

Statistics 452: Statistical Learning and Prediction

Chapter 3, Part 1: Simple Linear Regression

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2018-09-10

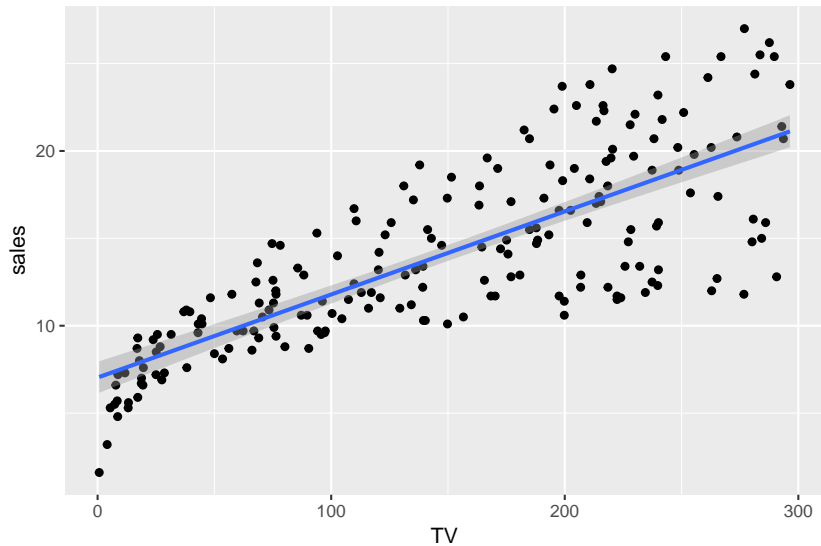
Example: Advertising Data

- Sales (in thousands of units), and advertising budgets in thousands of dollars for TV, radio and newspaper for 200 markets.

```
uu <- url("http://www-bcf.usc.edu/~gareth/ISL/Advertising.csv")
advert <- read.csv(uu,row.names=1)
head(advert)
```

##		TV	radio	newspaper	sales
## 1	230.1	37.8	69.2	22.1	
## 2	44.5	39.3	45.1	10.4	
## 3	17.2	45.9	69.3	9.3	
## 4	151.5	41.3	58.5	18.5	
## 5	180.8	10.8	58.4	12.9	
## 6	8.7	48.9	75.0	7.2	

```
ggplot(advert,aes(x=TV,y=sales)) + geom_point() +  
  geom_smooth(method="lm")
```



Simple Linear Regression Model

- ▶ Recall our general model from Chapter 2:

$$Y = f(X) + \epsilon$$

- ▶ Simple linear regression assumes the function f is linear in a single predictor X ; i.e., $f(X) = \beta_0 + \beta_1 X$.
 - ▶ β_0 is the intercept and
 - ▶ β_1 is the slope.

Fitting the line

- ▶ We use the method of least squares to fit the line.
- ▶ Goal: Using observed data $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ fit the model

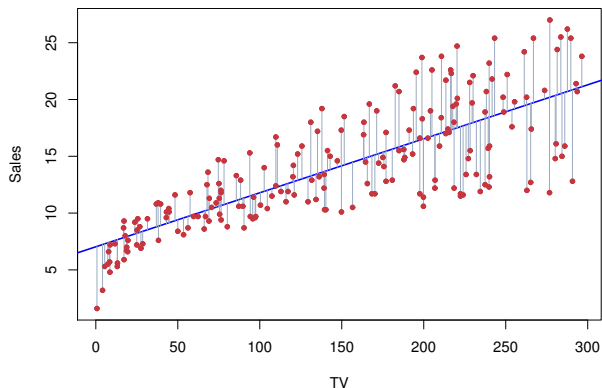
$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$$

where \hat{y}_i is the *predicted or fitted value* of Y for $X = x_i$.

- ▶ Idea: try all possible $\hat{\beta}_0$ and $\hat{\beta}_1$ until you find the line that fits the data the “best”; i.e. the \hat{y} ’s are as close to the y ’s as possible.
 - ▶ What is the criteria for best?

