LL(1) Parsing with Derivatives and Zippers

Efficient, Functional, and Formally Verified Approach to Parsing

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Abstract

In this paper, we present an efficient, functional, and formally verified parsing algorithm for LL(1) context-free expressions based on the concept of derivatives of formal languages. Parsing with derivatives is an elegant parsing technique, which, in the general case, suffers from cubic worst-case time complexity and slow performance in practice. We specialise the parsing with derivatives algorithm to LL(1) context-free expressions, where alternatives can be chosen given a single token of lookahead. We formalise the notion of LL(1) expressions and show how to efficiently check the LL(1) property. Next, we present a novel linear-time parsing with derivatives algorithm for LL(1) expressions operating on a zipperinspired data structure. We prove the algorithm correct in Coq and present an implementation as a part of Scallion¹, a parser combinators framework in Scala with enumeration and pretty printing capabilities.

Keywords Parsing, LL(1), Derivatives, Zipper, Formal proof

1 Introduction

In this paper, we propose a formally verified parsing approach for LL(1) languages based on derivatives. We present an implementation of the approach as a parsing combinator framework, which supports static checks that the grammar is LL(1), and provides not only parsing and semantic actions, but also enumeration and pretty-printing functionality. Our implementation remains functional yet efficient, which allows us to obtain an implementation and a proof that closely follow each other.

Whereas parsing is a well understood problem, recent years have seen a renewed interest in approaches that handle not just language recognition but also syntax tree construction, and that are proven correct formally. Such parsing techniques can then be leveraged to more productively construct efficient front ends for verified compilers such as CompCert [35] and CakeML [29]. Safe and correct parsers are also crucial for building serialization and deserialization layers of communication infrastructure, which has been a major target of high-impact security exploits [5].

Parsing traditionally uses context-free grammars as the starting specification formalism and proceeds using table and stack-based algorithms. Popular techniques include LR

parsing [13, 26, 31], LL parsing techniques [36, 49], recursive descent [8], Earley's algorithm [14], and the Cocke-Younger-Kasami (CYK) algorithm [9, 25, 58]. Due to the significant gap between implementation and specification in such approaches, the resulting proofs are often based on validation as opposed to proofs for the general case [24].

In 1964, Brzozowski introduced the concept of a derivatives of regular expressions [7]. This concept has proven successful in many formal proofs of parsing regular expressions and their generalisations [4, 44, 55, 56].

Derivatives of *context-free* expressions [34] generalize derivatives of regular expressions and have recently been used as an alternative principled approach to understanding context-free parsing [12, 39], avoiding explicit conversion into pushdown automata. Context-free expressions offer an algebraic view of context-free grammars. In addition to describing a language, context-free expressions also describe the value associated with each recognised input sequence, which makes integration into real-world parsers more natural. The concept of context-free expression derivatives was shown to naturally yield a parsing technique aptly named *parsing with derivatives* [39], which was later proved to have worst-case cubic complexity [1].

For integration into verifiable functional infrastructure, a particularly promising interface are parsing combinators [8, 15, 21, 22, 57]. Parsing combinator frameworks have been proposed for many functional programming languages, including Haskell [33] and Scala [18, 30]. Most implementations of parser combinators use recursive descent for parsing, which suffers from exponential worst-case complexity due to backtracking and can encounter stack overflows with deeply nested structures. Parsing expression grammars (PEGs) [17] are also popular in parsing combinators and have been formally verified [27]. In our experience, merging lexical and syntactic analysis is not helpful for performance, whereas the operational nature of PEGs (with asymmetrical alternative operator) makes it easy to write grammars that do not behave as expected.

In contrast, LL(1) parsing [36] is restricted to context-free grammars that can be non-ambiguously parsed given a single token of lookahead and runs in time linear in the input size. An appealing aspect of such grammars is that they can be algorithmically and efficiently analysed to prevent grammar design errors. In addition, they are known to provide

¹Freely available at https://github.com/epfl-lara/scallion

good performance and error localisation [2]. Previous parsing combinator libraries for LL(1) languages either do not perform LL(1) checks [53] or impose restrictions on emptiness when parsing sequences [28], beyond those necessary for the definition of LL(1) languages.

We show that by using the methodology of context-free expression derivatives, we can arrive at an efficient implementation of LL(1) parsing combinators, and so without introducing needless restrictions. We further show that, by embracing Huet's zipper [20, 37] data structure, parsing with derivatives on LL(1) languages can be implemented with linear time complexity. Our framework of derivatives leads to natural formal proofs in proof assistants. We have successfully proven the correctness of our algorithm for LL(1) parsing with derivatives using the Coq proof assistant. Thanks to the monoidal [38] interface and deep embedding of our combinators, our approach supports efficiently checking whether a syntax description is LL(1), ensuring the predictability of parsing, which we have also formally proven correct. The nature of the parser descriptions also enables enumeration of recognised sequences and pretty printing of values as token sequences, making it also suitable for use in grammardirected code completion.

Contributions

- We present a formalisation of context-free expressions (*syntaxes*) with the expressive power of context-free grammars but with an added ability to describe the values associated with recognised inputs. We then define *LL(1) syntaxes*, where all alternatives can be resolved given a single token of lookahead. We give formal definitions of productivity, nullability, first and should-not-follow sets, and show how to use them to check that a syntax is *LL(1)*.
- We show how propagator networks [46] can be used to compute properties of syntaxes in linear time.
- We present an algorithm for parsing with derivatives on LL(1) syntaxes. Compared to traditional parsing, the algorithm works directly at the level of syntaxes, not on a derived push-down automaton. We show a technique based on Huet's zipper [20] to make LL(1) parsing with derivatives efficient. We show that such zippy LL(1) parsing runs in time linear in the input.
- We present a Coq formalisation of syntaxes and prove the correctness of the zippy LL(1) parsing with derivatives algorithm and its auxiliary functions. For performing LL(1) checks, we formalise rule-based descriptions from which we can obtain both an inductive predicate and an equivalent propagator network. The Coq proofs are available at https://github.com/epfl-lara/scallion-proofs.
- We present Scallion, an implementation of syntaxes as a Scala parser combinators framework with a unique set of features, implementing LL(1) parsing using

derivatives and the zipper data structure. In addition to being reasonably efficient, the framework provides error reporting, recovery, enumeration of accepted sequences, as well as pretty printing. We benchmark the framework and show that its performance is comparable to that of the standard Scala Parser Combinators library [30], while avoiding stack overflows and providing more features. The framework is freely available under an open source license at https://github.com/epfl-lara/scallion.

2 Example

To give the flavour of our approach, Figure 1 presents a parser for JSON using Scallion, our parser combinators framework implemented in Scala. The sequencing combinator is denoted by infix ~, while disjunction is denoted by |. The parser runs efficiently, even though it does not rely on code generation: with our simple hand-written lexer it takes 40ms to parse 1MB of raw JSON data into a value of type Value, half of which is spent lexing. To provide a comparison point, an ANTLR-generated JSON parser [41–43] takes 13ms per 1MB to produce a parse tree (using its own lexer).

As the Scallion framework is embedded in Scala, we can use the Scala REPL to query the parser. The following snippets show an example REPL interaction with the framework. We start by checking the LL(1) property for the top-level jsonValue syntax, and then show its first set.

```
scala> jsonValue.isLL1
// true
scala> jsonValue.first
// Set(NullKind, SepKind('['), ...)
```

When we feed a valid sequence of tokens to the syntax, we obtain as expected a JSON value.

```
scala> val tokens = JSONLexer("[1,_2,_3]")
scala> jsonValue(tokens)
// Parsed(ArrayValue(...), ...)
```

When we feed it an invalid sequence, the syntax duly returns a parse error, indicating the first unrecognised token and providing the residual syntax at the point of error. We can then query the residual syntax for valid ways to continue the sequence, or even to resume parsing.

Figure 1. JSON Parser in Scala using Scallion, the parser combinator framework discussed in this paper.

```
scala> rest(JSONLexer(",_3]"))
// Parsed(ArrayValue(...), ...)
```

The call to trails showcases the enumeration capabilities of our framework. The method lazily enumerates all sequences of tokens recognised by the syntax, ordered by increasing length. The call to take reduces this infinite stream down to its first three elements, which we have then printed.

3 Algebraic Framework for Parsing

In this section, we formalise the notion of a *syntax* and describe their semantics as a relation between input token sequences and values. We use this semantics relation as a basis for the correctness of our approach.

We consider a set of values $\mathcal V$ and a set of types $\mathcal T$. For a value $v \in \mathcal V$ and a type $T \in \mathcal T$, we denote by v : T the fact that the value v has type T.

