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Maximal Ideals in an Algebra of Bounded Analytic Functions

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1. Introduction. Let D be the open unit disc in the complex plane, and let B be the algebra of bounded analytic functions on D. Using the uniform norm, B is a commutative Banach algebra which has attracted considerable attention in recent years. In this paper we shall present various results concerning the maximal ideal space of B. The results were obtained during the Conference on Analytic Functions held at the Institute for Advanced Study in 1957.

The structure of the paper is as follows. Section 2 introduces the space $\mathfrak{M}$ of complex homomorphisms (maximal ideals) of the algebra B, as well as the Gelfand isomorphism $f \to \hat{f}$ of B with a uniformly closed algebra of continuous functions on $\mathfrak{M}$. There is a natural *projection* π of $\mathfrak{M}$ onto the closed disc in the plane, obtained by sending each complex homomorphism into its value on the coordinate function z. This map π is one-one over the open disc D, and shows that the natural injection of D into $\mathfrak{M}$, which sends λ into the homomorphism "evaluation at λ", is a homeomorphism of D onto an open subset Δ of $\mathfrak{M}$. The remaining closed set of homomorphisms is mapped by π onto the unit circle C. This closed set $\mathfrak{M} - \Delta$ is decomposed by π into disjoint closed *fibers* $\mathfrak{M}_\alpha$, where for $|\alpha| = 1$

$$\mathfrak{M}_\alpha = \{\varphi \in \mathfrak{M}; \varphi(z) = \alpha\} = \pi^{-1}(\alpha).$$

Through the action of the rotation group of the plane on B, one sees that the fibers $\mathfrak{M}_\alpha$ are homeomorphic with one another.

In Section 3 we identify the Silov boundary for the algebra B. Its description is as follows. A theorem of FATOU enables one to identify B with a closed subalgebra of the algebra L^∞ of essentially bounded measurable functions on the unit circle. This gives a natural continuous map τ of the (extremally disconnected) space of maximal ideals of L^∞ into the space $\mathfrak{M}$. We show that τ is a homeomorphism, the range of which is the Silov boundary for B. It is observed that the Silov boundary is a subset of $\bar{\Delta} - \Delta$, but does not exhaust $\mathfrak{M} - \Delta$.

[1] A pseudonym for a large group of mathematicians who discussed these problems during the 1957 Conference on Analytic Functions sponsored by the Institute for Advanced Study under contract No. AF 18(603)-118 with the Air Force Office of Scientific Research.

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Section 4 contains further results on the fibers $\mathfrak{M}_\alpha$. In particular, for any f in B the range of the representing function $\hat{f}$ on $\mathfrak{M}_\alpha$ is the set of all cluster points of sequences $\{f(\lambda_n)\}$, where $\{\lambda_n\}$ is any sequence of points in D which converges to the boundary point α. Using such facts we show that the decomposition

$$\mathfrak{M} - \Delta = \bigcup_{\alpha \varepsilon C} \mathfrak{M}_\alpha$$

is not naturally homeomorphic to the topological product of the circle C and one of the fibers $\mathfrak{M}_\alpha$, even though the fibers are all homeomorphic.

Section 5 contains what is perhaps the most startling result; it is this. If α is any point on the circle, there is a homeomorphism ψ of the open disc D into the fiber $\mathfrak{M}_\alpha$ with this property: If we transfer the analytic structure of D to the image $\psi(D)$, then for each f in B the restriction of $\hat{f}$ to $\psi(D)$ is bounded and analytic, and every bounded analytic function on $\psi(D)$ arises in this way. Thus each fiber $\mathfrak{M}_\alpha$ contains a homeomorphic replica of the entire maximal ideal space $\mathfrak{M}$.

2. The Algebra of Bounded Analytic Functions. Let $D = \{\lambda; |\lambda| < 1\}$ be the open unit disc in the complex plane, and let $C = \{\lambda; |\lambda| = 1\}$ be the unit circle. We denote by B the complex linear algebra of all bounded analytic functions on D. Using the norm

$$||f|| = \sup_{\lambda \varepsilon D} |f(\lambda)|,$$

B is a commutative Banach algebra (with identity).

