SGN-11006. Basic Course in Signal Processing

Exercise # 9. Solutions

(I) $y[n] = a_1 x[n+k+1] + a_2 x[n+k] + a_2 [n+k-2] + a_1 x[n+k-3]$

$$Y(e^{j\omega}) = a_1 X(e^{j\omega})e^{(k+1)j\omega} + a_2 X(e^{j\omega})e^{kj\omega} + a_2 X(e^{j\omega})e^{(k-2)j\omega} + a_1 X(e^{j\omega})e^{(k-3)j\omega}$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \underline{a_1 e^{(k+1)j\omega} + a_2 e^{kj\omega} + a_2 e^{(k-2)j\omega} + a_1 e^{(k-3)j\omega}}$$

If $H(e^{j\omega})$ is a real function of ω then $H(e^{j\omega}) = H^*(e^{j\omega}) \Rightarrow$
$\Rightarrow H(e^{j\omega}) = H(e^{-j\omega}) \Rightarrow$

$$\Rightarrow a_1 e^{(k+1)j\omega} + a_2 e^{kj\omega} + a_2 e^{(k-2)j\omega} + a_1 e^{(k-3)j\omega} = a_1 e^{-(k+1)j\omega} + a_2 e^{-kj\omega} + a_2 e^{-(k-2)j\omega} + a_1 e^{(k-3)j\omega}$$

$$e^{(k-1)j\omega}(a_1 e^{2j\omega} + a_2 e^{j\omega} + a_2 e^{-j\omega} + a_1 e^{-2j\omega}) = e^{-(k-1)j\omega}(a_1 e^{-2j\omega} + a_2 e^{-j\omega} + a_2 e^{j\omega} + a_1 e^{2j\omega})$$

$$e^{(k-1)j\omega} = e^{-(k-1)j\omega}$$

$$k-1 = -k+1$$

$$2k = 2$$

$$\underline{k = 1.}$$

(II) $$H(z) = \frac{(5z+6)(2z-3)}{(z-0.3)(z+0.2)} = \frac{10z^2(1+\frac{6}{5}z^{-1})(1-\frac{3}{2}z^{-1})}{z^2(1-0.3z^{-1})(1+0.2z^{-1})} \quad ROC \equiv |z|>0.3$$

Poles: $z = 0.3$ and $z = -0.2$ are inside the unit circle $\Rightarrow$
$\Rightarrow H(z)$ is stable.

{ A causal stable TF with all zeroes inside/outside the unit circle is called a minimum/maximum-phase TF }

Zeroes: $z = -6/5$ and $z = 3/2$ are outside the unit circle $\Rightarrow$
$\Rightarrow H(z)$ is a maximum-phase TF.

For a maximum-phase TF $H(z)$ corresponding minimum-phase TF $G(z)$ such that $|H(e^{j\omega})| = |G(e^{j\omega})|$ can be found from the following equation: $H(z) = G(z)A(z)$. (*)
where $A(z)$ is a causal stable all pass TF.

{ All pass TF of degree N is defined as follows:
$A_N(z) = \pm \frac{z^{-N}(D_N(z^{-1}))}{D_N(z)}$ where $D_N(z) = 1 + d_1 z^{-1} + \ldots + d_{N-1} z^{-N+1} + d_N z^{-N}$
$z^{-N}(D_N(z^{-1}))$ is called a mirror-image polynom. }

Lets define $D(z^{-1})$ as $(5z+6)(2z-3)$ then
$D(z) = (5z^{-1}+6)(2z^{-1}-3)$ and $A(z) = \dfrac{z^{-2}(5z+6)(2z-3)}{(5z^{-1}+6)(2z^{-1}-3)}$

