

# Statistics 452: Statistical Learning and Prediction

## Chapter 3, Part 3: Other Considerations in Multiple Linear Regression

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# Topics

- ▶ Interaction and non-linear model terms
- ▶ Categorical variables as predictors
- ▶ Model diagnostics

## Example Data: Credit Card Balances

- Understand which variables are associated with credit card balance.

```
uu <- url("http://www-bcf.usc.edu/~gareth/ISL/Credit.csv")
credit <- read.csv(uu,row.names=1)
head(credit)
```

##		Income	Limit	Rating	Cards	Age	Education	Gender	Student	Married
## 1		14.891	3606	283	2	34	11	Male	No	Yes
## 2		106.025	6645	483	3	82	15	Female	Yes	Yes
## 3		104.593	7075	514	4	71	11	Male	No	No
## 4		148.924	9504	681	3	36	11	Female	No	No
## 5		55.882	4897	357	2	68	16	Male	No	Yes
## 6		80.180	8047	569	4	77	10	Male	No	No
##		Ethnicity	Balance							
## 1		Caucasian	333							
## 2		Asian	903							
## 3		Asian	580							
## 4		Asian	964							
## 5		Caucasian	331							
## 6		Caucasian	1151							

# Software Note

- ▶ In R, categorical variables should be stored as **factors**.

## Interaction and Non-Linear Model Terms

# Interaction and Non-Linear Model Terms

- ▶ We have already seen non-linear model terms when we modelled the relationship between income and education as a polynomial.
- ▶ We now discuss interaction.

# Statistical Interaction

- ▶ Start with two explanatory variables income ( $X_1$ ) and student status ( $X_2$ )
  - ▶ StudentYes=1 if the person is a student and 0 otherwise.
- ▶  $X_2$  is said to modify the effect of  $X_1$  on  $Y$  if the regression slope of the regression of  $Y$  on  $X_1$  differs in the  $X_2 = 0$  and  $X_2 = 1$  sub-groups.
  - ▶ If we stratify the analysis by student status and find different effects of income in the two groups, there is statistical interaction between income and student status.

# Model for Stratification by Student Status

- ▶  $f(X) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 X_2$  where
  - ▶  $X_1$  is income
  - ▶  $X_2$  is student status (1 is student, 0 is not)
  - ▶  $X_1 \times X_2$  is the statistical interaction between income and student status.
  - ▶  $\beta_1$ ,  $\beta_2$ , and  $\beta_3$  are the corresponding regression coefficients.
- ▶ This model allows separate lines for the two values of student status.
  - ▶ student status = 0 model: intercept  $\beta_0$  and slope  $\beta_1$
  - ▶ student status = 1 model: intercept  $\beta_0 + \beta_2$  and slope  $\beta_1 + \beta_3$
  - ▶ Interpret  $\beta_3$  as the difference between slopes.
  - ▶ If  $\beta_3 = 0$ , then student status does not modify effect of income on balance.
  - ▶ In practice, we test the hypothesis  $H_0 : \beta_3 = 0$ .



# Fitted Model

```
cfit <- lm(Balance ~ Income*Student,data=credit)
summary(cfit)$coefficients
```

##	Estimate	Std. Error	t value	Pr(> t )
## (Intercept)	200.623153	33.6983706	5.953497	5.789658e-09
## Income	6.218169	0.5920936	10.502003	6.340684e-23
## StudentYes	476.675843	104.3512235	4.567995	6.586095e-06
## Income:StudentYes	-1.999151	1.7312511	-1.154743	2.488919e-01

- ▶ The  $t$ -test of the hypothesis  $H_0 : \beta_3 = 0$  does not reject at any of the standard levels.
- ▶ That is, we retain the hypothesis that student status does not modify the effect of income on balance.

# Statistical Interaction More Generally

- ▶ Interaction terms are generally defined as products of two other model terms:
  - ▶  $f(X) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 X_2$
- ▶ In general we allow different lines for different values of  $X_2$ .
  - ▶  $X_2 = 0$  model: intercept  $\beta_0$  and slope  $\beta_1$
  - ▶  $X_2 = x_2$  model: intercept  $\beta_0 + \beta_2 x_2$  and slope  $\beta_1 + \beta_3 x_2$
  - ▶ Interpret  $\beta_3$  as the difference between slopes for a one-unit change in  $X_2$ .
  - ▶ If  $\beta_3 = 0$  then  $X_2$  does not modify effect of  $X_1$  on  $Y$  and *vice versa*.
  - ▶ In practice, test the hypothesis  $H_0 : \beta_3 = 0$ .