We shall denote by $\mathfrak{M} = \mathfrak{M}(B)$ the space of complex homomorphisms of the algebra B. It might be well if we review very briefly how the space $\mathfrak{M}$ is defined. The elements of $\mathfrak{M}$ are the homomorphisms of B onto the algebra of complex numbers, *i.e.*, the multiplicative linear functionals on B. If φ is such a homomorphism, then $|\varphi(f)| \leqq ||f||$ for all f in B. For if we had $|\varphi(f)| > ||f||$, the function $g(\lambda) = f(\lambda) - \varphi(f)$ would be invertible in B, leading us to the two contradictory statements

$$\varphi(g)\varphi\left(\frac{1}{g}\right) = \varphi(1) = 1,$$

$$\varphi(g) = 0.$$

Each point $\lambda \varepsilon D$ defines a complex homomorphism of B by

$$\varphi_\lambda(f) = f(\lambda).$$

From this one sees that

$$\sup_\varphi |\varphi(f)| = ||f||$$

because

$$||f|| \geqq \sup_\varphi |\varphi(f)| \geqq \sup_\lambda |\varphi_\lambda(f)| = ||f||.$$

With each f in B we associate a complex-valued function $\hat{f}$ on $\mathfrak{M}$ by

$$\hat{f}(\varphi) = \varphi(f).$$

As we have just pointed out,

$$\sup_{\mathfrak{M}} |\hat{f}(\varphi)| = ||f||.$$

From this one sees easily that $f \to \hat{f}$ is an isomorphism of B with an algebra $\hat{B} = \{\hat{f}; f \in B\}$ of complex-valued functions on $\mathfrak{M}$. We now topologize $\mathfrak{M}$ with the weakest topology which makes each $\hat{f}$ continuous. With this topology $\mathfrak{M}$ is a compact Hausdorff space and $\hat{B}$ is a uniformly closed algebra of continuous functions on $\mathfrak{M}$.

The space $\mathfrak{M}$ is often called the maximal ideal space of B. The terminology arises from this general fact about a commutative Banach algebra [1]. For each φ in $\mathfrak{M}$, the kernel of φ is a maximal ideal in the algebra B; conversely, every maximal ideal in B arises in this way. We have chosen to think of the points of $\mathfrak{M}$ as complex homomorphisms, rather than maximal ideals, because this will be a notational advantage.

Our object here is to relate some of the structure of the space $\mathfrak{M}$. At the outset the only complex homomorphisms of B which one can clearly identify are those which arise from points in the open disc D:

$$\varphi_\lambda(f) = f(\lambda). \tag{2.1}$$

Thus the task is to see what one can say about the remainder of the space $\mathfrak{M}$.

There is a natural *projection* or mapping of $\mathfrak{M}$ into the closed unit disc in the plane. If we denote by z the identity function on D,

$$z(\lambda) = \lambda, \tag{2.2}$$

the mapping we have in mind is simply the function $\hat{z}$. It is perhaps better to introduce a separate symbol for $\hat{z}$:

$$\pi(\varphi) = \varphi(z). \tag{2.3}$$

Theorem 2.1. *The projection π defined by* (2.3) *is a continuous map of $\mathfrak{M}$ onto the closed unit disc in the plane. If $\Delta = \pi^{-1}(D)$, then π maps the open set Δ homeomorphically onto the open disc D.*

Proof. By its very definition, $\pi = \hat{z}$ is a continuous complex-valued function on $\mathfrak{M}$. It maps $\mathfrak{M}$ into the closed disc because $||z|| = 1$. We have already observed (2.1) that each point of the open disc D is in the range of π:

$$\pi(\varphi_\lambda) = \lambda.$$

Since $\pi(\mathfrak{M})$ is compact and contains D, it contains all of the closed disc $D \cup C$.

Now let us observe that π is one-one over D. In other words, if $\varphi \in \mathfrak{M}$ and $\pi(\varphi) = \varphi(z) = \lambda$ with $|\lambda| < 1$, then $\varphi = \varphi_\lambda$. If $\varphi(z) = \lambda$, then $\varphi(f) = 0$ for every f of the form $f = (z - \lambda)g$. Thus $\varphi(f) = 0$ whenever $f(\lambda) = 0$, so that φ must be evaluation at λ.

If $\Delta = \pi^{-1}(D) = \{\varphi_\lambda ; \lambda \in D\}$, then Δ is an open subset of $\mathfrak{M}$. Either on Δ or on D the topology is the weak topology defined by the functions in B, and thus π is a homeomorphism of Δ onto D, *q.e.d.*

Of course, π maps $\mathfrak{M} - \Delta$ onto the circle C. One would not expect that π is one-one over C, and so it will be convenient to have a notation for the inverse image under π of a point of the circle. If $|\alpha| = 1$ let

$$\mathfrak{M}_\alpha = \pi^{-1}(\alpha) = \{\varphi \in \mathfrak{M}; \varphi(z) = \alpha\}. \tag{2.4}$$

Let us call $\mathfrak{M}_\alpha$ the *fiber* of $\mathfrak{M}$ *over* α. Of course, $\mathfrak{M}_\alpha$ is a closed subset of $\mathfrak{M}$. Intuitively, the elements of $\mathfrak{M}_\alpha$ are the complex homomorphisms of $\mathfrak{M}$ which behave something like "evaluation at α", that is, homomorphisms φ of B which send each f in B into some sort of limiting value of $f(\lambda)$ as λ approaches the boundary point α. We shall make this precise later. For now, let us content ourselves with some more elementary properties of the fibers $\mathfrak{M}_\alpha$ which will help to give us some picture of $\mathfrak{M} - \Delta$.

