Assignment 4 - MAT257

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October 7, 2013

1. Munkres Question 1

   Since $f'(a;u) = Df(a) \cdot u$, then $f'(a;cu) = Df(a) \cdot cu = c(Df(a) \cdot u) = cf'(a;u)$.

2. Munkres Question 2

   a) Note that the function $f : \mathbb{R}^2 \to \mathbb{R}$ can be rewritten like so:

   $$f((x,y)) = \begin{cases} \frac{xy}{\|(x,y)\|^2} & \text{if } (x,y) \neq 0 \\ 0 & \text{if } (x,y) = 0 \end{cases}$$

   Let $u = (h,k)$ be an arbitrary, non-zero vector. The directional derivative of f at origin with respect to u will then be:

   $$\begin{aligned} f'(0;u) &= \lim_{t \to 0} \frac{f(0+tu) - f(0)}{t} \\ &= \lim_{t \to 0} \frac{thtk}{\|t(h,k)\|^2} \frac{1}{t} \\ &= \lim_{t \to 0} \frac{t^2 hk}{t^3 \|(h,k)\|^2} \\ &= \lim_{t \to 0} \frac{hk}{t\|(h,k)\|^2} \end{aligned}$$

   This limit will only exist when h or k are zero. When this is the case, the limit will evaluate to zero.

   b) Since $D_1 f$ and $D_2 f$ are the directional derivatives of elementary unit vectors then k or h will be zero, and thus both of these will be zero.

   c) The function is not differentiable since some of the directional derivatives don't exist.

   d) The function is not continuous, since if $x = y$ then the limit of the function $\phi(x) = f((x,x)) = \frac{1}{2}$ as $x \to 0$ is $\frac{1}{2}$, however if $-x = y$ then the limit of the function $\phi(x) = f((x,-x)) = -\frac{1}{2}$ as $x \to 0$ is $-\frac{1}{2}$.

3. Munkres Question 3

   a) Note the function can be written as follows

   $$f((x,y)) = \begin{cases} \frac{x^2y^2}{x^2y^2 + (y-x)^2)} & \text{if } (x,y) \neq 0 \\ 0 & \text{if } (x,y) = 0 \end{cases}$$

   Let $u = (h,k)$ be an arbitrary, non-zero vector. The directional derivative of f at origin with respect to u will then be:

   $$f'(0;u) = \lim_{t \to 0} \frac{f(0+tu) - f(0)}{t}$$

$$= \lim_{t \to 0} \frac{th^2k^2}{t^2h^2k^2 + (k-h)^2}$$

Now assuming that $(k-h)^2 \neq 0$ then $\lim_{t\to 0} t^2h^2k^2 + (k-h)^2 = (k-h)^2 \neq 0$ so the limit becomes:

$$\begin{aligned}
&= \lim_{t \to 0} \frac{th^2k^2}{t^2h^2k^2 + (k-h)^2} \\
&= \frac{\lim_{t\to 0} th^2k^2}{\lim_{t\to 0} t^2h^2k^2 + (k-h)^2} \\
&= 0
\end{aligned}$$

Assuming that $(k-h)^2 = 0$ then $k = h$ so the limit becomes:

$$\begin{aligned}
&= \lim_{t \to 0} \frac{th^2k^2}{t^2h^2k^2 + (k-h)^2} \\
&= \lim_{t \to 0} \frac{tk^4}{t^2k^4} \\
&= \lim_{t \to 0} \frac{1}{t}
\end{aligned}$$

Which does not exist. Therefore the directional derivatives only exist when $k \neq h$.

b) Since $D_1 f$ and $D_2 f$ are the directional derivatives of elementary unit vectors then $k \neq h$, and thus both of these will be zero.

c) The function is not differentiable since some of the directional derivatives don't exist.

d) The function is continuous.

4. Munkres Question 4

   a) Note the function is to be written as follows

$$f((x,y)) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x,y) \neq 0 \\ 0 & \text{if } (x,y) = 0 \end{cases}$$

   Let $u = (h, k)$ be an arbitrary, non-zero vector. The directional derivative of f at origin with respect to u will then be:

$$\begin{aligned}
f'(0;u) &= \lim_{t \to 0} \frac{f(0+tu) - f(0)}{t} \\
&= \lim_{t \to 0} \frac{t^3h^2}{t^2(h^2+k^2)} \\
&= \lim_{t \to 0} \frac{th^2}{h^2+k^2}
\end{aligned}$$

   Which is always zero

   b) All directional derivatives exist, therefore $D_1 f$ and $D_2 f$ must exist.

   c) The function is differentiable since all of the partial derivatives are continuous.

   d) The function is continuous since it's differentiable.

5. Additional work, question 1

a) Note that the function $f : \mathbb{R}^2 \to \mathbb{R}$ can be rewritten like so:

$$f((x,y)) = \begin{cases} \frac{x|y|}{\|(x,y)\|} & \text{if } (x,y) \neq 0 \\ 0 & \text{if } (x,y) = 0 \end{cases}$$

Let $u = (h,k)$ be an arbitrary, non-zero vector. The directional derivative of f at origin with respect to u will then be:

$$\begin{aligned} f'(0;u) &= \lim_{t\to 0} \frac{f(0+tu) - f(0)}{t} \\ &= \lim_{t\to 0} \frac{th|tk|}{\|tu\|}\frac{1}{t} \\ &= \lim_{t\to 0} \frac{t|t|h|k|}{t|t|\|u\|} \\ &= \frac{h|k|}{\|u\|} \end{aligned}$$

Which is defined for all $u \neq 0$. Therefore all the directional derivatives exist for f.

b) If f were differentiable at the origin then $f(0;u) = ah + bk$ for some constant $a, b \in \mathbb{R}$, however there are no such constants that satisfy $ah + bk = \frac{h|k|}{\|u\|}$. Therefore f is not differentiable at the origin.

6. Additional work, question 2

a) $f(1,1) = 1^3 + 1^3 + 1 - 3 = 0$

b) Assuming that $f_y(x,y)$ means $D_2 f(x,y) = f((x,y);e_2)$, then we have;

$$\begin{aligned} f'((x,y);e_2) &= \lim_{t\to y} \frac{f((x,t)) - f((x,y))}{t-y} \\ &= \lim_{t\to y} \frac{x + t^3 + t - 3 - x - y^3 - y + 3}{t-y} \\ &= \lim_{t\to y} \frac{t^3 - y^3 + t - y}{t-y} \end{aligned}$$

Application of l'Hopital's rule gives:

$$\begin{aligned} \lim_{t\to y} \frac{t^3 - y^3 + t - y}{t-y} &= \lim_{t\to y} \frac{3t^2+1}{1} \\ &= 3y^2 + 1 \end{aligned}$$

Which is independent of x, so we only need to consider $y \in [0,2]$. Since $D_2 f((x,y))$ is an up facing parabola with vertical shift of $+1$, it will be greater than or equal than one over the entire number line. Therefore $D_2 f((x,y)) \geq 1$ for $(x,y) \in [0,2]^2$

c) From the above, we have that the functions $\phi(x) = f((x,0))$ and $\phi'2(x) = f((x,2))$ have no critical points, therefore we only need to show that $\phi(1-h), \phi(1+h) \leq -1$ and $\phi'(1-h), \phi'(1-h) \geq 1$ for some h. Taking $h = 1$ yields:

$$\begin{aligned} \phi(0) &= -3 \leq -1 \\ \phi(2) &= -1 \leq -1 \\ \phi'(0) &= 7 \geq 1 \\ \phi'(2) &= 9 \geq 1 \end{aligned}$$