On the elements of logic in quantum statistics

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Abstract

The problem of determining the rational context for the statistical principles of quantum theory (and the highly successful account it provides of subatomic behaviour) has long eluded a solution. This despite the earliest investigations into the problem coinciding with the articulation of these principles. The most learned conjectures in this regard indicate that a complete theoretical account of "fine-grained" and generally *non-differentiable* motion is needed before one may hope to answer the questions at the base of the problem. As the notion of "fine-graining", along with the related notions of "order", "scale" and "pathology", is most extensively developed (prior to the conception of quantum theory) in the realm of statistical mechanics, this last is reasonably regarded as a natural starting point of further inquiry. However, as quantum theory purports to provide a principle for the time evolution of simple systems rather than ensembles (as with a single particle in a given environment) it is fundamentally occupied with the elementary units of which an ensemble is an aggregate. While classical theory requires that the particles of a low density, high volume gas tend toward uniform motion in between collisions with other molecules, this assumption is *not* made with respect to subatomic particles. Hence the purported need for a separate *quantum* thermodynamics, founded on distinct ("quantum") dynamic principles. It is interesting to enquire as to the extent to which we can reasonably expect a consistent and coherent theory of non-differentiable motion to yield a dynamics capable of supporting the observations of quantum thermodynamics, in a manner analogous to the role of Brownian motion in classical thermodynamics.

Intuition appears to indicate that any consistent formulation of physical principles must result in descriptions of phenomena with complexity varying according to whether the system under study is elementary, or an *aggregate* of elementary systems. In classical dynamics, this principle is realised by laws that are first articulated in terms of *point particles* and then extended to describe aggregates. However, the abstraction here is less than straightforward: a point particle does not represent an object of zero dimensions, but, rather a system whose "internal" motions may be neglected in the context of a certain inquiry. This follows a common rule of thumb in science, whereby one frames a complex problem as a composition of simpler, instantly soluble ones. In this sense, the problem of analysing the behaviour of a fluid (say, a gas) of several molecules is relegated to the position of a corollary to the similar problem for a gas consisting of a single molecule, i.e. the dynamics of a particle in a given environment[1].

The solution to the latter constitues the bulk of classical mechanics, and endeavours to predict the motions of small numbers of particles in various circumstances, more or less exactly describable by differential equations. The resulting formulation appears to place restrictions on particle trajectories such as twice-differentiability, so as to enable the solution to the second problem[2] of mechanics: given a particle's motion $\boldsymbol{r}(t)$, to determine the *causes* of motion in the form of a *force law* $\boldsymbol{F}=\boldsymbol{F}(\boldsymbol{r},\dot{\boldsymbol{r}})$. The solution $m\ddot{\boldsymbol{r}}=\boldsymbol{F}$ requires differentiating the particle's trajectory $\boldsymbol{r}(t)$ twice in time t. However, these restrictions are recognised, in both theory and practice, as being wholly formal, as the solution of the inverse problem—the first problem of mechanics—is reduced to the

1. This corresponds to a logical sequence for the development of the associated ideas, though not a historical one.

2. We superficially distinguish between the two problems of mechanics in the order in which they are articulated by Newton: i) "...of motions resulting from any forces whatsoever..." and ii) "...of the forces required to produce any motions..."

general solution to the equation of motion. This general solution is a *prescription*, a method to take $\boldsymbol{r}(t)$ into $\boldsymbol{r}\,(t+\Delta\,t)$ by performing:

$$\dot{\boldsymbol{r}}\,(t+\Delta\,t) \leftarrow \dot{\boldsymbol{r}}(t) + \frac{\boldsymbol{F}(\boldsymbol{r}(t), \dot{\boldsymbol{r}}(t))}{m}\,\Delta\,t + O\,(\Delta\,t^2)$$

$$\boldsymbol{r}\,(t+\Delta\,t) \leftarrow \boldsymbol{r}(t) + \dot{\boldsymbol{r}}\,(t+\Delta\,t)\,\Delta\,t + \frac{\boldsymbol{F}(\boldsymbol{r}(t), \dot{\boldsymbol{r}}(t))}{2\,m}\,\Delta\,t^2 + O\,(\Delta\,t^3)$$

repeatedly. The entire utility of "exact" solutions lies in the relative unwieldiness of the above prescription. There are notable instances wherein they lose even this virtue, as in the study of planetary motion. But their chief shortcoming lies in the fact that the acceptance of an exact solution as "real" requires certain arbitrary assumptions about the nature of space, time and change which are not apriori true, and relate to questions about these not yet settled by either theory or experience.