The algebra B is rotation invariant. If $e^{i\theta}$ is a point of the circle, the corresponding rotation

$$(R_\theta f)(\lambda) = f(e^{i\theta}\lambda) \tag{2.5}$$

is an automorphism of B. The adjoint map R_θ^* defined by

$$(R_\theta^*\varphi)(f) = \varphi(R_\theta f) \tag{2.6}$$

is accordingly a homeomorphism of $\mathfrak{M}$ onto $\mathfrak{M}$. The rotation group thus acts as a discrete group of homeomorphisms of $\mathfrak{M}$. From this one sees that the various fibers $\mathfrak{M}_\alpha$ are homeomorphic. If α and β are points of the circle and

$$\beta = e^{i\theta}\alpha,$$

then the map R_θ^* carries $\mathfrak{M}_\alpha$ homeomorphically onto $\mathfrak{M}_\beta$. For $\varphi(z) = \alpha$ if and only if $(R_\theta^*\varphi)(z) = \varphi(e^{i\theta}z) = e^{i\theta}\alpha = \beta$.

3. The Silov Boundary. Let $L^\infty = L^\infty(C)$ be the space of all essentially bounded measurable complex-valued functions on the circle C. With the usual operations and the norm

$$||h||_\infty = \text{ess.sup } |h(e^{i\theta})|$$

L^∞ is a commutative Banach algebra. A classical theorem of Fatou enables us to associate with each bounded analytic function f an element F of L^∞ by

$$F(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}). \tag{3.1}$$

Fatou's theorem states that for each f in B the radial limit **(3.1)** exists for almost every θ and defines an element F of L^∞ such that

$$||f|| = ||F||_\infty . \tag{3.2}$$

One can recapture f from F by the Poisson formula

$$f(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} F(e^{i\theta})P_\lambda(\theta)\, d\theta. \tag{3.3}$$

The map $f \rightarrow F$ is an isometric isomorphism of B onto a closed subalgebra of L^∞. This subalgebra is often denoted H^∞. It consists of all F in L^∞ such that

$$\int_0^{2\pi} F(e^{i\theta})e^{in\theta}\, d\theta = 0, \qquad n = 1, 2, 3, \cdots . \tag{3.4}$$

The condition (3.4) is necessary and sufficient for (3.3) to define a bounded analytic function in the disc (when F is bounded).

Now L^∞ is not only a commutative Banach algebra under the essential sup norm, it is also closed under complex conjugation. Therefore L^∞ is isometrically isomorphic to the algebra $C(X)$ of all continuous complex-valued functions on a compact Hausdorff space X [see 2, p. 88]. The space X is just the space of complex homomorphisms (maximal ideals) of L^∞. For $G \in L^\infty$ we denote by $\hat{G}$ the corresponding continuous function on X. If M is a measurable subset of the circle C, we denote the characteristic function of M by K_M. Since the functions K_M generate L^∞, the functions $\hat{K}_M$ generate $C(X)$. Because $K_M^2 = K_M$, it follows that a basic open subset of X has the form

$$\{x \in X; \hat{K}_M(x) = 0\}$$

where M is a measurable subset of C. Such a set is also closed, and so X is totally disconnected. It is well known that X is even extremally disconnected.

There is a natural continuous map τ of X into $\mathfrak{M} = \mathfrak{M}(B)$ because we can identify B with the subalgebra H^∞ of L^∞. A point $x \in X$ is a complex homomorphism of L^∞, and by $\tau(x)$ we denote the complex homomorphism of B obtained by restricting x from L^∞ to H^∞ (identified with B).

Theorem 3.1. *The map τ is a homeomorphism of X into $\mathfrak{M}$. The range $\Gamma = \tau(X)$ is the Silov boundary for B, that is, Γ is the smallest closed subset of $\mathfrak{M}$ on which every function $\hat{f}$, $f \in B$, attains its maximum modulus.*

Proof. Let $x_0 \in X$. Let U be a basic neighborhood of the point x_0. Then U has the form

$$U = \{x; \hat{K}_M(x) = 0\}$$

where M is a measurable subset of the circle such that $\hat{K}_M(x_0) = 0$. Let

$$f(\lambda) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta} + \lambda}{e^{i\theta} - \lambda}\,[1 - K_M(e^{i\theta})]\, d\theta\right\},$$

so that f is a bounded analytic function, and the boundary function F of f satisfies

$$|F| = \exp\{1 - K_M\} \quad \text{a.e.}$$

On the space X, we thus have

$$|F| = \begin{cases} e & \text{on } U, \\ 1 & \text{on } X - U. \end{cases}$$

Now this argument shows us that the functions $\hat{F}$ for $F \in H^\infty$ separate the points of X, and also that there is no proper closed subset of X on which all such $\hat{F}$ attain their maximum modulus.

