control is similar to the minimum controllability problem of a linear system.

  Pinning control has been extensively applied to the **synchronization** of coupled oscillators and **flocking** of interacting agents, which are discussed as follows.

6.1.Synchronization of Coupled Oscillators

The general equation for a coupled oscillator system is:

$$\begin{aligned} \dot{x}_i &= f(x_i) + \sigma \sum_{j=1}^{N} a_{ij} w_{ij} [h(x_j) - h(x_i)] \\ &= f(x_i) - \sigma \sum_{j=1}^{N} g_{ij} h(x_j) \end{aligned} \tag{90}$$

where:

- $x_i \in \mathbb{R}^d$ is the d-dimensional state vector of the i-th node.
- $f(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ determines the individual dynamics of each node.
- σ is the coupling strength, also called the *coupling gain*.
- $A = (a_{ij}) \in \mathbb{R}^{N \times N}$ is the adjacency matrix.
- $w_{ij} \geq 0$ is the weight of link (i, j).
- $h(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is used to couple the oscillators and is identical for all oscillators.

> For example, if we use $h(\mathbf{x}) = (x, 0, 0)^T$ for a three-dimensional oscillator, it means that the oscillators are coupled only through their x components.

In general, $h(\cdot)$ can be any linear or nonlinear mapping of the state vector x.

- $\mathbf{G} = (g_{ij}) \in \mathbb{R}^{N\times N}$ is the **coupling matrix**, where
  - $g_{ij} = -a_{ij}w_{ij} \quad (i \neq j)$
  - $g_{ii} = -\sum_{j=1,j\neq i}^{N} g_{ij}$.

  If $w_{ij} = 1$ for all links, G is the Laplacian matrix L of the network. Note that G is not necessarily symmetric.

The system is synchronized when the trajectories of all nodes converge to a common trajectory, i.e., $\lim_{t\to\infty} |||x_i(t) - x_j(t)||| = 0$ for all i, j. Such synchronization behavior describes a continuous system that has a uniform movement.

The completely synchronized state $x_1(t) = x_2(t) = \cdots = x_N(t) = s(t)$ is a natural solution of Eq. (90). This also defines a linear invariant manifold, called the **synchronization manifold**, where all the oscillators evolve synchronously as $\dot{s} = f(s)$. Note that $s(t)$ may be an equilibrium point, a periodic orbit, or even a chaotic solution.

Our goal is to **identify the conditions under which the system can synchronize**. The best-known method, discussed next, is based on the calculation of the eigenvalues of the coupling matrix G.

6.1.1.Master Stability Formalism and Beyond

Consider the stability of the synchronization manifold with a small perturbation $x_i(t) = s(t) + \delta x_i(t)$. By expanding $f(x_i)$ and $h(x_i)$ to the first order of δx_i, we obtain a linear variational equation for $\delta x_i(t)$,

$$\delta\dot{x}_i = \mathcal{J}(s)\delta x_i - \sigma \sum_{j=1}^{N} g_{ij}\mathcal{E}(s)\delta x_j \tag{92}$$

with Jacobian matrices

- $\mathcal{J}(s) = \left.\frac{\partial f(x)}{\partial x}\right|_{x=s} \in \mathbb{R}^{d\times d}$
- $\mathcal{E}(s) = \left.\frac{\partial h(x)}{\partial x}\right|_{x=s} \in \mathbb{R}^{d\times d}$

Let $\delta X \equiv [\delta x_1, \ldots, \delta x_N]^T$, Then formally we have

> Question: Maybe δX should be $[\delta x_1^T, \delta x_2^T \cdots, \delta x_N^T]^T$ instead of $[\delta x_1, \delta x_2, \cdots, \delta x_N]^T$, since we usually treat x as a column vector with dimension $\mathbb{R}^{N\times 1}$. If we adopt the modification, then $\delta X \in \mathbb{R}^{Nd\times 1}$, which aligns with the dimension $\mathbb{R}^{Nd\times Nd}$ of $I \otimes \mathcal{J}(s)$ and $G \otimes \mathcal{E}(s)$.

$$\delta\dot{X} = [I \otimes \mathcal{J}(s) - \sigma G \otimes \mathcal{E}(s)]\delta X \tag{93}$$

where I is the $N \times N$ identity matrix and $\otimes$ is the Kronecker product (a.k.a. matrix direct product).

