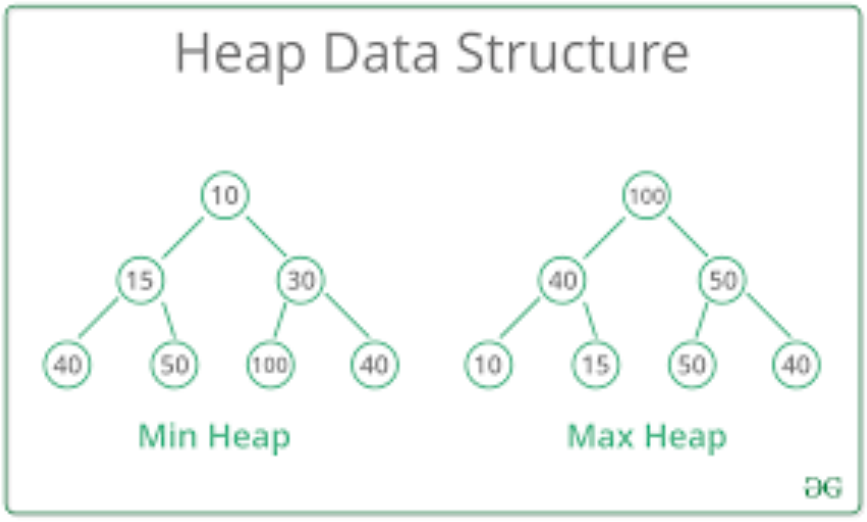
CSCI 3110 Heap, Priority Queue, and Heap Sort

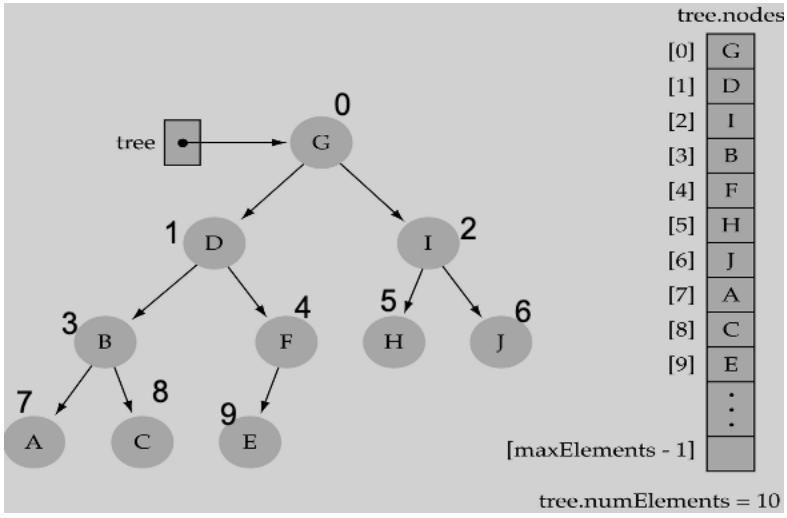
A Binary Heap: A heap is a complete binary tree stored in an array with the following ***heap property***: *the parent of a node always has a value greater than or equal to the value at the node*



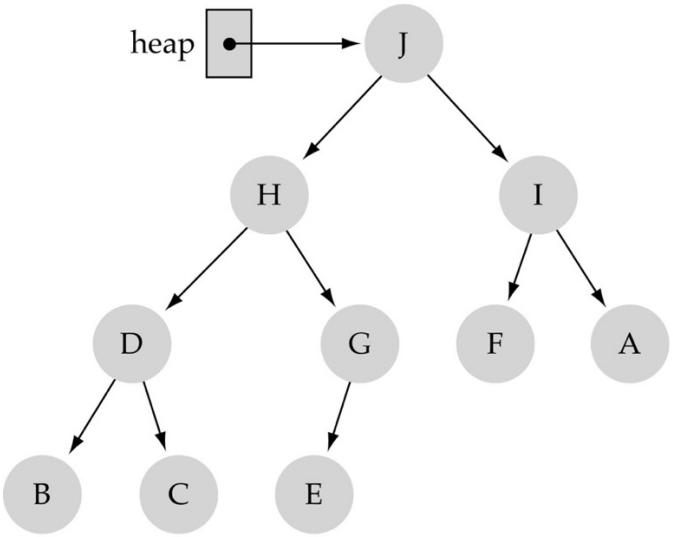
From the heap property, in Max Heap, the largest value of the heap is always stored at the root