The map τ is easily seen to be continuous. Since X is compact and we have just shown that τ is one-one, τ is a homeomorphism. For any $f \in B$

$$\sup_{\mathfrak{M}} |\hat{f}| = ||f|| = ||F||_\infty = \sup_X |\hat{F}|,$$

and since $\hat{f}(\tau(x)) = \hat{F}(x)$, we see that $\Gamma = \tau(X)$ is the smallest closed subset of $\mathfrak{M}$ on which each $\hat{f}$ attains its maximum modulus.

The picture which we now have of $\mathfrak{M}$ is the following. This space contains the "open disc" Δ. It also contains the closed set Γ which is the Silov boundary for B, and which is naturally homeomorphic with the extremally disconnected maximal ideal space of L^∞. It is not known whether Δ is dense in $\mathfrak{M}$; however, it is clear that the closure $\bar{\Delta}$ of Δ contains Γ. For, since

$$\sup_\Delta |\hat{f}| = ||f|| = \sup_{\mathfrak{M}} |\hat{f}|,$$

each $\hat{f}$ will attain its maximum absolute value on $\bar{\Delta}$. Of course, the maximum modulus principle tells us that Γ is actually contained in $\bar{\Delta} - \Delta$.

It is also easy to see that there are points in $\bar{\Delta} - \Delta$ which are not in the Silov boundary Γ. Consider the function f in B defined by

$$f(\lambda) = \exp\left\{\frac{\lambda + 1}{\lambda - 1}\right\}.$$

The boundary function F for f satisfies $|F| = 1$ almost everywhere. Thus F is invertible in L^∞, and so $\hat{F}$ does not vanish on the space X, *i.e.*, $\hat{f}$ does not vanish on Γ. On the other hand, $f(\lambda)$ tends to 0 as λ approaches 1 along the positive axis, and so $\hat{f}$ must vanish somewhere on $\bar{\Delta}$. Since $f(\lambda) \neq 0$ for λ in D, this zero of $\hat{f}$ occurs at a point of $\bar{\Delta} - \Delta$ not in Γ.

4. Values on the Fibers. In Section 2 we defined the fibers $\mathfrak{M}_\alpha$ of the maximal ideal space $\mathfrak{M}$ over the various points of the circle. If α is a point of the unit circle, the fiber $\mathfrak{M}_\alpha$ is by definition the set of all complex homomorphisms φ of B such that $\varphi(z) = \alpha$. From this it is apparent that for any f in B which is continuously extendable to the closed disc, the function $\hat{f}$ is constant on each fiber $\mathfrak{M}_\alpha$ because such an f is a uniform limit of polynomials in z. One can say more than this, namely, that the continuity of f at any one boundary point α implies that $\hat{f}$ is constant on $\mathfrak{M}_\alpha$.

Theorem 4.1. *Let f be in B and let α be a point of the unit circle C. Let λ_n be a sequence of points in the open disc D such that*

(i) $$\lambda_n \to \alpha,$$

(ii) $$\zeta = \lim_{n\to\infty} f(\lambda_n) \quad \textit{exists.}$$

Then there is a complex homomorphism φ of B which lies in the fiber $\mathcal{M}_\alpha$ and for which $\varphi(f) = \zeta$.

Proof. Let I be the set of all functions g in B such that $\lim_{n\to\infty} g(\lambda_n) = 0$. Then I is a proper closed ideal in the algebra B. Thus I is contained in a maximal ideal M. Let φ be the complex homomorphism of B of which M is the kernel. Now $z - \alpha$ is in I, as is $f - \zeta$. Thus

$$\varphi(z) = \alpha, \qquad \varphi(f) = \zeta.$$

So φ is the required homomorphism.

Theorem 4.2. *Let $f \in B$ and $\alpha \in C$. A necessary and sufficient condition that $\hat{f}$ be constant on $\mathcal{M}_\alpha$ is that f be continuously extendable to $D \cup \{\alpha\}$.*

Proof. First suppose f is continuously extendable to $D \cup \{\alpha\}$. This simply means that there is a complex number ζ such that $f(\lambda_n) \to \zeta$ whenever $\lambda_n \to \alpha$. The claim is, of course, that $\varphi(f) = \zeta$ for all $\varphi \in \mathcal{M}_\alpha$. We may as well assume that $\zeta = 0$. Let $g(\lambda) = \frac{1}{2}(1 + \lambda\alpha^{-1})$, so that $g(\alpha) = 1$ and $|g| < 1$ elsewhere on the closed disc. Since f is continuous at α if we define $f(\alpha) = 0$, it is easy to see that $(1 - g^n)f$ converges uniformly to f as $n \to \infty$. If $\varphi(z) = \alpha$, then $\varphi(g) = 1$ and $\varphi(1 - g^n) = 0$, so that $\varphi(f) = 0$.

