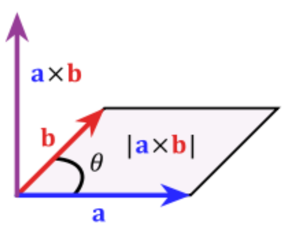
Mathematical foundations for Computer Graphics (1)

Basic terminologies

- Basic elements of geometry
  - scalars, vectors, points in 3 dimensional space
  - vector space: (a vector space V is a set that is closed under finite *vector addition* and *scalar multiplication*). The basic example is n-dimensional Euclidean space R^n, where every element is represented by a list of n real numbers, element addition is component-wise, and scalar-element multiplication is multiplication on each term separately.

  - Common vector operations:
    - vector (direction, magnitude) → directed line segments
      - length/magnitude
      - vector normalization and unit vectors
      - inverse
      - scalar-vector multiplication
      - vector-vector addition
      - vectors lack positions → need points
    - dot product of two vectors
      - compute the dot product of two vectors
      - compute the angle between two vectors
      - how do you tell if two vectors are perpendicular to each other?
    - cross product of two vectors
      - compute the vector perpendicular to the two vectors, i.e., the normal vector



$$u \times v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

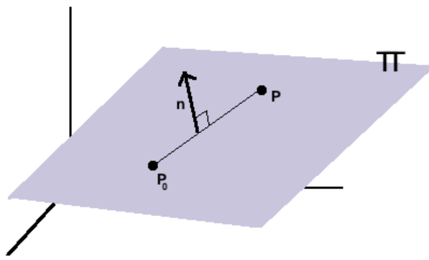
$$u \times v = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

  - ➔ affine space (Point + a vector space)
    - In an affine space, it is possible to fix a point and coordinate axis such that every point in the space can be represented as a n-tuple of its coordinates. Every ordered pair of points A and B in an affine space is then associated with a vector AB.
    - vector-vector addition, scalar-vector multiplication, point-vector addition, point-point subtraction
    - points:
      - point-point subtraction → vector
      - point-vector addition → point
  - affine space with orthogonal (aka perpendicular) vector space

- Lines: $P(\alpha)=P_0 + \alpha\, \mathbf{d}$ ← parametric form
  - Set of all points that pass through P_0 in the direction of the vector **d**
  - Another representation: $P(t)=P_0 + t*(P_1-P_0)$
  - tweening
- What is a ray?
- What is a line segment defined by two points P and Q?

- Planes: P(α, β) defined as:
  - a point + 2 vectors $P(\alpha, \beta)=P_0+\alpha*u+\beta*v$, or
  - Normals
    - In three dimensional spaces, every plane has a vector n perpendicular or orthogonal to it called the **normal vector**
    - How to compute the normal vector of a plane?
      - cross product
  - Planes defined with point-normal form

$$0 = \overrightarrow{P_0P} \cdot \vec{n}$$
$$0 = (x - x_0, y - y_0, z - z_0) \cdot (a, b, c)$$
$$0 = a(x - x_0) + b(y - y_0) + c(z - z_0)$$



  - Each triangle is defined with 3 points. How do these 3 points define the plane in parametric form and in point normal form?

Representations

- Linear Independence
  - A set of vectors $v_1, v_2, \ldots, v_n$ is *linearly independent* if
    $a_1v_1+a_2v_2+.. a_nv_n=0$ iff $a_1=a_2=\ldots a_n=0$
    - If a set of vectors is linearly independent, we cannot represent any one vector in the set in terms of any other vectors in the set
    - Are the following set of vectors linearly independent?
      - [2, 5] and [4, 10]
      - [2, 4, 1], [-2, -4, -1]
      - [2, 4, 2], [3, -1, 5], [-5, 3, -7]
      - [2, 4, 2], [3, -1, 5], [-5, 3, 5]
      - [12, -4], [-2, 6], [5, 1]
      - [0, 0, 2], [3, 0, 0],[0, 4, 0]

  - In a *n*-dimensional space, any set of n linearly independent vectors form a *basis* for the space

  - Given a basis $v_1, v_2, \ldots, v_n$, any vector v can be written as $v=a_1v_1+a_2v_2+\ldots+a_nv_n$
  where the $\{a_i\}$ are unique

