Assignment 14 - MAT257

David Knott
Student #999817685

September 18, 2013

1. Munkres, §1.1, Question 1

a) Consider the given $\|cx + dy\| \geq 0$ where $x, y \neq 0$, $c = \frac{1}{\|x\|}$ and $d = \frac{1}{\|y\|}$. If $\|x\| = \sqrt{\langle x, x\rangle}$ then we have:

$$\begin{aligned}
0 &\leq \|cx + dy\| \iff \\
0 &\leq \sqrt{\langle cx + dy, cx + dy\rangle} \iff \\
0 &\leq \sqrt{\langle cx, cx + dy\rangle + \langle dy, cx + dy\rangle} \iff \\
0 &\leq \sqrt{\langle cx, cx\rangle + \langle cx, dy\rangle + \langle dy, cx\rangle + \langle dy, dy\rangle} \iff \\
0 &\leq \sqrt{c^2\langle x, x\rangle + 2cd\langle x, y\rangle + d^2\langle y, y\rangle} \iff \\
0 &\leq \sqrt{c^2\|x\|^2 + 2cd\langle x, y\rangle + d^2\|y\|^2} \iff \\
0 &\leq \sqrt{2 + 2cd\langle x, y\rangle} \iff \\
0 &\leq 2 + 2cd\langle x, y\rangle \iff \\
2cd\langle x, y\rangle &\leq 2 \iff \\
\langle x, y\rangle &\leq \frac{1}{cd} \iff \\
\langle x, y\rangle &\leq \|x\| \cdot \|y\|
\end{aligned}$$

If x or y is zero we have equality.

b) Note that $\|x+y\|^2 = \langle x+y, x+y\rangle = 2\langle x, y\rangle + \|x\|^2 + \|y\|^2$. From (a) we have that $\langle x, y\rangle \leq \|x\| \cdot \|y\|$. Starting with this inequality, we find:

$$\begin{aligned}
\langle x, y\rangle &\leq \|x\| \cdot \|y\| \iff \\
2\langle x, y\rangle &\leq 2\|x\| \cdot \|y\| \iff \\
2\langle x, y\rangle + \|x\|^2 + \|y\|^2 &\leq 2\|x\| \cdot \|y\| + \|x\|^2 + \|y\|^2 \iff \\
\|x + y\|^2 &\leq 2\|x\| \cdot \|y\| + \|x\|^2 + \|y\|^2
\end{aligned}$$

By factoring the right side in terms of $\|x\|$ and $\|y\|$ then square rooting both sides we conclude:

$$\begin{aligned}
\|x + y\|^2 &\leq 2\|x\| \cdot \|y\| + \|x\|^2 + \|y\|^2 \iff \\
\|x + y\|^2 &\leq (\|x\| + \|y\|)^2 \iff \\
\|x + y\| &\leq \|x\| + \|y\|
\end{aligned}$$

c) We approach this the same way as (b). Note that $\|x-y\|^2 = \langle x-y, x-y\rangle = -2\langle x, y\rangle + \|x\|^2 + \|y\|^2$. From (a) we have that $\langle x, y\rangle \leq \|x\| \cdot \|y\|$. Starting with this inequality, we find:

$$\langle x, y\rangle \leq \|x\| \cdot \|y\| \iff$$

$$
\begin{aligned}
-2\langle x,y\rangle &\geq -2\|x\|\cdot\|y\| \iff \\
-2\langle x,y\rangle + \|x\|^2 + \|y\|^2 &\geq -2\|x\|\cdot\|y\| + \|x\|^2 + \|y\|^2 \iff \\
\|x-y\|^2 &\geq -2\|x\|\cdot\|y\| + \|x\|^2 + \|y\|^2
\end{aligned}
$$

By factoring the right side in terms of $\|x\|$ and $\|y\|$ then square rooting both sides we conclude:

$$
\begin{aligned}
&\|x-y\|^2 \geq -2\|x\|\cdot\|y\| + \|x\|^2 + \|y\|^2 \iff \\
&\|x-y\|^2 \geq (\|x\| - \|y\|)^2 \iff \\
&\|x-y\| \geq \|x\| - \|y\|
\end{aligned}
$$

