Assignment 5 - MAT257

David Knott
Student #999817685

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1. Munkres §2.6, Question 1

   Let $B = [0, 0]$. B is the derivative of f at $(0, 0)$ if the following holds:

   $$\frac{f(h) - f(0) - B \cdot h}{|h|} \to 0 \text{ as } h \to 0$$

   After expending out the terms we find that the above is equivalent to the following:

   $$\frac{|h_1 \cdot h_2|}{|h|} \to 0 \text{ as } h \to 0$$

   This can be proven with a simple delta-epsilon proof. Given any $\epsilon > 0$, if we take $\delta = \epsilon$ then given $k \in C(0, \delta)$ assuming that $k_1 \geq k_2$ then for some $0 \leq \gamma \leq 1$ we have $k_1 = \gamma k_2$. Also, because $k_1 \geq k_2$ then $|k| = k_1$. This implies the following:

   $$\frac{|k_1 \cdot k_2|}{|k|} = \frac{\gamma |k_1| \cdot |k_1|}{|k_1|} = \gamma |k_1|$$

   Since $|k| < \delta$ this implies $\gamma |k_1| < \delta$. Since $\delta = \epsilon$ we have that $\frac{|k_1 \cdot k_2|}{|k|} < \epsilon$. The case when $k_2 \geq k_1$ is essentially the same.

   The function is not of class C_1 for any neighborhood of 0, since for any $\epsilon > 0$ we have $(0, \epsilon) \in C(0, 2\epsilon)$. If we take $D_1 f((0, \epsilon)) = f'((0, \epsilon); e_1)$ then we must evaluate the limit:

   $$\lim_{t \to 0} \frac{f((0, \epsilon) + te_1) - f((0, \epsilon))}{t} = \lim_{t \to 0} \frac{|t\epsilon|}{t} = \lim_{t \to 0} \operatorname{sgn}(t)\epsilon$$

   Which doesn't exist since epsilon is positive and the sign of t does not converge to any value at zero. This means for any neighborhood of 0 there exists a point of discontinuity in the 1st partial derivative. Therefore no neighborhood around 0 is of class C_1.

2. Munkres §2.6, Question 2

   (a) Note that $f'(0) = \lim_{t \to 0} \frac{f(0+t) - f(0)}{t} = \lim_{t \to 0} \frac{t^2 \sin(1/t)}{t} = \lim_{t \to 0} t \sin(1/t)$. Since sine is bounded and t goes to zero, by the squeeze theorem the whole limit goes to zero. Therefore $f'(0) = 0$.

   (b) Since $t \neq 0$ we can apply the chain rule and the product rule to yield $f'(t) = 2t \sin(1/t) - \cos(1/t)$

   (c) If f' is continuous then $\lim_{t \to 0} f'(t) = 0$. However, if $\epsilon = 1/2$ then for any $\delta > 0$ there exists an $|x| < \delta$ such that $|2x \sin(1/x) - \cos(1/x)| > \epsilon$. This is because if we take x small enough the term $|2x \sin(1/x)|$ becomes negligible, but $|\cos(1/x)|$ is still greater than 1/2.

   (d) Since the derivative is not continuous at 0 the function is differentiable but not of class C_1

3. Munkres §2.6, Question 4

   Logically I'm parsing this problem like so:

   $$\exists \delta > 0. \exists \epsilon > 0. \forall j \in [1, m]. \forall m \in C(a, \delta). D_j f(m) \text{ exists} \land |D_j f(m)| < \epsilon \implies f \text{ is continuous at } a$$

   Which is the pedantic way to say that there's a delta sized neighborhood around a such that all the partial derivatives exist and are bounded by some epsilon. I intent to prove the contrapositive, which looks like so:

   $$f \text{ is discontinuous at } a \implies \forall \delta > 0. \forall \epsilon > 0. \exists j \in [1, m]. \exists m \in C(a, \delta). D_j f(m) \text{ doesn't exist} \lor |D_j f(m)| \geq \epsilon$$

   Which translates to: if f is discontinuous at a then for any neighborhood around a there is a point in the neighborhood where a partial derivative doesn't exist or given any epsilon there's a point in the neighborhood where a partial derivative's magnitude is greater than that epsilon.

   So assuming that f is discontinuous at a, there are three possibilities for the behavior of f around a:

   i. The discontinuity of f at a is a removable discontinuity. Either it doesn't exist or it's a value that's unequal to the limit as $h \to a$. In this case the derivative may be bounded, but it won't exist at a.
   ii. The discontinuity of f at a is a jump discontinuity. In this case limits approaching a from different directions or paths will exist but yield opposing numbers. The derivative may be bounded, but it won't exist at a for this reason.
   iii. The discontinuity of f at a is an asymptotic discontinuity, which is to say for any neighborhood around a and any number $\gamma > 0$ there's an m in that neighborhood such that $|f(m)| > \gamma$. In this case the derivative is not bounded nor does it exist at a, so both of the conditions are satisfied.

   This covers all the cases, and each one implies the consequent. Since this proves the contrapositive, the original implication follows.

4. Munkres §2.6, Question 10

   (a)
   (b)
   (c)
   (d)

5. Munkres §2.7, Question 1

   From the chain rule, we have that if $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^p$ then for some point $a \in \mathbb{R}^n$, if $f(a) = b$ then $D(g \circ f)(a) = Dg(b) \cdot Df(a)$

6. Additional Work, Question 1

   (a)
   (b)
   (c)
   (d)
   (e)
   (f)