The Carathéodory and Kobayashi Metrics and Applications in Complex Analysis

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0. PREFATORY THOUGHTS. In the late nineteenth century, Henri Poincaré (1854–1912) introduced the profoundly original idea of equipping the open unit disc D in the complex plane with a metric that is invariant under conformal self-maps[1] of D. One may recall (see [**12**]) that the conformal maps of the disc are generated by the rotations

$$\rho_\theta : \zeta \longmapsto e^{i\theta}\zeta$$

for $0 \le \theta < 2\pi$ and the Möbius transformations

$$\varphi_a : \zeta \longmapsto \frac{\zeta - a}{1 - \bar{a}\zeta}$$

for $a \in \mathbb{C}$, $|a| < 1$. While rotations certainly preserve Euclidean distance, the Möbius transformations do not—see Figure 1.

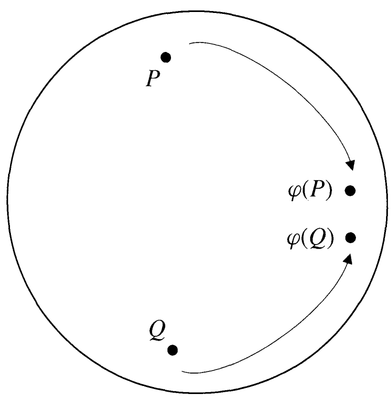

Figure 1. Möbius transformations do not preserve Euclidean distance.

It is most convenient to describe the Poincaré metric in infinitesimal form. In fact we set

$$\rho(\zeta) = \frac{1}{1 - |\zeta|^2}\,.$$

The *Poincaré length* of a vector ξ at base point P is then defined to be

$$\|\xi\|_{P,\mathrm{Poinc}} \equiv \rho(P) \cdot |\xi|\,,$$

[1]It is quite common in the literature to use the word "conformal map" to mean any map that is holomorphic. Our usage in this paper is somewhat more specific. For us, a conformal mapping is automatically holomorphic and one-to-one. Often it will be onto as well. This latter attribute will be made clear from context.

where $|\xi|$ denotes Euclidean length of the vector ξ. Throughout most of this paper we shall use Finsler metric notation, and we may as well begin now. In that notation, we write the Poincaré metric as

$$F_{\mathcal{P}}^{D}(P, \xi) = \frac{|\xi|}{1 - |P|^2} \, .$$

Now we define the length of a piecewise C^1 curve $\gamma : [a, b] \to D$ to be

$$L_{\mathcal{P}}(\gamma) \equiv \int_a^b F_{\mathcal{P}}^{D}(\gamma(t), \gamma'(t))\, dt \, .$$

The *Poincaré distance* of two points P and Q in the disc, denoted $d_{\mathcal{P}}(P, Q)$, is now declared to be the infinimum of the Poincaré lengths of all piecewise C^1 curves connecting P to Q.

The conformal invariance of the Poincaré metric is treated in detail in the reference [**22**]. We shall not discuss it here. Suffice it to say that Poincaré's construction is special to the disc. It is in fact a matter of some interest to equip virtually *any* domain[2] in the plane (or in higher-dimensional complex space) with a conformally or biholomorphically invariant metric. And there are various means of doing so. Certainly the most classical is to use the uniformization theorem. We shall discuss that approach in the next section. One of the first intrinsic techniques was developed by Stefan Bergman in 1922 (see [**1**] and references therein). Constantin Carathéodory [**3**] created another in 1927. One of the most recent is that developed in 1969 by S. Kobayashi [**19**]. See also the definitive references [**20**], [**18**], [**21**], [**27**].

Both Carathéodory's and Kobayashi's constructions have the advantage of being elementary, intrinsic, flexible, and immediately accessible. Their motivation from the proof of the Riemann mapping theorem is immediate. It is a lovely example of modern mathematics at work. The present paper is dedicated to the study of those two metrics.

It is a pleasure to thank the referees for very carefully reading my manuscript and contributing considerable wisdom and insight. The result is a cleaner and more precise presentation.

1. THE UNIFORMIZATION THEOREM. The uniformization theorem of Köbe and Poincaré is a remarkable generalization of the Riemann mapping theorem. We cannot prove it here, but we provide a brief discussion.

If X is a topological space satisfying certain technical hypotheses,[3] then it has a simply connected universal covering space $\widehat{X}$. The universal covering space is constructed by fixing a point $x_0 \in X$ and considering the space of all paths in X emanating from x_0. The covering map

$$\pi : \widehat{X} \to X$$

is a local homeomorphism. We refer the reader to [**29**] or [**16**] for details.

If X is a domain Ω in the complex plane, or more generally a Riemann surface, then the universal covering space $\widehat{X}$ will be a two-dimensional object (because π is a local homeomorphism), and $\widehat{X}$ can be endowed with a complex structure by local pullback

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[2]In the present paper, a domain in the plane is a connected, open set.

[3]The technical hypotheses are that the space be path connected, locally path connected, and semi-locally simply connected. These conditions are all trivially satisfied by any open subset of the plane, or by any Riemann surface. See [**17**] for details.

under π of the complex structure from X. So $\widehat{X}$ is a simply connected analytic object. What is it?

The uniformization theorem answers this question in a dramatic way. Before we present the answer, let us first restate the question—stripped of all the preliminary material that led up to it.

> **QUESTION:** Up to conformal equivalence, what are all the simply connected Riemann surfaces?

The answer is

> **ANSWER:** The only simply connected Riemann surfaces are **(i)** the disc D, **(ii)** the plane $\mathbb{C}$, and **(iii)** the Riemann sphere $\widehat{\mathbb{C}}$.

And in fact much more can be said. Let us return to the motivational discussion above. If the original analytic object X is a sphere, then it turns out that the universal covering space $\widehat{X}$ will be a sphere, and that is the *only* circumstance under which a sphere arises as the universal covering space.

If the original analytic object is a plane or a punctured plane or a torus or a cylinder, then the universal covering space $\widehat{X}$ is a plane, and these are the only circumstances in which the plane arises as the universal covering space (we note for the record that the punctured plane and a cylinder are conformally equivalent).

In all other circumstances, the universal covering space is the disc D. In other words,

> The universal covering space for any planar domain except $\mathbb{C}$ or $\mathbb{C} \setminus \{0\}$ is the disc D.

This is powerful information, and those who study Riemann surfaces have turned the result into an important tool (see [**6**]).

Suppose now that U is a planar domain that is neither the entire plane nor the punctured plane. Then the universal covering space is (conformally) equivalent to the disc and we have a covering map $\pi : D \to U$. Then we may push the Poincaré metric from the disc down to U—that is to say, measure the length of a tangent vector to U at $P \in U$ by pulling the vector back up to D by way of π. And so virtually *any* planar domain may be equipped with an invariant metric. We call such a domain *hyperbolic*—see [**22**]. One of the points of the present paper is that there are distinct advantages to constructing the invariant metric intrinsically. The constructions of Carathéodory and Kobayashi in fact generalize to a broad range of circumstances—even complex manifolds of many variables—and have proved to be powerful tools for function theory.

2. MOTIVATION BY WAY OF THE SCHWARZ AND SCHWARZ-PICK LEMMAS.

The construction of the Carathéodory and Kobayashi metrics is motivated in a natural way by Schwarz's lemma. The fact that an invariant metric is thereby constructed is closely related to the more general Schwarz-Pick lemma. We take this opportunity to review those ideas.

The classical Schwarz lemma is part of the grist of every complex analysis class. A version of it says this:

Lemma 1. *Let $f : D \to D$ be holomorphic. Assume that $f(0) = 0$. Then*

(a) $|f(z)| \leq |z|$ *for all $z \in D$;*

(b) $|f'(0)| \leq 1$.

At least as important as these two statements are the cognate uniqueness statements:

(c) *If* $|f(z)| = |z|$ *for some* $z \neq 0$, *then* f *is a rotation:* $f(z) = \lambda z$ *for some unimodular complex constant* λ*;*

(d) *If* $|f'(0)| = 1$, *then* f *is a rotation.*

There are a number of ways to prove this result. The classical argument is to consider $g(z) = f(z)/z$. On a circle $|z| = 1 - \epsilon$, we see that $|g(z)| \leq 1/(1-\epsilon)$. The maximum principle implies that this inequality persists on the disc $\overline{D}(0, 1-\epsilon)$. Thus $|f(z)| \leq |z|/(1-\epsilon)$ on that same disc. Since this inequality holds for all $\epsilon > 0$, part **(a)** follows. The Cauchy estimates show that $|f'(0)| \leq 1$.

For the uniqueness, if $|f(z)| = |z|$ for some $z \neq 0$, then $|g(z)| = 1$. The maximum modulus principle then forces $|f(z)| = |z|$ for all z. Thus g is a constant function and hence f is a rotation. If instead $|f'(0)| = 1$, then $|g(0)| = 1$ and again the maximum modulus principle yields that f is a rotation.

