# Assignment 3 - MAT257

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1. a) Since $A$ is closed this means $A^c$ will be open. This implies for every $x \notin A$ there is an epsilon-neighborhood around $x$ that does not intersect with $A$. This epsilon will be less than or equal to the distance between $x$ and any $y \in A$, since if the distance were less, then $y$ would be in the epsilon-neighborhood of $x$ that doesn't intersect with $A$ and that's a contradiction.

   b) Suppose there doesn't exist an $\epsilon > 0$. Consider the set $S_\epsilon = \{x \in \mathbb{R}^n \mid \forall y \in B.d(x,y) < \epsilon\}$. If there's no epsilon, then $S_\epsilon \cap A \neq \emptyset$ for any $\epsilon > 0$. If we take the sequence $\{x_n\}$ where $x_n$ is a point in the set $S_{1/n} \cap A$ then we have a convergent sequence to a point in the border of $B$. Since $A$ is closed, then the point that $\{x_n\}$ converges to must also be in $A$. But it's given that $B \cap A = \emptyset$. So we have a contradiction.

2. Since $K$ is compact, it must be closed and bounded. Since $K \subset U$, then all points $x$ in $K$ will have an epsilon-neighborhood around $x$ that's completely contained within $U$. Suppose $\epsilon_x$ denotes the size of the neighborhood around the point $x$ that satisfies the above condition. If we take the minimum of $\epsilon_x$ for all $x \in K$, call it $\epsilon_{min}$, then for all $x \in K$, $B(x, \epsilon_{min}) \subset U$.

   Notice that $\bigcup_{x \in K} B(x, \epsilon_{min})$ is an open set contained completely within $U$ that covers $K$. If we take the closure of that set, however, it might be that it's border lies outside of $U$. So, we prove the lemma that $\overline{B(x, \epsilon/2)} \subset B(x, \epsilon)$ for any $x$ or $\epsilon$.

   **Lemma 0.1** *At any point $x$, the closed ball of size $\epsilon$ is contained within an open ball of size $\epsilon'$ if $\epsilon < \epsilon'$.*

   **Proof** Assume that $\epsilon < \epsilon'$. Since the norm of every point in the closed ball is less than or equal to $\epsilon$ and $\epsilon < \epsilon'$, then every point in the closed ball must be in the open ball of larger size.

   Using this lemma, we show that $\bigcup_{x \in K} B(x, \epsilon_{min})$ covers $\bigcup_{x \in K} \overline{B(x, \epsilon_{min}/2)}$ (I'll call these sets $A$ and $B$ respectively.) Since any $x \in B$ is in a closed ball of radius $\epsilon/2$ centered around some point in $K$, it must be in the open ball of radius $\epsilon$ around the same point, and thus in $A$. Therefore $B \subset A$. Furthermore, since $A \subset U$, then $B \subset U$. Also $B$ bounded, since each point is only $\epsilon/2$ distance away from the furthest point in the bounded set $K$. Therefore, $B$ is a closed and bounded set, which makes it compact. Also, the interior of $B$ covers $K$. Therefore for any closed set $K$ in an open set $U$, there exists a closed set in $U$ who's interior covers $K$.

3. a) To prove that this function is a metric on $\mathbb{R}^n$ we must show that it satisfies the metric properties. Namely, we show that:

      i. $\rho(x,y) = \rho(y,x)$

      ii. $\rho(x,y) \geq 0$ with equality when $x = y$

      iii. $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$

   To prove i., note that since $d(x,y) = d(y,x)$ we can simply do the following:

$$\rho(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

$$
\begin{aligned}
&= \frac{d(y,x)}{1+d(y,x)} \\
&= \rho(y,x)
\end{aligned}
$$

For ii., we once again base our proof on the fact that $\rho$ is simply a function of $d$. If $x = y$ then $d(x,y) = 0$, and hence $\rho(x,y) = \frac{0}{1} = 0$. If $x \neq y$ then $d(x,y) > 0$ and so $\rho(x,y)$ must be greater than zero since it's a positive number divided by a positive number.

To prove iii., note that the function $f(x) = \frac{x}{x+1}$ is monotonically increasing for all $x \geq 0$. Also note that $f(x+y) \leq f(x) + f(y)$ since if $x = y = 0$ then $f(x+y) = f(x) + f(y) = 0$. If $x \neq 0$ and $y = 0$ then $f(x+y) = f(x) = f(x) + f(y)$. If $x, y \neq 0$ then the equation $\frac{xy(x+y+2)}{(x+1)(y+1)(x+y+1)}$ is positive, so the following holds:

$$
\begin{aligned}
f(x+y) &= \frac{x+y}{1+x+y} \\
&\leq \frac{x+y}{1+x+y} + \frac{xy(x+y+2)}{(x+1)(y+1)(x+y+1)} \\
&= \frac{x}{1+x} + \frac{y}{1+y} = f(x) + f(y)
\end{aligned}
$$

Also $f(d(x,y)) = \rho(x,y)$.

Since $d(x,y) \geq 0$ for any $x, y \in \mathbb{R}^n$, we can apply $f$ to both sides of the inequality $d(x,z) \leq d(x,y) + d(y,z)$ like so:

$$
\begin{aligned}
d(x,z) &\leq d(x,y) + d(y,z) \implies \\
f(d(x,z)) &\leq f(d(x,y) + d(y,z)) \implies \\
f(d(x,z)) &\leq f(d(x,y)) + f(d(y,z)) \implies \\
\rho(x,z) &\leq \rho(x,y) + \rho(y,z)
\end{aligned}
$$

So $\rho$ is a metric on $\mathbb{R}^n$

b) since $\rho(x,y) = f(d(x,y))$ and $d(x,y) \geq 0$ for all $x, y \in \mathbb{R}^2$ we simply show that $f$ is bounded for all $x \geq 0$. This is a simple task, since $f$ is a division of two positive numbers, where the denominator is larger than the numerator, $f$ must be always less than 1. Therefore, the metric $(\mathbb{R}^2, \rho)$ is bounded by 1.