If $\hat{f}$ has the constant value ζ on the fiber $\mathcal{M}_\alpha$, then Theorem 4.1 implies that $f(\lambda_n) \to \zeta$ whenever $\lambda_n \to \alpha$. If one defines $f(\alpha) = \zeta$, then f is continuous on $D \cup \{\alpha\}$.

Theorem 4.3. *Let $f \in B$ and $\alpha \in C$ and suppose there is a homomorphism φ in $\mathcal{M}_\alpha$ such that $\varphi(f) = 0$. Then there is a sequence $\{\lambda_n\}$ such that $\lambda_n \to \alpha$ and $f(\lambda_n) \to 0$.*

Proof. If no such sequence $\{\lambda_n\}$ exists, there is a neighborhood N of the point α such that $|f(\lambda)| \geqq \delta > 0$ for all $\lambda \in D \cap N$. Let

$$f_0(\lambda) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta} + \lambda}{e^{i\theta} - \lambda} \log |F(e^{i\theta})| \, d\theta\right\}. \tag{4.1}$$

Then f_0 is in B and $|F_0| = |F|$ almost everywhere on the circle C. We can write

$$f = f_i f_0 \tag{4.2}$$

where $f_i \in B$ and $|f_i| = 1$ almost everywhere on C. The functions f_i and f_0 were called by BEURLING the inner and outer parts of f, respectively. This factorization is well known [3], and it is also well known that, since $|f|$ is bounded away from 0 in a neighborhood of α, the inner function f_i is analytically continuable across that part of C which lies in N. Let

$$h(\lambda) = \exp\left\{\frac{1}{2\pi}\int_{N\cap C} \frac{e^{i\theta} + \lambda}{e^{i\theta} - \lambda} (-\log |F(e^{i\theta})|) \, d\theta\right\}. \tag{4.3}$$

Since $|f| \geqq \delta > 0$ on $N \cap C$, the function $- \log |F|$ is bounded on $N \cap C$, and so (4.3) defines a bounded analytic function h in the open disc. Also

$$f(\lambda)h(\lambda) = f_i(\lambda) \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + \lambda}{e^{i\theta} - \lambda} k(\theta)\, d\theta \right\}$$

where k is integrable and vanishes identically on $N \cap C$. Thus fh is analytically extendable across $N \cap C$ and $|fh| = 1$ on $N \cap C$. Accordingly, $(fh)\hat{}$ is constant on the fiber $\mathcal{H}_\alpha$, and the constant involved has modulus 1. If $\varphi(z) = \alpha$, then by Theorem 4.2

$$|\varphi(f) \cdot \varphi(h)| = 1,$$

which contradicts the assumption that $\varphi(f) = 0$ for some such φ.

Corollary. *Let $f \in B$ and $\alpha \in C$. The range of $\hat{f}$ on the fiber $\mathcal{H}_\alpha$ consists of those complex numbers ζ for which there is a sequence $\{\lambda_n\}$ in D with*

(i) $$\lambda_n \to \alpha,$$

(ii) $$f(\lambda_n) \to \zeta.$$

Now we should like to make some remarks about the topological nature of the decomposition

$$\mathcal{H} - \Delta = \bigcup_{\alpha \in C} \mathcal{H}_\alpha . \tag{4.4}$$

We showed in Section 2 that the fibers $\mathcal{H}_\alpha$ are all homeomorphic under the action of the rotation group. Thus one might expect that the decomposition (4.4) is simply a product decomposition, that is, that $\mathcal{H} - \Delta$ is naturally homeomorphic to the product space $C \times \mathcal{H}_\alpha$ for some fixed fiber $\mathcal{H}_\alpha$. As a point set, $\mathcal{H} - \Delta$ is identifiable in a natural way with $C \times \mathcal{H}_1$. If $\varphi \in \mathcal{H}_\alpha$ we associate with φ the ordered pair $(\alpha, R_{\bar{\alpha}}^* \varphi)$, where $R_{\bar{\alpha}}^*$ is the adjoint of the rotation induced by $\bar{\alpha}$ (see Section 2). Now

$$\varphi \to (\alpha, R_{\bar{\alpha}}^* \varphi)$$

is a one-one correspondence between $\mathcal{H} - \Delta$ and $C \times \mathcal{H}_1$; however, it is not a homeomorphism. If it were, then the orbit

$$\{R_\beta^* \varphi\}_{\beta \in C}$$

of any φ under the rotation group would be a continuous cross section of $\mathcal{H} - \Delta$ over C. This is impossible because such an orbit is not a closed subset of $\mathcal{H} - \Delta$. In fact, there are no continuous cross-sections of $\mathcal{H} - \Delta$ over C at all. To see this, let us list several observations about the topological nature of the decomposition (4.4).

