SARMA: Scalable Low-Rank High-Dimensional Autoregressive Moving Averages via Tensor Decomposition

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Abstract

Existing models for high-dimensional time series are overwhelmingly developed within the finite-order vector autoregressive (VAR) framework, whereas the more flexible vector autoregressive moving averages (VARMA) have been much less considered. This paper introduces a high-dimensional model for capturing VARMA dynamics, namely the Scalable ARMA (SARMA) model, by combining novel reparameterization and tensor decomposition techniques. To ensure identifiability and computational tractability, we first consider a reparameterization of the VARMA model and discover that this interestingly amounts to a Tucker-low-rank structure for the AR coefficient tensor along the temporal dimension. Motivated by this finding, we further consider Tucker decomposition across the response and predictor dimensions of the AR coefficient tensor, enabling factor extraction across variables and time lags. Additionally, we consider sparsity assumptions on the factor loadings to accomplish automatic variable selection and greater estimation efficiency. For the proposed model, we develop both rank-constrained and sparsity-inducing estimators. Algorithms and model selection methods are also provided. Simulation studies and empirical examples confirm the validity of our theory and advantages of our approaches over existing competitors.

Keywords: High-dimensional time series; Identifiability; Reduced-rank regression; Scalability; Tensor decomposition; $VAR(\infty)$; VARMA

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1 Introduction

The advent of the big data era has sparked a surge of interest in high-dimensional time series (HDTS) modelling. The goal is to build a single model to efficiently capture the dependence structure across both time and variables. Existing HDTS models are mostly developed within the framework of *finite-order* vector autoregression (VAR); see, e.g., Basu and Michailidis (2015); Wang et al. (2021b). Recently, empirical studies have shown that these models can be overly restrictive: The lag order typically has to be very large, or even grow with the sample size, in order to adequately fit HDTS data (Athanasopoulos and Vahid, 2008; Chan et al., 2016; Dias and Kapetanios, 2018; Wilms et al., 2021). However, this would entail a large number of coefficient matrices, which makes the fitted model rather cumbersome to interpret.

The limitation of the finite-order VAR reveals the paramount importance of the more general infinite-order VAR model, which is commonly parameterized as the vector autoregressive moving average (VARMA) model to ensure parsimony (Lütkepohl, 2005; Tsay, 2014; Peña and Tsay, 2021). For simplicity, consider the VARMA(1,1) model for an observed time series $\{y_t\}_{t=1}^T$ as follows:

$$\mathbf{y}_t = \mathbf{\Phi} \mathbf{y}_{t-1} + \mathbf{\varepsilon}_t - \mathbf{\Theta} \mathbf{\varepsilon}_{t-1}, \tag{1.1}$$

where $\boldsymbol{y}_t \in \mathbb{R}^N$, $\boldsymbol{\varepsilon}_t \in \mathbb{R}^N$ is the innovation term, and $\boldsymbol{\Phi}, \boldsymbol{\Theta} \in \mathbb{R}^{N \times N}$ are AR and MA coefficient matrices. Assuming invertibility, the model can be written into the following VAR(∞) form,

$$\boldsymbol{y}_{t} = \sum_{j=1}^{\infty} \boldsymbol{A}_{j} \boldsymbol{y}_{t-j} + \boldsymbol{\varepsilon}_{t}, \tag{1.2}$$

with

$$\mathbf{A}_j = \mathbf{A}_j(\mathbf{\Phi}, \mathbf{\Theta}) = \mathbf{\Theta}^{j-1}(\mathbf{\Phi} - \mathbf{\Theta}), \quad j \geqslant 1.$$
 (1.3)

Note that the exponential decay of A_j as $j \to \infty$ is driven by Θ , whose eigenvalues are all less than one in absolute value, so that (1.2) is well defined. This VAR(∞) form reveals that unlike the finite-order VAR, VARMA models can achieve very flexible temporal patterns with a much smaller number of parameters.

Despite the flexibility and parsimony of the VARMA model, it has enjoyed far less popularity than the finite-order VAR model in practice due to its (i) complicated identification issue, and (ii) heavy computation burden. Both problems become more cumbersome as the dimension N

increases. Take the VARMA(1,1) as an example. There generally exist many combinations of (Θ, Φ) that lead to the same values for $\{A_1, A_2, \dots\}$ and hence the same data generating process, unless suitable identification constraints on (Φ, Θ) are imposed. Moreover, the loss function for parameter estimation involves very high-order $N \times N$ matrix polynomials due to the form of $A_j(\Phi, \Theta)$; e.g., the degree is as high as 2(T-1) for the squared loss, where T is the sample size. Under a large dimension N, such matrix polynomials will incur substantial computation costs. Furthermore, the VARMA model falls short when it comes to model interpretation, as there is no intuitive interpretation based directly upon Φ and Θ . For instance, to understand the explicit relationship between y_t and its lags, one must rewrite the fitted VARMA model in the VAR(∞) form.

Instead of adhering to the original VARMA framework, this paper seeks a new approach to parsimoniously parameterizing VAR(∞) processes, which naturally leads to the development of the corresponding high-dimensional modelling strategy. In this paper, we first demonstrate the formulation of an alternative VAR(∞) model that essentially encompasses the VARMA model in the low-dimensional setup. This model emerges from a reparameterization of the VARMA model, with extra degrees of freedom introduced during the reparameterization. A distinctive advantage of this model is its identifiability: its AR coefficient matrices A_j are expressed using parameters that are identifiable without the need for any additional constraints. Moreover, we uncover a fascinating connection between the parameterization of these AR coefficient matrices and the tensor factorization. This connection allows us to gain deeper insights into how the model captures temporal patterns across an infinite number of lags using only a finite number of parameters. Specifically, consider the $N \times N \times \infty$ coefficient tensor \mathcal{A} formed by stacking the $N \times N$ AR coefficient matrices $\{A_1, A_2, \ldots\}$ across all lags. The parameterization of this alternative VAR(∞) model assumes that \mathcal{A} can be factorized along its third mode as follows:

$$\mathcal{A} = \mathcal{G} \times_3 \mathbf{L}(\boldsymbol{\omega}), \tag{1.4}$$

where \mathfrak{G} is an $N \times N \times d$ tensor of free parameters, with d being a fixed dimension, and $L(\cdot)$ is an $\infty \times d$ matrix-valued function parameterized by a fixed-dimensional parameter vector $\boldsymbol{\omega}$. Here \times_3 denotes the multiplication of a tensor by a matrix along its third mode; for details about tensor algebra, see the end of this section. Since the third mode of \mathcal{A} corresponds to the time lags, we call it the temporal mode (or dimension). Clearly, through the factorization in (1.4), the

dimension of the temporal mode of \mathcal{A} is reduced from ∞ to a fixed number, i.e., the dimension of ω . Thus, writing the parameterization in the form of (1.4) elucidates the mechanism underlying the parsimony of this VAR(∞) model along the temporal dimension.

However, to apply the VAR(∞) model with parameterization (1.4) to the high-dimensional setup, dimension reduction is still needed for the first two modes of the $N \times N \times \infty$ coefficient tensor \mathcal{A} . Note that the first two modes of \mathcal{A} arise from the rows and column dimensions of the $N \times N$ AR coefficient matrices \mathcal{A}_j 's. Thus, they further correspond to the dimensions of the response $\mathbf{y}_t \in \mathbb{R}^N$ and the lagged predictors $\mathbf{y}_{t-j} \in \mathbb{R}^N$, respectively; see the VAR(∞) form in (1.2). For convenience, we refer to them as the response and predictor modes (or dimensions) of \mathcal{A} , respectively. It is important to note that the factorization in (1.4) implies that \mathcal{A} has Tucker rank d at its temporal mode. We refer the readers to the end of this section for the definition of Tucker ranks and Section 3.2 for more details about the Tucker decomposition of a tensor. The low-Tucker-rank property at the temporal mode naturally motivates us to further assume that \mathcal{A} also has low Tucker ranks at the response and predictor modes. This enables a simultaneous dimension reduction for the $N \times N \times \infty$ coefficient tensor \mathcal{A} in three different directions, leading to an effective dimension of O(N).

As discussed in Section 3.3, the low-Tucker-rank structure for \mathcal{A} can be interpreted from the dynamic factor modelling perspective. Specifically, the low-rankness along the response and predictor dimensions (i.e., the first two modes) of \mathcal{A} implies latent factor structures. This means that the N-dimensional response \mathbf{y}_t and lagged predictors \mathbf{y}_{t-j} are summarized into \mathcal{R}_1 response factors and \mathcal{R}_2 predictor factors, respectively. Here, $\mathcal{R}_i \ll N$ represents the Tucker rank of \mathcal{A} at the ith mode for i = 1, 2. We name this low-Tucker-rank $VAR(\infty)$ model the scalable ARMA (SARMA) model to highlight its scalability across response, predictor and temporal dimensions, as well as its connection with the VARMA model.

In addition, in the ultra-high-dimensional setup where N may grow exponentially with the sample size T, we further consider a sparse low-Tucker-rank (SLTR) structure for \mathcal{A} by imposing entrywise-sparsity on the loadings of the response and predictor factors. This results in a more substantial dimension reduction and can be interpreted as an automatic selection of important variables into the response and predictor factors. For the proposed SARMA model, we introduce two estimators: (i) the rank-constrained estimator for the case with non-sparse factor loadings, and (ii) the SLTR estimator for the case with sparse factor loadings. For both estimators, we

derive nonasymptotic error bounds and develop a consistent estimator for the Tucker ranks. The algorithms for implementing the proposed methods are detailed in the supplementary file.

The rest of this paper is organized as follows. Section 2 outlines the motivations behind the proposed methods in simple settings. Section 3 introduces the low-dimensional VAR(∞) model, the high-dimensional SARMA model, and the dynamic factor interpretations of the latter. Section 4 develops estimation methods in both non-sparse and sparse cases, together with theoretical properties. Section 5 proposes a consistent estimator for the Tucker ranks. Simulation and empirical studies are provided in Sections 6 and 7, respectively. Section 8 concludes with a brief discussion. Algorithms and technical details are given in a separate supplementary file.

Unless otherwise specified, we denote scalars by lowercase letters x, y, \ldots , vectors by boldface lowercase letters x, y, \ldots , and matrices by boldface capital letters X, Y, \ldots . For any $a, b \in \mathbb{R}$, denote $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. For any vector x, denote its ℓ_2 norm by $\|x\|_2$. For any matrix $X \in \mathbb{R}^{d_1 \times d_2}$, let $\sigma_1(X) \geqslant \sigma_2(X) \geqslant \cdots \geqslant \sigma_{d_1 \wedge d_2}(X) \geqslant 0$ be its singular values in descending order. Let X', $\sigma_{\max}(X)$ (or $\sigma_{\min}(X)$), $\lambda_{\max}(X)$ (or $\lambda_{\min}(X)$), and rank(X) denote its transpose, largest (or smallest) singular value, largest (or smallest) eigenvalue, and rank, respectively. Its vectorization $\operatorname{vec}(X)$ is the long vector obtained by stacking all its columns. In addition, its operator norm, Frobenius norm, and nuclear norm are $\|X\|_{\operatorname{op}} = \sigma_{\max}(X)$, $\|X\|_F = \sqrt{\sum_{i,j} X_{ij}^2} = \sqrt{\sum_{k=1}^{d_1 \wedge d_2} \sigma_k^2(X)}$, and $\|X\|_* = \sum_{k=1}^{d_1 \wedge d_2} \sigma_k(X)$, respectively. For any two sequences x_n and y_n , denote $x_n \lesssim y_n$ (or $x_n \gtrsim y_n$) if there exists an absolute constant C > 0 such that $x_n \leqslant Cy_n$ (or $x_n \geqslant Cy_n$). Write $x_n = y_n$ if $x_n \lesssim y_n$ and $x_n \gtrsim y_n$. Let $\mathbb{I}_{\{\cdot\}}$ be the indicator function taking value one when the condition is true and zero otherwise. The capital letters C, C_g, \ldots and lowercase letters c, c_g, \ldots represent generic large and small positive absolute constants, respectively, whose values may vary from place to place.

This paper involves third-order tensors, a.k.a. three-way arrays, which are denoted by calligraphic capital letters. For example, a $d_1 \times d_2 \times d_3$ tensor is $\mathfrak{X} = (\mathfrak{X}_{i_1 i_2 i_3})_{1 \leqslant i_1 \leqslant d_1, 1 \leqslant i_2 \leqslant d_2, 1 \leqslant i_3 \leqslant d_3}$. It has three modes, with dimension d_i for mode i, for $1 \leqslant i \leqslant 3$. The Frobenius norm of the tensor is defined as $\|\mathfrak{X}\|_F = \sqrt{\sum_{i_1,i_2,i_3} \mathfrak{X}_{i_1 i_2 i_3}^2}$. The mode-3 product of \mathfrak{X} and a $K \times d_3$ matrix Y is the $d_1 \times d_2 \times K$ tensor given by $\mathfrak{X} \times_3 Y = (\sum_{i_3=1}^{d_3} \mathfrak{X}_{i_1 i_2 i_3} Y_{k i_3})_{1 \leqslant i_1 \leqslant d_1, 1 \leqslant i_2 \leqslant d_2, 1 \leqslant k \leqslant K}$. Similarly, the mode-i multiplication \times_i between \mathfrak{X} and a $K \times d_i$ matrix can be defined for i = 1, 2. The matricization along mode i of \mathfrak{X} results in a matrix where the mode i becomes the rows of the matrix, and the other modes are collapsed into the columns. The mode-i matricization is denoted

by $\mathfrak{X}_{(i)}$, and it can be shown that $\mathfrak{X}_{(1)} = (\boldsymbol{X}_1, \dots, \boldsymbol{X}_{d_3})$, $\mathfrak{X}_{(2)} = (\boldsymbol{X}_1', \dots, \boldsymbol{X}_{d_3}') \in \mathbb{R}^{d_2 \times d_1 d_3}$, and $\mathfrak{X}_{(3)} = (\operatorname{vec}(\boldsymbol{X}_1), \dots, \operatorname{vec}(\boldsymbol{X}_{d_3}))' \in \mathbb{R}^{d_3 \times d_1 d_2}$. The Tucker rank of \mathfrak{X} at mode i is the rank of $\mathfrak{X}_{(i)}$, i.e., $\mathcal{R}_i = \operatorname{rank}(\mathfrak{X}_{(i)})$ for $1 \leq i \leq 3$ (Tucker, 1966; De Lathauwer et al., 2000). Unlike row and column ranks of a matrix, $\mathcal{R}_1, \mathcal{R}_2$ and \mathcal{R}_3 in general are not identical.

2 Motivation for the SARMA model

2.1 Reparameterizing the VARMA model

For ease of understanding, we outline the main ideas behind the proposed SARMA model in this and the next subsection, before formally giving the definitions and properties of the model in Section 3.

Suppose that an N-dimensional time series $\{y_t\}$ is generated from the VAR (∞) process, $y_t = \sum_{j=1}^{\infty} A_j y_{t-j} + \varepsilon_t$, where $A_j \in \mathbb{R}^{N \times N}$ are the AR coefficient matrices. To overcome the parameter proliferation due to the infinite number of time lags, the VARMA model serves as a parsimonious parameterization of the VAR (∞) process; see (1.3). However, as mentioned in Section 1, this parameterization inherently introduce both identification and computational challenges. However, as we demonstrate via a simple example as follows, a reparameterization can resolve these issues.

For simplicity, consider the VARMA(1,1) model in (1.1), and suppose that its MA coefficient matrix Θ has $0 < r \le N$ distinct nonzero real eigenvalues $\lambda_1, \ldots, \lambda_r \in (-1,0) \cup (0,1)$ and no complex eigenvalues. Then, by Proposition 1 to be provided in Section 3, the AR coefficient matrices in (1.3) can be reparameterized as

$$A_j = \mathbf{\Theta}^{j-1}(\mathbf{\Phi} - \mathbf{\Theta}) = \mathbb{I}_{\{j=1\}}G_1 + \sum_{k=1}^r \mathbb{I}_{\{j\geqslant 2\}} \lambda_k^{j-1}G_{1+k}, \quad j \geqslant 1,$$

where $G_k \in \mathbb{R}^{N \times N}$ for $1 \le k \le 1 + r$ depend on Φ and the eigenvectors of Θ , and $\mathbb{I}_{\{\cdot\}}$ is the indicator function which equals one if the condition $\{\cdot\}$ is true and zero otherwise. Equivalently, this can be written as

$$\boldsymbol{A}_{j} = \sum_{k=1}^{1+r} \ell_{j,k}(\boldsymbol{\lambda}) \boldsymbol{G}_{k}, \quad \text{with} \quad \ell_{j,k}(\boldsymbol{\lambda}) = \begin{cases} \mathbb{I}_{\{j=1\}} & \text{if } k = 1, \\ \mathbb{I}_{\{j \geq 2\}} \lambda_{k-1}^{j-1} & \text{if } 2 \leq k \leq 1+r. \end{cases}$$

$$(2.1)$$

Now if we relax the dependence of $\lambda_1, \ldots, \lambda_r$ and G_1, \ldots, G_{1+r} on Φ and Θ , but rather treat

them as completely free parameters, then an alternative parsimonious parameterization for the VAR(∞) process follows. Compared with the original VARMA model, employing a VAR(∞) model with AR coefficient matrices parameterized in the form of (2.1) has two key advantages. First, its identifiability does not rely on any additional constraints; see Theorem 1 in Section 3. Second, it eliminates the need for computing high-order matrix polynomials due to Θ^{j-1} involved in $A_j(\Phi, \Theta)$, since each A_j is now simply a linear combination of the matrices G_k 's. This significantly lessens the computational burden compared with the original VARMA model.

Note that while the above example assumes that Θ has no complex eigenvalues, the key features of (2.1) carry over to the general case with both real and complex eigenvalues; see Section 3 for details. As an alternative framework for modelling VAR(∞) processes, this identifiable and computationally friendly model serves as the foundation for the proposed SARMA model for high-dimensional time series.

2.2 A tensor decomposition viewpoint

While Section 3.1 focuses on the temporal dimension, an interesting connection between the parameterization in (2.1) and the tensor decomposition motivates our strategies for reducing the cross-sectional dimensions in the proposed SARMA model.

Let \mathcal{A} be a tensor of size $N \times N \times \infty$ obtained by stacking the AR coefficient matrices $\{A_1, A_2, \ldots\}$, and likewise let \mathcal{G} be a tensor of size $N \times N \times (1+r)$ obtained by stacking $\{G_1, \ldots, G_{1+r}\}$. Since the third mode of \mathcal{A} corresponds to the time lags, we call it the temporal mode (or dimension). In addition, define the $\infty \times (1+r)$ matrix-valued function:

$$\boldsymbol{L}(\boldsymbol{\lambda}) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & \lambda_r \\ 0 & \lambda_1^2 & \cdots & \lambda_r^2 \\ \vdots & \vdots & & \vdots \end{pmatrix},$$

where $\lambda = (\lambda_1, \dots, \lambda_r)'$. Then it can be readily verified that (2.1) is equivalent to a factorization of \mathcal{A} along the temporal mode:

$$\mathcal{A} = \mathbf{9} \times_3 \mathbf{L}(\lambda). \tag{2.2}$$

Note that through the above factorization, the dimension of the temporal mode of \mathcal{A} is reduced from ∞ to r, i.e., the dimension of λ . This finding offers us a fresh angle to understand how

the temporal dimension for the VAR(∞) model is effectively reduced via parameterization (2.1). Simply speaking, by factoring out $L(\lambda)$, the essential temporal patterns are extracted along the temporal mode of \mathcal{A} , i.e., across time lags.

However, when the cross-sectional dimension N is large, we still need to conduct dimension reduction for the first two modes of \mathcal{A} , which we refer to as the response and predictor modes, respectively. These two modes arise from the rows and column dimensions of the $N \times N$ AR coefficient matrices \mathbf{A}_j 's, hence corresponding to the dimensions of the response $\mathbf{y}_t \in \mathbb{R}^N$ and the lagged predictor $\mathbf{y}_{t-j} \in \mathbb{R}^N$, respectively. Motivated by the temporal factorization in (2.2), it is natural to further factorize \mathcal{A} along the response and predictor modes, as we will show in (3.7). This dimension reduction scheme allows scalability across all three directions, leading to the formulation of the SARMA model to be proposed in Section 3.

3 Proposed SARMA model

3.1 The low-dimensional $VAR(\infty)$ model

In Section 2.1, we illustrate that an alternative $VAR(\infty)$ parameterization, with AR coefficient matrices parameterized as in (2.1), is motivated by a simple VARMA(1,1) model. When this idea is extended to the VARMA(p,q) model, a more general class of $VAR(\infty)$ models with AR coefficient matrices structured similarly to (2.1) is formulated.

For any VARMA(p,q) model in the form of $\mathbf{y}_t = \sum_{i=1}^p \mathbf{\Phi}_i \mathbf{y}_{t-i} + \boldsymbol{\varepsilon}_t - \sum_{j=1}^q \mathbf{\Theta}_j \boldsymbol{\varepsilon}_{t-j}$, the MA companion matrix (Lütkepohl, 2005) is defined as

$$oldsymbol{\Theta} = egin{pmatrix} \Theta_1 & \Theta_2 & \cdots & \Theta_{q-1} & \Theta_q \ I & 0 & \cdots & 0 & 0 \ 0 & I & \cdots & 0 & 0 \ dots & dots & \ddots & dots & dots \ 0 & 0 & \cdots & I & 0 \end{pmatrix},$$

which reduces to $\Theta = \Theta_1$ when q = 1. Suppose $\underline{\Theta}$ has exactly r nonzero real eigenvalues, $\lambda_k \in (-1,1)$ for $1 \leq k \leq r$, and s pairs of nonzero complex eigenvalues, $(\gamma_m e^{i\theta_m}, \gamma_m e^{-i\theta_m})$ with $\gamma_m \in (0,1)$ and $\theta_m \in (0,\pi)$ for $1 \leq m \leq s$.

Proposition 1. Consider the VARMA(p,q) process $\mathbf{y}_t = \sum_{i=1}^p \mathbf{\Phi}_i \mathbf{y}_{t-i} + \boldsymbol{\varepsilon}_t - \sum_{j=1}^q \mathbf{\Theta}_j \boldsymbol{\varepsilon}_{t-j}$. Suppose that the corresponding MA companion matrix $\underline{\mathbf{\Theta}}$ has r distinct nonzero real eigenvalues, $\lambda_k \in (-1,1)$ for $1 \leq k \leq r$, and s distinct conjugate pairs of nonzero complex eigenvalues, $(\gamma_k e^{i\theta_k}, \gamma_k e^{-i\theta_k})$ with $\gamma_k \in (0,1)$ and $\theta_k \in (0,\pi)$ for $1 \leq k \leq s$. Then $\{\mathbf{y}_t\}$ has the VAR(∞) representation $\mathbf{y}_t = \sum_{j=1}^\infty \mathbf{A}_j \mathbf{y}_{t-j} + \boldsymbol{\varepsilon}_t$ with $\mathbf{A}_j = \sum_{k=1}^d \ell_{j,k}(\boldsymbol{\omega}) \mathbf{G}_k$ for $j \geq 1$, and

$$\ell_{j,k}(\boldsymbol{\omega}) = \begin{cases} \mathbb{I}_{\{j=k\}} & \text{if } 1 \leqslant k \leqslant p, \\ \mathbb{I}_{\{j\geqslant p+1\}} \lambda_{m_k}^{j-p} & \text{if } p+1 \leqslant k \leqslant p+r \\ \mathbb{I}_{\{j\geqslant p+1\}} \gamma_{n_k}^{j-p} [\cos(j-p)\theta_{n_k} + \sin(j-p)\theta_{n_k}], & \text{if } p+r+1 \leqslant k \leqslant d, \end{cases}$$

where d = p + r + 2s, $m_k = k - p$, $n_k = \lceil \frac{k - p - r}{2} \rceil$, $\boldsymbol{\omega} = (\lambda_1, \dots, \lambda_r, \gamma_1, \theta_1, \dots, \gamma_s, \theta_s)'$, and $\boldsymbol{G}_1, \dots, \boldsymbol{G}_d \in \mathbb{R}^{N \times N}$ depend on the coefficient matrices $\boldsymbol{\Phi}_i$'s and $\boldsymbol{\Theta}_j$'s of the VARMA model.

Note that (2.1) is a special case of Proposition 1 with p = r = 1 and s = 0. While Proposition 1 originates from a VARMA process, it motivates an alternative class of VAR(∞) models which treat ω and G_1, \ldots, G_d as free parameters. For any given model orders (p, r, s), this multivariate time series model is defined as follows:

$$\mathbf{y}_t = \sum_{j=1}^{\infty} \mathbf{A}_j \mathbf{y}_{t-j} + \boldsymbol{\varepsilon}_t, \text{ with } \mathbf{A}_j = \mathbf{A}_j(\boldsymbol{\omega}, \boldsymbol{\mathcal{G}}) = \sum_{k=1}^{d} \ell_{j,k}(\boldsymbol{\omega}) \mathbf{G}_k,$$
 (3.1)

where d = p + r + 2s, $\mathbf{G}_k \in \mathbb{R}^{N \times N}$ for $1 \leq k \leq d$, the parameter space of $\boldsymbol{\omega}$ is

$$\Omega = \{ \boldsymbol{\omega} \in \mathbb{R}^{r+2s} \mid |\lambda_k|, \gamma_h \in (0,1), \theta_h \in (0,\pi) \text{ for } 1 \leqslant k \leqslant r \text{ and } 1 \leqslant h \leqslant s \},$$
(3.2)

and $\ell_{j,k}(\boldsymbol{\omega})$ is the (j,k)-th entry of the matrix

$$\boldsymbol{L}(\boldsymbol{\omega}) = \begin{pmatrix} \boldsymbol{I}_p & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\ell}^I(\lambda_1) & \cdots & \boldsymbol{\ell}^I(\lambda_r) & \boldsymbol{\ell}^{II}(\gamma_1, \theta_1) & \cdots & \boldsymbol{\ell}^{II}(\gamma_s, \theta_s) \end{pmatrix} \in \mathbb{R}^{\infty \times d}, \tag{3.3}$$

with

$$\boldsymbol{\ell}^{I}(\lambda) = (\lambda, \lambda^{2}, \lambda^{3}, \dots)' \quad \text{and} \quad \boldsymbol{\ell}^{II}(\gamma, \theta) = \begin{pmatrix} \gamma \cos(\theta) & \gamma^{2} \cos(2\theta) & \gamma^{3} \cos(3\theta) & \cdots \\ \gamma \sin(\theta) & \gamma^{2} \sin(2\theta) & \gamma^{3} \sin(3\theta) & \cdots \end{pmatrix}',$$

for any λ and (γ, θ) ; see also the concurrent work by Zheng (2024) which does not provide the

theoretical properties below. Given the model orders (p, r, s), the following theorem implies that the parameters ω and G_1, \ldots, G_d for this model are identifiable.

Theorem 1 (Identifiability). Suppose that $G_1, \ldots, G_d \neq 0$ and $\omega \in \Omega$. If $\lambda_1 < \cdots < \lambda_r$, and the pairs (γ_m, θ_m) 's are distinct and sorted in ascending order of γ_m 's and θ_m 's, then there is a one-to-one correspondence between matrices $\{A_1, A_2, \ldots\}$ and $\{\omega, G_1, \ldots, G_d\}$, where A_j 's are defined as in (3.1).

Since any VAR(∞) process is uniquely defined by its AR coefficient matrices $\{A_1, A_2, \dots\}$, Theorem 1 establishes the identifiability of ω and G_1, \dots, G_d up to a permutation. Thus, unlike the VARMA model, no additional parameter constraint is needed for the identification of the parameters $\{\omega, G_1, \dots, G_d\}$. Moreover, with A_j 's parameterized as linear combinations of matrices, the computation for this model avoids any high-order matrix polynomials, which substantially reduces the computational cost compared to the VARMA model.

The following theorem gives a sufficient condition for the weak (second-order) stationarity of the model.

Theorem 2 (Weak stationarity). Suppose that $\{\varepsilon_t\}$ is an i.i.d. sequence with $E(\|\varepsilon_t\|_2) < \infty$. If there exists $0 < \rho < 1$ such that

$$\max\{|\lambda_1|, \dots, |\lambda_r|, \gamma_1, \dots, \gamma_s\} \leqslant \rho \quad and \quad \sum_{k=1}^p \|\boldsymbol{G}_k\|_{\text{op}} + \frac{\rho}{1-\rho} \sum_{k=p+1}^d \|\boldsymbol{G}_k\|_{\text{op}} < 1, \qquad (3.4)$$

then there exists a unique weakly stationary solution to model (3.1), and it has the form of $\mathbf{y}_t = \mathbf{\varepsilon}_t + \sum_{j=1}^{\infty} \mathbf{\Psi}_j \mathbf{\varepsilon}_{t-j}$, where $\mathbf{\Psi}_j = \sum_{k=1}^{\infty} \sum_{j_1 + \dots + j_k = j} \mathbf{A}_{j_1} \cdots \mathbf{A}_{j_k}$ and $\mathbf{A}_j = \sum_{k=1}^d \ell_{j,k}(\boldsymbol{\omega}) \mathbf{G}_k$ for all $j \ge 1$.

3.2 The high-dimensional SARMA model

As discussed in Section 2.2, the parameterization in (2.1) can be viewed as a factorization of the $N \times N \times \infty$ coefficient tensor \mathcal{A} along the temporal mode, i.e., (2.2). This viewpoint can be directly generalized to model (3.1). Indeed, the second equation in (3.1) for $j \ge 1$ is equivalent to

$$\mathcal{A} = \mathcal{G} \times_3 \mathbf{L}(\boldsymbol{\omega}), \tag{3.5}$$

where \mathfrak{G} is the $N \times N \times d$ tensor formed by stacking $\{G_1, \ldots, G_d\}$, and $L(\omega)$ is the $\infty \times d$ matrix defined in (3.3), with d = p + r + 2s. By tensor algebra (Kolda and Bader, 2009), this factorization

implies that the Tucker rank of \mathcal{A} at its third mode, $\mathcal{R}_3 = \operatorname{rank}(\mathcal{A}_{(3)})$, is at most d. Thus, model (3.1) can be regarded as a dimension reduction scheme for the *temporal* mode of \mathcal{A} within the $\operatorname{VAR}(\infty)$ framework.

For high-dimensional time series, the above viewpoint motivates us to further conduct the dimension reduction for the *response* and *predictor* modes of \mathcal{A} . Specifically, we impose the low-Tucker-rank assumption on \mathcal{A} for its first two modes as follows:

$$\mathcal{R}_i = \operatorname{rank}(\mathcal{A}_{(i)}) \ll N, \quad i = 1, 2.$$

Note that $\operatorname{rank}(\mathcal{A}_{(i)}) = \operatorname{rank}(\mathcal{G}_{(i)})$ for i = 1, 2, as the factorizations along different modes of the tensor do not interfere with each other. Thus, this is also equivalent to assuming that \mathcal{G} has low Tucker ranks at its first two modes:

$$\mathcal{R}_i = \operatorname{rank}(\mathfrak{G}_{(i)}) \ll N, \quad i = 1, 2. \tag{3.6}$$

Then, under this assumption, there exist a small tensor $\mathbf{S} \in \mathbb{R}^{\mathcal{R}_1 \times \mathcal{R}_2 \times d}$ and full-rank matrices $\mathbf{U}_i \in \mathbb{R}^{N \times \mathcal{R}_i}$ for i = 1, 2 such that $\mathbf{G} = \mathbf{S} \times_1 \mathbf{U}_1 \times_2 \mathbf{U}_2$, which along with (3.5) implies that

$$\mathcal{A} = \underbrace{\mathbf{S} \times_1 \mathbf{U}_1 \times_2 \mathbf{U}_2}_{\mathbf{G}} \times_3 \mathbf{L}(\boldsymbol{\omega}) := [[\mathbf{S}; \mathbf{U}_1, \mathbf{U}_2, \mathbf{L}(\boldsymbol{\omega})]], \tag{3.7}$$

In tensor algebra, (3.7) is called the Tucker decomposition of the tensor \mathcal{A} , with \mathcal{S} termed the core tensor, and U_1, U_2 and $L(\omega)$ termed the factor matrices. Note that the factorization of \mathcal{G} is written mainly to facilitate the understanding of low-Tucker-rank assumption; see Section 3.3. The unknown parameters to be estimated are still ω and G_1, \ldots, G_d (i.e., the tensor \mathcal{G}).

Similar to the low-rankness of matrices, the low-Tucker-rank assumption enables a reduction in the number of parameters for the coefficient tensor: it reduces the effective dimension of \mathfrak{G} from N^2d to $O(N(\mathcal{R}_1 + \mathcal{R}_2) + \mathcal{R}_1\mathcal{R}_2d)$. From the viewpoint of VAR(∞) modelling, (3.7) reveals that a simultaneous dimension reduction is conducted across the response, predictor, and temporal modes of the AR coefficient tensor \mathcal{A} . To emphasize the resulting scalability across all three directions, we name model (3.1) with the low-Tucker-rank assumption in (3.6) for \mathfrak{G} the Scalable ARMA (SARMA) model.

3.3 Dynamic factor interpretation

In this section, we discuss the interpretation of the low-Tucker-rank assumption in (3.6) for the SARMA model and show that it implies low-dimensional dynamic factor structures underlying both the response y_t and the lagged predictor series y_{t-i} 's.

As the model is parameterized by ω and \mathcal{G} , it is not necessary to construct estimators for the components \mathcal{S} , U_1 and U_2 in the factorization of \mathcal{G} . Nonetheless, the representation in (3.7) facilitates our understanding of the low-Tucker-rank assumption on \mathcal{G} for the VAR(∞) model. It reveals that while $L(\omega)$ extracts essential patterns from the temporal mode of the coefficient tensor \mathcal{A} , the matrices U_1 and U_2 summarize information along the cross-sectional dimension of the response and lagged predictors, respectively. Note that

$$\mathbf{G} = \mathbf{S} \times_{1} \mathbf{U}_{1} \times_{2} \mathbf{U}_{2} = (\mathbf{S} \times_{1} \mathbf{O}_{1} \times_{2} \mathbf{O}_{2}) \times_{1} (\mathbf{U}_{1} \mathbf{O}_{1}^{-1}) \times_{2} (\mathbf{U}_{2} \mathbf{O}_{2}^{-1})$$
(3.8)

for any invertible matrices O_i with i = 1, 2, indicating the rotational and scale indeterminacies of the components. Without loss of generality, the normalization constraint $U'_iU_i = I_{\mathcal{R}_i}$ for i = 1, 2 can be imposed to facilitate interpretations.

Moreover, the SARMA model can be interpreted from the factor modelling perspective, with U_1 and U_2 representing loading matrices for the response factors and lagged predictor factors, respectively. To see this, first consider the simple example with Tucker ranks $\mathcal{R}_1 = \mathcal{R}_2 = 1$. In this case, $S = (s_1, \ldots, s_d)' \in \mathbb{R}^d := s$ and $U_i := u_i \in \mathbb{R}^N$ for i = 1, 2 all reduce to vectors, hence denoted by bold lowercase letters. Then, (3.7) implies $G_k = s_k u_1 u_2'$ for $1 \leq k \leq d$, which are rank-one matrices. As a result, $A_j = \sum_{k=1}^d \ell_{j,k}(\boldsymbol{\omega}) s_k u_1 u_2'$ for $j \geq 1$. Note that u_1 and u_2 capture patterns from the rows and columns of A_j 's, respectively. Consequently, with the normalization $u_i'u_i = 1$ for i = 1, 2, a single-factor model is implied as follows:

$$\underbrace{\boldsymbol{u}_1'\boldsymbol{y}_t}_{\text{single response factor}} = \sum_{j=1}^{\infty} \sum_{k=1}^{d} \ell_{j,k}(\boldsymbol{\omega}) s_k \underbrace{\boldsymbol{u}_2'\boldsymbol{y}_{t-j}}_{\text{single predictor factor}} + \boldsymbol{e}_t,$$

where $e_t = u'_1 \varepsilon_t$. For instance, suppose that y_t contains realized volatilities of N stocks in a market. Then $u'_1 y_t$ and $u'_2 y_{t-j}$ can be viewed as latent response and lagged predictor factors, respectively, which can also be regarded as two different market volatility indices. The predictor factor loading u_2 encapsulates how the the past signals from various stocks are absorbed into

the market, while the response factor loading u_1 summarizes the overall response of the *present* market to these signals; see also Section 7 for an empirical example.

For general Tucker ranks \mathcal{R}_1 and \mathcal{R}_2 , analogously we have

$$\boldsymbol{U}_{1}'\boldsymbol{y}_{t} = \sum_{j=1}^{\infty} \sum_{k=1}^{d} \ell_{j,k}(\boldsymbol{\omega}) \boldsymbol{S}_{k} \boldsymbol{U}_{2}' \boldsymbol{y}_{t-j} + \boldsymbol{e}_{t}.$$
(3.9)

Here, U'_1y_t represents \mathcal{R}_1 response factors, while U'_2y_{t-j} represents \mathcal{R}_2 lagged predictor factors. with the loading matrices being $U_i \in \mathbb{R}^{N \times \mathcal{R}_i}$ for i = 1 and 2, respectively. Thus, by imposing the low-Tucker-rank assumption on \mathbf{G} in (3.6), simultaneous dimension reduction is achieved by extracting factors across both the response and lagged predictors. For convenience, we call \mathcal{R}_1 and \mathcal{R}_2 the response and predictor ranks, respectively.

In addition, when N is extremely large, we may further assume that U_1 and U_2 are sparse matrices for more efficient dimension reduction. This implies that each factor contains only a small subset of variables. Take $U_1'y_t$ as an example. For $1 \le i \le N$ and $1 \le k \le \mathcal{R}_1$, if the (i,k)th entry of U_1 is nonzero, then it implies that the ith variable in y_t is selected into the kth response factor. This sparsity assumption, which is embedded in the Tucker decomposition, will make the estimation of the SARMA model more challenging; see Section 4.2 for details.

4 High-dimensional estimation

4.1 Rank-constrained estimator

We first introduce a rank-constrained approach to estimate the parameter vector ω and the low-Tucker-rank parameter tensor \mathfrak{g} . As will be shown in Section 4.3, this estimator is consistent under N = o(T), where T is the sample size; another estimation method applicable to the ultrahigh-dimensional case which allows $\log(N)/T \to 0$ will be introduced in Section 4.2.

Let $\boldsymbol{x}_t = (\boldsymbol{y}_{t-1}', \boldsymbol{y}_{t-2}', \dots)'$. Then the squared error loss function is $\mathbb{L}_T(\boldsymbol{\omega}, \boldsymbol{\mathfrak{G}}) = \sum_{t=1}^T \|\boldsymbol{y}_t - \boldsymbol{A}_{(1)}\boldsymbol{x}_t\|_2^2$, where $\boldsymbol{A}_{(1)} = (\boldsymbol{A}_1, \boldsymbol{A}_2, \dots)$ with $\boldsymbol{A}_j = \boldsymbol{A}_j(\boldsymbol{\omega}, \boldsymbol{\mathfrak{G}}) = \sum_{k=1}^d \ell_{j,k}(\boldsymbol{\omega})\boldsymbol{G}_k$ for $j \geq 1$. Since the loss depends on observations in the infinite past, initial values for $\{\boldsymbol{y}_t, t \leq 0\}$ are needed in practice. We set them to zero for simplicity, that is, let $\boldsymbol{\tilde{x}}_t = (\boldsymbol{y}_{t-1}', \dots, \boldsymbol{y}_1', 0, 0, \dots)'$ be the

initialized version of x_t , and define the feasible squared loss function:

$$\widetilde{\mathbb{L}}_{T}(\boldsymbol{\omega}, \boldsymbol{\mathcal{G}}) = \sum_{t=1}^{T} \|\boldsymbol{y}_{t} - \boldsymbol{\mathcal{A}}_{(1)} \widetilde{\boldsymbol{x}}_{t}\|_{2}^{2} = \sum_{t=1}^{T} \|\boldsymbol{y}_{t} - \sum_{j=1}^{t-1} \boldsymbol{A}_{j}(\boldsymbol{\omega}, \boldsymbol{\mathcal{G}}) \boldsymbol{y}_{t-j}\|_{2}^{2}.$$
(4.1)

The initialization effect will be accounted for in our theoretical analysis.

Suppose that the response and predictor ranks $(\mathcal{R}_1, \mathcal{R}_2)$ are known; see Section 5 for a datadriven selection procedure. When N is moderately large compared to T, we propose the rankconstrained estimator as follows:

$$(\widehat{\boldsymbol{\omega}}, \widehat{\boldsymbol{\mathcal{G}}}) = \underset{\boldsymbol{\omega} \in \Omega, \boldsymbol{\mathcal{G}} \in \Gamma(\mathcal{R}_1, \mathcal{R}_2)}{\arg \min} \widetilde{\mathbb{L}}_T(\boldsymbol{\omega}, \boldsymbol{\mathcal{G}}), \tag{4.2}$$

where Ω is defined in (3.2), and the parameter space of \mathfrak{G} is

$$\Gamma(\mathcal{R}_1, \mathcal{R}_2) = \{ \mathbf{G} \in \mathbb{R}^{N \times N \times d} \mid \operatorname{rank}(\mathbf{G}_{(1)}) \leqslant \mathcal{R}_1, \operatorname{rank}(\mathbf{G}_{(2)}) \leqslant \mathcal{R}_2 \}.$$

Then based on the results from (4.2), we can obtain $\hat{\mathcal{A}} = \hat{\mathcal{G}} \times_3 \boldsymbol{L}(\hat{\boldsymbol{\omega}})$; i.e., the corresponding AR coefficient matrices are estimated by $\hat{\boldsymbol{A}}_j = \sum_{k=1}^d \ell_{j,k}(\hat{\boldsymbol{\omega}}) \hat{\boldsymbol{G}}_k$ for $j \geq 1$.

Remark 1. Note that (4.2) does not require estimation of \mathbf{S} , U_1 and U_2 , i.e., the components in the Tucker decomposition of \mathbf{S} . Thus, the rotational and scale indeterminacies in (3.8) are not an issue. However, to interpret the underlying dynamic factor structure presented in (3.9), it is beneficial to conduct the Tucker decomposition of $\hat{\mathbf{S}}$ to obtain the corresponding estimated loading matrices $\hat{\mathbf{U}}_1$ and $\hat{\mathbf{U}}_2$ after the rank-constrained estimation in (4.2). A common approach to ensure the uniqueness of the Tucker decomposition is to employ the higher-order singular value decomposition (HOSVD), which is the special Tucker decomposition as follows (De Lathauwer et al., 2000). Specifically, to get the HOSVD, $\hat{\mathbf{S}} = \hat{\mathbf{S}} \times_1 \hat{\mathbf{U}}_1 \times_2 \hat{\mathbf{U}}_2$, the matrix $\hat{\mathbf{U}}_i$ is defined as the top \mathcal{R}_i left singular vectors of $\hat{\mathbf{S}}_{(i)}$ with the first element in each column of $\hat{\mathbf{U}}_i$ being positive, for i = 1, 2. This rules out both rotational and sign indeterminacies. In addition, by the orthonormality of $\hat{\mathbf{U}}_i$'s, we can compute $\hat{\mathbf{S}} = \hat{\mathbf{S}} \times_1 \hat{\mathbf{U}}_1' \times_2 \hat{\mathbf{U}}_2'$. Thus, the factor representation in (3.9) for the fitted model can be obtained. This will allow us to clearly interpret the dynamic factor structure based on the uniquely defined loading matrices $\hat{\mathbf{U}}_1$ and $\hat{\mathbf{U}}_2$.