We assume values and types are cartesian closed. We denote by $(v_1, v_2) \in \mathcal{V}$ the pair of the values v_1 and v_2 and by $(T_1, T_2) \in \mathcal{T}$ the pair of types T_1 and T_2 . We assume $(v_1, v_2) : (T_1, T_2)$ if and only if $v_1 : T_1$ and $v_2 : T_2$. We denote by $T_1 \to T_2$ the set of total functions from values of type T_1 to values of type T_2 .

We use $\langle \rangle$ to denote the empty sequence and use $xs_1 ++ xs_2$ to denote the concatenation of sequences xs_1 and xs_2 . We denote by x :: xs the prepending of x to xs.

3.1 Tokens and Kinds

We consider a single type amongst our types \mathcal{T} to be the type of tokens. We denote this type by Token. The values $v \in \mathcal{V}$

such that v: Token are called *tokens*. We will generally use the lower case letter t to denote such tokens. The task of parsing consists in turning a sequence of tokens into a value, or to fail when the sequence of tokens is invalid.

Each token is assigned to a single kind. Token kinds represent (potentially infinite) groups of tokens. We denote by \mathcal{K} the set of all kinds. While we generally will have infinitely many different possible tokens, we will only have a finite, relatively small, number of kinds.

Token kinds are meant to abstract away details in the tokens. As an example, the strings "hello world", "foo" and "bar" could be considered tokens, and string would be their token kind. The numbers 3, 17, 42 could be considered tokens, while number would be their associated kind. During parsing, the actual tokens are useful to build the resulting value, but whether or not a token is accepted can only be based on the token kind.

We denote by getKind(t) the kind of a token t. We assume that every kind has at least one associated token, and that equality between kinds is decidable.

3.2 Syntaxes

For every type $T \in \mathcal{T}$, we define the set S_T of syntaxes that associates token sequences with values of type T. Those sets are inductively defined by the rules in Figure 2.

The construct \bot , ε_v and $elem_k$ form the basic syntaxes. Intuitively, \bot represents failure, while ε_v represents the empty string, with associated value v. The syntax $elem_k$ represents a single token of kind k. The constructs $s_1 \lor s_2$ and $s_1 \cdot s_2$

$$\begin{array}{c|c}
T \in \mathcal{T} & v : T \\
\hline
\bot \in \mathcal{S}_{T} & s_{v} \in \mathcal{S}_{T}
\end{array}
\quad
\begin{array}{c|c}
k \in \mathcal{K} \\
elem_{k} \in \mathcal{S}_{\mathsf{Token}}
\end{array}$$

$$\begin{array}{c|c}
s_{1} \in \mathcal{S}_{T} & s_{2} \in \mathcal{S}_{T} \\
\hline
s_{1} \lor s_{2} \in \mathcal{S}_{T}
\end{array}
\quad
\begin{array}{c|c}
s_{1} \in \mathcal{S}_{T_{1}} & s_{2} \in \mathcal{S}_{T_{2}} \\
\hline
s_{1} \cdot s_{2} \in \mathcal{S}_{(T_{1}, T_{2})}
\end{array}$$

$$\begin{array}{c|c}
s \in \mathcal{S}_{T_{1}} & f \in T_{1} \to T_{2} \\
\hline
f \circledcirc s \in \mathcal{S}_{T_{2}}
\end{array}
\quad
\begin{array}{c|c}
x \in \Sigma_{T} \\
\hline
var_{x} \in \mathcal{S}_{T}
\end{array}$$

Figure 2. Definition of syntaxes.

$$\frac{k = \operatorname{getKind}(t)}{\operatorname{elem}_k \vdash \langle t \rangle \leadsto t}$$

$$\frac{s_1 \vdash ts \leadsto v}{s_1 \lor s_2 \vdash ts \leadsto v} \qquad \frac{s_2 \vdash ts \leadsto v}{s_1 \lor s_2 \vdash ts \leadsto v}$$

$$\frac{s_1 \vdash ts_1 \leadsto v_1 \qquad s_2 \vdash ts_2 \leadsto v_2}{s_1 \cdot s_2 \vdash ts_1 + ts_2 \leadsto (v_1, v_2)}$$

$$\frac{s \vdash ts \leadsto v}{f \circledcirc s \vdash ts \leadsto f(v)}$$

$$\frac{s = \operatorname{getDef}(x) \qquad s \vdash ts \leadsto v}{var_x \vdash ts \leadsto v}$$

Figure 3. Semantics of syntaxes.

respectively represent disjunction and sequencing. The construct $f \otimes s$ represents the application of the function f on values produced by s. Finally, the construct var_x represents a reference to a syntax defined in a global environment. The variables and the environment allow for mutually recursive syntaxes.

The global environment is a mapping from identifiers to syntaxes. We consider, for every type T, the finite set of identifiers Σ_T . We assume that the set Σ_T is non-empty only for a finite number of types T, that is that the environment is finite. For each identifier $x \in \Sigma_T$, the environment associates a unique syntax $s \in S_T$. We denote this syntax by getDef(x).

3.3 Semantics of Syntaxes

Syntaxes associate token sequences with values. The inductive predicate $s \vdash ts \leadsto v$ indicates that the syntax s associates the token sequence ts with the value v. The inductive predicate is defined by the rules in Figure 3.

Theorem 1 (Type correctness). For any type $T \in \mathcal{T}$, syntax $s \in \mathcal{S}_T$, token sequence ts and value $v \in \mathcal{V}$, if $s \vdash ts \rightsquigarrow v$ then v : T.

Remark. We do not present proofs of theorems in this paper and refer instead the reader to our formal proof in Coq, discussed in Section 7. Given the order of theorems we present, most proofs follow relatively straightforwardly by induction, with main insight being the choice of induction variable and schema.

4 Properties of Syntaxes

This section defines several computable properties of syntaxes which we use for LL(1) checking and parsing.

4.1 Productivity and Nullability

A syntax is said to be *productive* if it associates at least one sequence of tokens with a value. We derive productivity according to the rules in Figure 4a.

Theorem 2. For any syntax s:

$$PRODUCTIVE(s) \iff \exists ts, v. \ s \vdash ts \rightsquigarrow v$$

A syntax $s \in S_T$ is said to be *nullable with value v*, if it associates the empty sequence of tokens with the value v of type T. We will simply say s is *nullable* when we don't need to refer to the value that s is nullable with. We will use the function $nullable(_)$ to return a nullable value from a syntax, if such a value exists, or none otherwise. We derive nullability according to the rules in Figure 4b.

Theorem 3. For any syntax s and value v:

$$NULLABLE(s, v) \iff s \vdash \langle \rangle \rightsquigarrow v$$

4.2 First Set

The *first set* of a syntax *s* is the set containing the kinds of all tokens at the start of at least one sequence associated with some value by *s*. We define the first set inductively according to the rules shown in Figure 4c.

Theorem 4. The first set of a syntax s equals the set

$$\{ k \mid \exists t, ts, v. \ \text{getKind}(t) = k \land s \vdash t :: ts \leadsto v \}$$

4.3 Should-Not-Follow Set

The concept of a *should-not-follow set* is directly connected to the concept of LL(1) conflicts that we will later introduce. Intuitively, the should-not-follow set of a syntax is the set of kinds that would introduce an ambiguity if the first set of any syntax directly following that syntax was to contain that kind. The concept of should-not-follow set is used as an alternative to the concept of FOLLOW set generally used in the context of LL(1) parsing. While the FOLLOW set is a global property of a grammar, the should-not-follow set enjoys a local and more compositional nature. We define the should-not-follow set inductively according to the rules in Figure 4d. Our definition differs from the one used by Krishnaswami and Yallop [28] and introduced in earlier works [6, 23]. While we introduce elements to the set in the case of disjunctions, they do so in the case of sequences. Our definition seems

more appropriate: the previous work introduced additional restrictions on syntaxes, disallowing nullable expressions on left part of sequence, which is *not* needed in our approach (nor in conventional LL(1) definition for context-free grammars [3, Theorem 5.3, Page 343]).