Residuals

- ▶ The vertical distances $e_i = y_i - \hat{y}_i$ are called the residuals



Least Squares

- ▶ Least squares minimizes the sum of the squared residuals, known as the **residual sum of squares**

$$\text{RSS} = \sum_{i=1}^n e_i^2$$

- ▶ There are many visual demonstrations of the least squares idea on the internet; e.g.,
<http://www.dangoldstein.com/regression.html>
 - ▶ Try clicking the $-$ slope, $+$ slope, $-$ intercept, and $+$ intercept buttons to minimize the sum of squared distances, summarized by the blue square.
 - ▶ Then click “Fit and lock” to see the line that minimizes the sum of squares.

Least-Squares Regression

- ▶ The line that minimizes RSS has

$$\begin{aligned}\hat{\beta}_1 &= r \frac{s_y}{s_x} \\ \hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x}\end{aligned}$$

where

- ▶ r is the Pearson correlation between the x 's and y 's,
 - ▶ s_y is the sample SD of the y 's and s_x is the sample SD of the x 's.
- ▶ We will always use R to calculate these estimates.

Advertising Example

```
afit <- lm(sales ~ TV,data=advert)
summary(afit)$coefficients
```

##	Estimate	Std. Error	t value	Pr(> t)
## (Intercept)	7.03259355	0.457842940	15.36028	1.40630e-35
## TV	0.04753664	0.002690607	17.66763	1.46739e-42

Accuracy of the Coefficient Estimates

- ▶ Least squares is a good way to fit a line to a scatterplot, but if we want to assess the accuracy of the coefficient estimates we need assumptions about the distribution of errors.
- ▶ Recall the errors ϵ in $Y = f(X) + \epsilon$.
- ▶ Errors are assumed to be normally distributed with mean zero and SD σ .
 - ▶ The ϵ are the irreducible error terms, and σ quantifies the irreducible error.
- ▶ The SD of the error terms is assumed to be constant for all x .

Model Summary

- ▶ We can summarize the model assumptions by saying that:
 - ▶ the (X, Y) pairs are independent,
 - ▶ conditional on $X = x$, Y has a normal distribution $N(f(x), \sigma)$, with conditional mean $f(x) = \beta_0 + \beta_1 x$ and conditional standard deviation σ being a constant value (i.e. same for all x).

SD and SE of Coefficient Estimators

- ▶ Under the model assumptions, one can derive expressions for the SD of the sampling distribution of $\hat{\beta}_0$ and $\hat{\beta}_1$, which the text refers to as the standard error (SE).
 - ▶ Side note: What the text calls the SE is what I call the SD, and what the text calls the estimated SE is what I call the SE. I'll try to stick to their terminology, but may slip.
- ▶ One can derive expressions for the SE and estimated SE.
 - ▶ E.G., equation (3.8) of text, page 66.
 - ▶ Both SE and estimated SE denoted $SE(\hat{\beta}_i)$.
 - ▶ We will always use the computer to estimate SEs.

Simulation Example

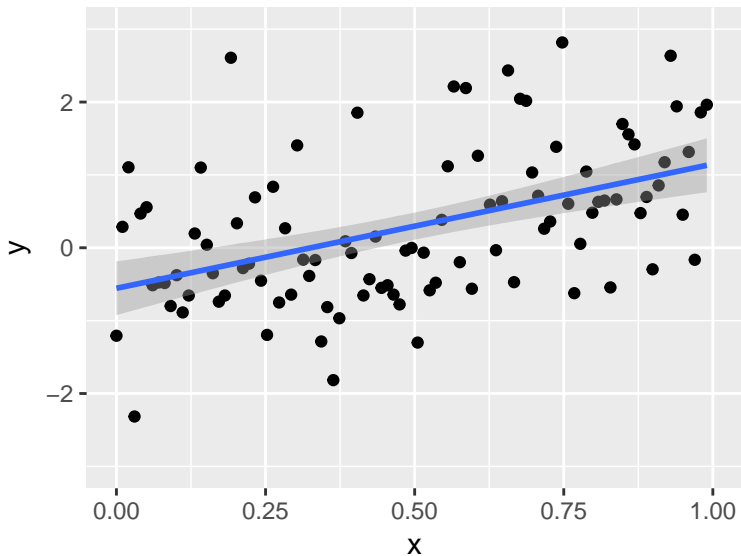
- Start R on your computer, choose your own random seed and run the following sequence of code chunks.

```
# simulation parameters
n <- 100; beta0 <- 0; beta1 <- 1; sd <- 1; NREPS <- 1000
x <- seq(from=0,to=1,length=n)
#simulation function
simdat <- function(x,n,beta0,beta1,sd) {
  f <- beta0 + beta1*x
  y <- f + rnorm(n,mean=0,sd=sd)
  return(list(dat = data.frame(x=x,y=y),coef=coefficients(lm(y~x))))
}
# Run the simulation
set.seed(1234)
```

```
# Do the following a few times
```

```
dat <- simdat(x,n,beta0,beta1,sd)$dat
```