The Schwarz-Pick lemma observes that there is no need to restrict to $f(0) = 0$. Once one comes up with the right formulation, the proof is straightforward:

Proposition 2. *Let* $f : D \to D$ *be holomorphic. Assume that* $a \neq b$ *are elements of* D *and that* $f(a) = \alpha$, $f(b) = \beta$. *Then*

(a) $\left|\dfrac{\beta - \alpha}{1 - \overline{\alpha}\beta}\right| \leq \left|\dfrac{b - a}{1 - \overline{a}b}\right|$;

(b) $\left|f'(a)\right| \leq \dfrac{1 - |\alpha|^2}{1 - |a|^2}$.

There is also a pair of uniqueness statements:

(c) *If* $\left|\dfrac{\beta - \alpha}{1 - \overline{\alpha}\beta}\right| = \left|\dfrac{b - a}{1 - \overline{a}b}\right|$, *then* f *is a conformal self-map of the disc* D*;*

(d) *If* $\left|f'(a)\right| = \dfrac{1 - |\alpha|^2}{1 - |a|^2}$, *then* f *is a conformal self-map of the disc* D.

Proof. We sketch the proof. Recall that, for a a complex number in D,

$$\varphi_a(\zeta) = \frac{\zeta - a}{1 - \overline{a}\zeta}$$

defines a *Möbius transformation*. This is a conformal self-map of the disc that takes a to 0. Note that φ_{-a} is the inverse mapping to φ_a.

Now, for the given f, consider

$$g(z) = \varphi_\alpha \circ f \circ \varphi_{-a}\,.$$

Then $g : D \to D$ and $g(0) = 0$. So the standard Schwarz lemma applies to g. By part **(a)** of that lemma,

$$|g(z)| \leq |z|\,.$$

Letting $z = \varphi_a(\zeta)$ yields

$$|(\varphi_\alpha \circ f)(\zeta)| \leq |\varphi_a(\zeta)|\,.$$

Writing this out, and setting $\zeta = b$, gives the conclusion

$$\left|\frac{\beta - \alpha}{1 - \overline{\alpha}\beta}\right| \le \left|\frac{b - a}{1 - \overline{a}b}\right| .$$

That is part **(a)**.

For part **(b)**, we certainly have that

$$\left|(\varphi_\alpha \circ f \circ \varphi_{-a})'(0)\right| \le 1 .$$

Using the chain rule, we may rewrite this as

$$\left|\varphi_\alpha'((f \circ \varphi_{-a})(0))\right| \cdot \left|f'(\varphi_{-a}(0))\right| \cdot \left|\varphi_{-a}'(0)\right| \le 1 . \tag{1}$$

Now of course

$$\varphi_a'(\zeta) = \frac{1 - |a|^2}{(1 - \overline{a}\zeta)^2} .$$

So we may rewrite (1) as

$$\left(\frac{1 - |\alpha|^2}{(1 - |\alpha|^2)^2}\right) \cdot |f'(a)| \cdot (1 - |a|^2) \le 1 .$$

Now part **(b)** follows.

We leave parts **(c)** and **(d)** as exercises for the reader. ■

The quantity

$$\rho(a, b) = \frac{|a - b|}{|1 - \overline{a}b|}$$

is called the *pseudohyperbolic metric*. It is actually a metric on D (details left to the reader). It is not identical to the Poincaré-Bergman metric. In fact it is not a Riemannian metric at all. But it is still true that conformal maps of the disc are distance-preserving in the pseudohyperbolic metric. Exercise: Use the Schwarz-Pick lemma to prove this last assertion.

One useful interpretation of the Schwarz-Pick lemma is that a holomorphic function f from the disc to the disc must take each disc $D(0, r)$, $0 < r < 1$, into (but not necessarily onto) the image of that disc under the linear fractional map

$$z \mapsto \frac{z + \alpha}{1 + \overline{\alpha} z} ,$$

where $f(0) = \alpha$. This image is in fact (in case $-1 < \alpha < 1$) a standard Euclidean disc with center on the real axis at α and diameter (in case $0 < \alpha < 1$) given by the interval

$$\left[\frac{\alpha - r}{1 - \alpha r}, \frac{\alpha + r}{1 + \alpha r}\right] .$$

The reader will see, when encountering the definitions of the Carathéodory and Kobayashi metrics, the Schwarz lemma acting as motivation. Certainly the Schwarz lemma arises frequently in the proofs of the basic results about these metrics.

3. BASIC FACTS ABOUT THE KOBAYASHI METRIC. A good reference for invariant metrics in complex analysis is [**18**]. Following the paradigm set in Section 0 (for the Poincaré metric), we shall define the Kobayashi metric at first on the infinitesimal level. That is to say, we shall specify the length of a tangent vector at each point. We will always let Ω denote a connected, open set, or a *domain*. We let $\Omega(D)$ denote the collection of holomorphic functions from D (the disc) to Ω. If $z \in \Omega$ then we further let $\Omega^z(D)$ denote the subcollection of elements f of $\Omega(D)$ which satisfy $f(0) = z$.

So let $\Omega \subseteq \mathbb{C}$ be a domain. Fix a point $P \in \Omega$ and a vector ξ which is thought of as being tangent to the plane at the point P. We define the infinitesimal Kobayashi or Kobayashi/Royden length of ξ at P to be

$$\begin{aligned} F_K^\Omega(P, \xi) &\equiv \inf\{\alpha : \alpha > 0 \text{ and } \exists f \in \Omega(D) \text{ with } f(0) = P,\ f'(0) = \xi/\alpha\} \\ &= \inf\left\{\frac{|\xi|}{|f'(0)|} : f \in \Omega^P(D)\right\}. \end{aligned}$$

It is not in general the case that F_K^Ω satisfies a triangle inequality in the second entry. Nonetheless we can, as indicated in our discussion of the Poincaré metric, construct from it a useful metric.

Remark. Recall the standard, modern proof of the Riemann mapping theorem (see [**12**]). We are given a simply connected domain Ω (not all of $\mathbb{C}$), and our goal is to construct a conformal mapping of D onto Ω. We fix a point $P \in \Omega$ and we consider the family $\mathcal{S}$ of holomorphic mappings $\varphi : D \to \Omega$ with $\varphi(0) = P$. A normal families argument is used to show that there is an element φ^* of $\mathcal{S}$ that maximizes the modulus of the derivative at 0. The function φ^* turns out to be the conformal mapping that we seek.

Now look at the definition of the Kobayashi/Royden metric. The metric at a point P in the direction ξ minimizes the expression $|\xi|/|f'(0)|$ over mappings f of the disc into Ω. This is the same as *maximizing* the quantity $|f'(0)|$. Thus we see the proof of the Riemann mapping theorem coming back to life in the definition of the Kobayashi/Royden metric.[4]

Definition 1. Let $\Omega \subseteq \mathbb{C}$ be open and $\gamma : [a, b] \to \Omega$ a piecewise C^1 curve. The *Kobayashi/Royden length* of γ is defined to be[5]

$$L_K^\Omega(\gamma) = L_K(\gamma) = \int_a^b F_K^\Omega(\gamma(t), \gamma'(t))dt.$$

Definition 2. Let $\Omega \subseteq \mathbb{C}$ be an open set and $z, w \in \Omega$. The *(integrated) Kobayashi/Royden distance* between z and w is defined to be

$$K^\Omega(z, w) = K(z, w) = \inf\{L_K(\gamma) : \gamma \text{ is a piecewise } C^1 \text{ curve connecting } z \text{ and } w\}.$$

Of course it must be noted that K^Ω need not be a distance function (that is, not a *metric*) in the classical sense of the term. As an instance, if Ω is the entire complex

[4]In fact the idea behind the Kobayashi metric has a long history. Even in the 1920s, T. Rado observed that the same extremal problem may be used to produce a proof of the uniformization theorem for planar domains. See [**11**, p. 256].

[5]A word needs to be said about why F_K^Ω is integrable. In fact it is not difficult to see that F_K^Ω is lower semicontinuous, since it is the infimum of continuous functions. And that is sufficient for the integrability.

plane then K^Ω is identically equal to zero. A domain for which K^Ω is a genuine (non-degenerate) distance is called *hyperbolic* (this is equivalent to our earlier use of the term "hyperbolic"). The book [**22**] has a concise treatment of hyperbolicity. See also [**20**]. For all practical purposes, hyperbolicity is an invariant version of boundedness—in other words, a hyperbolic domain has all the key properties of a bounded domain, but hyperbolicity has the additional advantage of being invariant under conformal mappings.

One of the most important, and most interesting, properties of the Kobayashi metric is that a holomorphic function is distance decreasing in the metric. We shall be able to make good use of it in the examples below.