It may be noted that the above framing of the general solution removes the restriction of twice differentiability on trajectories $\boldsymbol{r}(t)$, as the asymptotic terms in the above solution can be formally substituted by non-differentiable functions. Furthermore, classical dynamics requires that particles tend toward uniform rectilinear motion $\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}\,t$ in free space, and subsequently in between collisions in a low density, high volume gas. (Incidentally, the solution for the motion of a particle in free space is the only instance in which the differentiability of the resulting solution has a significance that is more than formal.)[3] As this requirement is abandoned in the quantum theory of a particle in free space, this forms a key motivation for the conjectures that the observations of quantum theory may be generated from a theory which admits of general "non-differentiable" motion and, perhaps, has particles describing such motion even in free space.

Statistical mechanics outlines ways in which to obtain formal descriptions of the motions of the constituent particles of an aggregate. The method of beginning from the dynamics of one particle $\boldsymbol{r}_i = (D^2)^{-1}\,\boldsymbol{F}_i$, and making macroscopic inferences about aggregates is fraught with technical difficulties, even after the replacement of the asymptotic expressions above with exact solutions. Instead, we have formal descriptions of this behaviour arrived at independently by purely statistical considerations, such as the trigonometric series developed to describe Brownian motion. By virtue of their general *non-differentiability* in time t, these last do not constitute well-formed inputs to the second problem of mechanics; that is to say they do not admit a quantitative account of the forces responsible for the motion in question within the framework of classical dynamics.

In mechanics, one may expect to remove this difficulty in a manner analogous to the handling of the apparently non-differentiable motions of particles exposed to an *impulsive force*, as with a particle reflecting against a potential wall. Such a particle with incident momentum $\boldsymbol{p}_i$ and reflecting momentum $\boldsymbol{p}_r$ should have experienced an impulsive force

$$\boldsymbol{F}(t) = [\boldsymbol{p}_i + (\boldsymbol{p}_r - \boldsymbol{p}_i)\,\theta\,(t-t_0)]' = (\boldsymbol{p}_r - \boldsymbol{p}_i)\,\delta\,(t-t_0).$$

The interpretation of this resut is that the 'real' force is an approximation to $\delta\,(t-t_0)$, as with

$$\boldsymbol{F}(t) = \frac{\boldsymbol{K}}{a^2\cosh^2 w\,(t-t_0) + b^2\sinh^2 w\,(t-t_0)}, \boldsymbol{r}(t) = (a\cosh w\,(t-t_0), b\sinh w\,(t-t_0))$$

i.e., the Coloumbic reflection of a fast moving particle repelled by a like charge (eg. Rutherord). The elementary constituents of any aggregate can then freely describe non-differentiable motions which may be processed and substituted as above. This is needed to satisfy the requirement of twice-differentiability which we see is, in practice, stringent and ultimately with very little conceptual support. This is the general way in which mechanics deals with high curvature paths, but the aforementioned formal difficulties of the classical theory arise in taking the curvature to infinity, where classical dynamics is found wanting in the facility to draw conclusions theoretically.

3. This principle is supplanted with the description of geodesics in a specific spacetime environment in general relativity.

1 Generalised kinematics

It is interesting to note that the manner in which the following achieves this facility is, at least tenuosuly, related to a general procedure in the statistical analysis of *fields*. As can be seen, the aforementioned prescription argues a local power series expansion, $\Sigma_{0\leq i\leq n}\frac{\boldsymbol{a}_i\,(\Delta\,t)^i}{i!}+O\,(\Delta\,t^{n+1})$. As trigonometric series have significantly greater facility for arguing high-curvature, this seems to lend greater weight to the similar procedure in which the coefficients of *trigonometric* series expansions are used as dynamical variables, as is already done in formulating the entropy of radiation.

