SGN-11006. Basic Course In Signal Processing

Exercise #7. Solutions.

(I). $H(z) = \dfrac{-6z - 3 + 2z^{-1}}{1 - 3z^{-1}}$ $\quad ROC \equiv |z| > 3$

$$H(z) = -6z \frac{1}{1-3z^{-1}} - 3\frac{1}{1-3z^{-1}} + 2z^{-1}\frac{1}{1-3z^{-1}} \Rightarrow \left\{ \alpha^n \mu[n] \overset{z}{\Leftrightarrow} \right.$$

$$\left. \overset{z}{\Leftrightarrow} \frac{1}{1-\alpha z^{-1}},\ |z| > |\alpha| + \text{time shifting and linearity properties} \right\} \Rightarrow$$

$$\underline{h[n] = -6 \cdot 3^{n+1}\mu[n+1] - 3 \cdot 3^n \mu[n] + 2 \cdot 3^{n-1}\mu[n-1]}$$

(II). $h[n] = 2^n \mu[-n]$, $\quad x[n] = (-0.5)^n \mu[n]$

$$h[n] = 2^n\mu[-n] \overset{z}{\Rightarrow} H(z) = \frac{1}{1-\frac{1}{2}z} \quad |z| < 2 \quad \left\{ \text{using time-reversal property} \right\}$$

$$x[n] = (-0.5)^n\mu[n] \overset{z}{\Rightarrow} X(z) = \frac{1}{1+\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}.$$

$$Y(z) = H(z)X(z) = \frac{1}{(1-\frac{1}{2}z)(1+0.5z^{-1})} = \frac{-2z^{-1}}{(1-2z^{-1})(1+0.5z^{-1})} \quad ROC \equiv \frac{1}{2} < |z| < 2$$

Residues:

$$p_1 = \left.\frac{-2z^{-1}}{1+0.5z^{-1}}\right|_{z=2} = -\frac{4}{5} \qquad p_2 = \left.\frac{-2z^{-1}}{1-2z^{-1}}\right|_{z=-0.5} = \frac{4}{5}$$

$$Y(z) = \frac{-4/5}{1-2z^{-1}} + \frac{4/5}{1+0.5z^{-1}} \Rightarrow \left\{ -\alpha^n\mu[-n-1] \overset{z}{\Leftrightarrow} \frac{1}{1-\alpha z^{-1}} \quad |z| < |\alpha| \right\}$$

$$\Rightarrow \underline{y[n] = \frac{4}{5} 2^n \mu[-n-1] + \frac{4}{5}\left(-\frac{1}{2}\right)^n \mu[n]}$$

(III) $X(z) = \dfrac{1 - 5z^{-3}}{(1-2z^{-1})(1+0.4z^{-1})}$ $\quad ROC \equiv 0.4 < |z| < 2$

$$X(z) = \underbrace{\frac{1}{(1-2z^{-1})(1+0.4z^{-1})}}_{G(z)} - 5z^{-3} \underbrace{\frac{1}{(1-2z^{-1})(1+0.4z^{-1})}}_{G(z)}$$

Lets find inverse z-transform of $G(z)$

Residues:

$$p_1 = \frac{1}{1+0.4z^{-1}}\Big|_{z=2} = \frac{5}{6} \qquad p_2 = \frac{1}{1-2z^{-1}}\Big|_{z=-0.4} = \frac{1}{6}$$

$$\begin{cases} G(z) = \frac{5/6}{1-2z^{-1}} + \frac{1/6}{1+0.4z^{-1}} \Rightarrow g[n] = -\frac{5}{6}2^n\mu[-n-1] + \frac{1}{6}(-0.4)^n\mu[n] \\ ROC \equiv 0.4<|z|<2 \end{cases}$$

Then, $X(z) = G(z) - 5z^{-3}G(z) \Rightarrow \{\text{time-shifting + linearity}\} \Rightarrow$
$\Rightarrow x[n] = g[n] - 5g[n-3]$

$$\underline{x[n] = -\frac{5}{6}2^n\mu[-n-1] + \frac{1}{6}(-0.4)^n\mu[n] + \frac{25}{6}2^{n-3}\mu[-n+2] - \frac{5}{6}(-0.4)^{n-3}\mu[n-3]}$$

(IV) $y[n] - y[n-1] + \frac{1}{4}y[n-2] = 3x[n] - 3x[n-1] - \frac{9}{4}x[n-2] + 2x[n-3] - \frac{1}{4}x[n-4]$

Let $Y(z) \overset{z}{\Leftrightarrow} y[n]$ and $X(z) \overset{z}{\Leftrightarrow} x[n]$, then using time-shifting and Linearity: $Y(z)(1 - z^{-1} + \frac{1}{4}z^{-2}) = X(z)(3 - 3z^{-1} - \frac{9}{4}z^{-2} + 2z^{-3} - \frac{1}{4}z^{-4})$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3 - 3z^{-1} - \frac{9}{4}z^{-2} + 2z^{-3} - \frac{1}{4}z^{-4}}{1 - z^{-1} + \frac{1}{4}z^{-2}} =$$

