Assignment 2 - MAT257

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1. Munkres, §3, Question 8
2. Munkres, §3, Question 9
3. Additional Work, Question 1

   Consider first arbitrary unions of open sets. Suppose S is the set of sets we're unioning. Since for every $x \in \cup S$ there is a $s \in S$ such that $x \in s$, and since all $s \in S$ is open, there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subset s$. Since $s \subseteq \cup S$ this implies for every $x \in \cup S$ there's an $\epsilon > 0$ such that $B(x, \epsilon) \subset \cup S$. Therefore, arbitrary unions of open sets are open.

   For finite intersections we note that if $x \in \cap S$ then for all $s \in S$ we have $x \in s$. Since all these sets are open, for all $s \in S$ there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subset s$. If we take the minimum of all these epsilons, call it ϵ', we have for all $s \in S$ that $B(x, \epsilon') \subset s$ since ϵ' is smaller than or equal to any of the epsilons. This is proof that there exists an $\epsilon > 0$ such that for all $x \in \cap S$ then $B(x, \epsilon) \subset \cap S$, since we simply take ϵ'.

   If we allow for arbitrary intersections of open sets it's possible to create a closed set. Consider for example the set of all sets that contain 0, which we will call S. Since the inclusion of 0 in a set does not imply the inclusion of any other points, for any point $x \neq 0$ there's a set in S that does not contain x. This implies that $x \notin \cap S$. So the only vector in the intersection is 0, which makes it a closed set.

   To prove the similar assertions for intersections and unions of closed sets, note that $\overline{S \cup S'} = \overline{S} \cap \overline{S'}$ and $\overline{S \cap S'} = \overline{S} \cup \overline{S'}$. Applying the rules we just proved we can see that arbitrary unions of the complement of sets is the same as arbitrary intersections of the set, so it directly implies that arbitrary intersections of closed sets are closed. The same goes for finite unions.

4. Additional Work, Question 2

   Let X and Y be metric spaces with metrics d_X and d_Y respectively. Let $f : A \to Y$ where $A \subset X$.

   If x_0 is an isolated point, then for some $\delta > 0$ there are no points in $B(x_0, \delta)$ other than x_0. Therefore, f is continuous at x_0 because the assertion $\forall x \in X \; . \; d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0))$ is vacuously true.

   Suppose for the rest of this section that x_0 is not an isolated point. In that case, x_0 must be a limit point.

   Assume that f is continuous at x_0. This means, by definition, that for any open subset V of Y containing $f(x_0)$ then there is an open subset U of A containing x such that $f(U) \subset V$. The statement "$f(x) \to f(x_0)$ as $x \to x_0$" means for any open subset V of Y containing $f(x_0)$ then there is an open subset U of X containing x such that $x \neq x_0 \wedge x \in U \cap A \implies f(x) \in V$.

5. Additional Work, Question 3