> The **matrix direct product**, also known as the **Kronecker product**, is often denoted by $\otimes$. Given $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ and $B = (b_{ij}) \in \mathbb{R}^{p\times q}$, we have
>
> $$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{mp\times nq}$$

The key idea of the master stability formalism is that we need to consider only variations that are **transverse** to the synchronization manifold, as variations along $s(t)$ leave the system in the synchronized state. If these transverse variations damp out, then the synchronization manifold is stable.

To separate out the transverse variations, we can project δX into the eigenspace spanned by the eigenvectors e_i of the coupling matrix G, i.e., $\delta X = (P \otimes I_d)\Xi$ with $P^{-1}GP = \hat{G} = \mathrm{Diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$. Then we have

$$\dot{\Xi} = [I \otimes \mathcal{J}(s) - \sigma \hat{G} \otimes \mathcal{E}(s)]\Xi \tag{94}$$

which results in a block diagonalized variational equation with N blocks, corresponding to N decoupled eigenmodes. Each block has the form

$$\dot{\xi}_i = [\mathcal{J}(s) - \sigma\lambda_i \mathcal{E}(s)]\xi_i \tag{95}$$

where $\xi_i \in \mathbb{R}^d$ is the eigenmode associated with the eigenvalue λ_i of G. Thus, each eigenmode of the perturbation is decoupled from the others and will damp out independently and simultaneously.

> If G is not diagonalizable, we can transform G into the Jordan canonical form, and there have been some further studies.

We can order the eigenvalues of G such that $0 = \lambda_1 \leq \operatorname{Re}\lambda_2 \leq \cdots \leq \operatorname{Re}\lambda_N$. Because the row sum of G is zero, the minimal eigenvalue λ_1 is always zero with the corresponding eigenvector $e_1 = (1, 1, \ldots, 1)^T$. Hence the first eigenmode $\xi_1 = \mathcal{J}(s)\xi_1$ corresponds to the perturbation parallel to the synchronization manifold. According to the Gerschgorin circle theorem, all other eigenvalues must have non-negative real parts. The corresponding $(N-1)$ eigenmodes are transverse to the synchronization manifold and must decay to have a stable synchronization manifold.

> Conclusion of the paragraph above: G must have a 0 eigenvalue, with the corresponding eigenvector $e_1 = (1, 1, \ldots, 1)^T$. The remaining $(N-1)$ eigenmodes are transverse to the synchronization manifold and must decay to have a stable synchronization manifold.

The form of each block in Eq. (95) is the same up to the scalar multiplier $\sigma\lambda_i$. This leads to the variational equation, called the master stability equation,

$$\dot{\xi} = [\mathcal{J} - (\alpha + i\beta)\mathcal{E}]\xi \tag{96}$$

For small ξ we have $\|\xi(t)\| \sim \exp[\Lambda(\alpha, \beta)t]$, which decays exponentially if the maximum Lyapunov characteristic exponent $\Lambda(\alpha, \beta) < 0$. Consequently, $\Lambda(\alpha, \beta)$ is called the master stability function (**MSF**).

Given a coupling strength σ [remember $\sigma\lambda_i = (\alpha + i\beta)$], the sign of the MSF in the point $\sigma\lambda_i$ in the complex plane reveals the stability of that eigenmode. If all eigenmodes are stable [i.e., $\Lambda(\sigma\lambda_i) < 0$ for all i's], then the synchronization manifold is stable at that coupling strength.

> Note: since the master stability formalism assesses only the linear stability of the synchronized state, it yields only the necessary but not the sufficient condition for synchronization.

> Appendix: For undirected and unweighted networks, the coupling matrix $\mathbf{G}$ is symmetric and all its eigenvalues are real, simplifying the stability analysis. In this case, the MSF $\Lambda(\alpha)$ can be classified as follows:
>
> - Bounded: $\Lambda(\alpha) < 0$ for $\alpha_1 < \alpha < \alpha_2$. The condition can be fulfilled if the eigen ratio R satisfies:
>
> $$R \equiv \frac{\lambda_N}{\lambda_2} < \frac{\alpha_2}{\alpha_1}$$
>
> - Unbounded: $\Lambda(\alpha) < 0$ for $\alpha > \alpha_1$. The condition can be fulfilled if
>
> $$\sigma > \sigma_{\min} = \alpha_1/\lambda_2$$

6.1.2.Pinning synchronizability

If a network of coupled oscillators cannot synchronize spontaneously, we can design controllers that applied to a **subset** of pinned nodes $\mathcal{C}$ to help synchronize the network.