# Interaction Between TV and Radio Advertising

```
afit <- lm(sales ~ TV*radio,data=advert)
summary(afit)$coefficients
```

##	Estimate	Std. Error	t value	Pr(> t )
## (Intercept)	6.750220203	0.2478713699	27.232755	1.541461e-68
## TV	0.019101074	0.0015041455	12.698953	2.363605e-27
## radio	0.028860340	0.0089052729	3.240815	1.400461e-03
## TV:radio	0.001086495	0.0000524204	20.726564	2.757681e-51

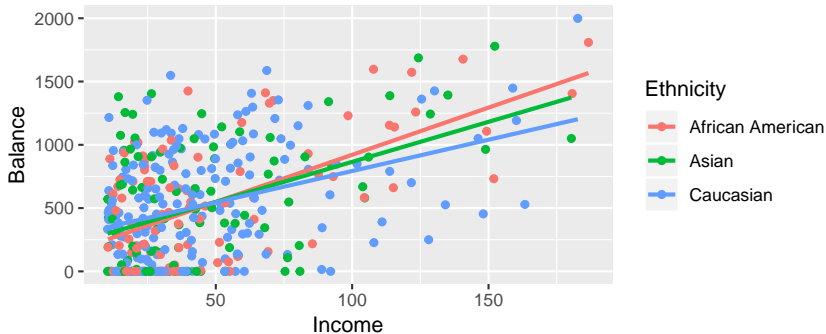
- ▶ There is statistical interaction between TV and radio ads.
  - ▶ TV modifies the effect of radio, or radio modifies the effect of TV

## Categorical Variables as Predictors

# Dummy Variables for Categorical Predictors

- ▶ We have seen dummy variables before.
  - ▶ A binary variable for student status, coded 0 or 1 to represent the two levels of a dichotomous variable.
- ▶ When the categorical variable has more than two values, or levels, we need more than one binary “dummy” variable.
  - ▶ Example: The *Ethnicity* variable from the credit data.

```
ggplot(credit,aes(x=Income,y=Balance,color=Ethnicity)) +  
  geom_point() + geom_smooth(method="lm",se=FALSE)
```



# Regression Model for Balance

```
cfit <- lm(Balance ~ Income*Ethnicity,data=credit)
summary(cfit)$coefficients
```

##	Estimate	Std. Error	t value	Pr(> t )
## (Intercept)	175.495201	65.192852	2.6919393	7.406626e-03
## Income	7.455728	1.062646	7.0161899	1.000371e-11
## EthnicityAsian	57.347367	91.784711	0.6248030	5.324620e-01
## EthnicityCaucasian	122.814867	81.198364	1.5125288	1.312011e-01
## Income:EthnicityAsian	-1.131111	1.559045	-0.7255149	4.685669e-01
## Income:EthnicityCaucasian	-2.510135	1.374387	-1.8263672	6.855154e-02

## Model Details

- ▶ Model requires too much notation to write out in detail; will explain in the context of this example.

```
coefficients(cfit)
```

```
##              (Intercept)              Income
##              175.495201              7.455728
##              EthnicityAsian      EthnicityCaucasian
##              57.347367              122.814867
##      Income:EthnicityAsian Income:EthnicityCaucasian
##              -1.131111              -2.510135
```

- ▶ The model uses African American as a “baseline”.
  - ▶ The intercept=175 and slope Income=7.46 terms are the fitted model for mean Balance in African Americans.
- ▶ The model for mean Balance in another ethnic group is the baseline plus ethnic-group-specific intercept and slope
  - ▶ E.G., for the Asians, add 57.3 to the African American intercept and  $-1.13$  to the African American slope



# Dummy Variables for Ethnic Group

- Create a binary variable for each non-baseline ethnic group that takes value 1 if the person is from that ethnic group and 0 otherwise.

