SGN-11006, Basic Course in Signal Processing

Exercise #10 (Test Exam). Solutions.

(I) $y[n] = \sum_{k=-\infty}^{n} x[k]$

a) Consider the input sequence $x'[n] = \alpha x_1[n] + \beta x_2[n]$, then

$$y'[n] = \sum_{k=-\infty}^{n} x'[k] = \sum_{k=-\infty}^{n} (\alpha x_1[k] + \beta x_2[k]) = \alpha \sum_{k=-\infty}^{n} x_1[k] + \beta \sum_{k=-\infty}^{n} x_2[k] =$$

$= \alpha y_1[k] + \beta y_2[k] \Rightarrow$ system is linear

b) Consider the delayed input $x_t[n] = x[n-t]$, then

$$y_t[n] = \sum_{k=-\infty}^{n} x_t[k] = \sum_{k=-\infty}^{n} x[k-t] = \sum_{\ell=-\infty}^{n-t} x[\ell] = y[n-t] \Rightarrow$$

$\{\ell = k - t\}$

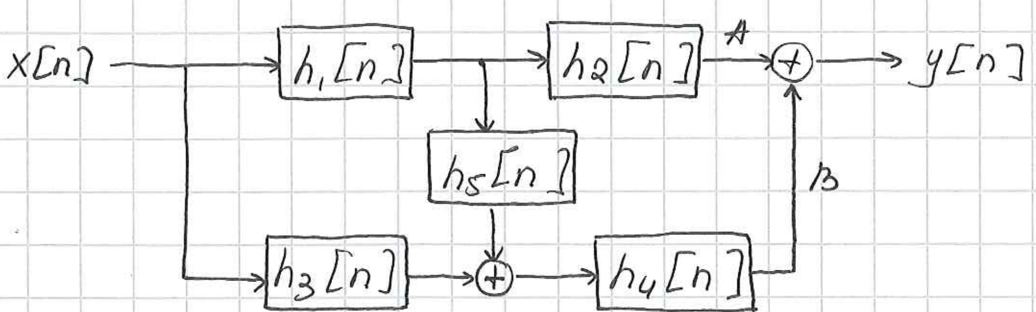
$\Rightarrow$ system is time-invariant.

c) If $|x[n]| \leq B$, then $|y[n]| \leq \sum_{k=-\infty}^{n} B = B \cdot \sum_{k=-\infty}^{n} 1$, which is not bounded $\Rightarrow$ system is not stable

d) Causal, because output $y[n]$ does not depend on future input.

e) Yes, because it is an LTI system

(II)



$A = x[n] \circledast h_1[n] \circledast h_2[n]$

$B = (x[n] \circledast h_1[n] \circledast h_5[n] + x[n] \circledast h_3[n]) \circledast h_4[n]$

a) $y[n] = A + B = x[n] \circledast \big(h_1[n] \circledast h_2[n] + h_1[n] \circledast h_5[n] \circledast h_4[n] +$
$+ h_3[n] \circledast h_4[n] \big)$

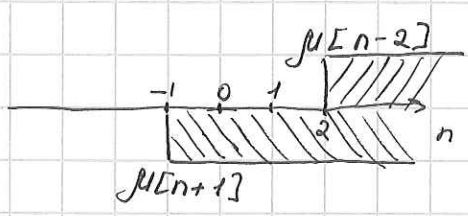
b) $h[n] = \dfrac{y[n]}{x[n]} = h_1[n] \circledast h_2[n] + h_1[n] \circledast h_5[n] \circledast h_4[n] + h_3[n] \circledast h_4[n]$

$$h_1[n] = 2\delta[n-2]$$
$$h_2[n] = 3\delta[n] + \delta[n-1]$$
$$h_3[n] = -2\delta[n+1]$$
$$h_4[n] = -\delta[n] + \delta[n+2]$$
$$h_5[n] = \delta[n+3]$$

