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Abstract. — We study the mapping properties of the maps $f(z) = \frac{\overline{z}-1}{z-1}$, $g(z) = |z|\, f(z)$ and $h(z) = -zf(z)$ with $|z| \leq 1$, $z \neq 1$.

6.4.2013

Introduction

In this paper we are concerned with the mapping properties of some non-holomorphic continuous functions on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and their behaviour at the boundary $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ of $\mathbb{D}$. Our first example is the function $f(z) = (\overline{z} - 1)/(z - 1)$ which played a prominent role in Earl's [**2**] constructive solution to the famous interpolation problem for bounded analytic functions, originally solved by L. Carleson [**1**], [**3**]. Earl considered finite Blaschke products of the form

$$B_n(z, \xi) = \prod_{k=1}^{n} \frac{z - \xi_k}{1 - \overline{\xi}_k z} \frac{1 - \overline{\xi}_k}{1 - \xi_k}.$$

In contrast to the usual rotational factors $-|\xi_k|/\xi_k$, these new unimodular factors $(1 - \overline{\xi}_k)/(1 - \xi_k)$ were chosen so that $B_n(z, \xi) = 1$ at $z = 1$, a fact fundamental for his solution to work. These factors reappeared in [**4**] in a similar context when studying the value distribution of interpolating Blaschke products. To see this, let

$$S(z) = \exp\left(-\frac{1 + z}{1 - z}\right)$$

2010 ***Mathematics Subject Classification.*** — Primary 30H50; Secondary 46J15.

be the atomic inner function. Choose $\sigma \in \mathbb{T}, \sigma \neq 1$, so that $S(\sigma) = 1$. Then the rotated Fostman shift

$$B(z) = \frac{S(\overline{\sigma} z) - b}{1 - \overline{b} S(\overline{\sigma} z)} \, \frac{1 - \overline{b}}{1 - b}$$

of S is an interpolating Blaschke product with singularity at σ that has the property that $B(1) = 1$. Moreover, as we did want that B additionally satisfies

$$\lim_{r \to 1} B(\sigma r) = a,$$

we were led to study the equation

$$-b \frac{1 - \overline{b}}{1 - b} = a.$$

(Note that $\lim_{r \to 1} S(r) = 0$.) This gave me the motivation to study in the present note the mapping properties of the function $h(z) = -z(\overline{z} - 1)/(z - 1)$.

It turns out that the map h also provides a solution (see Proposition 3.1) to the following question:

Does there exist continuous involutions of $\mathbb{D}$ onto itself (these are continuous functions ι for which $\iota \circ \iota = \mathrm{id}$, where id is the identity map), such that ι has a continuous extension with constant value at a largest possible subset of $\mathbb{T}$, namely $\mathbb{T} \setminus \{1\}$? [(1)] Note that the elliptic automorphisms $\varphi_a(z) = (a - z)/(1 - \overline{a} z)$ of $\mathbb{D}$ are involutions with $\phi_a(\mathbb{T}) = \mathbb{T}$; so these functions are more or less opposite to that class of functions we were looking for.

Now let us come back to the function $f(z) = (\overline{z} - 1)/(z - 1)$. It is clear that $|f(z)| = 1$ for every $z \in \mathbb{D}$. So in order to describe and better visualize the global mapping properties of f, I "added" the factor $|z|$. In this way we are led to study the function

$$g(z) = |z| \, \frac{\overline{z} - 1}{z - 1}.$$

As we shall see, g has a totally different behaviour than h. One striking fact, is that the image of $\mathbb{D}$ under g is no longer an open set. We will explicitely determine $g(\mathbb{D})$. It turns out that certain rhodonea curves (roses) as Dürer's folium, $r = \sin(\theta/2)$, play an important role in studying the image properties of g.

We include in our paper six figures that help to visualize and understand the calculations and results achieved.

(1) Later we shall see that one cannot achieve the constancy of the involution on the entire boundary of $\mathbb{D}$.