##	Ethnicity	Income	EthnicityAsian	EthnicityCaucasian
## 1	Caucasian	14.891	0	1
## 2	Asian	106.025	1	0
## 3	Asian	104.593	1	0
## 4	Asian	148.924	1	0
## 5	Caucasian	55.882	0	1
## 6	Caucasian	80.180	0	1

## Model with separate lines for each continent

##	Ethnicity	Income	EthnicityAsian	EthnicityCaucasian
## 1	Caucasian	14.891	0	1
## 2	Asian	106.025	1	0
## 3	Asian	104.593	1	0
## 4	Asian	148.924	1	0
## 5	Caucasian	55.882	0	1
## 6	Caucasian	80.180	0	1

##	Income.EthnicityAsian	Income.EthnicityCaucasian
## 1	0.000	14.891
## 2	106.025	0.000
## 3	104.593	0.000
## 4	148.924	0.000
## 5	0.000	55.882
## 6	0.000	80.180

# Multiple-Partial F-test for Interaction

```
cfitReduced <- lm(Balance ~ Income + Ethnicity, data=credit)
anova(cfitReduced, cfit)
```

```
## Analysis of Variance Table
##
## Model 1: Balance ~ Income + Ethnicity
## Model 2: Balance ~ Income * Ethnicity
##   Res.Df      RSS Df Sum of Sq    F Pr(>F)
## 1     396 66205558
## 2     394 65636921  2    568637 1.7067 0.1828
```

- There is little evidence that ethnicity modifies the effect of income on balance.

## Model Diagnostics based on Residuals

# Model Diagnostics based on Residuals

- ▶ Residuals are the primary tool for
  - ▶ checking model assumptions (correct linear model, constant error SD, and normal errors) and
  - ▶ identifying unusual observations.
- ▶ Residuals may also be useful for detecting correlation in the errors, but this is a more specialized topic not discussed.

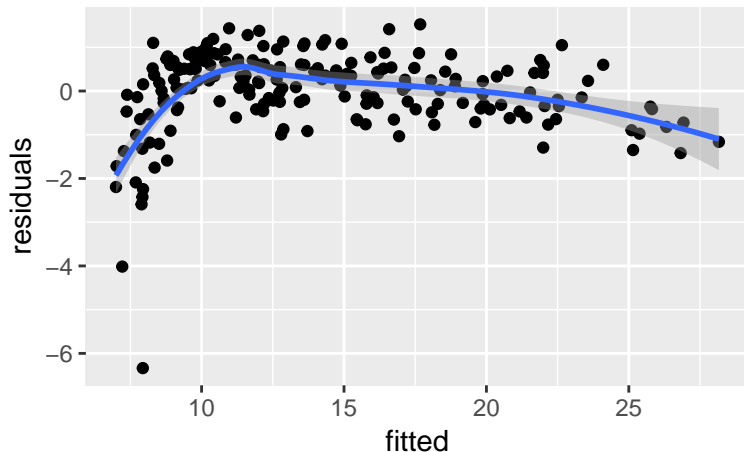
## Residuals versus fitted values

- ▶ Plot residuals vs fitted values to assess adequacy of the linear model and constant error SD.
  - ▶ A plot of  $\epsilon_i$  vs  $f(x_i)$  would show no pattern, because the  $\epsilon_i$ 's are random noise.
  - ▶ If the linear model is adequate, we should not see any trends or patterns in the residuals vs fitted values  $\hat{y}_i = \hat{f}(x_i)$ .
  - ▶ Also, if the error SD is constant, the variation in residuals vs  $\hat{y}_i$  should look roughly equal.
- ▶ We may also see outliers in the regression sense.

# Residuals versus Fitted Values - Advertising

- Use the `residual()` and `fitted()` extractor functions.

```
adAug <- data.frame(advert,fitted=fitted(afit),residuals=residuals(afit))  
ggplot(adAug,aes(x=fitted,y=residuals)) +  
  geom_point() + geom_smooth()
```



## Residual vs fitted – comments

- ▶ Horizontal line at zero is outside the error bands around smoother line.
  - ▶ Suggests we have missed a non-linear trend.
- ▶ Spread of residuals fairly constant over range of fitted values, so constant SD assumptions appears reasonable.