4.2 Sparse low-Tucker-rank estimator

When N is very large relative to the sample size T, the rank-constrained estimator can be inefficient, and a more substantial dimension reduction is needed. Motivated by the dynamic factor structure in (3.9), we additionally assume that the loadings U_1 and U_2 are sparse, and develop a high-dimensional estimator that simultaneously enforces the low-Tucker-rank and sparse structures. This not only improves the estimation efficiency but enhances the interpretability as it automatically selects only important variables into the factors.

However, unlike the rank-constrained estimator in (4.2), explicit factorization of \mathfrak{G} must be incorporated into the sparse estimation. Moreover, to ensure the identifiability of the sparsity patterns, we assume that U_i is the orthonormal matrix consisting of the top \mathcal{R}_i left singular vectors of $\mathfrak{G}_{(i)}$, for i = 1, 2. This implies that $\mathfrak{S} = \mathfrak{G} \times_1 U'_1 \times_2 U'_2$. Note that since U_i is orthonormal, it can be shown that $\mathfrak{S}_{(i)}$ is row-orthogonal, for i = 1, 2.

We consider the following ℓ_1 -regularized sparse low-Tucker-rank (SLTR) estimator:

$$(\widetilde{\boldsymbol{\omega}}, \widetilde{\boldsymbol{S}}, \widetilde{\boldsymbol{U}}_1, \widetilde{\boldsymbol{U}}_2) = \underset{\boldsymbol{\omega} \in \boldsymbol{\Omega}, \boldsymbol{S} \in \mathrm{RO}(\mathcal{R}_1, \mathcal{R}_2), \boldsymbol{U}_i' \boldsymbol{U}_i = \boldsymbol{I}_{\mathcal{R}_i}, i = 1, 2}{\mathrm{arg \, min}} \left\{ \widetilde{\mathbb{L}}_T(\boldsymbol{\omega}, \boldsymbol{S} \times_1 \boldsymbol{U}_1 \times_2 \boldsymbol{U}_2) + \lambda \sum_{i=1}^2 \|\boldsymbol{U}_i\|_1 \right\}, \quad (4.3)$$

where

$$RO(\mathcal{R}_1, \mathcal{R}_2) = \{ \mathbf{S} \in \mathbb{R}^{\mathcal{R}_1 \times \mathcal{R}_2 \times d} : \mathbf{S}_{(i)} \text{ is row-orthogonal, } i = 1, 2 \}.$$

Then it is straightforward to estimate \mathfrak{G} and $\widetilde{\mathcal{A}}$ by $\widetilde{\mathfrak{G}} = \widetilde{\mathfrak{S}} \times_1 \widetilde{U}_1 \times_2 \widetilde{U}_2$ and $\widetilde{\widetilde{\mathcal{A}}} = \widetilde{\mathfrak{G}} \times_3 L(\widetilde{\omega})$, respectively; i.e., the estimated coefficient matrices $\widetilde{G}_1, \ldots, \widetilde{G}_d$ and \widetilde{A}_j for $j \geqslant 1$ can be obtained.

4.3 Nonasymptotic error bounds

This section provides nonasymptotic error bounds for the proposed rank-constrained and SLTR estimators, in the non-sparse and sparse cases, respectively. We assume that the observed time series $\{\boldsymbol{y}_t\}_{t=1}^T$ is generated from a stationary SARMA model with response and predictor ranks $(\mathcal{R}_1, \mathcal{R}_2)$.

Let $\omega^* \in \Omega$ and $\mathfrak{G}^* \in \Gamma(\mathcal{R}_1, \mathcal{R}_2)$ denote the true values of ω and \mathfrak{G} , respectively. Similarly, \mathcal{A}^* , λ_k^* 's, γ_k^* 's, θ_k^* 's, etc., denote the true values of the corresponding parameters. To prove the consistency of the rank-constrained estimator, we make the following assumptions.

Assumption 1 (Sub-Gaussian error). Let $\varepsilon_t = \Sigma_{\varepsilon}^{1/2} \xi_t$, where ξ_t is a sequence of i.i.d. random

vectors with zero mean and $\operatorname{var}(\boldsymbol{\xi}_t) = \boldsymbol{I}_N$, and $\boldsymbol{\Sigma}_{\varepsilon}$ is a positive definite covariance matrix. In addition, the coordinates $(\boldsymbol{\xi}_{it})_{1 \leq i \leq N}$ within $\boldsymbol{\xi}_t$ are mutually independent and σ^2 -sub-Gaussian.

Assumption 2 (Parameters). (i) There exists an absolute constant $0 < \bar{\rho} < 1$ such that for all $\omega \in \Omega$, $|\lambda_1|, \ldots, |\lambda_r|, \gamma_1, \ldots, \gamma_s \in \Lambda$, where Λ is a compact subset of $(0, \bar{\rho})$; (ii) all λ_k^* 's are bounded away from each other, and all pairs (γ_m^*, θ_m^*) 's are bounded away from each other, for $1 \le k \le r$ and $1 \le m \le s$; and (iii) $\max_{1 \le k \le d} \|G_k^*\|_{\text{op}} \le C_9$ for some absolute constant $C_9 > 0$, and $\|G_k^*\|_{\text{F}} = \alpha$ for $p + 1 \le k \le d$, where $\alpha = \alpha(N) > 0$ may depend on the dimension N.

Assumption 1 is weaker than the commonly imposed Gaussian assumption in the literature on high-dimensional time series; see, e.g., Basu and Michailidis (2015) and Wilms et al. (2021). Assumption 2(i) requires $|\lambda_k|$'s and γ_m 's to be bounded away from one. Assumption 2(ii) ensures that different elements of ω^* can be distinguished in the estimation. While Assumption 2(iii) requires that $\|G_k^*\|_F$ for $p+1 \le k \le d$ have the same order of magnitude α , it is allowed to vary with N. This condition can be readily relaxed through a slightly more involved proof. In this case, the lower and upper bounds of $\|G_k^*\|_F$ will affect the error bounds.

While the proposed model is linear in \mathcal{A} , the loss function in (4.2) is nonconvex with respect to $\boldsymbol{\omega}$ and \mathcal{G} jointly. As an intermediate step to prove the consistency of the proposed estimators, the following lemma allows us to linearize \mathcal{A} with respect to $\boldsymbol{\omega}$ and \mathcal{G} within a constant-radius neighborhood of $\boldsymbol{\omega}^*$; see Remark 2 for more details about the radius $c_{\boldsymbol{\omega}}$.

Lemma 1. Under Assumption 2, for any $\mathcal{A} = \mathcal{G} \times_3 \mathbf{L}(\boldsymbol{\omega})$ with $\mathcal{G} \in \mathbb{R}^{N \times N \times d}$ and $\boldsymbol{\omega} \in \Omega$, if $\|\boldsymbol{\omega} - \boldsymbol{\omega}^*\|_2 \leqslant c_{\boldsymbol{\omega}}$, then $\|\mathcal{A} - \mathcal{A}^*\|_F \approx \|\mathcal{G} - \mathcal{G}^*\|_F + \alpha \|\boldsymbol{\omega} - \boldsymbol{\omega}^*\|_2$, where $c_{\boldsymbol{\omega}} > 0$ is a non-shrinking radius.

Note that any stationary VAR(∞) process admits the VMA(∞) representation, $\boldsymbol{y}_t = \boldsymbol{\Psi}_*(B)\boldsymbol{\varepsilon}_t$, where B is the backshift operator, and $\boldsymbol{\Psi}_*(B) = \boldsymbol{I}_N + \sum_{j=1}^{\infty} \boldsymbol{\Psi}_j^* B^j$; see Theorem 2 for a sufficient condition for the stationarity of the SARMA model. Here we suppress the dependence of $\boldsymbol{\Psi}_j^*$'s on \boldsymbol{A}_j^* 's and hence $\boldsymbol{\omega}^*$ and \boldsymbol{G}_k^* 's for brevity. Let $\mu_{\min}(\boldsymbol{\Psi}_*) = \min_{|z|=1} \lambda_{\min}(\boldsymbol{\Psi}_*(z)\boldsymbol{\Psi}_*^{\mathsf{H}}(z))$ and $\max_{|z|=1} \lambda_{\max}(\boldsymbol{\Psi}_*(z)\boldsymbol{\Psi}_*^{\mathsf{H}}(z))$, where $\boldsymbol{\Psi}_*^{\mathsf{H}}(z)$ is the conjugate transpose of $\boldsymbol{\Psi}_*(z)$ for $z \in \mathbb{C}$, and it can be verified that $\mu_{\min}(\boldsymbol{\Psi}_*) > 0$; see also Basu and Michailidis (2015). Then let $\kappa_1 = \lambda_{\min}(\boldsymbol{\Sigma}_{\varepsilon})\mu_{\min}(\boldsymbol{\Psi}_*) \min\{1, c_{\bar{\rho}}^2\}$ and $\kappa_2 = \lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})\mu_{\max}(\boldsymbol{\Psi}_*) \max\{1, C_{\bar{\rho}}^2\}$, where $c_{\bar{\rho}}, C_{\bar{\rho}} > 0$ are absolute constants defined in Lemma S.2 in the supplementary file.

Theorem 3 (Rank-constrained estimator). Let $d_{\mathcal{R}} = \mathcal{R}_1 \mathcal{R}_2 d + (\mathcal{R}_1 + \mathcal{R}_2) N$. Suppose that $\|\hat{\omega} - \omega^*\|_2 \leq c_{\omega}$ and $T \gtrsim (\kappa_2/\kappa_1)^2 d_{\mathcal{R}} \log(\kappa_2/\kappa_1)$. Then under Assumptions 1 and 2, with probability at least $1 - 4e^{-cd_{\mathcal{R}} \log(\kappa_2/\kappa_1)} - 8e^{-cN} - \{2 + \sqrt{\kappa_2/\lambda_{\max}(\Sigma_{\varepsilon})}\}\sqrt{N/\{(\mathcal{R}_1 + \mathcal{R}_2)T\}}$, we have the following estimation and prediction error bounds:

$$\|\widehat{\mathcal{A}} - \mathcal{A}^*\|_{\mathrm{F}} \lesssim \sqrt{\frac{\kappa_2 \lambda_{\max}(\Sigma_{\varepsilon}) d_{\mathcal{R}}}{\kappa_1^2 T}} \quad and \quad \frac{1}{T} \sum_{t=1}^T \left\| (\widehat{\mathcal{A}} - \mathcal{A}^*)_{(1)} \widetilde{\boldsymbol{x}}_t \right\|_2^2 \lesssim \frac{\kappa_2 \lambda_{\max}(\Sigma_{\varepsilon}) d_{\mathcal{R}}}{\kappa_1 T}.$$

Combining Theorem 3 with Lemma 1, we immediately have that with the same probability, $\|\widehat{\mathbf{G}} - \mathbf{G}^*\|_{\mathrm{F}} \lesssim \sqrt{\kappa_2 \lambda_{\max}(\mathbf{\Sigma}_{\varepsilon}) d_{\mathcal{R}}/(\kappa_1^2 T)} \text{ and } \|\widehat{\boldsymbol{\omega}} - \boldsymbol{\omega}^*\|_2 \lesssim \sqrt{\kappa_2 \lambda_{\max}(\mathbf{\Sigma}_{\varepsilon}) d_{\mathcal{R}}/(\alpha^2 \kappa_1^2 T)}.$

For the SLTR estimator, we make the following additional assumptions.

Assumption 3 (Sparsity). Each column of the matrix U_i^* has at most s_i nonzero entries, where i = 1, 2.

Assumption 4 (Restricted parameter space). The parameter spaces for S and U_i with i = 1 or 2 are $\Omega_S = \{S \in AO(\mathcal{R}_1, \mathcal{R}_2) : \sigma_1(S_{(i)}) \leq C_S < \infty, i = 1, 2\}$ and $U_i = \{U \in \mathbb{R}^{N \times \mathcal{R}_i} \mid U'U = I_{\mathcal{R}_i}, \text{ and } U_{j,m}^2 \geq \underline{u} > 0 \text{ or } U_{j,m}^2 = 0, \forall 1 \leq j \leq N, 1 \leq m \leq \mathcal{R}_i\}$, respectively, where \underline{u} is a uniform lower threshold, and $U_{j,m}$ is the (j,m)-th entry of the matrix U.

Assumption 5 (Relative spectral gap). The nonzero singular values of $\mathfrak{G}_{(i)}$ satisfy that $\sigma_{j-1}^2(\mathfrak{G}_{(i)}) - \sigma_j^2(\mathfrak{G}_{(i)}) \geqslant \beta \sigma_{j-1}^2(\mathfrak{G}_{(i)})$ for $2 \leqslant j \leqslant \mathcal{R}_i$ and i = 1, 2, where $\beta > 0$ is a constant.

Assumption 3 defines the entrywise sparsity of U_i^* 's. In Assumption 4, the upper bound condition on S is mild since large singular values in S could cause nonstationarity of the process. The lower threshold \underline{u} for U_i 's is needed to establish the restricted eigenvalue condition (Bickel et al., 2009). Since \underline{u} may shrink to zero as the dimension increases, this is not a stringent condition. Assumption 5 requires that the singular values of $S_{(i)}$'s are well separated to ensure identifiability. See Wang et al. (2021b) for similar assumptions. The consistency of the SLTR estimator is established as follows.

Theorem 4 (SLTR estimator). Let $d_{\mathcal{S}} = \mathcal{R}_1 \mathcal{R}_2 d + \sum_{i=1}^2 s_i \mathcal{R}_i \log(N\mathcal{R}_i)$. Suppose that $\|\widetilde{\boldsymbol{\omega}} - {\boldsymbol{\omega}}^*\|_2 \le c_{\omega}$ and $T \gtrsim d_{\mathcal{S}} + \underline{u}^{-1} \sum_{i=1}^2 \mathcal{R}_i \log(N\mathcal{R}_i) + \underline{u}^{-2} (s_1 \vee s_2) (\mathcal{R}_1 \vee \mathcal{R}_2) (d + \log N)$. Then under Assumptions 1-5, if $\lambda \gtrsim \sqrt{\kappa_2 \lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon}) d_{\mathcal{S}}/T}$, with probability at least $1 - 5e^{-cd_{\mathcal{S}} - c\underline{u}^{-1}(\mathcal{R}_1 + \mathcal{R}_2 + \log N\mathcal{R}_1 + \log N\mathcal{R}_2)} - c_{\omega} + c_{\omega}$

$$7e^{-cs_2\log N(\mathcal{R}_1\wedge\mathcal{R}_2)} - c\sqrt{(s_2+\underline{u}^{-1})\mathcal{R}_2/T}(1+\sqrt{(s_2+\underline{u}^{-1})\mathcal{R}_2/d_{\mathcal{S}}}), it holds$$

$$\|\widetilde{\mathcal{A}} - \mathcal{A}^*\|_{\mathrm{F}} \lesssim \frac{(\eta_1 + \eta_2)\sqrt{s_1 + s_2}\lambda}{\beta\kappa_1} \quad and \quad \frac{1}{T} \sum_{t=1}^T \|(\widetilde{\mathcal{A}} - \mathcal{A}^*)_{(1)}\widetilde{\boldsymbol{x}}_t\|_2^2 \lesssim \frac{(\eta_1 + \eta_2)^2(s_1 + s_2)\lambda^2}{\beta^2\kappa_1},$$

where $\eta_i = \sum_{j=1}^{R_i} \sigma_1^2(\mathbf{G}_{(i)}^*) / \sigma_j^2(\mathbf{G}_{(i)}^*)$ for i = 1, 2.

Taking $\lambda = \sqrt{\kappa_2 \lambda_{\max}(\Sigma_{\varepsilon}) d_{\mathcal{S}}/T}$, the estimation and prediction error bounds in Theorem 4 become $(\eta_1 + \eta_2) \sqrt{(s_1 + s_2) \kappa_2 \lambda_{\max}(\Sigma_{\varepsilon}) d_{\mathcal{S}}/(\beta^2 \kappa_1^2 T)}$ and $(\eta_1 + \eta_2)(s_1 + s_2) \kappa_2 \lambda_{\max}(\Sigma_{\varepsilon}) d_{\mathcal{S}}/(\beta^2 \kappa_1 T)$, respectively. Then, in view of Lemma 1, the high probability bounds for $\|\widetilde{\mathbf{G}} - \mathbf{G}^*\|_{\mathrm{F}}$ and $\|\widetilde{\boldsymbol{\omega}} - \boldsymbol{\omega}^*\|_{2}$ can be easily obtained.

In practice, the ranks \mathcal{R}_1 , \mathcal{R}_2 and model orders p, r, s are usually small. Then by fixing the constants $\lambda_{\min}(\Sigma_{\varepsilon})$, $\lambda_{\max}(\Sigma_{\varepsilon})$, $\mu_{\min}(\Psi_*)$, $\mu_{\max}(\Psi_*)$, η_1, η_2 and β , the estimation error bound for the rank-constrained estimator $\hat{\mathcal{A}}$ can be simplified to $\sqrt{N/T}$, while that for the SLTR estimator $\tilde{\mathcal{A}}$ reduces to $\sqrt{(s_1 + s_2)^2 \log(N)/T}$.

Remark 2. We give more details about the non-shrinking radius c_{ω} in Lemma 1. The result of Lemma 1 comes from the following first-order Taylor expansion: $\Delta(\omega, \mathfrak{G}) = \mathcal{A}(\omega, \mathfrak{G}) - \mathcal{A}^* = \mathcal{M}(\omega - \omega^*, \mathfrak{G} - \mathfrak{G}^*) \times_3 \mathbf{L}_{\text{stack}}(\omega^*) + remainder$, where $\mathcal{M}: \mathbb{R}^{r+2s} \times \mathbb{R}^{N \times N \times d} \to \mathbb{R}^{N \times N \times (d+r+2s)}$ is a bilinear function, and $\mathbf{L}_{\text{stack}}(\omega^*)$ is a $\infty \times (d+r+2s)$ constant matrix; see the proof of Lemma 1 in the supplementary file. The negligibility of the remainder term requires that ω lies within a constant radius of ω^* . In our proof, we derive the radius $c_{\omega} = \min \left\{2, \frac{c_{\mathfrak{G}}(1-\bar{\rho})\sigma_{\min,L}}{8\sqrt{2}C_L}\right\}$, where $\sigma_{\min,L} := \sigma_{\min}(\mathbf{L}_{\text{stack}}(\omega^*))$, $c_{\mathfrak{G}} := \min_{p+1 \leq k \leq d} \|\mathbf{G}_k^*\|_{\mathcal{F}} / \max_{p+1 \leq k \leq d} \|\mathbf{G}_k^*\|_{\mathcal{F}}$, and $C_L > 0$ is an absolute constant given in Lemma S.1 in the supplementary file. Note that Assumption 2 implies that $\sigma_{\min,L} > 0$ and $c_{\mathfrak{G}} > 0$ are both absolute constants: the former is shown by Lemma S.2 in the supplementary file, and the latter is a direct consequence of Assumption 2(iii). Thus, the radius c_{ω} is non-shrinking.

Remark 3. In the proofs of Theorems 3 and 4, we show that the effect of initial values for $\{y_t, t \leq 0\}$ has no contribution to the final estimation error rates; see the quantities $|S_i(\widehat{\Delta})|$ for $1 \leq i \leq 3$ in the supplementary file. We bound the initialization error terms by Markov's inequality, resulting in a nonexponential tail probability, which may be sharpened by employing more sophisticated concentration inequalities.

5 Selection of response and predictor ranks

As the response and predictor ranks are unknown in practice, we provide a data-driven method to select them and prove the consistency of the estimated ranks.

Denote the true values of the ranks by $(\mathcal{R}_1^*, \mathcal{R}_2^*)$. Suppose that $\widehat{\mathcal{A}}^{\text{init}}$ is a consistent initial estimator of \mathcal{A}^* ; see Remark 4 for a detailed discussion on its choice. Denote by $\widehat{\sigma}_j(i)$ and $\sigma_j^*(i)$ the jth largest singular value of $\widehat{\mathcal{A}}_{(i)}^{\text{init}}$ and $\mathcal{A}_{(i)}^*$, respectively, for i = 1 or 2. We adopt the ridge-type ratio estimator (Xia et al., 2015; Wang et al., 2021b):

$$\widehat{\mathcal{R}}_i = \underset{1 \le j \le N-1}{\arg \min} \frac{\widehat{\sigma}_{j+1}(i) + \tau}{\widehat{\sigma}_j(i) + \tau}, \quad i = 1, 2,$$

where τ is a parameter to be chosen such that Assumption 6 below is satisfied.

Let

$$\zeta_i = \frac{1}{\sigma_{\min}^*(i)} \cdot \max_{1 \le j \le \mathcal{R}_i^* - 1} \frac{\sigma_j^*(i)}{\sigma_{j+1}^*(i)}, \quad \text{for } i = 1, 2,$$

where $\sigma_{\min}^*(i)$ is the minimum singular value of $\mathcal{A}_{(i)}^*$. The following assumption is needed for the consistency of the rank selection method.

Assumption 6 (Signal strength). The parameter $\tau > 0$ is specified such that (i) $\|\widehat{\mathcal{A}}^{init} - \mathcal{A}^*\|_{F}/\tau = o_p(1)$; and (ii) $\tau \max\{\zeta_1, \zeta_2\} = o(1)$.

In Assumption 6, condition (i) requires that the estimation error of $\widehat{\mathcal{A}}^{\text{init}}$ is dominated by τ , and condition (ii) can be regarded as the minimal signal assumption which will simply reduce to $\tau = o(1)$ if σ_j^* for $1 \le j \le \mathcal{R}_i^*$ and i = 1, 2 are bounded above and away from zero by some absolute constant. Following Wang et al. (2021b), it is straightforward to establish the consistency of the estimator.

Theorem 5. Under Assumption 6, $\mathbb{P}(\hat{\mathcal{R}}_1 = \mathcal{R}_1^*, \hat{\mathcal{R}}_2 = \mathcal{R}_2^*) \to 1$ as $T \to \infty$.

Remark 4. We can obtain the initial estimator $\widehat{\mathcal{A}}^{init}$ through a VAR(P) approximation of the $VAR(\infty)$ process, where P scales with the sample size T (Lütkepohl, 2005). Let \mathcal{A}_{trim} be a truncated form of \mathcal{A} such that $(\mathcal{A}_{trim})_{(1)} = (\mathcal{A}_1, \ldots, \mathcal{A}_P)$. We begin by estimating $\widehat{\mathcal{A}}^{init}_{trim}$, and then append infinitely many zero matrices to $\widehat{\mathcal{A}}^{init}_{trim}$ to obtain $\widehat{\mathcal{A}}^{init}$ with $\widehat{\mathcal{A}}^{init}_{(1)} = ((\widehat{\mathcal{A}}^{init}_{trim})_{(1)}, \mathbf{0}_{N \times N}, \mathbf{0}_{N \times N}, \ldots)$. Following Proposition 4.2 in Wilms et al. (2021), under regularity conditions, the approximation error due to the truncation after lag P can be shown to be negligible if $P = T^{1/2-\epsilon}$, where $\epsilon \in$

(0,1/2). Some possible choices for $\hat{\mathbf{A}}^{init}$ are as follows: (a) The nuclear norm regularized estimator $\hat{\mathbf{A}}_{\mathcal{R},\text{trim}} = \arg\min_{\mathbf{A} \in \mathbb{R}^{N \times N \times P}} \sum_{t=P+1}^{T} \|\mathbf{y}_{t} - \sum_{j=1}^{P} \mathbf{A}_{j} \mathbf{y}_{t-j}\|_{2}^{2}/(T-P) + \lambda_{nuc} \sum_{i=1}^{2} \|(\mathbf{A}_{\text{trim}})_{(i)}\|_{*}$, where $\lambda_{nuc} > 0$, and the low-rankness of $(\mathbf{A}_{\text{trim}})_{(i)}$ for i=1,2 is enforced via the nuclear norm penalty; see, e.g., Gandy et al. (2011) and Raskutti et al. (2019); (b) the group-lasso estimator $\hat{\mathbf{A}}_{\mathcal{S},\text{trim}} = \arg\min_{\mathbf{A} \in \mathbb{R}^{N \times N \times P}} \sum_{t=P+1}^{T} \|\mathbf{y}_{t} - \sum_{j=1}^{P} \mathbf{A}_{j} \mathbf{y}_{t-j}\|_{2}^{2}/(T-P) + \lambda_{lag} \sum_{j=1}^{P} \|\mathbf{A}_{j}\|_{\mathrm{F}}$, which corresponds to the lag-sparse estimator in Nicholson et al. (2017); and (c) the spectral estimator in Han et al. (2022) which captures the low-Tucker-rank structure of \mathbf{A} . In practice, we suggest setting $P = T^{1/3}$ and choosing the regularization parameters λ_{nuc} and λ_{lag} chosen by the time series cross-validation method similar to that in Wilms et al. (2021). For the non-sparse case, we employ (a) to obtain the initialization for the rank-constrained estimator. Along the lines of the proofs of Theorem 2 in Wang et al. (2021a), under some regularity conditions, it can be shown that $\|\hat{\mathbf{A}}^{init} - \mathbf{A}^{*}\|_{\mathrm{F}} = O_{p}\{\sqrt{(R_{1}^{*} + R_{2}^{*})NP/(T-P)}\}$. For the sparse case, we recommend (b) for initializing the SLTR estimator, and it can be shown that $\|\hat{\mathbf{A}}^{init} - \mathbf{A}^{*}\|_{\mathrm{F}} = O_{p}\{\sqrt{N^{2}\log P/(T-P)}\}$.

Remark 5. In practice, the model orders (p, r, s) also need to be chosen. Given the Tucker ranks $(\hat{\mathcal{R}}_1, \hat{\mathcal{R}}_2)$ consistently estimated via the VAR(P) approximation approach in Remark 4, we can then select the model orders by minimizing the Bayesian information criterion (BIC), BIC(p, r, s) = $\log\{T^{-1}\sum_{t=1}^{T}\|\boldsymbol{y}_t - \sum_{j=1}^{t-1}\boldsymbol{A}_j(\boldsymbol{\omega}, \boldsymbol{\mathcal{G}})\boldsymbol{y}_{t-j}\|_2^2\} + T^{-1}cd_{\mathcal{M}}\log T$, where (p, r, s) is searched over the range $0 \leq p \leq p_{\text{max}}$, $0 \leq r \leq r_{\text{max}}$, and $0 \leq s \leq s_{\text{max}}$, for some predetermined upper bounds, c > 0 is a constant, and $\boldsymbol{\omega}$ and $\boldsymbol{\mathcal{G}}$ are the estimates obtained by fitting the model with orders (p, r, s) using either the rank-constrained estimator or the SLTR estimator. In addition, $d_{\mathcal{M}} = \mathcal{R}_1 \mathcal{R}_2 d + (\mathcal{R}_1 + \mathcal{R}_2)N$ for the former, and $d_{\mathcal{M}} = \mathcal{R}_1 \mathcal{R}_2 d + \sum_{i=1}^2 \mathcal{R}_i \log(N\mathcal{R}_i)$ for the latter. Then, the consistency of the selected model orders via the BIC can be established along the lines of Zheng (2024).

6 Simulation studies

In this section, we present simulation experiments to examine finite-sample performance of the proposed methods for the SARMA model with non-sparse or sparse factor matrices.

We consider the following two VARMA models as the data generating processes (DGPs),

- DGP1: the VMA(1) model $\boldsymbol{y}_t = \boldsymbol{\varepsilon}_t \boldsymbol{\Theta} \boldsymbol{\varepsilon}_{t-1}$, and
- DGP2: the VARMA(1,1) model $\boldsymbol{y}_t = \boldsymbol{\Phi} \boldsymbol{y}_{t-1} + \boldsymbol{\varepsilon}_t \boldsymbol{\Theta} \boldsymbol{\varepsilon}_{t-1}$,

which correspond to p = 0 and 1, respectively. For both DGPs, $\{\boldsymbol{\varepsilon}_t\}$ are *i.i.d.* $N(\mathbf{0}, \boldsymbol{I}_N)$, and we set $\boldsymbol{\Theta} = \boldsymbol{B}\boldsymbol{J}\boldsymbol{B}^{-1}$, where $\boldsymbol{J} = \operatorname{diag}\{\lambda_1, \dots, \lambda_r, \boldsymbol{C}(\gamma_1, \theta_1), \dots, \boldsymbol{C}(\gamma_s, \theta_s), \boldsymbol{0}\}$ is the real Jordan normal form, with each $\boldsymbol{C}(\gamma, \theta)$ being the 2×2 block defined as

$$C(\gamma, \theta) = \gamma \cdot \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix},$$

and \boldsymbol{B} is generated by a method to be specified below. For DGP2, we set $\boldsymbol{\Phi} = \boldsymbol{B}\boldsymbol{K}\boldsymbol{B}^{-1}$, where $\boldsymbol{K} = \operatorname{diag}\{\delta,0,\ldots,0\}$, with the entry $\delta \neq 0$. It is noteworthy that both DGPs can be written in the form of the SARMA model with orders (p,r,s) and Tucker ranks $\mathcal{R}_1 = \mathcal{R}_2 = r + 2s$. Moreover, to produce non-sparse and sparse factor matrices, we generate \boldsymbol{B} as follows:

- Non-sparse case: $\mathbf{B} \in \mathbb{R}^{N \times N}$ is a randomly generated orthogonal matrix.
- Sparse case: \boldsymbol{B} is obtained by inserting $N-\mathcal{S}$ zero rows into the randomly generated orthogonal matrix $\boldsymbol{B}_{\mathcal{S}} \in \mathbb{R}^{\mathcal{S} \times \mathcal{S}}$ and then concatenating the resulting $N \times \mathcal{S}$ matrix on the right with a $N \times (N-\mathcal{S})$ zero matrix. As a result, $(s_1, s_2) = (\mathcal{S}, \mathcal{S})$.

The non-sparse and sparse cases are fitted by the rank-constrained and SLTR estimators, respectively.

In the first experiment, we aim to verify the estimation error rates of the proposed estimators derived in Theorems 3 and 4. We set (r,s)=(1,0) and $\lambda_1=-0.7$ for both DGPs, $\delta=0.5$ for DGP2, and N=10, 20 or 40. The estimation is conducted via the algorithm in Section S1 or the ADMM Algorithm 2 in the supplementary file given the true ranks and model orders. For the nonsparse case, T is chosen such that $d_R/T \in \{0.05, 0.1, 0.15, 0.2, 0.25\}$. Figure 1 plots the estimation errors averaged over 500 replications against $\sqrt{d_R/T}$. In all settings, it can be observed that there exists a roughly linear relationship between the estimation errors and the theoretical rate, which confirms our theoretical results. For the sparse case, we set S=5 for both DGPs and choose T such that $d_S/T \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$. Figure 2 plots the estimation errors averaged over 500 replications against $\sqrt{d_S/T}$. Similar to the non-sparse case, we observe an approximately linear relationship between the estimation errors and the theoretical rate across all settings, although the estimation error for S might be influenced by algorithmic errors when N is large.

The second experiment examines the performance of the rank selection method in Section 5 and the model order selection criterion in Remark 5. Almost identical settings apply to both

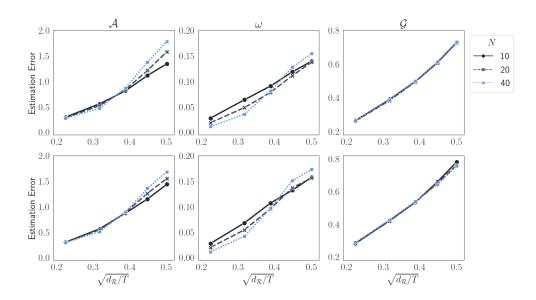


Figure 1: Plots of estimation errors $\|\widehat{\mathcal{A}} - \mathcal{A}^*\|_{\mathrm{F}}$ (left panel), $\|\widehat{\omega} - \omega^*\|_{2}$ (middle panel) and $\|\widehat{\mathcal{G}} - \mathcal{G}^*\|_{\mathrm{F}}$ (right panel) against $\sqrt{d_{\mathcal{R}}/T}$ for the rank-constrained estimator, where $(\mathcal{R}_1, \mathcal{R}_2, p, r, s) = (1, 1, 0, 1, 0)$ (top panel) or $(\mathcal{R}_1, \mathcal{R}_2, p, r, s) = (1, 1, 1, 1, 0)$ (bottom panel), and N = 10 (-*), 20 (-**-) or 40 (-*-).

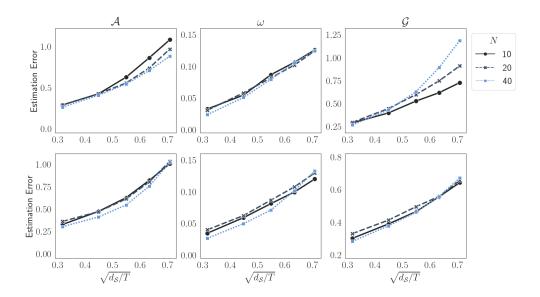


Figure 2: Plots of estimation errors $\|\widetilde{\mathcal{A}} - \mathcal{A}^*\|_{\mathrm{F}}$ (left panel), $\|\widetilde{\omega} - \omega^*\|_{2}$ (middle panel) and $\|\widetilde{\mathcal{G}} - \mathcal{G}^*\|_{\mathrm{F}}$ (right panel) against $\sqrt{d_{\mathcal{S}}/T}$ for the SLTR estimator, where $(\mathcal{R}_1, \mathcal{R}_2, p, r, s) = (1, 1, 0, 1, 0)$ (top panel) or $(\mathcal{R}_1, \mathcal{R}_2, p, r, s) = (1, 1, 1, 1, 0)$ (bottom panel), and N = 10 ($-\!\!\!\!\!-\!\!\!\!\!-\!\!\!\!\!-$), 20 ($-\!\!\!\!\!\!\!-\!\!\!\!\!-\!\!\!\!\!-$) or 40 ($-\!\!\!\!\!\!-\!\!\!\!\!-\!\!\!\!-$).

the non-sparse case and the sparse case. Specifically, we consider three cases under DGP1: $(\mathcal{R}_1, \mathcal{R}_2, r, s) = (1, 1, 1, 0) \pmod{A}$, $(2, 2, 0, 1) \pmod{B}$, and $(3, 3, 1, 1) \pmod{C}$. The results for DGP2 are similar and hence are omitted for brevity. For models B and C, we set

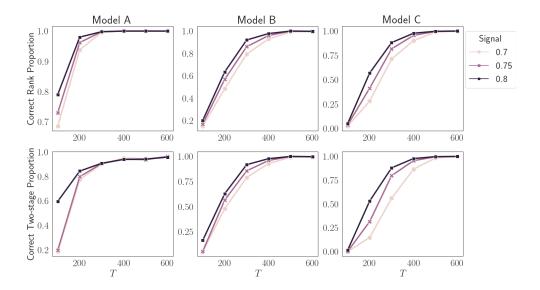


Figure 3: Proportions of correct rank selection (top panel) and two-stage selection (bottom panel) for models A (left panel), B (middle panel) and C (right panel) in the non-sparse case, where the signal strength is 0.7 (---), 0.75 (---) or 0.8 (---).

 $\theta_1 = \pi/4$. Note that $\mathcal{A}_{(1)}$ and $\mathcal{A}_{(2)}$ have the same singular values under DGP1. Moreover, when $r \leqslant 1$ and $s \leqslant 1$, the magnitude of the nonzero singular values are directly determined by $|\lambda_1|$ and γ_1 , which control the signal strength for the rank selection. We consider three levels of signal strength $\{0.7, 0.75, 0.8\}$, and set $-\lambda_1$ in model A, γ_1 in model B, and $-\lambda_1 = \gamma_1$ in model C to these values. In addition, we consider N = 10 and $T \in [100, 600]$. The initial estimator $\hat{\mathcal{A}}^{\text{init}}$ is obtained by the nuclear norm or lag group lasso regularized method in Remark 4 for the non-sparse or sparse case, respectively. For the model order selection, we minimize the BIC in Remark 5 with c = 0.1.

For the non-sparse case, the proportion of correct rank selection, $\{(\hat{\mathcal{R}}_1, \hat{\mathcal{R}}_2) = (\mathcal{R}_1^*, \mathcal{R}_2^*)\}$, and that of correct rank and model order selection, $\{(\hat{\mathcal{R}}_1, \hat{\mathcal{R}}_2, \hat{p}, \hat{r}, \hat{s}) = (\mathcal{R}_1^*, \mathcal{R}_2^*, p^*, r^*, s^*)\}$, based on the two-stage procedure are reported in Figure 3. It can be clearly seen that both proportions increase to one as T and the signal strength increases. For all models, the proportion that the ranks and model orders are correctly selected simultaneously is fairly close to one when $T \geq 400$ across all settings.

For the sparse case, we utilize the same data generation settings, with the only difference being that B is produced using a row sparsity of S = 5. The results are presented in Figure 4. Generally speaking, the patterns are similar to those in Figure 3. However, it is more evident that

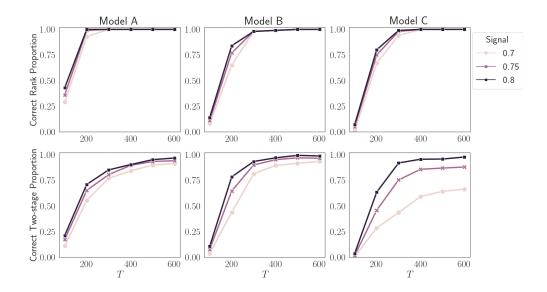


Figure 4: Proportions of correct rank selection (top panel) and two-stage selection (bottom panel) for models A (left panel), B (middle panel) and C (right panel) in the sparse case, where the signal strength is 0.7 (--), 0.75 (+-) or 0.8 (--).

when the signal strength ≤ 0.7 , model C requires a larger T to achieve comparable proportions of simultaneously correct ranks and model orders selection, since the model is more complex. Nevertheless, although not shown in the figure, the accuracy of two-stage selections for model C will continue to increase as T grows.

7 Two empirical examples

7.1 Macroeconomic dataset

This dataset contains observations of 20 quarterly macroeconomic variables from June 1959 to December 2019, with T=243, retrieved from FRED-QD (McCracken and Ng, 2016). These variables come from four categories: (i) stock market, (ii) exchange rates, (iii) money and credit, and (iv) interest rates. These categories are usually considered in the construction of financial condition index, since they reflect important factors that can affect the stance of monetary policy and aggregate demand conditions (Goodhart et al., 2001; Bulut, 2016; Hatzius et al., 2010). All series are transformed to be stationary, and standardized to have zero mean and unit variance; see Table S.1 in the supplementary file for more details of the variables and their transformations.

We first explore the factor structures of this dataset. As discussed in Section 3.2, U_1 and U_2

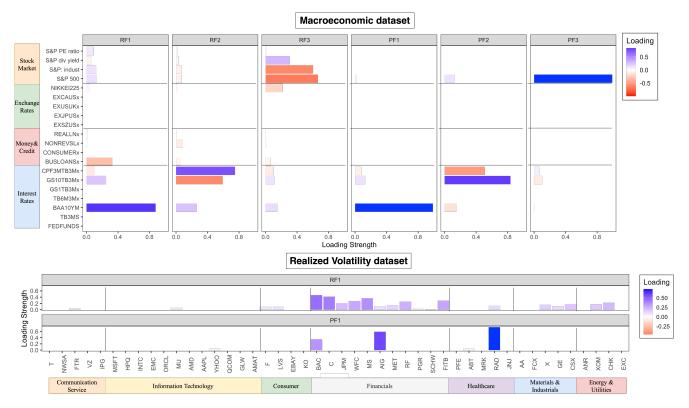


Figure 5: SLTR estimates of factor loadings in the proposed SARMA model for the macroeconomic and realized volatility datasets. Response factors ("RF"s) correspond to columns of U_1 while predictor factors ("PF"s) correspond to columns of U_2 .

capture response factor and predictor factor spaces, respectively. By the rank selection method in Section 5, we obtain $(\hat{\mathcal{R}}_1, \hat{\mathcal{R}}_2) = (3,3)$. Figure 5 displays $\hat{\boldsymbol{U}}_1$ and $\hat{\boldsymbol{U}}_2$ based on the proposed SLTR estimation in Section 3.2, where the regularization parameter is selected by cross-validation. Overall, it can be observed that the response factors (RFs) are mainly influenced by variables in categories (i) and (iv) and business loan indicator from category (iii), while the influence from categories (ii) is relatively weak. On the other hand, only the S&P 500 index from category (i) and category (iv) contributes significantly to the predictor factors (PFs).

We evaluate the performance of our method based on out-of-sample forecast accuracy. The following rolling forecast procedure is adopted: we first fit the models using historical data with the end point rolling from the fourth quarter of 2015 to the third quarter of 2019, and then conduct one-step-ahead forecasts based on the fitted models. In addition to the proposed rank-constrained (RC) and SLTR estimators, we consider five other existing methods, including three based on the VAR model and two based on the VARMA model. Specifically, for the VAR model, we consider (a) the Lasso method (Basu and Michailidis, 2015) and two methods in Wang et al. (2021b): (b) the multilinear low-rank (MLR) method and (c) the sparse higher-order reduced-rank (SHORR)

Table 1: Forecast errors for macroeconomic and realized volatility datasets. The smallest numbers in each row are marked in bold.

		VAR			VARMA		SARMA	
		(a) Lasso	(b) MLR	(c) SHORR	$\overline{(d) \ell_1}$	(e) HLag	RC	SLTR
Macroeconomic	MSFE	2.78	2.77	2.71	2.80	2.79	2.67	2.62
	MAFE	9.26	9.27	8.99	9.28	9.24	8.75	8.45
Realized Volatility	MSFE	5.17	4.93	4.87	5.19	5.19	4.78	4.74
	MAFE	21.58	19.02	18.22	21.70	21.70	16.45	16.99

method, which further imposes sparsity on the factor matrices in (b) using a slightly different regularizer than the method in this paper. For the VARMA model, we apply the method in Wilms et al. (2021) with (d) the ℓ_1 -penalty or (e) the HLag penalty. Note that (a) is used as the Phase-I estimator for the estimators in (d) and (e), and the AR order is selected according to Wilms et al. (2021). The AR order for (b) and (c) is chosen as in Wang et al. (2021b). For the proposed low-Tucker-rank SARMA model, the estimated model orders are $(\hat{p}, \hat{r}, \hat{s}) = (0, 1, 0)$. Throughout the rolling forecast procedure, the same model orders and ranks are used.

Table 1 reports the mean squared forecast error (MSFE) and mean absolute forecast error (MAFE) for all methods. It can be observed that the proposed methods achieve the smallest forecast errors among all competing ones. Compared to sparse but non-low-rank models, i.e., (a), (d) and (e), the proposed model can better capture the factor structure which is prominent in this dataset. Meanwhile, its higher flexibility than the VAR model is supported by its better forecasting performance than (b) and (c). In addition, note that imposing sparsity on the factor matrices generally results in smaller forecast errors for both VAR and SARMA models; see Figure 5.