The expressions used to describe the shape of the paths of particles executing Brownian motion are examples of non-differentiable entities, developed in the essentially *geometric* tradition which uses trigonometric series to generate patterns with varied degrees of pathology. Such entities have been, in the past, generated along two parallel lines of development, the former being the geometric and the latter deriving from an essentially *logical* or *symbolic* tradition that exploits the inherent pathologies of the real number system to simulate fine-grained behaviour. In particular, functions of this latter tradition tended to separately argue digits in the base 2 expansion of their real number arguments. It becomes apparent that these two aproaches are, in fact, not independent, when it is realised that the base 2 expansion of $t\in[0,\infty)$ has its n^{th} digit D_n oscillating between 0 and 1 with a period of 2^{n+1}, so that it is generally given by:

$$d_n(t)=\Sigma_{j=-\infty}^{\infty}a_j\exp\,(i\,2^{-n}\,\pi\,j\,t)$$

with $a_j=\frac{\exp\,(-i\,\pi\,j)-\exp\,(-i\,2\,\pi\,j)}{i\,2\,\pi\,j}$. The only deviations of D_n from d_n arise at each of the discontinuities of d_n, $t=2^n\,m, m\in\boldsymbol{W}$, where it takes the value $\frac{1}{2}$. However, it will be apparent in what follows that the functions $1-D_n(t)$ and $D_r(t)\,D_s(t)$ are elementary functions of t out of which functions of arbitrary complexity may be derived, and the substitution of d_n for D_n allows for a consistent use of these operations, especially so with the implicit understanding that when the result of evaluating d_n is neither 0 or 1, it must be rounded up. The technical difficulties can also be removed with the aid of a tedious formal expression

[4]

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Taking these (the digits) as the elementary objects of our analysis, we must be able to provide a logical description of the most general possible motion. Classical mechanics also describes the largest family of curves that can be said to correspond to an actual particle's motion, and it resembles the above expressions for $\boldsymbol{r}(t)$. Far from arbitrary, this choice of form is justfied by a theorem about any finite collection of data $\{(t_i,\boldsymbol{r}(t_i))\}_{i=1,\ldots,N}$. It may be proven that any such collection of data can be generated as $\boldsymbol{r}(t)=p(t)+O(t^{N+1})$, where $p(t)$ is a polynomial of degree N, whose coefficients are functions of the first members of the sequences

$$\boldsymbol{r}_i^{(n)}\equiv\boldsymbol{r}_{i+1}^{(n-1)}-\boldsymbol{r}_i^{(n-1)}$$

with $\boldsymbol{r}_i^{(0)}\equiv\boldsymbol{r}(t_i)$. It was initially conjectured that this choice of form was appropriate to all related inquiry, and, being the basis of calculus, the statement that the motion of a particle can be generated by an expression as above was *tautological,* and an additional notion was needed to obtain a positive principle. This was supplied in the form of the notion that the sequence $\boldsymbol{r}^{(2)}$ could be generated theoretically from a separate and simultaneous relation $\boldsymbol{r}^{(2)}=\boldsymbol{F}(\boldsymbol{r}^{(0)},\boldsymbol{r}^{(1)})$, i.e. the formulation of *forces*. This is rigorously extended into the systems of differential equations at the centre of the existing theory.

4. A little argument will go to demonstrate that the function

$$\Omega(t)=\Pi_{r=-\infty}^{\infty}(1-d_r(t))$$

yields 1 for $t=0$ and 0 otherwise. Rewriting this as: $\Omega(t)=\Pi_{r=-\infty}^{\infty}\Sigma_{s=-\infty}^{\infty}A_s\exp\,(i\,2^{-r}\,\pi\,s\,t)$, with $A_s\equiv\delta_{s0}-a_s$, we get a full expression for D_n as:

$$D_n(t)=\Sigma_{j=-\infty}^{\infty}a_j\exp\,(i\,2^{-n}\,\pi\,j\,t)-\frac{1}{2}\Sigma_{h=-\infty}^{\infty}\Omega\,(t-h\,2^n)$$

$$=\Sigma_{j=-\infty}^{\infty}a_j\exp\,(i\,2^{-n}\,\pi\,j\,t)-\tfrac{1}{2}\Sigma_{h=-\infty}^{\infty}\Pi_{r=-\infty}^{\infty}\Sigma_{s=-\infty}^{\infty}A_s\exp\,(i\,2^{-r}\,\pi\,s\,(t-h\,2^n))$$