1. The map $f(z) = (\overline{z} - 1)/(z - 1)$

Lemma 1.1. — *Consider for $z \in \mathbb{D}$ the function $f(z) = (\overline{z} - 1)/(z - 1)$ and let $0 < a < 1$. Then*

1. $\max_{|z|=a} \operatorname{Re} f(z) = 1$;
2. $\min_{|z|=a} \operatorname{Re} f(z) = 1 - 2a^2$;
3. $\max_{|z|=a} \operatorname{Im} f(z) = 1$ *if and only if* $\frac{1}{\sqrt{2}} \leq a < 1$ *and* $\max_{|z|=a} \operatorname{Im} f(z) = 2a\sqrt{1-a^2}$ *if and only if* $0 < a \leq \frac{1}{\sqrt{2}}$;
4. $\min_{|z|=a} \operatorname{Im} f(z) = -\max_{|z|=a} \operatorname{Im} f(z)$.

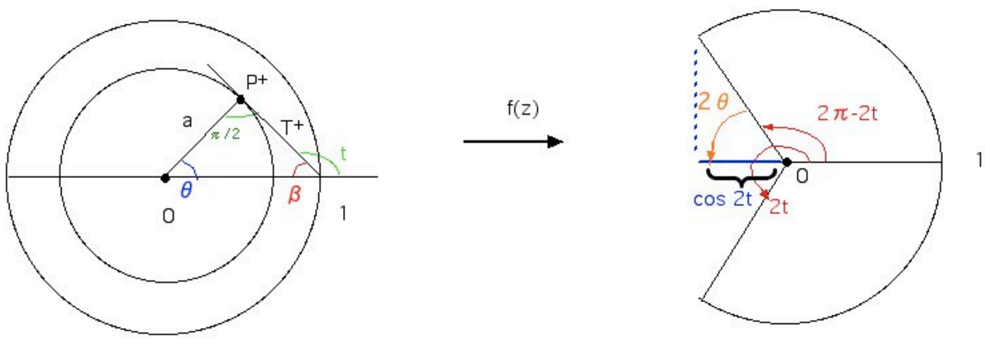

FIGURE 1. The domain of variation of t, t close to $\pi/2$.

Proof. — Let $z = 1 + \rho e^{it}$, $0 \leq t \leq 2\pi$. Then $f(z) = e^{-2it}$. Hence $\operatorname{Re} f(z) = \cos(2t)$ and $\operatorname{Im} f(z) = -\sin(2t)$. Let $T^{\pm}$ be the two tangents to the circle $|z| = a$ passing through the point 1. The intersection points of $T^{\pm}$ with the circle are given by

$$P_a^{\pm} = ae^{\pm i\theta} \tag{1.1}$$

for some $\theta \in [0, \pi/2]$. Consider the triangle Δ whose end-points are $0, 1$ and P_a^+ and let β be the angle formed by the segment $[0, 1]$ and the tangent T^+. Using that $\theta + \beta = \pi/2$, there exists $\rho > 0$ with $|1 + \rho e^{it}| = a$ if and only if $\pi - \beta \leq t \leq \pi + \beta$. (If $t \neq \pi \pm \beta$, then there are exactly two such radii ρ). The side-lengths of Δ are 1 (the hypotenuse), a and $L := |ae^{i\theta} - 1|$. Since $L^2 + a^2 = 1$, we see that $L = \sqrt{1 - a^2}$. On the other hand,

$$L^2 = a^2 + 1 - 2a\cos\theta.$$

Hence $a = \cos\theta$. Now let $t_{\max} := \pi - \beta$. Note that $t_{\max}$ is close to π if a is close to 0 and $t_{\max}$ is close to $\pi/2$ if a is close to 1.