Theorem 5. For any syntax s and kind k, if k is part of the should-not-follow set of s, then there exists a token t of kind k and (possibly empty) sequences of token ts_1 and ts_2 such that:

$$s \vdash ts_1 \leadsto v_1 \land s \vdash ts_1 ++ t :: ts_2 \leadsto v_2$$

4.4 LL(1) Conflicts

Finally, we introduce the notion of LL(1) conflicts. When a syntax has LL(1) conflicts, a choice between two alternatives can arise during parsing which can not be resolved given a single token of lookahead. Existence of LL(1) conflicts is formalised by the set of inductive rules presented in Figure 4e. Informally, LL(1) conflicts arise in three cases: 1) Both branches of a disjunction are nullable, which means that two potentially different values are associated with the empty string by the disjunction. 2) Branches of a disjunction have non-disjoint first sets, so both branches can accept a sequence starting with the same token. Given a single token of lookahead, a parser thus cannot decide which branch to choose. 3) The should-not-follow set of the left-hand side of a sequence and the first set of the right-hand side of that sequence both contain the same token kind, k. This means that there is a point in the left-hand side (after reading the sequence of tokens ts_1 from Theorem 5) where reading a token of kind k will make it impossible to decide whether we should stay in the left-hand side (and then read ts_2), or start parsing in the right-hand side of the sequence.

Definition 6. A syntax is LL(1) iff it has no LL(1) conflicts.

Theorem 7 (LL(1) syntaxes are non-ambiguous). For all LL(1) syntaxes s, token sequences ts and values v_1 and v_2 :

$$s \vdash ts \leadsto v_1 \land s \vdash ts \leadsto v_2 \implies v_1 = v_2$$

Productive LL(1) syntaxes can be shown to be non-left-recursive. We also have the following characterisation.

Theorem 8. Should-not-follow set of an LL(1) syntax s equals

$$\{ k \mid \exists t, ts_1, ts_2, v_1, v_2. \text{ getKind}(t) = k \land \\ s \vdash ts_1 \leadsto v_1 \land \\ s \vdash ts_1 + t :: ts_2 \leadsto v_2 \}$$

4.5 Computing with Propagator Networks

The definitions we introduced in this section are based on inductive rules. Due to the potentially cyclic nature of syntaxes arising from the variables and global environment, those definitions do not immediately give rise to recursive procedures. We propose using *propagator networks* [46, 52] to efficiently

compute the properties defined in the present section. The idea is to build a network of cells, one for each node in the syntax. For each identifier x, the var_x nodes share the same cell. Each cell has a mutable state which holds information about the properties of the corresponding syntax node. Information is then propagated through the network. To do so, the content of each cell is updated according to the inductive rules presented in Figure 4. A list of cells that need to be updated is maintained. The information propagation phase ends when such list is empty. Using this approach, we found that properties can be computed for a syntax and all its inner nodes in worst-case time linear in the size of the syntax, which was not obvious to us from the conventional fixpoint definitions of these concepts. The constant number of kinds also factors in the cost of computations of first and should-not-follow sets. We have proven the correctness of the approach in Coq, as further discussed in Section 7.

5 Derivatives of LL(1) Syntaxes

To devise a parsing algorithm for syntaxes, we use the concept of a *derivative*. The derivative of a syntax s with respect to a token t is a new syntax $\delta_t(s)$ which associates for every sequence ts the value v if and only if s associates t:: ts with v. The derivative of a syntax with respect to a token represents the state of the syntax after seeing the token t. Instead of defining the derivative for general syntaxes, we will only define it for LL(1) syntaxes. We define the derivative of a LL(1) syntax with respect to a token t recursively as follows:

$$\begin{split} \delta_t(\bot) &:= \bot \\ \delta_t(\varepsilon_v) &:= \bot \\ \delta_t(elem_k) &:= \begin{cases} \varepsilon_t & \text{if getKind(t)} = \mathbf{k} \\ \bot & \text{otherwise} \end{cases} \\ \delta_t(s_1 \lor s_2) &:= \begin{cases} \delta_t(s_1) & \text{if getKind}(t) \in \mathtt{FIRST}(s_1) \\ \delta_t(s_2) & \text{otherwise} \end{cases} \\ \delta_t(s_1 \cdot s_2) &:= \begin{cases} \varepsilon_v \cdot \delta_t(s_2) & \text{if nullable}(s_1) = \mathtt{some}(v) \\ & \text{and getKind}(t) \in \mathtt{FIRST}(s_2) \\ \delta_t(s_1) \cdot s_2 & \text{otherwise} \end{cases} \\ \delta_t(f \odot s) &:= f \odot \delta_t(s) \\ \delta_t(var_x) &:= \delta_t(\mathtt{getDef}(x)) \end{split}$$

The above definition makes good use of the fact that the syntax is LL(1). Compared to the original definition of derivatives of context-free expressions by Might et al. [39], our definition only performs recursive calls on at most one child syntax. The choice of which child to recursively derive is informed by first sets.

Theorem 9. The syntax $\delta_t(s)$ is well-defined for any productive LL(1) syntax s and token t.

Theorem 10 (Progress). For any productive LL(1) syntax s, token t, token sequence ts and value v we have that s associates

```
NULLABLE(\varepsilon_v, v)
 PRODUCTIVE(\varepsilon_v)
                             PRODUCTIVE(elem_k)
                                                                                       NULLABLE(s_1, v)
  PRODUCTIVE(s_1)
                                   PRODUCTIVE(s_2)
                                                                                    \frac{-}{\text{NULLABLE}(s_1 \vee s_2, v)}
                                                                                                                     NULLABLE(s_1 \lor s_2, v)
                                 PRODUCTIVE(s_1 \lor s_2)
PRODUCTIVE(s_1 \lor s_2)
                                                                                           NULLABLE(s_1, v_1) NULLABLE(s_2, v_2)
       PRODUCTIVE(s_1)
                              PRODUCTIVE(s_2)
                                                                                                  NULLABLE(s_1 \cdot s_2, (v_1, v_2))
                PRODUCTIVE(s_1 \cdot s_2)
                                                                                                        NULLABLE(s, v)
                   PRODUCTIVE(s)
                                                                                                    NULLABLE(f \otimes s, f(v))
                 PRODUCTIVE(f \otimes s)
                                                                                             s = getDef(x) NULLABLE(s, v)
        s = getDef(x) PRODUCTIVE(s)
                                                                                                      NULLABLE(var_x, v)
                 PRODUCTIVE(var_x)
                                                                                                    (b) Rules for nullability.
              (a) Rules for productivity.
                                                                                                                    k \in \text{SN-FOLLOW}(s_2)
                                                                                  k \in \text{SN-FOLLOW}(s_1)
                                                                               k \in \text{SN-FOLLOW}(s_1 \vee s_2) k \in \text{SN-FOLLOW}(s_1 \vee s_2)
                  k \in FIRST(elem_k)
                                                                                           k \in FIRST(s_1) NULLABLE(s_2, v)
 k \in \text{first}(s_1) \qquad \qquad k \in \text{first}(s_2)k \in \text{first}(s_1 \vee s_2) \qquad \qquad k \in \text{first}(s_1 \vee s_2)
                                                                                                k \in \text{SN-FOLLOW}(s_1 \vee s_2)
                                                                                           nullable(s_1, v) \qquad k \in first(s_2)
                                                                                                 k \in \text{SN-FOLLOW}(s_1 \vee s_2)
        k \in FIRST(s_1) PRODUCTIVE(s_2)
                  k \in FIRST(s_1 \cdot s_2)
                                                                                       k \in \text{SN-FOLLOW}(s_1) Nullable(s_2, v)
                                                                                                 k \in \text{SN-FOLLOW}(s_1 \cdot s_2)
        NULLABLE(s_1, v) k \in FIRST(s_2)
                  k \in \text{FIRST}(s_1 \cdot s_2)
                                                                                       PRODUCTIVE(s_1) k \in \text{SN-FOLLOW}(s_2)
                     k \in FIRST(s)
                                                                                                 k \in \text{SN-FOLLOW}(s_1 \cdot s_2)
                  k \in \text{FIRST}(f \otimes s)
                                                                                                    k \in \text{SN-FOLLOW}(s)
                                                                                                 k \in \text{SN-FOLLOW}(f \otimes s)
          s = getDef(x) k \in FIRST(s)
                   k \in FIRST(var_x)
                                                                                         s = getDef(x) k \in sn\text{-}follow(s)
        (c) Rules for inclusion in the first set.
                                                                                                  k \in \text{SN-FOLLOW}(var_x)
                                                                                  (d) Rules for inclusion in the should-not-follow set.
\text{NULLABLE}(s_1, v_1) \text{NULLABLE}(s_2, v_2)
                                                                                                  k \in \text{SN-FOLLOW}(s_1) k \in \text{FIRST}(s_2)
                                                     k \in FIRST(s_1) k \in FIRST(s_2)
        \text{HAS-CONFLICT}(s_1 \lor s_2)
                                                          \text{HAS-CONFLICT}(s_1 \lor s_2)
                                                                                                          HAS-CONFLICT(s_1 \cdot s_2)
    HAS-CONFLICT(s_1)
                                                                               HAS-CONFLICT(s_1)
                                                                                                                    \text{HAS-CONFLICT}(s_2)
                                        HAS-CONFLICT(s_2)
  HAS-CONFLICT(s_1 \lor s_2) HAS-CONFLICT(s_1 \lor s_2)
                                                                                                              HAS-CONFLICT(s_1 \cdot s_2)
                                                                          \texttt{HAS-CONFLICT}(s_1 \cdot s_2)
                        HAS-CONFLICT(s)
                                                                             s = getDef(x) HAS-CONFLICT(s)
                     \text{HAS-CONFLICT}(f \otimes s)
                                                                                      HAS-CONFLICT(var_x)
```

Figure 4. Inductive definitions of properties on syntaxes.