The relationships between nodes in the tree are as follows:

- **Parent** of a node i is stored at array location: **floor((i-1)/2)**
- **Left child** of a node i is stored at array location: **2*i+1**
- **Right child** of node i is stored at array location: **2*i+2**
- Leaf nodes: [numElements/2] to [numElements-1]



More examples of heap:



heap.elements

[0]	J
[1]	H
[2]	I
[3]	D
[4]	G
[5]	F
[6]	A
[7]	B
[8]	C
[9]	E

Practice exercise:
What is the tree representation of the heap: 20 10 8 5 2 4 ?

Priority Queue

- A Queue where each element being associated with a "priority"
- From the elements in the queue, the one with the highest priority is dequeued first
- Used in many applications, for example most graph search algorithms, AI informed search algorithms

- **Compare heaps with other priority queue representations**
  - Priority queue using sorted linked list
    - Remove the front queue item, i.e., the largest key ➔O(1)
    - Add a new queue item ➔ O(n)
  - Priority queue using heaps
    - Remove the front queue item, i.e., the largest key ➔ O(log n)
    - Add a new queue item ➔ O(log n)
  - Priority queue using binary search tree – *discussed after heap*
    - Remove the front queue item, i.e., the largest key ➔ O(n) worst case
    - Add a new queue item ➔ O(n) worst case
  - Priority queue using AVL tree – *discussed after heap*
    - Remove the front queue item, i.e., the largest key ➔ O(log n)
    - Add a new queue item ➔ O(log n)

Two main heap operations:

- Reheap up – add element to heap
- Reheap down – remove the root data from the heap

Reheaping Up to Add Elements

- When we enqueue an element in the priority queue, our heap becomes larger by one element.

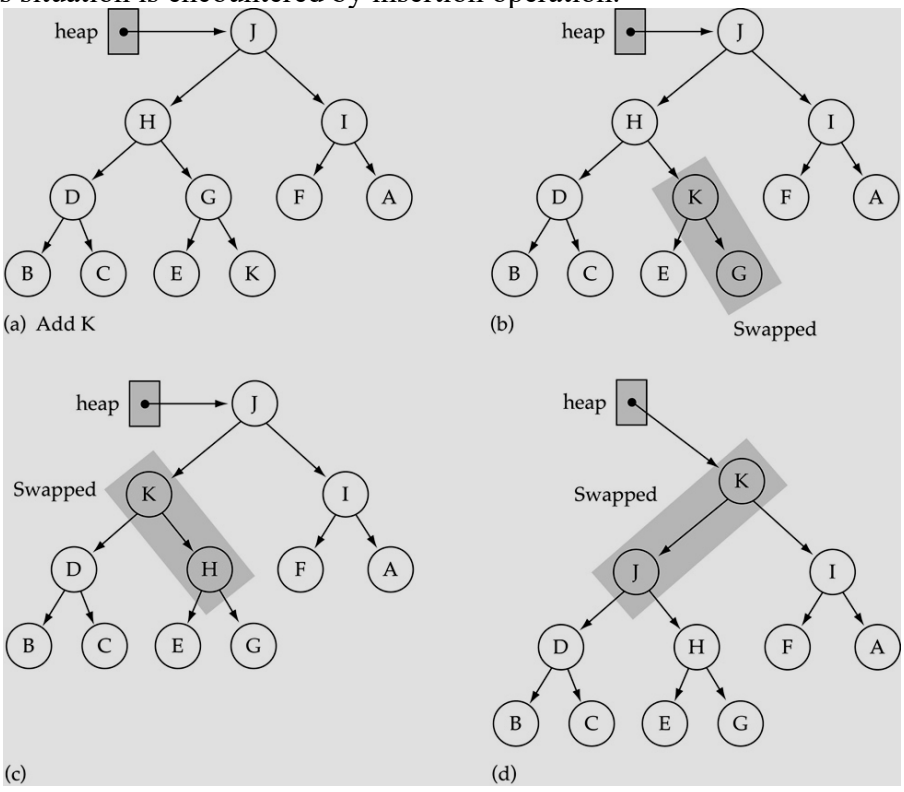
- We know exactly where the new element must go, in the next available array slot, so we can add it there, but this could destroy the heap property.