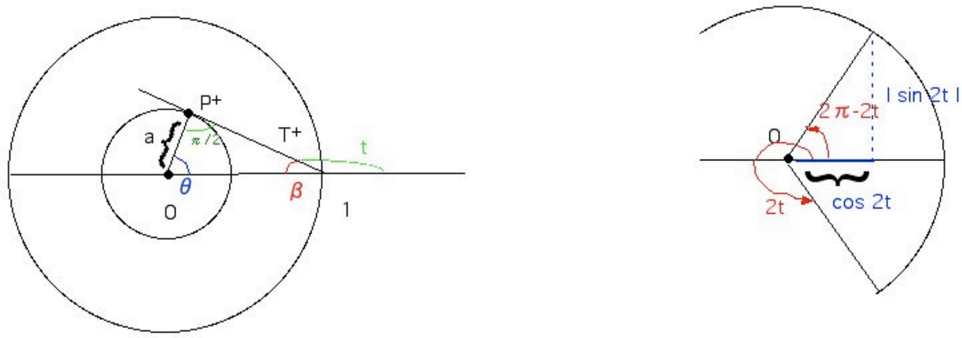

FIGURE 2. The domain of variation of t, t close to π

Since $t_{\max} = \theta + \pi/2$, we obtain

$$\cos(2t_{\max}) = \cos(2\theta + \pi) = -\cos(2\theta) = 1 - 2\cos^2(\theta) = 1 - 2a^2.$$

Thus $\min_{|z|=a} \mathrm{Re}\ f(z) = 1 - 2a^2$. The other identity $\max_{|z|=a} \mathrm{Re}\ f(z) = 1$ is clear by looking at the figure; it also follows from the fact that for $z = a$, $f(z) = 1$.

Now $\cos(2t_{\max}) = 0$ if $t_{\max} = 3\pi/4$. Hence

$$\max_{|z|=a} \mathrm{Im}\ f(z) = 1 \iff 1 - 2a^2 \leq 0 \iff \frac{1}{\sqrt{2}} \leq a < 1,$$

and

$$\max_{|z|=a} \mathrm{Im}\ f(z) = \sqrt{1 - (1 - 2a^2)^2} = 2a\sqrt{1 - a^2} \iff 0 < a \leq \frac{1}{\sqrt{2}}.$$

Finally, for all $a \in]0, 1[$,

$$\min_{|z|=a} \mathrm{Im}\ f(z) = -\max_{|z|=a} \mathrm{Im}\ f(z). \qquad \square$$

We can also use cartesian coordinates to find these extremal values: in fact, let $z = x + iy$, $|z| = a$. Then

$$\begin{aligned} \mathrm{Re}\ \tfrac{\overline{z}-1}{z-1} &= \mathrm{Re}\ \tfrac{(\overline{z}-1)^2}{|z-1|^2} = \tfrac{(x-1)^2-y^2}{x^2+y^2+1-2x} \\ &= \tfrac{x^2-2x+1-(a^2-x^2)}{a^2+1-2x} = 1 + \tfrac{2x^2-2a^2}{a^2+1-2x} \end{aligned}$$

Now

$$\left(\frac{x^2 - a^2}{a^2 + 1 - 2x}\right)' = \frac{2(x-1)(a^2-x)}{(a^2+1-2x)^2}$$

The zeros of this derivative are $x = 1$ and $x = a^2$. Since $-a \le x \le a$, we deduce that

$$\min_{|z|=a} \operatorname{Re} \frac{\overline{z}-1}{z-1} = 1 + \frac{2x^2 - 2a^2}{a^2+1-2x}\Big|_{x=a^2} = 1 - 2a^2$$

and

$$\max_{|z|=a} \operatorname{Re} \frac{\overline{z}-1}{z-1} = 1 + \frac{2x^2 - 2a^2}{a^2+1-2x}\Big|_{x=\pm a} = 1.$$

As a consequence, the cartesian coordinates of $P_a \pm$ are $(a^2, \pm a\sqrt{1-a^2})$.

Corollary 1.2. — *Let $0 < a < 1$. The image of the circle $|z| = a$ under the map*

$$f(z) = \frac{\overline{z}-1}{z-1}$$

is the arc

$$A := \{e^{i\sigma} : |\sigma| \le \pi - 2\arccos a\},$$

where $\arccos a \in \,]0, \pi/2[$.

Remark. — We also note that if τ runs from 0 to 2π, then $f(ae^{i\tau})$ runs on A from 1 to the upper end-point

$$E^+ := e^{i(\pi - 2\arccos a)} = 1 - 2a^2 + ia\sqrt{1-a^2}$$

of A, reaches this point when $\tau = \arccos a$ (that is $f(P_a^+) = E^+$)), then turns back, passes through the point 1 (when $\tau = \pi$) until it reaches the lower end-point

$$E^- := e^{-i(\pi - 2\arccos a)} = 1 - 2a^2 - ia\sqrt{1-a^2}$$

of A when $\tau = 2\pi - \arccos a$ (that is $f(P_a^-) = E^-$)), then turns back again up to the point 1, that is attained for $\tau = 2\pi$. In particular, with the exception of the two end-points of A, each point of A is traversed twice.

