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$X_1 \dots X_n$ are $n$ iid random variables whose density is:

$$f_\theta(x) = (k+1)\,\theta^{-k-1} \cdot x^k \cdot \mathbb{1}_{[0,\theta]}(x)$$

- $k$ is nonnegative integer fixed
- $\theta$ is an unknown positive parameter.
- $m = n(k+1)$

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$$L(\theta) = \prod_{i=1}^{n} f_\theta(x_i) = \prod_{i=1}^{n} (k+1) \cdot \theta^{-k-1} \cdot x_i^k \cdot \mathbb{1}_{[0,\theta]}(x_i)$$

$$= \begin{cases} (k+1)^n \, \theta^{n(-k-1)} \cdot \prod_{i=1}^{n} x_i^k & \text{if } x_i \in [0,\theta] \\ 0 & \text{if } x_i \notin [0,\theta] \end{cases}$$

take log:

$$\Longrightarrow \log L(\theta) =$$

$$= n \cdot \log(k+1) - n(k+1) \cdot \log\theta + k \cdot \sum_{i=1}^{n} \log(x_i)$$

$$\rightarrow \frac{d}{d\theta} \log L(\theta) = \frac{-n(k+1)}{\theta} + \begin{cases} 0 & \text{if } x_i \in [0,\theta[ \\ \frac{k \cdot n}{\theta} & \text{if } x_i = \theta \quad \left(\text{since } \frac{d}{d\theta}\left(k \cdot \sum_{i=1}^{n} \log\theta\right) = \frac{k \cdot n}{\theta}\right) \end{cases}$$

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Now we have two equations of $\frac{d}{d\theta} \log L(\theta)$:

$$\frac{d}{d\theta} \log L(\theta) = \begin{cases} \boxed{\frac{-n(k+1)}{\theta}} & \text{if } x_i \in [0,\theta[ \\ \frac{-n(k+1)}{\theta} + \frac{n \cdot k}{\theta} = \boxed{\frac{-n}{\theta}} & \text{if } x_i = \theta \end{cases}$$

So,

$$\frac{d}{d\theta}\log L(\theta) = \frac{-n(k+1)}{\theta} = 0 \quad , \text{ if } x_i \in [0,\theta[ \text{ where } i \in \{1..N\}$$

↳ We cannot solve it like this.

So, let's analyise more

For $x_i \notin [0,\theta]$ $L(\theta) = 0$

For $x_i \in [0,\theta]$

$$L(\theta) = (k+1)^n \, \theta^{-n(k+1)} \prod_{i=1}^{n} x_i^k$$

$x_i \leq \theta \quad \forall i \in [1,n]$

$$\text{so, } \hat{\theta} = \max_{1<i<n}(x_i)$$

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$$\text{Bias} = E\Big[\max_{1<i<n}(x_i)\Big] - \theta$$

$$E[\hat{\theta}] = \int_0^\theta x.f(x) = \int_0^\theta x.(k+1)\,\theta^{-k-1}\,x^k.dx = \Big[ (k+1)\,\theta^{-1}\,\frac{x^{k+2}}{(k+2)} \Big]_0^\theta$$

$$= \frac{\theta^{(k+1)}.(k+1)}{(k+2)}$$

$$\text{so, } E[\hat{\theta}] - \theta = \frac{\theta^{k+1}}{k+2}.(k+1) - \theta$$

- $\lambda_1 \cdot \hat{\theta}$ is unbiased

Bias was: $\dfrac{\theta^{k+1}}{k+2}(k+1) - \theta = 0$

$$\frac{\theta^{k+1}}{k+2}(k+1) = \theta \qquad \frac{\theta^{k}}{(k+2)}(k+1) = 1$$

$$\theta^{k} = \frac{(k+2)}{(k+1)}$$

$$\theta = \sqrt[k]{\frac{(k+2)}{(k+1)}}$$

so, $\lambda_1 = \sqrt[k]{\dfrac{(k+2)}{(k+1)}}$

- $\lambda_2 \hat{\theta}$: has the smallest quadratic error.

$\lambda_2 = 0$ so, it has the smallest error.