- We will fix the property by reheaping up.
  - We do this by comparing the element with its parent.
  - If the element is less than or equal to its parent then we're done.
  - Otherwise we swap the element and its parent and repeat the reheaping up (recursive) process with the parent.
  - We continue this way until we have satisfied the heap property.

- This reheaping up process may not stop until we reach the root. This means shifting up to O(log n) elements and takes **O(log n) time in the worst case.**

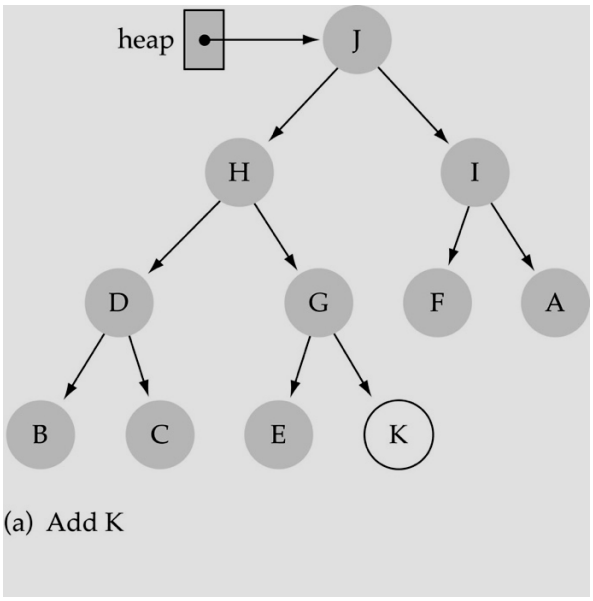
Reheap Up

In this case the heap property is violated at the rightmost node at the last level of the tree. This situation is encountered by insertion operation.

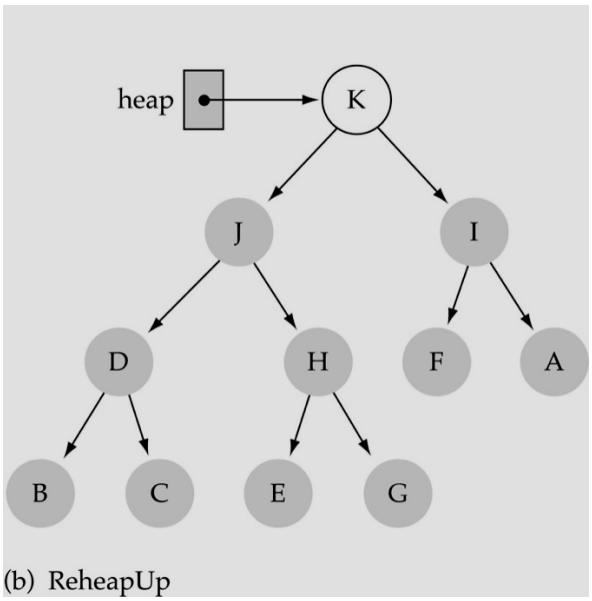


Practice exercise:

Suppose we add a new element 50 to this heap: 20 10 8 5 2 4 6, how many nodes will have a different value from before, after this operation?



(a) Add K



(b) ReheapUp

Reheap Up code:

```
template<class ItemType>
struct HeapType {
        void ReheapDown(int, int);
        void ReheapUp(int, int);
        ItemType *elements;  // allocate the array dynamically using the pointer
        int numElements; // heap elements
};

template<class ItemType>
void HeapType<ItemType>::ReheapUp(int root, int bottom) {
        int parent;

        if (bottom > root) { // tree is not empty
                parent = (bottom-1)/2;
                if (elements[parent] < elements[bottom]) {
                        Swap(elements, parent, bottom);
                        ReheapUp(root, parent);
                }
        }
}
```

Practice exercise:
Suppose we add a new element 10 to this heap: 25 15 8 9 2 4 , how many nodes will have a different value from before, after this operation?

