31251 – Data Structures and Algorithms

Week 6, Autumn 2020

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This time, in 31251 DSA:

- Binary Trees
- Tree Traversals
- Expression Trees (optional)
- Binary Search Trees
- AVL Trees (optional)

Binary Trees

Trees and Binary Trees

- A *Tree* is a connected graph with no cycles.
  - You can't walk through the graph and get back to your starting point without backtracking.
  - If a connected graph has n vertices and $n-1$ edges, it is a tree.
- A *Binary* Tree is a tree where every vertex has at most three neighbours.

3? What?

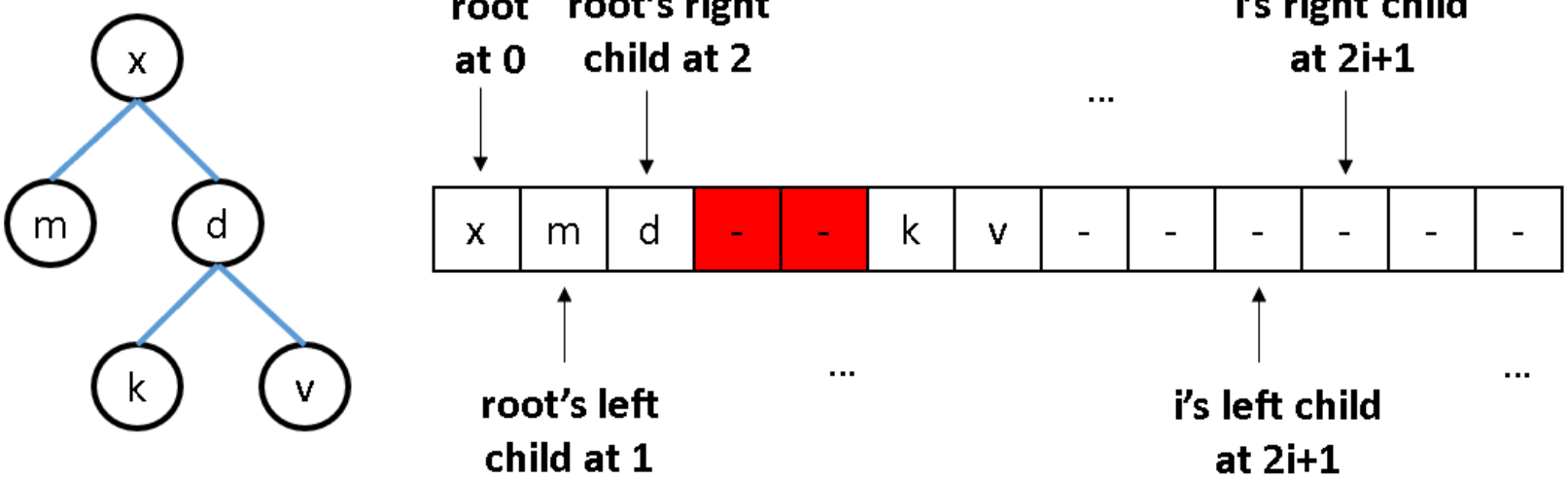
- If it has 3 at most 3 neighbours, why is it "binary"?
- We normally think of them as having an order:
  - One vertex is the root.
  - Each vertex has at most two children, and at most one parent.
  - Vertices with no children are called *leaves*.

Uses of Binary Trees

- Binary Trees find uses in many areas of computer science:
  - 3D rendering (binary space partition).
  - Networking (Binary Tries, Treaps).
  - Cryptology (GGM Trees).
  - Coding and Compression (Huffman Trees).
  - Hashing (Hash Trees).
  - Sorting (Heaps and Heapsort).
  - Searching (Binary Search Trees).
  - Parsing (Expression Trees)

How do we build them?

- There are two basic methods for building a binary tree:
  1. Kind of like a LinkedList with with two next pointers to left and right children—We saw this in Week 1.
  2. Embedded in an array, where the children of the vertex at index i are at indices $2i + 1$ and $2i + 2$.



How do we get around them?

- What if we have a tree, where each node has none or n children?
  - The children of the vertex in index i are at indices: $n * i + 1, \quad n * i + 2, \quad \cdots, \quad n * i + n$
- No matter which representation you choose, we can use versions of the same two traversals we saw with normal graphs:
  1. Breadth first - start with the root, then visit its children, then children's children, ...
  2. Depth first - go all the way to the bottom first, then backtrack.

Depth First Traversals

- Very amenable to recursive implementation.
- At each node we have three things to do:
  1. Deal with the current node,
  2. visit the left child,
  3. visit the right child.
- Gives three different traversals.
  1. Preorder (deal with the current node first)
  2. Inorder (deal with the current node between visiting the descendents)
  3. Postorder (deal with the current node last).

