Radial limits and boundary uniqueness

Arthur A. Danielyan

August 8, 2016

Abstract

The paper sheds a new light on the fundamental theorems of complex analysis due to P. Fatou, F. and M. Riesz, N. N. Lusin, I. I. Privalov, and A. Beurling. Only classical tools available at the times of Fatou are used. The proofs are very simple and in some cases - almost trivial.

1 Introduction and the main result.

Let U and T be the open unit disk and the unit circle in $\mathbb{C}$, recpetively. For a function f defined on U we denote by $f(e^{i\theta})$ the radial limit of f at $e^{i\theta}$ if the limit exists. The following classical theorems of P. Fatou [5] (cf. [4], Theorem 2.1) and of F. and M. Riesz [10] (cf. [4], Theorem 2.5) are among the most fundamental results of complex analysis.

Theorem A (P. Fatou, 1906). *Let f be analytic and bounded on U. Then for almost all $e^{i\theta}$ on T the radial limit $f(e^{i\theta})$ exists.*

Theorem B (F. and M. Riesz, 1918). *Let f be analytic and bounded on U such that $f(e^{i\theta}) = 0$ on a set E of positive measure on T. Then f is identically zero.*

For univalent functions the analogous results are due to A. Beurling [1] (cf. [4], Theorem 3.5).

Theorem C (A. Beurling, 1940). *Let f be univalent on U. Then: (i) at every point $e^{i\theta}$ of T, except possibly a set of zero (logarithmic) capacity, the radial limit $f(e^{i\theta})$ exists; (ii) $f(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$ cannot be zero on any positive capacity set on T.*

Theorem A presents the property of almost everywhere existence of the radial limits of bounded analytic functions, while Theorem B is the boundary uniqueness property of the

same functions. Both these properties (theorems) have been presented and extended in many books and countless papers. Under the lights of the influential original works of Fatou and F. and M. Riesz, quite naturally, Theorem A and Theorem B have been universally regarded as two fundamental, but different properties of analytic functions[1]. Undoubtedly the same persuasion has been the reason that the parts (i) and (ii) of Theorem C have been regarded as different properties; obviously Beurling did not even suspect that one of the parts of Theorem C might be a simple corollary of the other (for the proof of each part of Theorem C a special involved technic has been developed).

The present paper sheds a new light on the theory and in fact changes the mentioned viewpoint that Theorem A and Theorem B (or, part (i) and part (ii) of Theorem C) are presenting different properties. Instead, as we show, *the boundary uniqueness property is a direct corollary, or a simple particular case, of the property of the a.e. existence of the radial limits.* In fact, except for trivial cases, *the radial limits cannot be constant on a set E just because the radial limits exist on a large subset of E (or of T),* as the following main results (Theorem 1 and Corollary 1) of the paper show.

Theorem 1. *Let f be univalent (respectively, zero free, bounded analytic) on U such that $f(e^{i\theta}) = 0$ on a subset E of T. Then f generates a univalent (respectively, bounded analytic) function g on U such that g has no radial limit on E.*

Remark 1. If f in Theorem 1 is zero free, bounded analytic, the relation between f and g is especially simple and given explicitly by the equation $g(z) = e^{-i \log \log f(z)}$, as our below proof implies.

Even in a trivial case of measure zero E, as we will see, Theorem 1 implies a theorem of Lusin [7]. In contrast, the following corollary (of Theorem 1), in which f is not required to

---

[1]For instance the paper [3] by L. Carleson, emphasizing the difference between Theorem A and Theorem B, begins with the following sentences: "For a large number of classes C of functions $f(z)$ regular in the unit circle, we have very complete knowledge concerning the existence of a boundary function

$$F(\theta) = \lim_{r \to 1} f(re^{i\theta}),$$

the classical result being that of Fatou. However, very little is known about the properties of this boundary function $F(\theta)$, and in particular about the sets E associated with the class C, having the property that $f(z)$ vanishes identically if $F(\theta) = 0$ on E. ... Our whole knowledge in this direction seems to be contained in a classical result of F. and M. Riesz: E is a set of uniqueness for the class of bounded functions if and only if it has positive Lebesgue measure."

be zero free on U, loses its meaning for E of Lebesgue measure zero.