Proposition 3 (The Distance Decreasing Property of the Kobayashi Metric). *If Ω_1, Ω_2 are domains in $\mathbb{C}$, $z, w \in \Omega_1$, $\xi \in \mathbb{C}$, and if $f : \Omega_1 \to \Omega_2$ is holomorphic, then*

$$F_K^{\Omega_2}(f(z), f'(z)\xi) \le F_K^{\Omega_1}(z, \xi) \quad \textit{and} \quad K^{\Omega_2}(f(z), f(w)) \le K^{\Omega_1}(z, w)\,.$$

Remark. Observe that the Chain Rule demands that we put a factor of $f'(z)$ in front of the tangent vector when we calculate $F_K^{\Omega_2}(f(z), \cdot)$.

Proof of the Proposition. We prove the first inequality and leave the second for the reader.

Let $\varphi : D \to \Omega_1$ satisfy $\varphi(0) = z$. We call φ a *candidate mapping* for the Kobayashi metric at the point z on the domain Ω_1. Then $f \circ \varphi$ is a candidate mapping for the Kobayashi metric at the point $f(z)$ on the domain Ω_2. Thus

$$F_K^{\Omega_2}(f(z), f'(z)\xi) = \inf_{g \in \Omega_2^{f(z)}(D)} \frac{|f'(z)\xi|}{|g'(0)|} \le \frac{|f'(z)\xi|}{|(f \circ \varphi)'(0)|} = \frac{|\xi|}{|\varphi'(0)|}\,.$$

Now we take the infimum over all candidates φ to obtain

$$F_K^{\Omega_2}(f(z), f'(z)\xi) \le F_K^{\Omega_1}(z, \xi)\,. \qquad \blacksquare$$

Corollary 4. *If $f : \Omega_1 \to \Omega_2$ is conformal then f is an isometry in the Kobayashi/Royden metric. This means that f preserves distances:*

$$F_K^{\Omega_1}(z, \xi) = F_K^{\Omega_2}(f(z), f'(z)\xi) \quad \textit{and} \quad K^{\Omega_2}(f(z), f(w)) = K^{\Omega_1}(z, w)\,.$$

Remark. A caution is in order here. The reader who knows some differential geometry will be accustomed to the term "isometry", and will think of such a mapping as preserving distances in a strong (classical) sense. The "metrics" that we consider now may degenerate to 0, so our present use of the term "isometry" is somewhat more general.

Proof of the Corollary. Let us prove the first assertion. We leave the second to the reader. Now the proposition certainly tells us that

$$F_K^{\Omega_2}(f(z), f'(z)\xi) \le F_K^{\Omega_1}(z, \xi)\,. \tag{2}$$

But we may also apply the proposition to $f^{-1} : \Omega_2 \to \Omega_1$. The result is

$$F_K^{\Omega_1}(f^{-1}(a), [f^{-1}]'(a)\tau) \le F_K^{\Omega_2}(a, \tau)\,.$$

Now simply let $a = f(z)$ and $\tau = f'(f^{-1}(a))\xi$ to obtain

$$F_K^{\Omega_1}(z, \xi) \le F_K^{\Omega_2}(f(z), f'(z)\xi). \tag{3}$$

Combining (2) and (3) yields

$$F_K^{\Omega_1}(z, \xi) = F_K^{\Omega_2}(f(z), f'(z)\xi). \qquad \blacksquare$$

Corollary 5. *If $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{C}$ then, for any $z, w \in \Omega_1$, any $\xi \in \mathbb{C}$, we have*

$$F_K^{\Omega_1}(z, \xi) \ge F_K^{\Omega_2}(z, \xi) \qquad \textit{and} \qquad K^{\Omega_1}(z, w) \ge K^{\Omega_2}(z, w).$$

Proof. Simply apply the proposition to the inclusion mapping $i : \Omega_1 \to \Omega_2$. ■

4. BASIC FACTS ABOUT THE CARATHÉODORY METRIC. Following the model set in the last section, we shall define the Carathéodory metric at first on the infinitesimal level. That is to say, we shall specify the length of a tangent vector at each point. As usual, we let Ω denote a connected, open set, or a domain. Following tradition, we let $D(\Omega)$ denote the collection of holomorphic functions[6] from Ω to D. If $z \in \Omega$ then we further let $D^z(\Omega)$ denote the subcollection of elements g of $D(\Omega)$ such that $g(z) = 0$.

So let $\Omega \subseteq \mathbb{C}$ be a domain. Fix a point $P \in \Omega$ and a vector ξ which is thought of as being tangent to the plane at the point P. We define the infinitesimal Carathéodory length of ξ at P to be

$$F_C^{\Omega}(P, \xi) \equiv \sup_{\substack{f \in D^P(\Omega) \\ f(P)=0}} |f'(P)\xi|.$$

Remark. Refer to the discussion in the Remark following the definition at the beginning of Section 3 of the Kobayashi metric. It is worth noting that we could as well prove the Riemann mapping theorem by considering maps of the domain Ω into the disc D and maximizing the derivative at the point P. Now look at the definition of the Carathéodory metric. The metric at a point P in the direction ξ maximizes the expression $|f'(P)\xi|$ over mappings f of Ω into the disc. This is the same as *maximizing* the quantity $|f'(P)|$. Thus we see the proof of the Riemann mapping theorem coming back to life in the definition of the Carathéodory metric.

It is worth noting that an extremal function for the Carathéodory metric always exists, as can be seen with a normal families argument. The extremal function is often termed the *Ahlfors function*. It is, in many respects, a generalization of the Riemann mapping function (which is what it is in case Ω is simply connected). See **[7]** or **[24]** for a consideration of the Ahlfors function.

Definition 3. Let $\Omega \subseteq \mathbb{C}$ be open and $\gamma : [a, b] \to \Omega$ a piecewise C^1 curve. The *Carathéodory length* of γ is defined to be

$$L_C^{\Omega}(\gamma) = L_C(\gamma) = \int_a^b F_C^{\Omega}(\gamma(t), \gamma'(t))\, dt.$$

We note that F_C is integrable for reasons similar to the ones given for F_K in Section 1.

[6] Of course it is possible that $D(\Omega)$ is trivial—for example when Ω is the entire plane.

Next we are going to define the integrated Carathéodory distance in Ω. But now our approach will not parallel that for the Kobayashi metric. In fact we want the Carathéodory metric to have a certain "minimal property" among all metrics for which holomorphic functions are distance decreasing. This necessitates a new approach.

Definition 4. Let $\Omega \subseteq \mathbb{C}$ be an open set and $z, w \in \Omega$. The *Carathéodory distance* between z and w is defined to be

$$C^{\Omega}(z, w) = \sup_{f \in D(\Omega)} d_{\mathcal{P}}(f(z), f(w)),$$

where $d_{\mathcal{P}}$ is the Poincaré distance on D.

Remark. Of course the Carathéodory distance can be trivial—for instance if Ω is the entire plane.

One of the most important, and most interesting, properties of the Carathéodory metric is that a holomorphic function is distance decreasing in the metric. We shall be able to make good use of it in the examples below.

Proposition 6 (The Distance Decreasing Property of the Carathéodory Metric). *If Ω_1, Ω_2 are domains in $\mathbb{C}$, $z, w \in \Omega_1, \xi \in \mathbb{C}$, and if $f : \Omega_1 \to \Omega_2$ is holomorphic, then*

$$F_C^{\Omega_2}(f(z), f'(z)\xi) \leq F_C^{\Omega_1}(z, \xi) \quad \textit{and} \quad C^{\Omega_2}(f(z), f(w)) \leq C^{\Omega_1}(z, w).$$

Remark. Observe that the Chain Rule demands that we put a factor of $f'(z)$ in front of the tangent vector when we calculate $F_C^{\Omega_2}(f(z), \cdot)$.

Proof of the Proposition. We prove the first inequality and leave the second for the reader.