We may take a similar theorem as our own starting point. In the theory of logic, the process by which one arrives at a so-called *canonical disjunctive form* for any finite (input, output) combination as above is derived from one such theorem, which allows us to generate the (base 2) digits of the dependent $\boldsymbol{r}(t_i)$ by a combination of certain elementary functions on the digits of the independent t_i (or of i alone.) We have several equivalent choices for these elementary operations, and we may consider the following alone without loss of generality:

$$\neg d_m(t) := 1 - d_m(t) = \Sigma_{j=-\infty}^{\infty} A_j \exp\left(i\, 2^{-m}\, \pi\, j\, t\right), \text{where} \mathrm{A}_j \equiv \delta_{s0} - a_s$$

$$d_m(t) \wedge d_n(t) := d_m(t)\, d_n(t) = \left[\Sigma_{j=-\infty}^{\infty} a_j \exp\left(i\, 2^{-m}\, \pi\, j\, t\right)\right] \left[\Sigma_{j=-\infty}^{\infty} a_j \exp\left(i\, 2^{-n}\, \pi\, j\, t\right)\right]$$

$$= \Sigma_{j=-\infty}^{\infty} \alpha_j \exp\left(i\, 2^{-\max(m,n)}\, \pi\, j\, t\right), \text{where } \alpha_j \equiv \Sigma_{j_1 \in \boldsymbol{Z}}\, a_{j_1}\, a_{2^{-\max(m,n)} j - j_1}$$

The derivative operation $d_m(t) \vee d_n(t)$ is defined as $\neg(\neg d_m(t) \wedge \neg d_n(t))$. The form is obtained by taking every digit of the output, as $d_l(\boldsymbol{r}(t_i).\hat{\boldsymbol{e}_k})$, and expressing this as a combination of the above operations on the digits of t_i (or of i.) The appropriate combination is determined by following an unambiguous and well-defined method that operates upon all the data points $(t_i, \boldsymbol{r}(t_i))$. This method, crucially, identifies several digits of the argument of which the function is independent.

In the theory of computation, the resulting formulae are applied to phenomena which admit accurate modelling by a *deterministic automaton*. The theory enumerates these automata with the set of functions $\delta(s, f)$ where s is taken from a set of all possible *states* and f from a set of all possible *stimuli*[5]. The form of the result, δ, is said to change in accordance with the *computational power* of the automaton, which power specifies a hierarchy of progressively powerful automata. The least powerful are such that the past states and stimuli of the system are *not* recorded in any memory-facility the system may possess, such facility being solely reserved to note the system's state. Any higher device that satisfies this constraint can be simulated by one from the lowest level. Their special significance will become apparent below.

The necessary generalisation of this may be completed as follows. The choice of elementary objects and operations is apriori arbitrary, and natural phenomena are not obligated, apriori, to submit themselves to *efficient* modelling by a formal system of this type. We need only look at a particle describing a motion $\boldsymbol{r}(t) = \frac{-g\,t^2}{2}\,\hat{\boldsymbol{k}}$ close to the Earth's surface. This particle can be said to be executing a symbolic routine which computes this $\boldsymbol{r}(t)$ from an input t. This is a classical interpretation[6], but supplies us with the key suggestion: the computation of this value when t is stored symbolically and operated upon by an electronic logic array proceeds at a predictble rate[7] over a short period of time. However, the particle itself completes this computation instantly, as soon as each instant t is registered on a clock. If the array operates on a representation of t bounded in precision to M digits, it may occupy significantly less space that the actual magnitude of $\boldsymbol{r}(t)$, which is the "space" required by the particle to complete the computation. This is a special case of a more general occurence whereby the simulation of a natural phenomenon by a formal system requires time and space not identical to (and generally significantly different from) that required by the actual phenomenon. We may address this *by supplying our formal systems with infinite memory, and infinite time*, so that the general function may operate upon inputs admitting mapping to infinite precision real numbers (without loss of information) and may proceed indefinitely. Naturally this describes a family of functions that remains mostly imperceptible to analysis by virtue of either the infinite precision of their arguments or their non-termination. This is hardly a hindrance to further use of the above abstractions as evidenced by the presence of the same natural imperceptibilities in finite computing. The profusion of finitely describable methods to generate the real numbers, and other sets not attainable in a finite or discrete framework of perception, is also illustrative of the immense facility before us.