Then $G(z) = \dfrac{H(z)}{A(z)} = \dfrac{\cancel{(5z+6)}\cancel{(2z-3)}(5z^{-1}+6)(2z^{-1}-3)}{(z-0.3)(z+0.2)z^{-2}\cancel{(5z+6)}\cancel{(2z-3)}} =$

$$= \underline{\frac{(5+6z)(2-3z)}{(z-0.3)(z+0.2)}} = \underline{\frac{-18z^{+2}(1+5/6\,z^{-1})(1-2/3\,z^{-1})}{z^2(1-0.3z^{-1})(1+0.2z^{-1})}} \qquad \underline{ROC \equiv |z|>0.3}$$

Let $X(z) = \dfrac{1}{(1-0.3z^{-1})(1+0.2z^{-1})} = \dfrac{\rho_1}{1-0.3z^{-1}} + \dfrac{\rho_2}{1+0.2z^{-1}}$ $ROC \equiv |z|>0.3$

$$\rho_1 = \frac{1}{1+0.2z^{-1}}\Bigg|_{z=0.3} = \frac{3}{5} \qquad \rho_2 = \frac{1}{1-0.3z^{-1}}\Bigg|_{z=-0.2} = \frac{2}{5}$$

$$X(z) = \frac{3/5}{1-0.3z^{-1}} + \frac{2/5}{1+0.2z^{-1}} \qquad ROC \equiv |z|>0.3 \overset{z}{\Longleftrightarrow}$$

$$\overset{z}{\Longleftrightarrow} \quad x[n] = 0.6(0.3)^n\mu[n] + 0.4(-0.2)^n\mu[n].$$

$$H(z) = 10\left(1+\tfrac{6}{5}z^{-1}-\tfrac{3}{2}z^{-1}-\tfrac{9}{5}z^{-2}\right)X(z) = (10-3z^{-1}-18z^{-2})X(z) \overset{z}{\Longleftrightarrow}$$

$\overset{z}{\Longleftrightarrow}$ $\underline{h[n]} = 10x[n] - 3x[n-1] - 18x[n-2] = 6(0.3)^n\mu[n] + 4(-0.2)^n\mu[n] -$
$- 1.8(0.3)^{n-1}\mu[n-1] - 1.2(-0.2)^{n-1}\mu[n-1] - 10.8(0.3)^{n-2}\mu[n-2] - 7.2(-0.2)^{n-2}\mu[n-2].$

$$G(z) = -18\left(1+\tfrac{5}{6}z^{-1}-\tfrac{2}{3}z^{-1}-\tfrac{10}{18}z^{-2}\right)X(z) = (-18+3z^{-1}+10z^{-2})X(z) \overset{z}{\Longleftrightarrow}$$

$\overset{z}{\Longleftrightarrow}$ $\underline{g[n]} = -18x[n] + 3x[n-1] + 10x[n-2] = -10.8(0.3)^n\mu[n] - 7.2(-0.2)^n\mu[n] +$
$+ 1.8(0.3)^{n-1}\mu[n-1] + 1.2(-0.2)^{n-1}\mu[n-1] + 6(0.3)^{n-2}\mu[n-2] + 4(-0.2)^{n-2}\mu[n-2].$

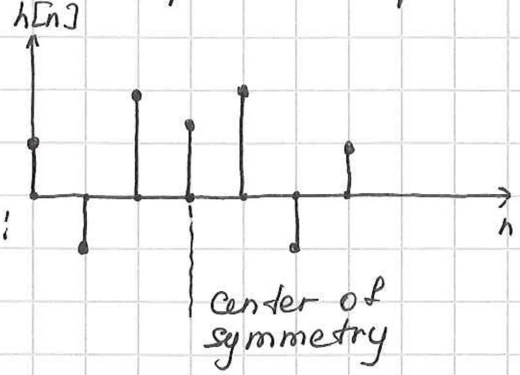
III Group delay can be computed from the phase shift as follows: $\tau(\omega) = -\dfrac{d}{d\omega}[\theta(\omega)]$ where $\theta(\omega)$ is the phase function.

