# Assignment 5 - MAT257

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1. Munkres §8, #2

   (a) Suppose that $f(x,y) = f(a,b)$ where $y, b \in (0, 2\pi)$. Then we have:

$$\begin{aligned} f(x,y) &= f(a,b) \implies \\ \|f(x,y)\| &= \|f(a,b)\| \implies \\ \sqrt{(e^x \cos y)^2 + (e^x \sin y)^2} &= \sqrt{(e^a \cos b)^2 + (e^a \sin b)^2} \implies \\ e^{2x}(\cos^2 y + \sin^2 y) &= e^{2a}(\cos^2 b + \sin^2 b) \implies \\ e^{2x} &= e^{2a} \implies \\ x &= a \end{aligned}$$

   Given $x = a$, the below follows:

$$\begin{aligned} f(x,y) &= f(a,b) \implies \\ e^x \cos y &= e^a \cos b \implies \\ \cos y &= \cos b \end{aligned}$$

   And:

$$\begin{aligned} f(x,y) &= f(a,b) \implies \\ e^x \sin y &= e^a \sin b \implies \\ \sin y &= \sin b \end{aligned}$$

   Since $y$ and $b$ are in $(0, 2\pi)$ then this implies $y = b$. Since $f(x,y) = f(a,b) \implies (x,y) = (a,b)$ the function is one to one.

   (b) $B = \mathbb{R}^2 - L$ where $L$ is the set $\{\, t(1,0) \mid t \in \mathbb{R}^+ \,\}$. This is because for any $(x,y) \in \mathbb{R}^2$ if we can find a $\theta \in (0, 2\pi)$ an $r \in \mathbb{R}^+$ such that $(x,y) = (r\cos\theta, r\sin\theta)$ unless $x \in L$, since in that case $\theta = 0$ or $2\pi$. In the context of our function, for any $(r, \theta)$ we can get $(x,y)$ through the function $f(\log r, \theta)$.

   (c) By the inverse function theorem, if $Df(x,y)$ is nonsingular, then the function is one to one in some neighborhood of $(x,y)$ and the inverse function $g$ has the derivative $Dg(f(x,y)) = [Df(x,y)]^{-1}$. Taking $Df(x,y)$ for any $(x,y) \in \mathbb{R}^2$ yields:

$$\begin{aligned} Df(x,y) &= \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix} \end{aligned}$$

The determinant of this matrix will be $(e^x \cos y)^2 + (e^x \sin y)^2 = e^{2x}$. Since $e^{2x} > 0$ for all $x$ this means the function's derivative is invertible anywhere.

Speaking generally again, if we take the formula for the inverse of a $2 \times 2$ matrix we find that $Df(x,y)^{-1}$ is:

$$\begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}^{-1} = \frac{1}{e^{2x}} \begin{bmatrix} e^x \cos y & e^x \sin y \\ -e^x \sin y & e^x \cos y \end{bmatrix}$$

To find $Dg(0,1)$ we must find an $(x,y)$ such that $f(x,y) = (0,1)$. Heuristically, we know that there exists a $\theta$ such that $\sin\theta = 1$ and $\cos\theta = 0$. If we let $x = 0$ and $y = \frac{\pi}{2}$ then $f(x,y) = (0,1)$. Plugging the vector $(0, \pi/2)$ into our inverted matrix formula yields:

$$\frac{1}{e^{2x}} \begin{bmatrix} e^x \cos y & e^x \sin y \\ -e^x \sin y & e^x \cos y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Which is $Dg(0,1)$.

2. Munkres §8, #3

   Expanding the function $f$ out into it's component functions:

$$f_1(x,y) = x(x^2 + y^2)$$
$$f_2(x,y) = y(x^2 + y^2)$$

   These are simple polynomials. This means $f$ is of class $C^\infty$

   Let $x, y \in B(0;1)$. Suppose $f(x) = f(y)$. Then $\|f(x)\| = \|f(y)\|$ and therefore:

$$\begin{aligned} \|\|x\|^2 \cdot x\| = \|\|y\|^2 \cdot y\| &\implies \\ \|x\|^2\|x\| = \|y\|^2\|y\| &\implies \\ \|x\| = \|y\| & \end{aligned}$$

   If $\|x\| = \|y\| = 0$ then $x = y = 0$. Otherwise, we have $\|x\|^2 \cdot x = \|y\|^2 \cdot y$. Multiplying both sides by $1/\|x\|^2$ yields $x = y$. Therefore $f$ is one to one on the unit ball.

   For any $(x,y) \in \mathbb{R}^2$, $Df(x,y)$ will be the matrix:

$$\begin{bmatrix} 3x^2 + y^2 & 2xy \\ 2xy & 3y^2 + x^2 \end{bmatrix}$$

   Since $f(0,0) = 0$, the inverse function $g$ must have a derivative that is the inverse of $f$ at 0. However, when $(x,y) = 0$ all of the terms in the matrix go to zero, and therefore it is not invertible. Hence, $g$ is not differentiable at 0.

3. Munkres §8, #5

4. Munkres §9, #3

5. Munkres §9, #4

6. Munkres §9, #5