This procedure is known as **pinning synchronization** (as opposed to spontaneous synchronization). In pinning synchronization, we choose the desired trajectory $s(t)$, aiming to achieve some desired control objective.

A controlled network is described by

$$\dot{x}_i = f(x_i) - \sigma \sum_{j=1}^{N} g_{ij} h(x_j) + \delta_i u_i(t) \tag{99}$$

where $\delta_i = 1$ for pinned nodes and 0 otherwise, and

$$u_i(t) = \sigma[p_i(s(t)) - p_i(x_i(t))] \tag{100}$$

is the d-dimensional linear feedback controller, $p_i(x(t))$ is the pinning function that controls the input of node i, and $s(t)$ is the desired synchronization trajectory satisfying $\dot{s}(t) = f(s(t))$.

Note that in the fully synchronized state $x_1(t) = x_2(t) = \cdots = x_N(t) = s(t)$, we have $u_i(t) = 0$ for all nodes. The form of the linear feedback controller (100) implies that the completely synchronized state is a natural solution of the controlled network (99).

6.1.2.1.Local Pinning Synchronizability

we first introduce a virtual node whose dynamics follows $\dot{s}(t) = f(s(t))$, representing the desired synchronization solution. The extended system now has $N+1$ nodes: $y_i(t) = x_i(t)$ for $i = 1, \cdots, N$, and $y_{N+1}(t) = s(t)$. The virtual node is connected to each pinned node.

We choose the pinning function with control gains:

$$p_i(x) = \kappa_i h(x)$$

Then we have $u_i(t) = \sigma\kappa_i[h(s(t)) - h(x_i(t))]$ and Eq.(99) becomes

$$\dot{y}_i = f(y_i) - \sigma \sum_{j=1}^{N+1} m_{ij} h(y_j) \tag{102}$$

where

$$M = \begin{bmatrix} g_{11} + \delta_1\kappa_1 & g_{12} & \cdots & g_{1N} & -\delta_1\kappa_1 \\ g_{21} & g_{22} + \delta_2\kappa_2 & \cdots & g_{3N} & -\delta_2\kappa_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{N1} & g_{N2} & \cdots & g_{NN} + \delta_N\kappa_N & -\delta_N\kappa_N \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

is the effective coupling matrix of the $(N+1)$-dimensional extended system. Apparently, $M = (m_{ij})_{(N+1)\times(N+1)}$ is a zero row-sum matrix; hence we can sort its eigenvalues as $0 = \lambda_1 \le \operatorname{Re}\lambda_2 \le \cdots \le \operatorname{Re}\lambda_{N+1}$. We can now apply the MSF approach to numerically explore the local stability of the synchronization manifold of the controlled network (102).

It has been proved that:

- Controlling more nodes always enhances the network pinning synchronizability, in line withour intuition.
- Selective pinning, when the nodes are chosen in the order of decreasing degree, yields better synchronizability than random pinning.

6.1.2.2.Global Pinning Synchronizability

It has been proved that

- If the desired asymptotic trajectory is an equilibrium point $\dot{s} = f(s) = 0$, we can derive sufficient conditions for globally stabilizing the pinning controlled network.
- For a more general desired trajectory, it has been shown that a single feedback controller can pin a complex network to a homogenous solution, without assuming symmetry, irreducibility, or linearityof the couplings.
- For a connected network, even for a limited number of pinned nodes, global pinning synchronizability can be achieved by properly selecting the coupling strength and the feedback gain.
- If $h(x) = \Gamma x$ and the oscillator dynamics $f(x)$ satisfies

$$(x-y)^T[f(x,t) - f(y,t)] \le (x-y)^T K\Gamma(x-y) \tag{105}$$

for a constant matrix K, sufficient conditions for pinning synchronizability can also be derived. Note that the condition (105) is so mild that many systems including RNN may satisfy this.

- Counterintuitively, it was found that for undirected networks the small-degree nodes, instead of hubs, should be pinned first when the coupling strength σ is small.
- For directed networks, nodes with very small in-degree or large out-degree should be pinned first.