If $\tau(\omega) = const$, then $\theta(\omega)$ must be a linear function of ω; $\theta(\omega) = c\omega + \beta$.
If $H(z)$ has a linear-phase, its frequency response must be of the form $H(e^{j\omega}) = e^{j(c\omega+\beta)}\breve{H}(\omega)$
where $\breve{H}(\omega)$ is amplitude response.
For $h[n] = a_0\delta[n] + \dots + a_6\delta[n-6] \overset{DTFT}{\Longrightarrow} H(e^{j\omega}) = a_0 + a_1e^{-j\omega} + a_2e^{-2j\omega} +$
$+ \dots + a_6e^{-6j\omega} = e^{j(c\omega+\beta)}\breve{H}(\omega)$

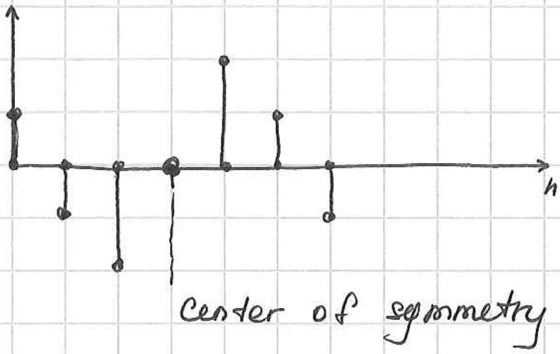
$h[n]$ is a real impulse response of length 7. $\Rightarrow$

$\Rightarrow$ Linear phase response is possible in this two cases:

Type 1:



Type 3:



Type 1: Symmetric impulse response with odd length

Because of symmetry: $h[0] = h[6]$, $h[1] = h[5]$, $h[2] = h[4]$ $\Rightarrow$

$$\Rightarrow \begin{cases} a_0 = a_6 \\ a_1 = a_5 \\ a_2 = a_4 \\ a_3 \text{ any real number} \end{cases}$$

$$\Rightarrow H(e^{j\omega}) = a_0(1+e^{-6j\omega}) + a_1(e^{-j\omega}+e^{-5j\omega}) + a_2(e^{-2j\omega}+e^{-4j\omega}) + a_3 e^{-3j\omega} =$$

$$= e^{-3j\omega}\left(a_0(e^{3j\omega}+e^{-3j\omega}) + a_1(e^{2j\omega}+e^{-2j\omega}) + a_2(e^{j\omega}+e^{-j\omega}) + a_3\right) =$$

$$= e^{-3j\omega}\underbrace{\left(2a_0\cos(3\omega) + 2a_1\cos(2\omega) + 2a_2\cos(\omega) + a_3\right)}_{\text{real function of } \omega.\ \text{amplitude response}} \Rightarrow$$

$$\Rightarrow \begin{cases} \theta(\omega) = -3\omega \\ \text{or} \\ \theta(\omega) = -3\omega \pm \pi \end{cases} \Rightarrow \tau(\omega) = -\frac{d\theta(\omega)}{d\omega} = 3.$$

{for the case when amplitude is negative, then $-1 = e^{\pm j\pi}$}

Type 2: Antisymmetric impulse response with odd length: $h[0] = -h[6]$, $h[1] = -h[5]$, $h[2] = -h[4]$, $h[3] = -h[3] = 0$. $\Rightarrow$

$$\Rightarrow \begin{cases} a_0 = -a_6 \\ a_1 = -a_5 \\ a_2 = -a_4 \\ a_3 = 0 \end{cases} \Rightarrow H(e^{j\omega}) = e^{-j3\omega} e^{j\pi/2}\left(2a_0\sin(3\omega) + 2a_1\sin(2\omega) + 2a_2\sin(\omega)\right) \Rightarrow$$

$$\Rightarrow \begin{cases} \theta(\omega) = -3\omega + \pi/2 \\ \text{or} \\ \theta(\omega) = -3\omega + \frac{3\pi}{2} \end{cases} \Rightarrow \tau(\omega) = -\frac{d\theta(\omega)}{d\omega} = 3.$$