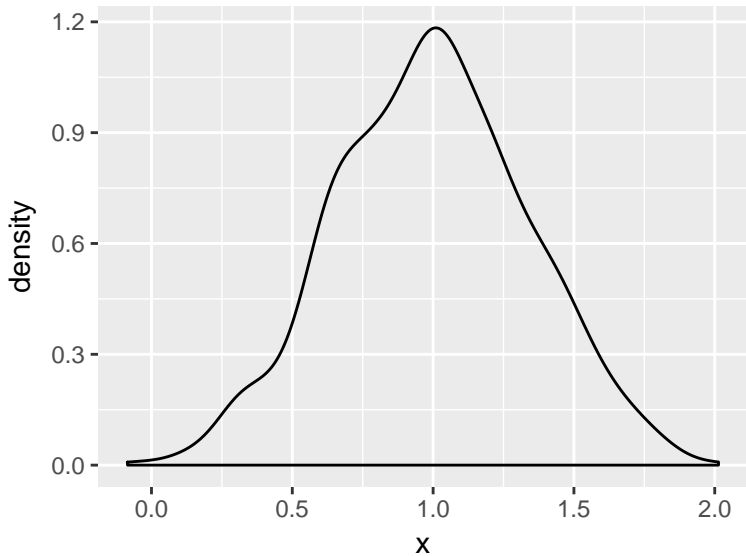
```
ggplot(dat,aes(x=x,y=y)) + geom_point() + geom_smooth(method="lm") + ylim(-3,3)
```



```
simcoef <- function(x,n,beta0,beta1,sd) {  
  return(simdat(x,n,beta0,beta1,sd)$coef)  
}  
simout <- replicate(NREPS,simcoef(x,n,beta0,beta1,sd))  
simout <- data.frame(t(simout))  
head(simout)
```

```
##    X.Intercept.      x  
## 1 -0.03538824  1.1532628  
## 2  0.31615149  0.6769044  
## 3  0.04367708  0.8964356  
## 4 -0.32101657  1.5984614  
## 5  0.01969292  0.6868602  
## 6  0.01532160  0.7936332
```

```
ggplot(simout,aes(x=x)) + geom_density()
```



Confidence Intervals

- ▶ The sampling distribution of the coefficients can be used to derive the following probability statement:
 - ▶ There is a 95% chance that the interval

$$(\hat{\beta}_1 - t^* SE(\hat{\beta}_1), \hat{\beta}_1 + t^* SE(\hat{\beta}_1))$$

contains the true value of β_1 .

- ▶ t^* is the upper critical value of a t distribution with $n - 2$ degrees of freedom (df).

Simulation Example

```
simCI <- function(x,n,beta0,beta1,sd) {  
  f <- beta0 + beta1*x  
  y <- f + rnorm(n,mean=0,sd=sd)  
  ci <- confint(lm(y~x))  
  ci["x",]  
}  
simCI(x,n,beta0,beta1,sd) # Does it contain true value beta1?
```

```
##      2.5 %      97.5 %  
## -0.3225214  1.0582357
```

```
# Exercise: Write code to repeat NREPS times and count  
# how many intervals include beta1.
```

Advertising Example

```
confint(afit)
```

```
##                2.5 %      97.5 %  
## (Intercept) 6.12971927 7.93546783  
## TV          0.04223072 0.05284256
```

- ▶ We say we are 95% confident that a \$1000 increase in TV advertising is associated with an increase in sales of between 42 and 53 units.