2. The map $g(z) = |z| f(z)$

Theorem 2.1. — *Let the map $g : \mathbb{D} \to \mathbb{C}$ be defined by*

$$g(z) = |z| \, \frac{\overline{z}-1}{z-1}.$$

Then g is a continuous map of $\mathbb{D}$ onto the set

$$\Omega = \mathbb{D} \setminus K^\circ,$$

where K is a closed region whose boundary is given by the curve

$$\gamma(a) = a(1-2a^2) \pm 2i \, a^2\sqrt{1-a^2}, \quad 0 \le a \le 1,$$

which is one half of the rhodonea (rose)

$$r = \sin(\theta/2),\ 0 \leq \theta \leq 2\pi.$$

Moreover, g is a homeomorphism of

$$H := \{z \in \mathbb{D} : |z - 0.5| > 0.5\} \text{ onto } \mathbb{D} \setminus K$$

and a homeomorphism of

$$\{z \in \mathbb{D} : |z - 0.5| < 0.5\} \text{ onto } \mathbb{D} \setminus K.$$

Let $C = \{z \in \mathbb{D} : |z - 0.5| = 0.5\}$. Then the function $g|_C$ has an injective continuous extension to the whole circle $\overline{C}$. The image of this extension coincides with ∂K (see figures 3 and 4).

Finally, for $|z| = 1$, $z \neq 1$, $g(z) = -\overline{z}$; thus g interchanges two points on the unit circle whenever they have same imaginary part.

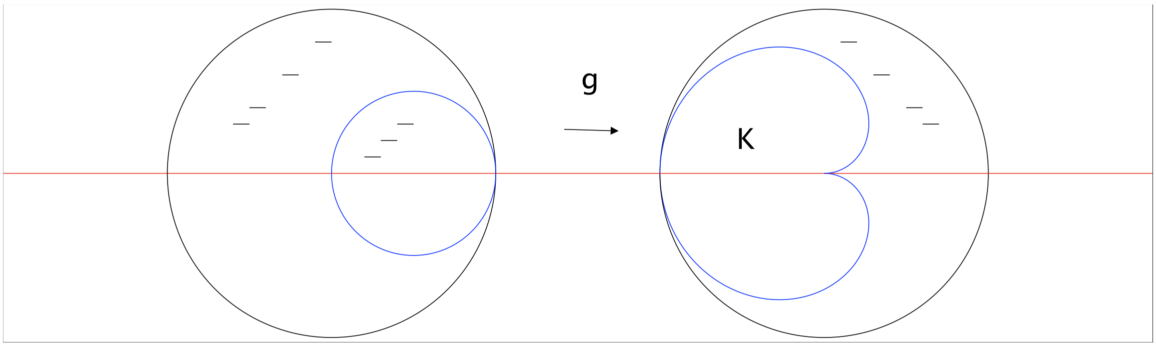

FIGURE 3. The mapping properties of g

Proof. — The first assertion on the image follows at once when we have noticed that by Lemma 1.1 and Corollary 1.2 the end-points of the image curve of $|z| = a$ under the map $(\overline{z} - 1)/(z - 1)$ are given by

$$1 - 2a^2 \pm i\sqrt{1 - (1 - 2a^2)^2} = 1 - 2a^2 \pm i\ 2a\sqrt{1 - a^2}$$

(see figure 4). Note also that the boundary of $g(\mathbb{D})$ is given by the set

$$\partial\mathbb{D} \cup R,$$

where R is parametrized by the curve

$$\gamma(a) = a(1 - 2a^2) \pm 2i\ a^2\sqrt{1 - a^2}, \quad 0 \leq a \leq 1.$$

Hence $g(\mathbb{D}) = \Omega$.

