SGN - 11006, Basic Course in Signal Processing

Exercise # 3, Solutions.

(I) $g_1[n] \xrightarrow{DTFT} G_1(e^{j\omega})$

$g_2[n] = g_1[n-2] + g_1[n-6]$, using linearity and time-shifting properties of DTFT:
$G_2(e^{j\omega}) = G_1(e^{j\omega}) \cdot e^{-2j\omega} + G_1(e^{j\omega}) \cdot e^{-6j\omega}$.

$g_3[n] = g_1[-n-1] + g_1[n]$, using linearity, time-shifting and time-reversal properties of DTFT:
$G_3(e^{j\omega}) = G_1(e^{-j\omega}) \cdot e^{j\omega} + G_1(e^{j\omega})$.

$g_4[n] = g_1[n+7] + g_1[-n]$, using linearity, time-shifting and time-reversal properties of DTFT:
$G_4(e^{j\omega}) = G_1(e^{j\omega}) \cdot e^{7j\omega} + G_1(e^{-j\omega})$.

(II) $x[n] = \alpha^n \mu[n]$, $|\alpha| < 1$ $\boxed{\text{Example from lecture slides}}$

$$\alpha^n \mu[n] \xrightarrow{DTFT} \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} =$$
$$= \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n = \frac{1}{1-\alpha e^{-j\omega}} \quad \text{as } |\alpha e^{-j\omega}| < 1.$$

a) $$\alpha^n \mu[n-1] = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n-1] e^{-j\omega n} = \sum_{n=1}^{\infty} \alpha^n e^{-j\omega n} =$$
$$= \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} - \alpha^0 e^{-j\omega \cdot 0} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} - 1 \overset{\{|\alpha|<1\}}{=} \frac{1}{1-\alpha e^{-j\omega}} - 1 =$$
$$= \frac{\alpha e^{-j\omega}}{1-\alpha e^{-j\omega}}.$$

b) $$\overset{x_2[n] \shortparallel}{n\alpha^n \mu[n+1]} = \sum_{n=-\infty}^{\infty} n\alpha^n \mu[n+1] e^{-j\omega n}$$

$$x_2[n] = \begin{cases} n \overbrace{\alpha^n \mu[n]}^{x[n]}, & n \geq 0 \\ -\frac{1}{\alpha}, & n = -1 \\ 0, & n < -1 \end{cases}$$

$$\sum_{n=-\infty}^{\infty} n\alpha^n \mu[n+1] e^{-j\omega n} = -\frac{e^{j\omega}}{\alpha} + \sum_{n=-\infty}^{\infty} \overbrace{n\alpha^n \mu[n]}^{n\,x[n]} =$$

$$= \left\{ \begin{array}{l} \text{differentiation in frequency} \\ \text{property } n g[n] \Rightarrow j \frac{dG(e^{j\omega})}{d\omega} \end{array} \right\} = -\frac{e^{j\omega}}{\alpha} + j\frac{dX(e^{j\omega})}{d\omega} =$$

$$= -\frac{e^{j\omega}}{\alpha} + j\frac{d(1-\alpha e^{-j\omega})^{-1}}{d\omega} = \{ d(f^n) = n f^{n-1} df \} =$$

$$= -\frac{e^{j\omega}}{\alpha} - j\frac{(1-\alpha e^{-j\omega})^{-2} d(1-\alpha e^{-j\omega})}{d\omega} = -\frac{e^{j\omega}}{\alpha} - j\frac{-\alpha d e^{-j\omega}}{(1-\alpha e^{-j\omega})^2 d\omega} =$$

$$= \left\{ \frac{de^u}{dx} = e^u \frac{du}{dx} \right\} = -\frac{e^{j\omega}}{\alpha} + j\frac{\alpha e^{-j\omega}\cdot(-j)\cancel{d\omega}}{(1-\alpha e^{-j\omega})^2 \cancel{d\omega}} =$$

$$= -\frac{e^{j\omega}}{\alpha} + \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2} = \frac{-e^{j\omega}(1-\alpha e^{-j\omega})^2 + \alpha^2 e^{-j\omega}}{\alpha(1-\alpha e^{-j\omega})^2} =$$

$$= \frac{-e^{j\omega} + 2\alpha - \alpha^2 e^{-j\omega} + \alpha^2 e^{-j\omega}}{\alpha(1-\alpha e^{-j\omega})^2} = \frac{-e^{j\omega}+2\alpha}{\alpha(1-\alpha e^{-j\omega})^2}.$$

c). $\alpha^n \mu[-n] = \left(\frac{1}{\alpha}\right)^{-n} \mu[-n] = \underbrace{\beta^{-n}\mu[-n]}_{x[-n]}, \quad |\beta| < 1.$

$$x[-n] \underset{\{\text{time-reversal}\}}{\xrightarrow{DTFT}} X(e^{-j\omega}) = \frac{1}{1-\beta e^{j\omega}} = \frac{\alpha}{\alpha - e^{j\omega}}.$$

III. a) $Y(e^{j\omega}) = H_2(e^{j\omega})\cdot(H_1(e^{j\omega})\cdot X(e^{j\omega}) + H_3(e^{j\omega})\cdot Y(e^{j\omega}))$

$$Y(e^{j\omega}) = H_2(e^{j\omega})\cdot H_1(e^{j\omega})\cdot X(e^{j\omega}) + H_2(e^{j\omega})\cdot H_3(e^{j\omega})\cdot Y(e^{j\omega}).$$

$$Y(e^{j\omega})\cdot(1 - H_2(e^{j\omega})\cdot H_3(e^{j\omega})) = H_2(e^{j\omega})\cdot H_1(e^{j\omega})\cdot X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{H_2(e^{j\omega})\cdot H_1(e^{j\omega})\cdot \cancel{X(e^{j\omega})}}{(1 - H_2(e^{j\omega})\cdot H_3(e^{j\omega}))\cdot \cancel{X(e^{j\omega})}} = \frac{H_2(e^{j\omega})\cdot H_1(e^{j\omega})}{1 - H_2(e^{j\omega}) H_3(e^{j\omega})}.$$

{ Here we use that convolution in time domain corresponds to multiplication in frequency domain, e.g. $y[n] = h[n] \circledast x[n] \Rightarrow Y(e^{j\omega}) = H(e^{j\omega})\cdot X(e^{j\omega})$. }

b) $h_1[n] = \frac{\sin(\pi n)}{\pi n} \underset{\{\text{sinc function}\}}{\xrightarrow{DTFT}} \text{rect}\left(\frac{\omega}{\pi}\right) = \begin{cases} 1, & |\omega| < \pi \\ 0, & |\omega| = \pi \end{cases}$

$$h_2[n] = 3^n \mu[-n] \underset{\{\text{using II. b}\}}{\xrightarrow{DTFT}} \frac{3}{3-e^{j\omega}}$$

$$h_3[n] = n\delta[n-2] = \sum_{n=-\infty}^{\infty} n\delta[n-2] e^{-j\omega n} \underset{\{n=2\}}{=} 2e^{-2j\omega}.$$

$$H(e^{j\omega}) = \frac{\frac{3}{3-e^{j\omega}} \cdot rect\left(\frac{\omega}{\pi}\right)}{1 - \frac{6e^{-2j\omega}}{3-e^{j\omega}}} = \frac{3 \cdot rect\left(\frac{\omega}{\pi}\right)}{3 - e^{j\omega} - 6e^{-2j\omega}} =$$

$$= \begin{cases} \frac{3}{3-e^{j\omega}-6e^{-2j\omega}} & , |\omega| < \pi \\ 0 & , |\omega| = \pi \end{cases}$$