Hypothesis Tests

- ▶ The sampling distribution of the coefficients can also be used to derive tests of hypotheses about the parameters.
- ▶ Under the null hypothesis that the true β_1 is 0 (no association),

$$t = \frac{\hat{\beta}_1}{SE(\hat{\beta}_1)} \sim t_{n-2}$$

- ▶ The usual alternative hypothesis is that $\beta_1 \neq 0$ (association). Then the p-value is the chance of $T > |t|$, where $T \sim t_{n-2}$.

Advertising Example

```
summary(afit)$coefficients
```

##		Estimate	Std. Error	t value	Pr(> t)
##	(Intercept)	7.03259355	0.457842940	15.36028	1.40630e-35
##	TV	0.04753664	0.002690607	17.66763	1.46739e-42

- ▶ There is very good evidence that increasing TV advertising increases sales.

Accuracy of the Model

- ▶ Two common measures of the ability of the model to explain variation in Y :

1. The residual SE (RSE) $\sqrt{RSS/(n-2)}$, which is an estimator of σ .
2. The $R^2 = \frac{TSS-RSS}{TSS}$, where TSS is the total sum of squares

$$\sum_i (y_i - \bar{y})^2$$

- ▶ R^2 is the proportion of variation in Y explained by the regression on X .
- ▶ The R^2 is more commonly used as a goodness-of-fit measure.

Advertising Example

```
summary(afit)
```

```
##
## Call:
## lm(formula = sales ~ TV, data = advert)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -8.3860 -1.9545 -0.1913  2.0671  7.2124
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  7.032594   0.457843   15.36  <2e-16 ***
## TV           0.047537   0.002691   17.67  <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.259 on 198 degrees of freedom
## Multiple R-squared:  0.6119, Adjusted R-squared:  0.6099
## F-statistic: 312.1 on 1 and 198 DF, p-value: < 2.2e-16
```

- ▶ TV advertising explains about 61% of the variation in sales.

Residual Plots

- ▶ Residuals are the primary tool for checking model assumptions.
- ▶ For example, a plot of residuals versus fitted values can show evidence of
 - ▶ departures from linearity – look for nonlinear trends
 - ▶ departures from constant SD – look for funnel shapes
 - ▶ outliers – unusually large residuals

Saving the Residuals and Fitted Values

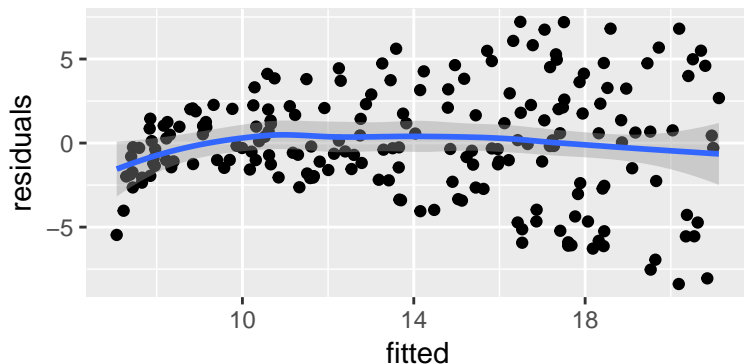
- Use the extractor functions `residuals()` and `fitted()`.

```
advertDiag <- data.frame(advert,residuals=residuals(afit),fitted=fitted(afit))  
head(advertDiag)
```

##	TV	radio	newspaper	sales	residuals	fitted
## 1	230.1	37.8	69.2	22.1	4.1292255	17.970775
## 2	44.5	39.3	45.1	10.4	1.2520260	9.147974
## 3	17.2	45.9	69.3	9.3	1.4497762	7.850224
## 4	151.5	41.3	58.5	18.5	4.2656054	14.234395
## 5	180.8	10.8	58.4	12.9	-2.7272181	15.627218
## 6	8.7	48.9	75.0	7.2	-0.2461623	7.446162

Plotting Residuals vs. Fitted Values

```
ggplot(advertDiag,aes(x=fitted,y=residuals)) +  
  geom_point() + geom_smooth()
```



- ▶ Some evidence of non-linearity on LHS of plot.
- ▶ Funnel from left to right.
- ▶ Consequences?