Corollary 1. *Let f be non-constant, bounded analytic on U such that $f(e^{i\theta}) = 0$ on a subset E of T. Then f generates a bounded analytic function g on U such that g has no radial limit on E except perhaps a subset of E of Lebesgue measure zero.*

The hypothesis of Theorem 1 automatically implies that f is non-constant. But in Corollary 1 one needs to exclude the (trivial) case of the constant function.

Obviously Theorem 1 implies that the part (ii) of Theorem C is a corollary of part (i) of Theorem C (nevertheless, the part (ii) itself is an important and frequently cited result). Similarly, Corollary 1 derives Theorem B from Theorem A.

Note that Theorem 1 (and Corollary 1) by implying both Theorem B and part (ii) of Theorem C, is giving the first unified proof of the boundary uniqueness property for univalent and merely analytic functions. Our approach greatly simplifies the case of univalent functions and the new proof of part (ii) of Theorem C below should be compared with its previous proof; cf. the proof of part (ii) of Theorem C in [4], pp. 61 - 64, or in [1].

In this paper we merely use classical theorems (of times of Fatou) and modulo to them the proof of Theorem 1 is simple and elementary.

Theorem 1 immediately implies also *the first elementary and very short proof* of the following theorem (see [7] or [8]), which is a classical converse of Theorem A.

Theorem D (N.N. Lusin, 1919). *Let E be a zero measure subset on T. Then there exists a bounded analytic function f on U such that the radial limit $f(e^{i\theta})$ does not exist at each $e^{i\theta} \in E$.*

We use the following classical result due to Privalov [12] (cf. [13], p. 295, or [14], p. 276).

Theorem E (I.I. Privalov, 1919). *Let E be a zero measure subset on T. Then there exists a zero free, bounded analytic, function f on U such that $f(z)$ tends to 0 as z approaches, in an arbitrary manner (in particlar, radially), any point of E.*

Since the function f existing by this theorem is zero free, bounded analytic, and $f(e^{i\theta}) = 0$ on E, it can serve as a function of hypotheses of Theorem 1, and thus, Theorem 1 readily implies Theorem D. Under the light of our simple (below) proof of Theorem 1, Theorem D

becomes nothing else but *an obvious corollary* of Theorem E. Since also Privalov's proof of Theorem E is a simple construction, we arrive to the first elementary self-contained proof of Theorem D, which is in a sharp contrast to its (very complex) original proof.

Lusin and Privalov have been collaborating for many years in topics involving their Theorem D and Theorem E, and Theorem D even appears again in their well known joint paper [8] (essentially with same original proof of 1919). However they did not notice that Theorem D immediately follows from the elementary Theorem E. (As a side effect of the present research, now one can finally include Theorem D with its new proof in the textbooks, and present it, along with Theorem A, in complex analysis graduate courses at the universities[2]).

In this paper we also use two classical results due to C. Carathéodory (1913) and F. Riesz (1923), respectively.

1.1   The theorems of Carathéodory and F. Riesz.

The prime end theorem of Carathéodory [2] (see for example, [9], p. 30) in fact is the first major result on the boundary behavior of univalent functions. It is on the extension of the Riemann mapping function to the boundary of a domain and has an especially simple formulation when the unit disc is mapped onto a Jordan domain. The general case uses the Carathéodory concept of a prime end of a simply connected domain G. We assume that the reader is familiar with it as well as with the concept of a null-chain representing the prime end (see for example [9], pp. 29-30). However we will use just a special case of Carathéodory's main theorem, which, as we will see, also can be derived from the Riemann mapping theorem avoiding the prime ends (and null-chains) altogether.

Theorem F (C. Carathéodory, 1913). *Let φ map the unit disc U conformally onto a simply connected domain G. Then there is a bijective mapping $\hat{\varphi}$ of the circle T onto the set of all prime ends of G such that, if $\zeta \in T$ and if $\{C_n\}$ is a null-chain representing the prime end $\hat{\varphi}(\zeta)$, then $\{f^{-1}(C_n)\}$ is a null-chain that separates 0 from ζ for large n.*

If a Jordan arc in U ends at a point $\zeta \in T$, then the image arc in G "approaches" (in the

---

[2]Theorem A is proved in the standard textbooks on complex analysis, but, as a rule, none of them (say, W. Rudin's comprehensive "Real and Complex Analysis") even mentions Theorem D. This leaves the reader wondering whether the conclusion of Theorem A is precise or not. An affirmative answer is provided by Theorem D and its presence in the textbooks seems highly desirable.

sense of prime ends) to the prime end $\hat{\varphi}(\zeta)$, and vice versa.