Reheaping Down to *Dequeue*

- We know that we want to return the element in the root, but we have to adjust the heap before we do that. We *move the element from the former last location into the root*.

- This may leave the root node violating the heap property. If so we must reheap down, swapping elements to move the element down until the heap property is restored.

- We want to move the element downward until it no longer has children that are too big. The way we do this is to swap it with the element from the larger of the element's children; it may go there because it is greater than (or equal to) its new children.

- The natural way to do this is recursively.

- **ReHeap Down process time complexity O(log n)**

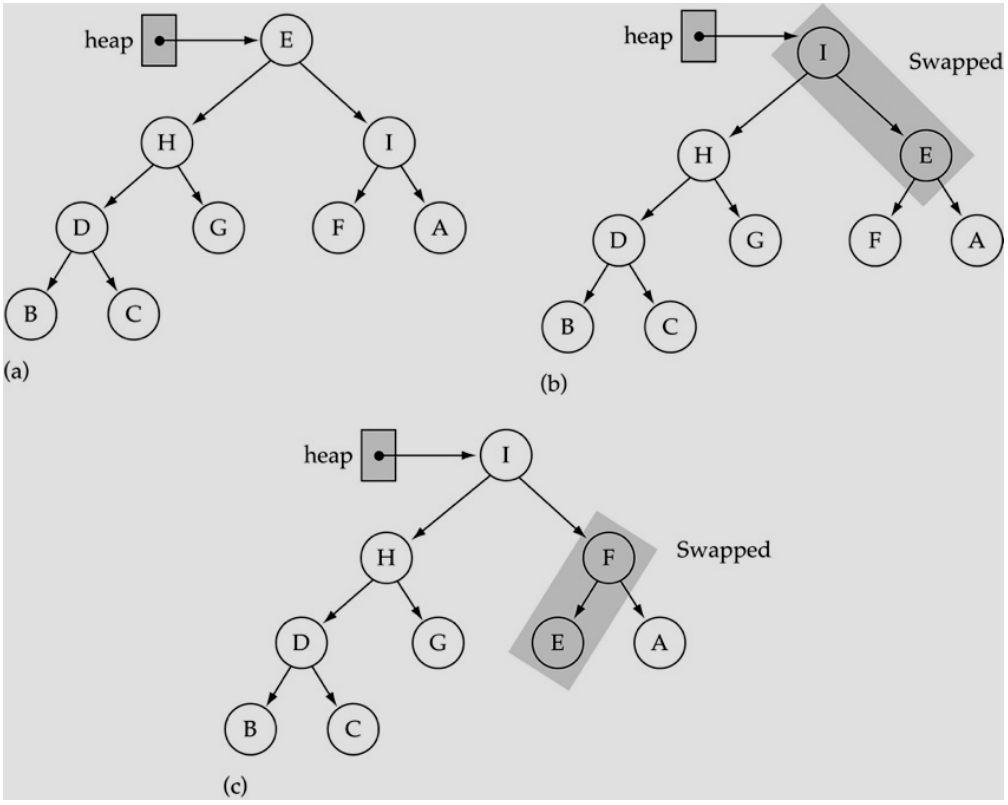
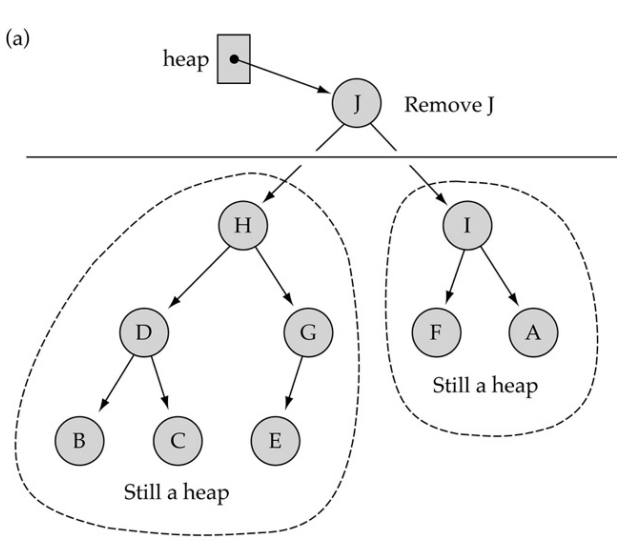
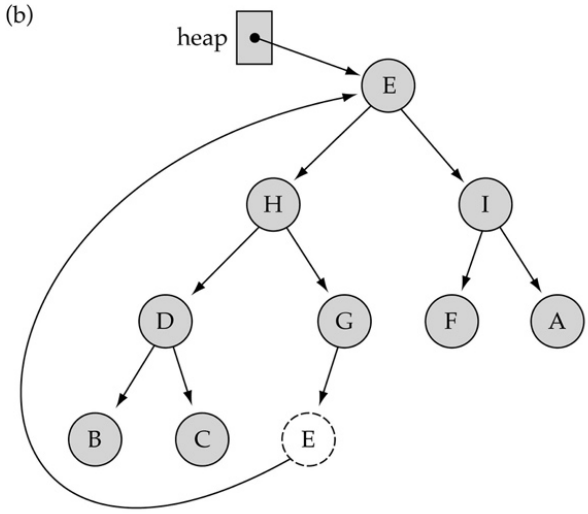
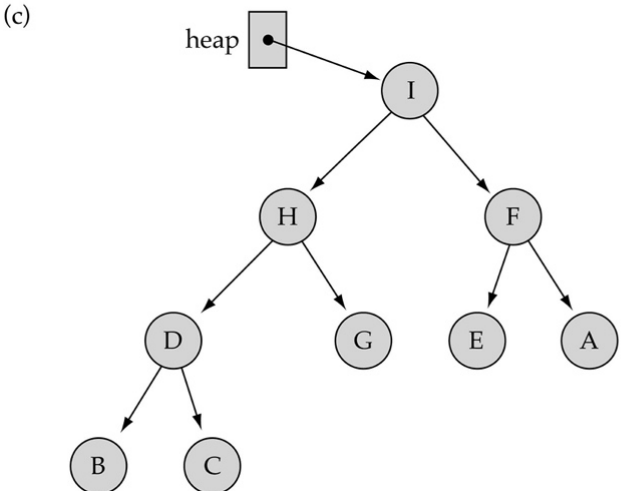