5. Here we have labelled the stimuli with an f in order to mark out the relation of these systems with those of Newtonian mechanics (and, to a large extent, all mechanics), wherein the states of material systems are modelled in essentially the same manner, the stimuli corresponding to the *forces*.

6. That dates back to Newton and Locke

7. Upto M^2 operations are performed on M digits in the limited precision representation of t

Ignoring those inevitable pathologies, we see that the general function is one that may be reached by repeated composition of the above functions. So that, a few definite observations can be made regarding the result of each composition:

i) Firstly, the frequency of the function reflects the place value of the most significant digit argued so far, as can be seen from the trigonometric formula for *conjunction* $(d_m \wedge d_n)$. If the function is not periodic, this place value must increase as the compositions proceed, and the frequencies in the spectrum multiply, with $\Delta f \to 0$. The summation at the end of the computation should amount to an "uncountable" addition for aperiodic funtions. The result can also be generated by a recurrence relation on d_n, and the significance of this becomes apparent below.

ii) The amplitudes are the result of succesive and repeated application of the operations $a_j \to \delta_{j0} - a_j$ and $a_j \to \Sigma_{j_1 \in \boldsymbol{Z}} a_{j_1} a_{2^{-\max(m,n)} j - j_1}$ in some order, starting from $a_j = \frac{\exp(-i\pi j) - \exp(-i 2\pi j)}{i 2 \pi j}$.

iii) As all functions describable within some formal framework are expressible in terms of the given operations, every such function can be expressed either a) as a trigonometric series $\Sigma_{r \in \boldsymbol{Z}} A_r \exp(i r 2^{-n} \pi t)$—if it is periodic or if there exists an upper bound on the absolute value of its real arguments, or their precision—or b) as the limit of a sequence of a sequence of trigonometric series as the smallest positive frequencies go to zero, or, equivalently, in terms of a recurrence relation on d_n as dscribed in (i).

2 Superposition

We are intuitively led to a description of a statistical mechanics founded on a dynamics that does not demand rectilinear motion in free space[8]. Essentially, the problem can be compred with one of evaluating the effect of increased curvature of the constituent particle motion on thermodynamic (and, more generally, macroscopic) measures. We see that for a gas in a conservative force field, curvature may be indexed by $C \sim |\ddot{\boldsymbol{q}}|$, allowing us to make a canonical transform that results in $\ddot{\boldsymbol{q}} = \frac{1}{m} \boldsymbol{F}(\boldsymbol{q})$ as one of the phase space variables. So that, the "high curvature system" corresponds to a system of higher entropy with

$$d S \propto d^n \boldsymbol{q} \times d^n \boldsymbol{p} \propto d^m \boldsymbol{F}$$

As higher entropy systems may be reached by the gradual removal of dynamic constraints on a system (and a general widening of a phase subspace), we can conceive of a process of gradual removal that has the effect of increasing individual particle curvature, in the classical theory. After absorbing the difficulties posed by intuitions about the principles of spatial and temporal locality, a program suggests itself to complete the description of subatomic aggregate systems. As the systems that most explicity defy classical accounting are superpositions of several base (or "pure") states, and since such states can be reached by the gradual removal of dynamic constraints on a system (and a general widening of a sample space), we may perfect the analogy with high entropy states within the framework of a new dynamics.

The process of successively developing the partial sum of a trigonometric series by adding components $\boldsymbol{a}_f e^{i 2\pi f t}$ results in a gradual increase or decrease of curvature according to the size and sign of the amplitude $\boldsymbol{a}_f$. In moving from the power series approach to the trigonometric series, we observe that the above argument may be repeated in terms of widening the frequency profile so as to include larger frequencies with sufficiently large amplitudes to scale them. In conventional analysis this is checked by the fact that the feasilbility of obtaining a trigonometric series depends on the convergence of an "energy" measure which diverges for diverging frequency profiles. We are also served well by the observation that a reduction of the dimensions of the constituent particles should enable, simultaneously, motion of generally increased curvature and, from the above, higher entropy relative to a system identical in other respects. Hence, perhaps, the relevance of increased curvature with respect to ensembles of subatomic particles.