Let D be a simply connected "double comb" domain in the w-plane obtained from the square

$$\{w = u + iv : 0 < u < 1, \ 0 < v < 1\}$$

by taking off the line segments $l_{2n} = \{u + iv : u = \frac{1}{2n}, \ 0 \le v \le \frac{3}{4}\}$ and $l_{2n+1} = \{u + iv : u = \frac{1}{2n+1}, \ \frac{1}{4} \le v \le 1\}$ for all values of n $(n = 1, 2, ...)$. Denote by AB the closed set $\{iv : 0 \le v \le 1\}$ (the left side of the original square). It contains no accessible boundary points of D. In other words, there is no Jordan arc in D ending at a point of AB (to approach to AB, a Jordan arc has to "oscillate"). Note that AB determines one prime end (of D), which we denote by P; more precisely, AB is the impression of P. It is the only prime end of D with an impression containing more than one point.

We have the following corollary of Theorem F.

Proposition 1. *Let φ map U conformally onto D. Then there exists a point ξ on T such that φ has no limit as z approaches ξ along any Jordan arc γ, $\gamma \setminus \{\xi\} \subset U$, ending at ξ.*

The existence of such ξ immediately follows from Theorem F; simply take as ξ the point, which corresponds to the above mentioned prime end P of D.

Let Γ be a halfopen Jordan arc in D, oscillating and approaching to AB asymptotically. For instance, as such Γ, one can take the polygonal in D joining the sequence of the points $M_1(\frac{1}{2}, \frac{7}{8})$, $M_2(\frac{1}{3}, \frac{1}{8})$, $M_3(\frac{1}{4}, \frac{7}{8})$, $M_4(\frac{1}{5}, \frac{1}{8})$,.... We may assume that Γ is given by an equation $w = w(t)$, $0 \le t < 1$, where $w(t)$ is continuous on $[0, 1)$ and $w(0) \equiv M_1(\frac{1}{2}, \frac{7}{8})$ is the initial point of Γ. Let Γ_1 be a Jordan arc in D having the same initial point $w(0)$ as Γ and ending at an accessible boundary point w_1 of D $(\Gamma_1 \setminus \{w_1\} \subset D)$, and such that $w(0)$ is the only common point of Γ and Γ_1.

The following proposition is obvious.

Proposition 2. *The set $\Gamma \cup \Gamma_1$ divides the domain D into two domains D_1 and D_2 such that the boundaries of both D_1 and D_2 contain either all segments l_{2n} or all segments l_{2n+1} except finitely many of such segments.*

Morera's theorem and the elementary (inner) uniqueness theorem immediately imply:

Proposition 3. *If f is continuous in a domain Ω and analytic in $\Omega \setminus L$ where L is a line segment, then f is analytic in Ω. If in addition $f(z) = c$ on L, then f is identically c on Ω.*

For Proposition 1 we now present a simple, direct proof, which avoids Theorem F and prime ends altogether, and only uses the Riemann mapping theorem and Theorem A.

Let $w = \varphi(z)$ be a conformal map of U onto D (as in Proposition 1). Denote by $z = \psi(w)$ the inverse of $w = \varphi(z)$. The image $\psi(\Gamma)$ of Γ is a halfopen Jordan arc in U given by the equation $w = \psi(w(t))$, $0 \le t < 1$. Note that $\psi(\Gamma)$ ends at a point $\xi \in T$, because otherwise $\psi(\Gamma)$ has to have two accumulating points a and b on T, and the function φ cannot have radial limits on one of the two complementary to a and b open arcs of T, which contradicts to Theorem A. Now we show that $\xi \in T$ has the property formulated in Proposition 1.

Let γ, $\gamma \setminus \{\xi\} \subset U$, be an arbitrary Jordan arc ending at ξ. If γ and $\psi(\Gamma)$ share points (other than ξ) at each neighborhood of ξ, then there is nothing to prove (because as Γ, the curve $\varphi(\gamma)$ too would be oscillating and approaching to AB in D). Thus, by deleting some initial portion of γ if necessary, we may assume that γ and $\psi(\Gamma)$ have no common point other than ξ. Let us join the initial points of γ and $\psi(\Gamma)$ by an arc $\delta \subset U$ such that δ has no other common point with γ or with $\psi(\Gamma)$. The curve $\psi(\Gamma) \cup \delta \cup \gamma$ divides U into two domains. One of them, let denote it by U_1, has only one point of T, namely ξ, on its boundary.

