Solution

Class 12 - Mathematics

Inverse Trigonometric functions

Section A

1. $\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$
   Put $x = a\sin\theta$
   $\tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right)$
   $\tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2\left(1-\sin^2\theta\right)}}\right)$
   $\tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right)$
   $\tan^{-1}(\tan\theta) = \theta$
   $= \sin^{-1}\left(\frac{x}{a}\right)$
2. Let $\cos ec^{-1}\left(-\sqrt{2}\right) = \theta$
   $\cos ec\theta = -\sqrt{2}$
   $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
   $\cos ec\theta = \cos ec\left(\frac{-\pi}{4}\right)$
   $\theta = \frac{-\pi}{4}$
   Principal value is $\frac{-\pi}{4}$
3. Given expression is: $\tan\left(\frac{\sin^{-1}x+\cos^{-1}x}{2}\right)$, when $x = \frac{\sqrt{3}}{2}$

   $\Rightarrow \tan\left(\frac{\sin^{-1}x+\cos^{-1}x}{2}\right) = \tan\left(\frac{\sin^{-1}\frac{\sqrt{3}}{2}+\cos^{-1}\frac{\sqrt{3}}{2}}{2}\right)$ Now, we know that $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

   $\therefore \tan\left(\frac{\sin^{-1}x+\cos^{-1}x}{2}\right) = \tan\frac{\pi}{4} = 1$
4. Using this formula $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$, we have

   The value of $\cot^{-1}(-x)$ for all x ∈ R in term of $\cot^{-1}x$ as $\pi - \cot^{-1}(x)$.
5. As $\sec^{-1}(\sec x) = x$
   Provided $x \in [0,\pi]-\left\{\frac{\pi}{2}\right\}$
   $\therefore$ we can write $\sec^{-1}\sec\left(\frac{\pi}{3}\right)$ as $\frac{\pi}{3}$

Section B

6. Let $\cos^{-1}\frac{5}{13} = y$
   $\Rightarrow \cos^{-1}\frac{5}{13} = y$ where $y \in \left[0, \frac{\pi}{2}\right]$
   To find : $\sin\left(\cos^{-1}\frac{5}{13}\right) = \sin y$
   As $\sin^2\theta + \cos^2\theta = 1$
   $\Rightarrow \sin y = \pm\sqrt{1-\cos^2 y}$
   As $y \in \left[0, \frac{\pi}{2}\right]$
   $\Rightarrow \sin y = \sqrt{1-\cos^2 y}$
   $\Rightarrow \sin y = \sqrt{1-\left(\frac{5}{13}\right)^2}$
   $\Rightarrow \sin y = \sqrt{1-\frac{25}{169}}$
   $\Rightarrow \sin y = \sqrt{\frac{144}{169}}$
   $\Rightarrow \sin y = \frac{12}{13}$
   $\Rightarrow \sin\left[\cos^{-1}\left(\frac{5}{13}\right)\right] = \frac{12}{13}$

Hence proved.

7. We have,

$\tan^{-1}\left\{\sqrt{\frac{1-\cos x}{1+\cos x}}\right\}$

$=\tan^{-1}\left\{\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right\}$

$=\tan^{-1}\left\{\sqrt{\tan^2\frac{x}{2}}\right\}$

$=\tan^{-1}\left\{\left|\tan\frac{x}{2}\right|\right\}$

$=\begin{cases}\tan^{-1}\left(-\tan\frac{x}{2}\right), & \text{if } -\pi<x<0\\ \tan^{-1}\left(\tan\frac{x}{2}\right), & \text{if } 0\le x<\pi\end{cases}$

$=\begin{cases}\tan^{-1}\left\{\tan\left(-\frac{x}{2}\right)\right\}=-\frac{x}{2}, \text{ if } -\pi<x<0\\ \tan^{-1}\left\{\tan\frac{x}{2}\right\}=\frac{x}{2}, \text{ if } 0<x<\pi\end{cases}$

8. According to question, we have

$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)$ = $\tan^{-1}\left[\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}\right]$

= $\tan^{-1}\left[\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}\right]$

= $\tan^{-1}\left[\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right]$

= $\tan^{-1}\left[\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right]$

= $\tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]=\frac{\pi}{4}+\frac{x}{2}$

9. We have ,sin x - $\cos^{-1}$ x = $\frac{\pi}{6}$

$\Rightarrow \sin^{-1}x-\left(\frac{\pi}{2}-\sin^{-1}x\right)=\frac{\pi}{6}$

$\Rightarrow$ 2 $\sin^{-1}$x - $\frac{\pi}{2}=\frac{\pi}{6}$ $\Rightarrow$ 2 $\sin^{-1}$x = $\frac{\pi}{2}+\frac{\pi}{6}$ $\Rightarrow$ 2 $\sin^{-1}$ x = $\frac{3\pi+\pi}{6}$

$\Rightarrow$ $\sin^{-1}$ x = $\frac{4\pi}{6\times 2}$ $\Rightarrow$ $\sin^{-1}$ x = $\frac{\pi}{3}$