$$h[n] = h_1[n] \circledast (3\delta[n] + \delta[n-1]) + h_1[n] \circledast \delta[n+3] \circledast h_4[n] +$$
$$+ h_3[n] \circledast (-\delta[n] + \delta[n+2]) = 3h_1[n] + h_1[n-1] +$$
$$+ h_1[n+3] \circledast h_4[n] - h_3[n] + h_3[n+2] = 6\delta[n-2] + 2\delta[n-3] +$$
$$+ 2\delta[n+1] \circledast (-\delta[n] + \delta[n+2]) + 2\delta[n+1] - 2\delta[n+3] =$$
$$= 6\delta[n-2] + 2\delta[n-3] - \cancel{2\delta[n+1]} + \cancel{2\delta[n+3]} + \cancel{2\delta[n+1]} - \cancel{2\delta[n+3]} =$$
$$= 6\delta[n-2] + 2\delta[n-3]$$

c) $H(e^{j\omega}) = 6e^{-2j\omega} + 2e^{-3j\omega}$

d) $x[n] = \mu[n-2] - \mu[n+1] =$
$= -\delta[n+1] - \delta[n] - \delta[n-1].$



$$y[n] = x[n] \circledast h[n] = -(6\delta[n-1] + 6\delta[n-2] + 6\delta[n-3] +$$
$$+ 2\delta[n-2] + 2\delta[n-3] + 2\delta[n-4]) =$$
$$= -6\delta[n-1] - 8\delta[n-2] - 8\delta[n-3] - 2\delta[n-4].$$

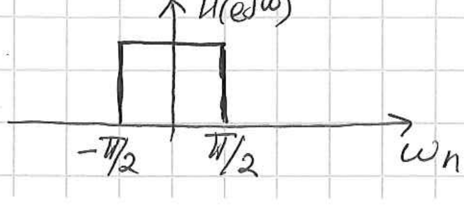
III $\left\{ \omega_n = \omega T = \dfrac{2\pi f}{f_s} \right\}$

Because the highest frequency in $x_a(t)$ is 20kHz, the minimum sampling frequency to avoid aliasing is $f_s = 40\,kHz$.

Relationship between continuous frequency f and discrete frequency ω_n is $\omega_n = \omega T = \dfrac{2\pi f}{f_s}$.

$\Rightarrow$ the frequency range $f \geq 10\,kHz$ corresponds to a digital frequency range $\omega_n \geq \dfrac{10\,kHz \cdot 2\pi}{40\,kHz} = \dfrac{\pi}{2}$

Corresponding digital filter is a low-pass filter that has frequency response :



(IV) $\{x[n]\} = \{-4.3,\ 5.2,\ 2.7,\ -3.4,\ 0,\ -2,\ 3.9,\ 4\}$ $N = 8$

$$\left\{ X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \right\} \quad \left\{ x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \right\}$$

$$\left\{ W_N = e^{-j2\pi/N} \right\}$$

a) $X[0] = \sum_{n=0}^{N-1} x[n] W_N^{0\cdot n} = \sum_{n=0}^{7} x[n] = 6.1$

b) $X[4] = \sum_{n=0}^{N-1} x[n] W_N^{4n} = \sum_{n=0}^{7} x[n] e^{-\frac{j2\pi\cdot 4n}{8}} = \sum_{n=0}^{7} x[n] e^{-j\pi n} = \sum_{n=0}^{7} (-1)^n x[n] = -1.5$

c) $\sum_{k=0}^{7} X[k] = \left\{ \text{IDFT! } x[0] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-k\cdot 0} = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \right\} = N \cdot x[0] = 8 \cdot (-4.3) = -34.4$

d) $\sum_{k=0}^{7} e^{-j(3\pi k/4)} X[k] = \sum_{k=0}^{7} \underbrace{e^{-3\frac{j2\pi k}{8}}}_{W_N^{3k} = W_N^{-(-3)k}} X[k] = N x[\langle -3 \rangle_8] = N x[5] = 8 \cdot (-2) = -16.$

e) $\sum_{k=0}^{7} |X[k]|^2 \overset{P.R}{=} N \sum_{n=0}^{N-1} |x[n]|^2 = 8 \cdot 99.59 = 796.72$

(V) $y[n] = \frac{1}{4} y[n-1] + \frac{1}{8} y[n-2] + x[n] - x[n-1]$

$y[n] - \frac{1}{4} y[n-1] - \frac{1}{8} y[n-2] = x[n] - x[n-1]$

Let $Y(z) \overset{z}{\Longleftrightarrow} y[n]$ and $X(z) \overset{z}{\Longleftrightarrow} x[n]$, then using time-shifting and linearity properties:

$Y(z)\left(1 - \frac{1}{4} z^{-1} - \frac{1}{8} z^{-2}\right) = X(z)\left(1 - z^{-1}\right)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - \frac{1}{4} z^{-1} - \frac{1}{8} z^{-2}} = \frac{1 - z^{-1}}{1 - \frac{1}{2} z^{-1} + \frac{1}{4} z^{-1} - \frac{1}{8} z^{-2}} = \frac{1 - z^{-1}}{(1 - \frac{1}{2} z^{-1}) + \frac{1}{4} z^{-1}(1 - \frac{1}{2} z^{-1})}$$

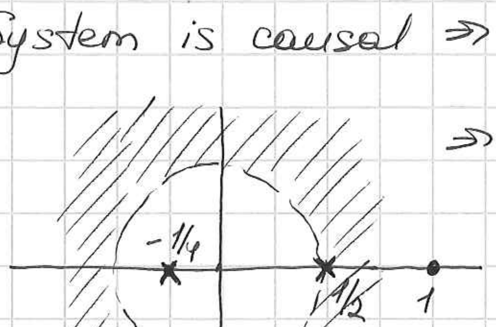
$$= \frac{1 - z^{-1}}{(1 - \frac{1}{2} z^{-1})(1 + \frac{1}{4} z^{-1})}$$

System is causal $\Rightarrow$ ROC: $\begin{cases} |z| > \frac{1}{2} \\ |z| > \frac{1}{4} \end{cases} \Rightarrow$

$\Rightarrow$ ROC $\equiv |z| > 1/2$.

Poles: $p_1 = \frac{1}{2}$, $p_2 = -\frac{1}{4}$

Zeroes: $z_1 = 1$.



System is stable because $|z| > 1/2$ includes the unit circle.

System is not minimum-phase as the zero $z_1 = 1$ is on the unit circle not incide

$$H(z) = \frac{p_1}{1-\frac{1}{2}z^{-1}} + \frac{p_2}{1+\frac{1}{4}z^{-1}} = \frac{-2/3}{1-\frac{1}{2}z^{-1}} + \frac{5/3}{1+\frac{1}{4}z^{-1}} \qquad ROC \equiv |z| > \frac{1}{2}$$

Residues:
$$p_1 = \left.\frac{1-z^{-1}}{1+\frac{1}{4}z^{-1}}\right|_{z=\frac{1}{2}} = \frac{1-2}{1+\frac{1}{2}} = -\frac{2}{3}$$

$$p_2 = \left.\frac{1-z^{-1}}{1-\frac{1}{2}z^{-1}}\right|_{z=-\frac{1}{4}} = \frac{1+4}{1+2} = \frac{5}{3}$$

$$h[n] = -\frac{2}{3}\left(\frac{1}{2}\right)^n \mu[n] + \frac{5}{3}\left(-\frac{1}{4}\right)^n \mu[n]$$

VI a) $x[n] = 4\delta[n] + 3\delta[n+2] + 3\delta[n-2]$

b) $x[n] = -\left(\frac{1}{2}\right)^n \mu[-n-1] + 3\left(\frac{1}{3}\right)^n \mu[n]$

c) $$X(z) = \frac{1}{1+z^{-1}+2z^{-1}+2z^{-2}} = \frac{1}{(1-z^{-1})+2z^{-1}(1+z^{-1})} =$$

$$= \frac{1}{(1+z^{-1})(1+2z^{-1})} = \frac{p_1}{1+z^{-1}} + \frac{p_2}{1+2z^{-1}} = \frac{-1}{1+z^{-1}} + \frac{2}{1+2z^{-1}}$$

$$p_1 = \left.\frac{1}{1+2z^{-1}}\right|_{z=-1} = \frac{1}{1-2} = -1$$

$ROC = 1 < |z| < 2 \Rightarrow$

$\underline{x[n] = -(-1)^n \mu[n] - 2(-2^n)\mu[-n-1]}$

$$p_2 = \left.\frac{1}{1+z^{-1}}\right|_{z=-2} = \frac{1}{1-\frac{1}{2}} = 2$$