Let $\varphi : \Omega_2 \to D$ satisfy $\varphi(f(z)) = 0$. We call φ a *candidate mapping* for the Carathéodory metric at the point $f(z)$ on the domain Ω_2. Then $\varphi \circ f$ is a candidate mapping for the Carathéodory metric at the point z on the domain Ω_1. Thus

$$F_C^{\Omega_1}(z, \xi) = \sup_{g \in D^z(\Omega_1)} |g'(z)\xi| \geq |(\varphi \circ f)'(z)\xi| = |\varphi'(f(z))| \cdot |f'(z)| \cdot |\xi|.$$

Now we take the supremum over all candidates φ to obtain

$$F_C^{\Omega_1}(z, \xi) \geq F_C^{\Omega_2}(f(z), f'(z)\xi). \qquad \blacksquare$$

Corollary 7. *If $f : \Omega_1 \to \Omega_2$ is conformal then f is an isometry in the Carathéodory metric. This means that f preserves distances:*

$$F_C^{\Omega_1}(z, \xi) = F_C^{\Omega_2}(f(z), f'(z)\xi) \quad \textit{and} \quad C^{\Omega_2}(f(z), f(w)) = C^{\Omega_1}(z, w).$$

Proof. Let us prove the second assertion. We leave the first to the reader. Now the proposition certainly tells us that

$$C^{\Omega_2}(f(z), f(w)) \leq C^{\Omega_1}(z, w). \tag{4}$$

But we may also apply the proposition to $f^{-1} : \Omega_2 \to \Omega_1$. The result is

$$C^{\Omega_1}(f^{-1}(a), f^{-1}(b)) \leq C^{\Omega_2}(a, b) .$$

Now simply let $a = f(z)$ and $b = f(w)$ to obtain

$$C^{\Omega_1}(z, w) \leq C^{\Omega_2}(f(z), f(w)) . \tag{5}$$

Combining (4) and (5) yields

$$C^{\Omega_1}(z, w) = C^{\Omega_2}(f(z), f(w)) . \qquad \blacksquare$$

Corollary 8. *If $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{C}$ then for any $z, w \in \Omega_1$, any $\xi \in \mathbb{C}$, we have*

$$F_C^{\Omega_1}(z, \xi) \geq F_C^{\Omega_2}(z, \xi) \qquad \text{and} \qquad C^{\Omega_1}(z, w) \geq C^{\Omega_2}(z, w).$$

Proof. Simply apply the proposition to the inclusion mapping $i : \Omega_1 \to \Omega_2$. ■

5. COMPARISON OF THE KOBAYASHI AND CARATHÉODORY METRICS.

First, it is always the case that the Kobayashi metric majorizes the Carathéodory metric:

Proposition 9. *Let $\Omega \subseteq \mathbb{C}$ be a domain. Let $P \in \Omega$ and let ξ be a vector. Then*

$$F_C^{\Omega}(P, \xi) \leq F_K^{\Omega}(P, \xi) .$$

Proof. Let $\varphi : D \to \Omega$ be a candidate mapping for the Kobayashi metric at $P \in \Omega$. Let $\psi : \Omega \to D$ be a candidate mapping for the Carathéodory metric at $P \in \Omega$. Then $h \equiv \psi \circ \varphi : D \to D$ and $h(0) = 0$. By Schwarz's lemma, $|h'(0)| \leq 1$. But this just says that

$$|\psi'(P)| \leq \frac{1}{|\varphi'(0)|} .$$

Now first take the infimum on the right over all candidate functions φ for the Kobayashi metric then take the supremum on the left over all candidate functions ψ for the Carathéodory metric. The result is

$$F_C^{\Omega}(P, \xi) \leq F_K^{\Omega}(P, \xi) . \qquad \blacksquare$$

We conclude the section with an interesting extremal property of C^{Ω}.

Theorem 10. *Let $\Omega \subseteq \mathbb{C}$ be an open set. Let d be any metric on Ω that satisfies $d(z, w) \geq d_{\mathcal{P}}(f(z), f(w))$ for all $f \in D(\Omega)$ and $z, w \in \Omega$. Then $d(z, w) \geq C^{\Omega}(z, w)$.*

Proof. Exercise. Use the definition of C^{Ω}. ■

It is worth noting that the Kobayashi metric satisfies an analogous extremal property:

Theorem 11. *Let $\Omega \subseteq \mathbb{C}$ be an open set. Let d be any metric on Ω that satisfies $d(f(z), f(w)) \leq d_{\mathcal{P}}(z, w)$ for all $f \in \Omega(D)$ and $z, w \in D$. Then $d(z, w) \leq K^{\Omega}(z, w)$.*

We leave the details of this last result for the interested reader. A useful reference is [**23**] or [**18**].

In the present paper the roles of the Carathéodory and Kobayashi metrics are virtually interchangeable. Any proof that uses the Carathéodory metric could just as well use the Kobayashi metric, and vice versa. But both metrics are interesting because they are defined in a dual manner, and because the one (the Kobayashi) always majorizes the other (the Carathéodory). As we have already noted, the Kobayashi metric is the *largest* metric in which holomorphic mappings are distance decreasing and the Carathéodory metric is the smallest. It is an interesting, and more recent, result of K. T. Hahn (see [**23**]) that the Bergman metric always majorizes the Carathéodory metric. It is also known, thanks to an example of Diederich and Fornæss [**4**], that the Bergman metric *cannot* in general be compared with the Kobayashi metric.

6. CALCULATION OF THE CARATHÉODORY AND KOBAYASHI METRICS.

Precious little is known about explicitly calculating the Kobayashi/Royden metric or the Carathéodory metric. For special domains such as the disc or the annulus, the automorphism group is a powerful tool for obtaining explicit formulas.[7] In many circumstances one can instead *estimate* the metrics, and that is sufficient for applications (see, for example, [**10**]). Let us now, just for illustrative purposes, calculate the Kobayashi metric on the disc.

EXAMPLE 5. We let $\Omega = D$ be the unit disc. We begin by calculating the infinitesimal Kobayashi metric at the origin 0. Let

$$f : D \to \Omega$$

be holomorphic and satisfy $f(0) = 0$. By Schwarz's lemma, we know that $|f'(0)| \leq 1$. But in fact the function $f_0(\zeta) \equiv \zeta$ maps D to Ω with $f_0(0) = 0$ and $f_0'(0) = 1$. We conclude that 1 is the extremal value and hence, for every ξ,

$$F_K^D(0, \xi) = \inf \left\{ \frac{|\xi|}{|f'(0)|} : f \in \Omega^0(D) \right\} = |\xi| .$$

Our next task is to derive a formula for F_K^D at an arbitrary base point $P \in D$. Now notice that the Möbius transformation

$$\varphi(\zeta) = \frac{\zeta - P}{1 - \overline{P}\zeta}$$

maps P to 0. Also

$$\varphi'(P) = \left. \frac{(1 - \overline{P}\zeta) \cdot 1 - (\zeta - P) \cdot (-\overline{P})}{(1 - \overline{P}\zeta)^2} \right|_{\zeta = P} = \frac{1}{1 - |P|^2} .$$

[7] Of course we *could* write down a formula for the Kobayashi or Carathéodory metric of the upper half plane, for example. But that is only because the half plane is—by way of the Cayley map—conformally equivalent to the unit disc. However there is an extensive literature on this subject. The paper [**28**] considers the Carathéodory metric on the annulus in some detail. Gehring and Palka [**10**] have developed a *quasihyperbolic metric* that can be used, with comparison arguments, to obtain estimates for the Kobayashi metric.

Therefore we may calculate, for any vector ξ, that

$$F_K^D(P,\xi) = F_K^D(\varphi(P), \varphi'(P)\xi) = F_K^D(0, \xi/(1-|P|^2))$$
$$= \frac{1}{1-|P|^2} \cdot F_K^D(0,\xi) = \frac{|\xi|}{1-|P|^2}.$$

Of course what we have just calculated is the *infinitesimal form* of the Kobayashi metric. It is certainly of interest to have a formula for the integrated form—as that will be a genuine metric in the classical sense. And it will be invariant under conformal mappings. We note that the conclusion of Example 5 already shows that, on the disc, the Kobayashi metric coincides with the Poincaré metric. And in fact a calculation nearly identical to the one we just performed shows that the infinitesimal Carathéodory metric coincides with the infinitesimal Kobayashi and Poincaré metrics on the unit disc.

Proposition 12. *The length of the curve $\gamma(t) = \gamma_\epsilon(t) = t + i0$, $0 \le t \le 1-\epsilon$ in the Kobayashi metric on the the disc D is*

$$L_K^D(\gamma_\epsilon) = \frac{1}{2} \cdot \log\left[\frac{2-\epsilon}{\epsilon}\right].$$

Remark. This proposition is particularly interesting, for it tells us that

$$\lim_{\epsilon\to 0^+} L_K^D(\gamma_\epsilon) = +\infty.$$

In other words, the distance from the origin to the boundary of D—at least along a straight line segment—is $+\infty$. If we can show that the straight line segment is the shortest curve in the Kobayashi metric from 0 to $(1-\epsilon) + i0$, then we will have proved that the distance of 0 to ∂D is $+\infty$. [We shall establish this latter contention in a moment. First we prove the proposition.] This will, in turn, say that the unit disc D is *complete* in the Kobayashi metric.

At first such a statement may seem bewildering. How can a bounded, open set be complete? It certainly does not appear to be closed in any sense; on the contrary, it is open! But think of the Euclidean plane in the ordinary Euclidean metric. It is certainly complete. And that is because the boundary is infinitely far away. That is exactly what is happening with the Kobayashi metric on the unit disc.