> This result can be understood by realizing that low in-degree nodes receive less information from other nodes and hence are less "influenced" by others. In the extreme case, nodes with zero in-degree will not be influenced by any other nodes, hence they must be pinned first. On the other hand, large out-degree nodes can influence many other nodes, hence it makes sense to pin them first.

6.1.3.Adaptive Pinning Control

Implementing the linear feedback pinning controller Eq. (100) requires detailed knowledge of the global network topology. Yet in practice we do not always have access to the global network topology.

Given this limitation, **adaptive control** has been proposed for pinning synchronization, in which case a controller adapts to a controlled system with parameters that **vary in time**, or are initially uncertain.

6.1.3.1.Adaptation of Control Gains

To adapt the control gain κ_i in Eq.(101), we choose the control input $u_i(t) = -\delta_i \kappa_i(t)[x_i(t) - s]$, and the control gains as

$$\dot{\kappa}_i(t) = q_i |e_i(t)| \tag{106}$$

In other words, the control gain κ_i varies in time and adapts to the error vector $e_i(t) \equiv s(t) - x_i(t)$.

6.1.3.2.Adaptation of Coupling Gains

The coupling gain σ_{ij}, defining the mutual coupling strength between node pair (i, j), can also be adapted using

$$\dot{\sigma}_{ij}(t) = \eta_{ij} |e_i(t) - e_j(t)|^2 \tag{107}$$

> However, these two adaptive strategies still require a prior selection of the pinned nodes based on some knowledge of the network topology. This limitation can be avoided by choosing pinned nodes in an adaptive fashion, as discussed next.

6.1.3.3.Adaptive Selection of Pinning Nodes

Adaptive pinning can be achieved by assuming the pinning node indicator δ_i to be neither fixed nor binary. A common approach is to introduce

$$\delta_i(t) = b_i^2(t) \tag{108}$$

where $b_i(t)$ satisfies the dynamics

$$\ddot{b}_i + \zeta \dot{b}_i + \frac{dU(b_i)}{db_i} = g(|e_i|) \tag{109}$$

In other words, $b_i(t)$ follows the dynamics of a unitary mass in a potential $U(b_i)$ subject to an external force g that is a function of the pinning error e_i and a linear damping term described by $\zeta \dot{b}_i$. This is termed as the edge-snapping mechanism.

For convenience, $U(\cdot)$ can be chosen as a double-well potential $U(z) = kz^2(z-1)^2$.

Eq. (109) has only two stable equilibria, 0 and 1, describing whether node i is pinned or not, respectively. So we do not have to choose the nodes we need to pin beforehand. Instead, we can select them in an adaptive fashion.

6.1.3.4. Adaptation of the Network Topology

We can ensure synchronization by adapting the network topology. Specially, we can set each off-diagonal element of the Laplacian matrix of the network as

$$\mathcal{L}_{ij}(t) = -\sigma_{ij}(t)\alpha_{ij}^2(t) \tag{110}$$

where $\sigma_{ij}(t)$ is the mutual coupling strength between node pair (i, j), which is adapted as in Eq.(107). The weight $\alpha_{ij}(t)$ is associated with every undirected edge of the target pinning edge and is adapted as

$$\begin{aligned} \dot{\alpha}_{ij} + \nu\dot{\alpha}_{ij} + \frac{dU(\alpha_{ij})}{d\alpha_{ij}} &= c(|e_{ij}|), \\ i, j = 1, \ldots, N, \quad i &\neq j, \end{aligned} \tag{111}$$

where $e_{ij}(t) = e_j(t) - e_i(t)$, and $U(\cdot)$ can be again chosen as a double-well potential so that Eq.(111) has only two stable equilibria, 0 and 1. In this case, the target network topology evolves in a decentralized way.

6.2.Flocking of Multiagent Dynamical System

The flocking of birds, shoaling of fish, swarming of insects, and herding of land animals are spectacular manifestations of a coordinated collective behavior of multiagent systems.

The Vicsek model and its variants can be interpreted as a decentralized feedback control system with time-varying network structure.

6.2.1.Vicsek Model and the Alignment Problem

The Vicsek model explains the origin of **alignment**, a key feature of flocking behavior. It is a discrete-time stochastic model in which autonomous agents move in a plane with a constant speed v_0, and following randomly chosen directions initially. The position x_i of agent i changes as

$$x_i(t+1) = x_i(t) + v_i(t+1)$$

where the velocity of each agent has the same absolute value v_0.

