

# Statistics 452: Statistical Learning and Prediction

## Chapter 9, Part 1: Introduction to Support Vector Machines

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# Overview

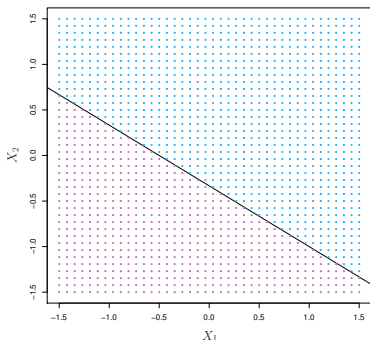
- ▶ The context is a binary classification problem.
  - ▶ We will discuss two classification rules that lead to the support vector machine.
1. Maximal margin classifier: If there exists a linear boundary in the space of explanatory variables that separates the classes in the training data, use this boundary as a classifier.
    - ▶ If one boundary exists, there will be many. Choose one that maximizes the “margin”, a minimum distance between predictors and the boundary.
    - ▶ Simple classifier, but requires that a linear boundary exist.
  2. Support vector classifier: Weaken the requirement of complete separation to a linear boundary that **best** separates the classes.
    - ▶ Better, but a linear boundary may not be the best choice.
- ▶ Support vector machine: Weaken the requirement of a linear boundary, and add a trick to prevent computations from becoming prohibitive.

## Linear Boundaries

- ▶ Linear boundaries can be represented as a “hyperplane”, which is a  $p - 1$ -dimensional subspace of  $p$  dimensions specified by a constraint

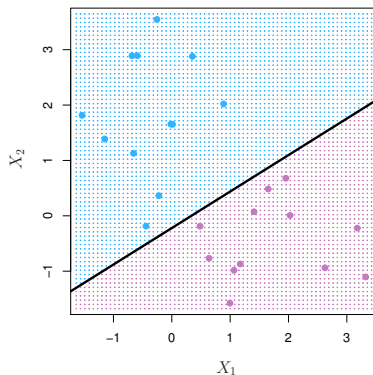
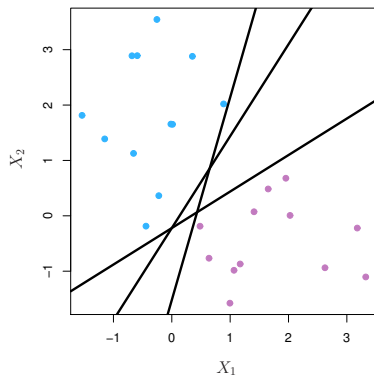
$$f(X) = \beta_0 + \beta_1 X_1 + \dots + \beta_p X_p = 0.$$

- ▶ Classify as blue class if  $f(X) > 0$  and red if  $f(X) < 0$ .



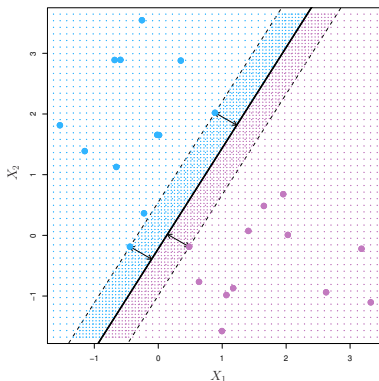
# Possible Hyperplanes

- ▶ When a separating hyperplane exists, there are many.
- ▶ Text, Figure 9.2:
  - ▶ Left panel, three separating hyperplanes
  - ▶ Right panel, one hyperplane and resulting decision rule



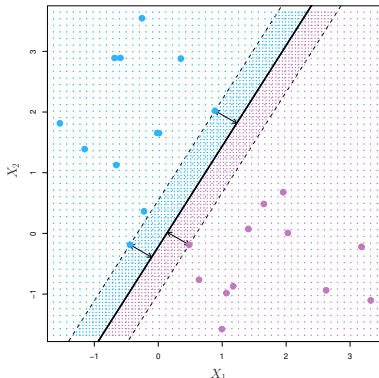
# Maximal Margin Classifier

- ▶ The maximal margin hyperplane is the separating hyperplane farthest from the training observations.
  - ▶ For each possible hyperplane, find the margin: The minimum of perpendicular distances between each point and the hyperplane.
  - ▶ Choose the hyperplane with the largest margin as the boundary (Fig. 9.3).