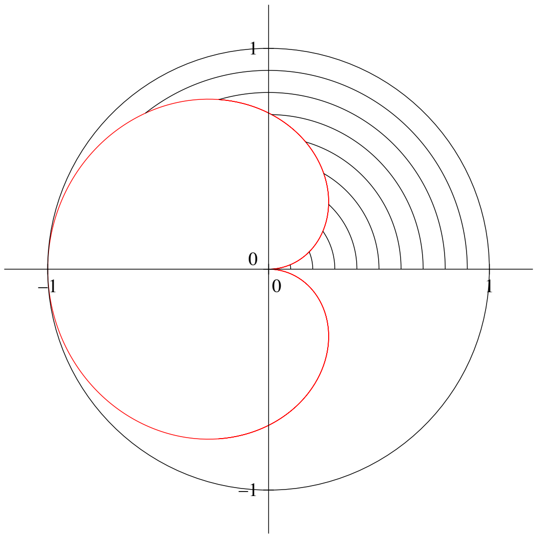

FIGURE 4. Creation of the image domain Ω

The locus of the points $P_a = ae^{i\arccos a}$, where $0 \le a \le 1$, equals the circle of center $1/2$ and radius $1/2$, because

$$\left|\frac{1}{2} - ae^{i\arccos a}\right| = \left|\frac{1}{2} - a\cos(\arccos a) - ia\sin(\arccos a)\right|$$

$$= \left|(\frac{1}{2} - a^2) - ia(\sqrt{1-a^2})\right| = \sqrt{(\frac{1}{2} - a^2)^2 + a^2(1-a^2)} = \frac{1}{2}.$$

By Corollary 1.2 and its remark,

$$g(ae^{i\arccos a}) = ae^{i(\pi - 2\arccos a)} = \gamma(a),\ a \ne 1.$$

Thus $g(C) = \partial K$. Moreover the open disk $|z-1/2| < 1/2$ is mapped bijectively onto Ω; the same holds for the set $\{z \in \mathbb{D} : |z - 1/2| > 1/2\}$.

It remains to show that $\gamma(a)$ coincides with (one part) of the rhodonea $r = \sin(\varphi/2)$, also called Dürer's folium, $0 \le \varphi \le 2\pi$.

So let $\gamma(a) = ae^{i\varphi}$, $0 \le \varphi \le 2\pi$. Note that $\gamma(a) = g(P_a^{\pm})$. Since $\cos\varphi = 1 - 2a^2$, we deduce that, in polar coordinates,

$$r(\varphi) = a = \sqrt{\frac{1}{2}(1-\cos\varphi)} = \sin\left(\frac{\varphi}{2}\right).$$

$\square$

At first glance (by looking at the picture), K seems to be a cardioid. This is not the case, though. The relation of K with the domain bounded by the classical cardioid, given by the parametrization

$$z(t) = -\frac{1}{2}(\cos\phi + 1)\cos\phi + i\frac{1}{2}(\cos\phi + 1)\sin\phi, \quad 0 \leq \phi \leq 2\pi$$

or in polar coordinates

$$r(\varphi) = \frac{1}{2}(1 - \cos\varphi)$$

is shown in the following figure (the cardioid is inside the domain K bounded by the "left part" of the rhodonea; the full rhodonea, called Dürer's folium, is given in the right picture.