The direction of agent i is updated using a local rule that depends on the average of its own direction and the directions of its "neighbors", i.e., all agents within a distance r from agent i (Fig. 37), and some random perturbations. These random perturbations
can be rooted in any stochastic or deterministic factors that
affect the motion of the flocking agents.

$$\theta_i(t+1) = \langle\theta_i(t)\rangle_r + \Delta_i(t)$$

- $\langle\theta_i(t)\rangle_r$ is the average direction of the agents (including agent i) within a circle of radius r.
- The perturbations are contained in $\Delta_i(t)$, which is a random number taken from a uniform distribution in the interval $[-\frac{\eta}{2}, \frac{\eta}{2}]$. These random perturbations can be rooted in any stochastic or deterministic factors.

The Vicsek model has three parameters:

- the agent density ρ (number of agents in the area L^2)
- the speed v_0
- the magnitude of perturbations η.

The model's order parameter is the normalized (vectorial) average velocity:

$$\phi \equiv \frac{1}{Nv_0} \sum_{i=1}^{N} |\mathbf{v}_i|$$

> If the system is in a completely synchronized state, then $\phi = 1$; if the system is in a completely disordered state, then $\phi = 0$.

For small speed v_0, if we decrease the magnitude of perturbations η, the Vicsek model displays a continuous phase transition from a disordered phase (Fig. 38(b)) to an ordered phase (Fig. 38(d)).

This phase transition takes place despite the fact that each agent's set of nearest neighbors changes with time and the absence of centralized coordination.

The Vicsek model raises a fundamental control problem: **Under what conditions can the multiagent system display a particular collective behavior**? There is a dynamically changing network, where two agents are connected if they interact. Since the agents are moving, the network of momentarily interacting units evolves in time in a complicated fashion.

Denote the interaction graph at time t as $G(t)$. If the interaction radius r is small,some agents are always isolated, implying that $G(t)$ is never connected. If r is large, then $G(t)$ is always a complete graph.

Theoretical Findings: If $G(t)$ is connected for all $t \geq 0$, we can prove that alignment will be asymptotically reached (Jadbabaie, Lin, and Morse, 2003). This result has been further extended by proving that if the collection of graphs is ultimately connected, i.e., there exists an initial time t_0 such that over the infinite interval $[t_0, \infty)$ the union graph $G = \bigcup_{t=t_0}^{\infty} G(t)$ is connected, then the alignment is asymptotically reached.

> Explanation of Graph Union: Union and Intersection Operation On Graph

Although the control theoretical analysis is deterministic, ignoring the presence of noise, it offers rigorous theoretical explanations based on the connectedness of the underlying graph. These control theoretical results suggest that to understand the effect of additive noise, we should focus on how noise inputs affect the connectivity of the associated neighbor graphs.

> For example, for a fixed noise beyond a critical agent density, all agents eventually become aligned. This can be adequately explained by percolation theory of random graphs.

6.2.2. Alignment Via Pinning

We can also ask if the spontaneously ordered phase can be induced externally. Therefore, we consider an effective pinning control strategy in which a single pinned node (agent) facilitates the alignment of the whole group. This is achieved by adding to the Vicsek model an additional agent, labeled 0, which acts as the group's **leader**.

Agent 0 moves at the same constant speed v_0 as its N followers but with a fixed direction θ_0, representing the desired direction for the whole system. If a follower is within a distance r of agent 0, it can be influenced by agent 0.

It has been proved that if the $N+1$ agents are linked together for each time interval, then alignment will be asymptotically reached. In other words, if the union of graphs of the $N+1$ agents encountered along each time interval is connected, then eventually all the follower agents will align with the leader.

6.2.3.Distributed Flocking Protocols

Alignment, addressed by the Vicsek model, is only one feature of flocking behavior. Indeed, there are three heuristic rules for flocking:

- Cohesion: an attempt to stay close to nearby flock mates;
- Separation: avoid collisions with nearby flock mates;
- Alignment: an attempt to match velocity with nearby flock mates.

We therefore need a general theoretical framework.

Denote the position as $\mathbf{q}$ and velocity as $\mathbf{p}$.