Let W_1 be the union of the fibers $\mathcal{H}_\alpha$ for $\text{Im}(\alpha) > 0$ and let W_2 be the union of all $\mathcal{H}_\alpha$ with $\text{Im}(\alpha) < 0$. Denote the closures of these sets by $\bar{W}_1$ and $\bar{W}_2$.

(i) The intersection of $\bar{W}_1$ and $\bar{W}_2$ is empty.

(ii) Each point in the fiber $\mathcal{H}_1$ is in $\bar{W}_1$ or in $\bar{W}_2$ or in neither. All three cases occur.

(iii) If $\{\varphi_n\}$ is any sequence of points of $\mathcal{H} - \Delta$ which converges, then all but a finite number of the φ_n lie in the same fiber $\mathcal{H}_\alpha$.

(iv) If S is any function from C into $\mathcal{H} - \Delta$ such that $\pi \circ S$ is the identity, then $S(C)$ is not closed. In particular, such a section S cannot be continuous.

To see (i), let u be the harmonic function in the disc with boundary values

$$u(\alpha) = \begin{cases} 0, & \text{Im}(\alpha) > 0, \\ 1, & \text{Im}(\alpha) < 0. \end{cases} \tag{4.5}$$

If v is a harmonic function conjugate to u, then $f = \exp\{u + iv\}$ is a bounded analytic function in the disc and $|f| = e^u$. We see from (4.5) and the corollary above that

$$|\hat{f}(\varphi)| = \begin{cases} 1 & \text{if} \quad \varphi \in W_1, \\ e & \text{if} \quad \varphi \in W_2. \end{cases}$$

This certainly shows that $\bar{W}_1$ and $\bar{W}_2$ are disjoint, because the continuous function $\hat{f}$ satisfies $|\hat{f}| = 1$ on $\bar{W}_1$ and $|\hat{f}| = e$ on $\bar{W}_2$.

For statement (ii) we need only observe the following. The sets $\pi(\bar{W}_1)$ and $\pi(\bar{W}_2)$ are compact, and so

$$\pi(\bar{W}_1) = \{\alpha \in C; \text{Im}(\alpha) \geqq 0\},$$
$$\pi(\bar{W}_2) = \{\alpha \in C; \text{Im}(\alpha) \leqq 0\}.$$

Thus there are points in the fiber $\mathcal{H}_1$ which lie in $\bar{W}_1$ and points of $\mathcal{H}_1$ which lie in $\bar{W}_2$. By (i) no point of $\mathcal{H}_1$ lies in both closures. To demonstrate that $\bar{W}_1$ and $\bar{W}_2$ do not cover $\mathcal{H}_1$, consider once again the function

$$g(\lambda) = \exp\left\{\frac{\lambda + 1}{\lambda - 1}\right\}.$$

If α is a point of the circle different from 1, then g is continuously (even analytically) extendable to α with $g(\alpha) = (\alpha + 1)(\alpha - 1)^{-1}$. Since $|g(\alpha)| = 1$, we see that $|\hat{g}| = 1$ on the fiber $\mathcal{H}_\alpha$. Let

$$U = \{\varphi \in \mathcal{H}; |\hat{g}(\varphi)| < 1\}.$$

Now U is an open subset of $\mathcal{H}$ and our remarks above show that $U \subset \Delta \cup {}_1\mathcal{H}$. Also $U \cap \mathcal{H}_1$ is non-empty because $\hat{g}$ vanishes at some φ in $\mathcal{H}_1$. Any φ in $U \cap \mathcal{H}_1$ lies neither in $\bar{W}_1$ nor in $\bar{W}_2$.