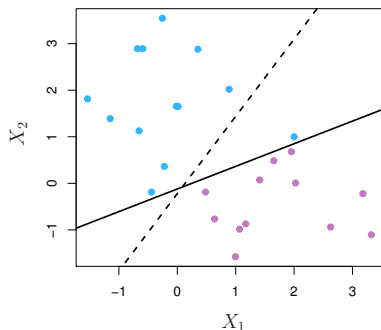
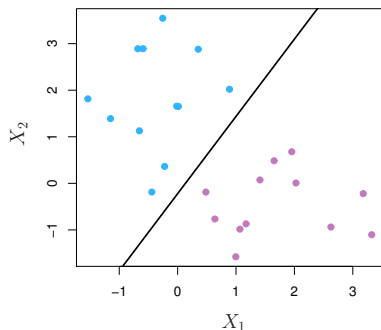
# Support Vectors

- ▶ The support vectors are the points ( $p$ -dimensional vectors) that determine the maximal margin hyperplane.
  - ▶ Support in the sense that if they move, so does the hyperplane.
  - ▶ Support vectors in the diagram are two blue points and one red with perpendicular distances indicated on the figure.
  - ▶ The classifier depends on the data through the support vectors **only**.



## Sensitivity to Support Vectors

- ▶ Classifier depends on data through the support vectors **only**.
- ▶ Adding/deleting a support vector can drastically change the maximal margin hyperplane.
- ▶ Example maximal margin hyperplane before (left panel) and after (right panel) adding a new support vector (text, Fig 9.5):



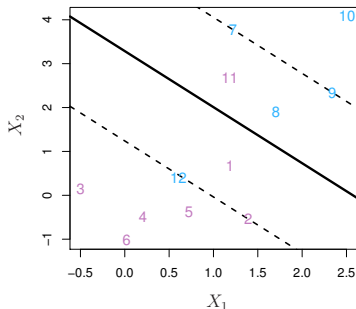
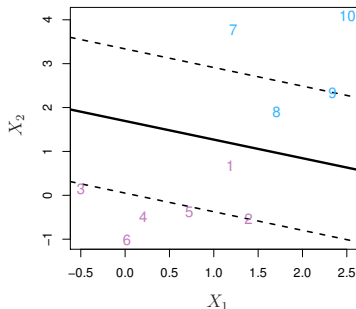
## A Budget of Margin Violations

- ▶ Notice that the margin in right panel of the previous example is very small.
- ▶ We might instead seek a larger margin, but allow some points to be on the wrong side.
  - ▶ Allow vector  $i$  to be a distance  $\epsilon_i$  on the wrong side of the margin, as long as  $\sum_{i=1}^n \epsilon_i \leq C$ , for some budget  $C$ .
  - ▶ By allowing points on the wrong side of the margin, and even wrong side of the hyperplane, we can accommodate the case where the vectors are not separable by a hyperplane.
- ▶ This is the support vector classifier.
  - ▶ The budget  $C$  is a tuning parameter.
- ▶ The support vectors are those on the margin or on the wrong side of the margin.
  - ▶ One can show that only these support vectors affect the choice of hyperplane and hence classification rule.



# Illustration of Support Vector Classifier

- ▶ In the left panel of the following Figure (Fig 9.6), one observation from each class is on the wrong side of the margin, but not the hyperplane.
- ▶ In the right panel, two more points are added that are also on the wrong side of the hyperplane.
- ▶ The support vectors are those on the margin or on the wrong side of the margin.



# Choosing the budget $C$

- ▶ Bias-variance tradeoff:
  - ▶ Small  $C$  means we require narrow margins, and are potentially over-fitting the data. Should have low bias but high variance
  - ▶ Large  $C$  means we allow wide margins and are not fitting the data as aggressively. Should have higher bias, but lower variance.
- ▶ Use cross-validation to select  $C$ .