Consider a gradient-based flocking protocol equipped with a velocity consensus mechanism, where each agent is steered by the control input:

$$\mathbf{u}_i = \mathbf{f}_i^g + \mathbf{f}_i^d \tag{121}$$

The first term

$$\mathbf{f}_i^g \equiv \nabla_{q_i} V_i(\mathbf{q}) \tag{122}$$

is gradient based and regulates the **distance** between agent i and its neighbors, avoiding the collision and cohesion of the agents. This term is derived from a smooth collective potential function $V_i(\mathbf{q})$, which has a unique minimum when each agent is at the same distance from all of its neighbors.

> In the ideal case, $\nabla_{q_i} V_i(\mathbf{q}) = 0$, so the first term has no effect on the motion of the agent.

The second term

$$\mathbf{f}_i^d \equiv \sum_{j \in \mathcal{N}_i(t)} a_{ij}(t)(\mathbf{p}_j - \mathbf{p}_i) \tag{123}$$

regulates the **velocity** of agent i to match the average velocity of its neighbors.

The flocking protocol Eq. (121) embodies all three rules. However, for a generic initial state and a large number of agents (e.g., $N > 100$), the protocol leads to fragmentation, rather than flocking. To resolve this fragmentation issue, we introduce a navigational feedback term to the control input of each agent:

$$\mathbf{u}_i = \mathbf{f}_i^g + \mathbf{f}_i^d + \mathbf{f}_i^\gamma \tag{124}$$

where

$$\mathbf{f}_i^\gamma \equiv -c_1(\mathbf{q}_i - \mathbf{q}_\gamma) - c_2(\mathbf{p}_i - \mathbf{p}_\gamma) \tag{125}$$

drives agent i to follow a **group objective**. The group objective can be considered as a virtual leader with the following equation of motion:

$$\dot{\mathbf{q}}_\gamma = \mathbf{p}_\gamma, \quad \dot{\mathbf{p}}_\gamma = f_\gamma(\mathbf{q}_\gamma, \mathbf{p}_\gamma) \tag{126}$$

where $\mathbf{q}_\gamma, \mathbf{p}_\gamma, \mathbf{f}_\gamma(\mathbf{q}_\gamma, \mathbf{p}_\gamma) \in \mathbb{R}^d$ are the position, velocity, and acceleration (control input) of the virtual leader, respectively. The new protocol Eq. (124) enables a group of agents to track a virtual leader that moves at a **constant velocity** and hence leads to flocking behavior (Fig. 41 (note that the scale of graph changes over time)).

If the virtual leader travels with a **varying velocity** $p_\gamma(t)$, the flocking protocol Eq. (124) enables all agents to eventually achieve a common velocity. Yet this common velocity is not guaranteed to match $p_\gamma(t)$. To resolve this issue, we update the third term of control input as follows:

$$f_i^\gamma = f_\gamma(q_\gamma, p_\gamma) - c_1(q_i - q_\gamma) - c_2(p_i - p_\gamma)$$

The resulting protocol enables the asymptotic tracking of the virtual leader with a varying velocity

Motivated by the idea of pinning control, it has been shown that, even when only a fraction of agents are informed, an uninformed agent will also move with the desired velocity, if it can be influenced by the informed agents from time to time.

Numerical simulations suggest that the larger the informed group is, the bigger fraction of agents will move with the desired velocity.

7.Outlook

Some major research topics of the control of complex systems include:

- **Stability of Complex Systems**:
  - Loosely speaking, a system is stable if its trajectories do not change too much under small perturbations of initial conditions. (**input stability**)
  - Another important notion of stability is **structural stability**, which concerns whether the qualitative behavior of the system trajectories will be affected by small perturbations of the system model itself. The notion of structural stability has not been well explored.
- **Controlling Adaptive Networks**:
  - **Adaptability** represents a system's ability to respond to changes in the external conditions.
  - **Adaptive networks**, also known as state-dependent dynamic networks, are collections of units that interact through a network, whose **topology evolves with time** as the state of the units changes.
  - A comprehensive analytical framework is needed to address the control of adaptive networks. This framework must recognize the network structure itself as a dynamical system, together with the nodal or edge dynamics on the network, capturing the feedback mechanisms linking the structure and dynamics.
- **Noise**: We lack a full understanding of the role of noise or stochastic fluctuations on the control of complex systems.
  - Complex systems are affected by two kinds of noise: the **intrinsic randomness** of individual events, and the **extrinsic influence** of changing environments.

- **Controlling Networks of Networks**
- **Controlling Quantum Networks**

8.Conclusion

(pass)