Proof of the Proposition. Now

$$L_K^D(\gamma) = \int_0^{1-\epsilon} F_K^D(\gamma(t), \gamma'(t))\,dt$$
$$= \int_0^{1-\epsilon} \frac{|\gamma'(t)|}{1-|\gamma(t)|^2}dt$$
$$= \int_0^{1-\epsilon} \frac{1}{1-t^2}dt$$
$$= \frac{1}{2}\log\left[\frac{2-\epsilon}{\epsilon}\right]. \qquad \blacksquare$$

Proposition 13. *Among all continuously differentiable curves of the form*

$$\mu(t) = t + iw(t)\,, \qquad 0 \le t \le 1 - \epsilon\,,$$

that satisfy $\mu(0) = 0$ *and* $\mu(1-\epsilon) = 1 - \epsilon + 0i$, *the one of least length in the Kobayashi metric is* $\gamma(t) = t$. *Here* $w(t)$ *is any continuously differentiable, real-valued function.*

Proof. In fact, for any such candidate μ, we have

$$\begin{aligned} L_K^D(\mu) &= \int_0^{1-\epsilon} F_K^D(\mu(t), \mu'(t))\, dt \\ &= \int_0^{1-\epsilon} \frac{1}{1 - |\mu(t)|^2} \cdot |\mu'(t)|\, dt \\ &= \int_0^{1-\epsilon} \frac{1}{1 - t^2 - [w(t)]^2} \cdot (1 + [w'(t)]^2)^{1/2}\, dt\,. \end{aligned}$$

However

$$\frac{1}{1 - t^2 - [w(t)]^2} \ge \frac{1}{1 - t^2} \qquad \text{and} \qquad (1 + [w'(t)]^2)^{1/2} \ge 1.$$

We conclude that

$$L_K^D(\mu) \ge \int_0^{1-\epsilon} \frac{1}{1 - t^2} dt = L_K^D(\gamma).$$

This is the desired result.

Notice that, with only small modifications, this argument can also be applied to *piecewise* continuously differentiable curves $t + iw(t)$. ■

In fact if a piecewise continuously differentiable curve connecting the point $0 \in D$ to $(1 - \epsilon) + 0i \in D$ is *not* of the form

$$\mu(t) = t + iw(t), \tag{6}$$

then it may cross itself. Of course we can eliminate the loops and thereby create a shorter curve. If the resulting curve is still not the graph of a function, then elementary comparisons show that it will be longer than a curve of the form (6) (see Figure 2). We may conclude that the curve γ in the proposition is the shortest of all curves connecting 0 to $(1 - \epsilon) + 0i$.

Of course we can use the last result to give an explicit formula for the Kobayashi or Carathéodory metric on the disc. This we now do.

Proposition 14. *The integrated Kobayashi or Carathéodory distance of two points P and Q in D is given by*

$$d(P, Q) = \frac{1}{2} \log \left(\frac{1 + \left|\frac{P-Q}{1-\overline{P}Q}\right|}{1 - \left|\frac{P-Q}{1-\overline{P}Q}\right|} \right).$$

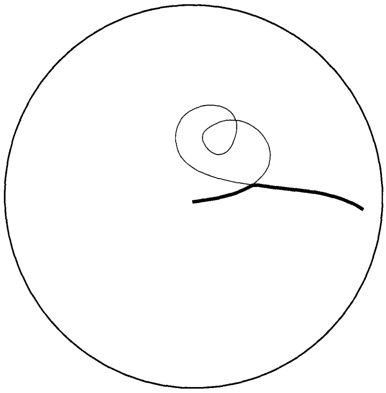

Figure 2. A curve that is not a graph.

Proof. First let us prove the result for the Kobayashi distance.

In case $P = 0$ and $Q = R + i0$, the result was already noted in Propositions 12 and 13. In the general case, note that we may define

$$\varphi(z) = \frac{z - P}{1 - \overline{P}z},$$

a Möbius transformation of the disc. Then, by Corollary 4 (letting d denote the Kobayashi distance),

$$d(P, Q) = d(\varphi(P), \varphi(Q)) = d(0, \varphi(Q)).$$

Next we have

$$d(0, \varphi(Q)) = d(0, |\varphi(Q)|) \tag{7}$$

since there is a rotation of the disc taking $\varphi(Q)$ to $|\varphi(Q)| + i0$. Finally,

$$|\varphi(Q)| = \left| \frac{P - Q}{1 - \overline{P}Q} \right|,$$

so that (7) together with the special case treated in the the first sentence of this paragraph gives the result.

Next observe that, by Example 5, the distance formula that we have just computed is valid also for the Poincaré distance.

Finally let d_P denote the integrated distance in the Poincaré metric. We apply the Schwarz-Pick lemma to notice that, for any holomorphic $f : D \to D$, $d_P(f(z), f(w)) \leq d_P(z, w)$ and equality holds only in the case of conformal mapping. It follows then that, in the definition of the Carathéodory distance, the supremum is achieved when f is a conformal self-mapping of the disc. Thus we see that the Carathéodory distance is the same as the Poincaré and Kobayashi distances. ∎

Notice the pseudohyperbolic metric appearing again in our calculations. It is a fundamental artifact of geometric function theory (see [**9**]).

The following is an interesting and nontrivial fact about the Carathéodory and Kobayashi metrics, one of the few that is valid for a large class of domains.

Theorem 15. *Let Ω be any bounded, planar domain that is not simply connected. Then the Kobayashi metric and the Carathéodory metric are unequal on Ω.*

Remark. We shall prove this result *not* by actually calculating the metrics, but rather by an indirect argument. It is a pleasing application of geometric analysis.

We shall first need a lemma.

Lemma 16. *Let Ω be as in the theorem and D the unit disc as usual. Then there do not exist holomorphic functions $\varphi : D \to \Omega$ and $\psi : \Omega \to D$ such that $\psi \circ \varphi(z) \equiv z$.*

Proof. We thank Paul Gauthier for the idea of this proof.

Obviously the function φ (if it exists) must be one-to-one, otherwise the composition could not be one-to-one. We claim that φ is onto. If that is the case, then φ is a conformal equivalence and hence, in particular, a homeomorphism. But the disc and Ω cannot be homeomorphic (their first homotopy groups are different, for example).

To prove the claim, suppose not. Then the image of φ is a proper open subset of Ω. Call the image $U = \varphi(D)$. Then U has a boundary point $p \in \Omega$. See Figure 3, which illustrates the idea when Ω is an annulus. Let $\{z_j\} \subseteq U$ be a sequence such that $z_j \to p$. Set $w_j = \varphi^{-1}(z_j)$. Then $w_j = \psi \circ \varphi(w_j) = \psi(z_j)$ for each j. Since $z_j \to p$ and ψ is continuous, $w_j = \psi(z_j) \to \psi(p) \equiv q \in D$. Similarly, the continuity of φ implies $z_j = \varphi(w_j) \to \varphi(q) \in U$. But we already know that $z_j \to p \notin U$, so this is a contradiction. ∎

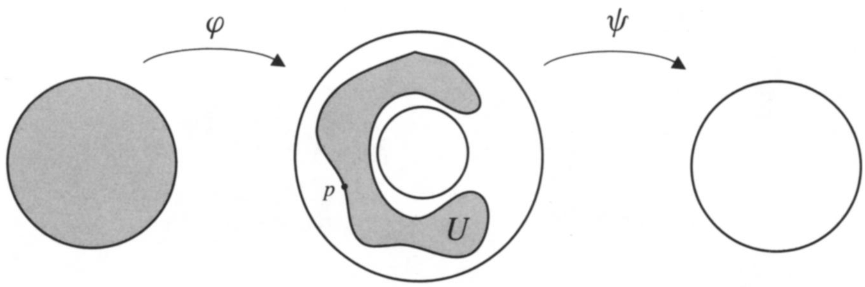

Figure 3. The identity on the disc does not factor through Ω (the case of the annulus).

Proof of the Theorem. Fix a point $p \in \Omega$. We claim that there is a constant $0 < c < 1$ such that if $\varphi : D \to \Omega$ is holomorphic with $\varphi(0) = p$ and $\psi : \Omega \to D$ is holomorphic with $\psi(p) = 0$ then

$$\left|(\psi \circ \varphi)'(0)\right| \le c. \tag{8}$$

Suppose not. Then, for each integer $j > 0$, there are holomorphic $\varphi_j : D \to \Omega$ and $\psi_j : \Omega \to D$, with $\varphi_j(0) = p$ and $\psi_j(p) = 0$, such that

$$\left|(\psi_j \circ \varphi_j)'(0)\right| > 1 - \frac{1}{j}.$$

Applying Montel's theorem, we may extract subsequences $\varphi_{j_k} \to \varphi_0$ (uniformly on compact sets) and $\psi_{j_k} \to \psi_0$ (uniformly on compact sets). Of course it will be the case that $\varphi_0 : D \to \Omega$, $\psi_0 : \Omega \to D$, φ_0 and ψ_0 are holomorphic, and $\varphi_0(0) = p$, $\psi_0(p) = 0$. And, what is most important,

$$\left|(\psi_0 \circ \varphi_0)'(0)\right| = 1\,.$$

By Schwarz's lemma, we may conclude that $\psi_0 \circ \varphi_0$ is a rotation. Postcomposing ψ_0 with the inverse of that rotation, we end up with a map from the disc to Ω and another map from Ω to the disc so that their composition from the disc to the disc is the identity. The lemma tells us that this is impossible.