8. As with gas in a a strong, but randomised gravitational field, i.e. in a high-curvature region of spacetime

3 Temporal and spatial locality

It should be noted that the measures of time, space and related magnitudes have always been treated as having infinite precision. Non-terminating routines exist in abundance in the classical framework in the form of the evaluation of real-valued functions. A good deal of the formal significance of the principles of locality and determinism–codified in differential equations–have consisted in the resolution they offer to these difficulties: they establish a means of generating the infinite data we require (eg. $\boldsymbol{r}(t)$ for $t \in \boldsymbol{R}$) from finite inputs (eg. $\boldsymbol{r}(0), \dot{\boldsymbol{r}}(0)$ with bounded precision). A few instructive observations may be readily made with regards the *prescriptive* solution to the equation of motion in the first section above. There are two logically equivalent, but otherwise highly distinct, ways in which to frame the input to this process. The first involves the specification of the quantity $\boldsymbol{r}$ on an open interval $[0, \Delta t]$, and the form and operation of the function $\boldsymbol{F}$. We can produce $\dot{\boldsymbol{r}}(0)$ from this. The second is a specification of the tuple $(\boldsymbol{r}(0), \dot{\boldsymbol{r}}(0))$ as an abstract entity at the point $t = 0$ alone. The first input is a *set*, of non-zero measure, and with cardinality $\aleph_1$. The second is a finite, discrete set of data, and $\dot{\boldsymbol{r}}(0)$ need not be interpreted as a derivative, as the asymptotic term in the solution need not be differentiable.

In shifting to a scheme to describe non-differentiable motion, as we have endeavoured to construct above, the first mode of input opens up great possibilities and potentialities for insight.

Let us begin by supposing that our new dynamic principles require, as input to the problem of predicting the motion $\boldsymbol{q}(t)$ of a particle, the data about the value of $\boldsymbol{q}(t)$ for $t \in [0, \Delta t]$. As it is no longer necessary for $\boldsymbol{q}$ to be differentiable on this interval, we must seek—in the place of a force law—a different type of additional input describing the causes of motion. This could then be substituted in the logical alternative to a differential equation. We have already explored this briefly when we noted, above, that trigonometric series can simulate behaviour of greater generality than power series, (and arguing greater—even infinite—degrees of freedom) for reasons already known in the context of field theory. We may proceed to model the data $\boldsymbol{q}([0, \Delta t])$ with a straightforward trigonometric series expansion $\boldsymbol{q} = \Sigma_j \boldsymbol{Q}_j e^{i\frac{2\pi}{\Delta t}jt}$, but then we run into the problem of convergence for the related computations, $\boldsymbol{Q}_j = \frac{1}{\Delta t}\int_{[0, \Delta t]} \boldsymbol{q}\, e^{-i\frac{2\pi}{\Delta t}jt}\, d t$. We are much better off modeling the data with a canonical disjunctive form, which will also result in a trigonometric series with a finite period. Since the given data can be taken as belonging to a function that is zero everywhere outside of $[0, \Delta t]$, the resulting function will be independent of digits with place value $> [\log_2 \Delta t] + 1$[9]. This means that the resulting series will have period $\sim 2^{[\log_2 \Delta t]}$. Straightforward extrapolation will yield a periodic $\boldsymbol{q}(t)$, *unless we have a relation to allow determination of subsequent (more significant, higher place value) digits from those in the interval* $(-\infty, [\log_2 \Delta t] + 1]$. This amounts to a requiement that there exists a relation connecting $\boldsymbol{Q}_j$'s,–entries in *the frequency domain* profile—or that the spectrum be generated by a recursive function that may begin with the data in the frequency profile of $\boldsymbol{q}$, as with $\{\boldsymbol{Q}_{2^{[\log_2 \Delta t]}m}\}_{m \in \boldsymbol{Z}}$. It is interesting to note that this was the point of departure of the main theoretical development of quantum theory, and is now shown to be a necessity in any thory that seeks to predict the behaviour of systems with infinite degrees of freedom. Furthermore, the presence of infinite degrees of freedom allows for the simulation of nondeterministic processes if these initial conditions cannot be generated by finite physical measurements,—there will be infinite variables whose values remain unknown and index the particle's "microstate".