Illustration of moving the element from the former last location into the root, then recursively apply reheap down:

- Copy the bottom level rightmost data into the root
- Delete the bottom rightmost node
- Restore the heap property by calling **Reheap Down**







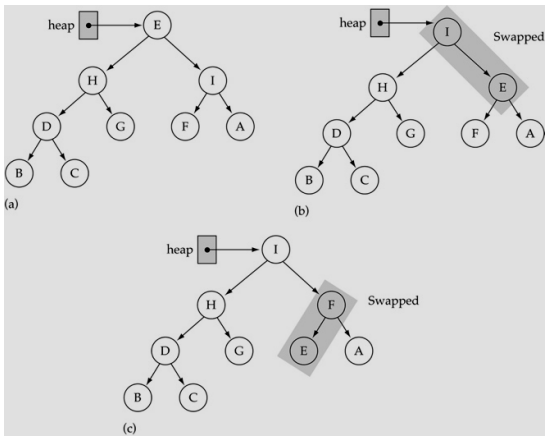
Code for Reheap Down

```
template<class ItemType>
struct HeapType {
        void ReheapDown(int, int);
        void ReheapUp(int, int);
        ItemType *elements;  // allocate the array dynamically using the pointer
        int numElements; // heap elements
};

template<class ItemType>
void HeapType<ItemType>::ReheapDown(int root, int bottom) {
        int maxChild, rightChild, leftChild;

        leftChild = 2*root+1;
        rightChild = 2*root+2;

        if(leftChild <= bottom) {  // left child is part of the heap
            if (leftChild == bottom) // only one child case
                maxChild = leftChild;
            else {
               if (elements[leftChild] <= elements[rightChild])
                    maxChild = rightChild;
               else
                     maxChild = leftChild;
            }// end if
            if (elements[root] < elements[maxChild]) {
               Swap(elements, root, maxChild);
               ReheapDown(maxChild, bottom);
            } // end if
         } // end if
}
```