Now inequality (8) tells us that

$$|\psi'(p)| \leq c \cdot \frac{1}{|\varphi'(0)|}\,.$$

Taking the infimum of the righthand side over φ and the supremum of the lefthand side over ψ as usual yields that

$$F_K^\Omega(p, 1) \leq c \cdot F_C^\Omega(p, 1)\,.$$

That is the desired result. ∎

We conclude this section by recording an interesting fact about our two invariant metrics. This will prove useful in the applications presented in the next section. The result has been somewhat anticipated in our discussion of the boundary behavior of the Kobayashi metric on the unit disc.

Proposition 17. *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with C^2 boundary (i.e., the boundary is locally the graph of a C^2 function). Then there are constants c, $C > 0$ such that, with $\delta(z)$ denoting the Euclidean distance to the boundary of Ω,*

$$\frac{c|\xi|}{\delta(z)} \leq F_K^\Omega(z, \xi) \leq \frac{C|\xi|}{\delta(z)}\,.$$

A similar set of estimates holds for the infinitesimal Carathéodory metric.

Remark. A glance at the proof of the proposition shows that the upper bound is true for *any* domain that is a proper subset of $\mathbb{C}$. For the disc $D(z, \delta(z))$ certainly lies in Ω, and then elementary comparison (Corollary 5) gives the result.

In the language of Gehring and Palka [**10**], Proposition 17 shows that the Kobayashi (or Carathéodory) metric is comparable to the quasihyperbolic metric.

Proof of the Proposition. It follows from elementary multivariable calculus that there are numbers r, $R > 0$ such that each point $p \in \partial\Omega$ has an *interior* circle $C(p', r)$ at p (so that $\overline{D}(p', r) \cap \partial\Omega = \{p\}$) and an *exterior* circle $C(p'', R)$ at p (so that $\overline{D}(p'', R) \cap \partial\Omega = \{p\}$).[8] See Figure 4.

Now if $z \in \Omega$ and z is sufficiently near the boundary then, by the tubular neighborhood theorem (see [**15**]), there is a unique nearest point $\pi(z) \in \partial\Omega$. Consider the interior circle $C(\pi(z)', r)$ at that point and its corresponding disc $D(\pi(z)', r)$. Then, by Corollary 5,

$$F_K^\Omega(z, \xi) \leq F_K^{D(\pi(z)', r)}(z, \xi) \approx \frac{C|\xi|}{\delta(z)}\,.$$

[8]These ideas are related to the classical calculus notion of "osculating circle" (see [**2**]) but are not necessarily the same.

That is one half of what we wish to prove. [Notice that we need only consider z near the boundary since the estimates are trivial in the interior.]

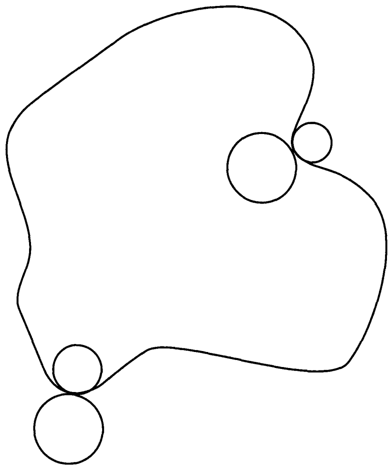

Figure 4. Interior and exterior circles.

For the other half, we again consider $z \in \Omega$, z near the boundary. Again let $\pi(z)$ be the nearest point in the boundary. Let $D(\pi(z)'', R)$ be the exterior disc at $\pi(z)$. Let $D(\pi(z)'', \widetilde{R})$ be a large disc centered at $\pi(z)''$ that contains the domain Ω. Now consider the region $U \equiv D(\pi(z)'', \widetilde{R}) \setminus \overline{D(\pi(z)'', R)}$. Then certainly $U \supseteq \Omega$. Hence, by Corollary 5,

$$F_K^{\Omega}(z, \xi) \geq F_K^U(z, \xi) .$$

But we may use a simple "reflection" map $\zeta \mapsto R^2/(\zeta - \pi(z)'')$ to compare the Kobayashi metric on U with the Kobayashi metric on a disc and then see that

$$F_K^U(z, \xi) \approx \frac{c|\xi|}{\delta(z)} .$$

Putting together the last two displayed lines yields

$$F_K^{\Omega}(z, \xi) \geq \frac{c|\xi|}{\delta(z)} . \qquad \blacksquare$$

It is an immediate corollary of Proposition 17 that the *Euclidean* diameter[9] of a Carathéodory or Kobayashi metric ball $B(p, r)$ (for $r > 0$ fixed) tends to 0 as p tends to the boundary of a C^2 bounded domain. We leave the details of this assertion for the interested reader.

Certainly one important upshot of Proposition 17 is that, on a domain with C^2 boundary, the Kobayashi metric is complete (a similar assertion holds for the Carathéodory metric). That is so because we know that the metric blows up like the reciprocal of the distance to the boundary. Thus we can see, with some tedious but straightforward calculations (just as we did on the unit disc), that the length of any curve tending to the boundary is infinite.

[9]The Euclidean diameter of a set is the supremum of Euclidean distances of pairs of points in the set.

7. SOME APPLICATIONS.

We now show how metric geometry can in fact inform our study of function theory.[10] The first result is due to Farkas and Ritt, but the proof is due to Earle and Hamilton [5]. It concerns fixed points for holomorphic functions. It is pleasing because it uses not only one of our invariant metrics, but it also uses a fixed-point theorem from functional analysis.

Theorem 18 (Farkas, Ritt). *Let $f : D \to D$ be holomorphic and assume that the image $M = \{f(z) : z \in D\}$ of f satisfies $\overline{M} \subseteq D$. Then there is a unique point $P \in D$ such that $f(P) = P$. We call P a fixed point for f.*

Remark. This result is actually amenable to a number of different proofs. If we take the image of f to lie in a disc $\overline{D}(0, r)$ for some $0 < r < 1$ then we may think of f as mapping $\overline{D}(0, r)$ to $\overline{D}(0, r)$ continuously. Thus the Brouwer fixed-point theorem applies and we find a fixed point (although this argument does not address the uniqueness question).

It is well known that *any* proof of a fixed-point theorem will involve argument principle considerations (that is what homotopy does for us in Brouwer's original proof; see also the proof in [8]). Thus it stands to reason that Rouché's theorem can be used to give the present result. That approach also does not address the uniqueness question. Our purpose here is to illustrate the utility of metric geometry, and also to derive the stronger uniqueness result for the fixed point.[11]

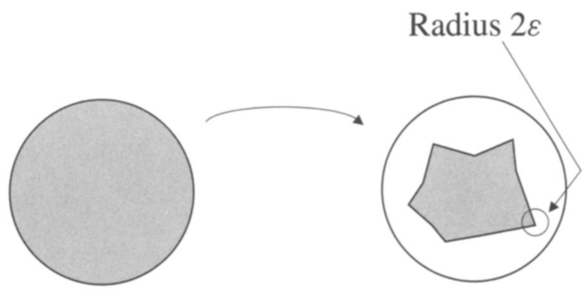

Figure 5. Relative compactness of the image of the mapping.

Proof. By hypothesis, there is an $\epsilon > 0$ such that if $m \in M$ and $|z| \geq 1$ then $|m - z| > 2\epsilon$. See Figure 5. Fix $z_0 \in D$ and define

$$g(z) = f(z) + \epsilon(f(z) - f(z_0)).$$

Then g is holomorphic and g still maps D into D. Also

$$g'(z_0) = (1 + \epsilon) f'(z_0).$$

Certainly g is distance-decreasing in the Kobayashi metric. Therefore

$$F_K^D(g(z_0), g'(z_0) \cdot \tau) \leq F_K^D(z_0, \tau)$$

[10] As noted earlier, the roles of the Carathéodory and the Kobayashi metrics are essentially interchangeable in these examples.