7.2 Realized volatility

As another example, we study daily realized volatilities for 46 stocks from January 2, 2012 to December 31, 2013, with T=495. These are the stocks of top S&P 500 companies ranked by trading volumes on the first day of 2013. Specifically, we obtain the tick-by-tick data from WRDS (https://wrds-www.wharton.upenn.edu) and compute the daily realized volatility from five-minute returns (Andersen et al., 2006). By examining the sample autocorrelation functions, we have confirmed the stationarity of all series. Each series is then standardized to have zero mean and unit variance. More information about the stocks is given in Table S.2 in the supplementary

file. We conduct the same rolling forecast procedure as in Section 7.1, where the last 10% of the sample is used as the forecast period. As shown in Table 1, the proposed methods considerably outperform the other ones in terms of forecast accuracy.

The estimated ranks and model orders are $(\hat{\mathcal{R}}_1, \hat{\mathcal{R}}_2, \hat{p}, \hat{r}, \hat{s}) = (1, 1, 0, 1, 0)$. As a result, the fitted model has the following factor structure: $\hat{u}'_1 y_t = 0.336 \sum_{j=1}^{\infty} 0.872^j \hat{u}'_2 y_{t-j} + e_t$, where the loadings \hat{u}_1 and \hat{u}_2 are displayed in Figure 5. We have several interesting findings. First, $\hat{\lambda} = 0.872$ indicates that the influence of the past on the present decays quite slowly. This lends support to the well-established fact that the volatility of asset returns is highly persistent, that is, the AR process of the volatility is nearly unit-root; see, e.g., Andersen et al. (2003). Second, it can be observed that the weights in \hat{u}_1 are more evenly spread out across four sectors, including Financials, Healthcare, Material & Industrials, and Energy & Utilities. However, the weights in \hat{u}_2 are more concentrated on a few stocks. As discussed in Section 3.3, $\hat{u}'_1 y_t$ and $\hat{u}'_2 y_t$ can be regarded as two different market volatility indices, with the loadings \hat{u}_1 and \hat{u}_2 capturing how the market responds to and picks up risks across stocks, respectively. Lastly, the estimated slope 0.336 signifies the overall association between y_t and its lags, after summarizing the information across all stocks into market indices, while taking into account the decaying temporal dependence over lags. It shows that the present and past volatilities are positively correlated, a phenomenon commonly known as the volatility clustering in the literature of financial time series (Tsay, 2010).

8 Conclusion and discussion

This paper contributes to the underdeveloped literature on high-dimensional VARMA models. First, the originally unwieldy VARMA form is turned into a much more tractable infinite-order VAR form. Second, building on the close connection between this form and the tensor decomposition for the AR coefficient tensor \mathcal{A} , a low-Tucker-rank structure is naturally considered, so that dimension reduction can be simultaneously performed across all time lags and variables. In summary, by combining the reparameterization and tensor decomposition techniques, this paper expands the available model family for high-dimensional time series from finite-order VAR to VARMA processes.

Moreover, a comprehensive high-dimensional estimation procedure is developed, together with theoretical properties and efficient algorithms that leverage the tractable form of the model. To the best of our knowledge, this is the first work addressing high-dimensional low-rank VARMA modelling in the literature. However, there are still many worthwhile questions that remain to be explored. Firstly, the convergence theory developed for the estimators in this paper focuses on the statistical error, whereas the optimization error of the algorithm is not studied. For the nonconvex estimation of low-rank tensor models, Han et al. (2022) establishes both the statistical error bound and the linear rate of computational convergence of their proposed algorithm. For our model, the main difficulty in conducting such an algorithmic analysis lies in the nonconvexity of the coefficient tensor \mathcal{A} with respect to ω . Second, it is important to develop high-dimensional statistical inference procedures for the proposed model. So far there have been limited studies on inference for low-rank tensor regression models. A recent work is Xia et al. (2022) which, however, focuses on i.i.d. low-Tucker-rank models with non-sparse factor matrices. Extensions of such asymptotic distributional results to the time series setting can be challenging. Moreover, when the factor matrices are sparse, the corresponding inference will be even more difficult, and debiasing techniques are likely inevitable. We leave these interesting problems to future research.

9 Supplementary material

The Supplementary Material contains algorithms for the proposed estimators, all technical details, and additional results for the simulation and empirical studies in this paper.

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Online Supplement for "SARMA: Scalable Low-Rank High-Dimensional Autoregressive Moving Averages via Tensor Decomposition"

Abstract

This supplementary file contains four sections. Sections S1 presents algorithms for the proposed rank-constrained and SLTR estimators. Section S2 provides descriptions of datasets for the empirical examples in the main paper. All technical proofs for the theoretical results in Sections 3 and 4 in main paper are provided in Sections S3 and S4, respectively.

S1 Algorithms

S1.1 Algorithm for the rank-constrained estimator

We first consider the algorithm for the rank-constrained estimator. By the factorization $\mathcal{G} = \mathbf{S} \times_1 \mathbf{U}_1 \times_2 \mathbf{U}_2$, the rank-constrained estimation in (4.2) can be rewritten as the unconstrained problem,

$$(\hat{\boldsymbol{\omega}}, \hat{\boldsymbol{S}}, \hat{\boldsymbol{U}}_1, \hat{\boldsymbol{U}}_2) = \arg\min \widetilde{\mathbb{L}}_T(\boldsymbol{\omega}, \boldsymbol{S}, \boldsymbol{U}_1, \boldsymbol{U}_2).$$
 (S1)

Then we have $\hat{\mathbf{G}} = \hat{\mathbf{S}} \times_1 \hat{\mathbf{U}}_1 \times_2 \hat{\mathbf{U}}_2$. Note that \mathbf{U}_i 's need not be subject to any orthogonality constraint in this minimization.

To implement (S1), we adopt an alternating minimization algorithm; see Algorithm 1. Note that the optimization for λ_k 's and (γ_k, θ_k) 's in lines 3–6 is efficient due to the following property:

$$\mathcal{A}_{(1)}\widetilde{\boldsymbol{x}}_t - \sum_{k=1}^p \boldsymbol{G}_k \boldsymbol{y}_{t-k} = \sum_{k=1}^r f^I(\widetilde{\boldsymbol{x}}_t; \lambda_k) + \sum_{k=1}^s f^{II}(\widetilde{\boldsymbol{x}}_t; \gamma_k, \theta_k),$$

where $f^I(\tilde{\boldsymbol{x}}_t; \lambda_k) = \sum_{j=1}^{t-p-1} \lambda_k^j \boldsymbol{G}_{p+k} \boldsymbol{y}_{t-p-j}$, and $f^{II}(\tilde{\boldsymbol{x}}_t; \gamma_k, \theta_k) = \sum_{j=1}^{t-p-1} \gamma_k^j [\cos(j\theta_k) \boldsymbol{G}_{p+r+2k-1} + \sin(j\theta_k) \boldsymbol{G}_{p+r+2k}] \boldsymbol{y}_{t-p-j}$. Note that each λ_k or (γ_k, θ_k) appears in only one summand. Thus, fixing all other parameters, the optimization problem for each λ_k or (γ_k, θ_k) will be only one- or two-dimensional, where the irrelevant summands will be treated as the intercept and absorbed into

Algorithm 1: Alternating minimization algorithm

```
1 Input: ranks (\mathcal{R}_1, \mathcal{R}_2), model orders (p, r, s), initialization \boldsymbol{\omega}^{(0)}, \boldsymbol{U}_1^{(0)}, \boldsymbol{U}_2^{(0)}, \boldsymbol{S}^{(0)} and \boldsymbol{\mathcal{G}}^{(0)}.
   2 repeat i = 0, 1, 2, \dots
                  for k = 1, ..., r:
   3
                      \lambda_k^{(i+1)} \leftarrow \underset{\lambda \in (-1,1)}{\operatorname{arg \, min}} \, \widetilde{\mathbb{L}}_T(\lambda_1^{(i+1)}, \dots, \lambda_{k-1}^{(i+1)}, \lambda, \lambda_{k+1}^{(i)}, \dots, \theta_s^{(i)}, \mathbf{G}^{(i)})
  4
                  for k = 1, ..., s:
  \mathbf{5}
                       (\gamma_k^{(i+1)}, \theta_k^{(i+1)}) \leftarrow \underset{\gamma \in (0,1), \theta \in (0,\pi)}{\arg \min} \widetilde{\mathbb{L}}_T(\lambda_1^{(i+1)}, \dots, \theta_{k-1}^{(i+1)}, \gamma, \theta, \gamma_{k+1}^{(i)}, \dots, \theta_s^{(i)}, \mathbf{S}^{(i)})
  6
                m{U}_{1}^{(i+1)} \leftarrow \operatorname*{arg\,min}_{m{U}_{1}} \sum_{t=1}^{T} \| m{y}_{t} - [\{ m{z}_{t}'(m{\omega}^{(i+1)}) (m{I}_{d} \otimes m{U}_{2}^{(i)}) m{S}_{(1)}^{(i)'} \} \otimes m{I}_{N}] \operatorname{vec}(m{U}_{1}) \|_{2}^{2}
  7
                m{U}_{2}^{(i+1)} \leftarrow \operatorname*{arg\,min}_{m{I}_{2}} \sum_{t=1}^{T} \| m{y}_{t} - m{U}_{1}^{(i+1)} m{S}_{(1)}^{(i)} \{ m{Z}_{t}'(m{\omega}^{(i+1)}) \otimes m{I}_{R_{2}} \} \operatorname{vec}(m{U}_{2}') \|_{2}^{2}
  8
                \mathbf{S}^{(i+1)} \leftarrow \underset{\mathbf{S}}{\arg\min} \sum_{t=1}^{T} \| \mathbf{y}_{t} - [\{ \mathbf{z}_{t}'(\boldsymbol{\omega}^{(i+1)}) (\mathbf{I}_{d} \otimes \mathbf{U}_{2}^{(i+1)}) \} \otimes \mathbf{U}_{1}^{(i+1)}] \operatorname{vec}(\mathbf{S}_{(1)}) \|_{2}^{2}\mathbf{S}^{(i+1)} = \mathbf{S}^{(i+1)} \times_{1} \mathbf{U}_{1}^{(i+1)} \times_{2} \mathbf{U}_{2}^{(i+1)}.
11 until convergence
```

the response. These problems can be solved efficiently by the Newton-Raphson method or even in parallel.

In lines 7–9 of Algorithm 1, the updates for U_1, U_2 and S are simple linear least squares problems. To see this, we can write

$$\mathcal{A}_{(1)}\widetilde{\boldsymbol{x}}_t = \left[\{ \boldsymbol{z}_t'(\boldsymbol{\omega}) (\boldsymbol{I}_d \otimes \boldsymbol{U}_2) \boldsymbol{S}_{(1)}' \} \otimes \boldsymbol{I}_N \right] \operatorname{vec}(\boldsymbol{U}_1) = \boldsymbol{U}_1 \boldsymbol{S}_{(1)} \{ \boldsymbol{Z}_t'(\boldsymbol{\omega}) \otimes \boldsymbol{I}_{R_2} \} \operatorname{vec}(\boldsymbol{U}_2')$$
$$= \left[\{ \boldsymbol{z}_t'(\boldsymbol{\omega}) (\boldsymbol{I}_d \otimes \boldsymbol{U}_2) \} \otimes \boldsymbol{U}_1 \right] \operatorname{vec}(\boldsymbol{S}_{(1)}),$$

where $z_t(\boldsymbol{\omega}) = (z'_{t,1}(\boldsymbol{\omega}), \dots, z'_{t,d}(\boldsymbol{\omega}))' = \{\boldsymbol{L}'(\boldsymbol{\omega}) \otimes \boldsymbol{I}_N\} \tilde{\boldsymbol{x}}_t$, with $z_{t,k}(\boldsymbol{\omega}) = \sum_{j=1}^{t-1} \ell_{j,k}(\boldsymbol{\omega}) \boldsymbol{y}_{t-j}$, and $\boldsymbol{Z}_t(\boldsymbol{\omega}) = (z_{t,1}(\boldsymbol{\omega}), \dots, z_{t,d}(\boldsymbol{\omega})) \in \mathbb{R}^{N \times d}$. Alternatively, when N is large and the computation of closed-form solutions is time-consuming, the gradient descent method can be used to further speed up the algorithm. In addition, Algorithm 1 can be applied to the basic SARMA model without any low-Tucker-rank constraint on \boldsymbol{G} ; see the supplementary file for a simulation study, which demonstrates its computational advantage over the VARMA model. In this case, lines 7, 8 and 10 will be dropped, and line 9 will be the update of $\boldsymbol{G}^{(i+1)}$, where both \boldsymbol{U}_1 and \boldsymbol{U}_2 are set to the $N \times N$ identity matrix.

Remark 6. We initialize Algorithm 1 in practice as follows. First, we apply the data-driven pro-

cedure in Section ?? to select the ranks and model orders. Then, given the selected $(\mathcal{R}_1, \mathcal{R}_2, p, r, s)$, we initialize the parameters $\boldsymbol{\omega}$, \boldsymbol{U}_1 , \boldsymbol{U}_2 , \boldsymbol{S} , and \boldsymbol{S} by the method described in Section S1.3, which exhibits reliable numerical performance in our simulations.

S1.2 Algorithm for the SLTR estimator

For the SLTR estimator, we adopt an alternating direction methods of multipliers (ADMM) algorithm (Boyd et al., 2011), where lines 7–9 in Algorithm 1 are revised to incorporate the ℓ_1 -penalties and orthogonality constraints. A similar approach is employed in Wang et al. (2021b).

Developing an efficient algorithm for the SLTR estimator involves two major challenges. The first challenge arises from the row-orthogonal constraint imposed on the mode-1 and mode-2 unfolding of the core tensor S, i.e. $S_{(1)}$ and $S_{(2)}$ in Assumption 4. This constraint cannot be handled in a straightforward manner. The second challenge is related to the joint imposition of l_1 -regularization and orthogonality constraints on U_i 's, as specified by the same assumption. The l_1 -regularization introduces non-smoothness to the algorithm, while the orthogonality constraints increase its nonconvexity. To address these challenges, we employ the ADMM algorithm which updates the variables U_i 's and S alternately. For a detailed step-by-step procedure, refer to Algorithm 2.

Firstly, to address the row-orthogonal constraint of $\mathbf{S}_{(j)}$ (where j=1 or 2), we decompose it using the equation $\mathbf{S}_{(j)} = \mathbf{D}_j \mathbf{V}_j'$. Here, $\mathbf{D}_j \in \mathbb{R}^{\mathcal{R}_j \times \mathcal{R}_j}$ represents a diagonal matrix, while $\mathbf{V}_1 \in \mathbb{R}^{\mathcal{R}_2 \mathcal{R}_3 \times \mathcal{R}_1}$ and $\mathbf{V}_2 \in \mathbb{R}^{\mathcal{R}_1 \mathcal{R}_3 \times \mathcal{R}_2}$ are orthogonormal matrices. These matrices satisfy the condition $\mathbf{V}_j' \mathbf{V}_j = \mathbf{I}_{\mathcal{R}_j}$ for j=1,2. By introducing these decompositions, we can then express the augmented Lagrangian corresponding to the objective functions given in (4.3) as follows:

Algorithm 2: ADMM algorithm for the SLTR estimator

1 Input: ranks $(\mathcal{R}_1, \mathcal{R}_2)$, model orders (p, r, s), initialization $\boldsymbol{\omega}^{(0)}, \boldsymbol{U}_1^{(0)}, \boldsymbol{U}_2^{(0)}, \boldsymbol{S}^{(0)}$ and $\boldsymbol{\mathcal{G}}^{(0)}$, hyperparameters $(\lambda, \varrho_1, \varrho_2)$.

2 repeat
$$i = 0, 1, 2, \dots$$

3 for
$$k = 1, ..., r$$
:

$$\mathbf{4} \qquad \lambda_k^{(i+1)} \leftarrow \underset{\lambda \in (-1,1)}{\arg\min} \ \widetilde{\mathbb{L}}_T(\lambda_1^{(i+1)}, \dots, \lambda_{k-1}^{(i+1)}, \lambda, \lambda_{k+1}^{(i)}, \dots, \lambda_r^{(i)}, \boldsymbol{\eta}^{(i)}, \boldsymbol{\mathcal{G}}^{(i)})$$

for
$$k = 1, ..., s$$
:

$$\mathbf{6} \qquad \boldsymbol{\eta}_k^{(i+1)} \leftarrow \underset{\boldsymbol{\eta} \in [0,1) \times (0,\pi)}{\arg\min} \widetilde{\mathbb{L}}_T(\boldsymbol{\lambda}^{(i+1)}, \boldsymbol{\eta}_1^{(i+1)}, \dots, \boldsymbol{\eta}_{k-1}^{(i+1)}, \boldsymbol{\eta}, \boldsymbol{\eta}_{k+1}^{(i)}, \dots, \boldsymbol{\eta}_s^{(i)}, \boldsymbol{\mathcal{G}}^{(i)})$$

$$\mathbf{7} \qquad \boldsymbol{U}_{1}^{(i+1)} \leftarrow \mathop{\arg\min}_{\boldsymbol{U}_{1}'\boldsymbol{U}_{1} = \boldsymbol{I}_{\mathcal{R}_{1}}} \sum_{t=1}^{T} \|\boldsymbol{y}_{t} - \left[\{\boldsymbol{z}_{t}'(\boldsymbol{\omega}^{(i+1)})(\boldsymbol{I}_{d} \otimes \boldsymbol{U}_{2}^{(i)})\boldsymbol{S}_{(1)}^{(i)'}\} \otimes \boldsymbol{I}_{N} \right] \operatorname{vec}(\boldsymbol{U}_{1})\|_{2}^{2} + \lambda \|\boldsymbol{U}_{1}\|_{1}$$

$$\mathbf{8} \qquad \boldsymbol{U}_{2}^{(i+1)} \leftarrow \mathop{\arg\min}_{\boldsymbol{U}_{2}'\boldsymbol{U}_{2} = \boldsymbol{I}_{R_{2}}} \sum_{t=1}^{T} \|\boldsymbol{y}_{t} - \boldsymbol{U}_{1}^{(i+1)}\boldsymbol{S}_{(1)}^{(i)} \{\boldsymbol{Z}_{t}'(\boldsymbol{\omega}^{(i+1)}) \otimes \boldsymbol{I}_{R_{2}}\} \operatorname{vec}(\boldsymbol{U}_{2}')\|_{2}^{2} + \lambda \|\boldsymbol{U}_{2}\|_{1}$$

9
$$\mathbf{S}^{(i+1)} \leftarrow \underset{\mathbf{S}}{\operatorname{arg\,min}} \sum_{t=1}^{T} \| \mathbf{y}_t - [\{ \mathbf{z}_t'(\boldsymbol{\omega}^{(i+1)}) (\mathbf{I}_d \otimes \mathbf{U}_2^{(i+1)}) \} \otimes \mathbf{U}_1^{(i+1)}] \operatorname{vec}(\mathbf{S}_{(1)}) \|_2^2$$

$$+ \sum_{j=1}^{2} \varrho_{j} \|\mathbf{S}_{(j)} - \mathbf{D}_{j}^{(i)} \mathbf{V}_{j}^{(i)\prime} + (\mathbf{C}_{j}^{(i)})_{(j)}\|_{\mathrm{F}}^{2}$$

11 for
$$j \in \{1, 2\}$$
 do

12
$$\boldsymbol{D}_{j}^{(i+1)} \leftarrow \operatorname{arg\,min}_{\boldsymbol{D}_{i} = \operatorname{diag}(\boldsymbol{d}_{i})} \|\boldsymbol{S}_{(j)}^{(i+1)} - \boldsymbol{D}_{j} \boldsymbol{V}_{j}^{(i)\prime} + (\boldsymbol{\mathcal{C}}_{j}^{(i)})_{(j)}\|_{F}^{2}$$

13
$$V_j^{(i+1)} \leftarrow \arg\min_{V_j'V_j = I_{\mathcal{R}_j}} \|\mathbf{S}_{(j)}^{(i+1)} - \mathbf{D}_j^{(i+1)}V_j' + (\mathbf{C}_j^{(i)})_{(j)}\|_{\mathcal{F}}^2$$

14
$$(\mathcal{C}_{j}^{(i+1)})_{(j)} \leftarrow (\mathcal{C}_{j}^{(i)})_{(j)} + \mathcal{S}_{(j)}^{(i+1)} - D_{j}^{(i+1)} V_{j}^{(i+1)\prime}$$

15
$$\mathbf{S}^{(i+1)} = \mathbf{S}^{(i+1)} \times_1 \mathbf{U}_1^{(i+1)} \times_2 \mathbf{U}_2^{(i+1)}$$
.

16 until convergence

$$\mathcal{L}_{\varrho}(\mathbf{S}, \{\boldsymbol{U}_{i}\}, \boldsymbol{\omega}, \{\boldsymbol{D}_{j}\}, \{\boldsymbol{V}_{j}\}; \{\boldsymbol{\mathfrak{C}}_{j}\}) = \widetilde{\mathbb{L}}_{T}(\boldsymbol{\omega}, \mathbf{S}, \boldsymbol{U}_{1}, \boldsymbol{U}_{2}) + \lambda \sum_{i=1}^{2} \|\boldsymbol{U}_{i}\|_{1}$$

$$+ 2 \sum_{j=1}^{2} \varrho_{j} \langle (\boldsymbol{\mathfrak{C}}_{j})_{(j)}, \boldsymbol{S}_{(j)} - \boldsymbol{D}_{j} \boldsymbol{V}_{j}' \rangle + \sum_{j=1}^{2} \varrho_{j} \|\boldsymbol{S}_{(j)} - \boldsymbol{D}_{j} \boldsymbol{V}_{j}'\|_{F}^{2}$$

where $C_j \in \mathbb{R}^{\mathcal{R}_1 \times \mathcal{R}_2 \times \mathcal{R}_3}$ are the tensor-valued dual variables, and $\varrho = (\varrho_1, \varrho_2)'$ is the set of regularization parameters, which in practice can be selected together with λ by a fine grid search with information criterion such as the BIC or its high-dimensional extensions, where the total number of nonzero parameters could be used as proxies for the degree of freedom. This brings

Algorithm 3: ADMM subroutine for sparse and orthogonal regression

- 1 Initialize: $B^{(0)} = W^{(0)}, M^{(0)} = 0$
- **2** repeat k = 0, 1, 2, ...
- $B^{(k+1)} \leftarrow \arg\min_{B'B=I} \{ \|y X \operatorname{vec}(B)\|_2^2 + \kappa \|B W^{(k)} + M^{(k)}\|_F^2 \}$ 3
- $\mathbf{W}^{(k+1)} \leftarrow \arg\min_{\mathbf{W}} \{ \kappa \| \mathbf{B}^{(k+1)} \mathbf{W} + \mathbf{M}^{(k)} \|_{\mathrm{F}}^{2} + \lambda \| \mathbf{W} \|_{1} \}$ $\mathbf{M}^{(k+1)} \leftarrow \mathbf{M}^{(k)} + \mathbf{B}^{(k+1)} \mathbf{W}^{(k+1)}$ 4
- 6 until convergence

us to Algorithm 2. It is important to note that the row-orthogonal constraint originally imposed on $S_{(1)}$ and $S_{(2)}$ has been effectively transferred to the matrices V_j 's in line 13. As a result, no explicit constraint is required for updating S in lines 9-10 of Algorithm 2.

Secondly, we discuss the update process for U_i . In (4.1), $\widetilde{\mathbb{L}}_T(\boldsymbol{\omega}, \boldsymbol{S}, \boldsymbol{U}_1, \boldsymbol{U}_2)$ represents a least squares loss function with respect to each U_i . Therefore, in lines 7-8 of Algorithm 2, the update steps for U_i involve solving ℓ_1 -regularized least squares problems under an orthogonality constraint. This can be expressed in a general form as follows:

$$\min_{\mathbf{B}} \{ \| \mathbf{y} - \mathbf{X} \operatorname{vec}(\mathbf{B}) \|_{2}^{2} + \lambda \| \mathbf{B} \|_{1} \}, \quad \text{s.t. } \mathbf{B}' \mathbf{B} = \mathbf{I}.$$
 (S2)

Since handling the ℓ_1 -regularization and the orthogonality constraint for **B** together is challenging, we employ an ADMM subroutine to separate them into two steps. To achieve this, we introduce a dummy variable W as a surrogate for B and rewrite the problem (S2) equivalently as:

$$\min_{\boldsymbol{B}, \boldsymbol{W}} \{ \| \boldsymbol{y} - \boldsymbol{X} \operatorname{vec}(\boldsymbol{B}) \|_{2}^{2} + \lambda \| \boldsymbol{W} \|_{1} \}, \text{ s.t. } \boldsymbol{B}' \boldsymbol{B} = \boldsymbol{I} \text{ and } \boldsymbol{B} = \boldsymbol{W}.$$

Then, the corresponding Lagrangian formulation is:

$$\min_{\boldsymbol{B}, \boldsymbol{W}} \{ \|\boldsymbol{y} - \boldsymbol{X} \operatorname{vec}(\boldsymbol{B})\|_{2}^{2} + \lambda \|\boldsymbol{W}\|_{1} + 2\kappa \langle \boldsymbol{M}, \boldsymbol{B} - \boldsymbol{W} \rangle + \kappa \|\boldsymbol{B} - \boldsymbol{W}\|_{F}^{2} \},$$
(S3)

where M represents the dual variable and κ is the regularization parameter. Algorithm 3 in Wang et al. (2021a) presents the ADMM subroutine for solving (S3), which provides solutions for the U_i -update subproblems in Algorithm 2. We also include it as Algorithm 3 here for sake of completeness.

Note that both the **B**-update step in Algorithm 3 and the V_i -update step in line 13 of Algorithm

2 involve solving least squares problems subject to an orthogonality constraint. These problems can be efficiently solved using the splitting orthogonality constraint (SOC) method (Lai and Osher, 2014). On the other hand, the W-update step in Algorithm 3 corresponds to an ℓ_1 -regularized minimization, which can be effectively addressed through explicit soft-thresholding. As for the S-and D_i -update steps in lines 9 and 12 of Algorithm 2, they simply entail solving straightforward least squares problems.

S1.3 Initialization for the algorithms

The proposed algorithms require suitable initial values for the parameters ω , U_1 , U_2 , S, and G. In this section, we provide an easy-to-implement method for initializing these values.

Given the initial value $\mathcal{A}^{(0)} = \widehat{\mathcal{A}}^{\text{init}}$ and pre-selected $(\mathcal{R}_1, \mathcal{R}_2, p, r, s)$, we are ready to initialize $\omega, U_1, U_2, \mathcal{S}$ and \mathcal{G} for the proposed algorithms as follows:

- Conduct the HOSVD of $\mathcal{A}^{(0)}$ for the first two modes, $\mathcal{A}^{(0)} = \mathcal{H}^{(0)} \times_1 U_1^{(0)} \times_2 U_2^{(0)}$, to obtain $\mathcal{H}^{(0)} \in \mathbb{R}^{\mathcal{R}_1 \times \mathcal{R}_2 \times \infty}$ and $U_i^{(0)} \in \mathbb{R}^{N \times \mathcal{R}_i}$ for i = 1, 2.
- Next we aim to obtain $S^{(0)}$ and $\omega^{(0)}$ such that $S^{(0)} \times_3 L(\omega^{(0)}) \approx \mathcal{H}^{(0)}$:
 - (i) To determine $\omega^{(0)}$, note that ω lies in the bounded parameter space Ω defined in (3.2). Thus, we consider a grid of initial values for each element of ω within the parameter space; e.g., $\lambda_k \in \{-0.75, -0.5, -0.25, 0.25, 0.5, 0.75\}$, $\gamma_h \in \{0.25, 0.5, 0.75\}$, and $\theta_h \in \{\pi/4, 4\pi/3\}$, for any $1 \le k \le r$ and $1 \le h \le s$. To ensure identifiability, we only consider the combinations with distinct λ_k 's and (γ_h, θ_h) 's.
 - (ii) For each choice of $\boldsymbol{\omega}^{(0)}$, we get the corresponding $\boldsymbol{\mathcal{S}}^{(0)} = \boldsymbol{\mathcal{H}}^{(0)} \times_3 \boldsymbol{L}^{\dagger}(\boldsymbol{\omega}^{(0)})$, where $\boldsymbol{L}^{\dagger}(\boldsymbol{\omega}^{(0)})$ is the left pseudo-inverse of $\boldsymbol{L}(\boldsymbol{\omega}^{(0)})$.
- Then, let $\mathbf{S}^{(0)} = \mathbf{S}^{(0)} \times_1 \mathbf{U}_1^{(0)} \times_2 \mathbf{U}_2^{(0)}$.
- Finally, among all choices of initial values, we select the one leading to the smallest value for the loss function.

S2 Descriptions of datasets

We provide more detailed descriptions of the variables and their transformations for the two datasets in Section 7 of the main paper through two tables. Table S.3 is for the quarterly macroeconomic dataset, and Table S.2 is for the daily realized volatilities dataset.

Table S.2: Forty six selected S&P 500 stocks. CODE: stock code in the New York Stock Exchange. NAME: name of company. G: group code, where 1 = communication service, 2 = information technology, 3 = consumer, 4 = financials, 5 = healthcare, 6 = materials and industrials, and 7 = energy and utilities.

CODE	NAME	G	CODE	NAME	G
$\overline{\mathrm{T}}$	AT&T Inc.		JPM	JPMorgan Chase & Co.	4
NWSA	News Corp	1	WFC	Wells Fargo & Company	4
FTR	Frontier Communications Parent Inc	1	MS	Morgan Stanley	4
VZ	Verizon Communications Inc.	1	AIG	American International Group Inc.	4
IPG	Interpublic Group of Companies Inc	1	MET	MetLife Inc.	4
MSFT	Microsoft Corporation	2	RF	Regions Financial Corp	4
HPQ	HP Inc	2	PGR	Progressive Corporation	4
INTC	Intel Corporation	2	SCHW	Charles Schwab Corporation	4
EMC	EMC Instytut Medyczny SA	2	FITB	Fifth Third Bancorp	4
ORCL	Oracle Corporation	2	PFE	Pfizer Inc.	5
MU	Micron Technology Inc.	2	ABT	Abbott Laboratories	5
AMD	Advanced Micro Devices Inc.	2	MRK	Merck & Co. Inc.	5
AAPL	Apple Inc.	2	RAD	Rite Aid Corporation	5
YHOO	Yahoo! Inc.	2	JNJ	Johnson & Johnson	5
QCOM	Qualcomm Inc	2	AA	Alcoa Corp	6
GLW	Corning Incorporated	2	FCX	Freeport-McMoRan Inc.	6
AMAT	Applied Materials Inc.	2	X	United States Steel Corporation	6
F	Ford Motor Company	3	GE	General Electric Company	6
LVS	Las Vegas Sands Corp.	3	CSX	CSX Corporation	6
EBAY	eBay Inc.	3	ANR	Alpha Natural Resources	7
KO	Coca-Cola Company	3	XOM	Exxon Mobil Corporation	7
BAC	Bank of America Corp	4	CHK	Chesapeake Energy	7
C	Citigroup Inc.	4	EXC	Exelon Corporation	7

Table S.3: Twenty quarterly macroeconomic variables. FRED MNEMONIC: mnemonic for data in FRED-QD. SW MNEMONIC: mnemonic in Stock and Watson (2012). T: data transformation, where 1 = no transformation, 2 = first difference, and 3 = first difference of log series. DESCRIPTION: brief definition of the data. G: Group code, where 1 = interest rate, 2 = money and credit, 3 = exchange rate, and 4 = stock market.

FRED MNEMONIC	SW MNEMONIC	Т	DESCRIPTION	G
FEDFUNDS	FedFunds	2	Effective Federal Funds Rate (Percent)	1
TB3MS	TB-3Mth	2	3-Month Treasury Bill: Secondary Market Rate (Percent)	1
BAA10YM	BAA_GS10	1	Moody's Seasoned Baa Corporate Bond Yield Relative to Yield on 10-Year Treasury Constant Maturity (Percent)	1
TB6M3Mx	$tb6m_tb3m$	1	6-Month Treasury Bill Minus 3-Month Treasury Bill, secondary market (Percent)	1
GS1TB3Mx	$GS1_tb3m$	1	1-Year Treasury Constant Maturity Minus 3-Month Treasury Bill, secondary market (Percent)	1
GS10TB3Mx	$GS10_{tb3m}$	1	10-Year Treasury Constant Maturity Minus 3-Month Treasury Bill, secondary market (Percent)	1
CPF3MTB3Mx	CP_Tbill Spread	1	3-Month Commercial Paper Minus 3-Month Treasury Bill, secondary market (Percent)	1
BUSLOANSx	Real C&Lloand	3	Real Commercial and Industrial Loans, All Commercial Banks (Billions of 2009 U.S. Dollars), deflated by Core PCE	2
CONSUMERX	Real ConsLoans	3	Real Consumer Loans at All Commercial Banks (Billions of 2009 U.S. Dollars), deflated by Core PCE	2
NONREVSLx	Real NonRevCredit	3	Total Real Nonrevolving Credit Owned and Securitized, Outstanding (Billions of Dollars), deflated by Core PCE	2
REALLNx	Real LoansRealEst	3	Real Real Estate Loans, All Commercial Banks (Billions of 2009 U.S. Dollars), deflated by Core PCE	2
EXSZUSx	Ex rate:Switz	3	Switzerland / U.S. Foreign Exchange Rate	3
EXJPUSx	Ex rate:Japan		Japan / U.S. Foreign Exchange Rate	3
EXUSUKx	Ex rate:UK	3	U.S. / U.K. Foreign Exchange Rate	3
EXCAUSx	EX rate:Canada		Canada / U.S. Foreign Exchange Rate	3
NIKKEI225		3	Nikkei Stock Average	4
S&P 500			S&P's Common Stock Price Index: Composite	4
S&P: indust			S&P's Common Stock Price Index: Industrials	4
S&P div yield			S&P's Composite Common Stock: Dividend Yield	4
S&P PE ratio		3	S&P's Composite Common Stock: Price-Earnings Ratio	4

S3 Proofs for Section 3 in the main paper

S3.1 Proof of Proposition 1

Proposition 1 is directly implied by the following more general result.

Proposition S2. Suppose that there are r distinct nonzero real eigenvalues of $\underline{\Theta}$, λ_j for $1 \le j \le r$, and s distinct conjugate pairs of nonzero complex eigenvalues of $\underline{\Theta}$, $(\lambda_{r+2k-1}, \lambda_{r+2k}) = (\gamma_k e^{i\theta_k}, \gamma_k e^{-i\theta_k})$ with $\gamma_k \in (0,1)$ and $\theta_k \in (-\pi/2, \pi/2)$ for $1 \le k \le s$. Moreover, the algebraic multiplicity of λ_j is n_j for $1 \le j \le r$, and that of $(\lambda_{r+2k-1}, \lambda_{r+2k})$ is m_k for $1 \le k \le s$, so there are R+2S nonzero eigenvalues of $\underline{\Theta}$ in total, where $R = \sum_{k=1}^r n_k$ and $S = \sum_{k=1}^s m_k$. Assume that the geometric multiplicities of all nonzero eigenvalues are one. Then for all $j \ge 1$, we have

$$\mathbf{A}_{j} = \sum_{k=1}^{p} \mathbb{I}_{\{j=k\}} \mathbf{G}_{k} + \sum_{k=1}^{r} \sum_{i=1}^{n_{k}} \mathbb{I}_{\{j \geqslant p+(i-1) \lor 1\}} \lambda_{k}^{j-p-i+1} {j-p \choose i-1} \mathbf{G}_{k,i}^{I}
+ \sum_{k=1}^{s} \sum_{i=1}^{m_{k}} \mathbb{I}_{\{j \geqslant p+(i-1) \lor 1\}} \gamma_{k}^{j-p-i+1} {j-p \choose i-1}
\cdot \left[\cos\{(j-p-i+1)\theta_{k}\} \mathbf{G}_{k,i}^{II,1} + \sin\{(j-p-i+1)\theta_{k}\} \mathbf{G}_{k,i}^{II,2} \right],$$
(S1)

where the first term is suppressed if p = 0, and $\mathbf{G}_{k,i}^{I}$'s, $\mathbf{G}_{k,i}^{II,1}$'s and $\mathbf{G}_{k,i}^{II,2}$'s are all determined jointly by $\widetilde{\mathbf{B}}$ and $\widecheck{\mathbf{B}}$. Moreover, for any fixed k and i, $\mathbf{G}_{k,i}^{II,h}$ for h = 1, 2 have the same row and column spaces, and $\operatorname{rank}(\mathbf{G}_{j,l}^{I}) \leqslant n_k$ and $\operatorname{rank}(\mathbf{G}_{k,i}^{II,h}) \leqslant 2m_k$ for all $1 \leqslant j \leqslant r$, $1 \leqslant k \leqslant s$, $1 \leqslant l \leqslant n_k$, $1 \leqslant i \leqslant m_k$, and h = 1, 2.

Proof of Proposition S2. Consider the general VARMA(p,q) model

$$oldsymbol{y}_t = \sum_{i=1}^p oldsymbol{\Phi}_i oldsymbol{y}_{t-i} + oldsymbol{arepsilon}_t - \sum_{j=1}^q oldsymbol{\Theta}_j oldsymbol{arepsilon}_{t-j}, \hspace{5mm} t \in \mathbb{Z}.$$

Note that it can be written equivalently as

$$\boldsymbol{\varepsilon}_t = \boldsymbol{\Theta}_1 \boldsymbol{\varepsilon}_{t-1} - \dots - \boldsymbol{\Theta}_q \boldsymbol{\varepsilon}_{t-q} + \boldsymbol{\Phi}(B) \boldsymbol{y}_t, \tag{S2}$$

where $\Phi(B) = \mathbf{I} - \sum_{i=1}^p \Phi_i B^i = -\sum_{i=0}^p \Phi_i B^i$, with $\Phi_0 = -\mathbf{I}$. Then we have

$$\underbrace{\begin{pmatrix} \boldsymbol{\varepsilon}_t \\ \boldsymbol{\varepsilon}_{t-1} \\ \boldsymbol{\varepsilon}_{t-2} \\ \vdots \\ \boldsymbol{\varepsilon}_{t-q+1} \end{pmatrix}}_{\underline{\boldsymbol{\varepsilon}}_t} = \underbrace{\begin{pmatrix} \boldsymbol{\Theta}_1 & \boldsymbol{\Theta}_2 & \cdots & \boldsymbol{\Theta}_{q-1} & \boldsymbol{\Theta}_q \\ \boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{I} & \boldsymbol{0} \end{pmatrix}}_{\underline{\boldsymbol{\varepsilon}}_t} \underbrace{\begin{pmatrix} \boldsymbol{\varepsilon}_{t-1} \\ \boldsymbol{\varepsilon}_{t-2} \\ \boldsymbol{\varepsilon}_{t-3} \\ \vdots \\ \boldsymbol{\varepsilon}_{t-q} \end{pmatrix}}_{\underline{\boldsymbol{v}}_t} + \underbrace{\begin{pmatrix} \boldsymbol{\Phi}(B) \boldsymbol{y}_t \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{\varepsilon}_{t-q} \end{pmatrix}}_{\underline{\boldsymbol{v}}_t},$$

where $\underline{\boldsymbol{\Theta}} \in \mathbb{R}^{Nq \times Nq}$ is the MA companion matrix. By recursion, we have $\underline{\boldsymbol{\varepsilon}}_t = \sum_{j=0}^{\infty} \underline{\boldsymbol{\Theta}}^j \underline{\boldsymbol{y}}_{t-j}$. Let $\boldsymbol{P} = (\boldsymbol{I}_N, \boldsymbol{0}_{N \times N(q-1)})$. Note that $\boldsymbol{P}\underline{\boldsymbol{\varepsilon}}_t = \boldsymbol{\varepsilon}_t$, and $\underline{\boldsymbol{y}}_t = \boldsymbol{P}' \boldsymbol{\Phi}(B) \boldsymbol{y}_t$. Thus,

$$\varepsilon_{t} = \sum_{j=0}^{\infty} \mathbf{P} \underline{\mathbf{\Theta}}^{j} \mathbf{P}' \Phi(B) \mathbf{y}_{t-j} = -\sum_{j=0}^{\infty} \mathbf{P} \underline{\mathbf{\Theta}}^{j} \mathbf{P}' \sum_{i=0}^{p} \Phi_{i} \mathbf{y}_{t-j-i} = -\sum_{k=0}^{\infty} \left(\sum_{i=0}^{p \wedge k} \mathbf{P} \underline{\mathbf{\Theta}}^{k-i} \mathbf{P}' \Phi_{i} \right) \mathbf{y}_{t-k}.$$
 (S3)

Since $\mathbf{PP'} = \mathbf{I}_N$, it follows from (S3) that the VAR(∞) representation of the VARMA(p,q) model can be written as

$$\boldsymbol{y}_{t} = \sum_{k=1}^{\infty} \left(\sum_{i=0}^{p \wedge k} \boldsymbol{P} \underline{\boldsymbol{\Theta}}^{k-i} \boldsymbol{P}' \boldsymbol{\Phi}_{i} \right) \boldsymbol{y}_{t-k} + \boldsymbol{\varepsilon}_{t}.$$
 (S4)

First, we simply set

$$G_{j} = \sum_{i=0}^{j} P \underline{\Theta}^{j-i} P' \Phi_{i} = A_{j}, \quad \text{for } 1 \leq j \leq p,$$
 (S5)

and then we only need to focus on the reparameterization of A_k for k > p. By (S4), for $j \ge 1$, we have

$$\mathbf{A}_{p+j} = \mathbf{P}\underline{\mathbf{\Theta}}^{j} \left(\sum_{i=0}^{p} \underline{\mathbf{\Theta}}^{p-i} \mathbf{P}' \mathbf{\Phi}_{i} \right). \tag{S6}$$

Next we derive an alternative parameterization for A_{p+j} with $j \ge 1$.