(e) Rules for existence of LL(1) conflicts.

the token sequence t :: ts with the value v if and only if $\delta_t(s)$ associates the token sequence ts with the same value v:

$$\forall s. \ PRODUCTIVE(s) \land \neg HAS\text{-}CONFLICT(s) \implies \forall t, ts, v. \ s \vdash t :: ts \leadsto v \iff \delta_t(s) \vdash ts \leadsto v$$

Theorem 11 (Preservation). For any productive LL(1) syntax s and token t, the syntax $\delta_t(s)$ is LL(1). In other words:

$$\forall s. \ \textit{PRODUCTIVE}(s) \land \neg \textit{HAS-CONFLICT}(s) \implies \\ \forall t. \ \neg \textit{HAS-CONFLICT}(\delta_t(s))$$

5.1 Simple Parsing with Derivatives

The derivation operation naturally leads to a parsing algorithm for LL(1) syntaxes:

$$\mathsf{sParse}(s,\langle\rangle) := \mathsf{nullable}(s)$$

$$\mathsf{sParse}(s,t::ts) := \begin{cases} \mathsf{sParse}(\delta_t(s),ts) & \text{if $\mathtt{PRODUCTIVE}(s)$} \\ \mathsf{none} & \text{otherwise} \end{cases}$$

Theorem 12 (Correctness). For any LL(1) syntax s, token sequence ts and value v:

$$sParse(s, ts) = v \iff s + ts \rightsquigarrow v$$

6 Zippy LL(1) Parsing with Derivatives

In this section, we demonstrate that the performance of simple parsing with derivatives for LL(1) syntaxes of Section 5.1 can degrade drastically on certain inputs. To alleviate this problem, we introduce the concept of *focused syntaxes*, which combine a syntax and a *context*. We then show that, using such "zipper" data structure [20], LL(1) parsing with derivatives takes linear time.

6.1 Inefficiency of Simple Parsing with Derivatives

While correct, parsing with derivatives as shown in the previous section is inefficient in practice. There are cases where the performance of the parser will degrade drastically. The reason is that, as we will show, the derivative of a syntax can grow larger than the original syntax. Partially created values, as well as continuation points, will tend to accumulate in the top layers of the syntax. With time, the syntax can grow arbitrarily large, and calls to the derive procedure will take longer and longer. Indeed, it can be shown that the parsing algorithm that we have described in the previous section takes time quadratic in the input size, whereas the typical push-down automaton-based parsing algorithm for LL(1) grammars only takes linear time [3]. Furthermore, simple parsing with derivatives can lead to stack overflows because derivation, when naturally defined as a recursive function, is not tail-recursive.

Example As a simple example to expose the problematic behaviour of the algorithm, we describe a syntax for the language $\{a^nb^n \mid n \in \mathcal{N}\}$. We assign to each recognised sequence the integer value that corresponds to half its length.

The tokens we will consider are a and b, while their respective kinds are A and B. To describe a syntax for this language, we consider the following environment:

$$x \mapsto f \otimes ((elem_A \cdot var_x) \cdot elem_B) \vee \varepsilon_0$$

where $f(((t_1, n), t_2)) = n + 1$

In this environment, the syntax that describes the proposed language is simply var_x . The syntax is LL(1).

To showcase the problematic behaviour, define the following sequence of syntaxes:

$$s_0 := var_x$$
 $s_{i+1} := \delta_a(s_i)$

The first element of the sequence s is the original syntax var_x , while subsequent elements are derivatives of the previous syntax with respect to a. This sequence models the state of the parsing with derivatives algorithm after encountering longer and longer strings of a's. We remark that each time a new a is encountered, additional layers of combinators are added on top of the previous syntax:

$$s_{i+1} = \delta_a(s_i) = f \odot ((\varepsilon_a \cdot s_i) \cdot elem_B)$$

The first layer around s_i , $\varepsilon_a \cdot _$, holds the value of the token that was just consumed. The second layer, $_\cdot$ elem_B indicates that an additional b must follow. Finally, the third layer, $f \odot$ indicates the function to compute the final value. To compute the derivative of s_{i+1} with respect to a, those layers have to be traversed until the syntax s_i inside is reached, at which point the derivative of s_i with respect to a is computed. This recursive process ends when the syntax $elem_A$ is finally encountered within var_x , deep inside all the extra layers of combinators. Finally, all the layers that have been traversed have to be re-applied to obtain the derivative syntax. Computing the derivative of s_i therefore takes time linear in i. In this particular case, the parsing algorithm that we have discussed in the previous section would require time quadratic in the input size. To tackle this phenomenon, we introduce focused syntaxes.

6.2 Focused Syntaxes

A focused syntax is simply a syntax with a focus on one of its nodes, in the spirit of zippers [20]. We define a *focused* syntax as a pair of a syntax s and a stack of *layers* c. Given a focused syntax (s, c), we call s the *focal point* and c the context

Layers are parameterised by two types, the *above* type and the *below* type. We denote by $\mathcal{L}_{T_2}^{T_1}$ the set of all layers with above type T_1 and below type T_2 . For all types T_1 and T_2 , the set of layers $\mathcal{L}_{T_2}^{T_1}$ is defined according to the following rules from Figure 5. Layers tell about the parent node of a syntax:

- apply(f) indicates that the parent node is $f \otimes _$.
- prepend(v) indicates that the parent node is $\varepsilon_v \cdot _$.
- follow-by(s) indicates that the parent node is $_\cdot s$.

Note that they correspond to the layers that can be created by the LL(1) derivation procedure shown in Section 5.

$$\begin{array}{c} f \in T_1 \to T_2 & v : T_1 \\ \\ \text{apply}(f) \in \mathcal{L}_{T_2}^{T_1} & \text{prepend}(v) \in \mathcal{L}_{(T_1, T_2)}^{T_2} \\ \\ \hline s \in \mathcal{S}_{T_2} \\ \\ \hline \text{follow-by}(s) \in \mathcal{L}_{(T_1, T_2)}^{T_1} \end{array}$$

Figure 5. Definition of layers.

The context is a stack of type-aligned layers. For any two consecutive layers in the stack, the below type of the first layer must match the above type of the second layer. For any types T_1 and T_2 , we denote by $C_{T_2}^{T_1}$ the set of all type-aligned contexts where T_1 is the above type of the first layer and T_2 is the below type of the last layer. For all types T we also include the empty stack $\langle \rangle$ in C_T^T . A focused syntax in \mathcal{F}_T is a pair of a syntax $s \in \mathcal{S}_{T'}$ and a context $c \in C_T^{T'}$ for some type T'. We define the function focus to focus the root node of a syntax:

```
focus(s) := (s, \langle \rangle)
```

Conversely, the unfocus function on a focused syntax (s, c) in \mathcal{F}_T unfocuses (s, c) by applying the layers in the context until the context is empty:

```
\begin{split} &\mathsf{unfocus}((s,c)) \coloneqq \mathbf{match} \ c \ \mathbf{with} \\ &| \ \langle \rangle \qquad \longrightarrow s \\ &| \ \mathsf{apply}(f) :: c' \qquad \to \mathsf{unfocus}((f \circledcirc s,c')) \\ &| \ \mathsf{prepend}(v) :: c' \qquad \to \mathsf{unfocus}((\varepsilon_v \cdot s,c')) \\ &| \ \mathsf{follow-by}(s') :: c' \to \mathsf{unfocus}((s \cdot s',c')) \end{split}
```

Definition 13. A focused syntax is LL(1) if its unfocused counterpart is LL(1).