2. Munkres, §1.1, Question 3

   Suppose, for the sake of contradiction, that the sup norm is derived from an inner product. Then, from the polarization identity, we have:

$$\langle x,y\rangle = \frac{|x+y|^2 - |x-y|^2}{4}$$

   To prove that the sup norm is not derived from an inner product it suffices to prove that the above definition violates one of the inner product properties. Indeed for the vectors $x=(1,0)$, $y=(0,1)$ and $z=(1,1)$ we find that:

$$\langle x+y,z\rangle \neq \langle x,z\rangle + \langle y,z\rangle$$

   Because

$$
\begin{aligned}
\langle x+y,z\rangle &= \frac{|x+y+z|^2 - |x+y-z|^2}{4} \\
&= \frac{|(2,2)|^2 - |(0,0)|^2}{4} \\
&= \frac{2^2}{4} \\
&= 1
\end{aligned}
$$

   But

$$
\begin{aligned}
\langle x,z\rangle + \langle y,z\rangle &= \frac{|x+z|^2 - |x-z|^2 + |y+z|^2 - |y-z|^2}{4} \\
&= \frac{|(2,1)|^2 - |(0,-1)|^2 + |(1,2)|^2 - |(-1,0)|^2}{4} \\
&= \frac{2^2 - 1^2 + 2^2 - 1^2}{4} \\
&= \frac{3}{2}
\end{aligned}
$$

   Therefore, the sup norm is not derived from an inner product.

3. Munkres, §1.1, Question 4

   b) For this question we will work backwards as to benefit from the generalization achieved by b. Consider the following:

$$\langle\ ,\ \rangle_{a,b,c} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

$$\langle x,y\rangle_{a,b,c} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= ax_1y_1 + b(y_1x_2 + x_1y_2) + cx_2y_2$$

$$\langle x, x\rangle_{a,b,c} = a(x_1)^2 + 2bx_1x_2 + c(x_2)^2$$

Despite the fact that it's not the square root, I will call the inner product of a vector with itself the vector's *norm*, for the lack of a better term.

To prove the iff assertion, we will first assume that the function $\langle x, y\rangle_{a,b,c}$ is a inner product of $\mathbb{R}^2$, then show that this implies $b^2 - ac < 0$ and $a > 0$ and $c > 0$. Note that $b^2 - ac < 0$ and $a > 0$ imply $c > 0$

Given the inner product $\langle x, y\rangle_{a,b,c}$ consider the point $x \in \mathbb{R}^2$ where $x = (x_1, x_2)$ and $x \neq 0$.

If $x_1 = 0$ then $x_2 \neq 0$, and the norm of x will be:

$$\begin{aligned}\langle x, x\rangle_{a,b,c} &= a(x_1)^2 + 2bx_1x_2 + c(x_2)^2 \\ &= c(x_2)^2\end{aligned}$$

Since the norm of x will be positive for all values $x \neq 0$ and x_2^2 must be positive, this implies c must also be positive. Repeating this process for $x_2 = 0$ yields the implication that a must also be only positive. Thus we conclude that both $a > 0$ and $c > 0$.

For the case that $x_1, x_2 \neq 0$, consider the function:

$$f_x : \mathbb{R} \to \mathbb{R}$$

$$\begin{aligned}f_x(t) &= \langle (x_1, t), (x_1, t)\rangle_{a,b,c} \\ &= a(x_1)^2 + 2bx_1t + ct^2\end{aligned}$$

Which parameterizes the norm of all vectors who's x_1 component is that of x. Since x_1 is non-zero the function will never take the norm of $(0, 0)$. This means the function will always be positive. Since f_x is an always positive polynomial with respect to t, it's discriminant should be less than zero. Taking the discriminant yields:

$$\begin{aligned}\Delta_t &= 4b^2(x_1)^2 - 4ac(x_1)^2 \\ &< 0\end{aligned}$$

Dividing both sides by $4(x_1)^2$ gives:

$$b^2 - ac < 0$$

We get the same result if we define $f_x(t) = \langle (t, x_2), (t, x_2)\rangle_{a,b,c}$

For the converse argument, we assume $b^2 - ac < 0$, $a > 0$ and $c > 0$. To prove that $\langle x, y\rangle_{a,b,c}$ is an inner product we show that the following properties hold:

i. $\langle x, y\rangle = \langle y, x\rangle$
ii. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
iii. $\langle cx, y\rangle = c\langle x, y\rangle = \langle x, cy\rangle$
iv. $\langle x, x\rangle > 0$ if $x \neq 0$

