SGN-11006, Basic Course in Signal Processing

Exercise #5. Solutions

(I.) a) $w[n] = \alpha x[\langle n-m_1\rangle_N] + \beta x[\langle n-m_2\rangle_N] \xRightarrow{DFT}$
{using linearity and circular time-shifting prop. of DFT}
$\xRightarrow{DFT} \alpha W_N^{km_1} X[k] + \beta W_N^{km_2} X[k]$

b) $g[n] = (-1)^n x[n] = e^{\pm j\pi n} x[n] = W_N^{\pm \frac{Nn}{2}} x[n] \xRightarrow{DFT}$
{using circular frequency shifting prop}
$\xRightarrow{DFT} X[\langle k \pm \frac{N}{2}\rangle_N]$

c) $y[n] = x[n] \times x[n] \xRightarrow{DFT}$ {modulation}
$\xRightarrow{DFT} \underbrace{\frac{1}{N}\sum_{m=0}^{N-1} X[m]X[\langle k-m\rangle_N]}_{N\text{-point circular convolution}} = \frac{1}{N} X[k] \overset{N}{\circledast} X[k]$

(II.) $x[n]$ is a real-valued sequence $\Rightarrow X[k] = X^*[\langle -k\rangle_N]$,

$X[0] = X^*[\langle N-0\rangle_N] = X^*[0] \Rightarrow 8.8 + j\alpha = 8.8 - j\alpha \Rightarrow \boxed{\alpha = 0}$

$X[k_1] = X^*[\langle N-k_1\rangle_N] = 2.1 - j4.5$, as all the samples except those that are given are equal to zero, we can observe that $X^*[\langle N-k_1\rangle_N] = X^*[108] \Rightarrow$
$\Rightarrow \boxed{k_1 = N - 108 = 130 - 108 = 22}$ and $\boxed{\varepsilon = 2.1}$

$X[k_2] = X^*[\langle N-k_2\rangle_N] = 3.6 + j1.3$, this is true only if $X^*[\langle N-k_2\rangle_N] = X^*[79] \Rightarrow \boxed{k_2 = 130 - 79 = 51}$ and $\boxed{\delta = -1.3}$

$X[65] = X^*[\langle 130-65\rangle_N] = X^*[65] \Rightarrow \boxed{\beta = 0}$

$X[k_3] = X^*[\langle N-k_3\rangle_N] = \gamma + j7.1$, this is true only if $X^*[\langle N-k_3\rangle_N] = X^*[55] \Rightarrow \boxed{k_3 = 130 - 55 = 75}$ and $\boxed{\gamma = 5}$

$X[k_4] = X^*[\langle N-k_4\rangle_N] = -3.7 - j2.2 \Rightarrow \boxed{k_4 = 130 - 13 = 117}$

$E = \sum_{n=0}^{N-1} |x[n]|^2 =$ {Perseval's relation} $= \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\cdot X^*[k] =$
$= \underline{667.94}$

III $\{x[n]\} = \{6.5, 8, -7.5, 8.5, -2.5, -8, -4.5, 1\}, N=8$

$$\left\{ X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \right\}$$

a) $X[0] = \sum_{n=0}^{N-1} x[n] W_N^{0\cdot n} = \sum_{n=0}^{N-1} x[n] = 6.5$

b) $X[4] = \sum_{n=0}^{N-1} x[n] W_N^{4n} = \sum_{n=0}^{7} x[n] e^{\frac{-j2\pi\cdot 4n}{8}} = \sum_{n=0}^{7} x[n] e^{-j\pi n} =$

$= \sum_{n=0}^{N-1} (-1)^n x[n] = 6.5 - 8 - 7.5 - 8.5 + 2.5 + 8 - 4.5 - 1 = -12.5$

c) $\sum_{k=0}^{N-1} X[k] = \left\{ \text{IDFT}: x[0] = \frac{1}{N} \sum_{k=0} X[k] W_N^{-k\cdot 0} = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \right\} =$

$= N x[0] = 8 \cdot 6.5 = 52$

d) $\sum_{k=0}^{N-1} e^{-11j\pi k/4} X[k] = \sum_{k=0}^{N-1} \underbrace{e^{-11\frac{j2\pi k}{8}}}_{W_N^{-(-11)k}} X[k] = N x[\langle -11 \rangle_N] =$

$= N x[5] = 8 \cdot (-8) = -64$

e) $\sum_{k=0}^{N-1} |X[k]|^2 \overset{\text{P.R.}}{=} N \sum_{n=0}^{N} |x[n]|^2 = 8 \cdot 326.25 = 2610.$

IV. $x_1 = \{-2, 0, 1, 3\}$ $x_2 = \{1, 4, -2, 0\}$

DFT can be done by combining the functions;

$x[n] = x_1[n] + j x_2[n] = \{-2+j, 4j, 1-2j, 3\}$

Using the matrix relation:

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = D_4 \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \cdot \begin{bmatrix} -2+j \\ 4j \\ 1-2j \\ 3 \end{bmatrix} = \begin{bmatrix} -2+j+4j+1-2j+3 \\ -2+j+4-1+2j+3j \\ -2+j-4j+1-2j-3 \\ -2+j-4-1+2j-3j \end{bmatrix} =$$

$$= \begin{bmatrix} 2+3j \\ 1+6j \\ -4-5j \\ -7 \end{bmatrix}$$

$X_1[k] = \frac{1}{2}\{X[\langle k \rangle_N] + X^*[\langle -k \rangle_N]\}$

$X_2[k] = \frac{1}{2j}\{X[\langle k \rangle_N] - X^*[\langle -k \rangle_N]\}.$

$X[\langle -k \rangle_N] = X[\langle 4-k \rangle_N] = \{X[0], X[3], X[2], X[1]\}$

$X^*[\langle -k \rangle_N] = \{2-3j, -7, -4+5j, 1-6j\}$

$$X_1[k] = \frac{1}{2}\{2+3j+2-3j,\ 1+6j-7,\ -4-5j-4+5j,\ -7+1-6j\} =$$

$$= \underline{\{2,\ -3+3j,\ -4,\ -3-3j\}}$$

$$X_2[k] = \frac{1}{2j}\{2+3j-2+3j,\ 1+6j+7,\ -4-5j+4-5j,\ -7-1+6j\} =$$

$$= \underline{\{3,\ 3-4j,\ -5,\ 3+4j\}}.$$