6.3 Operations on Focused Syntaxes

In this section, we define several operations on focused syntaxes, with the goal to define an efficient parsing procedure. The first operation we define on focused syntaxes is plug. The goal of this operation is to obtain a new focused syntax when the focal point reduces down to a value. This happens for instance when the focal point is an ε_v syntax. The function takes as input a value and a context, and returns a new focused syntax. Layers in the context are applied until either a follow-by(s) layer is encountered, or until the context is empty.

plug(v, c) := match c with

```
\begin{split} |\left\langle\right\rangle & \to (\varepsilon_v, \left\langle\right\rangle) \\ |\operatorname{apply}(f) :: c' & \to \operatorname{plug}(f(v), c') \\ |\operatorname{prepend}(v') :: c' & \to \operatorname{plug}((v', v), c') \\ |\operatorname{follow-by}(s) :: c' & \to (s, \operatorname{prepend}(v) :: c') \end{split}
```

Theorem 14. The focused syntax obtained by plugging a value v in a context c is equivalent to (ε_v, c) . Formally: $\forall ts, w$,

The next operation we define is locate, which takes as input a token kind and a focused syntax, and returns an optional focused syntax. The goal of the function is to move the focus towards a syntax node that can start with a given token kind, skipping nullable prefixes as needed.

```
locate(k, (s, c)) :=

if k \in \text{FIRST}(s) then some((s, c))

else match nullable(s) with

| \text{none} \rightarrow \text{none} |

| \text{some}(v) \rightarrow \text{if } c = \langle \rangle \text{ then none} |

else locate(k, \text{plug}(v, c))
```

In case the current focal point starts with the desired kind the current focused syntax is simply returned. Otherwise, the focus is to be moved to a consecutive syntax found within the context, at which point the operation is recursively applied. Note that the operation does not always succeed, and so for two reasons. First, in order to be able to skip the currently focused node, that node must be nullable. Second, the context might be empty, and therefore no consecutive syntax exists.

Theorem 15. When the locate function returns none, the focused syntax can not possibly start with the desired kind.

```
locate(k,(s,c)) = none \implies k \notin FIRST(unfocus((s,c)))
```

Theorem 16. For any focused syntax (s, c) and token kind k, when locate successfully returns a new focused syntax, the new focal point starts with the given token kind k.

```
locate(k,(s,c)) = some((s',c')) \implies k \in FIRST(s')
```

Theorem 17. For any focused syntax (s, c), token t and associated kind k, when locate successfully returns a new focused syntax, then that focused syntax is equivalent for all sequences that start with the token t.

```
\begin{split} & \mathsf{locate}(k,(s,c)) = \mathsf{some}((s',c')) \implies \forall ts,v. \\ & \mathsf{unfocus}((s',c')) \vdash t :: ts \leadsto v \Leftrightarrow \mathsf{unfocus}((s,c)) \vdash t :: ts \leadsto v \end{split}
```

The next operation we consider is pierce. Given a LL(1) syntax s and a token kind k where $k \in FIRST(s)$, the function returns the *context* around the unique $elem_k$ in a left-most position in s. An initial accumulator context is given to the function, and is only built upon by pierce.

```
pierce(k, s, c) := \mathbf{match} \ s \ \mathbf{with}

\mid elem_k \to c

\mid s_1 \lor s_2 \to \mathbf{if} \ k \in \mathsf{FIRST}(s_1) \ \mathbf{then} \ \mathsf{pierce}(k, s_1, c)

\mathbf{else} \ \mathsf{pierce}(k, s_2, c)

\mid s_1 \cdot s_2 \to \mathbf{sec}(k, s_2, c)
```

```
match nullable(s_1) with

| none → pierce(k, s_1, follow-by(s_2) :: c)

| some(v) →

if k \in \text{FIRST}(s_1) then pierce(k, s_1, follow-by(s_2) :: c)

else pierce(k, s_2, prepend(v) :: c)

| f \otimes s' \rightarrow \text{pierce}(k, s', apply(f) :: c)

| var_x \rightarrow \text{pierce}(k, getDef(x), c)
```

The definition of this operation has striking similarities with the definition of derivation on syntaxes that we have previously discussed. The function pierce can be thought of as computing the derivative of an LL(1) syntax, but instead of directly building the resulting syntax, the function returns an equivalent context.

Theorem 18. For any LL(1) focused syntax (s, c) and token kind k where $k \in FIRST(s)$, the following holds:

```
\forall ts, v. \text{ unfocus}((elem_k, pierce(k, s, c))) \vdash ts \leadsto v \iff unfocus((s, c)) \vdash ts \leadsto v
```

Finally, the function derive brings the various operations we have seen so far together. The function takes as argument a token t and an LL(1) focused syntax (s, c). The function returns a new focused syntax (s', c') that corresponds to the derivative of (s, c) with respect to t, or none if the token is not accepted by the focused syntax.

```
derive(t, (s, c)) :=
let k := \text{getKind}(t) in
match locate(k, (s, c)) with
| \text{none} \rightarrow \text{none} |
| \text{some}((s', c')) \rightarrow (\varepsilon_t, \text{pierce}(k, s', c'))
```

The operation first invokes locate to move the focus to a point which starts with the desired kind k, then, using pierce, moves the focus down to the left-most $elem_k$ within that syntax. Once focused on that particular $elem_k$ node, derivation is trivial, as it suffices to replace the focal point by an ε_t node.

Theorem 19. The derive operation preserves the LL(1)-ness of the focused syntax. In other words, for any LL(1) focused syntax (s, c), if its derivation exists, then the resulting focused syntax is also LL(1).

Theorem 20. When the derive operation returns none for a token t (of kind k) and a focused syntax (s, c) then the corresponding unfocused syntax doesn't start with k.

```
derive(t,(s,c)) = none \implies k \notin FIRST(unfocus((s,c)))
```

Theorem 21. For all LL(1) focused syntax (s,c) and token t, if the derivation returns a new focused syntax (s',c'), then (s',c') is the derivative of (s,c) with respect to t.

```
\begin{split} \operatorname{derive}(t,(s,c)) &= \operatorname{some}((s',c')) \implies \forall ts,v. \\ \operatorname{unfocus}((s',c')) &\vdash ts \leadsto v \Leftrightarrow \operatorname{unfocus}((s,c)) \vdash t :: ts \leadsto v \end{split}
```

The final piece of the puzzle is the result operation, which returns the value associated with the empty string by the focused syntax.

```
result((s, c)) := match nullable(s) with

| none \rightarrow none

| some(v) \rightarrow

if c = \langle \rangle then some(v)

else result(plug(v, c))
```

Theorem 22. For all LL(1) focused syntax (s, c):

```
result((s, c)) = nullable(unfocus((s, c)))
```

6.4 Zippy Parsing with Derivatives Algorithm

Using the previous definitions, we can finally present the zippy parsing with derivatives algorithm. Given a focused syntax (s, c) and a token sequence ts, the algorithm returns the value associated with the token sequence, if any.

```
\begin{aligned} \mathsf{parse}((s,c),ts) &\coloneqq \mathbf{match}\ ts\ \mathbf{with} \\ | \left< \right> &\to \mathsf{result}((s,c)) \\ | t :: ts' &\to \\ &\mathbf{match}\ \mathsf{derive}(t,(s,c))\ \mathbf{with} \\ | \ \mathsf{none} &\to \mathsf{none} \\ | \ \mathsf{some}((s',c')) &\to \mathsf{parse}((s',c'),ts') \end{aligned}
```

Theorem 23 (Correctness). The zippy LL(1) parsing with derivatives algorithm is correct. For any LL(1) syntax s, token sequence ts and value v:

```
parse(focus(s), ts) = some(v) \iff s + ts \rightsquigarrow v
```

6.5 Runtime Complexity of Parsers

In this section, we examine the time complexity of the zippy LL(1) parsing with derivatives algorithm. We argue that the algorithm runs in time linear in the number of input tokens (ignoring the cost of applying user-defined functions appearing in the syntax, which typically apply constant-time AST constructors). We rely on two key observations:

- The (non-epsilon) syntaxes stored and manipulated by the algorithm are always subtrees of the original syntax or syntaxes in the global environment. Indeed, no syntaxes are ever created by the algorithm, except for ε_v syntaxes.
- The call to pierce does not enter syntaxes in the environment multiple times. This property follows from the LL(1)-ness of the syntaxes that we consider. In particular, this means that the number of nodes traversed by a single invocation of pierce is bounded by a number which depends uniquely on the syntax.

The complexity can be shown to be linear by amortised analysis using the banker's method [11, Chapter 17]. When adding a layer to the context, we pay an extra 1 time unit for prepend(v) and apply(f) layers, and 2 units for

follow-by(s) layers. The cost of plug operations are entirely covered by the extra units paid.

