# The mapping properties of some non-holomorphic functions on the unit disk

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# THE MAPPING PROPERTIES OF SOME NON-HOLOMORPHIC FUNCTIONS ON THE UNIT DISK

*by*

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***Abstract.*** — We study the mapping properties of the maps $f(z) = \frac{\overline{z}-1}{z-1}$, $g(z) = |z|\, f(z)$ and $h(z) = -zf(z)$ with $|z| \leq 1$, $z \neq 1$.

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## Introduction

In this paper we are concerned with the mapping properties of some non-holomorphic continuous functions on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and their behaviour at the boundary $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ of $\mathbb{D}$. Our first example is the function $f(z) = (\overline{z} - 1)/(z - 1)$ which played a prominent role in Earl's [**2**] constructive solution to the famous interpolation problem for bounded analytic functions, originally solved by L. Carleson [**1**], [**3**]. Earl considered finite Blaschke products of the form

$$B_n(z, \xi) = \prod_{k=1}^{n} \frac{z - \xi_k}{1 - \overline{\xi}_k z} \frac{1 - \overline{\xi}_k}{1 - \xi_k}.$$

In contrast to the usual rotational factors $-|\xi_k|/\xi_k$, these new unimodular factors $(1 - \overline{\xi}_k)/(1 - \xi_k)$ were chosen so that $B_n(z, \xi) = 1$ at $z = 1$, a fact fundamental for his solution to work. These factors reappeared in [**4**] in a similar context when studying the value distribution of interpolating Blaschke products. To see this, let

$$S(z) = \exp\left(-\frac{1+z}{1-z}\right)$$

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be the atomic inner function. Choose $\sigma \in \mathbb{T}, \sigma \neq 1$, so that $S(\sigma) = 1$. Then the rotated Fostman shift

$$B(z) = \frac{S(\overline{\sigma}z) - b}{1 - \overline{b}S(\overline{\sigma}z)} \, \frac{1 - \overline{b}}{1 - b}$$

of $S$ is an interpolating Blaschke product with singularity at $\sigma$ that has the property that $B(1) = 1$. Moreover, as we did want that $B$ additionally satisfies

$$\lim_{r \to 1} B(\sigma r) = a,$$

we were led to study the equation

$$-b \frac{1 - \overline{b}}{1 - b} = a.$$

(Note that $\lim_{r\to 1} S(r) = 0$.) This gave me the motivation to study in the present note the mapping properties of the function $h(z) = -z(\overline{z} - 1)/(z - 1)$.

It turns out that the map $h$ also provides a solution (see Proposition 3.1) to the following question:

Does there exist continuous involutions of $\mathbb{D}$ onto itself (these are continuous functions $\iota$ for which $\iota \circ \iota = \text{id}$, where id is the identity map), such that $\iota$ has a continuous extension with constant value at a largest possible subset of $\mathbb{T}$, namely $\mathbb{T} \setminus \{1\}$? [1] Note that the elliptic automorphisms $\varphi_a(z) = (a - z)/(1 - \overline{a}z)$ of $\mathbb{D}$ are involutions with $\phi_a(\mathbb{T}) = \mathbb{T}$; so these functions are more or less opposite to that class of functions we were looking for.

Now let us come back to the function $f(z) = (\overline{z} - 1)/(z - 1)$. It is clear that $|f(z)| = 1$ for every $z \in \mathbb{D}$. So in order to describe and better visualize the global mapping properties of $f$, I "added" the factor $|z|$. In this way we are led to study the function

$$g(z) = |z| \, \frac{\overline{z} - 1}{z - 1}.$$

As we shall see, $g$ has a totally different behaviour than $h$. One striking fact, is that the image of $\mathbb{D}$ under $g$ is no longer an open set. We will explicitely determine $g(\mathbb{D})$. It turns out that certain rhodonea curves (roses) as Dürer's folium, $r = \sin(\theta/2)$, play an important role in studying the image properties of $g$.

We include in our paper six figures that help to visualize and understand the calculations and results achieved.

[1] Later we shall see that one cannot achieve the constancy of the involution on the entire boundary of $\mathbb{D}$.

