SGN - 11006 Basic Course in Signal Processing

Exercise #1. Solutions.

(I) Sinusoidal signal in general form: $u(t) = A\cos(2\pi f_0 t + \varphi)$, where A - amplitude, f_0 - frequency, φ - phase shift.

a) $x(t) = \cos(200\pi t) \Rightarrow A = 1, \quad f_0 = 100\text{ Hz}, \quad \varphi = 0$

b) $x(t) = 3.1\cos(740\pi t + 1.4) \Rightarrow A = 3.1, \quad f_0 = 370\text{ Hz}, \quad \varphi = 1.4$

c) $x(t) = 5\cos(300t - 1) \Rightarrow A = 5, \quad f_0 = \frac{150}{\pi}\text{ Hz}, \quad \varphi = -1.$

d) $x(t) = 32\cos(5\pi(t - 0.05)) = 32\cos(5\pi t - 0.25\pi) \Rightarrow A = 32, \; f_0 = 2.5\text{ Hz}, \; \varphi = -0.25\pi.$

e) $x(t) = \cos(2\pi(t-1)) + \cos(2\pi(t-1)) = 2\cos(2\pi t - 2\pi) = 2\cos(2\pi t)$

$\Rightarrow A = 2, \; f_0 = 1\text{ Hz}, \; \varphi = 0.$ due to periodicity of cos, $T = 2\pi$.

f) $x(t) = 3\cos(\pi(t-1) + \pi) = 3\cos(\pi t) \Rightarrow A = 3, \; f_0 = 0.5\text{ Hz}, \; \varphi = 0.$

(II) Linearity: system is linear if for input $x[n] = \alpha x_1[n] + \beta x_2[n]$ with arbitrary α and β, output is $y[n] = \alpha y_1[n] + \beta y_2[n]$.

a) $y[n] = \frac{1}{x[n]} + x[n-1].$

(1). Let $x[n] = \alpha x_1[n] + \beta x_2[n]$, then

$$y[n] = \frac{1}{\alpha x_1[n] + \beta x_2[n]} + \alpha x_1[n-1] + \beta x_2[n-1].$$

(2). $$\alpha y_1[n] + \beta y_2[n] = \alpha\left(\frac{1}{x_1[n]} + x_1[n-1]\right) + \beta\left(\frac{1}{x_2[n]} + x_2[n-1]\right) =$$
$$= \frac{\alpha}{x_1[n]} + \frac{\beta}{x_2[n]} + \alpha x_1[n-1] + \beta x_2[n-1].$$

(1) $\neq$ (2) $\Rightarrow$ system is <u>not linear.</u>

b) $y[n] = x[n] + 2x[n-5]$

(1). Let $x[n] = \alpha x_1[n] + \beta x_2[n]$, then
$$y[n] = \alpha x_1[n] + \beta x_2[n] + 2\alpha x_1[n-5] + 2\beta x_2[n-5].$$

(2). $$\alpha y_1[n] + \beta y_2[n] = \alpha(x_1[n] + 2x_1[n-5]) + \beta(x_2[n] + 2x_2[n-5]) =$$
$$= \alpha x_1[n] + \beta x_2[n] + 2\alpha x_1[n-5] + 2\beta x_2[n-5]$$

(1) = (2) $\Rightarrow$ system is <u>linear</u>

Time-invariance: system is time-invariant if for an input $x_1[n] = x[n-n_0]$, output is $y_1[n] = y[n-n_0]$.

a) Let $x_1[n] = x[n-n_0]$, then $y_1[n] = \frac{1}{x[n-n_0]} + x[(n-n_0)-1] = y[n-n_0] \Rightarrow$
$\Rightarrow$ system is time-invariant.

b) Let $x_1[n] = x[n-n_0]$, then $y_1[n] = x[n-n_0] + 2x[(n-n_0)-5] = y[n-n_0] \Rightarrow$
$\Rightarrow$ system is time-invariant.

Stability: system is stable if for a bounded input $x[n]$ ($|x[n]| \le B_x \; \forall n$) output $y[n]$ is also bounded ($|y[n]| \le B_y \; \forall n$).

a) Let $|x[n]| \le B_x \; \forall n$, then $|y[n]| = \left| \frac{1}{x[n]} + x[n-1] \right| \le$
$\le \left| \frac{1}{x[n]} \right| + |x[n-1]| \le \left| \frac{1}{x[n]} \right| + B_x$.

As $\left| \frac{1}{x[n]} \right| \to \infty$ when $x[n] \to 0 \Rightarrow$ system is not stable.

b) Let $|x[n]| \le B_x \; \forall n$, then $|y[n]| = |x[n] + 2x[n-5]| \le$
$\le |x[n]| + 2|x[n-5]| \le B_x + 2B_x = 3B_x = B_y \Rightarrow$ system is stable.

Causality: system is causal if output depends only on current or past input.
a) system is causal.
b) system is causal.

Impulse response: system's output is completely characterized by its impulse response if the system is linear and time-invariant, i.e. LTI.

a) System is not linear $\Rightarrow$ can not be defined by its impulse response.
b) System is linear and time-invariant $\Rightarrow$ its output can be determined using only its impulse response.

III. $x[n] = x[n+N]$, $\forall n$.

due to periodicity of $x[n]$

$y[n] = x[n] * h[n] \overset{def.}{=} \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[(n+N)-k] =$

$= y[n+N] \Rightarrow y[n]$ is periodic with $T = N$.

IV $y(t) = x(t) + \alpha x(t-\Delta)$, function $\varphi_{xx}(\vartheta)$ is given.

$$\varphi_{yy}(\vartheta) = \int_{-\infty}^{\infty} y(u)\,y(u-\vartheta)\,du = \int_{-\infty}^{\infty} (x(u) + \alpha x(u-\Delta))(x(u-\vartheta) + \alpha x(u-\vartheta-\Delta))\,du =$$

$$= \int_{-\infty}^{\infty} x(u)x(u-\vartheta) + \alpha\, x(u)x(u-\vartheta-\Delta) + \alpha x(u-\Delta)x(u-\vartheta) + \alpha^2 x(u-\Delta)x(u-\vartheta-\Delta)\,du =$$

$$= \int_{-\infty}^{\infty} x(u)x(u-\vartheta)\,du + \alpha\int_{-\infty}^{\infty} x(u)\,x(u-(\vartheta+\Delta))\,du + \alpha\int_{-\infty}^{\infty} x(u-\Delta)\,x(u-\Delta-(\vartheta-\Delta))\,du +$$

$$+ \alpha^2\int_{-\infty}^{\infty} x(u-\Delta)\,x((u-\Delta)-\vartheta)\,du \overset{v=u-\Delta}{=} \varphi_{xx}(\vartheta) + \alpha\cdot\varphi_{xx}(\vartheta+\Delta) + \alpha\int_{-\infty}^{\infty} x(v)x(v-(\vartheta-\Delta))\,dv +$$

$$+ \alpha^2\int_{-\infty}^{\infty} x(v)\,x(v-\vartheta) = \varphi_{xx}(\vartheta) + \alpha\,\varphi_{xx}(\vartheta+\Delta) + \alpha\,\varphi_{xx}(\vartheta-\Delta) + \alpha^2\varphi_{xx}(\vartheta).$$