Depth First Traversal – Implementation

Preorder traversal (recursive):

```
Function preorderTraversal(Node n)
    if n == null then
        return;
    visit(n);
    preorderTraversal(n.leftChild());
    preorderTraversal(n.rightChild());
```

Depth First Traversal – Implementation

We can switch from recursive to iterative by swapping the implicit use of the call stack with an explicit stack.

```
Function preorderTraversal(Node n)
    Stack<Node> s = new Stack<Node>();
    Node current = n;
    while current != null do
        visit(current);
        if current.rightChild() != null then
            s.push(current.rightChild());
        if current.leftChild() != null then
            s.push(current.leftChild());
        current = stack.pop();
```

Depth First Traversal – Implementation

Inorder traversal (recursive):

```
Function inorderTraversal(Node n)
    if n == null then
        return;
    inorderTraversal(n.leftChild());
    visit(n);
    inorderTraversal(n.rightChild());
```

Depth First Traversal – Implementation

Inorder traversal (iterative):

```
Function inorderTraversal(Node n)
    Stack<Node> s = new Stack<Node>();
    Node current = n;
    while current != null OR !s.isEmpty() do
        if current != null then
            s.push(current);
            current = current.leftChild();
        else
            current = s.pop();
            visit(current);
            current = current.rightChild();
```

Depth First Traversal – Implementation

Postorder traversal (recursive):

```
Function postorderTraversal(Node n)
    if n == null then
        return;
    postorderTraversal(n.leftChild());
    postorderTraversal(n.rightChild());
    visit(n);
```

Depth First Traversal – Implementation

Postorder traversal (iterative):

```
Function postorderTraversal(Node n)
    Stack<Node> s = new Stack<Node>();
    Node current = n;
    Node last = null;
    while current != null OR !s.isEmpty() do
        if current != null then
            s.push(current);
            current = current.leftChild();
        else
            if s.top().rightChild() != null && s.top().rightChild() != last
              then
                current = s.top().rightChild();
            else
                current = s.pop();
                visit(current);
                last = current;
```

Breadth First Traversal

- Visits vertices according to their level in the tree ("top to bottom, left to right").
- Simple to implement iteratively using a queue.

Breadth First Traversal - Implementation

```
Function breadthFirst(Node n)
    Queue<Node> q = new Queue<Node>();
    q.add(n);
    while !q.isEmpty() do
        Node current = q.front();
        q.pop();
        visit(current);
        if current.leftChild != null then
            q.add(current.leftChild());
        if current.rightChild != null then
            q.add(current.rightChild());
```

A Brief Summary

Only inorder traversal produces the node sequences that can be used to rebuild the original tree.

You can easily find trees for preorder, postorder, and BFS, respectively, from which they will generate the same sequences.

Binary Search Trees

Binary Search Trees

- Binary Search Tree (BST) is a simple data structure that allows fast *insertion*, *removal*, and *lookup* of elements.
- It is an ordered data structure but does not require reshuffling everything to put something new in (Unlike array-like data structures).
- They follow a simple rule to find or insert:
  - If what you're looking for (or what you have) has a smaller key than the current vertex, go to the left; otherwise, go to the right. (Special case: duplicate keys)
- Insertion then is just traversing the tree to the bottom and adding the new element wherever you stop.
- Finding something mimics binary search, if you get to the bottom without finding it, it's not there.

Complexity of Binary Search Trees

Operation	Average Case	Worst Case
Space	$O(n)$	$O(n)$[1]
Insert	$O(\log n)$	$O(n)$
Remove	$O(\log n)$	$O(n)$
Find	$O(\log n)$	$O(n)$

At the cost of more complicated code, there are several self-balancing BST data structures, which reduce the worst case insert, remove and find to $O(\log n)$: 2-3 Trees, Red-Black Trees, AVL Trees, Splay Trees and others.

[1] Using the array representation, this can actually end up as $O(2^n)$.

Some Uses of Binary Trees

Representing Grammars with Trees

- Many things in computer science can be expressed in terms of *Formal Grammars*.
  - We don't need to know what they are though.
- In particular, expressions that have syntax can be modelled with grammars.
  - e.g., syntax compiler of programming languages.
- One way of showing how a concrete expression derives from a grammar is by using a tree.
  - ... and for certain types of expression grammars, binary trees!

Boolean Expressions

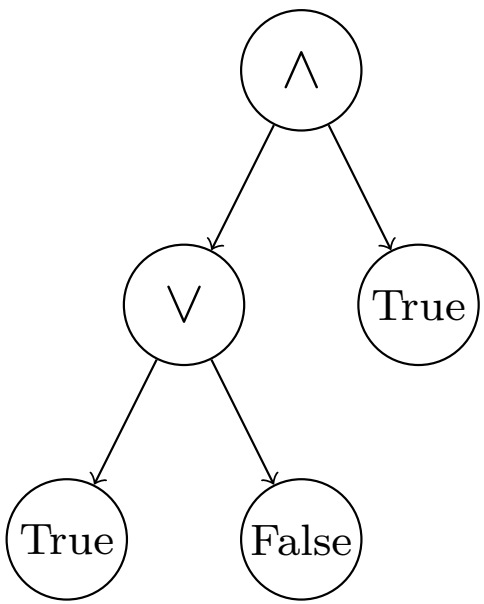
- We can take Boolean expressions in mathematical logic as an example. They have simple rules:
  1. True is a Boolean expression.
  2. False is a Boolean expression.
  3. If A a Boolean expression, $\neg A$ is a Boolean expression.
  4. If A and B are Boolean expressions, $A \wedge B$ is a Boolean expression.
  5. If A and B are Boolean expressions, $A \vee B$ is a Boolean expression.