6.6 Connections to Traditional LL(1) Parsing

The zippy LL(1) parsing with derivatives that we have presented in this section bears striking similarities with the traditional LL(1) parsing algorithm. Immediately, we can observe that both algorithms maintain a stack of rules to be applied on subsequent input. Interestingly, we arrived at that stack rather naturally by introducing a focus within our syntaxes. Furthermore, our derive procedure corresponds to the table-based lookup procedure of the traditional algorithm. Instead of storing the transitions in a table, our transitions are obtained by calling pierce on individual nodes of the syntax. If we were to pre-compute the layers added by pierce for every kind k in the first set of nodes of syntaxes, we would arrive at an almost identical approach (with a new and formal proof of correctness).

7 Coq Proofs

We formalised the parsing with derivatives algorithm with zippy syntaxes in Coq (around 8900 lines). The Coq proofs are freely available at https://github.com/epfl-lara/scallion-proofs. We defined the recursive functions that require non-trivial measures using the Equations library [50]. There are two main parts in the formalism: one to define the functions corresponding to the basic properties of syntaxes, and one to define the parsing algorithm based on zippy syntaxes (and its correctness).

In the first part, we defined for each function the inductive rules as described in Figure 4 and a corresponding propagator network that gives a way to compute the function. We defined a uniform way to specify these rules on syntaxes using the notion of *description* (see file Description.v).

We then made a generic construction that takes a syntax and a description, and builds a propagator network that computes the function corresponding to the description on the syntax. This propagator network has one cell per node in the syntax, and each cell is updated using the inductive rules based on the cells corresponding to the children of the syntax. We proved soundness and completeness of this construction (see DescriptionToFunctionSoundness.v and DescriptionToFunctionCompleteness.v). Here, soundness means that if the network computes a certain value, then this value is actually related to the syntax by the inductive rules. Completeness means that if there exists a value related to the syntax by the inductive rules, the network will compute a value for this syntax (not necessarily the same value, e.g. for a nullable syntax, the network will compute some value v such that NULLABLE(s, v) holds). Our Coq definitions of propagator networks (and their termination guarantees) are general and can be reused independently of this paper and independently of syntaxes.

In the propagator networks, we made use of safe *casts*, i.e. converting a term from a type A to a type B when A = B. This is needed when A and B are not definitionally equal, but only propositionally equal (two notions of equality that Coq distinguishes, with the former being stronger than the latter). For instance, when computing nullable(s) on a syntax s of type T, we look up the cell associated with s in the network, and cast the state of that cell to the type option T. This is possible because we have proven separately that, after constructing the networks and after the computations, the type of this cell is propositionally equal to option T (yet not definitionally equal). To do the soundness and completeness proofs involving these casts, we included the unicity of identity proofs axiom, which states that for any two terms x and y, any two proofs p_1 and p_2 of x = y are equal $(p_1 = p_2)$. This axiom is consistent with the calculus of constructions [10] and is among the weaker extensions that are useful in practice [54] (weaker than proof irrelevance for arbitrary proofs).

In the second part, we defined zippy syntaxes, the functions plug, locate, pierce, derive and proved all the necessary properties to show the correctness of parsing as stated in Theorem 23. In particular, we proved that these functions terminate, that they do not introduce conflicts, and that they produce syntaxes that recognise the expected languages (Theorem 21).

8 Parsing and Printing Combinators

In this section, we discuss the implementation of syntaxes as a parsing and printing combinators framework in Scala. The framework is freely available under an open source license². The Scala implementation closely follows the Coq formalism. For performance reasons, we did not mechanically extract an implementation from the formalisation.

8.1 Syntax Definition

Syntaxes are defined as a generalised algebraic datatype. Each construct of the formalism straightforwardly corresponds to one constructor of the datatype.

```
sealed trait Syntax[A]
case class Eps[A](value: A) extends Syntax[A]
case class Fail[A]() extends Syntax[A]
case class Elem(kind: Kind) extends Syntax[Token]
case class Seq[A, B](1: Syntax[A], r: Syntax[B])
  extends Syntax[A ~ B]
case class Dis[A](1: Syntax[A], r: Syntax[A])
  extends Syntax[A]
case class Map[A, B](f: A => B, i: B => List[A],
  s: Syntax[A]) extends Syntax[B]
sealed trait Rec[A] extends Syntax[A] {
  def inner: Syntax[A] }
object Rec {
  def create[A](syntax: => Syntax[A]) =
    new Rec[A] {
      override lazy val inner = syntax } }
```

²The framework is available at https://github.com/epfl-lara/scallion

The ε_v construct is represented by the Eps(v) constructor, \bot is represented by Fail() and $elem_k$ by Elem(k). The disjunction operator $s_1 \lor s_2$ corresponds to the constructor Dis(s_1 , s_2), while the sequencing operator $s_1 \cdot s_2$ corresponds to the constructor Seq(s_1 , s_2). Pairs are denoted by A \sim B instead of (A, B) for easier pattern matching. The $f \odot s$ construct is represented by the Map constructor. The constructor contains an extra argument for (a subset of) the inverse of the function applied on parsed values. The inverse is solely used for pretty printing. The syntax variables and environment of the formalisation correspond to Rec instances. The syntax associated with the variable is stored in the lazy field inner of the Rec instance.

8.2 Computing Properties of Syntaxes

Properties of syntaxes (productivity, nullability, first sets etc.) are stored as public fields of Syntax instances. In addition to being used for LL(1) checking and parsing, the fields can be accessed by the users of the framework for debugging or error reporting purposes. For instance, the first set of a syntax can be used to suggest fixes in case of parse errors. Propagator networks [46, 52] are used to initialise the fields, as explained in Section 4.5.

The LL(1) property of syntaxes can be checked via a simple method call. In case a syntax is not LL(1), the list of conflicts can be obtained and their root causes identified. Coupled with the enumeration capabilities of the framework, users of the framework can easily get examples of token sequences which lead to conflicts. In our experience, this feature is of great help to programmers.

8.3 Parsing

Parsing is performed via the apply method of Syntax[A]. The method takes as input an Iterator of tokens and returns a value of type ParseResult[A], which can be:

- 1. Parsed(value, descr), which indicates that the given value (of type A) was successfully parsed.
- UnexpectedToken(token, descr), indicating that token was not expected. Values from the input iterator are not consumed beyond that token.
- 3. UnexpectedEnd(descr), which indicates that the end of input was not expected.

In each case, a residual focused syntax descr is also returned. This syntax represents the state at the end of parsing, respectively at the point of error. Importantly, this syntax can be queried and used as any other syntax. For instance, it can be used for error reporting and recovery. Such a syntax is available for free due to our use of parsing with derivatives.

The framework faithfully implements the zippy LL(1) parsing with derivatives presented in Section 6. The methods plug, locate and pierce are tail-recursive, which ensures that the call stack of underlying virtual machine does not

overflow during parsing. The framework also supports memoisation of calls to pierce. The additional layers of context returned by pierce are stored in reverse order for fast concatenation.

8.4 Enumeration and Pretty Printing

Our framework also supports pretty printing, that is, the enumeration of token sequences that would be parsed into given values. To support this feature, the Map combinator accepts an extra argument for the inverse of the function to be applied on produced values. Whenever local Map inverses are correct, all generated pretty printed sequences are guaranteed to parse and generate a given value. Pretty printed representations are enumerated in the order of increasing length, typically resulting in the first having, e.g., the fewest number of parentheses.

8.5 Library of Combinators

A library of useful combinators is offered to programmers, such as repetition combinators (many, many1), repetition with separators combinators (repsep, rep1sep), optional combinator (opt), tagged disjunctions (infix method | |) and many others. Higher level combinators, such as combinators for infix operators with multiple priority levels and associativities are also available in the library. All combinators are expressed in terms of the primitive syntaxes and combinators shown in section 8.1, and have support for pretty printing out of the box.

9 Experimental Evaluation

We compare the performance of the presented zippy LL(1) parsing with derivatives algorithm with the simple (non-zippy) LL(1) parsing with derivatives and with the Scala Parser Combinators [30] library. The latter is a widely adopted parser combinators library in Scala, which uses recursive descent parsing by default, but also supports packrat parsing.

Table 1 shows the performance of the three approaches for parsing JSON files of size ranging from 100KB to 10MB. Each JSON file contains a single large array of objects, each containing several string and array fields. The JSON files were randomly generated using an online JSON generator [40]. The benchmarks were run on a MacBook Pro with Core i7 CPU@2.2GHz and 16 GB RAM, running Scala 2.12.8 and Java 1.8 on the HotSpot™ JVM. We used ScalaMeter [45] as the benchmarking tool. All three approaches were given tokens from the same lexer. Lexing time is not reported. The table reports the mean values of 36 measurements.