The property i. is equivalent to the assertion $\langle x, y\rangle - \langle y, x\rangle = 0$. Expanding this out, we get:

$$\begin{aligned}0 &= \langle x, y\rangle - \langle y, x\rangle \\ &= ax_1y_1 + by_1x_2 + bx_1y_2 + cx_2y_2 \\ &\quad - (ay_1x_1 + bx_1y_2 + by_1x_2 + cy_2x_2)\end{aligned}$$

All the terms cancel out to give 0.
For ii., recall the matrix definition $\langle x, y\rangle = x \cdot \mathbf{C} \cdot y$ where $\mathbf{C} = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ and x and y are in column and row representations respectively. Applying matrix properties on $\langle x + y, z\rangle$ gives:

$$\begin{aligned}\langle x + y, z\rangle &= (x + y) \cdot \mathbf{C} \cdot z \\ &= x \cdot \mathbf{C} \cdot z + y \cdot \mathbf{C} \cdot z \\ &= \langle x, z\rangle + \langle y, z\rangle\end{aligned}$$

By again inheriting matrix multiplication properties, we prove iii. like so:

$$\begin{aligned}c\langle x, y\rangle &= c(x \cdot \mathbf{C} \cdot y) \\ &= (cx) \cdot \mathbf{C} \cdot y \\ &= \langle cx, y\rangle \\ c\langle x, y\rangle &= c(x \cdot \mathbf{C} \cdot y) \\ &= x \cdot \mathbf{C} \cdot (cy) \\ &= \langle x, cy\rangle\end{aligned}$$

iv. is a bit trickier, and we fall back on the function $f_x(t)$ from the proof of the converse. If we let $x \in \mathbb{R}^2$ and $x \neq 0$, then

a) Since $2 > 0$, $1 > 0$ and $(-1)^2 - 2 \cdot 1 = -1 < 0$, by application of (b) we have that $\langle x, y\rangle$ is an inner product of $\mathbb{R}^2$

4. Additional work. Question 1

a) Consider the integral:

$$\begin{aligned}\int_a^b \big(f(t) - \lambda g(t)\big)^2 \, dt &= \int_a^b f(t)^2 \, dt - \lambda \int_a^b f(t)g(t) \, dt + \lambda^2 \int_a^b g(t)^2 \, dt \\ &\geq 0\end{aligned}$$

Since it's the integral of a square. It is also a polynomial in terms of λ. Since it has at most one root, then it's discriminant in terms of λ must be non-positive. This means we have:

$$\begin{aligned}\Delta_\lambda &\leq 0 \iff \\ \left(\int_a^b f(t)g(t) \, dt\right)^2 - \left(\int_a^b f(t)^2 \, dt\right) \cdot \left(\int_a^b g(t)^2 \, dt\right) &\leq 0 \iff \\ \left(\int_a^b f(t)g(t) \, dt\right)^2 &\leq \left(\int_a^b f(t)^2 \, dt\right) \cdot \left(\int_a^b g(t)^2 \, dt\right)\end{aligned}$$

b) Suppose $f = \lambda g$, then we have:

$$\begin{aligned}\left(\int_a^b \lambda g(t)^2 \, dt\right)^2 &\leq \left(\int_a^b \lambda^2 g(t)^2 \, dt\right) \cdot \left(\int_a^b g(t)^2 \, dt\right) \iff \\ \lambda^2\left(\int_a^b g(t)^2 \, dt\right)^2 &\leq \lambda^2\left(\int_a^b g(t)^2 \, dt\right) \cdot \left(\int_a^b g(t)^2 \, dt\right)\end{aligned}$$

And equality is obviously true.

Now instead suppose that, for some continuous f, g, we have:

$$\left(\int_a^b f(t)g(t)\ dt\right)^2 = \left(\int_a^b f(t)^2\ dt\right) \cdot \left(\int_a^b g(t)^2\ dt\right)$$