## Q-Q Plots

- ▶ A quantile-quantile (Q-Q) plot is a plot of the quantiles of one distribution to another.
  - ▶ If the two distributions have the similar shape, the points should fall roughly on a straight line.
- ▶ Our interest is in comparing the quantiles of the distribution of residuals to the quantiles of the distribution they should have under normal errors.
  - ▶ One can argue that the residuals don't have the same distribution (those closer to the centre of the plot are slightly more variable).
- ▶ However, Studentized residuals do – they have a  $t$  distribution with  $n - k - 2$  degrees of freedom.
  - ▶ The Studentized residual for the  $i$ th case is

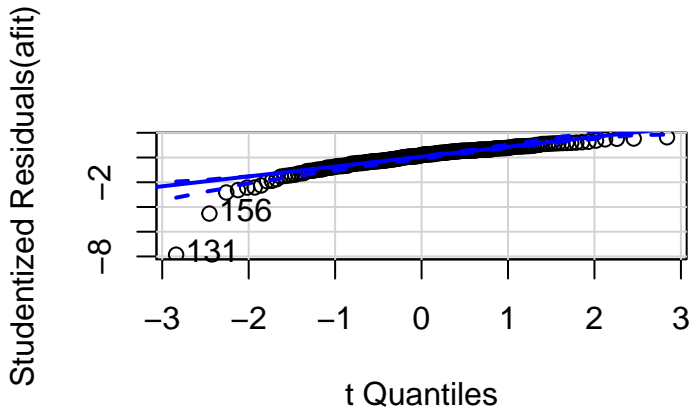
$$\frac{y_i - \hat{y}_i}{\hat{\sigma}_{-i} \sqrt{1 - h_{ii}}}.$$

where  $h_{ii}$  is the leverage or hat value and  $\hat{\sigma}_{-i}$  is the estimate of the error SD **without** case  $i$

## Q-Q plot of Studentized residuals

- ▶ The `qqPlot()` function from the `car` package plots Studentized residuals against quantiles of the appropriate  $t$  distribution.
  - ▶ Also adds error bands: If all points fit within bands, it is plausible that the sample is from the  $t$ .

```
library(car) # Use install.packages("car") to install
qqPlot(afit)
```



# Identifying unusual observations

- ▶ Studentized residuals can identify outliers
  - ▶ Rule-of-thumb: Residuals beyond  $\pm 2$  are moderate outliers and beyond  $\pm 3$  are serious outliers.
- ▶ Leverage ( $h_i$ ) is a measure of how atypical an observation's  $X$  values are.
  - ▶ Rule-of-thumb:  $h_i > 2(p + 1)/n$  is somewhat high leverage and  $h_i > 3(p + 1)/n$  is very high leverage.
- ▶ Cook's distance measures the influence of an observation; i.e., how much the estimated regression coefficients change when the observation is removed.
  - ▶ Rule-of-thumb: Cook's distance  $> 0.5$  is moderately influential, and  $> 1$  is highly influential

# Identify Unusual Observations in Advertising Data

- ▶ Augment the dataset and `View()` to identify cases that are unusual according to our rules-of-thumb.
  - ▶ Several “moderate” residuals, and two severe residuals of  $-4.5$  and  $-7.9$ .
  - ▶ For leverage,  $p = 2$ ,  $n = 200$ ,  $2(p + 1)/n = 0.03$ ,  $3(p + 1)/n = 0.045$ : 27 cases with moderate leverage, 12 with very high leverage!
  - ▶ One moderately influential case.

```
adAug <- data.frame(advert, studRes = rstudent(afit),  
                    hats = hatvalues(afit),  
                    cooks = cooks.distance(afit))  
  
# Now View(adAug)
```

## Correlated Predictors, or Collinearity

- ▶ The distribution of  $X$ 's can affect stability of the least squares estimates.
  - ▶ For simple linear regression one can show that:

$$SE(\hat{\beta}_1) = \frac{\hat{\sigma}}{S_X \sqrt{n-1}}$$

where  $S_X$  is the SD of the  $X$ 's in the dataset.

- ▶ Implies that the larger the  $S_X$  the smaller the SE (i.e., the more stable the fit).
- ▶ In general can think of the positioning of  $X$ 's as a “foundation” that supports the least squares surface – the broader the base, the more stable the estimates.
- ▶ Collinearity, or correlation between predictors, yields an unstable foundation and hence unstable estimates.

## More on SEs

- ▶ With two explanatory variables,  $X_1$  and  $X_2$ , can show that

$$SE(\hat{\beta}_1) = \frac{\hat{\sigma}}{S_{X_1} \sqrt{n-1} \sqrt{1-r_{12}^2}}$$

and

$$SE(\hat{\beta}_2) = \frac{\hat{\sigma}}{S_{X_2} \sqrt{n-1} \sqrt{1-r_{12}^2}}$$

where  $r_{12}$  is the correlation between  $X_1$  and  $X_2$ .