[11] It would be appropriate to note that Schwarz-Pick certainly gives the uniqueness immediately.

for any tangent vector τ. Now if $\gamma : [a, b] \to D$ is any continuously differentiable curve, and if we take $t \in [a, b]$, $z_0 = \gamma(t)$, and $\tau = \gamma'(t)$, then we may conclude that

$$F_K^D(g(\gamma(t)), g'(\gamma(t)) \cdot \gamma'(t)) \leq F_K^D(\gamma(t), \gamma'(t)) .$$

Writing this out gives

$$(1 + \epsilon) F_K^D(f(\gamma(t)), f'(\gamma(t)) \cdot \gamma'(t)) \leq F_K^D(\gamma(t), \gamma'(t)) .$$

Integrating both sides from a to b, we conclude that

$$L_K^D(f \circ \gamma) \leq (1 + \epsilon)^{-1} L_K^D(\gamma) .$$

If P and Q are elements of D then we see that

$$K(f(P), f(Q)) \leq (1 + \epsilon)^{-1} K(P, Q) .$$

We conclude that f is a contraction in the Kobayashi metric. Recall that in Section 6 we proved that the disc D is a complete metric space when equipped with the Kobayashi metric. By the contraction mapping fixed-point theorem (see [**25**]), f has a unique fixed point. ∎

Now we shift gears and look at the boundary behavior of holomorphic functions. Complete background may be found in [**23**, Ch. 8] or [**24**]. We begin by reviewing some terminology. Let f be a function (not necessarily holomorphic) on the unit disc D. Let $p = e^{i\theta}$ be a point in the boundary of the disc. We say that f has *radial boundary limit* ℓ at p if

$$\lim_{r \to 1^-} f(rp) = \ell .$$

As a counterpoint, let us now consider a more general notion of limit. For $p = e^{i\theta} \in \partial D$ and $\alpha > 1$, let us define

$$\Gamma_\alpha(p) = \{z \in D : |z - p| < \alpha(1 - |z|)\} .$$

See Figure 6. In fact an analogous definition works just as well on any domain with C^2 boundary (with $(1 - |z|)$ replaced by $\delta(z)$, the distance of z to the boundary).

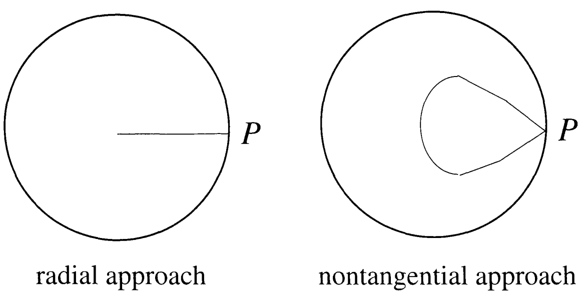

Figure 6. Radial convergence and nontangential convergence.

We say that f on D has *nontangential* limit ℓ at p if

$$\lim_{\Gamma_\alpha(p) \ni z \to p} f(z) = \ell$$

for each $\alpha > 1$. Our purpose now is to compare and relate radial convergence with nontangential convergence. We shall work on domains with C^2 boundary, which simply means that the boundary is locally the graph of a twice continuously differentiable function.

Theorem 19. *Let Ω_1, Ω_2 be bounded domains with C^2 boundary and let*

$$f : \Omega_1 \longrightarrow \Omega_2$$

be holomorphic. If $P \in \partial\Omega_1$, $Q \in \partial\Omega_2$, and f has radial limit Q at P then f has nontangential limit Q at P.

Remark. This is a version of a classical result that is known as the *Lindelöf principle*. The usual proof of that result uses a normal families argument (see, for example, [**23**] or [**24**]). That argument is lurking in the background of the more geometric argument that we present here.

Proof. If z is an element of one of our domains Ω_j and if $s > 0$ then we let $B_{\Omega_j}(z, s)$ denote the metric ball with center z and radius s in the Carathéodory metric for Ω_j. For $P \in \partial\Omega_1$, we let ν_P denote the unit outward normal at P to the boundary $\partial\Omega_1$. If $r_0 > 0$ and $\beta > 0$ are fixed, we define

$$\mathcal{M}_\beta^{r_0}(P) = \bigcup_{0<r<r_0} B_{\Omega_1}(P - r\nu_P, \beta)\,.$$

Observe that we use a subscript on B to indicate in what domain the metric ball lives. We usually suppress the superscript r_0 since it is of little interest.

The estimate

$$F_C^{\Omega}(z, \xi) \approx \frac{k|\xi|}{\mathrm{dist}(z, \partial U)} \tag{9}$$

from Proposition 17 makes it a tedious but not difficult exercise to calculate that the regions $\mathcal{M}_\beta$ are comparable to the regions Γ_α (in this last formula, and in what follows, "dist" means Euclidean distance).[12] By this we mean that, for each $\alpha > 1$, there is some $\beta > 0$ so that $\Gamma_\alpha(P) \subseteq \mathcal{M}_\beta(P)$. [A similar conclusion in the opposite direction is also valid.]

In point of fact, suppose that z lies in for some $\Gamma_\alpha(p)$, some $p \in \partial\Omega_1$. Let us denote by τ_p the *inward* normal *segment* (*not* the vector!) of length r_0 emanating from that boundary point p. Using the estimate (9), one can then estimate that $C^{\Omega_1}(z, \tau_p) \leq C \cdot \alpha$. For the converse estimate, assume that $z \notin \Gamma_\alpha(p)$ and the same estimate shows that $C^{\Omega_1}(z, \tau_p) \geq C \cdot \alpha$.

Thus we see that

$$\lim_{\Gamma_\alpha(P)\ni z \to P} f(z) = \ell, \qquad \forall \alpha > 1$$

[12]There is a subtle fact at play here. According to our definition, the Carathéodory *distance* is formally not the same as that distance that one obtains by integrating the infinitesimal Carathéodory metric. And in fact the two are genuinely different—as an examination of the domain $B(0, 2) \setminus \overline{B}(0, 1)$ (the set-theoretic difference of two balls) will show. We cannot provide the details here, but see [**23**]. It is always the case that the metric defined as we have done so in Definition 4 is less than or equal to the metric that is obtained by integrating the infinitesimal metric. In the current application, near the boundary of a smoothly bounded domain, the two notions of distance are comparable.

iff

$$\lim_{\mathcal{M}_\beta(P)\ni z\to P} f(z) = \ell, \qquad \forall \beta > 0. \tag{10}$$

Thus it is enough to prove (10).

Fix $\beta > 0$. Since

$$\mathcal{M}_\beta(P) = \bigcup_{0<r<r_0} B_{\Omega_1}(P - r\nu_P, \beta)\,,$$

the distance-decreasing property of f with respect to the Carathéodory metric implies that

$$f(\mathcal{M}_\beta(P)) \subseteq \bigcup_{0<r<r_0} B_{\Omega_2}(f(P - r\nu_P), \beta).$$

Pick $\epsilon > 0$. By the radial limit hypothesis, there is a $\delta > 0$ such that if $0 < t < \delta$ then

$$|f(P - t\nu_P) - Q| < \epsilon.$$

For such a t, if $z \in B_{\Omega_1}(P - t\nu_P, \beta)$ then

$$f(z) \in B_{\Omega_2}(f(P - t\nu_P), \beta).$$

But

$$\mathrm{dist}(f(P - t\nu_P), \partial\Omega_2) \le \mathrm{dist}(f(P - t\nu_P), Q) < \epsilon.$$

Therefore the estimate (9) implies that the metric ball $B_{\Omega_2}(f(P - t\nu_P), \beta)$ has Euclidean diameter not exceeding $C \cdot \epsilon$. Here C depends on β, but β has been fixed once and for all. Thus

$$|f(z) - f(P - t\nu_P)| < C\epsilon, \qquad \forall z \in B_{\Omega_1}(P - t\nu_P, \beta).$$

We conclude that

$$\begin{aligned} |f(z) - Q| &\le |f(z) - f(P - t\nu_P)| + |f(P - t\nu_P) - Q| \\ &\le C\epsilon + \epsilon = C'\epsilon\,. \end{aligned}$$

This is the desired conclusion. ∎

Our final application concerns automorphism groups. Some preliminary discussion is in order. If Ω is a planar domain then we consider the collection of all conformal self-maps $\varphi : \Omega \to \Omega$. To be explicit, we demand that φ be holomorphic, one-to-one, and onto, and have a holomorphic inverse. This collection is a group when equipped with the binary operation of composition of functions. We call this the *automorphism group* of Ω, and we denote it by $\mathrm{Aut}(\Omega)$.

We endow the automorphism group with the topology of uniform convergence on compact sets. This is equivalent with the compact-open topology. It is a fact that, with this topology, the automorphism group of a bounded domain is a real Lie group (see [**20**]). We shall not need that information here.

One of the ways that we can understand a domain is by understanding its automorphism group. This may entail studying the group's algebraic properties, or studying its topological properties, or perhaps considering some combination of the two. The next result illustrates this symbiosis.

Theorem 20. *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with C^2 boundary. If* $\mathrm{Aut}(\Omega)$ *is noncompact then Ω is conformally equivalent to the unit disc.*

Remark. It is certainly known (see [**26**]) that a finitely connected domain with connectivity at least 3 (i.e., at least two holes) has only finitely many conformal self-maps. This is a nontrivial result. Our approach gives another way to think about that classical result (which was proved in sharp form by Maurice Heins in the 1940s—see [**13**], [**14**]).

We prove this theorem with a sequence of lemmas, each of which has intrinsic interest.

Lemma 21. *Let $\Omega \subseteq \mathbb{C}$ be bounded. The group* $\mathrm{Aut}(\Omega)$ *is compact if and only if, for each $P \in \Omega$, there is a compact $K^P \subseteq \Omega$ such that $\varphi(P) \in K^P$ for all $\varphi \in$* $\mathrm{Aut}(\Omega)$.