The zippy LL(1) parsing with derivatives outperforms the simple variant by orders of magnitude. The speed of the simple LL(1) parsing with derivatives algorithm degrades with the number of tokens, unlike the speed of the zippy

File size (KB)	Tokens	Parse time (ms)				Speed (token/ms)			
		Simple	Zippy	Zippy*	SPC	Simple	Zippy	Zippy*	SPC
100	9649	99.9	3.4	2.8	2.3	96.6	2829.6	3446	4195.2
1000	97821	7069.2	20.4	14.3	19.0	13.8	4804.6	6840.6	5159.3
10000	971501	†	177.2	150.2	166.0	†	5482.5	6468.0	5852.4

Table 1. Performance comparison between simple LL(1) parsing with derivatives (Simple), zippy LL(1) parsing with derivatives (Zippy), zippy LL(1) parsing with derivatives with caching (Zippy*), and Scala Parser Combinators (SPC) for parsing JSON. Entries marked with † encountered a stack overflow. Entries correspond to the mean of 36 measurements on a hot JVM.

variant. Moreover, the simple parsing algorithm encounters a stack overflow on large files.

The performance of the zippy LL(1) parsing with derivatives is comparable to the performance of the recursive descent algorithm implemented by the Scala Parser Combinators library. Our algorithm is faster than recursive descent on larger files when calls to pierce are cached. The recursive descent algorithm however suffers from potential stack overflows when parsing deeply nested structures. Since parsers are often exposed to user inputs, an attacker could exploit this vulnerability to cause crashes, and so with a relatively small input JSON file (as small as 2616 bytes in our tests). Our implementation also offers more comprehensive error reporting and recovery, in part because it does not rely on recursion in the host language.

We also benchmarked the performance of Parseback [51], a recent Scala implementation of the parsing with derivatives algorithm [39] by one of the original authors, with performance optimisations from [1]. The results are not reported in Table 1 as the parser encounters a stack overflow in each of the benchmarks. The largest file we managed to parse with that library was 1387 bytes long, and it took 1388ms.

In addition to the JSON parser, we have developed parsers for several other non-trivial languages. We used the presented framework to build a parser and pretty printer for a first-order logic formulas quasiquoter, a parser and pretty printer for lambda-calculus, a parser for an expression language with infix, prefix and postfix operators, as well as several other examples. In addition, we have used the parser combinators framework in a third-year bachelor compiler construction course with over 40 students. As a semester-long project, students build a compiler for a subset of Scala. Students successfully used the presented framework to build their parsers, and so with reasonable ease, in part thanks to the debugging capabilities of the framework.

10 Related Work

Ford [16] presents *packrat* parsing, a parsing technique for *parsing expression grammars* (PEGs). Packrat parsers are non-ambiguous and guaranteed to run in linear time through heavy use of memoisation but tend to be slower than many other linear-time parsing techniques. Whereas PEGs disallow ambiguities through biased choices, LL(1) approaches such

as ours support detecting ambiguities before parsing starts. We believe that it is better to detect and report ambiguities rather than to hide them. Our combinators also enjoy more natural algebraic properties, with our disjunctions being commutative and associative, which is not the case in PEGs, making the composition of PEGs trickier.

Ramananandro et al. [47] demonstrate the importance of parsers in security and present combinators for building verified high-performance parser for lower-level encodings of data formats. In contrast, we focus on parsing generalisations of context-free grammars. Formally verified parsers are of special interest to verified compilers such as CompCert [35] and CakeML [29]. Koprowski and Binsztok [27] present a formally verified Coq parser interpreter for PEGs. In recent work authors Lasser et al. [32] present a Coq-verified LL(1) parser generator. The generated parser uses the traditional table-based LL(1) algorithm, and relies on fixpoint computations for properties such as nullability, first sets and others. While these works operate at the level PEGs or context-free grammars, our work works on value-aware context-free expressions. As an alternative approach, Jourdan et al. [24] developed a validator (implemented and verified in Coq) for LR(1) parsers. Their approach works by verifying a posteriori that an automaton-based parser faithfully implements a context-free grammar, while we present a general correctness proof of a parser operating directly on context-free expressions. Swierstra and Duponcheel [53] propose parser combinators for LL(1) languages. Due to their approach based on a shallow embedding of combinators, they are unable to check for LL(1) conflicts a priori. The parsing procedure they use is based on lookup tables, as opposed to our parsing approach based on derivatives.

Our implementation supports mutually inverse parsing and pretty printing, which is also present in Rendel and Ostermann [48] based on *syntactic descriptions* and using recursive descent parsing (instead of using derivatives).

Krishnaswami and Yallop [28] propose a type-system for LL(1) context-free expressions. They use the usual conversion to push-down automata for parsing, and rely on codegeneration for good performance. In their approach, the various properties of context-free expressions (nullability, first sets, etc.) are obtained via fixpoint computations, as opposed to our approach based on propagator networks. They

use a weaker definition of *should-not-follow* set (which they call *follow-last* set, abbreviated as FLAST). Their type system is more restrictive than ours as it does not allow nullable expressions to appear on the left of sequences.

Might et al. [39] present a parsing algorithm for context-free expressions based on derivatives. Compared to our paper, their approach is not restricted to only LL(1) expressions, but is applicable to a wider family of context-free expressions. The worse-case complexity of their approach is cubic in general [1], and can be shown to be quadratic for LL(1) expressions. Our approach is limited to LL(1) languages but has guaranteed linear time complexity thanks to the use of a zipper-like data structure. Henriksen et al. [19] show a parsing technique based on derivatives for context-free grammars. They show that their approach is equivalent to Earley's algorithm [14] and argue that parsing with derivatives has deep connections with traditional parsing techniques. In this paper, we reinforce such connection, linking traditional LL(1) parsing to efficient parsing with derivatives.

References

- [1] Michael D. Adams, Celeste Hollenbeck, and Matthew Might. 2016. On the Complexity and Performance of Parsing with Derivatives. In Proceedings of the 37th ACM SIGPLAN Conference on Programming Language Design and Implementation (PLDI '16). ACM, New York, NY, USA, 224–236. https://doi.org/10.1145/2908080.2908128
- [2] Alfred V. Aho, Monica S. Lam, Ravi Sethi, and Jeffrey D. Ullman. 2006. Compilers: Principles, Techniques, and Tools (2nd Edition). Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA.
- [3] Alfred V. Aho and Jeffrey D. Ullman. 1972. The theory of parsing, translation, and compiling. 1: Parsing. Prentice-Hall.
- [4] Fahad Ausaf, Roy Dyckhoff, and Christian Urban. 2016. POSIX Lexing with Derivatives of Regular Expressions. Archive of Formal Proofs (May 2016). http://isa-afp.org/entries/Posix-Lexing.html, Formal proof development.
- [5] Cloudflare Blog. 2019. Incident report on memory leak caused by Cloudflare parser bug. https://blog.cloudflare.com/incident-reporton-memory-leak-caused-by-cloudflare-parser-bug/.
- [6] Anne Brüggemann-Klein and Derick Wood. 1992. Deterministic regular languages. In Annual Symposium on Theoretical Aspects of Computer Science. Springer, 173–184.
- [7] Janusz A Brzozowski. 1964. Derivatives of regular expressions. In *Journal of the ACM*. Citeseer.
- [8] William H Burge. 1975. Recursive programming techniques. (1975).
- [9] John Cocke. 1969. Programming languages and their compilers: Preliminary notes. (1969).
- [10] Thierry Coquand. 1989. Metamathematical investigations of a calculus of constructions. Technical Report.
- [11] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Cliff Stein. 2009. Introduction to Algorithms (Third Edition). MIT Press.
- [12] Nils Anders Danielsson. 2010. Total Parser Combinators. In Proceedings of the 15th ACM SIGPLAN International Conference on Functional Programming (ICFP '10). ACM, New York, NY, USA, 285–296. https://doi.org/10.1145/1863543.1863585
- [13] Franklin Lewis DeRemer. 1969. Practical translators for LR (k) languages. Ph.D. Dissertation. Massachusetts Institute of Technology.
- [14] Jay Earley. 1970. An efficient context-free parsing algorithm. Commun. ACM 13, 2 (1970), 94–102.
- [15] Jeroen Fokker. 1995. Functional parsers. In International School on Advanced Functional Programming. Springer, 1–23.