Let K = R + 2S, where $R = \sum_{k=1}^{r} n_k$ and $S = \sum_{k=1}^{s} m_k$. Under the conditions of this

proposition, the real Jordan form (Horn and Johnson, 2012, Chap. 3) of $\underline{\Theta}$ can be written as

$$oldsymbol{\Theta} = oldsymbol{B} oldsymbol{J} B^{-1} = oldsymbol{B} egin{pmatrix} oldsymbol{J}_1 & & & & & & \\ & \ddots & & & & & \\ & & oldsymbol{J}_{r+1} & & & & \\ & & & \ddots & & & \\ & & & oldsymbol{J}_{r+s} & & & \\ & & & & oldsymbol{O}(Nq-K) imes(Nq-K) \end{pmatrix} oldsymbol{B}^{-1}, \qquad (S7)$$

where $\boldsymbol{B} \in \mathbb{R}^{Nq \times Nq}$ is an invertible matrix, each \boldsymbol{J}_k with $1 \leq k \leq r$ is the $n_k \times n_k$ Jordan block corresponding to λ_k ,

$$\boldsymbol{J}_{k} = \begin{pmatrix} \lambda_{k} & 1 & & \\ & \lambda_{k} & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_{k} \end{pmatrix}, \quad 1 \leqslant k \leqslant r,$$

and each J_{r+k} for $1 \le k \le s$ is the $2m_k \times 2m_k$ real Jordan block corresponding to the conjugate pair $(\lambda_{r+2k-1}, \lambda_{r+2k})$,

$$\boldsymbol{J}_{r+k} = \begin{pmatrix} \boldsymbol{C}_k & \boldsymbol{I}_2 & & \\ & \boldsymbol{C}_k & \ddots & \\ & & \ddots & \boldsymbol{I}_2 \\ & & & \boldsymbol{C}_k \end{pmatrix} \quad \text{with} \quad \boldsymbol{C}_k = \gamma_k \cdot \begin{pmatrix} \cos(\theta_k) & \sin(\theta_k) \\ -\sin(\theta_k) & \cos(\theta_k) \end{pmatrix}, \quad 1 \leqslant k \leqslant s.$$

Denote $\widetilde{\boldsymbol{B}} = \boldsymbol{P}\boldsymbol{B}$ and $\widetilde{\boldsymbol{B}}_{-} = \boldsymbol{B}^{-1}\left(\sum_{i=0}^{p} \underline{\boldsymbol{\Theta}}^{p-i}\boldsymbol{P}'\boldsymbol{\Phi}_{i}\right)$. Note that when p = q = 1, $\widetilde{\boldsymbol{B}} = \boldsymbol{B}$ and $\widetilde{\boldsymbol{B}}_{-} = \boldsymbol{B}^{-1}(\boldsymbol{\Phi}_{1} - \boldsymbol{\Theta}_{1})$. Then by (S6) and (S7), for $j \geqslant 1$, we have

$$\boldsymbol{A}_{p+j} = \widetilde{\boldsymbol{B}} \boldsymbol{J}^j \widetilde{\boldsymbol{B}}_{-}. \tag{S8}$$

According to the block form of J in (S7), we can partition the $N \times Nq$ matrix \tilde{B} vertically and the $Nq \times N$ matrix \tilde{B}_{-} horizontally as

$$\widetilde{\boldsymbol{B}} = (\widetilde{\boldsymbol{B}}_1, \dots, \widetilde{\boldsymbol{B}}_{r+s}, \widetilde{\boldsymbol{B}}_{r+s+1})$$
 and $\widetilde{\boldsymbol{B}}_- = (\widetilde{\boldsymbol{B}}_{-1}, \dots, \widetilde{\boldsymbol{B}}_{-(r+s)}, \widetilde{\boldsymbol{B}}_{-(r+s+1)})'$,

where $\widetilde{\boldsymbol{B}}_k$ and $\widetilde{\boldsymbol{B}}_{-k}$ are $N \times n_k$ matrices for $1 \leq k \leq r$, $\widetilde{\boldsymbol{B}}_{r+k}$ and $\widetilde{\boldsymbol{B}}_{-(r+k)}$ are $N \times 2m_k$ matrices

for $1 \leq k \leq s$, and $\widetilde{\boldsymbol{B}}_{r+s+1}$ and $\widetilde{\boldsymbol{B}}_{-(r+s+1)}$ are $N \times (Nq - K)$ matrices. Notice that for any $j \geq 1$,

$$\boldsymbol{J}_{k}^{j} = \begin{pmatrix} \lambda_{k}^{j} & \binom{j}{1} \lambda_{k}^{j-1} & \binom{j}{2} \lambda_{k}^{j-2} & \cdots & \binom{j}{n_{k}-1} \lambda_{k}^{j-n_{k}+1} \\ 0 & \lambda_{k}^{j} & \binom{j}{1} \lambda_{k}^{j-1} & \cdots & \binom{j}{n_{k}-2} \lambda_{k}^{j-n_{k}+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{k}^{j} \end{pmatrix}, \quad 1 \leqslant k \leqslant r,$$

and

$$\boldsymbol{J}_{r+k}^{j} = \begin{pmatrix} \boldsymbol{C}_{k}^{j} & \binom{j}{1} \boldsymbol{C}_{k}^{j-1} & \binom{j}{2} \boldsymbol{C}_{k}^{j-2} & \cdots & \binom{j}{m_{k}-1} \boldsymbol{C}_{k}^{j-m_{k}+1} \\ \boldsymbol{0} & \boldsymbol{C}_{k}^{j} & \binom{j}{1} \boldsymbol{C}_{k}^{j-1} & \cdots & \binom{j}{m_{k}-2} \boldsymbol{C}_{k}^{j-m_{k}+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{C}_{k}^{j} \end{pmatrix}, \quad 1 \leqslant k \leqslant s,$$

with

$$C_k^j = \gamma_k^j \cdot \begin{pmatrix} \cos(j\theta_k) & \sin(j\theta_k) \\ -\sin(j\theta_k) & \cos(j\theta_k) \end{pmatrix}.$$

Let $\widetilde{\boldsymbol{b}}_{k}^{(i)}$ and $\widetilde{\boldsymbol{b}}_{-k}^{(i)}$ be the *i*-th columns of $\widetilde{\boldsymbol{B}}_{k}$ and $\widetilde{\boldsymbol{B}}_{-k}$, respectively. In addition, denote $\boldsymbol{\eta}_{k} = (\gamma_{k}, \theta_{k})'$ for $1 \leq k \leq s$. Then by (S8), for $j \geq 1$, we can show that

$$\boldsymbol{A}_{p+j} = \sum_{k=1}^{r+s} \widetilde{\boldsymbol{B}}_{k} \boldsymbol{J}_{k}^{j} \widetilde{\boldsymbol{B}}_{-k}^{\prime} = \sum_{k=1}^{r} \sum_{i=1}^{n_{k}} \ell_{i,j}^{I}(\lambda_{k}) \boldsymbol{G}_{k,i}^{I} + \sum_{k=1}^{s} \sum_{i=1}^{m_{k}} \left\{ \ell_{i,j}^{II,1}(\boldsymbol{\eta}_{k}) \boldsymbol{G}_{k,i}^{II,1} + \ell_{i,j}^{II,2}(\boldsymbol{\eta}_{k}) \boldsymbol{G}_{k,i}^{II,2} \right\}, \quad (S9)$$

where $\ell_{i,j}^{I}(\cdot)$, $\ell_{i,j}^{II,1}(\cdot)$, and $\ell_{i,j}^{II,2}(\cdot)$ are real-valued functions defined as

$$\ell_{i,j}^{I}(\lambda) = \lambda^{j-i+1} \binom{j}{i-1} \mathbb{I}_{\{j \geqslant i-1\}},$$

$$\ell_{i,j}^{II,1}(\boldsymbol{\eta}) = \gamma^{j-i+1} \binom{j}{i-1} \cos\{(j-i+1)\theta\} \mathbb{I}_{\{j \geqslant i-1\}},$$

$$\ell_{i,j}^{II,2}(\boldsymbol{\eta}) = \gamma^{j-i+1} \binom{j}{i-1} \sin\{(j-i+1)\theta\} \mathbb{I}_{\{j \geqslant i-1\}},$$
(S10)

for any $\lambda \in (-1,0) \cap (0,1)$ and $\boldsymbol{\eta} = (\gamma,\theta)' \in (0,1) \times (-\pi/2,\pi/2)$, and

$$\mathbf{G}_{k,i}^{I} = \sum_{h=i}^{n_{k}} \widetilde{\mathbf{b}}_{k}^{(h-i+1)} \widetilde{\mathbf{b}}_{-k}^{(h)'}, \quad 1 \leq k \leq r, \ 1 \leq i \leq n_{k},
\mathbf{G}_{k,i}^{II,1} = \sum_{h=i}^{m_{k}} \left(\widetilde{\mathbf{b}}_{r+k}^{(2h-2i+1)} \widetilde{\mathbf{b}}_{-(r+k)}^{(2h-1)'} + \widetilde{\mathbf{b}}_{r+k}^{(2h-2i+2)} \widetilde{\mathbf{b}}_{-(r+k)}^{(2h)'} \right), \quad 1 \leq k \leq s, \ 1 \leq i \leq m_{k},
\mathbf{G}_{k,i}^{II,2} = \sum_{h=i}^{m_{k}} \left(\widetilde{\mathbf{b}}_{r+k}^{(2h-2i+1)} \widetilde{\mathbf{b}}_{-(r+k)}^{(2h)'} - \widetilde{\mathbf{b}}_{r+k}^{(2h-2i+2)} \widetilde{\mathbf{b}}_{-(r+k)}^{(2h-1)'} \right), \quad 1 \leq k \leq s, \ 1 \leq i \leq m_{k}.$$
(S11)

Note that for any fixed k and i, $G_{k,i}^{II,h}$ for h = 1, 2 have the same row and column spaces. Moreover, $\operatorname{rank}(G_{j,l}^I) \leq n_k$ and $\operatorname{rank}(G_{k,i}^{II,h}) \leq 2m_k$ for all $1 \leq j \leq r$, $1 \leq k \leq s$, $1 \leq l \leq n_k$, $1 \leq i \leq m_k$, and h = 1, 2. Finally, combining (S5) and (S9), we accomplish the proof of this proposition.

S3.2 Proof of Theorem 1

Let $\mathcal{L}(\cdot): \Omega \to \mathbb{R}^{(r+2s)\times(r+2s)}$ and $\mathcal{T}(\cdot): \Omega \to \mathbb{C}^{(r+2s)\times(r+2s)}$ be two matrix-valued functions such that for any $\omega \in \Omega$, $\mathcal{L}(\omega) = (\mathcal{L}_1(\omega), \dots, \mathcal{L}_{r+2s}(\omega))'$ and $\mathcal{T}(\omega) = (\mathcal{T}_1(\omega), \dots, \mathcal{T}_{r+2s}(\omega))'$, where for all $1 \leq j \leq r+2s$,

$$\mathcal{L}_{j}(\boldsymbol{\omega}) = (\lambda_{1}^{j}, \dots, \lambda_{r}^{j}, \gamma_{1}^{j} \cos(j\theta_{1}), \gamma_{1}^{j} \sin(j\theta_{1}), \dots, \gamma_{s}^{j} \cos(j\theta_{s}), \gamma_{s}^{j} \sin(j\theta_{s}))' \text{ and}$$

$$\mathcal{T}_{j}(\boldsymbol{\omega}) = (\lambda_{1}^{j}, \dots, \lambda_{r}^{j}, (\gamma_{1}e^{i\theta_{1}})^{j}, (\gamma_{1}e^{-i\theta_{1}})^{j}, \dots, (\gamma_{s}e^{i\theta_{s}})^{j}, (\gamma_{s}e^{-i\theta_{s}})^{j})'. \tag{S12}$$

Then, let $\mathcal{F}: \mathcal{L}(\cdot) \to \mathcal{T}(\cdot)$ be a functional mapping such that for any $\boldsymbol{\omega} \in \Omega$,

$$\mathcal{T}(\boldsymbol{\omega}) = \mathcal{F}(\mathcal{L}(\boldsymbol{\omega})) = \mathcal{L}(\boldsymbol{\omega})\boldsymbol{F}, \text{ with } \boldsymbol{F} = \begin{pmatrix} \boldsymbol{I}_r \\ \boldsymbol{I}_s \otimes \boldsymbol{F}_c \end{pmatrix} \text{ and } \boldsymbol{F}_c = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}.$$
 (S13)

Since \mathbf{F} is invertible, $\mathcal{F}(\cdot)$ is a bijective map. Set $(x_1, \ldots, x_{r+2s}) = \mathcal{T}_1(\boldsymbol{\omega})$ such that $x_k = \lambda_k$ for $1 \leq k \leq r$, while $x_{r+2k-1} = \gamma_k e^{i\theta_k}$ and $x_{r+2k} = \gamma_k e^{-i\theta_k}$ for $1 \leq k \leq s$, where we suppress x_k 's dependence on $\boldsymbol{\omega}$ for notation simplicity.

For any $x \in \mathbb{C}$, we next define a vector-valued function $\mathbf{v}(x) = (x, x^2, \dots, x^{r+2s})' \in \mathbb{C}^{r+2s}$. For any $\boldsymbol{\omega} \in \Omega$ satisfying the conditions of this lemma, we first show that $\mathcal{T}(\boldsymbol{\omega})$ is invertible. It holds trivially $\mathcal{T}(\boldsymbol{\omega}) = (\mathbf{v}(x_1), \dots, \mathbf{v}(x_{r+2s}))$. Suppose that there exists $\mathbf{c} \in \mathbb{C}^{r+2s}$ such that $\mathcal{T}(\boldsymbol{\omega})\mathbf{c} = \mathbf{0}$. By the condition that $x_k \neq x_\ell$ for all $1 \leq k \neq \ell \leq r+2s$, this implies that the (r+2s)-order

polynomial

$$poly(x) = c_1 x + c_2 x^2 + \dots + c_{r+2s} x^{r+2s} = x(c_1 + c_2 x + \dots + c_{r+2s} x^{r+2s-1})$$
(S14)

has (r+2s) non-zero, distinct roots. Since the polynomial at (S14) can have at most (r+2s-1) non-zero, distinct roots, it must be a zero polynomial, i.e. $\mathbf{c} = \mathbf{0}$. As a result, $\mathcal{T}(\boldsymbol{\omega})$ has linearly independent columns, i.e. it is invertible. Moreover, by (S13), $\mathcal{L}(\boldsymbol{\omega})$ is also invertible.

Then, let $L_{[1:d]}(\boldsymbol{\omega})$ be a square matrix consisting of the first d = p + r + 2s rows of $L(\boldsymbol{\omega})$, and it follows that

$$oldsymbol{L}_{[1:d]}(oldsymbol{\omega}) = egin{pmatrix} oldsymbol{I}_p & & & \ & \mathcal{L}(oldsymbol{\omega}) \end{pmatrix} \in \mathbb{R}^{d imes d}$$

is an invertible matrix. Subsequently, the matrix $(\boldsymbol{G}_1,\ldots,\boldsymbol{G}_d)$ can be uniquely defined as

$$(\boldsymbol{G}_1,\ldots,\boldsymbol{G}_d)=(\boldsymbol{A}_1,\ldots,\boldsymbol{A}_d)\left([\boldsymbol{L}_{[1:d]}(\boldsymbol{\omega})']^{-1}\otimes\boldsymbol{I}_N\right),$$

where the inverse of $L_{[1:d]}(\omega)$ is well-defined.

It remains show that ω is unique, i.e. there does not exist $\widetilde{\omega} \neq \omega$ such that $A_j(\omega, \mathfrak{G}) = A_j(\widetilde{\omega}, \widetilde{\mathfrak{G}})$ for all $j \geq 1$, where $\widetilde{\mathfrak{G}}$ may be different from \mathfrak{G} but still satisfies the condition that all $\widetilde{\mathfrak{G}}_k$'s are non-zero matrices. Suppose that such $\widetilde{\omega}$ does exist and further set $(\widetilde{x}_1, \ldots, \widetilde{x}_{r+2s}) = \mathcal{T}_1(\widetilde{\omega})$ where $\mathcal{T}_1(\cdot)$ is defined in (S12). There must exists some non-zero $\widetilde{x}_k \notin \{x_1, \ldots, x_{r+2s}\}$. Suppose that there exists $c \in \mathbb{C}^{r+2s+1}$ such that

$$(\boldsymbol{v}(x_1),\ldots,\boldsymbol{v}(x_{r+2s}),\boldsymbol{v}(\widetilde{x}_k))\boldsymbol{c}=\mathbf{0}.$$

This implies that the polynomial at (S14) has (r + 2s + 1) non-zero distinct roots, which only holds if $\mathbf{c} = 0$, i.e. $\mathbf{v}(\widetilde{x}_k)$ is linearly independent of $\mathbf{v}(x_\ell)$ for all $1 \leq \ell \leq r + 2s$. By (S13), this implies that the k-th column of $\mathcal{L}(\widetilde{\boldsymbol{\omega}})$ does not belong to the column space of $\mathcal{L}(\boldsymbol{\omega})$. Then, it is implied from

$$(oldsymbol{A}_{p+1},\ldots,oldsymbol{A}_d)=(oldsymbol{G}_r,\ldots,oldsymbol{G}_d)\,(oldsymbol{\mathcal{L}}(oldsymbol{\omega})'\otimesoldsymbol{I}_N)=(oldsymbol{\widetilde{G}}_r,\ldots,oldsymbol{\widetilde{G}}_d)\,(oldsymbol{L}(oldsymbol{\widetilde{\omega}})'\otimesoldsymbol{I}_N)$$

that $\widetilde{\boldsymbol{G}}_k = \boldsymbol{0}$, leading to a contradiction.

Therefore, if the parameters ω and \mathfrak{G} satisfy the conditions of the lemma, they are uniquely

identified for any A_1, A_2, \ldots

S3.3 Proof of Theorem 2

We first prove the existence. Using iteratively model (3.1), we get

$$\boldsymbol{y}_{t} = \boldsymbol{\varepsilon}_{t} + \sum_{k=1}^{\infty} \sum_{j_{1},\dots,j_{k} \geq 1} \boldsymbol{A}_{j_{1}} \cdots \boldsymbol{A}_{j_{k}} \boldsymbol{\varepsilon}_{t-j_{1}-\dots-j_{k}}, \quad \boldsymbol{A}_{j} = \sum_{k=1}^{d} \ell_{j,k}(\boldsymbol{\omega}) \boldsymbol{G}_{k},$$
 (S15)

where the first equation can be written as $\boldsymbol{y}_t = \boldsymbol{\varepsilon}_t + \sum_{j=1}^{\infty} \boldsymbol{\Psi}_j \boldsymbol{\varepsilon}_{t-j}$ with $\boldsymbol{\Psi}_j = \sum_{k=1}^{\infty} \sum_{j_1 + \dots + j_k = j} \boldsymbol{A}_{j_1} \cdots \boldsymbol{A}_{j_k}$. Note that \boldsymbol{y}_t takes value in $[-\infty, \infty]^N$.

By the triangle inequality, we have

$$\|oldsymbol{y}_t\|_2\leqslant \|oldsymbol{arepsilon}_t\|_2+\sum_{k=1}^\infty\sum_{j_1,...,j_k\geqslant 1}\|oldsymbol{A}_{j_1}\|_{ ext{op}}\cdots \|oldsymbol{A}_{j_k}\|_{ ext{op}}\|oldsymbol{arepsilon}_{t-j_1-...-j_k}\|_2.$$

Denote $S = \sum_{j=1}^{\infty} \|\boldsymbol{A}_j\|_{\text{op}}$. Since $\boldsymbol{A}_j = \boldsymbol{G}_j$ for $1 \leq j \leq p$ and $\boldsymbol{A}_j = \sum_{k=p+1}^d \ell_{j,k}(\boldsymbol{\omega})\boldsymbol{G}_k$ for $j \geq p+1$, under condition (3.4), we have

$$S \leq \sum_{k=1}^{p} \|\boldsymbol{G}_{j}\|_{\text{op}} + \sum_{j=1}^{\infty} \sum_{k=p+1}^{d} |\ell_{j+p,k}(\boldsymbol{\omega})| \|\boldsymbol{G}_{k}\|_{\text{op}} \leq \sum_{k=1}^{p} \|\boldsymbol{G}_{j}\|_{\text{op}} + \sum_{j=1}^{\infty} \sum_{k=p+1}^{d} \rho^{j} \|\boldsymbol{G}_{k}\|_{\text{op}} < 1.$$
 (S16)

Since ε_t are *i.i.d.* with $E(\|\varepsilon_t\|_2) < \infty$, this leads to

$$\mathbb{E}(\|\boldsymbol{y}_t\|_2) \leqslant \mathbb{E}(\|\boldsymbol{\varepsilon}_t\|_2)(1 + \sum_{k=1}^{\infty} S^k) = \frac{\mathbb{E}(\|\boldsymbol{\varepsilon}_t\|_2)}{1 - S} < \infty.$$
 (S17)

Thus, the VMA(∞) process $\{y_t\}$ is weakly stationary. This proves the existence of a weakly stationary solution to model (3.1).

To prove the uniqueness, suppose that $\{y_t, t \in \mathbb{Z}\}$ is a weakly stationary and causal solution to model (3.1). Then, applying recurrence relation (3.1) m times, we obtain

$$oldsymbol{y}_t = oldsymbol{arepsilon}_t + \sum_{k=1}^m \sum_{j_1,...,j_k \geqslant 1} oldsymbol{A}_{j_1} \cdots oldsymbol{A}_{j_k} oldsymbol{arepsilon}_{t-j_1-...-j_k} oldsymbol{arepsilon}_{t-j_1-...-j_k} + oldsymbol{r}_{t,m},$$

where

$$oldsymbol{r}_{t,m} = \sum_{j_1,...,j_{m+1}\geqslant 1} oldsymbol{A}_{j_1} \cdots oldsymbol{A}_{j_k} oldsymbol{y}_{t-j_1-...-j_{m+1}}.$$

As shown in (S16) and (S17), under the conditions of this theorem, $0 \le S < 1$ and $\mathbb{E}(\|\boldsymbol{y}_t\|_2) < \infty$. As a result,

$$\mathbb{E}(\|\boldsymbol{r}_{t,m}\|_{2}) \leqslant \sum_{j_{1},\dots,j_{m+1}\geqslant 1} \|\boldsymbol{A}_{j_{1}}\|_{\text{op}} \cdots \|\boldsymbol{A}_{j_{m+1}}\|_{\text{op}} \mathbb{E}(\|\boldsymbol{y}_{t-j_{1}-\dots-j_{m+1}}\|_{2}) \leqslant S^{m} \mathbb{E}(\|\boldsymbol{y}_{t}\|_{2}) \to 0,$$

as $m \to \infty$. By the Borel-Cantelli Lemma, as $m \to \infty$, $\|\boldsymbol{r}_{t,m}\|_2 \to 0$ almost surely, that is, $\boldsymbol{r}_{t,m} \to \boldsymbol{0}$ almost surely. Thus, \boldsymbol{y}_t satisfies (S15) almost surely, and the uniqueness is verified.

S4 Proofs for Section 4 in the main paper

S4.1 Useful properties of $L(\omega)$

According to the definition of $L(\omega)$, for $j \ge 1$, denote the jth entry of $\ell^I(\lambda_i)$ by $\ell^I_j(\lambda_i) = \lambda_i^j$ with $1 \le i \le r$, and denote the transpose of the jth row of the $\infty \times 2$ matrix $\ell^{II}(\boldsymbol{\eta}_k)$ by $\ell^{II}_j(\boldsymbol{\eta}_k) := (\ell^{II,1}_j(\boldsymbol{\eta}_k), \ell^{II,2}_j(\boldsymbol{\eta}_k))' = (\gamma_k^j \cos(j\theta_k), \gamma_k^j \sin(j\theta_k))'$ with $1 \le k \le s$. Let $L^I(\lambda) = (\ell^I(\lambda_1), \cdots, \ell^I(\lambda_r))$ and $L^{II}(\boldsymbol{\eta}) = (\ell^{II}(\boldsymbol{\eta}_1), \cdots, \ell^{II}(\boldsymbol{\eta}_s))$. In addition, define the following matrix by augmenting $L(\omega)$ with (r+2s) extra columns consisting of first-order derivatives:

$$\boldsymbol{L}_{\text{stack}}(\boldsymbol{\omega}) = \begin{pmatrix} \boldsymbol{I}_p & & \\ & \boldsymbol{L}^I(\boldsymbol{\lambda}) & \boldsymbol{L}^{II}(\boldsymbol{\eta}) & \nabla \boldsymbol{L}^I(\boldsymbol{\lambda}) & \nabla_{\theta} \boldsymbol{L}^{II}(\boldsymbol{\eta}) \end{pmatrix}, \tag{S1}$$

where $\nabla \boldsymbol{L}^{I}(\boldsymbol{\lambda}) = (\nabla \boldsymbol{\ell}^{I}(\lambda_{1}), \cdots, \nabla \boldsymbol{\ell}^{I}(\lambda_{r}))$ and $\nabla_{\theta} \boldsymbol{L}^{II}(\boldsymbol{\eta}) = (\nabla_{\theta} \boldsymbol{\ell}^{II}(\boldsymbol{\eta}_{1}), \cdots, \nabla_{\theta} \boldsymbol{\ell}^{II}(\boldsymbol{\eta}_{s}))$. We can similarly define $\nabla_{\gamma} \boldsymbol{L}^{II}(\boldsymbol{\eta})$. Note that from (S11), it holds $\operatorname{colsp}\{\nabla_{\gamma} \boldsymbol{L}^{II}(\boldsymbol{\eta})\} = \operatorname{colsp}\{\nabla_{\theta} \boldsymbol{L}^{II}(\boldsymbol{\eta})\}$, which is why $\nabla_{\gamma} \boldsymbol{L}^{II}(\boldsymbol{\eta})$ is not included in $\boldsymbol{L}_{\operatorname{stack}}(\boldsymbol{\omega})$. Denote

$$\sigma_{\min,L} = \sigma_{\min}(\boldsymbol{L}_{\mathrm{stack}}(\boldsymbol{\omega}^*))$$
 and $\sigma_{\max,L} = \sigma_{\max}(\boldsymbol{L}_{\mathrm{stack}}(\boldsymbol{\omega}^*))$,

where ω^* is the true value of ω . Lemma S.1 below gives some exponential decay properties induced by the parametric form of $L(\omega)$, which will be used repeatedly in our theoretical analysis. Then, based on Lemma S.1(ii), we can show that $\sigma_{\min,L} = 1$ and $\sigma_{\max,L} = 1$; this is stated in Lemma S.2.

Lemma S.1. Suppose that Assumption 2(i) holds. Then (i) there exists an absolute constant

 $C_L > 0$ such that for all $\omega \in \Omega$ and $j \ge 1$,

$$\max_{1\leqslant i\leqslant r,1\leqslant k\leqslant s,1\leqslant h\leqslant 2}\{|\nabla \ell_j^I(\lambda_i)|,\|\nabla \ell_j^{II,h}(\boldsymbol{\eta}_k)\|_2,|\nabla^2 \ell_j^I(\lambda_i)|,\|\nabla^2 \ell_j^{II,h}(\boldsymbol{\eta}_k)\|_{\mathrm{F}}\}\leqslant C_L\bar{\rho}^j;$$

and (ii) there exists an absolute constant $C_* > 0$ such that $\|\mathbf{A}_j^*\|_{op} \leqslant C_* \bar{\rho}^j$ for all $j \geqslant 1$ if Assumption 2(iii) further holds.

Lemma S.2. Let J = 2(r+2s). Denote $x_k^* = \lambda_k^*$ for $1 \le k \le r$ and $x_{r+2k-1}^* = \gamma_k^* e^{i\theta_k^*}$, $x_{r+2k}^* = \gamma_k^* e^{-i\theta_k^*}$ for $1 \le k \le s$, and let $\nu_1^* = \min\{|x_k^*|, 1 \le k \le r+2s\}$ and $\nu_2^* = \min\{|x_j^* - x_k^*|, 1 \le j < k \le r+2s\}$. Under Assumptions 2(i) and 2(ii), the matrix $\mathbf{L}_{\text{stack}}(\boldsymbol{\omega}^*)$ has full rank, and its maximum and minimum singular values satisfy

$$\min\{1, c_{\bar{\rho}}\} \leqslant \sigma_{\min, L} \leqslant \sigma_{\max, L} \leqslant \max\{1, C_{\bar{\rho}}\}.$$

where
$$C_{\bar{\rho}} = C_1 \bar{\rho} \sqrt{J} (1 - \bar{\rho})^{-1} \approx 1$$
 and $c_{\bar{\rho}} = 0.25^s (\nu_1^*)^{3J/2} (\nu_2^*)^{J(J/2-1)} / C_{\bar{\rho}}^{J-1} \approx 1$.

S4.2 Notations and main idea: linearization of parametric structure

For simplicity, denote the perturbations of $\boldsymbol{\omega}^*$, $\boldsymbol{\mathcal{G}}^*$ and $\boldsymbol{\mathcal{A}}^*$ by $\delta_{\boldsymbol{\omega}} = \|\boldsymbol{\omega} - \boldsymbol{\omega}^*\|_2$, $\delta_{\boldsymbol{\mathcal{G}}} = \|\boldsymbol{\mathcal{G}} - \boldsymbol{\mathcal{G}}^*\|_F$ and $\delta_{\boldsymbol{\mathcal{A}}} = \|\boldsymbol{\mathcal{A}} - \boldsymbol{\mathcal{A}}^*\|_F = \|\boldsymbol{\Delta}\|_F$, respectively. Let

$$\Upsilon = \left\{ \boldsymbol{\Delta} = \boldsymbol{\mathcal{A}} - \boldsymbol{\mathcal{A}}^* \in \mathbb{R}^{N \times N \times \infty} \mid \boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{G}} \times_3 \boldsymbol{L}(\boldsymbol{\omega}), \boldsymbol{\mathcal{G}} \in \Gamma(\mathcal{R}_1, \mathcal{R}_2), \boldsymbol{\omega} \in \Omega, \delta_{\boldsymbol{\omega}} \leqslant c_{\boldsymbol{\omega}} \right\},$$

where $\Gamma(\mathcal{R}_1, \mathcal{R}_2) = \{ \mathcal{G} \in \mathbb{R}^{N \times N \times d} \mid \operatorname{rank}(\mathcal{G}_{(1)}) \leqslant \mathcal{R}_1, \operatorname{rank}(\mathcal{G}_{(2)}) \leqslant \mathcal{R}_2 \}$. It is noteworthy that under the conditions of Theorem 3, $\hat{\Delta} := \hat{\mathcal{A}} - \mathcal{A}^* \in \Upsilon$.

A crucial intermediate step for our theoretical analysis is to establish the following linear approximation within a fixed local neighborhood of ω^* ,

$$\Delta(\omega, \mathcal{G}) = \mathcal{A}(\omega, \mathcal{G}) - \mathcal{A}^* \approx \mathcal{M}(\omega - \omega^*, \mathcal{G} - \mathcal{G}^*) \times_3 \mathbf{L}_{\text{stack}}(\omega^*), \tag{S2}$$

where $\mathbf{M}: \mathbb{R}^{r+2s} \times \mathbb{R}^{N \times N \times d} \to \mathbb{R}^{N \times N \times (d+r+2s)}$ is a bilinear function defined as follows:

$$\mathfrak{M}(\boldsymbol{a}, \boldsymbol{\mathcal{B}}) = \operatorname{stack}\left(\boldsymbol{\mathcal{B}}, \left\{a_{i}\boldsymbol{G}_{i}^{I*}\right\}_{1 \leq i \leq r}, \left\{a_{r+2k}\boldsymbol{G}_{k}^{II,1*} - \frac{a_{r+2k-1}}{\gamma_{k}^{*}}\boldsymbol{G}_{k}^{II,2*}, a_{r+2k-1}\boldsymbol{G}_{k}^{II,2*} + \frac{a_{r+2k}}{\gamma_{k}^{*}}\boldsymbol{G}_{k}^{II,1*}\right\}_{1 \leq k \leq s}\right),$$
(S3)

for any $\mathbf{a} = (a_1, \dots, a_{r+2s})' \in \mathbb{R}^{r+2s}$ and $\mathbf{B} \in \mathbb{R}^{N \times N \times d}$, with the true values $\boldsymbol{\omega}^*$ and \mathbf{G}^* fixed. The linear approximation in (S2) will be formalized in the proof of Lemma 1; in particular, see (S13) and (S14) for the linear form and the remainder term, respectively.

In addition, the following notations will be used in the proof of Lemma 1. First, for the convenience of notation in the proof, according to the block form of $\boldsymbol{L}(\boldsymbol{\omega})$, we partition $\boldsymbol{\mathfrak{G}} \in \mathbb{R}^{N \times N \times d}$ as $\boldsymbol{\mathfrak{G}} = \operatorname{stack}(\boldsymbol{\mathfrak{G}}^{\operatorname{AR}}, \boldsymbol{\mathfrak{G}}^{\operatorname{MA}}) = (\boldsymbol{\mathfrak{G}}^{\operatorname{AR}}, \boldsymbol{\mathfrak{G}}^{I}, \boldsymbol{\mathfrak{G}}^{II})$, where $\boldsymbol{\mathfrak{G}}^{\operatorname{AR}} = \operatorname{stack}(\boldsymbol{G}_1, \dots, \boldsymbol{G}_p)$, $\boldsymbol{\mathfrak{G}}^I = \operatorname{stack}(\boldsymbol{G}_1^{I}, \dots, \boldsymbol{G}_r^{I})$, and $\boldsymbol{\mathfrak{G}}^{II} = \operatorname{stack}(\boldsymbol{G}_1^{II,1}, \boldsymbol{G}_1^{II,2}, \dots, \boldsymbol{G}_s^{II,1}, \boldsymbol{G}_s^{II,2})$ are $N \times N \times p$, $N \times N \times r$, and $N \times N \times 2s$ tensors, respectively. Here, $\boldsymbol{G}_i^I = \boldsymbol{G}_{p+i}$ for $1 \leq i \leq r$, and $\boldsymbol{G}_k^{II,1} = \boldsymbol{G}_{p+r+2k-1}$ and $\boldsymbol{G}_k^{II,2} = \boldsymbol{G}_{p+r+2k}$ for $1 \leq k \leq s$. Then, for any $\boldsymbol{\mathcal{A}} = \boldsymbol{\mathfrak{G}} \times_3 \boldsymbol{L}(\boldsymbol{\omega})$, we have $\boldsymbol{A}_k = \boldsymbol{G}_k$ for $1 \leq k \leq p$, and

$$\mathbf{A}_{p+j} = \sum_{i=1}^{r} \ell_{j}^{I}(\lambda_{i}) \mathbf{G}_{i}^{I} + \sum_{k=1}^{s} \left\{ \ell_{j}^{II,1}(\boldsymbol{\eta}_{k}) \mathbf{G}_{k}^{II,1} + \ell_{j}^{II,2}(\boldsymbol{\eta}_{k}) \mathbf{G}_{k}^{II,2} \right\}, \text{ for } j \geqslant 1.$$
 (S4)

Moreover, for simplicity, let

$$\mathcal{G}_{\mathrm{stack}} = \mathcal{M}(\omega - \omega^*, \mathcal{G} - \mathcal{G}^*)$$

Equivalently, we can express $\mathfrak{G}_{\text{stack}} = \text{stack}(\mathfrak{G} - \mathfrak{G}^*, \mathfrak{D}(\boldsymbol{\omega}))$ as the $N \times N \times (d + r + 2s)$ tensor formed by augmenting $\mathfrak{G} - \mathfrak{G}^*$ with the $N \times N \times (r + 2s)$ tensor

$$\mathcal{D}(\boldsymbol{\omega}) = \operatorname{stack}\left(\left\{(\lambda_{i} - \lambda_{i}^{*})\boldsymbol{G}_{i}^{I*}\right\}_{1 \leq i \leq r},\right.$$

$$\left\{(\theta_{k} - \theta_{k}^{*})\boldsymbol{G}_{k}^{II,1*} - \frac{\gamma_{k} - \gamma_{k}^{*}}{\gamma_{k}^{*}}\boldsymbol{G}_{k}^{II,2*}, \left(\theta_{k} - \theta_{k}^{*}\right)\boldsymbol{G}_{k}^{II,2*} + \frac{\gamma_{k} - \gamma_{k}^{*}}{\gamma_{k}^{*}}\boldsymbol{G}_{k}^{II,1*}\right\}_{1 \leq k \leq s}\right).$$

Lastly, note that for every $\Delta(\omega, \mathfrak{G}) \in \Upsilon$, its corresponding $\mathfrak{G}_{\text{stack}} \in \Xi$, where

$$\boldsymbol{\Xi} = \left\{ \boldsymbol{\mathcal{M}}(\boldsymbol{a}, \boldsymbol{\mathcal{B}}) \in \mathbb{R}^{N \times N \times (d+r+2s)} \mid \boldsymbol{a} \in \mathbb{R}^{r+2s}, \boldsymbol{\mathcal{B}} \in \Gamma(2\mathcal{R}_1, 2\mathcal{R}_2) \right\}.$$
 (S5)

S4.3 Proof of Lemma 1

By Assumption 2(iii), without loss of generality, let $\max_{p+1 \leq k \leq d} \|\boldsymbol{G}_k^*\|_{\mathrm{F}} = \alpha$ and $\min_{p+1 \leq k \leq d} \|\boldsymbol{G}_k^*\|_{\mathrm{F}} = c_{\mathtt{g}}\alpha$, where $c_{\mathtt{g}} > 0$ is an absolute constant.

Let $\Delta = \mathcal{A} - \mathcal{A}^* = \mathcal{G} \times_3 \mathbf{L}(\omega) - \mathcal{G}^* \times_3 \mathbf{L}(\omega^*)$. Denote by Δ_j with $j \ge 1$ the frontal slices of Δ , i.e. $\Delta_{(1)} = (\Delta_1, \Delta_2, \dots)$. Then $\Delta_j = \mathbf{G}_j - \mathbf{G}_j^*$ for $1 \le j \le p$. For $j \ge 1$, by (S4) and the Taylor expansion,

$$\Delta_{p+j} = \boldsymbol{A}_{p+j} - \boldsymbol{A}_{p+j}^{*}$$

$$= \sum_{k=1}^{r} \left\{ \ell_{j}^{I}(\lambda_{k}^{*}) + \nabla \ell_{j}^{I}(\lambda_{k}^{*})(\lambda_{k} - \lambda_{k}^{*}) + \frac{1}{2} \nabla^{2} \ell_{j}^{I}(\widetilde{\lambda}_{k})(\lambda_{k} - \lambda_{k}^{*})^{2} \right\} \boldsymbol{G}_{k}^{I}$$

$$+ \sum_{k=1}^{s} \left\{ \ell_{j}^{II,1}(\boldsymbol{\eta}_{k}^{*}) + (\boldsymbol{\eta}_{k} - \boldsymbol{\eta}_{k}^{*})' \nabla \ell_{j}^{II,1}(\boldsymbol{\eta}_{k}^{*})$$

$$+ \frac{1}{2} (\boldsymbol{\eta}_{k} - \boldsymbol{\eta}_{k}^{*})' \nabla^{2} \ell_{j}^{II,1}(\widetilde{\boldsymbol{\eta}}_{k})(\boldsymbol{\eta}_{k} - \boldsymbol{\eta}_{k}^{*}) \right\} \boldsymbol{G}_{k}^{II,1}$$

$$+ \sum_{k=1}^{s} \left\{ \ell_{j}^{II,2}(\boldsymbol{\eta}_{k}^{*}) + (\boldsymbol{\eta}_{k} - \boldsymbol{\eta}_{k}^{*})' \nabla \ell_{j}^{II,2}(\boldsymbol{\eta}_{k}^{*})$$

$$+ \frac{1}{2} (\boldsymbol{\eta}_{k} - \boldsymbol{\eta}_{k}^{*})' \nabla^{2} \ell_{j}^{II,2}(\widetilde{\boldsymbol{\eta}}_{k})(\boldsymbol{\eta}_{k} - \boldsymbol{\eta}_{k}^{*}) \right\} \boldsymbol{G}_{k}^{II,2} - \boldsymbol{A}_{p+j}^{*}$$

$$:= \boldsymbol{H}_{j} + \boldsymbol{R}_{j}, \tag{S6}$$

where $\widetilde{\lambda}_k$ lies between λ_k^* and λ_k for $1 \leq k \leq r$, $\widetilde{\boldsymbol{\eta}}_k$ lies between $\boldsymbol{\eta}_k^*$ and $\boldsymbol{\eta}_k$ for $1 \leq k \leq s$,

$$\mathbf{H}_{j} = \sum_{k=1}^{r} \ell_{j}^{I}(\lambda_{k}^{*}) (\mathbf{G}_{k}^{I} - \mathbf{G}_{k}^{I*}) + \sum_{k=1}^{s} \sum_{h=1}^{2} \ell_{j}^{II,h} (\boldsymbol{\eta}_{k}^{*}) (\mathbf{G}_{k}^{II,h} - \mathbf{G}_{k}^{II,h*})
+ \sum_{k=1}^{r} (\lambda_{k} - \lambda_{k}^{*}) \nabla \ell_{j}^{I}(\lambda_{k}^{*}) \mathbf{G}_{k}^{I*} + \sum_{k=1}^{s} \sum_{h=1}^{2} (\boldsymbol{\eta}_{k} - \boldsymbol{\eta}_{k}^{*})' \nabla \ell_{j}^{II,h} (\boldsymbol{\eta}_{k}^{*}) \mathbf{G}_{k}^{II,h*},$$
(S7)

and

$$\mathbf{R}_{j} = \sum_{i=1}^{r} \nabla \ell_{j}^{I}(\lambda_{k}^{*})(\lambda_{k} - \lambda_{k}^{*})(\mathbf{G}_{k}^{I} - \mathbf{G}_{k}^{I*})
+ \sum_{i=1}^{s} \sum_{h=1}^{2} (\boldsymbol{\eta}_{k} - \boldsymbol{\eta}_{k}^{*})' \nabla \ell_{j}^{II,h}(\boldsymbol{\eta}_{k}^{*})(\mathbf{G}_{k}^{II,h} - \mathbf{G}_{k}^{II,h*})
+ \frac{1}{2} \sum_{k=1}^{r} \nabla^{2} \ell_{j}^{I}(\widetilde{\lambda}_{k})(\lambda_{k} - \lambda_{k}^{*})^{2} \mathbf{G}_{k}^{I}
+ \frac{1}{2} \sum_{k=1}^{s} \sum_{h=1}^{2} (\boldsymbol{\eta}_{k} - \boldsymbol{\eta}_{k}^{*})' \nabla^{2} \ell_{j}^{II,h}(\widetilde{\boldsymbol{\eta}}_{k})(\boldsymbol{\eta}_{k} - \boldsymbol{\eta}_{k}^{*}) \mathbf{G}_{k}^{II,h}.$$
(S8)

We first handle the terms in \mathbf{R}_j , and denote $\mathbf{R}_j = \mathbf{R}_{1j} + \mathbf{R}_{2j} + \mathbf{R}_{3j}$, where

$$\mathbf{R}_{1j} = \sum_{k=1}^{r} \nabla \ell_{j}^{I}(\lambda_{k}^{*})(\lambda_{k} - \lambda_{k}^{*})(\mathbf{G}_{k}^{I} - \mathbf{G}_{k}^{I*})
+ \sum_{k=1}^{s} \sum_{h=1}^{2} (\boldsymbol{\eta}_{k} - \boldsymbol{\eta}_{k}^{*})' \nabla \ell_{j}^{II,h}(\boldsymbol{\eta}_{k}^{*})(\mathbf{G}_{k}^{II,h} - \mathbf{G}_{k}^{II,h*}),
\mathbf{R}_{2j} = \frac{1}{2} \sum_{k=1}^{r} \nabla^{2} \ell_{j}^{I}(\widetilde{\lambda}_{k})(\lambda_{k} - \lambda_{k}^{*})^{2}(\mathbf{G}_{k}^{I} - \mathbf{G}_{k}^{I*})
+ \frac{1}{2} \sum_{k=1}^{s} \sum_{h=1}^{2} (\boldsymbol{\eta}_{k} - \boldsymbol{\eta}_{k}^{*})' \nabla^{2} \ell_{j}^{II,h}(\widetilde{\boldsymbol{\eta}}_{k})(\boldsymbol{\eta}_{k} - \boldsymbol{\eta}_{k}^{*})(\mathbf{G}_{k}^{II,h} - \mathbf{G}_{k}^{II,h*}),
\mathbf{R}_{3j} = \frac{1}{2} \sum_{k=1}^{r} \nabla^{2} \ell_{j}^{I}(\widetilde{\lambda}_{k})(\lambda_{k} - \lambda_{k}^{*})^{2} \mathbf{G}_{k}^{I*}
+ \frac{1}{2} \sum_{k=1}^{s} \sum_{h=1}^{2} (\boldsymbol{\eta}_{k} - \boldsymbol{\eta}_{k}^{*})' \nabla^{2} \ell_{j}^{II,h}(\widetilde{\boldsymbol{\eta}}_{k})(\boldsymbol{\eta}_{k} - \boldsymbol{\eta}_{k}^{*}) \mathbf{G}_{k}^{II,h*}.$$
(S9)