To prove (iii), suppose $\{\varphi_n\}$ is a sequence of points of $\mathcal{H} - \Delta$. If there are more than a finite number of points of the circle among the images $\alpha_n = \pi(\varphi_n)$, we wish to show that $\{\varphi_n\}$ does not converge. By passing to a subsequence, we may assume that the points α_n are distinct. Choose open arcs A_n about α_n which are disjoint, and define

$$u_0(\alpha) = \begin{cases} (-1)^n & \text{if} \quad \alpha \in A_n, \\ 0 & \text{otherwise}. \end{cases}$$

We extend u_0 to a harmonic function in the disc, select a conjugate harmonic function v_0, and define

$$f_0 = \exp\{u_0 + iv_0\}.$$

Now $f_0 \varepsilon B$ and $|f_0| = e^{u_0}$. Thus

$$\varphi_n(f_0) = \begin{cases} e, & n \text{ even}, \\ \frac{1}{e}, & n \text{ odd}, \end{cases}$$

which shows that $\{\varphi_n\}$ cannot converge.

Statement (iv) follows immediately from (iii). If S is a function from C into $\mathfrak{M} - \Delta$ such that $\pi \circ S$ is the identity, let $\varphi_0 = S(1)$. Choose a sequence of distinct points α_n on the circle such that $\alpha_n \to 1$, and let $\varphi_n = S(\alpha_n)$. If $S(C)$ were closed, then φ_0 would be the only cluster point of the sequence $\{\varphi_n\}$. But by (iii) the sequence $\{\varphi_n\}$ cannot converge to φ_0 because the α_n are distinct.

Statement (ii) gives some indication of the bizarre nature of the topology on $\mathfrak{M} - \Delta$. Roughly speaking, it says that in any given fiber $\mathfrak{M}_\alpha$ some of the points can be approached from points in fibers $\mathfrak{M}_\beta$ with β a little way to one side of α, some points are approachable from points in $\mathfrak{M}_\beta$ with β on the other side of α, and some points of $\mathfrak{M}_\alpha$ cannot be approached from points in any other fibers $\mathfrak{M}_\beta$. These are three mutually exclusive possibilities which all occur. One can show that no point of the Silov boundary is of the third type. It is an unsolved problem whether $\mathfrak{M}_\alpha$ has any interior. This is equivalent to the question of the density of Δ in $\mathfrak{M}$, which we mentioned earlier.

Two other topological problems which we have left unresolved are these. Since $\mathfrak{M} - \Delta$ contains an extremally disconnected set Γ, it seems interesting to ask whether $\mathfrak{M} - \Delta$ is connected, and whether each fiber $\mathfrak{M}_\alpha$ is connected.

5. Embedding a Disc in a Fiber. A mapping ψ from D into $\mathfrak{M}$ is called analytic if $\hat{f} \circ \psi$ is analytic on D for each $f \varepsilon B$. If $\{\psi_n\}$ is a sequence of analytic maps of D into $\mathfrak{M}$, the compactness of $\mathfrak{M}$ guarantees that there is a cluster point ψ of $\{\psi_n\}$ in the space of maps of D into $\mathfrak{M}$. It is easy to see that ψ is also analytic because for each $f \varepsilon B$ the sequence $\hat{f} \circ \psi_n$ is uniformly bounded and hence uniformly equicontinuous on each compact subset of D.

We are going to construct an analytic map ψ of D into the space $\mathfrak{M}$ which is a homeomorphism and actually maps D into the fiber $\mathfrak{M}_1$. The "disc" $\psi(D)$ in the fiber $\mathfrak{M}_1$ will also have the property that the restriction of $\hat{B}$ to $\psi(D)$ consists exactly of all bounded analytic functions on this "disc".

Let L be the linear fractional transformation

$$L(\lambda) = \frac{\lambda + i(\lambda - 1)}{1 + i(\lambda - 1)}. \tag{5.1}$$

Then L maps D conformally onto itself and maps the closed disc $D \cup C$ homeomorphically onto itself. The point $\lambda = 1$ is the single fixed point of L in the closed disc. If $L^{(n)}$ denotes the n^{th} composition of L with itself, then one can readily verify that

$$(5.2) \qquad L^{(n)}(\lambda) = \frac{\lambda + ni(\lambda - 1)}{1 + ni(\lambda - 1)}, \qquad n = 0, \pm 1, \pm 2, \cdots .$$

Let ψ_n be the map of D into $\mathfrak{M}$ defined by

$$(5.3) \qquad \psi_n(\lambda) = \pi^{-1}(L^{(2^n)}(\lambda)).$$

In other words, ψ_n maps D into Δ, and $\psi_n(\lambda)$ is the complex homomorphism of B which evaluates each f in B at the point $L^{(2^n)}(\lambda)$ in D. It is certainly clear that ψ_n is an analytic map of D into $\mathfrak{M}$ (even into Δ). Let ψ be a cluster point of the sequence of maps $\{\psi_n\}$, so that ψ is an analytic map of D into $\mathfrak{M}$. Now it is easy to see that ψ must map D into the fiber $\mathfrak{M}_1$. For if we fix a number λ in D,

$$(5.4) \qquad \lim_{n\to\infty} L^{(n)}(\lambda) = 1.$$

This shows us that $\pi(\psi(\lambda)) = 1$ for each λ in D.