- ▶ In addition to  $S_{X_1}$  and  $S_{X_2}$  we must consider the correlation between  $X_1$  and  $X_2$ .
- ▶ The larger the squared correlation, the larger the SEs

# Variance Inflation Factors (VIFs)

- ▶ In a multiple regression with  $X_1, \dots, X_p$ ,

$$SE(\hat{\beta}_j) = \frac{\hat{\sigma}}{S_{X_j} \sqrt{n-1} \sqrt{1-R_j^2}}$$

where  $R_j^2$  is the  $R^2$  from the regression of  $X_j$  on  $X_{(-j)}$ .

- ▶ The term  $1/\sqrt{1-R_j^2}$ , is the factor by which the SE of  $\hat{\beta}_j$  is inflated over the SE from a simple linear regression by correlation between  $X_j$  and the other  $X$ 's.
- ▶ The variance inflation factor for  $X_j$ ,  $VIF_j$ , is defined to be  $1/(1-R_j^2)$ ; i.e., the SE of  $\hat{\beta}_j$  is inflated by  $\sqrt{VIF_j}$ .
- ▶ High VIFs indicate instability.
  - ▶ One rule of thumb is that a  $VIF_j > 10$  is cause for concern.

# VIFs and Other Diagnostics in the car Package

- ▶ If you haven't already done so, install the R package car.

```
library(car)
vif(afit)
```

```
##          TV      radio TV:radio
## 3.727848 3.907651 6.937860
```

- ▶ The VIFs suggest the interaction is quite well predicted by TV and radio, but the VIF is less than our threshold so we are not concerned.



# Collinearity with Polynomial Terms

```
uu <- url("http://www-bcf.usc.edu/~gareth/ISL/Income1.csv")
income <- read.csv(uu,row.names=1)
ifit<- lm(Income ~ Education + I(Education^2) + I(Education^3), data=income)
vif(ifit)
```

```
##      Education I(Education^2) I(Education^3)
##      5612.306      23764.744      6449.139
```

# Remedies for collinearity

- ▶ When the collinearity arises from explanatory variables that are products of other variables, centering can help.

```
centre <- function(x) { x - mean(x) }  
income <- data.frame(income, cEducation = centre(income$Education))  
ifit <- lm(Income ~ cEducation + I(cEducation^2) + I(cEducation^3), data=income)  
vif(ifit)
```

```
##      cEducation I(cEducation^2) I(cEducation^3)  
##      6.275936      1.000000      6.275936
```

## Collinearity: If centering doesn't help

- ▶ May need to exclude a variable.
  - ▶ Sounds drastic, but high  $R_j^2$  indicates  $X_j$  is very well predicted by  $X_{(-j)}$ , so nothing really lost.
- ▶ Which variable to exclude?
- ▶ First use common sense:
  - ▶ If one variable is a surrogate for another, drop the surrogate.
  - ▶ For example, if we are modeling house prices with (i) size of the house in square feet and (ii) the number of bedrooms, we may think bedrooms is just a surrogate for size.
- ▶ If no obvious candidate to drop, use model selection.

# Fit of the Advertising Data Revisited

```
library(dplyr)
advert <- mutate(advert, cTV = TV - mean(TV),
                  cRadio = radio - mean(radio))
afitC <- lm(sales ~ cTV*cRadio+I(cTV^2) +I(cTV^3)+I(cTV^4),
            data=advert)
summary(afitC)$coefficients
```

##	Estimate	Std. Error	t value	Pr(> t )
## (Intercept)	1.449416e+01	6.224612e-02	232.852415	2.089992e-238
## cTV	3.355736e-02	8.898603e-04	37.710818	5.634012e-91
## cRadio	1.960573e-01	2.130075e-03	92.042410	2.118337e-161
## I(cTV^2)	1.829559e-05	1.766454e-05	1.035724	3.016270e-01
## I(cTV^3)	7.812590e-07	6.237580e-08	12.525033	9.967540e-27
## I(cTV^4)	-6.750627e-09	8.823011e-10	-7.651160	9.242163e-13
## cTV:cRadio	1.041128e-03	2.468919e-05	42.169394	2.478978e-99

```
vif(afitC)
```

##	cTV	cRadio	I(cTV^2)	I(cTV^3)	I(cTV^4)	cTV:cRadio
##	5.961561	1.021525	13.337249	5.992502	13.370396	1.020922

```
mutate(advert,fitted=fitted(afitC),residuals=residuals(afitC)) %>%  
  ggplot(aes(x=fitted,y=residuals)) + geom_point() + geom_smooth()
```