Proof. Assume that $\mathrm{Aut}(\Omega)$ is compact. Fix $P \in \Omega$. If there is no set K^P as claimed then there exist $\varphi_j \in \mathrm{Aut}(\Omega)$ such that $\varphi_j(P) \to w \in \partial\Omega$, for some w. But Ω is bounded so that $\{\varphi_j\}$ is a normal family; thus there is a subsequence φ_{j_k} and a holomorphic limit function φ_0 such that

$$\varphi_{j_k} \longrightarrow \varphi_0$$

normally.

Notice that the image of each φ_j lies in Ω hence the image of φ_0 lies in the *closure* $\overline{U}$ of Ω. If φ_0 is nonconstant then it satisfies the open mapping principle. But

$$\varphi_0(P) = \lim_{k\to\infty} \varphi_{j_k}(P) = w,$$

hence the image of φ_0 contains the accumulation point $w \in \partial\Omega$, so it contains a neighborhood of w. This is impossible because w is in the boundary of the image of φ_0. Therefore φ_0 must be constantly equal to w; thus $\varphi_0 \notin \mathrm{Aut}(\Omega)$. The sequence φ_{j_k} therefore violates the compactness of $\mathrm{Aut}(\Omega)$. We conclude that K^P must exist.

For the converse, fix $P \in \Omega$ and let K^P be the corresponding compact set in Ω whose existence we assume. Let $\{\varphi_j\} \subseteq \mathrm{Aut}(\Omega)$ be any sequence. Since Ω is bounded, there is a normally converging subsequence φ_{j_k} with holomorphic limit function φ_0. As in the first half of the proof, if the image of φ_0 contains any boundary point w then φ_0 must be constantly equal to w. But the image of P under φ_0 must lie in K^P, so this possibility is ruled out. We conclude that the image of φ_0 lies in Ω.

Next notice that each φ_{j_k} has an inverse ψ_{j_k}. Passing to another subsequence, we may suppose that the $\psi_{j_{k_\ell}}$ converge to a limit function ψ_0. Just as for φ_0, we can be sure that the image of ψ_0 lies in Ω. In the next paragraph we shall see that ψ_0 is nonconstant.

Now we have

$$z \equiv \lim_{\ell\to\infty} \varphi_{j_{k_\ell}} \circ \psi_{j_{k_\ell}}(z) = \varphi_0 \circ \psi_0(z).$$

Since $\mathbf{i}(z) \equiv z$ is onto, so is φ_0. Also, by the argument principle, the image of ψ_0 is open, closed, and nonempty. Therefore ψ_0 is surjective. Since $\mathbf{i}(z)$ is injective, it now follows that ψ_0 is injective. Therefore $\psi_0 \in \mathrm{Aut}(\Omega)$ and it follows that $\phi_0 \in \mathrm{Aut}(\Omega)$. We conclude that

$$\mathrm{Aut}(\Omega) \ni \varphi_{j_{k_\ell}} \longrightarrow \varphi_0 \in \mathrm{Aut}(\Omega),$$

and $\mathrm{Aut}(\Omega)$ is compact. ∎

Remark. It may be noted that, in the proof of the converse direction of Lemma 21, only one compact set K^P was needed.

Lemma 22. *Let $\Omega \subseteq \mathbb{C}$ be a bounded domain with C^2 boundary. Suppose that $P \in \Omega$, $\{\varphi_j\}$ are holomorphic maps from Ω to Ω, and*

$$\varphi_j(P) \longrightarrow w \in \partial\Omega.$$

If K is compact in Ω and V is a neighborhood of w then there exists a positive number J such that if $j \geq J$ then

$$\varphi_j(K) \subseteq V.$$

See Figure 7.

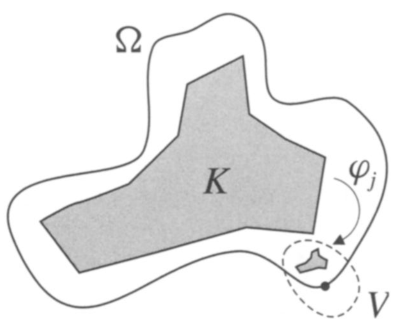

Figure 7. Noncompact group action on a compact set.

Proof. Since Ω, when equipped with the Carathéodory metric, is a metric space[13] and since K is compact, there is a positive number R such that the metric ball $B(P, R)$ contains K. Let $Q_j = \varphi_j(P)$. Since each φ_j is distance-decreasing in the Carathéodory metric, it follows that $\varphi_j(B(P, R)) \subseteq B(Q_j, R)$. We claim that there is a positive J such that, if $j \geq J$, then $B(Q_j, R) \subseteq V$. Assuming the claim, we would then have

$$\varphi_j(K) \subseteq \varphi_j(B(P, R)) \subseteq B(Q_j, R) \subseteq V,$$

as required.

To prove the claim, recall (because the Carathéodory metric on Ω is complete) that the Euclidean radii of the metric balls $B(Q_j, R)$ must tend to 0. Choose $\epsilon > 0$

[13]It is worth noting that, on compact sets, the Carathéodory metric is comparable to the Euclidean metric (which one sees just be comparing the domain to an interior ball and an including ball). As a result, open sets in the Carathéodory metric are the same as open sets in the Euclidean metric.

such that the Euclidean disc of center w and diameter 2ϵ lies in V. We select J so large that when $j > J$, both the Euclidean distance of Q_j to w is less than ϵ and the Euclidean diameter of $B(Q_j, R)$ is less than ϵ. The claim now follows from the triangle inequality. ■

Proof of Theorem 20. If $\mathrm{Aut}(\Omega)$ is not compact then, by Lemma 21, there is a sequence $\varphi_j \in \mathrm{Aut}(\Omega)$ and a $P \in \Omega$ such that

$$\varphi_j(P) \longrightarrow w \in \partial\Omega ,$$

for some $w \in \partial\Omega$.

Let

$$\gamma : [0, 1] \longrightarrow \Omega$$

be any continuous closed curve in Ω. Since $\partial\Omega$ is C^2 there is a neighborhood V of w such that $\Omega \cap V$ is simply connected (see Figure 8—the existence of the interior circle, or the tubular neighborhood, provided by the proof of Proposition 17 makes this assertion clear).

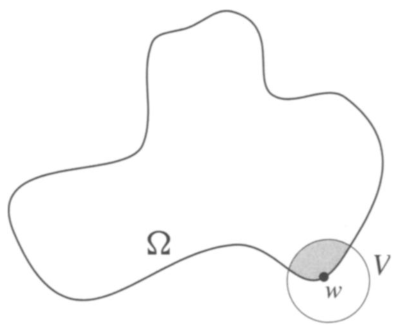

Figure 8. Simply connected boundary neighborhood.

Let

$$K = \{\gamma(t) : 0 \leq t \leq 1\}.$$

Then K is compact. By Lemma 22, there is a $J \geq 0$ such that $j \geq J$ implies $\varphi_j(K) \subseteq \Omega \cap V$. Thus $\varphi_j \circ \gamma$ is a continuous, closed curve in $\Omega \cap V$. The simple connectivity of $\Omega \cap V$ implies that $\varphi_j \circ \gamma$ may be continuously deformed to the point $(\varphi_j \circ \gamma)(0)$; that is, there is a homotopy

$$\Psi : [0, 1] \times [0, 1] \longrightarrow \Omega \cap V$$

such that

$$\Psi(0, t) = (\varphi_j \circ \gamma)(t), \qquad \forall t \in [0, 1]$$

and

$$\Psi(1, t) = (\varphi_j \circ \gamma)(0), \qquad \forall t \in [0, 1].$$

But then

$$(\varphi_j)^{-1} \circ \Psi$$

is a homotopy of the curve γ to the point $\gamma(0)$. It follows that Ω is simply connected. By the Riemann mapping theorem, Ω is conformally equivalent to the disc. ■

8. CONCLUDING REMARKS. Certainly the interaction of metric geometry with function theory has been one of the seminal developments of twentieth century complex analysis. There have many vectors in this activity: **(i)** Poincaré, Bergman, Carathéodory, and Kobayashi (among several others) have provided us with a family of extremely useful conformally invariant metrics, **(ii)** Lars Ahlfors has shown that the Schwarz lemma may be understood in terms of curvature of a suitable conformal metric (see [**22**]), and **(iii)** many of the phenomena of function theory have been given very natural interpretations in terms of the geometry of Kähler manifolds. Surely other authors would emend or modify this list.

The result of all these new ideas has been a subject enriched with new results, and with new interpretations of old results. Even the deep Picard theorems may be given rather direct and quick proofs using metric geometry (see [**22**] for the details). Each of the applications presented in the present paper can actually be proved with classical techniques. But the metric geometry proofs are natural, enlightening, and fun.

We hope that this excursion into the world of complex analysis and geometry has provided the reader with adequate motivation to explore further. The result will be both edifying and rewarding.

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