Note that for any matrix $\boldsymbol{Y} = \sum_{k=1}^{d} a_k \boldsymbol{X}_k$, it holds

$$\|\boldsymbol{Y}\|_{\text{op}} \leqslant \|\boldsymbol{Y}\|_{\text{F}} \leqslant (\sum_{k=1}^{d} \|\boldsymbol{X}_{k}\|_{\text{F}}^{2})^{1/2} (\sum_{k=1}^{d} a_{k}^{2})^{1/2} = \|\boldsymbol{X}\|_{\text{F}} \|\boldsymbol{a}\|_{2},$$

and $\sum_{k=1}^{d} a_k^4 \leq (\sum_{k=1}^{d} a_k^2)^2$, where $\boldsymbol{a} = (a_1, \dots, a_d)' \in \mathbb{R}^d$, and $\boldsymbol{\mathfrak{X}}$ is a tensor with frontal slices \boldsymbol{X}_k 's such that $\boldsymbol{\mathfrak{X}}_{(1)} = (\boldsymbol{X}_1, \dots, \boldsymbol{X}_d)$. Then, by Lemma S.1(i),

$$\|\boldsymbol{R}_{1j}\|_{\mathrm{F}} \leqslant C_L \bar{\rho}^j \sqrt{\|\boldsymbol{\lambda} - \boldsymbol{\lambda}^*\|_2^2 + 2\|\boldsymbol{\eta} - \boldsymbol{\eta}^*\|_2^2} \cdot \|\boldsymbol{G}^{\mathrm{MA}} - \boldsymbol{G}^{\mathrm{MA}*}\|_{\mathrm{F}}$$

$$\leqslant \sqrt{2} C_L \bar{\rho}^j \delta_{\boldsymbol{\omega}} \cdot \|\boldsymbol{G}^{\mathrm{MA}} - \boldsymbol{G}^{\mathrm{MA}*}\|_{\mathrm{F}} \leqslant \sqrt{2} C_L \bar{\rho}^j \delta_{\boldsymbol{\omega}} \delta_{\mathrm{G}},$$

and

$$\|\boldsymbol{R}_{2j}\|_{F} \leq \frac{\sqrt{2}}{2} C_{L} \bar{\rho}^{j} \delta_{\boldsymbol{\omega}}^{2} \sqrt{\sum_{k=1}^{r} \|\boldsymbol{G}_{k}^{I} - \boldsymbol{G}_{k}^{I*}\|_{F}^{2} + \sum_{k=1}^{s} \sum_{h=1}^{2} \|\boldsymbol{G}_{k}^{II,h} - \boldsymbol{G}_{k}^{II,h*}\|_{F}^{2}}$$

$$\leq \frac{\sqrt{2}}{2} C_{L} \bar{\rho}^{j} \delta_{\boldsymbol{\omega}}^{2} \cdot \|\boldsymbol{G}^{MA} - \boldsymbol{G}^{MA*}\|_{F} \leq \frac{\sqrt{2}}{2} C_{L} \bar{\rho}^{j} \delta_{\boldsymbol{\omega}}^{2} \delta_{S}.$$

Moreover, by Assumption 2(iii) and Lemma S.1(i), we can show that

$$\|\mathbf{R}_{3j}\|_{\mathrm{F}} \leqslant C_L \alpha \bar{\rho}^j \delta_{\boldsymbol{\omega}}^2.$$

As a result,

$$\|\mathbf{R}_{j}\|_{F} \leq \|\mathbf{R}_{1j}\|_{F} + \|\mathbf{R}_{2j}\|_{F} + \|\mathbf{R}_{3j}\|_{F}$$

$$\leq C_{L}\bar{\rho}^{j}\delta_{\omega} \left(\sqrt{2}\delta_{g} + \frac{\sqrt{2}}{2}\delta_{\omega}\delta_{g} + \alpha\delta_{\omega}\right). \tag{S10}$$

Now consider \mathbf{H}_j in (S7). Notice that for any $j \ge 1$ and $1 \le k \le s$,

$$\nabla_{\gamma} \ell_{j}^{II,1}(\boldsymbol{\eta}_{k}) = j \gamma_{k}^{j-1} \cos(j\theta_{k}) = \frac{1}{\gamma_{k}} \nabla_{\theta} \ell_{j}^{II,2}(\boldsymbol{\eta}_{k}),$$

$$\nabla_{\gamma} \ell_{j}^{II,2}(\boldsymbol{\eta}_{k}) = j \gamma_{k}^{j-1} \sin(j\theta_{k}) = -\frac{1}{\gamma_{k}} \nabla_{\theta} \ell_{j}^{II,1}(\boldsymbol{\eta}_{k}).$$
(S11)

Thus, the last term on the right side of (S7) can be simplified to

$$\sum_{k=1}^{s} \sum_{h=1}^{2} (\boldsymbol{\eta}_{k} - \boldsymbol{\eta}_{k}^{*})' \nabla \ell_{j}^{II,h} (\boldsymbol{\eta}_{k}^{*}) \boldsymbol{G}_{k}^{II,h*}
= \sum_{k=1}^{s} \left[(\theta_{k} - \theta_{k}^{*}) \boldsymbol{G}_{k}^{II,1*} - \frac{1}{\gamma_{k}^{*}} (\gamma_{k} - \gamma_{k}^{*}) \boldsymbol{G}_{k}^{II,2*} \right] \nabla_{\theta} \ell_{j}^{II,1} (\boldsymbol{\eta}_{k}^{*})
+ \sum_{k=1}^{s} \left[(\theta_{k} - \theta_{k}^{*}) \boldsymbol{G}_{k}^{II,2*} + \frac{1}{\gamma_{k}^{*}} (\gamma_{k} - \gamma_{k}^{*}) \boldsymbol{G}_{k}^{II,1*} \right] \nabla_{\theta} \ell_{j}^{II,2} (\boldsymbol{\eta}_{k}^{*}).$$
(S12)

Let $\mathcal{H} = \operatorname{stack}(\boldsymbol{H}_1, \boldsymbol{H}_2, \dots)$ and $\mathcal{R} = \operatorname{stack}(\boldsymbol{R}_1, \boldsymbol{R}_2, \dots)$. Then by (S7) and (S12), it can be

verified that

$$\widetilde{\mathcal{H}} := \operatorname{stack}(\boldsymbol{G}_{1} - \boldsymbol{G}_{1}^{*}, \cdots, \boldsymbol{G}_{p} - \boldsymbol{G}_{p}^{*}, \boldsymbol{\mathcal{H}})
= (\boldsymbol{G} - \boldsymbol{G}^{*}) \times_{3} \boldsymbol{L}(\boldsymbol{\omega}^{*}) + \boldsymbol{\mathcal{D}}(\boldsymbol{\omega}) \times_{3} (\nabla \boldsymbol{L}^{I}(\boldsymbol{\lambda}^{*}), \nabla_{\boldsymbol{\theta}} \boldsymbol{L}^{II}(\boldsymbol{\eta}^{*}))
= \boldsymbol{G}_{\operatorname{stack}} \times_{3} \boldsymbol{L}_{\operatorname{stack}}(\boldsymbol{\omega}^{*}),$$
(S13)

where $\mathbf{D}(\boldsymbol{\omega}) \in \mathbb{R}^{N \times N \times (r+2s)}$ and $\mathbf{G}_{\text{stack}} \in \mathbb{R}^{N \times N \times (d+r+2s)}$ are defined in Appendix S4.2. Note that

$$\Delta = \widetilde{\mathcal{H}} + \operatorname{stack}(\mathbf{0}_{N \times N \times p}, \mathcal{R}). \tag{S14}$$

Moreover,

$$\|\mathbf{D}(\boldsymbol{\omega})\|_{F}^{2} = \sum_{i=1}^{r} (\lambda_{i} - \lambda_{i}^{*})^{2} \|\boldsymbol{G}_{i}^{I*}\|_{F}^{2} + \sum_{k=1}^{s} \left\| (\theta_{k} - \theta_{k}^{*}) \boldsymbol{G}_{k}^{II,1*} - \frac{\gamma_{k} - \gamma_{k}^{*}}{\gamma_{k}^{*}} \boldsymbol{G}_{k}^{II,2*} \right\|_{F}^{2}$$

$$+ \sum_{k=1}^{s} \left\| (\theta_{k} - \theta_{k}^{*}) \boldsymbol{G}_{k}^{II,2*} + \frac{\gamma_{k} - \gamma_{k}^{*}}{\gamma_{k}^{*}} \boldsymbol{G}_{k}^{II,1*} \right\|_{F}^{2}$$

$$= \sum_{i=1}^{r} (\lambda_{i} - \lambda_{i}^{*})^{2} \|\boldsymbol{G}_{i}^{I*}\|_{F}^{2} + \sum_{k=1}^{s} (\theta_{k} - \theta_{k}^{*})^{2} (\|\boldsymbol{G}_{k}^{II,1*}\|_{F}^{2} + \|\boldsymbol{G}_{k}^{II,2*}\|_{F}^{2})$$

$$+ \sum_{k=1}^{s} \frac{(\gamma_{k} - \gamma_{k}^{*})^{2}}{\gamma_{k}^{*2}} (\|\boldsymbol{G}_{k}^{II,1*}\|_{F}^{2} + \|\boldsymbol{G}_{k}^{II,2*}\|_{F}^{2}),$$
(S15)

which, together with Assumption 2(iii), leads to

$$\sqrt{2}c_{9}\alpha\delta_{\omega} \leqslant \|\mathbf{\mathcal{D}}(\omega)\|_{F} \leqslant \frac{\sqrt{2}\alpha\delta_{\omega}}{\min_{1\leqslant k\leqslant s}\gamma_{k}^{*}}.$$
(S16)

By the simple inequalities $(|x| + |y|)/2 \le \sqrt{x^2 + y^2} \le |x| + |y|$, we have $0.5(\delta_g + \|\mathcal{D}(\boldsymbol{\omega})\|_F) \le \|\mathcal{G}_{\text{stack}}\|_F \le \delta_g + \|\mathcal{D}(\boldsymbol{\omega})\|_F$, and thus in view of (S16) we further have

$$0.5 \left(\delta_{g} + \sqrt{2} c_{g} \alpha \delta_{\omega} \right) \leq \|\mathbf{g}_{\text{stack}}\|_{F} \leq \delta_{g} + \frac{\sqrt{2} \alpha \delta_{\omega}}{\min_{1 \leq k \leq s} \gamma_{k}^{*}}, \tag{S17}$$

where $\delta_{\mathfrak{g}} = \|\mathbf{g} - \mathbf{g}^*\|_{F}$. By Lemma S.2, $\sigma_{\min,L} = \sigma_{\min}(\mathbf{L}_{\text{stack}}(\boldsymbol{\omega}^*)) > 0$. Then it follows from (S17) that

$$0.5\sigma_{\min,L}\left(\delta_{9} + \sqrt{2}c_{9}\alpha\delta_{\boldsymbol{\omega}}\right) \leqslant \|\widetilde{\boldsymbol{\mathcal{H}}}\|_{\mathrm{F}} \leqslant \sigma_{\max,L}\left(\delta_{9} + \frac{\sqrt{2}\alpha\delta_{\boldsymbol{\omega}}}{\min_{1\leqslant k\leqslant s}\gamma_{k}^{*}}\right).$$

Combining this with (S10), (S14), (S16), as well as the fact that $\|\mathbf{g}^{MA} - \mathbf{g}^{MA*}\|_{F} \leq \delta_{g}$, we have

$$\begin{split} \|\boldsymbol{\Delta}\|_{\mathrm{F}} &\leqslant \|\widetilde{\boldsymbol{\mathcal{H}}}\|_{\mathrm{F}} + \|\boldsymbol{\mathcal{R}}\|_{\mathrm{F}} \\ &\leqslant \left\{\sigma_{\mathrm{max},L} + \frac{\sqrt{2}C_L}{1-\bar{\rho}}\left(\delta_{\boldsymbol{\omega}} + \frac{\delta_{\boldsymbol{\omega}}^2}{2}\right)\right\}\delta_{\mathrm{g}} + \left(\frac{\sqrt{2}\sigma_{\mathrm{max},L}}{\min_{1\leqslant k\leqslant s}\gamma_k^*} + \frac{C_L}{1-\bar{\rho}}\delta_{\boldsymbol{\omega}}\right)\alpha \end{split}$$

and

$$\|\boldsymbol{\Delta}\|_{\mathrm{F}} \geqslant \|\widetilde{\boldsymbol{\mathcal{H}}}\|_{\mathrm{F}} - \|\boldsymbol{\mathcal{R}}\|_{\mathrm{F}}$$

$$\geqslant \left\{ 0.5\sigma_{\min,L} - \frac{\sqrt{2}C_L}{1-\bar{\rho}} \left(\delta_{\boldsymbol{\omega}} + \frac{\delta_{\boldsymbol{\omega}}^2}{2}\right) \right\} \delta_{9} + \left(\frac{c_{9}\sigma_{\min,L}}{\sqrt{2}} - \frac{C_L}{1-\bar{\rho}}\delta_{\boldsymbol{\omega}}\right) \alpha.$$

Thus, by taking

$$\delta_{\omega} \leqslant c_{\omega} = \min \left\{ 2, \frac{c_{\mathsf{g}}(1-\bar{\rho})\sigma_{\min,L}}{8\sqrt{2}C_{L}} \right\},$$

we can show that

$$c_{\Delta} \left(\delta_{\mathfrak{g}} + \alpha \delta_{\boldsymbol{\omega}} \right) \leqslant \| \boldsymbol{\Delta} \|_{\mathcal{F}} \leqslant C_{\Delta} \left(\delta_{\mathfrak{g}} + \alpha \delta_{\boldsymbol{\omega}} \right),$$

where

$$c_{\Delta} = c_l \cdot \sigma_{\min,L} \approx 1$$
 and $C_{\Delta} = c_u \cdot \max \left\{ \sigma_{\max,L}, (1 - \bar{\rho})^{-1} \right\} \approx 1$,

with $c_l = 0.25 \min\{1, \sqrt{2}c_9\}$ and $c_u = 1 + \sqrt{2}(\min_{1 \le k \le s} \gamma_k^*)^{-1} + (4\sqrt{2} + 2)C_L$. By Lemma S.2, we have $c_{\omega} = 1$, $c_{\Delta} = 1$, and $C_{\Delta} = 1$. The proof of this lemma is complete.

S4.4 Proof of Theorem 3

Let $\boldsymbol{x}_t = (\boldsymbol{y}_{t-1}', \boldsymbol{y}_{t-2}', \dots)'$. Denote by $\boldsymbol{\Delta}_j$ with $j \geqslant 1$ the frontal slices of $\boldsymbol{\Delta}$, i.e., $\boldsymbol{\Delta}_{(1)} = (\boldsymbol{\Delta}_1, \boldsymbol{\Delta}_2, \dots)$. Denote

$$S_{1}(\boldsymbol{\Delta}) = \frac{2}{T} \sum_{t=1}^{T} \langle \sum_{j=1}^{\infty} \boldsymbol{\Delta}_{j} \boldsymbol{y}_{t-j}, \sum_{k=t}^{\infty} \boldsymbol{\Delta}_{k} \boldsymbol{y}_{t-k} \rangle,$$

$$S_{2}(\boldsymbol{\Delta}) = \frac{2}{T} \sum_{t=1}^{T} \langle \sum_{j=t}^{\infty} \boldsymbol{A}_{j}^{*} \boldsymbol{y}_{t-j}, \sum_{k=1}^{t-1} \boldsymbol{\Delta}_{k} \boldsymbol{y}_{t-k} \rangle,$$

$$S_{3}(\boldsymbol{\Delta}) = \frac{2}{T} \sum_{t=1}^{T} \langle \boldsymbol{\varepsilon}_{t}, \sum_{j=t}^{\infty} \boldsymbol{\Delta}_{j} \boldsymbol{y}_{t-j} \rangle.$$
(S18)

The following three lemmas are sufficient for the proof of Theorem 3.

Lemma S.3 (Strong convexity and smoothness properties). Under Assumptions 1 and 2, if $T \gtrsim (\kappa_2/\kappa_1)^2 d_{\mathcal{R}} \log(\kappa_2/\kappa_1)$, then with probability at least $1 - 2e^{-cd_{\mathcal{R}} \log(\kappa_2/\kappa_1)} - 3e^{-cN}$,

$$\kappa_1 \|\mathbf{\Delta}\|_{\mathrm{F}}^2 \lesssim \frac{1}{T} \sum_{t=1}^T \|\mathbf{\Delta}_{(1)} \boldsymbol{x}_t\|_2^2 \lesssim \kappa_2 \|\mathbf{\Delta}\|_{\mathrm{F}}^2, \quad \forall \mathbf{\Delta} \in \boldsymbol{\Upsilon}.$$

Lemma S.4 (Deviation bound). Under the conditions of Lemma S.3, with probability at least $1 - 2e^{-cd_R \log(\kappa_2/\kappa_1)} - 5e^{-cN}$,

$$\left| \frac{1}{T} \left| \sum_{t=1}^{T} \langle \boldsymbol{\varepsilon}_t, \boldsymbol{\Delta}_{(1)} \boldsymbol{x}_t \rangle \right| \lesssim \sqrt{\frac{\kappa_2 \lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon}) d_{\mathcal{R}}}{T}} \|\boldsymbol{\Delta}\|_{\mathrm{F}}, \quad \forall \boldsymbol{\Delta} \in \boldsymbol{\Upsilon}.$$

Lemma S.5 (Effects of initial values). Under Assumptions 1 and 2, if $T \gtrsim (\kappa_2/\kappa_1)d_{\mathcal{R}}$, then with probability at least $1 - \{2 + \sqrt{\kappa_2/\lambda_{\max}(\Sigma_{\varepsilon})}\}\sqrt{N/\{(\mathcal{R}_1 + \mathcal{R}_2)T\}}$,

$$|S_1(\boldsymbol{\Delta})| \lesssim \kappa_1 \|\boldsymbol{\Delta}\|_{\mathrm{F}}^2, \quad |S_i(\boldsymbol{\Delta})| \lesssim \sqrt{\frac{\kappa_2 \lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon}) d_{\mathcal{R}}}{T}} \|\boldsymbol{\Delta}\|_{\mathrm{F}}, \quad i = 2, 3, \quad \forall \boldsymbol{\Delta} \in \boldsymbol{\Upsilon}.$$

Now we prove Theorem 3. Note that $\sum_{j=1}^{t-1} \mathbf{A}_j \mathbf{y}_{t-j} = \mathbf{A}_{(1)} \widetilde{\mathbf{x}}_t$. Due to the optimality of $\widehat{\mathbf{A}}$, we have

$$\sum_{t=1}^T \|oldsymbol{y}_t - oldsymbol{\mathcal{A}}^*_{(1)} \widetilde{oldsymbol{x}}_t - \widehat{oldsymbol{\Delta}}_{(1)} \widetilde{oldsymbol{x}}_t \|_2^2 \leqslant \sum_{t=1}^T \|oldsymbol{y}_t - oldsymbol{\mathcal{A}}^*_{(1)} \widetilde{oldsymbol{x}}_t \|_2^2,$$

Then, since $\boldsymbol{y}_t - \boldsymbol{\mathcal{A}}_{(1)}^* \widetilde{\boldsymbol{x}}_t = \boldsymbol{\varepsilon}_t + \sum_{j=t}^{\infty} \boldsymbol{A}_j^* \boldsymbol{y}_{t-j}$, it follows that

$$\frac{1}{T} \sum_{t=1}^{T} \|\widehat{\boldsymbol{\Delta}}_{(1)} \widetilde{\boldsymbol{x}}_{t}\|_{2}^{2} \leqslant \frac{2}{T} \sum_{t=1}^{T} \langle \boldsymbol{\varepsilon}_{t}, \widehat{\boldsymbol{\Delta}}_{(1)} \widetilde{\boldsymbol{x}}_{t} \rangle + \underbrace{\frac{2}{T} \sum_{t=1}^{T} \langle \sum_{j=t}^{\infty} \boldsymbol{A}_{j}^{*} \boldsymbol{y}_{t-j}, \widehat{\boldsymbol{\Delta}}_{(1)} \widetilde{\boldsymbol{x}}_{t} \rangle}_{S_{2}(\widehat{\boldsymbol{\Delta}})}$$

$$= \frac{2}{T} \sum_{t=1}^{T} \langle \boldsymbol{\varepsilon}_{t}, \widehat{\boldsymbol{\Delta}}_{(1)} \boldsymbol{x}_{t} \rangle + S_{2}(\widehat{\boldsymbol{\Delta}}) - S_{3}(\widehat{\boldsymbol{\Delta}}), \tag{S19}$$

where $S_2(\cdot)$ and $S_3(\cdot)$ are defined as in (S18), $\hat{\boldsymbol{\Delta}}_{(1)}\tilde{\boldsymbol{x}}_t = \sum_{k=1}^{t-1} \hat{\boldsymbol{\Delta}}_k \boldsymbol{y}_{t-k}$, and $\hat{\boldsymbol{\Delta}}_{(1)}\boldsymbol{x}_t = \sum_{k=1}^{\infty} \hat{\boldsymbol{\Delta}}_k \boldsymbol{y}_{t-k}$. Moreover, applying the inequality $\|\boldsymbol{a} - \boldsymbol{b}\|_2^2 \ge \|\boldsymbol{a}\|_2^2 - 2\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ with $\boldsymbol{a} = \hat{\boldsymbol{\Delta}}_{(1)}\boldsymbol{x}_t = \sum_{j=1}^{\infty} \hat{\boldsymbol{\Delta}}_j \boldsymbol{y}_{t-j}$ and $\boldsymbol{b} = \sum_{k=t}^{\infty} \hat{\boldsymbol{\Delta}}_k \boldsymbol{y}_{t-k}$, we can lower bound the left-hand side of (S19) to further obtain that

$$\frac{1}{T} \sum_{t=1}^{T} \|\widehat{\boldsymbol{\Delta}}_{(1)} \boldsymbol{x}_t\|_2^2 - S_1(\widehat{\boldsymbol{\Delta}}) \leqslant \frac{2}{T} \sum_{t=1}^{T} \langle \boldsymbol{\varepsilon}_t, \widehat{\boldsymbol{\Delta}}_{(1)} \boldsymbol{x}_t \rangle + S_2(\widehat{\boldsymbol{\Delta}}) - S_3(\widehat{\boldsymbol{\Delta}}), \tag{S20}$$

where $S_1(\cdot)$ is defined as in (S18). It is worth pointing out that $S_i(\widehat{\Delta})$ for $1 \leq i \leq 3$ capture the initialization effect of $\mathbf{y}_s = \mathbf{0}$ for $s \leq 0$ on the estimation.

Note that $\widehat{\Delta} = \widehat{\mathcal{A}} - \mathcal{A}^* \in \Upsilon$ and $\kappa_2 \geqslant \kappa_1$. Suppose that the high probability events in Lemmas S.3–S.5 hold. Then we can derive the estimation error bound from (S20):

$$\kappa_1 \| \widehat{\Delta} \|_{\mathrm{F}}^2 \lesssim \sqrt{\frac{\kappa_2 \lambda_{\max}(\Sigma_{\varepsilon}) d_{\mathcal{R}}}{T}} \| \widehat{\Delta} \|_{\mathrm{F}}, \quad \text{or} \quad \| \widehat{\Delta} \|_{\mathrm{F}} \lesssim \sqrt{\frac{\kappa_2 \lambda_{\max}(\Sigma_{\varepsilon}) d_{\mathcal{R}}}{\kappa_1^2 T}}.$$

Furthermore, applying Lemma S.5 again, we can derive the prediction error bound from (S19) and the above result as follows:

$$\frac{1}{T} \sum_{t=1}^{T} \|\widehat{\boldsymbol{\Delta}}_{(1)} \widetilde{\boldsymbol{x}}_t\|_2^2 \lesssim \frac{\kappa_2 \lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon}) d_{\mathcal{R}}}{\kappa_1 T}.$$

The proof of this theorem is complete.

S4.5 Proof of Lemma S.1

Proof of (i): By definition, $\ell_j^I(\lambda_i) = \lambda_i^j$ for $1 \leq i \leq r$, and $\ell_j^{II,1}(\boldsymbol{\eta}_k) = \gamma_k^j \cos(j\theta_k)$ and $\ell_j^{II,2}(\boldsymbol{\eta}_k) = \gamma_k^j \sin(j\theta_k)$ for $1 \leq k \leq s$. Then the first-order derivatives are $\nabla \ell_j^I(\lambda_i) = j\lambda_i^{j-1}$, $\nabla_{\gamma} \ell_j^{II,1}(\boldsymbol{\eta}_k) = j\gamma_k^{j-1}\cos(j\theta_k)$, $\nabla_{\theta} \ell_j^{II,1}(\boldsymbol{\eta}_k) = -j\gamma_k^j \sin(j\theta_k)$, $\nabla_{\gamma} \ell_j^{II,2}(\boldsymbol{\eta}_k) = j\gamma_k^{j-1}\sin(j\theta_k)$, and $\nabla_{\theta} \ell_j^{II,2}(\boldsymbol{\eta}_k) = j\gamma_k^j \cos(j\theta_k)$. The second-order derivatives are $\nabla^2 \ell_j^I(\lambda_i) = j(j-1)\lambda_i^{j-2}$, $\nabla_{\gamma}^2 \ell_j^{II,1}(\boldsymbol{\eta}_k) = j(j-1)\gamma_k^{j-2}\cos(j\theta_k)$, $\nabla_{\gamma\theta}^2 \ell_j^{II,1}(\boldsymbol{\eta}_k) = -j^2\gamma_k^j \sin(j\theta_k)$, $\nabla_{\theta}^2 \ell_j^{II,1}(\boldsymbol{\eta}_k) = -j^2\gamma_k^j \cos(j\theta_k)$, $\nabla_{\gamma}^2 \ell_j^{II,2}(\boldsymbol{\eta}_k) = j(j-1)\gamma_k^{j-2}\sin(j\theta_k)$, $\nabla_{\gamma\theta}^2 \ell_j^{II,2}(\boldsymbol{\eta}_k) = j^2\gamma_k^{j-1}\cos(j\theta_k)$, and $\nabla_{\theta}^2 \ell_j^{II,2}(\boldsymbol{\eta}_k) = -j^2\gamma_k^j \sin(j\theta_k)$. By Assumption 2(i), there exists $\rho_1 > 0$ such that $\max\{|\lambda_1|, \dots, |\lambda_r|, \gamma_1, \dots, \gamma_s\} \leq \rho_1 < \bar{\rho}$. Thus,

$$\max_{1 \leq i \leq r, 1 \leq k \leq s, 1 \leq h \leq 2} \{ |\nabla \ell_j^I(\lambda_i)|, \|\nabla \ell_j^{II,h}(\boldsymbol{\eta}_k)\|_2, |\nabla^2 \ell_j^I(\lambda_i)|, \|\nabla^2 \ell_j^{II,h}(\boldsymbol{\eta}_k)\|_F \} \leqslant C_L \bar{\rho}^j.$$

by choosing C_L dependent on ρ_1 and $\bar{\rho}$ such that $C_L \ge 2j^2(\rho_1/\bar{\rho})^{j-2}\bar{\rho}^{-2}$ for all $j \ge 1$. Note that C_L exists and is an absolute constant.

Proof of (ii): By Assumption 2, $\max\{|\lambda_1^*|, \dots, |\lambda_r^*|, \gamma_1^*, \dots, \gamma_s^*\} \leq \bar{\rho}$, and there exists an absolute constant $C_9 > 0$ such that $\|G_k^*\|_{\text{op}} \leq C_9$ for all $1 \leq k \leq p$. Then, by a method similar to (S16),

we can show that $\|\boldsymbol{A}_{j}^{*}\|_{\text{op}} = \|\boldsymbol{G}_{j}^{*}\|_{\text{op}} \leqslant C_{9}$ for $1 \leqslant j \leqslant p$, and

$$\|\boldsymbol{A}_{j}^{*}\|_{\text{op}} \leq \sum_{k=p+1}^{d} |\ell_{j,k}(\boldsymbol{\omega}^{*})| \|\boldsymbol{G}_{k}^{*}\|_{\text{op}} \leq \bar{\rho}^{j-p} \sum_{k=p+1}^{d} \|\boldsymbol{G}_{j}^{*}\|_{\text{op}} \leq \bar{\rho}^{j-p} (r+2s) C_{9}$$

for $j \ge p+1$. Then, taking $C_* = \bar{\rho}^{-p} \max\{(r+2s)C_{\mathfrak{g}}, 1\}$, we accomplish the proof of (ii).

S4.6 Proof of Lemma S.2

Let J = 2(r+2s). Consider the following partitions of the $\infty \times (p+J)$ matrix $\boldsymbol{L}_{\text{stack}}(\boldsymbol{\omega})$:

$$m{L}_{
m stack}(m{\omega}) = \left(egin{array}{cc} m{I}_p & & \ & m{L}_{
m stack}^{
m MA}(m{\omega}) \end{array}
ight) = \left(m{I}_p & & \ & m{L}_{[1:J]}(m{\omega}) \ & m{L}_{
m Rem}(m{\omega}) \end{array}
ight),$$

where $\boldsymbol{L}_{\mathrm{stack}}^{\mathrm{MA}}(\boldsymbol{\omega}) = (\boldsymbol{L}^{I}(\boldsymbol{\lambda}), \boldsymbol{L}^{II}(\boldsymbol{\eta}), \nabla \boldsymbol{L}^{I}(\boldsymbol{\lambda}), \nabla_{\theta} \boldsymbol{L}^{II}(\boldsymbol{\eta}))$ is further partitioned into two blocks, the $J \times J$ block $\boldsymbol{L}_{[1:J]}(\boldsymbol{\omega})$ and the $\infty \times J$ remainder block $\boldsymbol{L}_{\mathrm{Rem}}(\boldsymbol{\omega})$. Note that for $1 \leqslant j \leqslant J$, the jth row of $\boldsymbol{L}_{[1:J]}(\boldsymbol{\omega})$ is

$$oldsymbol{L}_{j}(oldsymbol{\omega}) := \left(\left(oldsymbol{\ell}_{j}^{I}(oldsymbol{\lambda})
ight)', \left(oldsymbol{\ell}_{j}^{II}(oldsymbol{\eta})
ight)', \left(
abla oldsymbol{\ell}_{j}^{I}(oldsymbol{\lambda})
ight)', \left(
abla_{ heta}oldsymbol{\ell}_{j}^{II}(oldsymbol{\eta})
ight)'\right),$$

where $\boldsymbol{\ell}_{j}^{I}(\boldsymbol{\lambda}) = (\lambda_{1}^{j}, \dots, \lambda_{r}^{j})', \, \nabla \boldsymbol{\ell}_{j}^{I}(\boldsymbol{\lambda}) = (j\lambda_{1}^{j-1}, \dots, j\lambda_{r}^{j-1})', \, \text{and}$

$$\boldsymbol{\ell}_{j}^{II}(\boldsymbol{\eta}) = \left(\gamma_{1}^{j}\cos(j\theta_{1}), \gamma_{1}^{j}\sin(j\theta_{1}), \dots, \gamma_{s}^{j}\cos(j\theta_{s}), \gamma_{s}^{j}\sin(j\theta_{s})\right)',
\nabla_{\theta}\boldsymbol{\ell}_{j}^{II}(\boldsymbol{\eta}) = \left(-j\gamma_{1}^{j}\sin(j\theta_{1}), j\gamma_{1}^{j}\cos(j\theta_{1}), \dots, -j\gamma_{s}^{j}\sin(j\theta_{s}), j\gamma_{s}^{j}\cos(j\theta_{s})\right)'.$$

For $j \ge 1$, the jth row of $\mathbf{L}_{\text{Rem}}(\boldsymbol{\omega})$ is $\mathbf{L}_{J+j}(\boldsymbol{\omega})$.

By Lemma S.1(i), we have
$$\|\boldsymbol{L}_{\mathrm{stack}}^{\mathrm{MA}}(\boldsymbol{\omega})\|_{\mathrm{F}} \leqslant \sqrt{J\sum_{j=1}^{\infty}C_{L}^{2}\bar{\rho}^{2j}} \leqslant C_{L}\sqrt{J}\bar{\rho}(1-\bar{\rho})^{-1} = C_{\bar{\rho}}$$
. Then

$$\sigma_{\max}(\boldsymbol{L}_{\text{stack}}(\boldsymbol{\omega})) \leqslant \max\left\{1, \sigma_{\max}(\boldsymbol{L}_{\text{stack}}^{\text{MA}}(\boldsymbol{\omega}))\right\} \leqslant \max\left\{1, \|\boldsymbol{L}_{\text{stack}}^{\text{MA}}(\boldsymbol{\omega})\|_{\text{F}}\right\} \leqslant \max\{1, C_{\bar{\rho}}\}$$
 (S21)

and

$$\sigma_{\max}(\boldsymbol{L}_{[1:J]}(\boldsymbol{\omega})) \leqslant \|\boldsymbol{L}_{[1:J]}(\boldsymbol{\omega})\|_{F} \leqslant \|\boldsymbol{L}_{\text{stack}}^{\text{MA}}(\boldsymbol{\omega})\|_{F} \leqslant C_{\bar{\rho}}.$$
 (S22)

It remains to derive a lower bound of $\sigma_{\min}(\boldsymbol{L}_{\text{stack}}(\boldsymbol{\omega}))$. To this end, we first derive a lower bound of $\sigma_{\min}(\boldsymbol{L}_{[1:J]}(\boldsymbol{\omega}))$ by lower bounding the determinant of $\boldsymbol{L}_{[1:J]}(\boldsymbol{\omega})$. For any $(\gamma, \theta) \in [0, 1) \times (-\pi/2, \pi/2)$, it can be verified that

$$\left(\gamma^{j}\cos(j\theta), \gamma^{j}\sin(j\theta)\right)\underbrace{\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}}_{:=C_{1}} = \left((\gamma e^{i\theta})^{j}, (\gamma e^{-i\theta})^{j}\right)$$

and

$$\left(-j\gamma^{j}\sin(j\theta),j\gamma^{j}\cos(j\theta)\right)\underbrace{\begin{pmatrix}-i&i\\1&1\end{pmatrix}}_{:=\mathbf{C}_{2}} = \left(j(\gamma e^{i\theta})^{j},j(\gamma e^{-i\theta})^{j}\right).$$

Let $P_1 = \text{diag}(I_r, C_1, \dots, C_1, I_r, C_2, \dots, C_2)$ be a $J \times J$ block diagonal matrix consisting of two identity matrices I_r and s repeated blocks of C_1 and C_2 . We then have $\det(P_1) = (-2i)^{2s} = 4^s$, and

$$\boldsymbol{L}_{[1:J]}(\boldsymbol{\omega})\boldsymbol{P}_{1} = \begin{pmatrix} x_{1} & x_{2} & \cdots & x_{r+2s} & x_{1} & x_{2} & \cdots & x_{r+2s} \\ x_{1}^{2} & x_{2}^{2} & \cdots & x_{r+2s}^{2} & 2x_{1}^{2} & 2x_{2}^{2} & \cdots & 2x_{r+2s}^{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{1}^{J} & x_{2}^{J} & \cdots & x_{r+2s}^{J} & Jx_{1}^{J} & Jx_{2}^{J} & \cdots & Jx_{r+2s}^{J} \end{pmatrix} := \boldsymbol{P}_{2} \in \mathbb{R}^{J \times J},$$

where $x_i = \lambda_i$ for $1 \le i \le r$, while $x_{r+2k-1} = \gamma_k e^{i\theta_k}$ and $x_{r+2k} = \gamma_k e^{-i\theta_k}$ for $1 \le k \le s$, and i is the imaginary unit.

We subtract the jth column of P_2 from its (r + 2s + j)th column, for all $1 \le j \le r + 2s$, and obtain a matrix with the same determinant as P_2 as follows,

$$\boldsymbol{P}_{3} = \begin{pmatrix} x_{1} & x_{2} & \cdots & x_{r+2s} & 0 & 0 & \cdots & 0 \\ x_{1}^{2} & x_{2}^{2} & \cdots & x_{r+2s}^{2} & x_{1}^{2} & x_{2}^{2} & \cdots & x_{r+2s}^{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{1}^{J} & x_{2}^{J} & \cdots & x_{r+2s}^{J} & (J-1)x_{1}^{J} & (J-1)x_{2}^{J} & \cdots & (J-1)x_{r+2s}^{J} \end{pmatrix}.$$

Note that $P_3 = P_4 P_5$, where

$$\boldsymbol{P}_{4} = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ x_{1} & x_{2} & \cdots & x_{r+2s} & x_{1} & x_{2} & \cdots & x_{r+2s} \\ x_{1}^{2} & x_{2}^{2} & \cdots & x_{r+2s}^{2} & 2x_{1}^{2} & 2x_{2}^{2} & \cdots & 2x_{r+2s}^{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{1}^{J-1} & x_{2}^{J-1} & \cdots & x_{r+2s}^{J-1} & (J-1)x_{1}^{J-1} & (J-1)x_{2}^{J-1} & \cdots & (J-1)x_{r+2s}^{J-1} \end{pmatrix}$$

is a generalized Vandermonde matrix (Li and Tan, 2008), and $\mathbf{P}_5 = \operatorname{diag}\{x_1, \dots, x_{r+2s}, x_1, \dots, x_{r+2s}\}$. By Li and Tan (2008), $|\operatorname{det}(\mathbf{P}_4)| = \prod_{i=1}^{r+2s} x_i \prod_{1 \leq k < j \leq r+2s} (x_j - x_k)^4$. As a result,

$$|\det(\mathbf{P}_2)| = |\det(\mathbf{P}_3)| = |\det(\mathbf{P}_4)| |\det(\mathbf{P}_5)| = \prod_{j=1}^{r+2s} |x_j|^3 \prod_{1 \le j < k \le r+2s} (x_j - x_k)^4 \ge \nu_1^{3J/2} \nu_2^{J(J/2-1)},$$

where $\nu_1 = \min\{|x_k|, 1 \le k \le r + 2s\}$, $\nu_2 = \min\{|x_j - x_k|, 1 \le j < k \le r + 2s\}$, and J = 2(r + 2s). It follows that

$$|\det(\mathbf{L}_{[1:J]}(\boldsymbol{\omega}))| = \frac{|\det(\mathbf{P}_2)|}{|\det(\mathbf{P}_1)|} \ge 0.25^s \nu_1^{3J/2} \nu_2^{J(J/2-1)} > 0,$$
 (S23)

and hence $L_{[1:J]}(\omega)$ is full-rank. Moreover, combining (S22) and (S23), we have

$$\sigma_{\min}(\mathbf{L}_{[1:J]}(\boldsymbol{\omega})) \geqslant \frac{|\det(\mathbf{L}_{[1:J]}(\boldsymbol{\omega}))|}{\sigma_{\max}^{J-1}(\mathbf{L}_{[1:J]}(\boldsymbol{\omega}))} \geqslant \frac{0.25^{s} \nu_{1}^{3J/2} \nu_{2}^{J(J/2-1)}}{C_{\tilde{\rho}}^{J-1}}.$$
 (S24)

Note that the right side of (S24) when $\omega = \omega^*$ is $c_{\bar{\rho}}$, and Assumption 2(ii) ensures that $c_{\bar{\rho}} > 0$ is an absolute constant.

Finally, similar to (S21), by the Courant-Fischer theorem, it can be shown that

$$\sigma_{\min}(\boldsymbol{L}_{\mathrm{stack}}(\boldsymbol{\omega})) \geqslant \min\left\{1, \sigma_{\min}(\boldsymbol{L}_{\mathrm{stack}}^{\mathrm{MA}}(\boldsymbol{\omega}))\right\} \geqslant \min\left\{1, \sigma_{\min}(\boldsymbol{L}_{[1:J]}(\boldsymbol{\omega}))\right\},$$

which, together with (S24), leads to a lower bound of $\sigma_{\min}(\mathbf{L}_{\text{stack}}(\boldsymbol{\omega}))$. Applying this and the upper bound in (S21) to $\boldsymbol{\omega} = \boldsymbol{\omega}^*$ under Assumption 2(ii), we accomplish the proof of this lemma.

S4.7 Proof of Lemma S.3

It suffices to show that the event stated in this lemma holds uniformly over the intersection of Υ and the sphere $S(\delta) := \{ \Delta \in \mathbb{R}^{N \times N \times \infty} \mid ||\Delta||_F = \delta \}$, for some radius $\delta > 0$ to be chosen such that $\Upsilon \cap S(\delta) \neq \emptyset$, since the event will remain true if we multiply Δ by any nonzero real number.