## 1. The map $f(z) = (\overline{z} - 1)/(z - 1)$

***Lemma 1.1.*** — *Consider for $z \in \mathbb{D}$ the function $f(z) = (\overline{z} - 1)/(z - 1)$ and let $0 < a < 1$. Then*

1. $\max_{|z|=a} \operatorname{Re} f(z) = 1$;
2. $\min_{|z|=a} \operatorname{Re} f(z) = 1 - 2a^2$;
3. $\max_{|z|=a} \operatorname{Im} f(z) = 1$ *if and only if* $\frac{1}{\sqrt{2}} \leq a < 1$ *and* $\max_{|z|=a} \operatorname{Im} f(z) = 2a\sqrt{1-a^2}$ *if and only if* $0 < a \leq \frac{1}{\sqrt{2}}$;
4. $\min_{|z|=a} \operatorname{Im} f(z) = -\max_{|z|=a} \operatorname{Im} f(z)$.

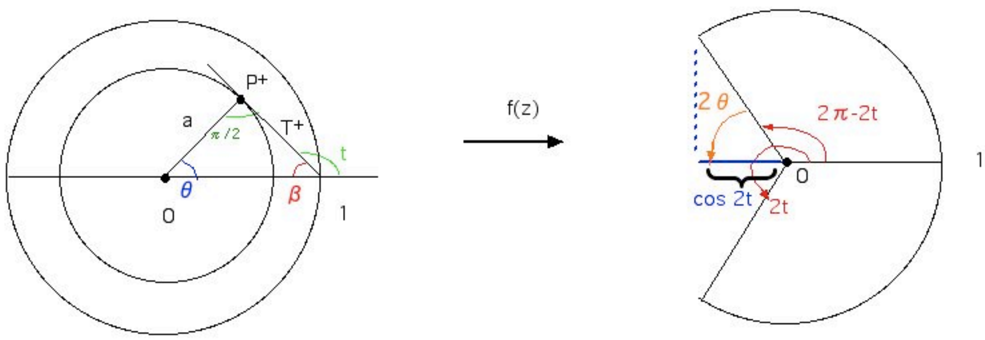


FIGURE 1. The domain of variation of $t$, $t$ close to $\pi/2$.

*Proof.* — Let $z = 1 + \rho e^{it}$, $0 \leq t \leq 2\pi$. Then $f(z) = e^{-2it}$. Hence $\operatorname{Re} f(z) = \cos(2t)$ and $\operatorname{Im} f(z) = -\sin(2t)$. Let $T^{\pm}$ be the two tangents to the circle $|z| = a$ passing through the point 1. The intersection points of $T^{\pm}$ with the circle are given by

$$P_a^{\pm} = ae^{\pm i\theta} \tag{1.1}$$

for some $\theta \in [0, \pi/2]$. Consider the triangle $\Delta$ whose end-points are $0, 1$ and $P_a^+$ and let $\beta$ be the angle formed by the segment $[0,1]$ and the tangent $T^+$. Using that $\theta + \beta = \pi/2$, there exists $\rho > 0$ with $|1 + \rho e^{it}| = a$ if and only if $\pi - \beta \leq t \leq \pi + \beta$. (If $t \neq \pi \pm \beta$, then there are exactly two such radii $\rho$). The side-lengths of $\Delta$ are 1 (the hypotenuse), $a$ and $L := |ae^{i\theta} - 1|$. Since $L^2 + a^2 = 1$, we see that $L = \sqrt{1-a^2}$. On the other hand,

$$L^2 = a^2 + 1 - 2a\cos\theta.$$

Hence $a = \cos\theta$. Now let $t_{\max} := \pi - \beta$. Note that $t_{\max}$ is close to $\pi$ if $a$ is close to 0 and $t_{\max}$ is close to $\pi/2$ if $a$ is close to 1.