- Just for those that are interested, a grammar for that looks like:

$$S \to \neg S \mid S \wedge S \mid S \vee S \mid \mathrm{True} \mid \mathrm{False}$$

- You can build more complicated ones that build in operator precedence and associativity too.

- We can convert these rules to a binary tree structure.
- For example, the expression $(\mathrm{True} \vee \mathrm{False}) \wedge \mathrm{True}$ can be represented by:



Why is this useful?

- Given an expression as text, it is not immediately obvious how to begin computing its value.
- You need to read the whole thing, then decide which bits you are going to do first.
  - Operator precedence is important! ($3 \times 2 - 1$ vs. $3 \times (2 - 1)$)
- If we can quickly build a tree like this, then we can *recursively* evaluate it!
  - It becomes a simple divide and conquer algorithm!
- We can also traverse it looking for other properties (very useful for compiling optimisation).

Prefix, Infix and Postfix

- Another thing we can do is turn the expression back into a string in different ways.
  1. Prefix notation results from a preorder traversal of the tree.
  2. Infix notation results from an inorder traversal (Programmers use infix notations!).
  3. Postfix notation results from a postorder traversal of the tree.
- When representing grammars, prefix and postfix are unambiguous, and don't require operator precedence or associativity rules to parse.
  - Compiler will convert an infix expression to postfix or prefix can make things easier.

A More Balanced Binary Search Tree - AVL Tree

Better Balance = Less Falling Over

- Binary Search Trees work well if the tree is balanced.
- ... but this doesn't always happen.
- We can correct this by doing some extra work to maintain balance.

AVL Tree does extra work at each insertion and deletion, to maintain good running time no matter what:

Operation	Average Case	Worst Case
Space	$O(n)$	$O(n)^2$
Insert	$O(\log n)$	$O(\log n)$
Remove	$O(\log n)$	$O(\log n)$
Find	$O(\log n)$	$O(\log n)$

[2] Same caveat as the BST.

How do they balance things?

- AVL trees keep things balanced by maintaining an invariant at each vertex: the balance factor.
- Balance factor of a vertex v =
  left subtree's height – right subtree's height.
- The balance factor of a vertex can be -1, 0 or 1. Anything else, and the tree needs rebalancing.
- Because inserting a new element only adds one vertex, the balance factor will only get as bad as -2 or 2, before something is done.

Insertion into AVL Trees

- First we insert as normal with a BST.
- Then we check the balance factor of the ancestor of the newly inserted node.
- We can store the height at each vertex, then we only need to update the relevent ones when we add a new child.

Tree Rotations

- Tree rotation is an order-invariant operation on binary trees.
  - It changes the structure but preserve the traversal order.
- A tree rotation can be left or right; they are the reverse operation to each other.
- Given a vertex x, with a right subtree γ, and a left child y which has subtrees α and β, we can rotate right to get y with left subtree α, and right child x which has subtrees β and γ.

Further Reading

If we get a vertex x with balance factor 2:

1. We look at the left child y (this must be the larger one).
2. If it has balance factor -1, it "leans to the right".
    1. We rotate left the child y and its right child z. This makes z the left child of x.
    2. Then we rotate right z and x to get a balanced tree.
3. Otherwise
    1. Rotate right with y and x to get a balanced tree.

The -2 case is analogous:

1. If y has balance factor 1
   1. Rotate right with z and y.
   2. Rotate left with z and x.
2. Else
   1. Rotate left with y and x.

- This leaves the subtree with balance factor -1, 0 or 1, depending on the exact balance factors of its subtrees (but entirely predictable).
- Then we continue up the tree checking.
- So we only need to follow a path to the root, doing at most two rotations at each step – turns out to not be that expensive.

- We need to keep track of a couple of things:
  1. Let x be the vertex we want to delete.
  2. y is a vertex whose value we will move.
  3. and z is the actual vertex we remove.

Deletion

1. If x is a leaf, or has one child $z := x$.
2. Otherwise
   1. Find the largest value in the left subtree, or the smallest in the right subtree of x, this will be y.
   2. Copy the value at y over the value at x.
   3. Now $z := y$.
3. If z has a subtree, attach it to z's parent in its place, and delete z (or set the child to the root if z is the root).
4. Starting with z's parent, rebalance the tree.