Recall that any $\Delta \in \Upsilon$ can be written as $\Delta = \mathcal{A} - \mathcal{A}^* = \mathcal{G} \times_3 L(\omega) - \mathcal{A}^* := \Delta(\omega, \mathcal{G})$, for some $\mathcal{G} \in \mathbb{R}^{N \times N \times d}$ and $\omega \in \Omega$ dependent on Δ with $\|\omega - \omega^*\|_2 \leqslant c_{\omega}$. In addition, by (S13) and (S14), we can further write

$$\Delta = \mathcal{G}_{\text{stack}} \times_3 L_{\text{stack}}(\omega^*) + \text{stack}(\mathbf{0}_{N \times N \times p}, \mathcal{R}).$$

Note that throughout the proofs of Lemmas S.3–S.5, we will suppress the dependence of $\mathfrak{G}_{\text{stack}}$ and \mathfrak{R} on Δ . Thus,

$$\boldsymbol{\Delta}_{(1)}\boldsymbol{x}_t = \{\boldsymbol{\mathfrak{G}}_{\mathrm{stack}} \times_3 \boldsymbol{L}_{\mathrm{stack}}(\boldsymbol{\omega}^*)\}_{(1)}\boldsymbol{x}_t + \boldsymbol{\mathfrak{R}}_{(1)}\boldsymbol{x}_{t-p} = (\boldsymbol{\mathfrak{G}}_{\mathrm{stack}})_{(1)}\boldsymbol{z}_t + \boldsymbol{\mathfrak{R}}_{(1)}\boldsymbol{x}_{t-p},$$

where

$$\boldsymbol{x}_t = (\boldsymbol{y}_{t-1}', \boldsymbol{y}_{t-2}', \dots)'$$
 and $\boldsymbol{z}_t = \{\boldsymbol{L}_{\text{stack}}'(\boldsymbol{\omega}^*) \otimes \boldsymbol{I}_N\} \boldsymbol{x}_t.$ (S25)

For simplicity, denote $X = (x_T, ..., x_1)$, $X_{-p} = (x_{T-p}, ..., x_{1-p})$ and $Z = (z_T, ..., z_1)$. Then $\Delta_{(1)}X = (\mathfrak{G}_{\text{stack}})_{(1)}Z + \mathfrak{R}_{(1)}X_{-p}$, and hence $\|(\mathfrak{G}_{\text{stack}})_{(1)}Z\|_F - \|\mathfrak{R}_{(1)}X_{-p}\|_F \leqslant \|\Delta_{(1)}X\|_F \leqslant \|(\mathfrak{G}_{\text{stack}})_{(1)}Z\|_F + \|\mathfrak{R}_{(1)}X_{-p}\|_F$, i.e.,

$$\left(\sum_{t=1}^{T} \|\boldsymbol{\Delta}_{(1)}\boldsymbol{x}_{t}\|_{2}^{2}\right)^{1/2} \geqslant \left(\sum_{t=1}^{T} \|(\boldsymbol{\mathcal{G}}_{\text{stack}})_{(1)}\boldsymbol{z}_{t}\|_{2}^{2}\right)^{1/2} - \left(\sum_{t=1}^{T} \|\boldsymbol{\mathcal{R}}_{(1)}\boldsymbol{x}_{t-p}\|_{2}^{2}\right)^{1/2}$$
(S26)

and

$$\left(\sum_{t=1}^{T} \|\boldsymbol{\Delta}_{(1)}\boldsymbol{x}_{t}\|_{2}^{2}\right)^{1/2} \leqslant \left(\sum_{t=1}^{T} \|(\boldsymbol{\mathcal{G}}_{\text{stack}})_{(1)}\boldsymbol{z}_{t}\|_{2}^{2}\right)^{1/2} + \left(\sum_{t=1}^{T} \|\boldsymbol{\mathcal{R}}_{(1)}\boldsymbol{x}_{t-p}\|_{2}^{2}\right)^{1/2}. \tag{S27}$$

Now we restrict our attention to $\Delta \in \Upsilon \cap \mathcal{S}(\delta)$, where $\delta > 0$ will be specified later. If $\Delta \in \Upsilon \cap \mathcal{S}(\delta)$, since $\|\Delta\|_F = \delta$, then it follows from Lemma 1 that

$$\delta C_{\Delta}^{-1} \leqslant \delta_{S} \leqslant \delta c_{\Delta}^{-1} \quad \text{and} \quad \frac{\delta C_{\Delta}^{-1}}{\alpha} \leqslant \delta_{\omega} \leqslant \frac{\delta c_{\Delta}^{-1}}{\alpha},$$
 (S28)

where $\delta_{\omega} = \|\omega - \omega^*\|_2$ and $\delta_{\mathfrak{g}} = \|\mathfrak{g} - \mathfrak{g}^*\|_F$. To guarantee that $\Upsilon \cap \mathcal{S}(\delta) \neq \emptyset$, it is sufficient to

choose $\delta > 0$ such that

$$\frac{\delta c_{\Delta}^{-1}}{\alpha} \leqslant c_{\omega}. \tag{S29}$$

Furthermore, by (S17) and (S28), we can obtain the following bounds of $\|\mathcal{G}_{\text{stack}}\|_F$ by restricting the corresponding $\Delta \in \Upsilon \cap \mathcal{S}(\delta)$:

$$\sup_{\Delta \in \Upsilon \cap \mathcal{S}(\delta)} \|\mathbf{G}_{\text{stack}}\|_{F} \leq \delta c_{\Delta}^{-1} \left(1 + \frac{\sqrt{2}}{\min_{1 \leq k \leq s} \gamma_{k}^{*}} \right) \approx \delta,$$

$$\inf_{\Delta \in \Upsilon \cap \mathcal{S}(\delta)} \|\mathbf{G}_{\text{stack}}\|_{F} \geq 0.5 \delta C_{\Delta}^{-1} (1 + \sqrt{2}c_{g}) \approx \delta.$$
(S30)

Next we establish the following union bounds that hold for all $\Delta \in \Upsilon \cap \mathcal{S}(\delta)$:

(i) If $T \gtrsim (\kappa_2/\kappa_1)^2 d_{\mathcal{R}} \log(\kappa_2/\kappa_1)$, then

$$\mathbb{P}\left\{\forall \boldsymbol{\Delta} \in \boldsymbol{\Upsilon} \cap \mathcal{S}(\delta) : \frac{c_{\mathbf{M}}\delta^{2}\kappa_{1}}{8} \lesssim \frac{1}{T} \sum_{t=1}^{T} \|(\boldsymbol{9}_{\mathrm{stack}})_{(1)}\boldsymbol{z}_{t}\|_{2}^{2} \lesssim 6C_{\mathbf{M}}\delta^{2}\kappa_{2}\right\} \geqslant 1 - 2e^{-cd_{\mathcal{R}}\log(\kappa_{2}/\kappa_{1})}.$$

(ii) if $T \gtrsim N$, then

$$\mathbb{P}\left\{\sup_{\boldsymbol{\Delta}\in\boldsymbol{\Upsilon}\cap\mathcal{S}(\delta)}\frac{1}{T}\sum_{t=1}^{T}\|\boldsymbol{\mathcal{R}}_{(1)}\boldsymbol{x}_{t-p}\|_{2}^{2}\lesssim\delta^{2}\delta_{\boldsymbol{\omega}}^{2}\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})\mu_{\max}(\boldsymbol{\Psi}_{*})\right\}\geqslant1-3e^{-N\log9}.$$

Proof of (i): Note that by (S30), for every $\Delta \in \Upsilon \cap \mathcal{S}(\delta)$, its corresponding $\mathcal{G}_{\text{stack}}$ has a bounded Frobenius norm in the order of δ . This relationship between Δ and $\mathcal{G}_{\text{stack}}$ allows us to convert the problem of finding union bounds over all $\Delta \in \Upsilon \cap \mathcal{S}(\delta)$ to that over all $\mathcal{G}_{\text{stack}}$ with Frobenius norm in the order of δ .

Moreover, note that for all $\Delta \in \Upsilon$, it holds $\mathcal{G}_{\text{stack}} \in \Xi$, where Ξ is defined as in (S5), and hence

$$\frac{\mathcal{G}_{\text{stack}}}{\|\mathcal{G}_{\text{stack}}\|_{F}} \in \Xi_{1} = \{ \mathcal{M} \in \Xi \mid \|\mathcal{M}\|_{F} = 1 \}. \tag{S31}$$

Then, by taking $\mathbf{M} = \mathbf{g}_{\text{stack}}/\|\mathbf{g}_{\text{stack}}\|_{\text{F}}$, it follows directly from (S44) in Lemma S.6 that, if $T \gtrsim (\kappa_2/\kappa_1)^2 d_{\mathcal{R}} \log(\kappa_2/\kappa_1)$,

$$\mathbb{P}\left\{\forall \boldsymbol{\Delta} \in \boldsymbol{\Upsilon} : \frac{c_{\mathbf{M}}\kappa_{1}}{8}\|\boldsymbol{\mathcal{G}}_{\mathrm{stack}}\|_{\mathrm{F}}^{2} \leqslant \frac{1}{T}\sum_{t=1}^{T}\|(\boldsymbol{\mathcal{G}}_{\mathrm{stack}})_{(1)}\boldsymbol{z}_{t}\|_{2}^{2} \leqslant 6C_{\mathbf{M}}\kappa_{2}\|\boldsymbol{\mathcal{G}}_{\mathrm{stack}}\|_{\mathrm{F}}^{2}\right\} \geqslant 1 - 2e^{-cd_{\mathcal{R}}\log(\kappa_{2}/\kappa_{1})}.$$

Thus, by restricting our attention to $\Delta \in \Upsilon \cap \mathcal{S}(\delta)$ and combining (S30) with the above result, we can obtain (i).

Proof of (ii): First note that $\mathcal{R}_{(1)}x_{t-p} = \sum_{j=1}^{\infty} \mathbf{R}_j \mathbf{y}_{t-p-j}$, where $\mathcal{R} = \operatorname{stack}(\mathbf{R}_1, \mathbf{R}_2, \dots)$. For all $\Delta \in \Upsilon \cap \mathcal{S}(\delta)$, it follows from (S28) and the choice of δ in (S29) that

$$\delta_{\mathcal{S}} \leqslant \delta c_{\Delta}^{-1}, \quad \alpha \delta_{\omega} \leqslant \delta c_{\Delta}^{-1}, \quad \text{and} \quad \delta_{\omega} \leqslant c_{\omega}.$$
 (S32)

Combining the above bounds with (S10), we have

$$\|\mathbf{R}_{j}\|_{F} \leq \|\mathbf{R}_{1j}\|_{F} + \|\mathbf{R}_{2j}\|_{F} + \|\mathbf{R}_{3j}\|_{F} \leq \delta C_{\mathcal{R}} \bar{\rho}^{j} \delta_{\omega},$$
 (S33)

where $C_{\mathcal{R}} = C_L c_{\Delta}^{-1} (\sqrt{2} + \sqrt{2} c_{\omega}/2 + 1) \approx 1$. Then

$$\frac{1}{T} \sum_{t=1}^{T} \|\mathbf{\mathcal{R}}_{(1)} \mathbf{x}_{t-p}\|_{2}^{2} = \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mathbf{y}'_{t-p-i} \mathbf{R}'_{i} \mathbf{R}_{j} \mathbf{y}_{t-p-j}
= \frac{1}{T} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\sum_{t=1}^{T} \langle \mathbf{R}_{i} \mathbf{y}_{t-p-i}, \mathbf{R}_{j} \mathbf{y}_{t-p-j} \rangle \right)
\leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\frac{1}{T} \sum_{t=1}^{T} \|\mathbf{R}_{j} \mathbf{y}_{t-p-i}\|_{2}^{2} \right)^{1/2} \left(\frac{1}{T} \sum_{t=1}^{T} \|\mathbf{R}_{j} \mathbf{y}_{t-p-j}\|_{2}^{2} \right)^{1/2}
= \left\{ \sum_{j=1}^{\infty} \left(\frac{1}{T} \sum_{t=1}^{T} \|\mathbf{R}_{j} \mathbf{y}_{t-p-j}\|_{2}^{2} \right)^{1/2} \right\}^{2}.$$
(S34)

In addition, we can show that

$$\frac{1}{T}\sum_{t=1}^{T}\|\boldsymbol{R}_{j}\boldsymbol{y}_{t-p-j}\|_{2}^{2} = \operatorname{tr}\left\{\boldsymbol{R}_{j}\left(\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{y}_{t-p-j}\boldsymbol{y}_{t-p-j}'\right)\boldsymbol{R}_{j}'\right\} \leqslant \|\boldsymbol{R}_{j}\|_{F}^{2}\left\|\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{y}_{t-p-j}\boldsymbol{y}_{t-p-j}'\right\|_{\operatorname{op}}.$$

As a result,

$$\frac{1}{T} \sum_{t=1}^{T} \| \mathbf{\mathcal{R}}_{(1)} \mathbf{x}_{t-p} \|_{2}^{2} \leq \left(\sum_{j=1}^{\infty} \| \mathbf{R}_{j} \|_{F} \left\| \frac{1}{T} \sum_{t=1}^{T} \mathbf{y}_{t-p-j} \mathbf{y}'_{t-p-j} \right\|_{\text{op}}^{1/2} \right)^{2}.$$
 (S35)

Then, by (S33), (S35) and (S46) in Lemma S.7, if $T \gtrsim N$, with probability at least $1 - 3e^{-N\log 9}$,

we have

$$\sup_{\boldsymbol{\Delta} \in \boldsymbol{\Upsilon} \cap \mathcal{S}(\delta)} \frac{1}{T} \sum_{t=1}^{T} \|\boldsymbol{\mathcal{R}}_{(1)} \boldsymbol{x}_{t-p}\|_2^2 \lesssim \delta^2 \delta_{\boldsymbol{\omega}}^2 \lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon}) \mu_{\max}(\boldsymbol{\Psi}_*) \left(\sum_{j=1}^{\infty} \bar{\rho}^j \sqrt{j\sigma^2 + 1} \right)^2 \lesssim \delta^2 \delta_{\boldsymbol{\omega}}^2 \lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon}) \mu_{\max}(\boldsymbol{\Psi}_*),$$

where the second inequality follows from the fact that $\sum_{j=1}^{\infty} \bar{\rho}^j \sqrt{j\sigma^2 + 1} \approx 1$. Thus (ii) is verified.

Finally, it remains to combine (S26) and (S27) with the results in (i) and (ii). To ensure that the lower bound of $T^{-1} \sum_{t=1}^{T} \|(\mathbf{G}_{\text{stack}})_{(1)} \mathbf{z}_{t}\|_{2}^{2}$ in (i) dominates the upper bound of $T^{-1} \sum_{t=1}^{T} \|\mathbf{\mathcal{R}}_{(1)} \mathbf{x}_{t-p}\|_{2}^{2}$ in (ii), we only need $\delta_{\omega} \lesssim \sqrt{c_{\mathbf{M}} \kappa_{1}/\{\lambda_{\max}(\mathbf{\Sigma}_{\varepsilon})\mu_{\max}(\mathbf{\Psi}_{*})\}}$. In view of (S28), this can be guaranteed by choosing $\delta > 0$ such that

$$\frac{\delta c_{\Delta}^{-1}}{\alpha} \lesssim \sqrt{\frac{c_{\mathcal{M}} \kappa_1}{\lambda_{\max}(\Sigma_{\varepsilon}) \mu_{\max}(\Psi_*)}}.$$
 (S36)

Combining the above condition with (S29), we can obtain the desirable δ . Then we have

$$\mathbb{P}\left\{\forall \boldsymbol{\Delta} \in \boldsymbol{\Upsilon} \cap \mathcal{S}(\delta) : c_{\mathbf{M}} \delta^{2} \kappa_{1} \lesssim \frac{1}{T} \sum_{t=1}^{T} \|\boldsymbol{\Delta}_{(1)} \boldsymbol{x}_{t}\|_{2}^{2} \lesssim C_{\mathbf{M}} \delta^{2} \kappa_{2}\right\} \geqslant 1 - 2e^{-cd_{\mathcal{R}} \log(\kappa_{2}/\kappa_{1})} - 2e^{-N\log 9}.$$

Since $c_{\mathfrak{M}}$ and $C_{\mathfrak{M}}$ are absolute constants, and $\|\Delta\|_{\mathrm{F}} = \delta$ for all $\Delta \in \Upsilon \cap \mathcal{S}(\delta)$, the proof of this lemma is complete.

S4.8 Proof of Lemma S.4

Similar to Lemma S.3, it suffices to prove that the result of Lemma S.4 holds uniformly over the intersection of Υ and the sphere $S(\delta) := \{\Delta \in \mathbb{R}^{N \times N \times \infty} \mid \|\Delta\|_F = \delta\}$, where the radius $\delta > 0$ is chosen to satisfy condition (S29) to ensure that $\Upsilon \cap S(\delta) \neq \emptyset$. Specifically, we will prove that

$$\mathbb{P}\left\{\sup_{\boldsymbol{\Delta}\in\boldsymbol{\Upsilon}\cap\mathcal{S}(\delta)}\frac{1}{T}\left|\sum_{t=1}^{T}\langle\boldsymbol{\Delta}_{(1)}\boldsymbol{x}_{t},\boldsymbol{\varepsilon}_{t}\rangle\right|\lesssim\delta\sqrt{\frac{\kappa_{2}\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})d_{\mathcal{R}}}{T}}\right\}\geqslant1-2e^{-cd_{\mathcal{R}}\log(\kappa_{2}/\kappa_{1})}-5e^{-cN}.\tag{S37}$$

From the proof of Lemma S.3, for every $\Delta = \Delta(\omega, \mathcal{G}) \in \Upsilon \cap \mathcal{S}(\delta)$, the corresponding \mathcal{G}_{stack} and $\mathcal{R} = stack(\mathbf{R}_1, \mathbf{R}_2, \dots)$ satisfy

$$\|\mathbf{\mathcal{G}}_{\text{stack}}\|_{\text{F}} \lesssim \delta, \quad \frac{\mathbf{\mathcal{G}}_{\text{stack}}}{\|\mathbf{\mathcal{G}}_{\text{stack}}\|_{\text{F}}} \in \mathbf{\Xi}_{1}, \quad \text{and} \quad \|\mathbf{R}_{j}\|_{\text{F}} \leqslant \delta c_{\omega} C_{\Re} \bar{\rho}^{j} \quad \text{for all } j \geqslant 1;$$
 (S38)

see (S29)–(S33). Moreover, by the definition of \mathbf{R}_j in (S9), it can be verified that, for all $j \ge 1$,

$$\operatorname{colsp}(\boldsymbol{R}_j) \subseteq \operatorname{colsp}(\{\boldsymbol{\mathfrak{G}}_{(1)},\boldsymbol{\mathfrak{G}}_{(1)}^*\}) \quad \text{ and } \quad \operatorname{colsp}(\boldsymbol{R}_j') \subseteq \operatorname{colsp}(\{\boldsymbol{\mathfrak{G}}_{(2)},\boldsymbol{\mathfrak{G}}_{(2)}^*\}),$$

where $\operatorname{colsp}(\boldsymbol{U})$ represents the column space of a matrix \boldsymbol{U} , and $\operatorname{colsp}(\{\boldsymbol{U},\boldsymbol{V}\})$ represents the span of all column vectors of the matrices \boldsymbol{U} and \boldsymbol{V} . Since for any $\boldsymbol{\Delta} = \boldsymbol{\Delta}(\boldsymbol{\omega},\boldsymbol{\mathcal{G}}) \in \boldsymbol{\Upsilon}$ it holds $\operatorname{rank}(\boldsymbol{\mathcal{G}}_{(i)}) \leqslant \mathcal{R}_i$ and $\operatorname{rank}(\boldsymbol{\mathcal{G}}_{(i)}^*) \leqslant \mathcal{R}_i$ for i = 1, 2, we then have

$$\operatorname{rank}(\mathbf{R}_j) \leq 2(\mathcal{R}_1 \wedge \mathcal{R}_2) := 2\mathcal{R}_{\wedge}, \quad \text{for all } j \geqslant 1.$$
 (S39)

Note that $\Delta_{(1)} \boldsymbol{x}_t = (\boldsymbol{\mathcal{G}}_{\text{stack}})_{(1)} \boldsymbol{z}_t + \boldsymbol{\mathcal{R}}_{(1)} \boldsymbol{x}_{t-p}$ and $\boldsymbol{\mathcal{R}}_{(1)} \boldsymbol{x}_{t-p} = \sum_{j=1}^{\infty} \boldsymbol{R}_j \boldsymbol{y}_{t-p-j}$. As a result, by (S38) and (S39), we have

$$\sup_{\boldsymbol{\Delta} \in \boldsymbol{\Upsilon} \cap \mathcal{S}(\delta)} \frac{1}{T} \left| \sum_{t=1}^{T} \langle \boldsymbol{\Delta}_{(1)} \boldsymbol{x}_{t}, \boldsymbol{\varepsilon}_{t} \rangle \right| \leq \sup_{\boldsymbol{\Delta} \in \boldsymbol{\Upsilon} \cap \mathcal{S}(\delta)} \frac{1}{T} \left| \sum_{t=1}^{T} \langle (\boldsymbol{G}_{\text{stack}})_{(1)} \boldsymbol{z}_{t}, \boldsymbol{\varepsilon}_{t} \rangle \right| + \sum_{j=1}^{\infty} \sup_{\boldsymbol{\Delta} \in \boldsymbol{\Upsilon} \cap \mathcal{S}(\delta)} \frac{1}{T} \left| \sum_{t=1}^{T} \langle \boldsymbol{R}_{j} \boldsymbol{y}_{t-p-j}, \boldsymbol{\varepsilon}_{t} \rangle \right|$$

$$\leq \delta \sup_{\boldsymbol{M} \in \boldsymbol{\Xi}_{1}} \frac{1}{T} \sum_{t=1}^{T} \langle \boldsymbol{M}_{(1)} \boldsymbol{z}_{t}, \boldsymbol{\varepsilon}_{t} \rangle + \delta \sum_{j=1}^{\infty} \bar{\rho}^{j} \sup_{\boldsymbol{M} \in \boldsymbol{\Pi}(2\mathcal{R}_{\wedge})} \frac{1}{T} \sum_{t=1}^{T} \langle \boldsymbol{M} \boldsymbol{y}_{t-p-j}, \boldsymbol{\varepsilon}_{t} \rangle.$$

Applying (S82) and (S47) in Lemmas S.6 and S.7, respectively, we can show that

$$\sup_{\boldsymbol{\Delta} \in \boldsymbol{\Upsilon} \cap \mathcal{S}(\delta)} \frac{1}{T} \left| \sum_{t=1}^{T} \langle \boldsymbol{\Delta}_{(1)} \boldsymbol{x}_{t}, \boldsymbol{\varepsilon}_{t} \rangle \right| \lesssim \delta \sqrt{\frac{\kappa_{2} \lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon}) d_{\mathcal{R}}}{T}} + \delta \lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon}) \sqrt{\frac{2\mu_{\max}(\boldsymbol{\Psi}_{*}) N \mathcal{R}_{\wedge}}{T}} \sum_{j=1}^{\infty} \bar{\rho}^{j} (2j\sigma^{2} + 1)$$

$$\lesssim \delta \sqrt{\frac{\kappa_{2} \lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon}) d_{\mathcal{R}}}{T}}$$

with probability at least $1 - e^{-cd_{\mathcal{R}}} - 2e^{-cd_{\mathcal{R}}\log(\kappa_2/\kappa_1)} - 4e^{-N\log 9}$, where the second inequality follows from the fact that $d_{\mathcal{R}} \gtrsim N\mathcal{R}_{\wedge}$, $\kappa_2 \gtrsim \lambda_{\max}(\Sigma_{\varepsilon})\mu_{\max}(\Psi_*)$ and $\sum_{j=1}^{\infty} \bar{\rho}^j(2j\sigma^2 + 1) \approx 1$. Since $e^{-cd_{\mathcal{R}}} \leqslant e^{-cN}$, (S37) holds and the proof of this lemma is complete.

S4.9 Proof of Lemma S.5

Similar to Lemmas S.3 and S.3, it suffices to prove that the result of Lemma S.5 holds uniformly over the intersection of Υ and the sphere $S(\delta) := \{ \Delta \in \mathbb{R}^{N \times N \times \infty} \mid \|\Delta\|_F = \delta \}$, where the radius $\delta > 0$ is chosen to satisfy condition (S29) to ensure that $\Upsilon \cap S(\delta) \neq \emptyset$.

Recall from the proof of Lemma S.3 that for every $\Delta = \Delta(\omega, \mathfrak{G}) \in \Upsilon \cap \mathcal{S}(\delta)$,

$$\delta_{\mathfrak{g}} \leqslant \delta c_{\Delta}^{-1}, \quad \alpha \delta_{\omega} \leqslant \delta c_{\Delta}^{-1}, \quad \text{and} \quad \|\mathbf{R}_{j}\|_{F} \leqslant \delta c_{\omega} C_{\mathfrak{R}} \bar{\rho}^{j},$$

where $\delta_{\omega} = \|\omega - \omega^*\|_2$ and $\delta_9 = \|\mathbf{G} - \mathbf{G}^*\|_F$; see (S32) and (S33). Combining this with (S6), (S7), Lemma S.1(i) and Assumption 2(iii), we can show that

$$\|\boldsymbol{H}_i\|_{\mathrm{F}} \leqslant \|\boldsymbol{\ell}^{(j)}(\boldsymbol{\omega}^*)\|_2 \delta_{\mathrm{S}} + \|\nabla \boldsymbol{\ell}^{(j)}(\boldsymbol{\omega}^*)\|_2 \alpha \delta_{\boldsymbol{\omega}} \leqslant \delta c_{\Lambda}^{-1} (1 + C_L) \sqrt{r + 2s} \bar{\rho}^j,$$

where $\ell^{(j)}(\omega^*)$ represents the jth row of the matrix $L^{\text{MA}}(\omega) = (\ell^I(\lambda_1), \dots, \ell^I(\lambda_r), \ell^{II}(\eta_1), \dots, \ell^{II}(\eta_s))$, and further that

$$\|\Delta_{j+p}\|_{F} \leq \|H_{j}\|_{F} + \|R_{j}\|_{F} \leq \delta \left\{ c_{\Delta}^{-1} (1 + C_{L}) \sqrt{r + 2s} + c_{\omega} C_{\Re} \right\} \bar{\rho}^{j}, \quad j \geq 1.$$

Note that $\|\boldsymbol{\Delta}_j\|_{\mathrm{F}} = \|\boldsymbol{G}_j - \boldsymbol{G}_j^*\|_{\mathrm{F}} \leqslant \delta c_{\Delta}^{-1}$ for $1 \leqslant j \leqslant p$. Then, we have

$$\|\Delta_j\|_{\text{op}} \leq \|\Delta_j\|_{\text{F}} \leq \delta C_1 \bar{\rho}^j, \quad \text{for all } j \geq 1,$$
 (S40)

where $C_1 = \{c_{\Delta}^{-1}(1 + C_L)\sqrt{r + 2s} + c_{\omega}C_{\Re}\}\,\bar{\rho}^{-p} \approx 1.$

In addition, by Lemma S.11,

$$\mathbb{E}(\|\boldsymbol{y}_t\|_2^2) = \operatorname{tr}(\boldsymbol{\Sigma}_y) \leqslant N\lambda_{\max}(\boldsymbol{\Sigma}_y) \leqslant N\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})\mu_{\max}(\boldsymbol{\Psi}_*).$$

Combining this with Lemma S.1(ii) and (S40), for all $j \ge 1$, we have

$$\mathbb{E}(\|\boldsymbol{A}_{j}^{*}\boldsymbol{y}_{t-j}\|_{2}) \leqslant \left\{\mathbb{E}(\|\boldsymbol{A}_{j}^{*}\boldsymbol{y}_{t-j}\|_{2}^{2})\right\}^{1/2} \leqslant C_{*}\bar{\rho}^{j}\sqrt{\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})\mu_{\max}(\boldsymbol{\Psi}_{*})N}$$
(S41)

and

$$\mathbb{E}\left(\sup_{\boldsymbol{\Delta}\in\boldsymbol{\Upsilon}\cap\mathcal{S}(\delta)}\|\boldsymbol{\Delta}_{j}\boldsymbol{y}_{t-j}\|_{2}\right)\leqslant\left\{\mathbb{E}\left(\sup_{\boldsymbol{\Delta}\in\boldsymbol{\Upsilon}\cap\mathcal{S}(\delta)}\|\boldsymbol{\Delta}_{j}\boldsymbol{y}_{t-j}\|_{2}^{2}\right)\right\}^{1/2}\leqslant\delta C_{1}\bar{\rho}^{j}\sqrt{\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})\mu_{\max}(\boldsymbol{\Psi}_{*})N}.$$
(S42)

By the Cauchy-Schwarz inequality and (S42),

$$\mathbb{E}\left\{\sup_{\boldsymbol{\Delta}\in\boldsymbol{\Upsilon}\cap\mathcal{S}(\delta)}|S_1(\boldsymbol{\Delta})|\right\}\leqslant \frac{2}{T}\sum_{t=1}^T\sum_{j=1}^\infty\sum_{k=t}^\infty\delta^2C_1^2\bar{\rho}^{j+k}\lambda_{\max}(\boldsymbol{\Sigma}_\varepsilon)\mu_{\max}(\boldsymbol{\Psi}_*)N\leqslant \frac{\delta^2C_2\kappa_2N}{T},$$

where $C_2 = 2C_1^2 \bar{\rho}^2/(1-\bar{\rho})^3 \approx 1$. Similarly, by (S41) and (S42),

$$\mathbb{E}\left\{\sup_{\boldsymbol{\Delta}\in\boldsymbol{\Upsilon}\cap\mathcal{S}(\delta)}|S_2(\boldsymbol{\Delta})|\right\}\leqslant \frac{2}{T}\sum_{t=1}^T\sum_{j=t}^\infty\sum_{k=1}^{t-1}\delta C_*C_1\bar{\rho}^{j+k}\lambda_{\max}(\boldsymbol{\Sigma}_\varepsilon)\mu_{\max}(\boldsymbol{\Psi}_*)N\leqslant \frac{\delta C_3\kappa_2N}{T},$$

where $C_3 = 2C_*C_1\bar{\rho}^2/(1-\bar{\rho})^3 \approx 1$. Moreover, note that $\mathbb{E}(\|\boldsymbol{\varepsilon}_t\|_2) \leqslant \sqrt{\mathbb{E}(\|\boldsymbol{\varepsilon}_t\|_2^2)} \leqslant \sqrt{\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})N}$. Then by (S42) and a method similar to the above,

$$\mathbb{E}\left\{\sup_{\boldsymbol{\Delta}\in\boldsymbol{\Upsilon}\cap\mathcal{S}(\delta)}|S_3(\boldsymbol{\Delta})|\right\}\leqslant \frac{2}{T}\sum_{t=1}^T\sum_{j=t}^\infty\delta C_1\bar{\rho}^j\lambda_{\max}(\boldsymbol{\Sigma}_\varepsilon)\sqrt{\mu_{\max}(\boldsymbol{\Psi}_*)}N\leqslant \frac{\delta C_4\sqrt{\kappa_2\lambda_{\max}(\boldsymbol{\Sigma}_\varepsilon)}N}{T},$$

where $C_4 = 2C_1\bar{\rho}/(1-\bar{\rho})^2 \approx 1$. By Markov's inequality, we can show that

$$\mathbb{P}\left\{\sup_{\boldsymbol{\Delta}\in\boldsymbol{\Upsilon}\cap\mathcal{S}(\delta)}|S_1(\boldsymbol{\Delta})|\geqslant \delta^2C_2\kappa_1\right\}\leqslant \frac{\mathbb{E}\{\sup_{\boldsymbol{\Delta}\in\boldsymbol{\Upsilon}\cap\mathcal{S}(\delta)}|S_1(\boldsymbol{\Delta})|\}}{\delta^2C_2\kappa_1}\leqslant \frac{\kappa_2N}{\kappa_1T}\leqslant \sqrt{\frac{N}{(\mathcal{R}_1+\mathcal{R}_2)T}},\quad (S43)$$

$$\mathbb{P}\left\{\sup_{\boldsymbol{\Delta}\in\boldsymbol{\Upsilon}\cap\mathcal{S}(\delta)}|S_2(\boldsymbol{\Delta})|\geqslant \delta C_3\sqrt{\frac{\kappa_2\lambda_{\max}(\boldsymbol{\Sigma}_\varepsilon)d_{\mathcal{R}}}{T}}\right\}\leqslant \sqrt{\frac{\kappa_2N^2}{\lambda_{\max}(\boldsymbol{\Sigma}_\varepsilon)Td_{\mathcal{R}}}}\leqslant \sqrt{\frac{\kappa_2N}{\lambda_{\max}(\boldsymbol{\Sigma}_\varepsilon)(\mathcal{R}_1+\mathcal{R}_2)T}},$$

and

$$\mathbb{P}\left\{\sup_{\boldsymbol{\Delta}\in\boldsymbol{\Upsilon}\cap\mathcal{S}(\delta)}|S_3(\boldsymbol{\Delta})|\geqslant \delta C_4\sqrt{\frac{\kappa_2\lambda_{\max}(\boldsymbol{\Sigma}_\varepsilon)d_{\mathcal{R}}}{T}}\right\}\leqslant \sqrt{\frac{N^2}{Td_{\mathcal{R}}}}\leqslant \sqrt{\frac{N}{(\mathcal{R}_1+\mathcal{R}_2)T}},$$

where the last inequality in (S43) uses the condition that $T \gtrsim (\kappa_2/\kappa_1)d_{\mathcal{R}}$, and $d_{\mathcal{R}} = \mathcal{R}_1\mathcal{R}_2d + (\mathcal{R}_1 + \mathcal{R}_2)N$. Then the sum of the above three tail probabilities is $(2+\sqrt{\kappa_2/\lambda_{\max}(\Sigma_{\varepsilon})})\sqrt{N/\{(\mathcal{R}_1 + \mathcal{R}_2)T\}}$. Thus, we have proved the result of this lemma for all $\Delta \in \Upsilon \cap \mathcal{S}(\delta)$. Replacing δ by $\|\Delta\|_F$ in the above inequalities, we accomplish the proof for all $\Delta \in \Upsilon$.

S4.10 Auxiliary lemmas for the proofs of Lemmas S.3 and S.4

Lemmas S.3 and S.4 are established based on the auxiliary results below. In particular, Lemmas S.6 and S.7 are both used directly in the proofs of Lemmas S.3 and S.4. The covering and discretization results in Lemma S.8 play an essential role in the proof of Lemma S.6. The last

three lemmas are useful results for the proofs of both Lemmas S.6 and S.7: Lemmas S.9 and S.10 give high-probabilitity concentration and Hanson-Wright inequalities for stationary time series, respectively; and Lemma S.11 provides deterministic bounds for covariance matrices of stationary time series. As in (S31), let $\Xi_1 = \{ \mathbf{M} \in \Xi \mid \|\mathbf{M}\|_F = 1 \}$. Note that the following definition is used in Lemma S.8.

Definition 1 (Generalized ϵ -net of Ξ_1). For any $\epsilon > 0$, we say that $\bar{\Xi}(\epsilon)$ is a generalized ϵ -net of Ξ_1 if $\bar{\Xi}(\epsilon) \subset \Xi$, and for any $\mathbf{M}(\mathbf{a}, \mathbf{B}) \in \Xi_1$, there exists $\mathbf{M}(\bar{\mathbf{a}}, \bar{\mathbf{B}}) \in \bar{\Xi}(\epsilon)$ such that $\|\mathbf{M}(\mathbf{a}, \mathbf{B}) - \mathbf{M}(\bar{\mathbf{a}}, \bar{\mathbf{B}})\|_{\mathrm{F}} \leq \epsilon$. However, $\bar{\Xi}(\epsilon)$ is not required to be a subset of Ξ_1 ; that is, $\bar{\Xi}(\epsilon)$ may not be an ϵ -net of Ξ_1 .

Lemma S.6. Suppose that Assumptions 2 and 1 hold and $T \gtrsim (\kappa_2/\kappa_1)^2 d_{\mathcal{R}} \log(\kappa_2/\kappa_1)$. Let $\mathbf{z}_t = \{\mathbf{L}'_{\text{stack}}(\boldsymbol{\omega}^*) \otimes \mathbf{I}_N\} \mathbf{x}_t$ be defined as in (S25). Then

$$\mathbb{P}\left(\frac{c_{\mathbf{M}}\kappa_1}{8} \leqslant \inf_{\mathbf{M} \in \mathbf{\Xi}_1} \frac{1}{T} \sum_{t=1}^{T} \|\mathbf{M}_{(1)} \boldsymbol{z}_t\|_2^2 \leqslant \sup_{\mathbf{M} \in \mathbf{\Xi}_1} \frac{1}{T} \sum_{t=1}^{T} \|\mathbf{M}_{(1)} \boldsymbol{z}_t\|_2^2 \leqslant 6C_{\mathbf{M}}\kappa_2\right) \geqslant 1 - 2e^{-cd_{\mathcal{R}}\log(\kappa_2/\kappa_1)}. \quad (S44)$$

and

$$\mathbb{P}\left\{\sup_{\mathbf{M}\in\mathbf{\Xi}_{1}}\frac{1}{T}\sum_{t=1}^{T}\langle\mathbf{M}_{(1)}\boldsymbol{z}_{t},\boldsymbol{\varepsilon}_{t}\rangle\lesssim\sqrt{\frac{\kappa_{2}\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})d_{\mathcal{R}}}{T}}\right\}\geqslant1-e^{-cd_{\mathcal{R}}}-2e^{-cd_{\mathcal{R}}\log(\kappa_{2}/\kappa_{1})}.$$
(S45)

Lemma S.7. Suppose that Assumptions 2 and 1 hold and $T \gtrsim N$. Then

$$\mathbb{P}\left\{\forall j \geqslant 1 : \left\| \frac{1}{T} \sum_{t=1}^{T} \boldsymbol{y}_{t-p-j} \boldsymbol{y}'_{t-p-j} \right\|_{\text{op}} \leqslant 2\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon}) \mu_{\max}(\boldsymbol{\Psi}_{*}) (j\sigma^{2} + 1) \right\} \geqslant 1 - 3e^{-N\log 9} \qquad (S46)$$

and moreover, if $T \gtrsim N\mathcal{R}$,

$$\mathbb{P}\left\{\forall j \geqslant 1: \sup_{\boldsymbol{M} \in \boldsymbol{\Pi}(\mathcal{R})} \frac{1}{T} \sum_{t=1}^{T} \langle \boldsymbol{M} \boldsymbol{y}_{t-p-j}, \boldsymbol{\varepsilon}_{t} \rangle \leqslant 24\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})(2j\sigma^{2}+1)\sqrt{\frac{\mu_{\max}(\boldsymbol{\Psi}_{*})N\mathcal{R}}{T}}\right\} \geqslant 1 - 4e^{-N\mathcal{R}\log 9}.$$
(S47)

Lemma S.8 (Covering number and discretization for Ξ_1). For any $0 < \epsilon < 2/3$, let $\bar{\Xi}(\epsilon)$ be a minimal generalized ϵ -net of Ξ_1 .

(i) The cardinality of $\bar{\Xi}(\epsilon)$ satisfies

$$\log |\bar{\Xi}(\epsilon)| \lesssim d_{\mathcal{R}} \log(1/\epsilon),$$

where $d_{\mathcal{R}} = \mathcal{R}_1 \mathcal{R}_2 d + (\mathcal{R}_1 + \mathcal{R}_2) N$.

- (ii) There exist absolute constants $c_{\mathbf{M}}, C_{\mathbf{M}} > 0$ independent of ϵ such that for any $\mathbf{M} \in \bar{\Xi}(\epsilon)$, it holds $c_{\mathbf{M}} \leq \|\mathbf{M}\|_{\mathrm{F}} \leq C_{\mathbf{M}}$.
- (iii) For any $\mathbf{X} \in \mathbb{R}^{N \times N(d+r+2s)}$ and $\mathbf{Z} \in \mathbb{R}^{N(d+r+2s) \times T}$, it holds

$$\sup_{\mathbf{M} \in \mathbf{\Xi}_1} \langle \mathbf{M}_{(1)}, \mathbf{X} \rangle \leqslant (1 - 1.5\epsilon)^{-1} \max_{\mathbf{M} \in \mathbf{\Xi}(\epsilon)} \langle \mathbf{M}_{(1)}, \mathbf{X} \rangle, \tag{S48}$$

$$\sup_{\mathbf{M} \in \mathbf{\Xi}_1} \|\mathbf{M}_{(1)} \mathbf{Z}\|_{\mathrm{F}} \leqslant (1 - 1.5\epsilon)^{-1} \max_{\mathbf{M} \in \bar{\mathbf{\Xi}}(\epsilon)} \|\mathbf{M}_{(1)} \mathbf{Z}\|_{\mathrm{F}}.$$
 (S49)

In Lemmas S.9–S.11 below, we adopt notations as follows. Let $\{\boldsymbol{w}_t\}$ be a time series taking values in \mathbb{R}^M , where M is an arbitrary positive integer. If $\{\boldsymbol{w}_t\}$ is stationary with mean zero, then we denote the covariance matrix of \boldsymbol{w}_t by $\boldsymbol{\Sigma}_w = \mathbb{E}(\boldsymbol{w}_t \boldsymbol{w}_t')$. In addition, let $\underline{\boldsymbol{w}}_T = (\boldsymbol{w}_T', \dots, \boldsymbol{w}_1')'$, and denote its covariance matrix by

$$\underline{\Sigma}_w = \mathbb{E}(\underline{\boldsymbol{w}}_T \underline{\boldsymbol{w}}_T') = (\boldsymbol{\Sigma}_w(j-i))_{1 \leq i,j \leq T},$$

where $\Sigma_w(\ell) = \mathbb{E}(\boldsymbol{w}_t \boldsymbol{w}'_{t-\ell})$ is the lag- ℓ autocovariance matrix of \boldsymbol{w}_t for $\ell \in \mathbb{Z}$, and $\Sigma_w(0) = \Sigma_w$. For the time series $\{\boldsymbol{y}_t\}$, accordingly we define $\Sigma_y = \mathbb{E}(\boldsymbol{y}_t \boldsymbol{y}'_t)$ and $\underline{\Sigma}_y = \mathbb{E}(\underline{\boldsymbol{y}}_T \underline{\boldsymbol{y}}'_T) = (\Sigma_y(j-i))_{1 \leq i,j \leq T}$, where $\underline{\boldsymbol{y}}_T = (\boldsymbol{y}'_T, \dots, \boldsymbol{y}'_1)'$, $\Sigma_y(\ell) = \mathbb{E}(\boldsymbol{y}_t \boldsymbol{y}'_{t-\ell})$ is the lag- ℓ covariance matrix of \boldsymbol{y}_t for $\ell \in \mathbb{Z}$, and $\Sigma_y = \Sigma_y(0)$.

Lemma S.9 (Concentration bound for stationary time series). Suppose that Assumption 1 holds for $\{\varepsilon_t\}$, and $\{\boldsymbol{w}_t\}$ is a zero-mean stationary time series. Assume that \boldsymbol{w}_t is \mathscr{F}_{t-1} -measurable, where $\mathscr{F}_t = \sigma\{\varepsilon_t, \varepsilon_{t-1}, \ldots\}$ for $t \in \mathbb{Z}$ is a filtration. Then, for any a, b > 0, we have

$$\mathbb{P}\left\{\sum_{t=1}^{T} \langle \boldsymbol{w}_{t}, \boldsymbol{\varepsilon}_{t} \rangle \geqslant a, \sum_{t=1}^{T} \|\boldsymbol{w}_{t}\|^{2} \leqslant b\right\} \leqslant \exp\left\{-\frac{a^{2}}{2\sigma^{2}\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})b}\right\}.$$

Lemma S.10 (Hanson-Wright inequalities for stationary time series). Suppose that Assumption 1 holds for $\{\varepsilon_t\}$, and $\{w_t\}$ has the vector $MA(\infty)$ representation

$$oldsymbol{w}_t = \sum_{j=1}^\infty oldsymbol{\Psi}_j^w oldsymbol{arepsilon}_{t-j},$$

where $\Psi_j^w \in \mathbb{R}^{M \times N}$ for all j, and $\sum_{j=1}^{\infty} \|\Psi_j^w\|_{\text{op}} < \infty$. Let T_0 be a fixed integer.

(i) Then, $\{w_t\}$ is a zero-mean stationary time series, and

$$\mathbb{P}\left\{\left|\frac{1}{T}\sum_{t=T_0+1}^{T_0+T}\|\boldsymbol{w}_t\|_2^2 - \mathbb{E}\left(\|\boldsymbol{w}_t\|_2^2\right)\right| \geqslant \frac{M}{\sqrt{T}}\sigma^2\lambda_{\max}(\underline{\Sigma}_w)\right\} \leqslant 2e^{-cM}.$$

(ii) For any $\mathbf{M} \in \mathbb{R}^{Q \times M}$ with $Q \geqslant 1$ and any $\delta > 0$, it holds

$$\mathbb{P}\left\{\left|\frac{1}{T}\sum_{t=T_0+1}^{T_0+T}\|\boldsymbol{M}\boldsymbol{w}_t\|_2^2 - \mathbb{E}\left(\|\boldsymbol{M}\boldsymbol{w}_t\|_2^2\right)\right| \geqslant \delta\sigma^2\lambda_{\max}(\underline{\boldsymbol{\Sigma}}_w)\|\boldsymbol{M}\|_{\mathrm{F}}^2\right\} \leqslant 2e^{-c\min(\delta,\delta^2)T}.$$

Lemma S.11 (Deterministic bounds for covariance matrices of stationary time series). Suppose that Assumptions 2 and 1 hold, and the vector $MA(\infty)$ representation of $\{\boldsymbol{y}_t\}$ is $\boldsymbol{y}_t = \boldsymbol{\Psi}_*(B)\boldsymbol{\varepsilon}_t$, where $\boldsymbol{\Psi}_*(B) = \boldsymbol{I}_N + \sum_{j=1}^{\infty} \boldsymbol{\Psi}_j^* B^j$. Let $\mu_{\min}(\boldsymbol{\Psi}_*) = \min_{|z|=1} \lambda_{\min}(\boldsymbol{\Psi}_*(z)\boldsymbol{\Psi}_*^{\mathsf{H}}(z))$ and $\mu_{\max}(\boldsymbol{\Psi}_*) = \max_{|z|=1} \lambda_{\max}(\boldsymbol{\Psi}_*(z)\boldsymbol{\Psi}_*^{\mathsf{H}}(z))$, where $\boldsymbol{\Psi}_*^{\mathsf{H}}(z)$ is the conjugate transpose of $\boldsymbol{\Psi}_*(z)$.