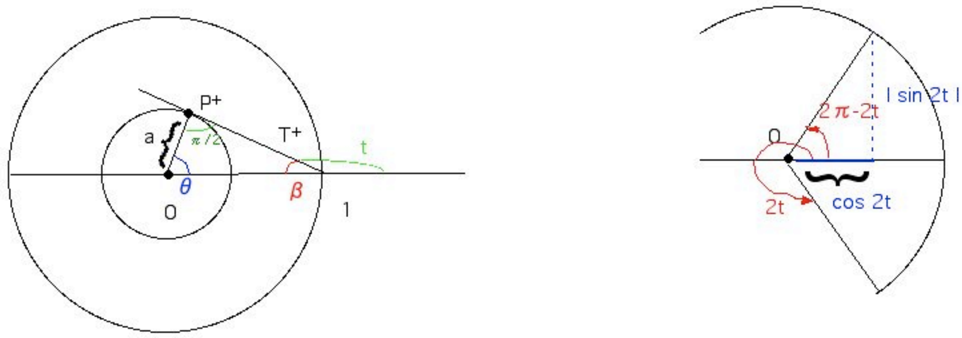


FIGURE 2. The domain of variation of $t$, $t$ close to $\pi$

Since $t_{\max} = \theta + \pi/2$, we obtain

$$\cos(2t_{\max}) = \cos(2\theta + \pi) = -\cos(2\theta) = 1 - 2\cos^2(\theta) = 1 - 2a^2.$$

Thus $\min_{|z|=a} \operatorname{Re} f(z) = 1 - 2a^2$. The other identity $\max_{|z|=a} \operatorname{Re} f(z) = 1$ is clear by looking at the figure; it also follows from the fact that for $z = a$, $f(z) = 1$.

Now $\cos(2t_{\max}) = 0$ if $t_{\max} = 3\pi/4$. Hence

$$\max_{|z|=a} \operatorname{Im} f(z) = 1 \iff 1 - 2a^2 \leq 0 \iff \frac{1}{\sqrt{2}} \leq a < 1,$$

and

$$\max_{|z|=a} \operatorname{Im} f(z) = \sqrt{1-(1-2a^2)^2} = 2a\sqrt{1-a^2} \iff 0 < a \leq \frac{1}{\sqrt{2}}.$$

Finally, for all $a \in ]0, 1[$,

$$\min_{|z|=a} \operatorname{Im} f(z) = -\max_{|z|=a} \operatorname{Im} f(z). \qquad \square$$

We can also use cartesian coordinates to find these extremal values: in fact, let $z = x + iy$, $|z| = a$. Then

$$\begin{aligned} \operatorname{Re} \tfrac{\bar{z}-1}{z-1} &= \operatorname{Re} \tfrac{(\bar{z}-1)^2}{|z-1|^2} = \tfrac{(x-1)^2 - y^2}{x^2+y^2+1-2x} \\ &= \tfrac{x^2-2x+1-(a^2-x^2)}{a^2+1-2x} = 1 + \tfrac{2x^2-2a^2}{a^2+1-2x} \end{aligned}$$

Now

$$\left(\frac{x^2 - a^2}{a^2 + 1 - 2x}\right)' = \frac{2(x-1)(a^2-x)}{(a^2+1-2x)^2}$$

The zeros of this derivative are $x = 1$ and $x = a^2$. Since $-a \le x \le a$, we deduce that

$$\min_{|z|=a} \operatorname{Re} \frac{\overline{z}-1}{z-1} = 1 + \frac{2x^2-2a^2}{a^2+1-2x}\Big|_{x=a^2} = 1 - 2a^2$$

and

$$\max_{|z|=a} \operatorname{Re} \frac{\overline{z}-1}{z-1} = 1 + \frac{2x^2-2a^2}{a^2+1-2x}\Big|_{x=\pm a} = 1.$$

As a consequence, the cartesian coordinates of $P_a\pm$ are $(a^2, \pm a\sqrt{1-a^2})$.

***Corollary 1.2.*** — *Let $0 < a < 1$. The image of the circle $|z| = a$ under the map*

$$f(z) = \frac{\overline{z}-1}{z-1}$$

*is the arc*

$$A := \{e^{i\sigma} : |\sigma| \le \pi - 2\arccos a\},$$

*where* $\arccos a \in \,]0, \pi/2[$.

***Remark.*** — We also note that if $\tau$ runs from 0 to $2\pi$, then $f(ae^{i\tau})$ runs on $A$ from 1 to the upper end-point

$$E^+ := e^{i(\pi - 2\arccos a)} = 1 - 2a^2 + ia\sqrt{1-a^2}$$

of $A$, reaches this point when $\tau = \arccos a$ (that is $f(P_a^+) = E^+$)), then turns back, passes through the point 1 (when $\tau = \pi$) until it reaches the lower end-point

$$E^- := e^{-i(\pi - 2\arccos a)} = 1 - 2a^2 - ia\sqrt{1-a^2}$$

of $A$ when $\tau = 2\pi - \arccos a$ (that is $f(P_a^-) = E^-$)), then turns back again up to the point 1, that is attained for $\tau = 2\pi$. In particular, with the exception of the two end-points of $A$, each point of $A$ is traversed twice.