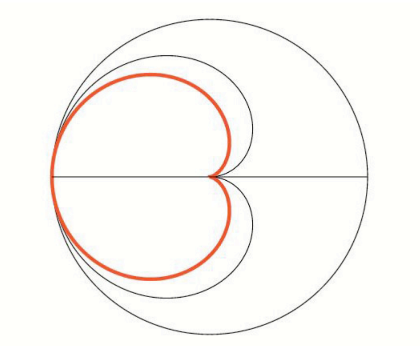

FIGURE 5.
Cardioid, rhodonea and unit cirle

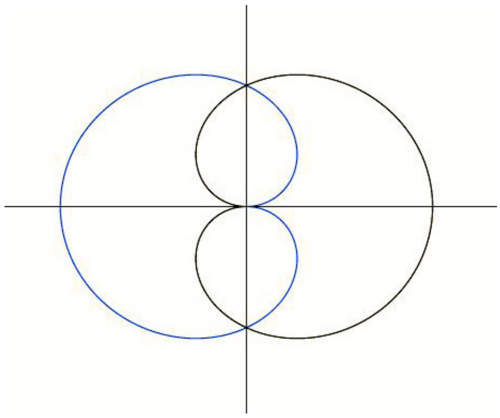

FIGURE 6. Dürer's folium

3. The map $h(z) = -z\frac{\overline{z}-1}{z-1}$

If one replaces in the definition of

$$g(z) = |z|\,\frac{\overline{z} - 1}{z - 1},$$

the factor $|z|$ by $-z$, then the new function has a very different behaviour. Part of the following result is from my previous joint work with P. Gorkin [**4**]. For the readers convenience, we recapture its short proof here. Recall that the cluster set, $C(u, \alpha)$, of a continuous function $u : \mathbb{D} \to \mathbb{C}$ at the point $\alpha \in \mathbb{T}$ is the set of all values $w \in \hat{\mathbb{C}}$ such there exists a sequence (z_n) in $\mathbb{D}$ for which $u(z_n) \to w$ as $n \to \infty$.

Proposition 3.1. — *Let* $h : \mathbb{D} \to \mathbb{D}$ *be given by*

$$h(z) = -z\frac{\overline{z} - 1}{z - 1}.$$

Then h *is a bijective involution (that is* $h \circ h = \text{id}$*) of* $\mathbb{D}$ *onto* $\mathbb{D}$*. The map* h *has a continuous extension to* $\overline{\mathbb{D}} \setminus \{0\}$ *with constant value* 1*. The cluster set* $C(h, 1)$ *of* h *at* 1 *equals the unit circle* $\mathbb{T}$*.*

Proof. — The first assertion follows from the fact that $h(z) = a$ implies $|z| = |a|$ and the following equivalences:

$$-z\frac{\overline{z} - 1}{z - 1} = a \iff -z + |z|^2 - a + az = 0 \iff$$

$$-z + |a|^2 - a + az = 0 \iff z = -a\frac{\overline{a} - 1}{a - 1}.$$

If $|z| = 1$, $z \neq 1$, then $-z\dfrac{\overline{z} - 1}{z - 1} = \dfrac{-1 + z}{z - 1} = 1$. Thus we may define $h(\lambda) = 1$ whenever $|\lambda| = 1, \lambda \neq 1$.

Since the cluster set of h at 1 is a decreasing intersection of continua, namely,

$$C(h, 1) = \bigcap_{n=1}^{\infty} \overline{h(D_n)}^{\mathbb{C}},$$

where $D_n = \{z \in \mathbb{D} : |z - 1| \leq 1/n\}$, we see that $C(h, 1)$ is a nonvoid connected compact set. Now $\lim\limits_{\substack{x \to 1 \\ 0 < x < 1}} h(x) = -1$ and $\lim\limits_{\theta \to 0} h(e^{i\theta}) = 1$.

Since $\mu \in C(h, 1)$ if and only if $\overline{\mu} \in C(h, 1)$ (note that $h(\overline{z}) = \overline{h(z)}$), and $|h(z)| = |z| \to 1$ if $z \to 1$, we conclude that $C(h, 1) = \mathbb{T}$. □

We note that a continuous involution F of $\mathbb{D}$ onto $\mathbb{D}$ is an open map. Therefore, F cannot have a continuous extension to $\mathbb{T}$ that is constant there. In fact, if this would be the case, say $F \equiv 1$ on $\mathbb{T}$, then we choose a sequence $w_n \in F(\mathbb{D})$ converging to a boundary point, β, of $F(\mathbb{D})$ different from 1. Let $z_n \in \mathbb{D}$ satisfy $F(z_n) = w_n$ for all n. We may assume, by passing to a subsequence if necessary, that (z_n) converges to $a \in \overline{\mathbb{D}}$. Since we have assumed that F has a continuous extension to $\overline{\mathbb{D}}$, we conclude that $F(a) = \beta$. Because $\beta \neq 1$, the constancy of F on $\mathbb{T}$ implies that $a \in \mathbb{D}$. But this contradicts the fact that F is an open map.

Acknowledgements

I thank the referee for drawing my attention to the class of curves, called roses (rhodonea), and Jérôme Noël for having realized figure 4 with TEXgraph.

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