Practice Exercise :

Insert 35 to the heap(25 15 8 9 2 4) as the new element at the bottom of the heap → trace Reheap Up

Heap Sort

Basic steps:
1. Build the heap from an array of data
2. Remove the current root of the heap and add it to the sorted part of the array being built
3. Restore heap property
4. Repeat steps 2-3 until the heap is empty

```
// Main function to do heap sort
void HeapSort(int arr[], int N){

    // Build max heap
    for (int i = N / 2 - 1; i >= 0; i--)
        heapify(arr, i, N);

    // Heap sort
    for (int i = N - 1; i >= 0; i--) {
        swap(arr[0], arr[i]);
       // Heapify root element to get highest element at root again
       heapify(arr, 0, i);
    }
}

// To heapify a subtree rooted with node I, which is an index in arr[]. n is size of the heap
void heapify(int arr[], int i, int N){
    // Find largest among root, left child and right child
    int largest = i; // Initialize largest as root
    int left = 2 * i + 1;     // left = 2*i + 1
    int right = 2 * i + 2; // right = 2*i + 2

    // If left child is larger than root
    if (left < N && arr[left] > arr[largest])
        largest = left;

    // If right child is larger than largest so far
    if (right < N && arr[right] > arr[largest])
        largest = right;

    // Swap and continue heapifying if root is not largest
    if (largest != i) { // If largest is not root
        swap(arr[i], arr[largest]);
        heapify(arr, largest, N);        // Recursively heapify the affected sub-tree
    }
}
```

What is the time complexity of Heap Sort? O(n log n)

Example :
Sort the following data using Heap Sort. arr: 46, 81, 98, 28, 37, 32. ← N is 6

Function call: HeapSort(array, N)

Step 1: Build the max heap:
i=2, Heapify(arr, 2, 6) → no change
i=1, Heapify(arr, 1, 6) → no change
i=0, Heapify(arr, 0, 6) → swap(a[0], a[2]), arr: 98 81 46 28 37 32

Step 2: Heap Sort
i=5, swap(arr[0], arr[5]), arr: 32 81 46 28 37 || 98. (98 is in its sorted position)
Heapify(arr, 0, 5)
Swap(arr[0], arr[1]), swap(arr[1], arr[4])
arr: 81 37 46 28 32 || 98

i=4, swap(arr[0], arr[4]), arr: 32. 37. 46. 28. || 81 98 (81 and 98 in their sorted position)
Heapify(arr, 0, 4)
Swap(arr[0], arr[2])
arr: 46 37 32 28 || 81 98

i=3, swap(arr[0], arr[3]), arr: 28 37 32 || 46 81 98
Heapify(arr, 0, 3)
Swap(arr[0], arr[1])
arr: 37 28 32 || 46 81 98

i=2, swap(arr[0], arr[2]), arr: 32 28 || 37 46 81 98

_____Heapify(arr, 0, 2)
No change
arr: 32 28 || 37 46 81 98

i=1, swap(arr[0], arr[1]), arr: 28 || 32 37 46 81 98. ➔ complete!

Heap Sort analysis

Building max heap:
Repeat Heapify (N/2-1) times
Each Heapify is O(logN)
Total O(NlogN) time

Heap sort loop
Repeat N-1 times:
1. Swap out the first array element
2. Heapify

Total f(N) = (N-1)*(1+logN)
O(NlogN)

Therefore, the time complexity of Heap sort is O(NlogN).

Practice Exercise:
Sort the following array of data into descending order using Heap sort:
46 81 98 28 37 32