## 2. The map $g(z) = |z|f(z)$

***Theorem 2.1.*** — *Let the map $g : \mathbb{D} \to \mathbb{C}$ be defined by*

$$g(z) = |z|\,\frac{\overline{z}-1}{z-1}.$$

*Then $g$ is a continuous map of $\mathbb{D}$ onto the set*

$$\Omega = \mathbb{D} \setminus K^\circ,$$

*where $K$ is a closed region whose boundary is given by the curve*

$$\gamma(a) = a(1-2a^2) \pm 2i\, a^2\sqrt{1-a^2}, \quad 0 \le a \le 1,$$

*which is one half of the rhodonea (rose)*

$$r = \sin(\theta/2),\ 0 \leq \theta \leq 2\pi.$$

*Moreover, $g$ is a homeomorphism of*

$$H := \{z \in \mathbb{D} : |z - 0.5| > 0.5\} \text{ onto } \mathbb{D} \setminus K$$

*and a homeomorphism of*

$$\{z \in \mathbb{D} : |z - 0.5| < 0.5\} \text{ onto } \mathbb{D} \setminus K.$$

*Let $C = \{z \in \mathbb{D} : |z - 0.5| = 0.5\}$. Then the function $g|_C$ has an injective continuous extension to the whole circle $\overline{C}$. The image of this extension coincides with $\partial K$ (see figures 3 and 4).*

*Finally, for $|z| = 1$, $z \neq 1$, $g(z) = -\overline{z}$; thus $g$ interchanges two points on the unit circle whenever they have same imaginary part.*

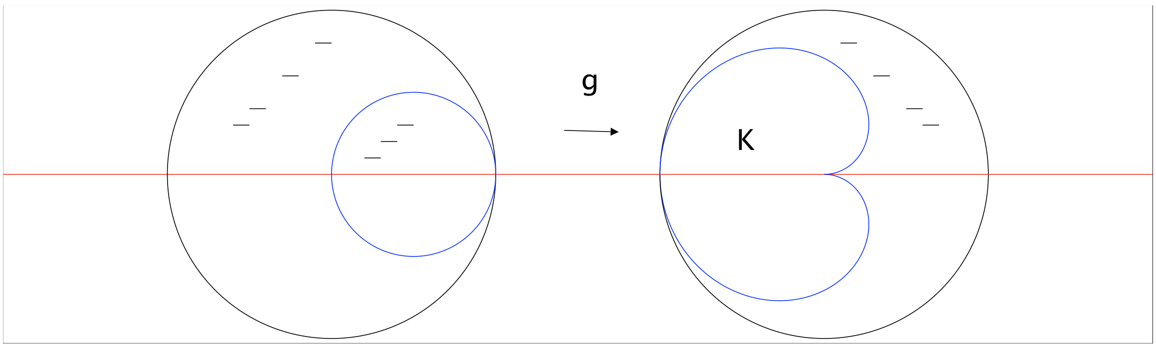


FIGURE 3. The mapping properties of $g$

*Proof.* — The first assertion on the image follows at once when we have noticed that by Lemma 1.1 and Corollary 1.2 the end-points of the image curve of $|z| = a$ under the map $(\overline{z} - 1)/(z - 1)$ are given by

$$1 - 2a^2 \pm i\sqrt{1 - (1 - 2a^2)^2} = 1 - 2a^2 \pm i\ 2a\sqrt{1 - a^2}$$

(see figure 4). Note also that the boundary of $g(\mathbb{D})$ is given by the set

$$\partial\mathbb{D} \cup R,$$

where $R$ is parametrized by the curve

$$\gamma(a) = a(1 - 2a^2) \pm 2i\ a^2\sqrt{1 - a^2}, \quad 0 \leq a \leq 1.$$

Hence $g(\mathbb{D}) = \Omega$.

