A Note on the Paper of I. J. Schark

KENNETH HOFFMAN

Communicated by A. M. GLEASON

This note is really an addendum to the foregoing paper of I. J. SCHARK. We shall carry over the notation of that paper.

In SCHARK's paper there were mentioned three open questions. Is the open disc Δ dense in the space $\mathfrak{M}$ of maximal ideals of the algebra B of bounded analytic functions? Is $\mathfrak{M} - \Delta$ connected? Is each fiber $\mathfrak{M}_\alpha$ of $\mathfrak{M} - \Delta$ connected? The first question is very difficult, and we shall not deal with it. We should like to prove here that the answer to both the second and third questions is yes.

For the proofs, we shall utilize results from [1], as well as some of the results of I. J. SCHARK.

Theorem 1. *Each fiber $\mathfrak{M}_\alpha$ is connected.*

Proof. Let B_α be the restriction of the algebra $\hat{B}$ to the fiber $\mathfrak{M}_\alpha$. Then B_α is a uniformly closed algebra of continuous functions on $\mathfrak{M}_\alpha$, and the maximal ideal space of B_α is $\mathfrak{M}_\alpha$. This is proved in [1]. It is an easy consequence of the fact that the function

$$g = \tfrac{1}{2}(1 + \bar{\alpha}z)$$

satisfies $\hat{g} = 1$ on $\mathfrak{M}_\alpha$ and $|\hat{g}| < 1$ elsewhere on $\mathfrak{M}$.

Suppose $\mathfrak{M}_\alpha$ is not connected. Since $\mathfrak{M}_\alpha$ is the maximal ideal space of B_α, a theorem of ŠILOV [2] states that B_α must contain a non-trivial idempotent function. This means that there is an $f \in B$ such that on $\mathfrak{M}_\alpha$ the range of $\hat{f}$ consists exactly of the numbers 0 and 1. By the corollary in Section 4 of SCHARK's paper, if $\{\lambda_n\}$ is a sequence of points of the open disc which converges to α, the only cluster points of $\{f(\lambda_n)\}$ are 0 or 1. This means that f must map a sufficiently small neighborhood of α onto a disconnected set, and this is absurd.

Theorem 2. *Let $\mathfrak{a}$ be the uniformly closed algebra of continuous functions on $\mathfrak{M} - \Delta$ which is generated by $\hat{B}$ and the conjugate of $\hat{z}$. The maximal ideal space of $\mathfrak{a}$ is $\mathfrak{M} - \Delta$.*

Proof. This is also proved in [1]. The idea of the proof in this. The function $\hat{z}$ is constant on each fiber $\mathfrak{M}_\alpha$, and hence its conjugate is also. Thus the restriction of the algebra $\mathfrak{a}$ to any fiber $\mathfrak{M}_\alpha$ will be identical with the restriction of $\hat{B}$

Journal of Mathematics and Mechanics, Vol. 10, No. 5 (1961).

to that fiber. Using the functions $g = \frac{1}{2}(1 + \bar{\alpha}z)$, one can show that any homomorphism of B in $\mathcal{H} - \Delta$ is actually a homomorphism of the restriction of $\hat{B}$ to one of the fibers $\mathcal{H}_\alpha$. Thus B and $\mathfrak{a}$ have the same complex homomorphisms, except that the adjunction of $\bar{z}$ to B deletes the open disc Δ, $(z - \lambda)^{-1}$ being a limit of polynomials in z and $\bar{z}$ for $|\lambda| < 1$.

Theorem 3. *The space $\mathcal{H} - \Delta$ is connected.*

Proof. By Theorem 2, $\mathcal{H} - \Delta$ is the maximal ideal space of the algebra $\mathfrak{a}$. Suppose $\mathcal{H} - \Delta$ is not connected. Again using Silov's theorem [2], $\mathfrak{a}$ must, contain a non-trivial idempotent function e. Now for some α, the restriction of e to $\mathcal{H}_\alpha$ must be non-constant. For, if e is either constantly 0 or constantly 1 on each $\mathcal{H}_\alpha$, the sets

$$S_0 = \{\alpha \in C; e = 0 \quad \text{on} \quad \mathcal{H}_\alpha\}, \qquad S_1 = \{\alpha \in C; e = 1 \quad \text{on} \quad \mathcal{H}_\alpha\}$$

disconnect the unit circle C.

Thus there is an α such that e is non-constant on $\mathcal{H}_\alpha$. But, by Theorem 1, $\mathcal{H}_\alpha$ is connected and cannot support a non-trivial idempotent continuous function. This contradiction shows that $\mathcal{H} - \Delta$ is connected.

Bibliography

[1] Hoffman, K. & Singer, I. M., *Maximal Algebras of Continuous Functions*, Acta Mathematica, **103** (1960), pp. 217–241.

[2] Silov, G. E., *On the decomposition of a commutative normed ring into a direct sum of ideals*, Mat. Sbornik T. 32, **2** (1953), pp. 353–364 (Russian).

Massachusetts Institute of Technology
Cambridge, Massachusetts