We want to verify that ψ is a one-one analytic map of D into $\mathfrak{M}_1$. We define

$$(5.5) \qquad f(\lambda) = \lambda \cdot \prod_{k=0}^{\infty} L^{(-2^k)}(\lambda).$$

From (5.2) one can see that on any compact subset of the disc

$$(5.6) \qquad |L^{(n)} - 1| \leqq K\left(\frac{1}{|n|}\right), \qquad |n| > 0.$$

And, since $|L^{(n)}| \leqq 1$, this shows that the infinite product (5.5) converges uniformly on compact subsets of D to a function f in B with $\|f\| \leqq 1$. We shall use f to show that $\psi: D \to \mathfrak{M}_1$ is a homeomorphism. We claim that

$$(5.7) \qquad \hat{f}(\psi(\lambda)) = \lambda, \qquad \lambda \in D.$$

For, using (5.6) for λ in a compact subset of D,

$$\begin{aligned}
|f(L^{(2^n)}(\lambda)) - \lambda| &= \left| L^{(2^n)}(\lambda) \prod_{k=0}^{\infty} L^{(2^n - 2^k)}(\lambda) - \lambda \right| \\
&= |\lambda| \left| L^{(2^n)}(\lambda) \prod_{k=0}^{n-1} L^{(2^n - 2^k)}(\lambda) \cdot \prod_{k=n+1}^{\infty} L^{(2^n - 2^k)}(\lambda) - 1 \right| \\
&\leqq |\lambda| \left[|L^{(2^n)}(\lambda) - 1| + \sum_{k=0}^{n-1} |L^{(2^n - 2^k)}(\lambda) - 1| + \sum_{k=n+1}^{\infty} |L^{(2^n - 2^k)}(\lambda) - 1| \right] \\
&\leqq |\lambda| \cdot K \left[\frac{1}{2^n} + \sum_{k=0}^{n-1} \frac{1}{2^n - 2^k} + \sum_{k=n+1}^{\infty} \frac{1}{2^k - 2^n} \right] \\
&\leqq |\lambda| \cdot K \left[\frac{1}{2^n} + \sum_{k=0}^{n-1} \frac{1}{2^{n-1}} + \sum_{k=n+1}^{\infty} \frac{1}{2^{k-1}} \right] \\
&\leqq |\lambda| \cdot K \left[\frac{n+2}{2^{n-1}} \right] \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

In passing from line 2 to line 3 we used the fact that if $\{\lambda_k\}$ is a sequence of points in D, then

$$\left|\prod_{k=1}^{\infty} \lambda_k - 1\right| \leqq \sum_{k=1}^{\infty} |\lambda_k - 1|.$$

This proves (5.7), and thus shows that ψ is a homeomorphism. For the continuous function $\hat{f}$ is the inverse of ψ. Also, if g is a bounded analytic function on the disc D, there is an $h \varepsilon B$ such that

$$\hat{h}(\psi(\lambda)) = g(\lambda).$$

For, using (5.7), we have only to take $h = g \circ \hat{f}$.

In summary, we have constructed a homeomorphism ψ of the open disc D into the fiber $\mathcal{H}_1$, and ψ is analytic, in the sense that $\hat{f} \circ \psi$ is analytic for every $f \varepsilon B$. Thus the disc $\psi(D)$ has a natural analytic structure, and when we restrict the algebra $\hat{B}$ to this disc, we obtain the algebra of all bounded analytic functions on $\psi(D)$. It is easy to see that the uniformly closed restriction algebra $\hat{B}|_{\psi(D)}$ will have as its maximal ideal space the subset S of $\mathcal{H}$ defined by

$$S = \{\varphi \varepsilon \mathcal{H};\, |\varphi(f)| \leqq \sup_{\psi(D)} |\hat{f}|, \quad \text{all} \quad f \varepsilon B\}.$$

This set S is contained in $\mathcal{H}_1$, as we see by considering $f(\lambda) = \frac{1}{2}(1 + \lambda)$. Since the restriction algebra is isomorphic to the algebra of bounded analytic functions in the disc, the set S must be homeomorphic to the entire maximal ideal space $\mathcal{H}$.

The maximum modulus principle makes it clear that $\psi(D)$ lies in $\mathcal{H}_1 - \Gamma$, and so we see more vividly than before that $\bar{\Delta} - \Delta \neq \Gamma$. One sees from the above discussion that the space $\mathcal{H}$ "reproduces" itself in any given fiber *ad infinitum*. Because in S there are fibers attached to the disc $\psi(D)$ in each of which is a closed set homeomorphic to $\mathcal{H}$, and so on.

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