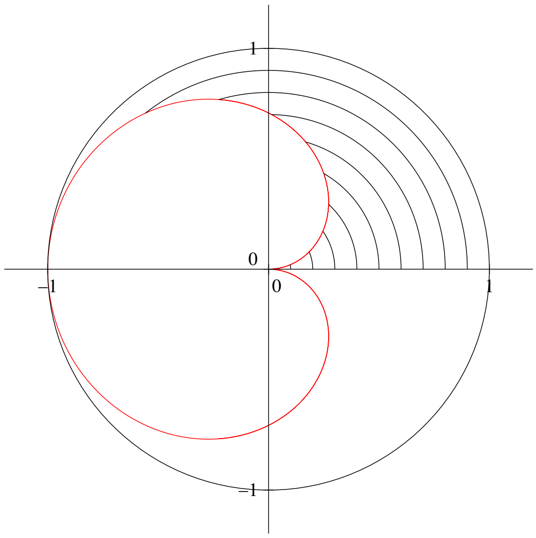


FIGURE 4. Creation of the image domain $\Omega$

The locus of the points $P_a = ae^{i\arccos a}$, where $0 \le a \le 1$, equals the circle of center $1/2$ and radius $1/2$, because

$$\left|\frac{1}{2} - ae^{i\arccos a}\right| = \left|\frac{1}{2} - a\cos(\arccos a) - ia\sin(\arccos a)\right|$$

$$= \left|(\frac{1}{2} - a^2) - ia(\sqrt{1-a^2})\right| = \sqrt{(\frac{1}{2} - a^2)^2 + a^2(1-a^2)} = \frac{1}{2}.$$

By Corollary 1.2 and its remark,

$$g(ae^{i\arccos a}) = ae^{i(\pi - 2\arccos a)} = \gamma(a),\ a \ne 1.$$

Thus $g(C) = \partial K$. Moreover the open disk $|z-1/2| < 1/2$ is mapped bijectively onto $\Omega$; the same holds for the set $\{z \in \mathbb{D} : |z-1/2| > 1/2\}$.

It remains to show that $\gamma(a)$ coincides with (one part) of the rhodonea $r = \sin(\varphi/2)$, also called Dürer's folium, $0 \le \varphi \le 2\pi$.

So let $\gamma(a) = ae^{i\varphi}$, $0 \le \varphi \le 2\pi$. Note that $\gamma(a) = g(P_a^{\pm})$. Since $\cos\varphi = 1 - 2a^2$, we deduce that, in polar coordinates,

$$r(\varphi) = a = \sqrt{\frac{1}{2}(1-\cos\varphi)} = \sin\left(\frac{\varphi}{2}\right).$$

$\square$

At first glance (by looking at the picture), $K$ seems to be a cardioid. This is not the case, though. The relation of $K$ with the domain bounded by the classical cardioid, given by the parametrization

$$z(t) = -\frac{1}{2}(\cos\phi + 1)\cos\phi + i\frac{1}{2}(\cos\phi + 1)\sin\phi, \quad 0 \leq \phi \leq 2\pi$$

or in polar coordinates

$$r(\varphi) = \frac{1}{2}(1 - \cos\varphi)$$

is shown in the following figure (the cardioid is inside the domain $K$ bounded by the "left part" of the rhodonea; the full rhodonea, called Dürer's folium, is given in the right picture.

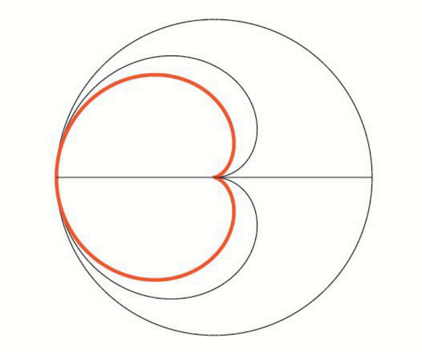

FIGURE 5.
Cardioid, rhodonea and unit cirle

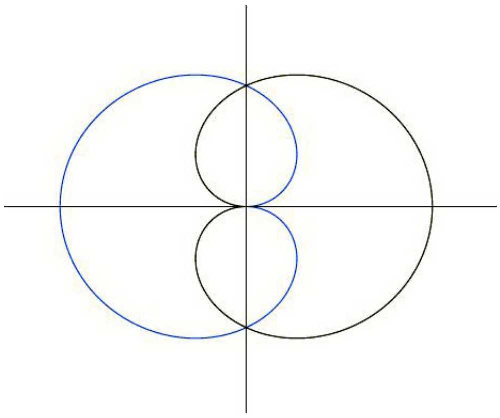

FIGURE 6. Dürer's folium

## 3. The map $h(z) = -z\frac{\overline{z}-1}{z-1}$

If one replaces in the definition of

$$g(z) = |z|\,\frac{\overline{z}-1}{z-1},$$

the factor $|z|$ by $-z$, then the new function has a very different behaviour. Part of the following result is from my previous joint work with P. Gorkin [**4**]. For the readers convenience, we recapture its short proof here. Recall that the cluster set, $C(u, \alpha)$, of a continuous function $u : \mathbb{D} \to \mathbb{C}$ at the point $\alpha \in \mathbb{T}$ is the set of all values $w \in \hat{\mathbb{C}}$ such there exists a sequence $(z_n)$ in $\mathbb{D}$ for which $u(z_n) \to w$ as $n \to \infty$.