(i) It holds

$$\lambda_{\min}(\Sigma_{\varepsilon})\mu_{\min}(\Psi_*) \leqslant \lambda_{\min}(\underline{\Sigma}_y) \leqslant \lambda_{\max}(\underline{\Sigma}_y) \leqslant \lambda_{\max}(\Sigma_{\varepsilon})\mu_{\max}(\Psi_*)$$

and

$$\lambda_{\min}(\Sigma_{\varepsilon})\mu_{\min}(\Psi_*) \leqslant \lambda_{\min}(\Sigma_y) \leqslant \lambda_{\max}(\Sigma_y) \leqslant \lambda_{\max}(\Sigma_{\varepsilon})\mu_{\max}(\Psi_*).$$

(ii) Define the time series $\{\boldsymbol{w}_t\}$ by $\boldsymbol{w}_t = \boldsymbol{W}\boldsymbol{x}_t = \sum_{i=1}^{\infty} \boldsymbol{W}_i \boldsymbol{y}_{t-i}$, where $\boldsymbol{x}_t = (\boldsymbol{y}_{t-1}', \boldsymbol{y}_{t-2}', \dots)'$, $\boldsymbol{W} = (\boldsymbol{W}_1, \boldsymbol{W}_2, \dots) \in \mathbb{R}^{M \times \infty}$, and \boldsymbol{W}_i 's are $M \times N$ blocks such that $\sum_{i=1}^{\infty} \|\boldsymbol{W}_i\|_{\text{op}} < \infty$. Then, $\{\boldsymbol{w}_t\}$ is a zero-mean stationary time series. Moreover,

$$\lambda_{\min}(\Sigma_{\varepsilon})\mu_{\min}(\Psi_{*})\sigma_{\min}^{2}(\boldsymbol{W}) \leqslant \lambda_{\min}(\Sigma_{w}) \leqslant \lambda_{\max}(\Sigma_{w}) \leqslant \lambda_{\max}(\Sigma_{\varepsilon})\mu_{\max}(\Psi_{*})\sigma_{\max}^{2}(\boldsymbol{W}) \quad (S50)$$

and

$$\lambda_{\max}(\underline{\Sigma}_w) \leqslant \lambda_{\max}(\Sigma_{\varepsilon})\mu_{\max}(\Psi_*) \left(\sum_{i=1}^{\infty} \|\boldsymbol{W}_i\|_{\text{op}}\right)^2.$$
 (S51)

S4.11 Proof of Lemma S.6

Proof of (S44): Denote $\boldsymbol{W} = \boldsymbol{L}'_{\text{stack}}(\boldsymbol{\omega}^*) \otimes \boldsymbol{I}_N$, and let $\boldsymbol{L}_{\text{stack}}^{(j)}(\boldsymbol{\omega}^*)$ be the jth row of $\boldsymbol{L}_{\text{stack}}(\boldsymbol{\omega}^*)$ for $j \geq 1$. Then $\boldsymbol{z}_t = \boldsymbol{W} \boldsymbol{x}_t = \sum_{j=1}^{\infty} \boldsymbol{W}_j \boldsymbol{y}_{t-j}$ and $\boldsymbol{W} = (\boldsymbol{W}_1, \boldsymbol{W}_2, \dots)$, where $\boldsymbol{x}_t = (\boldsymbol{y}'_{t-1}, \boldsymbol{y}'_{t-2}, \dots)'$

and $W_j = L_{\text{stack}}^{(j)}(\omega^*) \otimes I_N$ for $j \ge 1$. By the definition of $L_{\text{stack}}(\omega^*)$ and Lemma S.1(i), we have

$$\|\boldsymbol{L}_{\mathrm{stack}}^{(j)}(\boldsymbol{\omega}^*)\|_2 = 1$$
 for $1 \leq j \leq p$ and $\|\boldsymbol{L}_{\mathrm{stack}}^{(j)}(\boldsymbol{\omega}^*)\|_2 \leq C_L \sqrt{J}\bar{\rho}^j$ for $j \geq p+1$,

where J = 2(r + 2s). Thus,

$$\sum_{j=1}^{\infty} \|\boldsymbol{W}_j\|_{\text{op}} = \sum_{j=1}^{\infty} \|\boldsymbol{L}_{\text{stack}}^{(j)}(\boldsymbol{\omega}^*)\|_2 \leqslant C_{\bar{\rho}},$$

where $C_{\bar{\rho}} = C_L \sqrt{J} \bar{\rho} (1 - \bar{\rho})^{-1}$. In addition, by Lemma S.2,

$$\min\{1, c_{\bar{\rho}}\} \leqslant \sigma_{\min,L} = \sigma_{\min}(\boldsymbol{W}) \leqslant \sigma_{\max}(\boldsymbol{W}) = \sigma_{\max,L} \leqslant \max\{1, C_{\bar{\rho}}\}.$$

Then, by setting $\boldsymbol{w}_t = \boldsymbol{z}_t$, it follows from Lemma S.11(ii) that

$$\kappa_1 \leqslant \lambda_{\min}(\Sigma_w) \leqslant \lambda_{\max}(\Sigma_w) \leqslant \kappa_2$$
(S52)

and

$$\lambda_{\max}(\underline{\Sigma}_w) \leqslant \lambda_{\max}(\Sigma_{\varepsilon})\mu_{\min}(\Psi_*)C_{\bar{\varrho}}^2 \leqslant \kappa_2, \tag{S53}$$

where $\kappa_1 = \lambda_{\min}(\mathbf{\Sigma}_{\varepsilon})\mu_{\min}(\mathbf{\Psi}_*)\min\{1, c_{\bar{\rho}}^2\}$ and $\kappa_2 = \lambda_{\max}(\mathbf{\Sigma}_{\varepsilon})\mu_{\max}(\mathbf{\Psi}_*)\max\{1, C_{\bar{\rho}}^2\}$.

Furthermore, since $\mathbf{z}_t = \mathcal{W}(B)\mathbf{y}_t = \mathcal{W}(B)\mathbf{\Psi}_*(B)\boldsymbol{\varepsilon}_t$ is a zero-mean and stationary time series, where $\mathcal{W}(B) = \sum_{i=1}^{\infty} \mathbf{W}_i B^i$, we can apply Lemma S.10(ii) with $T_0 = 0$, $\mathbf{w}_t = \mathbf{z}_t$, and $\delta = \kappa_1/(2\sigma^2\kappa_2)$, in conjunction with (S52) and (S53), to obtain

$$\mathbb{P}\left\{\left|\frac{1}{T}\sum_{t=1}^{T}\|\mathbf{M}_{(1)}\boldsymbol{z}_{t}\|_{2}^{2}-\mathbb{E}\left(\|\mathbf{M}_{(1)}\boldsymbol{z}_{t}\|_{2}^{2}\right)\right|\geqslant0.5\kappa_{1}\|\mathbf{M}\|_{\mathrm{F}}^{2}\right\}\leqslant2e^{-c_{\sigma}(\kappa_{1}/\kappa_{2})^{2}T},$$

where $c_{\sigma} = c \min\{0.5\sigma^{-2}, 0.25\sigma^{-4}\}$. Note that by (S52) and $\boldsymbol{w}_t = \boldsymbol{z}_t$, we can show that

$$\kappa_1 \|\mathbf{\mathcal{M}}\|_{\mathrm{F}}^2 \leqslant \lambda_{\min}(\mathbf{\Sigma}_w) \|\mathbf{\mathcal{M}}\|_{\mathrm{F}}^2 \leqslant \mathbb{E}\left(\|\mathbf{\mathcal{M}}_{(1)}\boldsymbol{z}_t\|_2^2\right) \leqslant \lambda_{\max}(\mathbf{\Sigma}_w) \|\mathbf{\mathcal{M}}\|_{\mathrm{F}}^2 \leqslant \kappa_2 \|\mathbf{\mathcal{M}}\|_{\mathrm{F}}^2.$$

As a result, for any $\mathbf{M} \in \mathbb{R}^{N \times N \times (d+r+2s)}$, we have

$$\mathbb{P}\left(0.5\kappa_{1}\|\mathbf{M}\|_{F}^{2} \leqslant \frac{1}{T}\sum_{t=1}^{T}\|\mathbf{M}_{(1)}\boldsymbol{z}_{t}\|_{2}^{2} \leqslant 1.5\kappa_{2}\|\mathbf{M}\|_{F}^{2}\right) \geqslant 1 - 2e^{-c_{\sigma}(\kappa_{1}/\kappa_{2})^{2}T}.$$
 (S54)

Next we strengthen (S54) to union bounds that hold for all $\mathbf{M} \in \mathbf{\Xi}_1$. For simplicity, denote $\mathbf{Z} = (\mathbf{z}_T, \dots, \mathbf{z}_1)$, and then

$$\frac{1}{T} \sum_{t=1}^{T} \| \mathbf{M}_{(1)} \mathbf{z}_t \|_2^2 = \frac{1}{T} \| \mathbf{M}_{(1)} \mathbf{Z} \|_{\mathrm{F}}^2.$$

We consider a minimal generalized ϵ_0 -net $\bar{\Xi}(\epsilon_0)$ of Ξ_1 , where $0 < \epsilon_0 < 2/3$ will be chosen later. By Lemma S.8(ii), any $\mathbf{M} \in \bar{\Xi}(\epsilon_0)$ satisfies $c_{\mathbf{M}} \leq \|\mathbf{M}\|_{\mathrm{F}} \leq C_{\mathbf{M}}$. Define the event

$$\mathscr{E}(\epsilon_0) = \left\{ \forall \mathbf{M} \in \bar{\mathbf{\Xi}}(\epsilon_0) : \sqrt{0.5c_{\mathbf{M}}\kappa_1} < \frac{1}{\sqrt{T}} \|\mathbf{M}_{(1)}\mathbf{Z}\|_{\mathrm{F}} < \sqrt{1.5C_{\mathbf{M}}\kappa_2} \right\}.$$

Then, by the pointwise bounds in (S54) and the covering number in Lemma S.8(i), we have

$$\mathbb{P}\{\mathscr{E}^{\complement}(\epsilon_{0})\} \leqslant e^{Cd_{\mathcal{R}}\log(1/\epsilon_{0})} \max_{\mathbf{M} \in \tilde{\Xi}(\epsilon_{0})} \mathbb{P}\left[\left\{0.5c_{\mathbf{M}}\kappa_{1} \leqslant \frac{1}{T} \|\mathbf{M}_{(1)}\mathbf{Z}\|_{F}^{2} \leqslant 1.5C_{\mathbf{M}}\kappa_{2}\right\}^{\complement}\right]
\leqslant 2 \exp\left\{-c_{\sigma}(\kappa_{1}/\kappa_{2})^{2}T + Cd_{\mathcal{R}}\log(1/\epsilon_{0})\right\}.$$
(S55)

By Lemma S.8(iii), it holds

$$\mathscr{E}(\epsilon_0) \subset \left\{ \max_{\mathbf{M} \in \tilde{\Xi}(\epsilon_0)} \frac{1}{\sqrt{T}} \| \mathbf{M}_{(1)} \mathbf{Z} \|_{F} \leqslant \sqrt{1.5C_{\mathbf{M}}\kappa_2} \right\} \subset \left\{ \sup_{\mathbf{M} \in \Xi_1} \frac{1}{\sqrt{T}} \| \mathbf{M}_{(1)} \mathbf{Z} \|_{F} \leqslant \frac{\sqrt{1.5C_{\mathbf{M}}\kappa_2}}{1 - 1.5\epsilon_0} \right\}. \quad (S56)$$

Moreover, for any $\mathfrak{M} \in \Xi_1$ and its corresponding $\bar{\mathfrak{M}} \in \bar{\Xi}(\epsilon_0)$ defined as in the proof of Lemma S.8(iii), similarly to (S63), we can show that

$$\frac{1}{\sqrt{T}} \|\mathbf{M}_{(1)} \mathbf{Z}\|_{F} \geqslant \frac{1}{\sqrt{T}} \|\bar{\mathbf{M}}_{(1)} \mathbf{Z}\|_{F} - \frac{1}{\sqrt{T}} \|(\mathbf{M} - \bar{\mathbf{M}})_{(1)} \mathbf{Z}\|_{F}$$

$$\geqslant \min_{\bar{\mathbf{M}} \in \bar{\mathbf{\Xi}}(\epsilon)} \frac{1}{\sqrt{T}} \|\bar{\mathbf{M}}_{(1)} \mathbf{Z}\|_{F} - \frac{1}{\sqrt{T}} \sum_{i=1}^{4} \|(\mathbf{M}_{i})_{(1)} \mathbf{Z}\|_{F}$$

$$\geqslant \min_{\bar{\mathbf{M}} \in \bar{\mathbf{\Xi}}(\epsilon)} \frac{1}{\sqrt{T}} \|\bar{\mathbf{M}}_{(1)} \mathbf{Z}\|_{F} - \sum_{i=1}^{4} \|\mathbf{M}_{i}\|_{F} \sup_{\mathbf{M} \in \mathbf{\Xi}_{1}} \frac{1}{\sqrt{T}} \|\mathbf{M}_{(1)} \mathbf{Z}\|_{F}$$

$$\geqslant \min_{\bar{\mathbf{M}} \in \bar{\mathbf{\Xi}}(\epsilon)} \frac{1}{\sqrt{T}} \|\bar{\mathbf{M}}_{(1)} \mathbf{Z}\|_{F} - 1.5\epsilon_{0} \sup_{\mathbf{M} \in \mathbf{\Xi}_{1}} \frac{1}{\sqrt{T}} \|\mathbf{M}_{(1)} \mathbf{Z}\|_{F}.$$

Taking the infimum over all $\mathbf{M} \in \mathbf{\Xi}_1$ and combining the result with (S56), we can show that on the event $\mathscr{E}(\epsilon_0)$, it holds

$$\inf_{\mathbf{M} \in \mathbf{\Xi}_1} \frac{1}{\sqrt{T}} \|\mathbf{M}_{(1)} \mathbf{Z}\|_{\mathrm{F}} \geqslant \sqrt{0.5 c_{\mathbf{M}} \kappa_1} - 1.5 \epsilon_0 \cdot \frac{\sqrt{1.5 C_{\mathbf{M}} \kappa_2}}{1 - 1.5 \epsilon_0} \geqslant \sqrt{0.5 c_{\mathbf{M}} \kappa_1} - 3 \epsilon_0 \sqrt{1.5 C_{\mathbf{M}} \kappa_2}$$

if $0 < \epsilon_0 \le 1/3$. Thus, by setting

$$\epsilon_0 = \min \left\{ \frac{1}{6} \sqrt{\frac{c_{\mathcal{M}} \kappa_1}{3 C_{\mathcal{M}} \kappa_2}}, \frac{1}{3} \right\},\,$$

we have

$$\mathscr{E}(\epsilon_0) \subset \left\{ \inf_{\mathbf{M} \in \mathbf{\Xi}_1} \frac{1}{\sqrt{T}} \| \mathbf{M}_{(1)} \mathbf{Z} \|_{\mathbf{F}} \geqslant \frac{\sqrt{0.5 c_{\mathbf{M}} \kappa_1}}{2} \right\}. \tag{S57}$$

As a result, with the above choice of ϵ_0 , we have

$$\mathscr{E}(\epsilon_0) \subset \left\{ \frac{c_{\mathsf{M}}\kappa_1}{8} \leqslant \inf_{\mathsf{M} \in \Xi_1} \frac{1}{T} \sum_{t=1}^T \|\mathbf{M}_{(1)} \mathbf{z}_t\|_2^2 \leqslant \sup_{\mathsf{M} \in \Xi_1} \frac{1}{T} \sum_{t=1}^T \|\mathbf{M}_{(1)} \mathbf{z}_t\|_2^2 \leqslant 6C_{\mathsf{M}}\kappa_2 \right\},\,$$

which, together with (S55) and the condition that $T \gtrsim (\kappa_2/\kappa_1)^2 d_{\mathcal{R}} \log(\kappa_2/\kappa_1)$, leads to (S44).

Proof of (S82): Consider a minimal generalized 1/3-net $\bar{\Xi}(1/3)$ of Ξ_1 . By (S55) and (S56), we have

$$\mathbb{P}\left\{\max_{\mathbf{M}\in\tilde{\boldsymbol{\Xi}}(1/3)}\frac{1}{\sqrt{T}}\|\boldsymbol{\mathcal{M}}_{(1)}\boldsymbol{Z}\|_{\mathrm{F}}>\sqrt{1.5C_{\mathbf{M}}\kappa_{2}}\right\}\leqslant 2e^{-cd_{\mathcal{R}}\log(\kappa_{2}/\kappa_{1})},$$

under the condition that $T \gtrsim (\kappa_2/\kappa_1)^2 d_{\mathcal{R}} \log(\kappa_2/\kappa_1)$. Note that $\sum_{t=1}^T \langle \mathbf{M}_{(1)} \mathbf{z}_t, \boldsymbol{\varepsilon}_t \rangle = \langle \mathbf{M}_{(1)}, \sum_{t=1}^T \boldsymbol{\varepsilon}_t \mathbf{z}_t' \rangle$. Then by Lemma S.8, for any K > 0, we have

$$\begin{split} & \mathbb{P}\left\{\sup_{\mathbf{M}\in\mathbf{\Xi}_{1}}\frac{1}{T}\sum_{t=1}^{T}\langle\mathbf{M}_{(1)}\boldsymbol{z}_{t},\boldsymbol{\varepsilon}_{t}\rangle\geqslant K\right\} \\ & \leqslant \mathbb{P}\left\{\max_{\mathbf{M}\in\bar{\mathbf{\Xi}}(1/3)}\frac{1}{T}\sum_{t=1}^{T}\langle\mathbf{M}_{(1)}\boldsymbol{z}_{t},\boldsymbol{\varepsilon}_{t}\rangle\geqslant\frac{K}{2}\right\} \\ & \leqslant \mathbb{P}\left\{\max_{\mathbf{M}\in\bar{\mathbf{\Xi}}(1/3)}\frac{1}{T}\sum_{t=1}^{T}\langle\mathbf{M}_{(1)}\boldsymbol{z}_{t},\boldsymbol{\varepsilon}_{t}\rangle\geqslant\frac{K}{2},\;\max_{\mathbf{M}\in\bar{\mathbf{\Xi}}(1/3)}\frac{1}{T}\sum_{t=1}^{T}\|\mathbf{M}_{(1)}\boldsymbol{z}_{t}\|_{2}^{2}\leqslant1.5C_{\mathbf{M}}\kappa_{2}\right\}+2e^{-cd_{\mathcal{R}}\log(\kappa_{2}/\kappa_{1})} \\ & \leqslant e^{Cd_{\mathcal{R}}\log3}\max_{\mathbf{M}\in\bar{\mathbf{\Xi}}(1/3)}\mathbb{P}\left\{\frac{1}{T}\sum_{t=1}^{T}\langle\mathbf{M}_{(1)}\boldsymbol{z}_{t},\boldsymbol{\varepsilon}_{t}\rangle\geqslant\frac{K}{2},\;\frac{1}{T}\sum_{t=1}^{T}\|\mathbf{M}_{(1)}\boldsymbol{z}_{t}\|_{2}^{2}\leqslant1.5C_{\mathbf{M}}\kappa_{2}\right\}+2e^{-cd_{\mathcal{R}}\log(\kappa_{2}/\kappa_{1})}, \end{split}$$

where the first inequality follows from (S48), and the last from the covering number in Lemma S.8(i). For any $\mathbf{M} \in \mathbb{R}^{N \times N \times (d+r+2s)}$, we can apply Lemma S.9 with $\mathbf{w}_t = \mathbf{M}_{(1)} \mathbf{z}_t$ to obtain the following pointwise bound:

$$\mathbb{P}\left\{\frac{1}{T}\sum_{t=1}^{T}\langle\boldsymbol{\mathcal{M}}_{(1)}\boldsymbol{z}_{t},\boldsymbol{\varepsilon}_{t}\rangle\geqslant\frac{K}{2},\ \frac{1}{T}\sum_{t=1}^{T}\|\boldsymbol{\mathcal{M}}_{(1)}\boldsymbol{z}_{t}\|_{2}^{2}\leqslant1.5C_{\mathbf{M}}\kappa_{2}\right\}\leqslant\exp\left\{-\frac{K^{2}T}{12\sigma^{2}C_{\mathbf{M}}\kappa_{2}\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})}\right\}.$$

Then, choosing K such that $K^2T/\{12\sigma^2C_{\mathfrak{M}}\kappa_2\lambda_{\max}(\Sigma_{\varepsilon})\} \gtrsim d_{\mathcal{R}}$, i.e., $K \simeq \sqrt{\kappa_2\lambda_{\max}(\Sigma_{\varepsilon})d_{\mathcal{R}}/T}$, we have

$$\mathbb{P}\left\{\sup_{\mathbf{M}\in\mathbf{\Xi}_{1}}\frac{1}{T}\sum_{t=1}^{T}\langle\mathbf{M}_{(1)}\boldsymbol{z}_{t},\boldsymbol{\varepsilon}_{t}\rangle\geqslant K\right\}\leqslant e^{-cd_{\mathcal{R}}}+2e^{-cd_{\mathcal{R}}\log(\kappa_{2}/\kappa_{1})},$$

and hence (S82). The proof of Lemma S.6 is complete.

S4.12 Proof of Lemma S.7

The following result for low-rank matrices is used in the proof of Lemma S.7.

Lemma S.12 (Covering number and discretization for low-rank matrices). Let $\Pi(\mathcal{R}) = \{ M \in \mathbb{R}^{N \times N} \mid ||M||_{F} = 1, \operatorname{rank}(M) \leqslant \mathcal{R} \}$, and let $\bar{\Pi}(\mathcal{R})$ be a minimal 1/2-net of $\Pi(\mathcal{R})$ in the Frobenius norm. Then the cardinality of $\bar{\Pi}(\mathcal{R})$ satisfies

$$\log |\bar{\mathbf{\Pi}}(\mathcal{R})| \le (2N+1)\mathcal{R}\log 18.$$

Moreover, for any $X \in \mathbb{R}^{N \times N}$, it holds

$$\sup_{\boldsymbol{M}\in \boldsymbol{\Pi}(\mathcal{R})} \langle \boldsymbol{M}, \boldsymbol{X} \rangle \leqslant 4 \max_{\boldsymbol{M}\in \bar{\boldsymbol{\Pi}}(\mathcal{R})} \langle \boldsymbol{M}, \boldsymbol{X} \rangle.$$

Proof of Lemma S.12. The covering number is given by Lemma 3.1 in Candes and Plan (2011). For any $M \in \Pi(\mathcal{R})$, there exists $\bar{M} \in \bar{\Pi}(\mathcal{R})$ satisfying $\|M - \bar{M}\|_{\mathrm{F}} \leqslant 1/2$. Note that the rank of $M - \bar{M}$ is at most $2\mathcal{R}$. Based on the singular value decomposition of $M - \bar{M}$, we can find two matrices $M^{(1)}$ and $M^{(2)}$ with rank at most \mathcal{R} such that $M - \bar{M} = M^{(1)} + M^{(2)}$ and $\langle M^{(1)}, M^{(2)} \rangle = 0$. Then it holds $\|M^{(1)}\|_{\mathrm{F}} + \|M^{(2)}\|_{\mathrm{F}} \leqslant \sqrt{2}\|M - \bar{M}\|_{\mathrm{F}} \leqslant \sqrt{2}/2$. Hence, for any $X \in \mathbb{R}^{N \times N}$, we have

$$\begin{split} \langle \boldsymbol{M}, \boldsymbol{X} \rangle &= \langle \bar{\boldsymbol{M}}, \boldsymbol{X} \rangle + \sum_{i=1}^{2} \langle \boldsymbol{M}^{(i)}, \boldsymbol{X} \rangle \leqslant \max_{\bar{\boldsymbol{M}} \in \bar{\boldsymbol{\Pi}}(\mathcal{R})} \langle \bar{\boldsymbol{M}}, \boldsymbol{X} \rangle + \sum_{i=1}^{2} \| \boldsymbol{M}^{(i)} \|_{\mathrm{F}} \sup_{\boldsymbol{M} \in \boldsymbol{\Pi}(\mathcal{R})} \langle \boldsymbol{M}, \boldsymbol{X} \rangle \\ &\leqslant \max_{\bar{\boldsymbol{M}} \in \bar{\boldsymbol{\Pi}}(\mathcal{R})} \langle \bar{\boldsymbol{M}}, \boldsymbol{X} \rangle + \frac{\sqrt{2}}{2} \sup_{\boldsymbol{M} \in \boldsymbol{\Pi}(\mathcal{R})} \langle \boldsymbol{M}, \boldsymbol{X} \rangle. \end{split}$$

Taking supremum with respect to $M \in \Pi(\mathcal{R})$ on both sides of the last inequality, we accomplish the proof of this lemma.

Proof of (S46): Denote $S^{N-1} = \{ \boldsymbol{u} \in \mathbb{R}^N \mid \|\boldsymbol{u}\|_2 = 1 \}$, and let \bar{S}^{N-1} be a minimal (1/4)-net of

 S^{N-1} in the Euclidean norm. Fix $j \ge 1$. By Lemma 5.4 in Vershynin (2010),

$$\left\| \frac{1}{T} \sum_{t=1}^{T} \boldsymbol{y}_{t-p-j} \boldsymbol{y}'_{t-p-j} \right\|_{\text{op}} \leq 2 \max_{\boldsymbol{u} \in \bar{S}^{N-1}} \boldsymbol{u}' \left(\frac{1}{T} \sum_{t=1}^{T} \boldsymbol{y}_{t-p-j} \boldsymbol{y}'_{t-p-j} \right) \boldsymbol{u} = 2 \max_{\boldsymbol{u} \in \bar{S}^{N-1}} \frac{1}{T} \sum_{t=1}^{T} (\boldsymbol{u}' \boldsymbol{y}_{t-p-j})^{2}.$$

Then for any K > 0,

$$\mathbb{P}\left(\left\|\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{y}_{t-p-j}\boldsymbol{y}_{t-p-j}'\right\|_{\text{op}} \geqslant K\right) \leqslant \mathbb{P}\left\{\max_{\boldsymbol{u}\in\bar{S}^{N-1}}\frac{1}{T}\sum_{t=1}^{T}(\boldsymbol{u}'\boldsymbol{y}_{t-p-j})^{2} \geqslant K/2\right\}$$

$$\leqslant 9^{N}\max_{\boldsymbol{u}\in S^{N-1}}\mathbb{P}\left\{\frac{1}{T}\sum_{t=1}^{T}(\boldsymbol{u}'\boldsymbol{y}_{t-p-j})^{2} \geqslant K/2\right\}, \tag{S58}$$

where we used the fact that the cardinality of \bar{S}^{N-1} satisfies $|\bar{S}^{N-1}| \leq 9^N$. For any $\boldsymbol{u} \in S^{N-1}$, applying Lemma S.10(ii) with $\boldsymbol{M} = \boldsymbol{u}'$, $\boldsymbol{w}_t = \boldsymbol{y}_{t-1}$, and $T_0 = 1 - p - j$, together with the result

$$\lambda_{\max}(\underline{\Sigma}_w) = \lambda_{\max}(\underline{\Sigma}_y) \leqslant \lambda_{\max}(\Sigma_\varepsilon) \mu_{\max}(\Psi_*)$$

as implied by Lemma S.11(i), we can show that

$$\mathbb{P}\left\{\left|\frac{1}{T}\sum_{t=1}^{T}(\boldsymbol{u}'\boldsymbol{y}_{t-p-j})^{2} - \mathbb{E}\{(\boldsymbol{u}'\boldsymbol{y}_{t-p-j})^{2}\}\right| \geqslant \delta\sigma^{2}\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})\mu_{\max}(\boldsymbol{\Psi}_{*})\right\} \leqslant 2e^{-c\min(\delta,\delta^{2})T}.$$

holds for any $\delta > 0$. In addition, by Lemma S.11(i),

$$\mathbb{E}\{(\boldsymbol{u}'\boldsymbol{y}_{t-n-i})^2\} \leqslant \lambda_{\max}(\boldsymbol{\Sigma}_{\boldsymbol{v}}) \leqslant \lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})\mu_{\max}(\boldsymbol{\Psi}_{*}).$$

In view of the above result and taking $\delta = j$, we further have

$$\mathbb{P}\left\{\frac{1}{T}\sum_{t=1}^{T}(\boldsymbol{u}'\boldsymbol{y}_{t-p-j})^{2} \geqslant \lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})\mu_{\max}(\boldsymbol{\Psi}_{*})(j\sigma^{2}+1)\right\} \leqslant 2e^{-cjT}.$$
 (S59)

Combining (S58) and (S59), if $T \ge 2N \log 9/c$, then

$$\mathbb{P}\left\{\left\|\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{y}_{t-p-j}\boldsymbol{y}_{t-p-j}'\right\|_{\text{op}} \geqslant 2\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})\mu_{\max}(\boldsymbol{\Psi}_{*})(j\sigma^{2}+1)\right\} \leqslant 2e^{-jN\log 9}.$$
 (S60)

By considering the union bound over all $j \ge 1$, we have

$$\mathbb{P}\left\{\exists j \geqslant 1: \left\|\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{y}_{t-p-j}\boldsymbol{y}_{t-p-j}'\right\|_{\text{op}} \geqslant 2\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})\mu_{\max}(\boldsymbol{\Psi}_{*})(j\sigma^{2}+1)\right\} \leqslant \sum_{j=1}^{\infty}2e^{-jN\log 9} \leqslant 3e^{-N\log 9}.$$

and hence (S46).

Proof of (S47): We first fix $j \ge 1$. Note that $\sum_{t=1}^{T} \langle \boldsymbol{M} \boldsymbol{y}_{t-p-j}, \boldsymbol{\varepsilon}_{t} \rangle = \langle \boldsymbol{M}, \sum_{t=1}^{T} \boldsymbol{\varepsilon}_{t} \boldsymbol{y}'_{t-p-j} \rangle$. Moreover, it can be verified that

$$\frac{1}{T} \sum_{t=1}^{T} \| \boldsymbol{M} \boldsymbol{y}_{t-p-j} \|_2^2 \leqslant \| \boldsymbol{M} \|_{\mathrm{F}}^2 \left\| \frac{1}{T} \sum_{t=1}^{T} \boldsymbol{y}_{t-p-j} \boldsymbol{y}_{t-p-j}' \right\|_{\mathrm{op}}.$$

Thus, by (S60) and Lemma S.12, for any K > 0, we have

$$\mathbb{P}\left\{\sup_{\boldsymbol{M}\in\boldsymbol{\Pi}(\mathcal{R})}\frac{1}{T}\sum_{t=1}^{T}\langle\boldsymbol{M}\boldsymbol{y}_{t-p-j},\boldsymbol{\varepsilon}_{t}\rangle\geqslant K\right\}$$

$$\leq \mathbb{P}\left\{\max_{\boldsymbol{M}\in\boldsymbol{\Pi}(\mathcal{R})}\frac{1}{T}\sum_{t=1}^{T}\langle\boldsymbol{M}\boldsymbol{y}_{t-p-j},\boldsymbol{\varepsilon}_{t}\rangle\geqslant \frac{K}{4}, \left\|\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{y}_{t-p-j}\boldsymbol{y}_{t-p-j}'\right\|_{\text{op}}\leq 2\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})\mu_{\max}(\boldsymbol{\Psi}_{*})(j\sigma^{2}+1)\right\}$$

$$+2e^{-jN\log 9}$$

$$\leq e^{9N\mathcal{R}}\max_{\boldsymbol{M}\in\boldsymbol{\Pi}(\mathcal{R})}\mathbb{P}\left\{\frac{1}{T}\sum_{t=1}^{T}\langle\boldsymbol{M}\boldsymbol{y}_{t-p-j},\boldsymbol{\varepsilon}_{t}\rangle\geqslant \frac{K}{4}, \frac{1}{T}\sum_{t=1}^{T}\|\boldsymbol{M}\boldsymbol{y}_{t-p-j}\|_{2}^{2}\leq 2\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})\mu_{\max}(\boldsymbol{\Psi}_{*})(j\sigma^{2}+1)\right\}$$

$$+2e^{-jN\log 9}.$$

Then, applying Lemma S.9 with $\boldsymbol{w}_t = \boldsymbol{M} \boldsymbol{y}_{t-p-j}$, we have the pointwise bound for any $\boldsymbol{M} \in \mathbb{R}^{N \times N}$ as follows:

$$\mathbb{P}\left\{\frac{1}{T}\sum_{t=1}^{T}\langle \boldsymbol{M}\boldsymbol{y}_{t-p-j}, \boldsymbol{\varepsilon}_{t}\rangle \geqslant \frac{K}{4}, \frac{1}{T}\sum_{t=1}^{T}\|\boldsymbol{M}\boldsymbol{y}_{t-p-j}\|_{2}^{2} \leqslant 2\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})\mu_{\max}(\boldsymbol{\Psi}_{*})(j\sigma^{2}+1)\right\}$$

$$\leqslant \exp\left\{-\frac{K^{2}T}{64\lambda_{\max}^{2}(\boldsymbol{\Sigma}_{\varepsilon})\mu_{\max}(\boldsymbol{\Psi}_{*})(j\sigma^{2}+1)\sigma^{2}}\right\} = \exp\{-9N\mathcal{R}(j+1)\},$$

if we choose

$$K = 24\lambda_{\max}(\mathbf{\Sigma}_{\varepsilon})\sqrt{\frac{\mu_{\max}(\mathbf{\Psi}_{*})(j\sigma^{2}+1)(j\sigma^{2}+\sigma^{2})N\mathcal{R}}{T}}.$$

Consequently, we have

$$\mathbb{P}\left\{\sup_{\boldsymbol{M}\in\boldsymbol{\Pi}(\mathcal{R})}\frac{1}{T}\sum_{t=1}^{T}\langle\boldsymbol{M}\boldsymbol{y}_{t-p-j},\boldsymbol{\varepsilon}_{t}\rangle\geqslant 24\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})\sqrt{\frac{\mu_{\max}(\boldsymbol{\Psi}_{*})(j\sigma^{2}+1)(j\sigma^{2}+\sigma^{2})N\mathcal{R}}{T}}\right\}$$

$$\leqslant e^{-9jN\mathcal{R}}+2e^{-jN\log 9}\leqslant 3e^{-jN\log 9}.$$

Taking the union bound over all $j \ge 1$ as in the proof of (S46) and noting that $(j\sigma^2 + 1)(j\sigma^2 + \sigma^2) \le (2j\sigma^2 + 1)^2$, we can verify (S47).

S4.13 Proof of Lemma S.8

The following covering result for low-Tucker-rank tensors is used in the proof of Lemma S.8.

Lemma S.13 (Covering number for low-Tucker-rank tensors). Let $\Pi(\mathcal{R}_1, \mathcal{R}_2) = \{ \mathfrak{T} \in \mathbb{R}^{p_1 \times p_2 \times p_3} : \|\mathfrak{T}\|_{\mathrm{F}} \leq 1, \operatorname{rank}(\mathfrak{T}_{(i)}) \leq \mathcal{R}_i, i = 1, 2 \}$. For any $\epsilon > 0$, let $\bar{\Pi}(\epsilon; \mathcal{R}_1, \mathcal{R}_2)$ be a minimal ϵ -net for $\Pi(\mathcal{R}_1, \mathcal{R}_2)$ in the Frobenius norm. Then the cardinality of $\bar{\Pi}(\epsilon; \mathcal{R}_1, \mathcal{R}_2)$ satisfies

$$|\bar{\mathbf{\Pi}}(\epsilon; \mathcal{R}_1, \mathcal{R}_2)| \leq (9/\epsilon)^{\mathcal{R}_1 \mathcal{R}_2 p_3 + p_1 \mathcal{R}_1 + p_2 \mathcal{R}_2}$$

Proof of Lemma S.13. The proof of this lemma is straightforward given the proof of Lemma 2 in Rauhut et al. (2017). \Box

Proof of (i): Notice that the results for $\mathfrak{G}_{\text{stack}}$ in (S17) can be generalized to any $\mathfrak{M}(\boldsymbol{a}, \boldsymbol{\mathcal{B}})$ with $\boldsymbol{a} \in \mathbb{R}^{r+2s}$ and $\boldsymbol{\mathcal{B}} \in \mathbb{R}^{N \times N \times (r+2s)}$, where $\mathfrak{G}_{\text{stack}} = \mathfrak{M}(\boldsymbol{\omega} - \boldsymbol{\omega}^*, \mathfrak{G} - \mathfrak{G}^*)$. That is, by a method similar to that for (S17), we can show that for any $\boldsymbol{a} \in \mathbb{R}^{r+2s}$ and $\boldsymbol{\mathcal{B}} \in \mathbb{R}^{N \times N \times (r+2s)}$,

$$0.5(\|\mathbf{B}\|_{F} + \varpi_{1}\|\mathbf{a}\|_{2}) \leq \|\mathbf{M}(\mathbf{a}, \mathbf{B})\|_{F} \leq \|\mathbf{B}\|_{F} + \varpi_{2}\|\mathbf{a}\|_{2}, \tag{S61}$$

where

$$\varpi_1 = \sqrt{2}c_9\alpha \quad \text{and} \quad \varpi_2 = \frac{\sqrt{2}\alpha}{\min_{1 \leqslant k \leqslant s} \gamma_k^*}.$$

Thus, if $\|\mathbf{M}(\boldsymbol{a}, \mathbf{B})\|_{\mathrm{F}} = 1$, then $1 \leq \|\mathbf{B}\|_{\mathrm{F}} \leq 2$ and $\varpi_2^{-1} \leq \|\boldsymbol{a}\|_2 \leq \varpi_1^{-1}$. As a result,

$$\mathbf{\Xi}_1 \subset {\{\mathbf{M}(\boldsymbol{a}, \mathbf{\mathcal{B}}) \mid \boldsymbol{a} \in \boldsymbol{\Pi}^{(1)}, \mathbf{\mathcal{B}} \in \boldsymbol{\Pi}^{(2)}\}},$$

where

$$\mathbf{\Pi}^{(1)} = \left\{ oldsymbol{a} \in \mathbb{R}^{r+2s} \mid arpi_2^{-1} \leqslant \|oldsymbol{a}\|_2 \leqslant arpi_1^{-1}
ight\}$$

and

$$\mathbf{\Pi}^{(2)} = \{ \mathbf{\mathcal{B}} \in \mathbb{R}^{N \times N \times d} \mid \mathbf{\mathcal{B}} \in \mathbf{\Gamma}(2\mathcal{R}_1, 2\mathcal{R}_2), 1 \leqslant \|\mathbf{\mathcal{B}}\|_{\mathrm{F}} \leqslant 2 \}.$$

Hence, the problem of covering Ξ_1 can be converted into that of covering $\Pi^{(1)}$ and $\Pi^{(2)}$.

For any fixed $\epsilon > 0$, let $\bar{\Pi}^{(1)}(\epsilon)$ be a minimal $\epsilon/(2\varpi_2)$ -net for $\Pi^{(1)}$ in the Euclidean norm, and let $\bar{\Pi}^{(2)}(\epsilon)$ be a minimal $\epsilon/2$ -net for $\Pi^{(2)}$ in the Frobenius norm. Then denote

$$\bar{\boldsymbol{\Xi}}(\epsilon) = \left\{ \boldsymbol{\mathcal{M}}(\boldsymbol{a},\boldsymbol{\mathcal{B}}) \in \mathbb{R}^{N \times N \times (d+r+2s)} \mid \boldsymbol{a} \in \bar{\boldsymbol{\Pi}}^{(1)}(\epsilon), \boldsymbol{\mathcal{B}} \in \bar{\boldsymbol{\Pi}}^{(2)}(\epsilon) \right\}.$$

Thus, for every $\mathbf{M}(\boldsymbol{a}, \mathbf{B}) \in \mathbf{\Xi}_1$, there exists $\mathbf{M}(\bar{\boldsymbol{a}}, \bar{\mathbf{B}}) \in \bar{\mathbf{\Xi}}(\epsilon)$ with $\bar{\boldsymbol{a}} \in \bar{\boldsymbol{\Pi}}^{(1)}(\epsilon)$ and $\bar{\mathbf{B}} \in \bar{\boldsymbol{\Pi}}^{(2)}(\epsilon)$ such that

$$\|\boldsymbol{a} - \bar{\boldsymbol{a}}\|_{2} \leqslant \epsilon/(2\varpi_{2}) \quad \text{and} \quad \|\boldsymbol{\mathcal{B}} - \bar{\boldsymbol{\mathcal{B}}}\|_{F} \leqslant \epsilon/2.$$
 (S62)

Since $\mathcal{M}(a, \mathcal{B}) - \mathcal{M}(\bar{a}, \bar{\mathcal{B}}) = \mathcal{M}(a - \bar{a}, \mathcal{B} - \bar{\mathcal{B}})$, it follows from (S61) and (S62) that

$$\|\mathbf{\mathcal{M}}(\boldsymbol{a},\mathbf{\mathcal{B}}) - \mathbf{\mathcal{M}}(\bar{\boldsymbol{a}},\bar{\mathbf{\mathcal{B}}})\|_{\mathrm{F}} = \|\mathbf{\mathcal{M}}(\boldsymbol{a} - \bar{\boldsymbol{a}},\mathbf{\mathcal{B}} - \bar{\mathbf{\mathcal{B}}})\|_{\mathrm{F}} \leqslant \|\mathbf{\mathcal{B}} - \bar{\mathbf{\mathcal{B}}}\|_{\mathrm{F}} + \varpi_2\|\boldsymbol{a} - \bar{\boldsymbol{a}}\|_2 \leqslant \epsilon.$$

In addition, note that $\bar{\Xi}(\epsilon) \subset \Xi$. Therefore, $\bar{\Xi}(\epsilon)$ is a generalized ϵ -net of Ξ_1 . Moreover, by a standard volumetric argument (see also Corollary 4.2.13 in Vershynin (2018) for details) and Lemma S.13, the cardinalities of $\bar{\Pi}^{(1)}(\epsilon)$ and $\bar{\Pi}^{(2)}(\epsilon)$ satisfy

$$\log |\bar{\mathbf{\Pi}}^{(1)}(\epsilon)| \leq (r+2s) \log \{6\varpi_2/(\varpi_1\epsilon)\} \quad \text{and} \quad \log |\bar{\mathbf{\Pi}}^{(2)}(\epsilon)| \leq \{4\mathcal{R}_1\mathcal{R}_2d + 2(\mathcal{R}_1 + \mathcal{R}_2)N\} \log(18/\epsilon).$$

Noting that $\varpi_1\varpi_2^{-1} \approx 1$ is independent of ϵ , we have

$$\log |\bar{\Xi}(\epsilon)| \leq \log |\bar{\Pi}^{(1)}(\epsilon)| + \log |\bar{\Pi}^{(2)}(\epsilon)| \leq \{\mathcal{R}_1 \mathcal{R}_2 d + (\mathcal{R}_1 + \mathcal{R}_2) N\} \log(1/\epsilon).$$

Proof of (ii): Since $\bar{\Pi}^{(1)}(\epsilon) \subset \Pi^{(1)}$ and $\bar{\Pi}^{(2)}(\epsilon) \subset \Pi^{(2)}$, we have

$$\bar{\boldsymbol{\Xi}}(\epsilon) \subset \left\{\boldsymbol{\mathcal{M}}(\boldsymbol{a},\boldsymbol{\mathcal{B}}) \in \mathbb{R}^{N \times N \times (d+r+2s)} \mid \boldsymbol{a} \in \boldsymbol{\Pi}^{(1)}, \boldsymbol{\mathcal{B}} \in \boldsymbol{\Pi}^{(2)} \right\}.$$

Thus, by (S61), for any $\mathbf{M} \in \bar{\mathbf{\Xi}}(\epsilon)$, it holds

$$c_{\mathfrak{M}} := 0.5(1 + \varpi_1 \varpi_2^{-1}) \leqslant \|\mathbf{M}\|_{\mathcal{F}} \leqslant 2 + \varpi_2 \varpi_1^{-1} := C_{\mathfrak{M}}.$$

Since $\varpi_1\varpi_2^{-1}=c_9\min_{1\leqslant k\leqslant s}\gamma_k^*\approx 1$ is independent of $\epsilon,$ (ii) is proved.