## Example Support Vector Classifier for the Heart Data

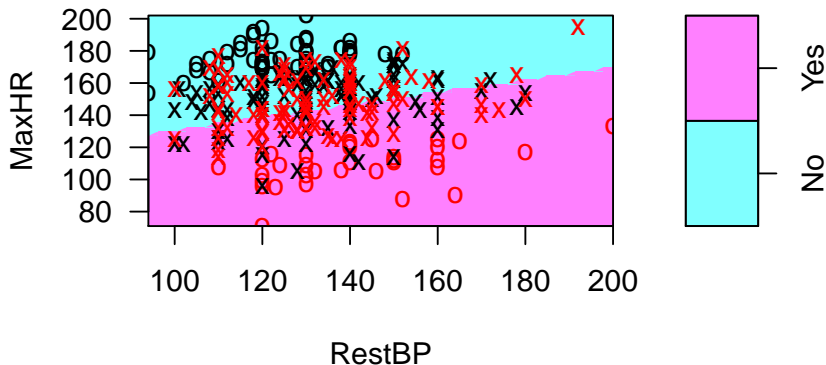
- ▶ Use the `svm()` function from the `e1071` package fits support vector machines.
  - ▶ Set argument `kernel=linear` for the support vector classifier.

```
uu <- url("http://www-bcf.usc.edu/~gareth/ISL/Heart.csv")
Heart <- read.csv(uu,row.names=1)
Heart <- na.omit(Heart)
library(dplyr)
Heart <- select(Heart,AHD,MaxHR,RestBP)
library(e1071)
svc.heart <- svm(AHD ~ .,type="C-classification",cost=1,
                 data=Heart,kernel="linear")
```

# Visualize the Classifier

```
plot(svc.heart,Heart)
```

## SVM classification plot



- (Zoom for better view) Classes are colour-coded; O means on right side and X means wrong side of margin.

## Choose the cost by CV

- The `tune()` function does the CV over a user-supplied grid of costs.

```
set.seed(123)
tune.heart <- tune(svm,AHD ~ .,data=Heart,kernel="linear",
                  ranges=list(cost=c(10^{-3:3})))
summary(tune.heart)$performances
```

```
##      cost      error dispersion
## 1 1e-03 0.4618391 0.08397328
## 2 1e-02 0.3237931 0.08230515
## 3 1e-01 0.3166667 0.06829012
## 4 1e+00 0.3100000 0.07484525
## 5 1e+01 0.3133333 0.08140333
## 6 1e+02 0.3166667 0.07994958
## 7 1e+03 0.3166667 0.07994958
```

- According to these CV results, we should use cost 1.

# Non-linear Decision Boundaries

- ▶ By expanding the features to include polynomial terms, the support vector classifier will have non-linear decision boundaries in the original feature space.
  - ▶ In the expanded feature space, say  $X_1, X_1^2, \dots, X_p, X_p^2$ , the boundary will be a curve of the form

$$f(X) = \beta_0 + \beta_1 X_1 + \beta_2 X_1^2 + \dots + \beta_{2p-1} X_p + \beta_p X_p^2 = 0.$$

- ▶ We could consider higher-order polynomials, but eventually we would have so many features that computation time would become a problem.
- ▶ The support vector machine uses a computational trick to allow non-linear decision boundaries without prohibitive computation.
  - ▶ To describe the trick, recast the computation for the support vector classifier in terms of “kernels”.

# Support Vector Classifier *via* Kernels

- ▶ It turns out that the classification rule for a point  $x$  depends on the sign of

$$f(x) = \beta_0 + \sum_{i=1}^n \alpha_i K(x, x_i)$$

where

- ▶  $K(x, x_i) = \sum_{j=1}^p x_j x_{ij}$  and
- ▶  $\beta_0$  and the  $\alpha_i$ 's depend on the data only through the  $n(n-1)/2$  values of  $K(x_i, x_{i'})$ .
- ▶ The function  $K(\cdot, \cdot)$  is called the linear kernel.

# Support Vector Machine

- ▶ We can extend the approach by choosing other kernel functions  $K(\cdot, \cdot)$ .
  - ▶ In general, kernels are functions that measure similarity between two feature vectors.