There are many ways in which the data in $\boldsymbol{q}([0, \Delta t])$ can be generated from lesser data. In the first place, the data may prove to be dependent on only a finite number of digits of the argument, a fact that can be gleaned from the canonical disjunctive form. Besides this, if the rule by which the frequency profile can be generated allows for the direct substitution of the limited measurable data in this rule, the probability distribution of $\boldsymbol{q}$ in space represents a genuine constraint on the form of $\boldsymbol{q}([0, \Delta t])$. We see this in classical theory, where the definition of the probability distribution

$$p(\boldsymbol{x})\, d^n \boldsymbol{x} = \mu\, \{t\colon |\boldsymbol{q}(t) - \boldsymbol{x}| \leq |d\, \boldsymbol{x}|\}$$

9. where we use the older $[x]$ operation that coincides with the floor operation $\lfloor x \rfloor$for $x \geq 0$ and with the ceiling operation $\lceil x \rceil$ otherwise. "Place value" is here being measured logarithmically, with the 2^n-th place indexed by n.

can be used to frame a differential equation. The simplest instance is in a single dimension where we have

$$p(x)\,d\,x = \mu\,\{t\colon q(t)\in[x, x+d\,x]\} = {q^{-1}}'(x)\,d\,x$$

$$\Rightarrow p(x) = {q^{-1}}'(x) \Rightarrow q(t) = \left[\int p(\chi)\,d\,\chi\right]^{-1}(t)$$

when q and $\int p\,d\,\chi$ are invertible over appropriate intervals.

Even further, we can experimentally examine the subset of the frequency profile, $\{\boldsymbol{Q}_{2^{[\log_2 \Delta t]}m}\}_{m\in\boldsymbol{Z}}$, for insight into the full data $\boldsymbol{q}([0, \Delta\,t])$, even proceed with the above analysis by fitting this data on some recognisable curve. While quantum theory tends to proceed by making inferences about *charges* from the radiation spectrum, this is mainly done due to the common formal constraint that the charges go to compose separate oscillators in space. This does *not*, however, preclude studying the inversion of the frequency profile: this inversion should enable the study of the actual *field* in space and time.

4 Transition function; Conclusion

Even for finite data $\{(t_i, q_i)\}_{i=1,\ldots,N}$ the process of obtaining the canonical form grows progressively difficult for large N. Our only hope to make predictions theoretically lies in perfecting the anlogy with differential equations, and the postulation of the existence of force laws. In this context we establish the logical parallel to differential equations—the specification of the form and action of the *transition function*, $\delta(s, f)$. The parallel becomes clear on examining the justification for the formulation of *states* in both classical and quantum theory: the theory of computation appears to provide a logical basis for the identification of states—physical elements that carry no memory of their past activity. This is absorbed in the fact that devices which store and manipulate data inescapably consist of these elementary memory-free devices in their basic structure. But as the action of greater devices on a given initial configuration stored in memory can be described as a sequence of the above elementary logical operations[10], performed on various memory elements—and as we may better model phenomena by admitting non-terminating routines arguing infinite memory—our initial formulation absorbs all modelling by higher devices, even of "inifinite" analogues of them. In finite computing, constraints such as locality and determinism on the behaviour of systems translates to constraints on the automata representing the behaviour of these systems. As the former are encoded in differential equations, it follows that the latter would amount to constraining these automata to the lowest level of the hierarchy

[11]

. As such a hierarchy can scarce be distinguished in the infinite precision model we use above, the constraints lose their formal meaning.

The transition function δ plays an even more fundamental logical role in this formulation, since—if we use $\delta$to encode all the finitely expressible data from which we can generate the required infinite data—the form of δ identifies an *expeiment*; theoretical speculation results in the identification of a *natural* automaton, and experiment helps determine the entries of its $(s, f)\rightarrow s'$ table.

Rather than contemplate the full import of utilising a specification of δ in the place of differential equations by means of several illustrations, we may confine ourselves to a few key observations:

i) All functions that can be evaluated for a given argument can be described in terms of a symbolic routine, and therefore admits generation from a transition function δ.