Assume by contrary that $\varphi(z)$ has a limit equal to q as z approaches ξ along γ. This means that q is an accessible boundary point of D (and the Jordan arc $\varphi(\gamma)$ ends at q). The Jordan arc $\varphi(\delta)$ joins in D the initial points of Γ and $\varphi(\gamma)$, and $\varphi(\delta)$ has no other common point with them. Let us take $\varphi(\gamma) \cup \varphi(\delta)$ as Γ_1 and apply Proposition 2; we conclude that there exists either a segment l_{2n+1} or a segment l_{2n} lying on the boundary of the image $\varphi(U_1)$ of U_1. For briefness, denote this segment by l.

Because ξ is the only boundary point of U_1 belonging to T, the conformal mapping $\psi(w)$ of $\varphi(U_1)$ onto U_1 will be continuously extended to the set l once we put $\psi(w) = \xi$ on l. Now Proposition 3 implies that $\psi(w)$ is identically equal to ξ, which is impossible since ψ is a univalent function. This contradiction completes the proof of Proposition 1.

Next, we formulate a classical result on the radial limits of Blaschke products. By Theorem A, of course, the radial limits of a Blaschke product exist a.e. on T. In 1923 F. Riesz [11] (cf. [4], Theorem 2.11) proved the following result.

6

Theorem G (F. Riesz, 1923). *A Blaschke product B possesses radial limits of modulus 1 for almost all $e^{i\theta}$ on T*

An elementary proof of this theorem can be found in K. Hoffman's book[3] (see [6], page bottom of 65 - top of 66). Without going into details, we describe the main steps of this proof. First, some simple integral estimates imply that for a (convergent) Blaschke product B the sequence of partial products B_n converge to B in H^2 on the unit circle T. In particular, B_n converges to B in L^2 norm, and therefore, a subsequence of B_n converges pointwise a. e. on T to B. Since each $|B_n|$ is identically 1 on T, $|B|$ is 1 a. e. on T, as Theorem G claims.

We close this subsection with some remarks on our proof of Theorem 1. First of all we show that Proposition 1 easily implies Theorem 1. But, in fact, Proposition 1 is needed for Theorem 1 only for the case of univalent functions; for the case of just analytic functions we present yet another proof which instead of Proposition 1 merely uses an elementary argument. Also, Theorem G immediately reduces Corollary 1 to Theorem 1, and this is the only occasion when we use Theorem G in our proofs.

Thus, for the case of analytic functions, Theorem 1 is independent of Proposition 1, and, in fact, Theorem 1 becomes almost trivial in this case (similarly, since f provided by Theorem E is zero free, Theorem D is a trivial corollary of Theorem E, as mentioned above). For the case of univalent functions Theorem 1 follows almost trivially from Proposition 1. And, Corollary 1 is almost trivial if we assume that Theorem G is granted.

Thus in all cases the property of the existence of the radial limits almost trivially implies the boundary uniqueness property (assuming at most Theorem G and Proposition 1 granted). (Based on below proofs, we hope it does make sense to use here the wording "almost trivially", but the final judgement on this is reserved for the reader.) Let us stress again that Proposition 1, for which we presented also a direct simple proof, is an obvious corollary of Carathéodory's Theorem F of 1913, while Theorem G of F. Riesz is of 1923 and has an elementary proof presented in [6] as we noted above.

---

[3]I am indebt to Don Marshall for calling my attention to this proof.

2 Proofs.

In the formulation of Theorem 1 in general case the univalent function f is not required to be bounded. However, as it is well known, one can reduce this to the case of bounded f using some elementary mappings (cf. [4], p. 57). Therefore, in the following proof with no loss of generality we assume that f is bounded by M (on U) also for the case of univalent f.

2.1 Proof of Theorem 1.

Let f be bounded, univalent (or analytic and zero free) on U, and let $f(e^{i\theta}) = 0$ on E. (Since $f(e^{i\theta}) = 0$ on E, of course, f is zero free on U also in case if f is univalent.) We may assume f is bounded by 1. Then $f(z) = e^{h(z)}$, where h is univalent (or analytic), $\Re h(z) < 0$ on U and $h(e^{i\theta}) = \infty$ on E. Let the univalent function φ and the point $\xi \in T$ be as in Proposition 1, and let $\psi(\zeta)$ be a fractional-linear mapping of the left half plane onto the unit disk under which ∞ corresponds to ξ. Then the function $g(z) = \varphi(\psi(h(z)))$ does not have radial limits on E. Next, g is bounded analytic, and if f is univalent, then with h also g is univalent. The proof is over.