$\Rightarrow$ x = sin $\frac{\pi}{3}$

$\Rightarrow x=\frac{\sqrt{3}}{2}$

10. We have,

$\tan^{-1}x-\cot^{-1}x=\tan^{-1}\frac{1}{\sqrt{3}}$

$\Rightarrow \quad \tan^{-1}x-\cot^{-1}x=\frac{\pi}{6}$ ...(i)

We know that

$\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$ ...(ii)

Adding (i) and (ii), we get

$2\tan^{-1}x=\frac{2\pi}{3}$

$\Rightarrow \quad \tan^{-1}x=\frac{\pi}{3}\Rightarrow x=\tan\frac{\pi}{3}=\sqrt{3}$

Section C

11. Here we need to find the value of x .

Now, given equation is tan $^{-1}$ x + 2 cot $^{-1}$ x = $\frac{2\pi}{3}$.

Therefore , the given equation can be written as $\tan^{-1}x+2\tan^{-1}\left(\frac{1}{x}\right)=\frac{2\pi}{3}$

$\left[\because \cot^{-1}x=\tan^{-1}\frac{1}{x}, x>0\right]$

$\Rightarrow$ $\tan^{-1}$ x+ $\tan^{-1}$ $\left(\frac{2\times\frac{1}{x}}{1-\frac{1}{x^2}}\right)=\frac{2\pi}{3}$ $\left[\because 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right);-1<x<1\right]$

$\Rightarrow$ $\tan^{-1}$ x+ $\tan^{-1}$$\left(\frac{\frac{2}{x}}{\frac{x^2-1}{x^2}}\right)=\frac{2\pi}{3}$

$\Rightarrow$ tan^{-1} x+ tan$^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{2\pi}{3}$

$\Rightarrow$ tan$^{-1}\left(\frac{x+\frac{2x}{x^2-1}}{1-\frac{2x^2}{x^2-1}}\right) = \frac{2\pi}{3}\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right);xy<1\right]$

$\Rightarrow \frac{x^3-x+2x}{x^2-1-2x^2} = \tan\frac{2\pi}{3}$

$\Rightarrow \frac{x^2+x}{-1-x^2} = \tan\left(\pi-\frac{\pi}{3}\right) \Rightarrow \frac{x^3+x}{-(1+x^2)} = -\tan\frac{\pi}{3}\left[\because \tan(\pi-\theta)=-\tan\theta\right]$

$\therefore \frac{x(1+x^2)}{-(1+x^2)} = -\sqrt{3}\left[\because \tan\frac{\pi}{3}=\sqrt{3}\right]$

$\Rightarrow x = \sqrt{3}$

12. Put x = tanθ

$\Rightarrow \theta$ = tan^{-1}(x)

tan^{-1}\{$\sqrt{1+\tan^2\theta}$–tanθ \}

= tan^{-1}\{$\sqrt{\sec^2\theta}$–tanθ \}

= tan^{-1}\{secθ –tanθ \}

= tan$^{-1}\left\{\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}\right\}$

= tan$^{-1}\left\{\frac{1-\sin\theta}{\cos\theta}\right\}$

Sinθ = $2\times\sin\frac{\theta}{2}\times\cos\frac{\theta}{2}$,cosθ = $\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}$, using these formulla

= tan$^{-1}\left\{\frac{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}-2\times\sin\frac{\theta}{2}\times\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}\right\}$

= tan$^{-1}\left\{\frac{\left(\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)\times\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)}\right\}$

= tan$^{-1}\left\{\frac{\left(\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right)}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}\right\}$

Dividing by $\cos\frac{\theta}{2}$ we get

= tan$^{-1}\left\{\frac{\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}-\frac{\cos\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)}{\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}+\frac{\cos\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)}\right\}$

= tan$^{-1}\left(\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right)$

tan$^{-1}\left(\frac{\tan\frac{\pi}{4}-\tan\frac{\theta}{2}}{1+\tan\frac{\pi}{4}\tan\frac{\theta}{2}}\right)$

tan(x–y) = $\frac{\tan x-\tan y}{1+\tan x\tan y}$

= tan$^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right)$

= $\frac{\pi}{4}-\frac{\theta}{2}$

From 1 we get

= $\frac{\pi}{4}-\frac{\tan^{-1}x}{2}$.

Therefore, the simplification of given equation is $\frac{\pi}{4}-\frac{\tan^{-1}x}{2}$.