- [16] Bryan Ford. 2002. Packrat Parsing:: Simple, Powerful, Lazy, Linear Time, Functional Pearl. In Proceedings of the Seventh ACM SIGPLAN International Conference on Functional Programming (ICFP '02). ACM, New York, NY, USA, 36–47. https://doi.org/10.1145/581478.581483
- [17] Bryan Ford. 2004. Parsing Expression Grammars: A Recognition-based Syntactic Foundation. In Proceedings of the 31st ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL '04). ACM, New York, NY, USA, 111–122. https://doi.org/10.1145/964001.964011
- [18] Li Haoyi. 2019. FastParse 2.1.3. http://www.lihaoyi.com/fastparse/.
- [19] Ian Henriksen, Gianfranco Bilardi, and Keshav Pingali. 2019. Derivative Grammars: A Symbolic Approach to Parsing with Derivatives. *Proc.* ACM Program. Lang. 3, OOPSLA, Article 127 (Oct. 2019), 28 pages. https://doi.org/10.1145/3360553
- [20] Gérard Huet. 1997. The zipper. Journal of functional programming 7, 5 (1997), 549–554.
- [21] Graham Hutton. 1992. Higher-order functions for parsing. Journal of functional programming 2, 3 (1992), 323–343.
- [22] Graham Hutton and Erik Meijer. 1996. Monadic parser combinators. (1996).
- [23] Adrian Johnstone and Elizabeth Scott. 1998. Generalised recursive descent parsing and follow-determinism. In *International Conference* on Compiler Construction. Springer, 16–30.
- [24] Jacques-Henri Jourdan, François Pottier, and Xavier Leroy. 2012. Validating LR (1) parsers. In European Symposium on Programming. Springer, 397–416.
- [25] Tadao Kasami. 1966. An efficient recognition and syntax-analysis algorithm for context-free languages. Coordinated Science Laboratory Report no. R-257 (1966).
- [26] Donald E Knuth. 1965. On the translation of languages from left to right. *Information and control* 8, 6 (1965), 607–639.
- [27] Adam Koprowski and Henri Binsztok. 2010. TRX: A formally verified parser interpreter. In European Symposium on Programming. Springer, 345–365
- [28] Neelakantan R. Krishnaswami and Jeremy Yallop. 2019. A Typed, Algebraic Approach to Parsing. In Proceedings of the 40th ACM SIG-PLAN Conference on Programming Language Design and Implementation (PLDI 2019). ACM, New York, NY, USA, 379–393. https: //doi.org/10.1145/3314221.3314625
- [29] Ramana Kumar, Magnus O. Myreen, Michael Norrish, and Scott Owens. 2014. CakeML: A Verified Implementation of ML. In Proceedings of the 41st ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL '14). ACM, New York, NY, USA, 179–191. https://doi.org/10.1145/2535838.2535841
- [30] LAMP EPFL and Lightbend, Inc. 2019. Scala Parser Combinators. https://github.com/scala/scala-parser-combinators.
- [31] Bernard Lang. 1974. Deterministic techniques for efficient nondeterministic parsers. In *International Colloquium on Automata, Lan*guages, and Programming. Springer, 255–269.
- [32] Sam Lasser, Chris Casinghino, Kathleen Fisher, and Cody Roux. 2019. A Verified LL (1) Parser Generator. In 10th International Conference on Interactive Theorem Proving (ITP 2019). Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik.
- [33] Daan Leijen and Erik Meijer. 2001. Parsec: Direct style monadic parser combinators for the real world. (2001).
- [34] Haas Leiß. 1991. Towards Kleene algebra with recursion. In International Workshop on Computer Science Logic. Springer, 242–256.
- [35] Xavier Leroy. 2009. Formal verification of a realistic compiler. Commun. ACM 52, 7 (2009), 107–115.
- [36] P. M. Lewis, II and R. E. Stearns. 1968. Syntax-Directed Transduction. J. ACM 15, 3 (July 1968), 465–488. https://doi.org/10.1145/321466.321477
- [37] Conor McBride. 2001. The Derivative of a Regular Type is its Type of One-Hole Contexts (Extended Abstract).
- [38] Conor McBride and Ross Paterson. 2008. Applicative programming with effects. *Journal of functional programming* 18, 1 (2008), 1–13.

- [39] Matthew Might, David Darais, and Daniel Spiewak. 2011. Parsing with Derivatives: A Functional Pearl. In Proceedings of the 16th ACM SIGPLAN International Conference on Functional Programming (ICFP '11). ACM, New York, NY, USA, 189–195. https://doi.org/10.1145/ 2034773.2034801
- [40] Vazha Omanashvili. 2019. JSON Generator. https://www.json-generator.com. Accessed 2019-11-20.
- [41] Terence Parr. 2013. The definitive ANTLR 4 reference. Pragmatic Bookshelf.
- [42] Terence Parr. 2019. Grammars written for ANTLR v4; expectation that the grammars are free of actions. https://github.com/antlr/grammars-v4/tree/master/json. Accessed 2019-11-22.
- [43] Terence Parr and Kathleen Fisher. 2011. LL(*): the foundation of the ANTLR parser generator. In Proceedings of the 32nd ACM SIGPLAN Conference on Programming Language Design and Implementation, PLDI 2011, San Jose, CA, USA, June 4-8, 2011. 425–436. https://doi.org/10. 1145/1993498.1993548
- [44] Benjamin C. Pierce, Arthur Azevedo de Amorim, Chris Casinghino, Marco Gaboardi, Michael Greenberg, Cătălin Hriţcu, Vilhelm Sjöberg, and Brent Yorgey. 2018. *Logical Foundations*. Electronic textbook. Version 5.5. http://www.cis.upenn.edu/~bcpierce/sf.
- [45] Aleksandar Prokopec. 2019. Scalameter: Automate your performance testing today. https://scalameter.github.io/. Accessed 2019-11-20.
- [46] Alexey Radul. 2009. Propagation networks: A flexible and expressive substrate for computation. (2009).
- [47] Tahina Ramananandro, Antoine Delignat-Lavaud, Cédric Fournet, Nikhil Swamy, Tej Chajed, Nadim Kobeissi, and Jonathan Protzenko. 2019. EverParse: Verified Secure Zero-Copy Parsers for Authenticated Message Formats. In 28th USENIX Security Symposium, USENIX Security 2019, Santa Clara, CA, USA, August 14-16, 2019. 1465–1482. https://www.usenix.org/conference/usenixsecurity19/ presentation/delignat-lavaud

- [48] Tillmann Rendel and Klaus Ostermann. 2010. Invertible Syntax Descriptions: Unifying Parsing and Pretty Printing. In Proceedings of the Third ACM Haskell Symposium on Haskell (Haskell '10). ACM, New York, NY, USA, 1–12. https://doi.org/10.1145/1863523.1863525
- [49] Elizabeth Scott and Adrian Johnstone. 2010. GLL parsing. Electronic Notes in Theoretical Computer Science 253, 7 (2010), 177–189.
- [50] Matthieu Sozeau and Cyprien Mangin. 2019. Equations reloaded: high-level dependently-typed functional programming and proving in Coq. Proceedings of the ACM on Programming Languages 3, ICFP (2019), 86.
- [51] Daniel Spiewak. 2018. Parseback. https://github.com/djspiewak/ parseback.
- [52] Guy L Steele Jr. 1980. The definition and implementation of a computer programming language based on constraints. (1980).
- [53] S Doaitse Swierstra and Luc Duponcheel. 1996. Deterministic, errorcorrecting combinator parsers. In *International School on Advanced Functional Programming*. Springer, 184–207.
- [54] The Coq Team 2019. The Logic of Coq. https://github.com/coq/coq/ wiki/The-Logic-of-Coq.
- [55] Dmitriy Traytel. 2015. Derivatives of Logical Formulas. Archive of Formal Proofs (May 2015). http://isa-afp.org/entries/Formula_Derivatives. html, Formal proof development.
- [56] Dmitriy Traytel and Tobias Nipkow. 2014. Decision Procedures for MSO on Words Based on Derivatives of Regular Expressions. Archive of Formal Proofs (June 2014). http://isa-afp.org/entries/MSO_Regex_ Equivalence.html, Formal proof development.
- [57] Philip Wadler. 1985. How to replace failure by a list of successes a method for exception handling, backtracking, and pattern matching in lazy functional languages. In Conference on Functional Programming Languages and Computer Architecture. Springer, 113–128.
- [58] Daniel H Younger. 1967. Recognition and parsing of context-free languages in time n^3 . *Information and control* 10, 2 (1967), 189–208.