- ▶ Examples of alternative kernels include

- ▶ the polynomial kernel, for given degree  $d$ :

$$K(x_i, x_{i'}) = \left(1 + \sum_{j=1}^p x_{ij} x_{i'j}\right)^d$$

- ▶ and the radial kernel for given  $\gamma > 0$ :

$$K(x_i, x_{i'}) = \exp\left(-\gamma \sum_{j=1}^p (x_{ij} - x_{i'j})^2\right).$$

- ▶ The extension of the support vector classifier to non-linear boundaries, with an expanded feature space and computations done *via* kernels is called the support vector machine.

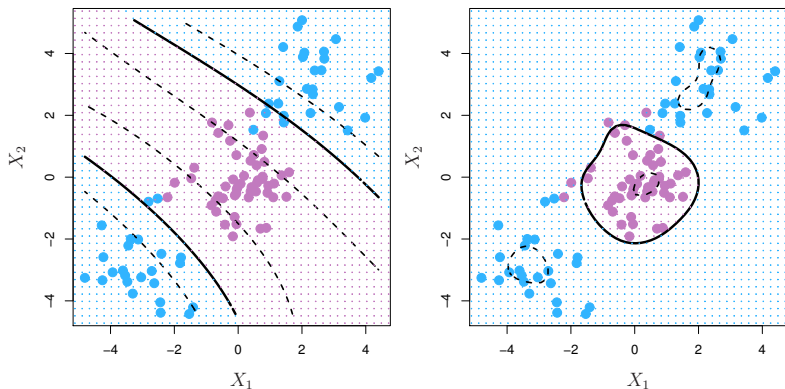


# Computation for the SVM

- ▶ Key point: No matter how large the dimension of the expanded feature space, the computations only rely on the data through  $n(n - 1)/2$  values of  $K(x_i, x_{i'})$ .

# Example Decision Boundaries for Polynomial and Radial Kernels

- ▶ Figure 9.9. Left panel is from the polynomial kernel with  $d = 3$ ; right panel is from a radial kernel with unknown  $\gamma$ .

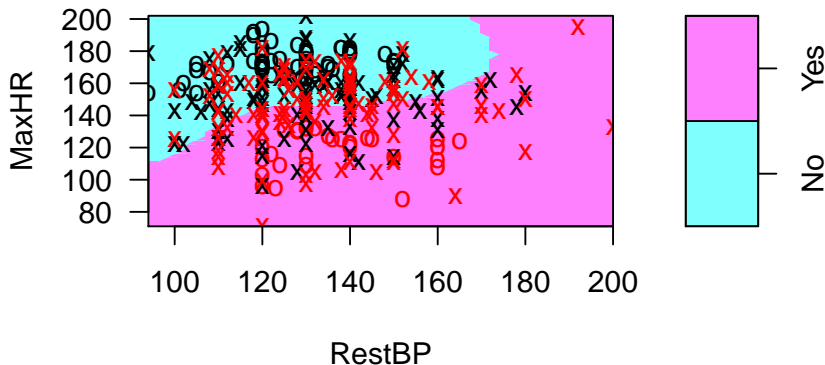


# SVM for Heart Data

- Use a radial kernel.

```
svm.heart <- svm(AHD ~ ., type="C-classification", cost=1,  
                 data=Heart, kernel="radial", gamma=1/2)  
plot(svm.heart, Heart)
```

**SVM classification plot**



## Choose the Cost and $\gamma$ by CV

- The `tune()` function can do the CV over a grid of costs **and** other tuning parameters such as  $\gamma$  for the radial kernel.

```
set.seed(123)
tune2.heart <- tune(svm,AHD ~ .,data=Heart,kernel="radial",
                   ranges=list(cost=c(10^{-3:3}),
                               gamma=c(.5,1:4)))
perf<-summary(tune2.heart)$performances
perf[which.min(perf$error),]
```

```
##   cost gamma   error dispersion
## 4    1    0.5 0.3029885 0.09734697
```

# Classifications

- ▶ Here we predict the training data.
  - ▶ In your weekly exercises you will split the Heart data into training and test sets and will predict the test set.

```
preds <- predict(svm.heart)
table(preds,Heart$AHD)
```

```
##
## preds  No  Yes
##    No 127  47
##    Yes  33  90
```

- ▶ About a 27% misclassification.
  - ▶ Looks pretty bad, but we're using just MaxHR and RestBP at this point.