13. To solve this we use substitution

Let $\cos^{-1}\frac{4}{5}=x$ and $\tan^{-1}\frac{2}{3}=y$

$\Rightarrow \cos x=\frac{4}{5}$ and $\tan y=\frac{2}{3}$

where $\mathrm{x,y}\in\left[0,\frac{\pi}{2}\right]$

Now, LHS is reduced to: tan(x+y)

$\Rightarrow \tan(\mathrm{x}+\mathrm{y}) = \frac{\tan\mathrm{x}+\tan\mathrm{y}}{1-\tan\mathrm{x}\cdot\tan\mathrm{y}}$ ..equation (i)

As $\tan x=\sqrt{\sec^2x-1}$ where $\mathrm{x}\in\left[0,\frac{\pi}{2}\right]$

$\Rightarrow \tan x = \sqrt{\frac{1}{\cos^2 x} - 1}$
$\Rightarrow \tan x = \sqrt{\left(\frac{5}{4}\right)^2 - 1}$
$\Rightarrow \tan x = \sqrt{\frac{9}{16}}$
$\Rightarrow \tan x = \frac{3}{4}$
Now putting the values of tan x and tan y in equation (i)
$\Rightarrow \tan(x+y) = \left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\left(\frac{3}{4}\right)\left(\frac{2}{3}\right)}\right)$
$\Rightarrow \tan(x+y) = \frac{17}{6}$
= RHS

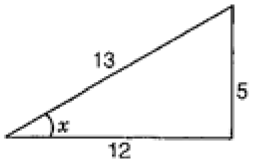
14. To prove: $\cos^{-1}(4x^3 - 3x) = 3\cos^{-1}x$, $\frac{1}{2} \leq x \leq 1$
Formula used: $\cos 3A = 4\cos^3 A - 3\cos A$
we will show LHS = RHS..
LHS = $\cos^{-1}(4x^3 - 3x)$ ...(i)
Let x = cos A (ii)
Substituting (ii) in (i),
LHS = $\cos^{-1}(4\cos^3 A - 3\cos A)$
= $\cos^{-1}(\cos 3A)$
= 3A
From (ii), A = $\cos^{-1} x$,
3A = $3\cos^{-1} x$
= RHS
Therefore, LHS = RHS
Hence proved

15. $\sin^{-1}(1-x) = \frac{\pi}{2} + 2\sin^{-1}x$
$(1-x) = \sin\left(\frac{\pi}{2} + 2\sin^{-1}x\right)$
1 - x = $\cos(2\sin^{-1}x)$
1 - x = $\cos 2(\sin^{-1}x)$ $\left[\cos 2\theta = 1 - 2\sin^2\theta\right]$
= $1 - 2\sin^2(\sin^{-1}x)$
= $1 - 2[\sin(\sin^{-1}x)]^2$
1 - x = $1 - 2x^2$
$2x^2 - x = 0$
x(2x - 1) = 0
x = 0, $x = \frac{1}{2}$

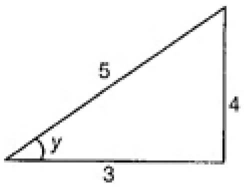
Section D

16. $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right]$
Put $x = \tan\theta,\ y = \tan\phi$
$= \tan\frac{1}{2}\left[\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta} + \cos^{-1}\frac{1-\tan^2\phi}{1+\tan^2\phi}\right]$
$= \tan\frac{1}{2}\left[\sin^{-1}(\sin 2\theta) + \cos^{-1}(\cos 2\phi)\right]$
$= \tan\frac{1}{2}[2\theta + 2\phi]$
$= \tan\frac{1}{2}2(\theta + \phi)$
$= \frac{\tan\theta + \tan\phi}{1 - \tan\theta . \tan\phi} = \frac{x+y}{1-xy}$

17. We have,
$\sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5} = \tan^{-1}\frac{63}{16}$ .....(i)



Let
$sin^{-1}\frac{5}{13}=x$
$\Rightarrow \sin x=\frac{5}{13}$
And $\cos^2$x = 1 - $\sin^2$x
$=1-\frac{25}{169}=\frac{144}{169}$
$\Rightarrow \cos x=\sqrt{\frac{144}{169}}=\frac{12}{13}$
$\therefore \tan x=\frac{\sin x}{\cos x}=\frac{\frac{5}{13}}{\frac{12}{13}}=\frac{5}{12}$ ...(ii)
$\Rightarrow \tan x=\frac{5}{12}$ ...(iii)
Again, Let
$\cos^{-1}\frac{3}{5}=y\Rightarrow \cos y=\frac{3}{5}$
$\therefore \sin y=\sqrt{1-\cos^2 y}$
$=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{1-\frac{9}{25}}$
$\sin y=\sqrt{\frac{16}{25}}=\frac{4}{5}$
$\Rightarrow \tan y=\frac{\sin y}{\cos y}=\frac{\frac{4}{5}}{\frac{3}{5}}=\frac{4}{3}$ ...(iv)



We know that,
$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x.\tan y}$
$\Rightarrow \tan(x+y)=\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\cdot\frac{4}{3}}\Rightarrow \tan(x+y)=\frac{\frac{15+48}{36}}{\frac{36-20}{36}}$
$\Rightarrow \tan(x+y)=\frac{\frac{63}{36}}{\frac{16}{36}}$
$\Rightarrow \tan(x+y)=\frac{63}{16}$
$\Rightarrow x+y=\tan^{-1}\frac{63}{16}$
$\Rightarrow \tan^{-1}\frac{5}{12}+\tan^{-1}\frac{4}{3}=\tan^{-1}\frac{63}{16}$
Hence proved.