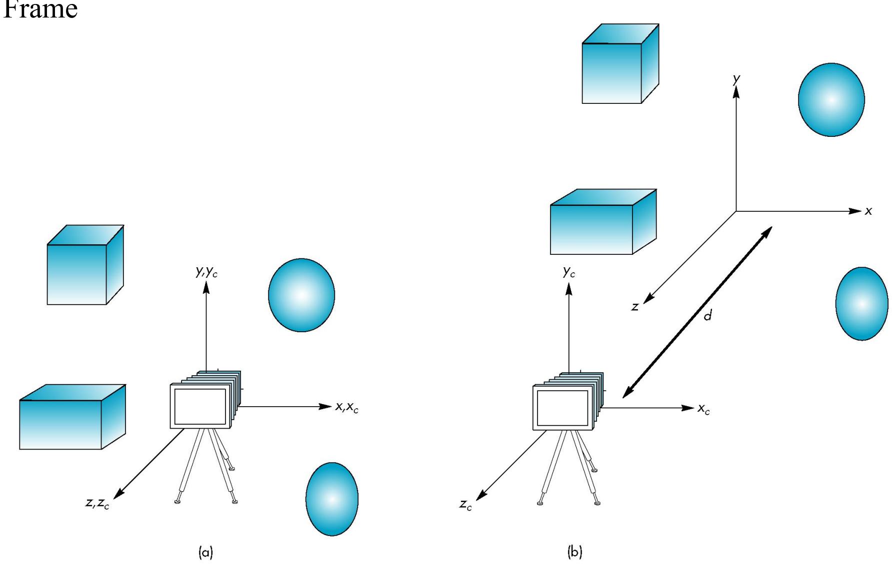
- Coordinate systems
  - Consider a basis $v_1, v_2, \ldots, v_n$, a vector in this system is written as $v=a_1v_1+a_2v_2+\ldots+a_nv_n$, where the list of scalars $\{a_1, a_2, \ldots a_n\}$ is the *representation* of v with respect to the given basis.
    - Not unique
  - Need a frame of reference to relate points and objects to our physical world.
- Frame
  - What is a coordinate system? world coordinate? camera coordinate? …

  - Frame is determined by: point + basis vectors for n dimensions in the space: (P_0, v_1, v_2, v_3), where v_1, v_2, v_3 are the basis for the space, P_0 is the origin point of the frame.

- Within this frame, every vector can be written as: $v = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$
- Every point can be written as: $P = P_0 + b_1 v_1 + b_2 v_2 + \ldots + b_n v_n$
- Problem: points and vectors may be easily confused using the normal 3-d representation

- **Homogeneous Coordinates**
  - Four-dimensional *homogeneous coordinate* representation
    - $\mathbf{v} = [a_1\ a_2\ a_3\ 0]^T$
    - $\mathbf{p} = [b_1\ b_2\ b_3\ 1]^T$
  - Homogeneous coordinates are key to all computer graphics systems
    - All standard transformations (rotation, translation, scaling) can be implemented with matrix multiplications using 4 x 4 transformation matrices
    - Hardware pipeline works with 4 dimensional representations
    - For orthographic viewing, we can maintain w=0 for vectors and w=1 for points
    - For perspective we need a *perspective division*

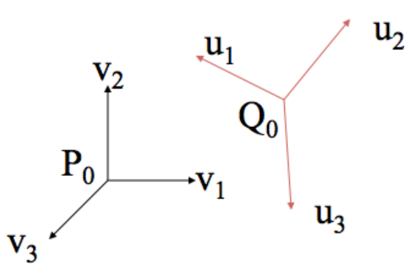
- Change of Frame



Moving the camera (Change from the world frame to the camera frame)
- Initially these frames overlap (V=Identity)
- Since objects are on both sides of z=0 plane, we must move the camera
- Afterwards, the objects need to be transformed into the camera frame by changing their locations in the world frame to the camera frame using the view matrix

How to transform the object positions from the world frame into the camera frame?

- Given two frames:



$u_1 = \gamma_{11}v_1 + \gamma_{12}v_2 + \gamma_{13}v_3$

$u_2 = \gamma_{21}v_1 + \gamma_{22}v_2 + \gamma_{23}v_3$

$u_3 = \gamma_{31}v_1 + \gamma_{32}v_2 + \gamma_{33}v_3$

$Q_0 = \gamma_{41}v_1 + \gamma_{42}v_2 + \gamma_{43}v_3 + \gamma_{44}P_0$

M=

$$\begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & 0 \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & 0 \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & 1 \end{bmatrix}$$

- Given a point or a vector **a**=[a_1 a_2 a_3 a_4] in the left frame, what will be the coordinates/representation of this point/vector when the frame is changed into the right frame?
    - How to multiply a vector of size 4 by a 4x4 matrix?

- Given a point or a vector **b**=[b_1 b_2 b_3 b_4] in the right frame, what will be the coordinates/representation of this point/vector when the frame is changed into the left frame?
    - How to compute the transpose of a 4x4 matrix M?

- Affine Transformations
    - Every linear transformation is equivalent to a change in frames
    - Every affine transformation preserves lines
    - An affine transformation has only 12 *degrees of freedom* because 4 of the elements in the matrix are fixed and are a subset of all possible 4 x 4 linear transformations