MAT 180 Theoretical Homework 1

Problem 1: Recall the outer product of two vectors $v \in \mathbb{R}^m, w \in \mathbb{R}^n$, denoted $\text{outer}(v, w)$ consists of the matrix $(v_i w_j) \in \mathbb{R}^{m \times n}$. Show that for all matrices $A \in \mathbb{R}^{m \times k}$ and $B \in \mathbb{R}^{k \times n}$ we have

$$AB = \sum_{i=1}^{k} \text{outer}(a_{*,i}, b_{i,*})$$

where $a_{*,j}$ denotes the jth column of A and $b_{i,*}$ denotes the ith row of B.

Hint: write A as a sum of its columns (embedded in matrices the same size as A with all zeros outside a single column) and similarly B as a sum of its rows. Use the distributive property of matrix multiplication and notice that most terms are zero matrices.

Problem 2: Given the matrix

$$M = \begin{pmatrix} 3 & 0 \\ 1 & 0 \\ 0 & -2 \end{pmatrix}$$

Compute an SVD for M by finding the eigenvectors of $M^T M$.

Problem 3: Let Q be an orthogonal matrix. Use an SVD argument to determine the operator 2-norm of Q.

Problem 4: To finish our discussion of PCA from lecture, prove by induction on k that

$$V_k = \underset{C:C^T C = I_k}{\text{argmax}} \, \text{Tr}(C^T X^T X C)$$

where the columns of V_k are the eigenvectors corresponding to the largest k eigenvalues of $X^T X$.