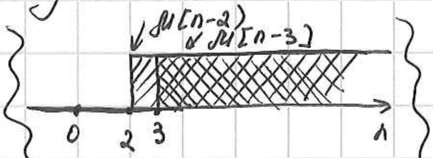
$$= \frac{3(1 - z^{-1} - 1/4\, z^{-2}) - 12/4\, z^{-2} + 2z^{-3} - 1/4\, z^{-4}}{1 - z^{-1} + 1/4\, z^{-2}} = 3 + \frac{-3z^{2} + 2z^{-3} - 1/4\, z^{-4}}{1 - z^{-1} + 1/4\, z^{-2}} =$$

$$= 3 + \frac{-2z^{-2}(1 - z^{-1} + 1/4\, z^{-2}) - z^{-2} + 1/4\, z^{-4}}{1 - z^{-1} + 1/4\, z^{-2}} = 3 - 2z^{-2} + \frac{-z^{-2}(1 + 1/2\, z^{-1})}{(1 - 1/2\, z^{-1})^2}$$

$$= 3 - 2z^{-2} - \frac{z^{-2}}{1 - 1/2\, z^{-1}} - \frac{1/2\, z^{-3}}{1 - 1/2\, z^{-1}} \qquad ROC \equiv |z| > \frac{1}{2}. \ \{\text{system is causal}\}$$

$$\Rightarrow h[n] = 3\delta[n] - 2\delta[n-2] - \left(\frac{1}{2}\right)^{n-2}\mu[n-2] - \frac{1}{2}\left(\frac{1}{2}\right)^{n-3}\mu[n-3] =$$

$$= 3\delta[n] - 2\delta[n-2] - \left(\frac{1}{2}\right)^{n-2}\left(\mu[n-2] + \mu[n-3]\right) =$$



$$= 3\delta[n] - 2\delta[n-2] - \delta[n-2] - 2\cdot\left(\frac{1}{2}\right)^{n-2}\mu[n-3] =$$

$$= \underline{3\delta[n] - 3\delta[n-2] - \left(\frac{1}{2}\right)^{n-3}\mu[n-3]}.$$

System is stable because $|z| > \frac{1}{2}$ includes the unit circle.

Another Solution: $\left\{ G(z) = \dfrac{1/2\, z^{-1}}{(1-1/2 z^{-1})^2} \Rightarrow g[n] = n\left(\frac{1}{2}\right)^n \mu[n] \quad ROC \equiv |z| > \frac{1}{2} \right\}$

$$H(z) = 6zG(z) - 6G(z) - \frac{9}{2}z^{-1}G(z) + 4z^{-2}G(z) - \frac{1}{2}z^{-3}G(z) \overset{z}{\Rightarrow}$$

$$\Rightarrow 6(n+1)\left(\frac{1}{2}\right)^{n+1}\mu[n+1] - 6n\left(\frac{1}{2}\right)^n\mu[n] - \frac{9}{2}(n-1)\left(\frac{1}{2}\right)^{n-1}\mu[n-1] + 4(n-2)\left(\frac{1}{2}\right)^{n-2}\mu[n-2] -$$

$$- \frac{1}{2}(n-3)\left(\frac{1}{2}\right)^{n-3}\mu[n-3] = 6\sum_{k=-1}^{\infty}(k+1)\left(\frac{1}{2}\right)^{k+1}\delta[n-k] - 6\sum_{k=0}^{\infty}k\left(\frac{1}{2}\right)^k\delta[n-k] -$$

$$-\frac{9}{2}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k-1}(k-1)\delta[n-k] + 4\sum_{k=2}^{\infty}(k-2)\left(\frac{1}{2}\right)^{k-2}\delta[n-k] - \sum_{k=3}^{\infty}(k-3)\left(\frac{1}{2}\right)^{k-2}\delta[n-k] =$$

$$= 3\delta[n] + 3\delta[n-1] + \frac{9}{4}\delta[n-2] + 6\sum_{k=3}^{\infty}(k+1)\left(\frac{1}{2}\right)^{k+1}\delta[n-k] - 3\delta[n-1] -$$

$$- 3\delta[n-2] - 6\sum_{k=3}^{\infty}k\left(\frac{1}{2}\right)^k\delta[n-k] - \frac{9}{4}\delta[n-2] - \frac{9}{2}\sum_{k=3}^{\infty}\left(\frac{1}{2}\right)^{k-1}(k-1)\delta[n-k] +$$

$$+ 4\sum_{k=3}^{\infty}(k-2)\left(\frac{1}{2}\right)^{k-2}\delta[n-k] - \sum_{k=3}^{\infty}(k-3)\left(\frac{1}{2}\right)^{k-2}\delta[n-k] =$$

$$= 3\delta[n] - 3\delta[n-2] + \sum_{k=3}^{\infty}\left(\frac{1}{2}\right)^{k-2}\left(\frac{6}{8}(k+1) - \frac{6}{4}k - \frac{9}{4}(k-1) + 4(k-2) - k + 3\right)\delta[n-k] =$$

$$= 3\delta[n] - 3\delta[n-2] + \sum_{k=3}^{\infty}\left(\frac{1}{2}\right)^{k-2}\left(\frac{3}{4}k + \frac{3}{4} - \frac{6}{4}k - \frac{9}{4}k + \frac{9}{4} + 4k - 8 - k + 3\right)\delta[n-k] =$$

$$= 3\delta[n] - 3\delta[n-2] + \sum_{k=3}^{\infty}\left(\frac{1}{2}\right)^{k-2}(6-8)\delta[n-k] = \underline{3\delta[n] - 3\delta[n-2] - \left(\frac{1}{2}\right)^{n-3}\mu[n-3]}.$$