Note that since this implies the discriminant for the polynomial from part (a) is zero, this means the polynomial has exactly one root. In other words, there exists a λ such that:

$$\int_a^b \big(f(t) - \lambda g(t)\big)^2\ dt = 0$$

To continue we must prove that $\int f^2 = 0 \implies f = 0$ if f is a continuous function.
To do this suppose, for contradiction, that there is some f such that $\int f^2 = 0$ but $f \neq 0$. Hence, there is some $\alpha \in [a, b]$ such that $f(\alpha) \neq 0$. Since f is continuous there will be some ϵ-neighborhood around α such that for all $\delta \in [\alpha - \epsilon, \alpha + \epsilon]$ we have $f(\delta) \neq 0$.
Since the lower bound on any integral is the lower sum for any partition of $[a, b]$, we can simply choose the partition $\{a, \alpha - \epsilon/2, \alpha, b\}$. It's easy to see that the infimum of f within the interval $[\alpha - \epsilon/2, \alpha]$ will be non-zero, and hence f^2 will be positive and hence the lower bound on the integral will be positive. This contradicts the assumption that $\int f^2 = 0$.
So, this means that:

$$\begin{aligned}
\int_a^b \big(f(t) - \lambda g(t)\big)^2\ dt = 0 &\implies \\
f(t) - \lambda g(t) = 0 &\implies \\
f(t) = \lambda g(t)&
\end{aligned}$$

c) Consider the functions f, g such that if $x \in \big((k-1), k\big)$ for some $k = 1, 2, \ldots, n$:

$$f(x) = a_k$$

$$g(x) = b_k$$

Note that for f (and likewise, g), integrating over $[0, n]$ yields:

$$\begin{aligned}
\int_0^n f(t)\ dt &= \sum_{k=1}^n \int_{k-1}^k f(t)\ dt \\
&= \sum_{k=1}^n a_k \\
\int_0^n f(t)^2\ dt &= \sum_{k=1}^n \int_{k-1}^k f(t)^2\ dt \\
&= \sum_{k=1}^n (a_k)^2
\end{aligned}$$

Furthermore, we have:

$$\int_0^n f(t) \cdot g(t)\ dt = \sum_{k=1}^n \int_{k-1}^k f(t) \cdot g(t)\ dt$$

$$= \sum_{k=1}^{n} a_k \cdot b_k$$

So from our proven inequalities, we find that:

$$\left(\int_0^n f(t)g(t)\ dt \right)^2 \leq \left(\int_0^n f(t)^2\ dt \right) \cdot \left(\int_0^n g(t)^2\ dt \right) \iff$$
$$\left(\sum_{k=1}^{n} a_k \cdot b_k \right)^2 \leq \left(\sum_{k=1}^{n} (a_k)^2 \right) \cdot \left(\sum_{k=1}^{n} (b_k)^2 \right)$$

5. Additional work. Question 2

a) First assume that $\langle x, y \rangle = \langle T(x), T(y) \rangle$ for all $x, y \in \mathbb{R}^n$. Then, for any x:

$$\begin{aligned} \|x\| &= \sqrt{\langle x, x \rangle} \\ &= \sqrt{\langle T(x), T(x) \rangle} \\ &= \|T(x)\| \end{aligned}$$

So inner product preservation implies norm preservation. Going the other way, using the polarization identity we can resolve the inner product from the norm. Assuming for all $x \in \mathbb{R}^n$ that $\|x\| = \|T(x)\|$ we find:

$$\begin{aligned} \langle x, x \rangle &= \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2) \\ &= \frac{1}{4}(\|T(x + y)\|^2 - \|T(x - y)\|^2) \\ &= \frac{1}{4}(\|T(x) + T(y)\|^2 - \|T(x) - T(y)\|^2) \\ &= \langle T(x), T(y) \rangle \end{aligned}$$

So the converse is also true.

b) The proof of this relies on the rank-nullity theorem, which, for our purposes, implies that if the nullity of T is zero then the rank is n, and thus T is invertible.

The nullity of a transformation is the dimension of the space of vectors x such that $T(x) = 0$. If there are no vectors other than 0 that satisfy this, then the nullity of T must be zero. Indeed, since we've assumed that T is norm-preserving, we find that if there was an $x \neq 0$ where $T(x) = 0$, then $0 < \|x\| = \|T(x)\| = 0$, which is a contradiction. Thus, the rank of T must be n.

This means if a linear transformation is norm preserving then it must be invertible, and since it's invertible it must be 1-1.

Because it's 1-1, for all $y \in \mathbb{R}$ there's a corresponding $x \in \mathbb{R}$ such that $y = T(x)$ or $T^{-1}(y) = x$. Since $\|y\| = \|T(x)\| = \|x\|$, norming both sides of the second equation yields $\|T^{-1}(y)\| = \|x\| = \|y\|$. So the inverse is norm preserving, too.