10. That is to say that they can be "unrolled" into a sequence of successive operations of $\neg$ and $\wedge$ on various memory elements

11. It is of some significant interest to note that this question already occupied the first authorities of mechanics, for we hear from Newton that-

⟨error|compound quoteenv⟩

ii) All specifications of δ in finite computing can be mapped to a unique *recursive function*, the most abstract generalisation of differential equations, difference equations and recurrence relations as of yet established.

iii) The specification of δ in the above context generally corresponds to the specification of a method of generating the digits $d_n(x)$ of real arguments.

iv) The generation of a *measure* of a set is, effectively, accomplished by the solution of a differential equation, as with $\mu\,\{(x,\,y){:}\,x\in[a,\,b],\,y\in[0,\,f(x)]\}=\int_a^b f(x)\,d\,x=(D^{-1}\,f)|_{[a,b]}$. The analogous operation may be expected to generalise evaluation of a measure in instructive ways. In particular, the measure of sets with non-differentiable boundaries may be assigned a new and meaningful value.

Staying with this last observation, we are led to contemplate the implications for *probability* measures. A little experimentation with the system here discussed raises the possibility that non-convergent limits in calculus may be assigned a meaningful interpretation within this formal system—perhaps due to the rationalisation of non-terminating routines[12]. For instance, if we interpret a continuous function that takes $\boldsymbol{R}$ into $\boldsymbol{R}^2$ bijectively, as the trajectory of a particle moving in a plane, we can explicitly express it in closed-form in our framework. We get a simple expression for the similar function taking [0,1] into $[0,1]^2$:

$$\boldsymbol{r}(t)=(\Sigma_{m\in\boldsymbol{N}}\,2^{-m}\,\Sigma_{j\in\boldsymbol{Z}}\,a_j\exp{(i\,2^{2m-1}\,\pi\,j\,t)},\Sigma_{m\in\boldsymbol{N}}\,2^{-m}\,\Sigma_{j\in\boldsymbol{Z}}\,a_j\exp{(i\,2^{2m}\,\pi\,j\,t)})$$

We do not need to attempt to differentiate the expression explicitly to see that it is not differentiable: from the definition of the function we can immediately conclude that $\boldsymbol{r}\,(t_0+2^{-(2m-1)})-\boldsymbol{r}(t_0)$ will be parallel to the x-axis for any $m\in\boldsymbol{N}$ and $\boldsymbol{r}\,(t_0+2^{-2m})-\boldsymbol{r}(t_0)$ will be parallel to the y-axis, so that this vector will continue to oscillate between two perpendicular directions as $m\to\infty$. However, despite the non-convergence of $\lim_{\Delta t\to 0}\frac{\boldsymbol{r}\,(t_0+\Delta\,t)-\boldsymbol{r}(t_0)}{\Delta\,t}$, we can evaluate the expression inside the limit for all $\Delta\,t$. More than that, we can obtain a probability distribution $P(\boldsymbol{r})$ for this expression by varying $\Delta\,t$ over some small interval $[0,T]$, with different probabilities resulting from different rates of variation of $\Delta\,t$. As a similar experiment may be performed with integrating a similarly defined function $\boldsymbol{v}(t)$, we are inevitably led into analogies with differentiating and integrating the time and frequency domain profiles of a function, and their associated uncertainty principles. Our own prescription for generating functions from their specifications on restricted subdomains, it will be noted, requires the relation of unknown amplitudes of certain frequency components with those of larger frequency components.

As for the evaluation of probability measures, there are at least two contexts in which the convergence of a limit challenges the formulation of a probability distribution—the first being in the obvious context of the correspondence between this measure and a differential equation, and the second argues the definition of probability distributions as the limit of a function of the results of some experiment: $P(\boldsymbol{r})=\lim_{N\to\infty}\frac{n(\boldsymbol{r})}{N}$. As there are, indeed, meaningful extensions of these computations to be made in the present formulation, then there may be defined separate criteria to judge the *independence* of probabilistic events,—and this last is the central formulation upon which all demonstrations of essential nonlocality are predicated.

12. This is highly similar in principle to the generalisation of "rate" that calculus achieves, from the straightforward definition only applicable to linear functions