***Proposition 3.1.*** — *Let $h : \mathbb{D} \to \mathbb{D}$ be given by*

$$h(z) = -z\frac{\overline{z} - 1}{z - 1}.$$

*Then $h$ is a bijective involution (that is $h \circ h = \mathrm{id}$) of $\mathbb{D}$ onto $\mathbb{D}$. The map $h$ has a continuous extension to $\overline{\mathbb{D}} \setminus \{\ \}1$with constant value $1$. The cluster set $C(h, 1)$ of $h$ at $1$ equals the unit circle $\mathbb{T}$.*

*Proof.* — The first assertion follows from the fact that $h(z) = a$ implies $|z| = |a|$ and the following equivalences:

$$-z\frac{\overline{z} - 1}{z - 1} = a \iff -z + |z|^2 - a + az = 0 \iff$$

$$-z + |a|^2 - a + az = 0 \iff z = -a\frac{\overline{a} - 1}{a - 1}.$$

If $|z| = 1$, $z \neq 1$, then $-z\dfrac{\overline{z} - 1}{z - 1} = \dfrac{-1 + z}{z - 1} = 1$. Thus we may define $h(\lambda) = 1$ whenever $|\lambda| = 1, \lambda \neq 1$.

Since the cluster set of $h$ at $1$ is a decreasing intersection of continua, namely,

$$C(h, 1) = \bigcap_{n=1}^{\infty} \overline{h(D_n)}^{\mathbb{C}},$$

where $D_n = \{z \in \mathbb{D} : |z-1| \leq 1/n\}$, we see that $C(h, 1)$ is a nonvoid connected compact set. Now $\lim\limits_{\substack{x \to 1 \\ 0<x<1}} h(x) = -1$ and $\lim\limits_{\theta \to 0} h(e^{i\theta}) = 1$.

Since $\mu \in C(h, 1)$ if and only if $\overline{\mu} \in C(h, 1)$ (note that $h(\overline{z}) = \overline{h(z)}$), and $|h(z)| = |z| \to 1$ if $z \to 1$, we conclude that $C(h, 1) = \mathbb{T}$. $\square$

We note that a continuous involution $F$ of $\mathbb{D}$ onto $\mathbb{D}$ is an open map. Therefore, $F$ cannot have a continuous extension to $\mathbb{T}$ that is constant there. In fact, if this would be the case, say $F \equiv 1$ on $\mathbb{T}$, then we choose a sequence $w_n \in F(\mathbb{D})$ converging to a boundary point, $\beta$, of $F(\mathbb{D})$ different from 1. Let $z_n \in \mathbb{D}$ satisfy $F(z_n) = w_n$ for all $n$. We may assume, by passing to a subsequence if necessary, that $(z_n)$ converges to $a \in \overline{\mathbb{D}}$. Since we have assumed that $F$ has a continuous extension to $\overline{\mathbb{D}}$, we conclude that $F(a) = \beta$. Because $\beta \neq 1$, the constancy of $F$ on $\mathbb{T}$ implies that $a \in \mathbb{D}$. But this contradicts the fact that $F$ is an open map.

**Acknowledgements**

I thank the referee for drawing my attention to the class of curves, called roses (rhodonea), and Jérôme Noël for having realized figure 4 with TEXgraph.

## References

[1] L. Carleson, An interpolation problem for bounded analytic functions, Amer. J. Math. 80 (1958), 921-930.

[2] J.P. Earl, On the interpolation of bounded sequences by bounded functions, J. London Math. Soc. 2 (1970), 544-548.

[3] J.B. Garnett *Bounded Analytic Functions*, Academic Press, New York, 1981.

[4] P. Gorkin, R. Mortini, Value distribution of interpolating Blaschke products, J. London Math. Soc. 72 (2005), 151-168.

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