2.2 Elementary proof of Theorem 1 for the case of zero free, bounded analytic f.

We may assume f is bounded by 1. Then $f(z) = e^{h(z)}$, where h is analytic, $\Re h(z) < 0$ on U and $h(e^{i\theta}) = \infty$ on E. We have an analytic $\log h(z) = \log |h(z)| + i \arg h(z)$ on U with $\pi/2 < \arg h(z) < 3\pi/2$. Then $g(z) = e^{-i \log h(z)} = e^{\arg h(z)}(\cos \log |h(z)| - i \sin \log |h(z)|)$ is analytic and bounded by $e^{3\pi/2}$. On each radius ending on E the oscillation of g exceeds $e^{\pi/2}$. The proof is over.

2.3 Elementary proof of Lusin's Theorem D.

For a given zero measure set $E \subset T$ let f be the function existing by Theorem E. Repeating the previous paragraph for this f proves Theorem D.

2.4 Proof of Corollary 1.

Let f be as in Corollary 1. Then $f(z) = B(z)f_1(z)$ where B is a Blashke product and f_1 is analytic, bounded, and zero free on U. By Theorem G we have $|B(e^{i\theta})| = 1$ a.e. on T, and thus $f_1(e^{i\theta}) = 0$ on some $E_1 \subset E$ such that $E \setminus E_1$ is of Lebesgue measure zero. By Theorem 1 for f_1 there exists a bounded analytic function g which does not have radial limits on E_1. The proof is over.

Remark 2. Since f_1 is zero free, $g(z) = e^{-i \log \log f_1(z)}$ as in Remark 1, and thus we have $g(z) = e^{-i \log \log \frac{f(z)}{B(z)}}$ as an explicit equation which connects f and g of Corollary 1.

Acknowledgment. The author thanks Tom Carroll, Don Marshall, and Larry Zalcman for very helpful discussions of a preliminary version of this paper.

References

[1] A. Beurling, Ensembles exceptionnels, Acta Math., V. 72, 1-13, 1940.

[2] C. Carathéodory, Über die Begrenzung einfach zusammenhängender Gebiete, Math. Ann., V. 73, 323-370, 1913.

[3] L. Carleson, Sets of uniqueness for functions regular in the unit circle, Acta Math., V. 87, 325-345, 1952.

[4] E.F. Collingwood and A.J. Lohwater, The Theory of Cluster sets, Cambridge University Press, 1966.

[5] P. Fatou, Séries trigonométriques et séries de Taylor, Acta Math., V. 30, 335-400, 1906.

[6] K. Hoffman, Banach Spaces of Analytic Functions, Prentice Hall, Englewood Cliffs, New Jersey, 1962.

[7] N.N. Lusin, Sur la représentation conforme, Bull. Ivanovo-Vozn. Politech. Inst., V. 2, 77-80, 1919.

[8] N. N. Lusin and I. I. Privalov, Sur l'unicité et la multiplicité des fonctions analytiques, Ann. Sci. École Norm. Sup., V. 42, 143-191, 1925.

[9] Ch. Pommerenke, Boundary Behavior of Conformal Maps, Springer-Verlag, Berlin, 1992.

[10] F. Riesz and M. Riesz, Über die Randwerte einer analytischen Funktionen, Quatrième Congrès des Math. Scand. Stockholm, 27-44, 1916.

[11] F. Riesz, Über die Randwerte einer analytischen Funktionen, Math. Zeit., V. 18, 87-95, 1923.

[12] I.I. Privalov, Intégrale de Chauchy (In Russian), Saratov, 1919.

[13] I.I. Privalov, Boundary Properties of Analytic Functions, Second ed., GITTL, Moscow-Leningrad, 1950 (in Russian).

[14] A. Zygmund, Trigonometric series, V. 1, Cambridge, 1959.

Arthur A. Danielyan
Department of Mathematics and
Statistics
University of South Florida
Tampa, Florida 33620
USA
e-mail: adaniely@usf.edu