Proof of (iii): From the proof of (i), for every $\mathbf{M} := \mathbf{M}(\boldsymbol{a}, \mathbf{B}) \in \Xi_1$, there exists $\bar{\mathbf{M}} := \mathbf{M}(\bar{\boldsymbol{a}}, \bar{\mathbf{B}}) \in \bar{\Xi}(\epsilon)$ such that $\bar{\boldsymbol{a}} \in \bar{\Pi}^{(1)}(\epsilon)$ and $\bar{\mathbf{B}} \in \bar{\Pi}^{(2)}(\epsilon)$ satisfy (S62). Since $\bar{\Pi}^{(2)}(\epsilon) \subset \Pi^{(2)}$, we have $\mathbf{B} - \bar{\mathbf{B}} \in \Gamma(4\mathcal{R}_1, 4\mathcal{R}_2)$. Then by considering the higher-order singular value decomposition for $\mathbf{B} - \bar{\mathbf{B}}$, we can find four tensors $\mathbf{B}_i \in \Gamma(2\mathcal{R}_1, 2\mathcal{R}_2)$ with $1 \leq i \leq 4$ such that $\mathbf{B} - \bar{\mathbf{B}} = \sum_{i=1}^4 \mathbf{B}_i$ and $\langle \mathbf{B}_i, \mathbf{B}_j \rangle = 0$ for all $i \neq j$. As a result, we can show that

$$\mathcal{M} - \bar{\mathcal{M}} = \mathcal{M}(\boldsymbol{a}, \mathcal{B}) - \mathcal{M}(\bar{\boldsymbol{a}}, \bar{\mathcal{B}}) = \mathcal{M}(\boldsymbol{a} - \bar{\boldsymbol{a}}, \mathcal{B} - \bar{\mathcal{B}}) = \mathcal{M}\left(\sum_{i=1}^{4} \frac{\boldsymbol{a} - \bar{\boldsymbol{a}}}{4}, \sum_{i=1}^{4} \mathcal{B}_i\right) = \sum_{i=1}^{4} \mathcal{M}_i,$$

where $\mathbf{M}_i = \mathbf{M}((\mathbf{a} - \bar{\mathbf{a}})/4, \mathbf{B}_i) \in \mathbf{\Xi}$. Moreover, by (S61), (S62) and the Cauchy-Schwarz inequality, it holds

$$\sum_{i=1}^{4} \|\mathbf{\mathcal{M}}_i\|_{\mathrm{F}} \leqslant \sum_{i=1}^{4} \left\{ \|\mathbf{\mathcal{B}}_i\|_{\mathrm{F}} + \frac{\varpi_2}{4} \|\boldsymbol{a} - \bar{\boldsymbol{a}}\|_2 \right\} \leqslant 2 \|\mathbf{\mathcal{B}} - \bar{\mathbf{\mathcal{B}}}\|_{\mathrm{F}} + \varpi_2 \|\boldsymbol{a} - \bar{\boldsymbol{a}}\|_2 \leqslant 1.5\epsilon.$$

Therefore, for any $\mathfrak{M} \in \Xi_1$, we can show that

$$\begin{split} \langle \mathbf{M}_{(1)}, \boldsymbol{X} \rangle &= \langle \bar{\mathbf{M}}_{(1)}, \boldsymbol{X} \rangle + \langle (\mathbf{M} - \bar{\mathbf{M}})_{(1)}, \boldsymbol{X} \rangle \leqslant \max_{\bar{\mathbf{M}} \in \bar{\mathbf{\Xi}}(\epsilon)} \langle \bar{\mathbf{M}}_{(1)}, \boldsymbol{X} \rangle + \sum_{i=1}^{4} \langle (\mathbf{M}_{i})_{(1)}, \boldsymbol{X} \rangle \\ &\leqslant \max_{\bar{\mathbf{M}} \in \bar{\mathbf{\Xi}}(\epsilon)} \langle \bar{\mathbf{M}}_{(1)}, \boldsymbol{X} \rangle + \sum_{i=1}^{4} \| \mathbf{M}_{i} \|_{\mathrm{F}} \sup_{\mathbf{M} \in \mathbf{\Xi}_{1}} \langle \mathbf{M}_{(1)}, \boldsymbol{X} \rangle \\ &\leqslant \max_{\bar{\mathbf{M}} \in \bar{\mathbf{\Xi}}(\epsilon)} \langle \bar{\mathbf{M}}_{(1)}, \boldsymbol{X} \rangle + 1.5\epsilon \sup_{\mathbf{M} \in \mathbf{\Xi}_{1}} \langle \mathbf{M}_{(1)}, \boldsymbol{X} \rangle, \end{split}$$

Taking supremum over all $\mathfrak{M} \in \Xi_1$ on both sides, we accomplish the proof of (S48). The proof of (S49) follows the same arguments as those for (S48) except that the above inequalities are revised

to

$$\|\mathbf{\mathcal{M}}_{(1)}\mathbf{Z}\|_{F} \leq \|\bar{\mathbf{\mathcal{M}}}_{(1)}\mathbf{Z}\|_{F} + \|(\mathbf{\mathcal{M}} - \bar{\mathbf{\mathcal{M}}})_{(1)}\mathbf{Z}\|_{F} \leq \max_{\bar{\mathbf{\mathcal{M}}} \in \bar{\mathbf{\Xi}}(\epsilon)} \|\bar{\mathbf{\mathcal{M}}}_{(1)}\mathbf{Z}\|_{F} + \sum_{i=1}^{4} \|(\mathbf{\mathcal{M}}_{i})_{(1)}\mathbf{Z}\|_{F}$$

$$\leq \max_{\bar{\mathbf{\mathcal{M}}} \in \bar{\mathbf{\Xi}}(\epsilon)} \|\bar{\mathbf{\mathcal{M}}}_{(1)}\mathbf{Z}\|_{F} + \sum_{i=1}^{4} \|\mathbf{\mathcal{M}}_{i}\|_{F} \sup_{\mathbf{\mathcal{M}} \in \mathbf{\Xi}_{1}} \|\mathbf{\mathcal{M}}_{(1)}\mathbf{Z}\|_{F}$$

$$\leq \max_{\bar{\mathbf{\mathcal{M}}} \in \bar{\mathbf{\Xi}}(\epsilon)} \|\bar{\mathbf{\mathcal{M}}}_{(1)}\mathbf{Z}\|_{F} + 1.5\epsilon \sup_{\mathbf{\mathcal{M}} \in \mathbf{\Xi}_{1}} \|\mathbf{\mathcal{M}}_{(1)}\mathbf{Z}\|_{F}. \tag{S63}$$

The proof of Lemma S.8 is complete.

S4.14 Proof of Lemma S.9

Note that $\langle \boldsymbol{w}_t, \boldsymbol{\varepsilon}_t \rangle = \langle \boldsymbol{\Sigma}_{\epsilon}^{1/2} \boldsymbol{w}_t, \boldsymbol{\xi}_t \rangle$, where $\boldsymbol{\xi}_t$ is mean-zero and σ^2 -sub-Gaussian. Moreover, it holds $\|\boldsymbol{\Sigma}_{\varepsilon}^{1/2} \boldsymbol{w}_t\|_2^2 \leq \lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon}) \|\boldsymbol{w}_t\|_2^2$. Then by a straightforward multivariate generalization of Lemma 4.2 in Simchowitz et al. (2018), we can show that

$$\mathbb{P}\left\{\sum_{t=1}^{T} \langle \boldsymbol{w}_{t}, \boldsymbol{\varepsilon}_{t} \rangle \geqslant a, \sum_{t=1}^{T} \|\boldsymbol{w}_{t}\|^{2} \leqslant b\right\} \leqslant \mathbb{P}\left\{\sum_{t=1}^{T} \langle \boldsymbol{\Sigma}_{\varepsilon}^{1/2} \boldsymbol{w}_{t}, \boldsymbol{\xi}_{t} \rangle \geqslant a, \sum_{t=1}^{T} \|\boldsymbol{\Sigma}_{\varepsilon}^{1/2} \boldsymbol{w}_{t}\|^{2} \leqslant \lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})b\right\}$$
$$\leqslant \exp\left\{-\frac{a^{2}}{2\sigma^{2}\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})b}\right\}.$$

The proof is complete.

S4.15 Proof of Lemma S.10

Proof of (i): First it is obvious that $\{w_t\}$ is a zero-mean stationary time series. Without loss of generality, we let $T_0 = 0$ in what follows.

It is worth noting that under Assumption 1, $\varepsilon_t = \Sigma_{\varepsilon}^{1/2} \boldsymbol{\xi}_t$, and all coordinates of the vector $\boldsymbol{\xi} = (\boldsymbol{\xi}'_{T-1}, \boldsymbol{\xi}'_{T-2}, \dots)'$ are independent and σ^2 -sub-Gaussian with mean zero and variance one. In addition, by the vector $\mathrm{MA}(\infty)$ representation of \boldsymbol{w}_t , we have $\underline{\boldsymbol{w}}_T = \underline{\boldsymbol{\Psi}}^w \boldsymbol{\xi}$, where

$$\underline{\underline{\Psi}}_{TM\times\infty}^{w} = \begin{pmatrix}
\Psi_{1}^{w} \Sigma_{\varepsilon}^{1/2} & \Psi_{2}^{w} \Sigma_{\varepsilon}^{1/2} & \Psi_{3}^{w} \Sigma_{\varepsilon}^{1/2} & \cdots & \Psi_{T}^{w} \Sigma_{\varepsilon}^{1/2} & \cdots \\
\Psi_{1}^{w} \Sigma_{\varepsilon}^{1/2} & \Psi_{2}^{w} \Sigma_{\varepsilon}^{1/2} & \cdots & \Psi_{T-1}^{w} \Sigma_{\varepsilon}^{1/2} & \cdots \\
& \ddots & & & & \\
& & \Psi_{1}^{w} \Sigma_{\varepsilon}^{1/2} & \cdots
\end{pmatrix}.$$
(S64)

Then, it holds

$$\underline{\Sigma}_{w} = \mathbb{E}(\underline{\boldsymbol{w}}_{T}\underline{\boldsymbol{w}}_{T}') = \underline{\boldsymbol{\Psi}}^{w}(\underline{\boldsymbol{\Psi}}^{w})'. \tag{S65}$$

Observe that $\sum_{t=1}^{T} \|\boldsymbol{w}_{t}\|_{2}^{2} = \underline{\boldsymbol{w}}_{T}^{\prime}\underline{\boldsymbol{w}}_{T} = \boldsymbol{\xi}^{\prime}(\underline{\boldsymbol{\Psi}}^{w})^{\prime}\underline{\boldsymbol{\Psi}}^{w}\boldsymbol{\xi}$. Since $\boldsymbol{\xi}$ is a vector with independent, zero-mean and sub-Gaussian coordinates, we can apply the Hanson-Wright inequality (Vershynin, 2018) to obtain that for any $\iota > 0$,

$$\mathbb{P}\left(\left|\sum_{t=1}^{T} \|\boldsymbol{w}_{t}\|_{2}^{2} - T\mathbb{E}\left(\|\boldsymbol{w}_{t}\|_{2}^{2}\right)\right| \geqslant \iota\right) \leqslant 2 \exp\left\{-c \min\left(\frac{\iota}{\sigma^{2} \|(\underline{\boldsymbol{\Psi}}^{w})'\underline{\boldsymbol{\Psi}}^{w}\|_{\operatorname{op}}}, \frac{\iota^{2}}{\sigma^{4} \|(\underline{\boldsymbol{\Psi}}^{w})'\underline{\boldsymbol{\Psi}}^{w}\|_{F}^{2}}\right)\right\}. \quad (S66)$$

Note that by (S65), $\|(\underline{\Psi}^w)'\underline{\Psi}^w\|_{\text{op}} = \|\underline{\Psi}^w(\underline{\Psi}^w)'\|_{\text{op}} = \lambda_{\max}(\underline{\Sigma}_w)$, and $\underline{\Sigma}_w$ is a $TM \times TM$ matrix. Then

$$\|(\underline{\Psi}^w)'\underline{\Psi}^w\|_{\mathrm{F}} = \|\underline{\Psi}^w(\underline{\Psi}^w)'\|_{\mathrm{F}} = \|\underline{\Sigma}_w\|_{\mathrm{F}} \leqslant \sqrt{TM}\lambda_{\max}(\underline{\Sigma}_w).$$

Taking $\iota = \sigma^2 \sqrt{T} M \lambda_{\max}(\underline{\Sigma}_w)$ in (S66), the proof of (i) is complete.

Proof of (ii): Define the vector $\underline{\boldsymbol{m}}_T = ((\boldsymbol{M}\boldsymbol{w}_T)', \dots, (\boldsymbol{M}\boldsymbol{w}_1)')' = (\boldsymbol{I}_T \otimes \boldsymbol{M})\underline{\boldsymbol{w}}_T$. Then $\underline{\boldsymbol{m}}_T = \boldsymbol{P}\boldsymbol{\xi}$, where $\boldsymbol{P} = (\boldsymbol{I}_T \otimes \boldsymbol{M})\underline{\boldsymbol{\Psi}}^w$. As a result, $\sum_{t=1}^T \|\boldsymbol{M}\boldsymbol{w}_t\|_2^2 = \underline{\boldsymbol{m}}_T'\underline{\boldsymbol{m}}_T = \boldsymbol{\xi}'\boldsymbol{P}'\boldsymbol{P}\boldsymbol{\xi}$. Similar to (S65), it follows from the Hanson-Wright inequality that for any $\iota > 0$,

$$\mathbb{P}\left(\left|\sum_{t=1}^{T} \|\boldsymbol{M}\boldsymbol{w}_{t}\|_{2}^{2} - T\mathbb{E}\left(\|\boldsymbol{M}\boldsymbol{w}_{t}\|_{2}^{2}\right)\right| \geqslant \iota\right) \leqslant 2\exp\left\{-c\min\left(\frac{\iota}{\sigma^{2}\|\boldsymbol{P}'\boldsymbol{P}\|_{\mathrm{op}}}, \frac{\iota^{2}}{\sigma^{4}\|\boldsymbol{P}'\boldsymbol{P}\|_{\mathrm{F}}^{2}}\right)\right\}. \tag{S67}$$

By (S65), we have $\|\boldsymbol{P'P}\|_{\text{op}} = \|\boldsymbol{PP'}\|_{\text{op}} \leqslant \|\boldsymbol{MM'}\|_{\text{op}} \|\underline{\boldsymbol{\Psi}}^w(\underline{\boldsymbol{\Psi}}^w)'\|_{\text{op}} \leqslant \lambda_{\max}(\underline{\boldsymbol{\Sigma}}_w)\|\boldsymbol{M}\|_{\text{F}}^2$. Moreover,

$$tr(\mathbf{P}'\mathbf{P}) = tr(\mathbf{P}\mathbf{P}') = tr\{(\mathbf{I}_T \otimes \mathbf{M})\underline{\Sigma}_w(\mathbf{I}_T \otimes \mathbf{M}')\}$$
$$= vec(\mathbf{I}_T \otimes \mathbf{M})'(\underline{\Sigma}_w \otimes \mathbf{I}_{TQ})vec(\mathbf{I}_T \otimes \mathbf{M}) \leqslant T\lambda_{\max}(\underline{\Sigma}_w)\|\mathbf{M}\|_F^2,$$

where the second equality follows from (S65). As a result,

$$\|\boldsymbol{P}'\boldsymbol{P}\|_{\mathrm{F}} \leqslant \sqrt{\|\boldsymbol{P}'\boldsymbol{P}\|_{\mathrm{op}}\operatorname{tr}(\boldsymbol{P}'\boldsymbol{P})} \leqslant \sqrt{\|\boldsymbol{P}\boldsymbol{P}'\|_{\mathrm{op}}\operatorname{tr}(\boldsymbol{P}\boldsymbol{P}')} \leqslant \sqrt{T}\lambda_{\max}(\underline{\boldsymbol{\Sigma}}_w)\|\boldsymbol{M}\|_{\mathrm{F}}^2.$$

Taking $\iota = \delta \sigma^2 T \lambda_{\max}(\underline{\Sigma}_w) \| \boldsymbol{M} \|_{\mathrm{F}}^2$ in (S67), the proof of (ii) is complete.

S4.16 Proof of Lemma S.11

Proof of (i): Consider the spectral density of $\{y_t\}$,

$$\boldsymbol{f}_{\boldsymbol{y}}(\boldsymbol{\theta}) = (2\pi)^{-1}\boldsymbol{\Psi}_{*}(\boldsymbol{e}^{-i\boldsymbol{\theta}})\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}\boldsymbol{\Psi}_{*}^{\mathsf{H}}(\boldsymbol{e}^{-i\boldsymbol{\theta}}), \hspace{0.5cm} \boldsymbol{\theta} \in [-\pi,\pi].$$

Let

$$\mathcal{M}(\boldsymbol{f}_y) = \max_{\boldsymbol{\theta} \in [-\pi,\pi]} \lambda_{\max}(\boldsymbol{f}_y(\boldsymbol{\theta})) \quad \text{and} \quad \boldsymbol{m}(\boldsymbol{f}_y) = \min_{\boldsymbol{\theta} \in [-\pi,\pi]} \lambda_{\min}(\boldsymbol{f}_y(\boldsymbol{\theta}))$$

Along the lines of Basu and Michailidis (2015), it holds

$$2\pi \textit{m}(\textit{\textbf{f}}_y) \leqslant \lambda_{\min}(\underline{\Sigma}_y) \leqslant \lambda_{\max}(\underline{\Sigma}_y) \leqslant 2\pi \mathcal{M}(\textit{\textbf{f}}_y),$$

$$2\pi m(\boldsymbol{f}_y) \leqslant \lambda_{\min}(\boldsymbol{\Sigma}_y) \leqslant \lambda_{\max}(\boldsymbol{\Sigma}_y) \leqslant 2\pi \mathcal{M}(\boldsymbol{f}_y),$$

and

$$\lambda_{\min}(\boldsymbol{\Sigma}_{\varepsilon})\mu_{\min}(\boldsymbol{\Psi}_{*}) \leqslant 2\pi m(\boldsymbol{f}_{y}) \leqslant 2\pi \mathcal{M}(\boldsymbol{f}_{y}) \leqslant \lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})\mu_{\max}(\boldsymbol{\Psi}_{*}); \tag{S68}$$

see Proposition 2.3 therein. Thus, (i) is proved.

Proof of (ii): First, since $\sum_{i=1}^{\infty} \| \boldsymbol{W}_i \|_{\text{op}} < \infty$ and $\{ \boldsymbol{y}_t \}$ is stationary with mean zero, the time series $\boldsymbol{w}_t = \mathcal{W}(B) \boldsymbol{y}_t = \mathcal{W}(B) \boldsymbol{\Psi}_*(B) \boldsymbol{\varepsilon}_t$ is also zero-mean and stationary, where $\mathcal{W}(B) = \sum_{i=1}^{\infty} \boldsymbol{W}_i B^i$.

For any $\ell \in \mathbb{Z}$, denote by $\Sigma_y(\ell) = \mathbb{E}(\boldsymbol{y}_t \boldsymbol{y}_{t-\ell}')$ the lag- ℓ covariance matrix of \boldsymbol{y}_t , and then $\Sigma_y(\ell) = \int_{-\pi}^{\pi} \boldsymbol{f}_y(\theta) e^{i\ell\theta} d\theta$. For any fixed $\boldsymbol{u} \in \mathbb{R}^N$ with $\|\boldsymbol{u}\|_2 = 1$,

$$\mathbf{u}' \mathbf{\Sigma}_{w} \mathbf{u} = \mathbf{u}' \mathbb{E} \left(\sum_{j=1}^{\infty} \mathbf{W}_{j} \mathbf{y}_{t-j} \sum_{k=1}^{\infty} \mathbf{W}'_{k} \mathbf{y}_{t-k} \right) \mathbf{u}$$

$$= \mathbf{u}' \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \mathbf{W}_{j} \mathbf{\Sigma}_{y} (k-j) \mathbf{W}'_{k} \mathbf{u}$$

$$= \int_{-\pi}^{\pi} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \mathbf{u}' \mathbf{W}_{j} \mathbf{f}_{y} (\theta) e^{-i(j-k)\theta} \mathbf{W}'_{k} \mathbf{u} d\theta$$

$$= \int_{-\pi}^{\pi} \mathbf{u}' \mathcal{W} (e^{-i\theta}) \mathbf{f}_{y} (\theta) \mathcal{W}^{\mathsf{H}} (e^{-i\theta}) \mathbf{u} d\theta, \qquad (S69)$$

where $\mathscr{W}(z) = \sum_{j=1}^{\infty} \mathbf{W}_{j} z^{j}$ for $z \in \mathbb{C}$, and $\mathscr{W}^{\mathsf{H}}(e^{-i\theta}) = \left\{\mathscr{W}(e^{i\theta})\right\}'$ is the conjugate transpose of $\mathscr{W}(e^{-i\theta})$. Since $\mathbf{f}_{y}(\theta)$ is Hermitian, $\mathbf{u}'\mathscr{W}(e^{-i\theta})\mathbf{f}_{y}(\theta)\mathscr{W}^{\mathsf{H}}(e^{-i\theta})\mathbf{u}$ is real for all $\theta \in [-\pi, \pi]$. Then it

is easy to see that

$$\textit{m}(\boldsymbol{f}_y) \cdot \boldsymbol{u}' \mathcal{W}(e^{-i\theta}) \mathcal{W}^{\mathsf{H}}(e^{-i\theta}) \boldsymbol{u} \leqslant \boldsymbol{u}' \mathcal{W}(e^{-i\theta}) \boldsymbol{f}_y(\theta) \mathcal{W}^{\mathsf{H}}(e^{-i\theta}) \boldsymbol{u} \leqslant \mathcal{M}(\boldsymbol{f}_y) \cdot \boldsymbol{u}' \mathcal{W}(e^{-i\theta}) \mathcal{W}^{\mathsf{H}}(e^{-i\theta}) \boldsymbol{u}.$$

Moreover, since $\int_{-\pi}^{\pi} e^{i\ell\theta} d\theta = 0$ for any $\ell \neq 0$, we can show that

$$\int_{-\pi}^{\pi} \mathbf{u}' \mathcal{W}(e^{-i\theta}) \mathcal{W}^{\mathsf{H}}(e^{-i\theta}) \mathbf{u} d\theta = \int_{-\pi}^{\pi} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \mathbf{u}' \mathbf{W}_{j} e^{-i(j-k)\theta} \mathbf{W}'_{k} \mathbf{u} d\theta$$
$$= 2\pi \mathbf{u}' \mathbf{W} \mathbf{W}' \mathbf{u}.$$

which, together with the fact of $\|\boldsymbol{u}\|_2 = 1$, implies that

$$2\pi\sigma_{\min}^{2}(\boldsymbol{W}) \leqslant \int_{-\pi}^{\pi} \boldsymbol{u}' \mathcal{W}(e^{-i\theta}) \mathcal{W}^{\mathsf{H}}(e^{-i\theta}) \boldsymbol{u} \, d\theta \leqslant 2\pi\sigma_{\max}^{2}(\boldsymbol{W}). \tag{S70}$$

In view of (S68)–(S70), we accomplish the proof of (S50).

To verify (S51), note that the spectral density of $\{\boldsymbol{w}_t\}$ is

$$\boldsymbol{f}_{w}(\theta) = \mathcal{W}(e^{-i\theta})\boldsymbol{f}_{u}(\theta)\mathcal{W}^{\mathsf{H}}(e^{-i\theta}), \quad \theta \in [-\pi, \pi];$$

see Section 9.2 of Priestley (1981). Then

$$\begin{split} \mathcal{M}(\boldsymbol{f}_w) &= \max_{\boldsymbol{\theta} \in [-\pi,\pi]} \lambda_{\max}(\boldsymbol{f}_w(\boldsymbol{\theta})) \leqslant \mathcal{M}(\boldsymbol{f}_y) \max_{\boldsymbol{\theta} \in [-\pi,\pi]} \lambda_{\max} \{ \mathcal{W}(e^{-i\boldsymbol{\theta}}) \mathcal{W}^{\mathsf{H}}(e^{-i\boldsymbol{\theta}}) \} \\ &= \mathcal{M}(\boldsymbol{f}_y) \max_{\boldsymbol{\theta} \in [-\pi,\pi]} \left\| \sum_{j=1}^{\infty} \boldsymbol{W}_j e^{-ij\boldsymbol{\theta}} \right\|_{\mathrm{op}}^2 \\ &\leqslant \mathcal{M}(\boldsymbol{f}_y) \left(\sum_{j=1}^{\infty} \| \boldsymbol{W}_j \|_{\mathrm{op}} \right)^2 \end{split}$$

In addition, by a method similar to the proof of Proposition 2.3 in Basu and Michailidis (2015), we can show that

$$\lambda_{\max}(\underline{\Sigma}_w) \leqslant 2\pi \mathcal{M}(\boldsymbol{f}_w).$$

Combining the above results with (S68), the proof of (S51) is complete.

S4.17 Proof of Theorem 4

For a matrix $\boldsymbol{X} \in \mathbb{R}^{p_1 \times p_2}$, we denote by $\|\boldsymbol{X}\|_0$ the number of nonzero elements in \boldsymbol{X} , $\|\boldsymbol{X}\|_{2,0}$ the number of nonzero rows in \boldsymbol{X} , and define $\|\boldsymbol{X}\|_1 = \|\operatorname{vec}(\boldsymbol{X})\|_1$ and $\|\boldsymbol{X}\|_{2,1} = \sum_{i=1}^{p_1} \|\boldsymbol{x}_i\|_2$, where \boldsymbol{x}_i 's are the row vectors of \boldsymbol{X} . Let

$$\Upsilon_{\mathbb{S}} = \left\{ \Delta = \mathcal{A} - \mathcal{A}^* \in \mathbb{R}^{N \times N \times \infty} \mid \mathcal{A} = \mathcal{G} \times_3 L(\omega), \mathcal{G} \in \Gamma_{\mathbb{S}}, \omega \in \Omega, \delta_{\omega} \leqslant c_{\omega} \right\},$$

where $\Gamma_{\mathbb{S}} = \{ \mathcal{G} = \mathcal{S} \times_1 U_1 \times_2 U_2 \mid \mathcal{S} \in \Omega_{\mathcal{S}}, U_i \in \mathcal{U}_i, i = 1 \text{ or } 2 \}$. It is noteworthy that under the conditions of Theorem 4, $\widetilde{\Delta} := \widetilde{\mathcal{A}} - \mathcal{A}^* \in \Upsilon_{\mathbb{S}}$. For simplicity, we further denote the perturbation of \mathcal{S}^* by $\delta_{\mathcal{S}} = \|\mathcal{S} - \mathcal{S}^*\|_{\mathcal{F}}$. Moreover, we denote

$$\Gamma_{\mathcal{S}}(s_1, s_2, \mathcal{R}_1, \mathcal{R}_2) = \{ \mathcal{M} = \mathcal{S} \times_1 \mathcal{U}_1 \times_2 \mathcal{U}_2 \mid \mathcal{S} \in \mathbb{R}^{\mathcal{R}_1 \times \mathcal{R}_2 \times d}, \mathcal{U}_i \in \mathcal{U}_{\mathcal{S},i}, i = 1, 2 \},$$

where $U_{S,i} = \{ U \in \mathbb{R}^{N \times \mathcal{R}_i} \mid U'U = I_{\mathcal{R}_i}, ||U||_0 \leq s_i \}$, and then define

$$\Xi_{S}(s_{1}, s_{2}, \mathcal{R}_{1}, \mathcal{R}_{2}) = \left\{ \mathcal{M}(\boldsymbol{a}, \boldsymbol{\mathcal{B}}) \in \mathbb{R}^{N \times N \times (d+r+2s)} \mid \boldsymbol{a} \in \mathbb{R}^{r+2s}, \boldsymbol{\mathcal{B}} \in \Gamma_{S}(s_{1}, s_{2}, \mathcal{R}_{1}, \mathcal{R}_{2}) \right\}, \quad (S71)$$

and
$$\Xi_{S,1}(s_1, s_2, \mathcal{R}_1, \mathcal{R}_2) = \Xi_S(s_1, s_2, \mathcal{R}_1, \mathcal{R}_2) \cap \{ \mathbf{M} \in \mathbb{R}^{N \times N \times (d+r+2s)} \mid \| \mathbf{M} \|_F = 1 \}.$$

The proof of Theorem 4 depends directly on the following three lemmas.

Lemma S.14 (Strong convexity and smoothness properties for the sparse model). Under Assumptions 1–5, if $T \gtrsim (\kappa_2/\kappa_1)^2 \bar{d}_S \log(\kappa_2/\kappa_1)$, then with probability at least $1-2e^{-c\bar{d}_S \log(\kappa_2/\kappa_1)}-3e^{-c\bar{s}_2 \log N}$,

$$\kappa_1 \| \boldsymbol{\Delta} \|_{\mathrm{F}}^2 \lesssim \frac{1}{T} \sum_{t=1}^T \| \boldsymbol{\Delta}_{(1)} \boldsymbol{x}_t \|_2^2 \lesssim \kappa_2 \| \boldsymbol{\Delta} \|_{\mathrm{F}}^2, \quad \forall \boldsymbol{\Delta} \in \boldsymbol{\Upsilon}_{\mathbb{S}},$$

where $\bar{d}_{\mathcal{S}} = \mathcal{R}_1 \mathcal{R}_2 d + \sum_{i=1}^2 \bar{s}_i \mathcal{R}_i (1 + \log N \mathcal{R}_i)$ and $\bar{s}_i = (s_i + \underline{u}^{-1}) \mathcal{R}_i$, with i = 1 or 2.

Lemma S.15 (Deviation bound for the sparse model). Under the conditions of Lemma S.14 and if $T \gtrsim (\kappa_2/\kappa_1)^2 d_{\mathcal{S}} \log(\kappa_2/\kappa_1)$, given that $\lambda \gtrsim \sqrt{\kappa_2 \lambda_{\max}(\Sigma_{\varepsilon}) d_{\mathcal{S}}/T}$,

$$\left| \frac{1}{T} \left| \sum_{t=1}^{T} \langle \boldsymbol{\varepsilon}_{t}, \boldsymbol{\Delta}_{(1)} \boldsymbol{x}_{t} \rangle \right| \lesssim \lambda \left(\delta_{8} + \alpha \delta_{\boldsymbol{\omega}} + \sum_{i=1}^{2} \|\boldsymbol{\Delta}_{\boldsymbol{U}_{i}}\|_{1} \right) / 4 + \tau \|\boldsymbol{\Delta}_{\boldsymbol{U}_{1}}\|_{1} \|\boldsymbol{\Delta}_{\boldsymbol{U}_{2}}\|_{1}, \quad \forall \boldsymbol{\Delta} \in \boldsymbol{\Upsilon}_{\mathbb{S}}$$

holds with probability at least $1 - 3e^{-cd_{\mathcal{S}}\log(\kappa_2/\kappa_1)} - 4e^{-cs_2\log N(\mathcal{R}_1 \wedge \mathcal{R}_2)}$, where $d_{\mathcal{S}} = \mathcal{R}_1\mathcal{R}_2d + 2e^{-cs_2\log N(\mathcal{R}_1 \wedge \mathcal{R}_2)}$

$$\sum_{i=1}^{2} s_i \mathcal{R}_i (1 + \log N \mathcal{R}_i) \text{ and } \tau = \sqrt{(d + \log N)/T}.$$

Lemma S.16 (Effects of initial values). Under Assumptions 1-4, if $T \gtrsim \bar{s}_2$, then with probability at least $1 - c\sqrt{\bar{s}_2/T}(1 + \sqrt{\bar{s}_1/d_S})$,

$$|S_1(\boldsymbol{\Delta})| \lesssim \kappa_1 \|\boldsymbol{\Delta}\|_{\mathrm{F}}^2 / 4, \quad |S_i(\boldsymbol{\Delta})| \lesssim \sqrt{\frac{\kappa_2 \lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon}) d_{\mathcal{S}}}{T}} \|\boldsymbol{\Delta}\|_{\mathrm{F}} / 4, \quad i = 2, 3, \quad \forall \boldsymbol{\Delta} \in \boldsymbol{\Upsilon}_{\mathbb{S}} \cap \mathcal{S}(\delta),$$

where $\bar{d}_{\mathcal{S}}$ and $d_{\mathcal{S}}$ are defined in Lemmas S.14 and S.15, and $\bar{s}_i = (s_i + \underline{u}^{-1})\mathcal{R}_i$ for i = 1 or 2.

Now we give the proof of Theorem 4. Denote $\widetilde{\Delta} = \widetilde{\mathcal{A}} - \mathcal{A}^*$. Note that $\sum_{j=1}^{t-1} A_j y_{t-j} = \mathcal{A}_{(1)} \widetilde{x}_t$. Due to the optimality of $\widetilde{\mathcal{A}}$, we have

$$\sum_{t=1}^T \|\boldsymbol{y}_t - \boldsymbol{\mathcal{A}}_{(1)}^* \widetilde{\boldsymbol{x}}_t - \widetilde{\boldsymbol{\Delta}}_{(1)} \widetilde{\boldsymbol{x}}_t \|_2^2 + \lambda \sum_{i=1}^2 \|\widetilde{\boldsymbol{U}}_i\|_1 \leqslant \sum_{t=1}^T \|\boldsymbol{y}_t - \boldsymbol{\mathcal{A}}_{(1)}^* \widetilde{\boldsymbol{x}}_t \|_2^2 + \lambda \sum_{i=1}^2 \|\boldsymbol{U}_i^*\|_1,$$

Then, since $\boldsymbol{y}_t - \boldsymbol{\mathcal{A}}_{(1)}^* \widetilde{\boldsymbol{x}}_t = \boldsymbol{\varepsilon}_t + \sum_{j=t}^{\infty} \boldsymbol{A}_j^* \boldsymbol{y}_{t-j}$ and $\widetilde{\boldsymbol{\Delta}}_{(1)} (\boldsymbol{x}_t - \widetilde{\boldsymbol{x}}_t) = \sum_{k=t}^{\infty} \widetilde{\boldsymbol{\Delta}}_k \boldsymbol{y}_{t-k}$, it follows from (S19) and (S20) that

$$\frac{1}{T} \sum_{t=1}^{T} \|\widetilde{\boldsymbol{\Delta}}_{(1)} \boldsymbol{x}_{t}\|_{2}^{2} \leqslant \frac{2}{T} \sum_{t=1}^{T} \langle \boldsymbol{\varepsilon}_{t}, \widetilde{\boldsymbol{\Delta}}_{(1)} \boldsymbol{x}_{t} \rangle + \sum_{k=1}^{3} S_{k}(\widetilde{\boldsymbol{\Delta}}) + \lambda \sum_{i=1}^{2} \left(\|\boldsymbol{U}_{i}^{*}\|_{1} - \|\widetilde{\boldsymbol{U}}_{i}\|_{1} \right), \tag{S72}$$

where $S_k(\cdot)$ for $1 \leq k \leq 3$ are the initialization error terms defined as in (S18), and $\widetilde{\Delta}_{(1)}\boldsymbol{x}_t = \sum_{k=1}^{\infty} \widetilde{\Delta}_k \boldsymbol{y}_{t-k}$. Let $\widetilde{\Delta}_s = \widetilde{S} - S^*$, $\widetilde{\Delta}_{\omega} = \widetilde{\omega} - \omega^*$, and $\widetilde{\Delta}_{\boldsymbol{U}_i} = \widetilde{\boldsymbol{U}}_i - \boldsymbol{U}_i^*$ for i = 1 or 2. On the right-hand side of (S72), denote the event that the first term are bounded by $\lambda(\|\widetilde{\Delta}_s\|_F + \alpha\|\widetilde{\Delta}_{\omega}\|_2 + \sum_{i=1}^2 \|\widetilde{\Delta}_{\boldsymbol{U}_i}\|_1)$ and $\tau \|\widetilde{\Delta}_{\boldsymbol{U}_1}\|_1 \|\widetilde{\Delta}_{\boldsymbol{U}_2}\|_1$ as \mathcal{I}_1 ,

$$\mathcal{I}_{1} = \left\{ \frac{1}{T} \left| \sum_{t=1}^{T} \langle \boldsymbol{\varepsilon}_{t}, \widetilde{\boldsymbol{\Delta}}_{(1)} \boldsymbol{x}_{t} \rangle \right| \lesssim \lambda \left(\|\widetilde{\boldsymbol{\Delta}}_{8}\|_{F} + \alpha \|\widetilde{\boldsymbol{\Delta}}_{\omega}\|_{2} + \sum_{i=1}^{2} \|\widetilde{\boldsymbol{\Delta}}_{\boldsymbol{U}_{i}}\|_{1} \right) / 4 + \tau \|\widetilde{\boldsymbol{\Delta}}_{\boldsymbol{U}_{1}}\|_{1} \|\widetilde{\boldsymbol{\Delta}}_{\boldsymbol{U}_{2}}\|_{1} \right\}.$$

On event \mathcal{I}_1 , if we multiply 2 to both sides of (S72) we have

$$\frac{2}{T} \sum_{t=1}^{T} \|\widetilde{\boldsymbol{\Delta}}_{(1)} \boldsymbol{x}_{t}\|_{2}^{2} \leq \lambda \left(\|\widetilde{\boldsymbol{\Delta}}_{8}\|_{F} + \alpha \|\widetilde{\boldsymbol{\Delta}}_{\omega}\|_{2} + \sum_{i=1}^{2} \|\widetilde{\boldsymbol{\Delta}}_{\boldsymbol{U}_{i}}\|_{1} + 2 \sum_{i=1}^{2} \|\boldsymbol{U}_{i}^{*}\|_{1} - 2 \sum_{i=1}^{2} \|\widetilde{\boldsymbol{U}}_{i}\|_{1} \right) + 2 \sum_{k=1}^{3} S_{k}(\widetilde{\boldsymbol{\Delta}}) + 4\tau \|\widetilde{\boldsymbol{\Delta}}_{\boldsymbol{U}_{1}}\|_{1} \|\widetilde{\boldsymbol{\Delta}}_{\boldsymbol{U}_{2}}\|_{1}.$$

Denote by \mathbb{S}_i the index set of the nonzero entries of U_i^* , and by \mathbb{S}_i^C the complement of \mathbb{S}_{U_i} , for

i=1 or 2. By the elementwise sparsity of each \boldsymbol{U}_{i}^{*} in Assumption 3, the cardinality of the index set $|\mathbb{S}_{\boldsymbol{U}_{i}}| = \|\boldsymbol{U}_{i}^{*}\|_{0} \leqslant s_{i}\mathcal{R}_{i}$ for i=1 or 2. Note that $\|\widetilde{\boldsymbol{U}}_{i}\|_{1} \geqslant \|(\widetilde{\boldsymbol{\Delta}}_{\boldsymbol{U}_{i}})_{\mathbb{S}_{i}^{C}} + (\boldsymbol{U}_{i}^{*})_{\mathbb{S}_{i}}\|_{1} - \|(\widetilde{\boldsymbol{\Delta}}_{\boldsymbol{U}_{i}})_{\mathbb{S}_{i}}\|_{1} = \|(\widetilde{\boldsymbol{\Delta}}_{\boldsymbol{U}_{i}})_{\mathbb{S}_{i}^{C}}\|_{1} + \|(\boldsymbol{U}_{i}^{*})_{\mathbb{S}_{i}}\|_{1} - \|(\widetilde{\boldsymbol{\Delta}}_{\boldsymbol{U}_{i}})_{\mathbb{S}_{i}}\|_{1}, (\boldsymbol{U}_{i}^{*})_{\mathbb{S}_{i}} = \boldsymbol{U}_{i}^{*} \text{ and } \|\widetilde{\boldsymbol{\Delta}}_{\boldsymbol{U}_{i}}\|_{1} \leqslant \|(\widetilde{\boldsymbol{\Delta}}_{\boldsymbol{U}_{i}})_{\mathbb{S}_{i}}\|_{1} + \|(\widetilde{\boldsymbol{\Delta}}_{\boldsymbol{U}_{i}})_{\mathbb{S}_{i}^{C}}\|_{1}, \text{ so we have}$

$$\frac{2}{T} \sum_{t=1}^{T} \|\widetilde{\boldsymbol{\Delta}}_{(1)} \boldsymbol{x}_{t}\|_{2}^{2} \leq \lambda (\|\widetilde{\boldsymbol{\Delta}}_{\mathbf{S}}\|_{F} + \alpha \|\widetilde{\boldsymbol{\Delta}}_{\omega}\|_{2}) + 3\lambda \sum_{i=1}^{2} \|(\widetilde{\boldsymbol{\Delta}}_{\boldsymbol{U}_{i}})_{\mathbb{S}_{i}}\|_{1} - \lambda \sum_{i=1}^{2} \|(\widetilde{\boldsymbol{\Delta}}_{\boldsymbol{U}_{i}})_{\mathbb{S}_{i}^{C}}\|_{1} + 2\sum_{k=1}^{3} S_{k}(\widetilde{\boldsymbol{\Delta}}) + 4\tau \|\widetilde{\boldsymbol{\Delta}}_{\boldsymbol{U}_{1}}\|_{1} \|\widetilde{\boldsymbol{\Delta}}_{\boldsymbol{U}_{2}}\|_{1}.$$

Next, we assume that there is a lower bound for $T^{-1}\sum_{t=1}^{T} \|\widetilde{\boldsymbol{\Delta}}_{(1)}\boldsymbol{x}_{t}\|_{2}^{2}$ and then define the event $\mathcal{I}_{2} = \{T^{-1}\sum_{t=1}^{T} \|\widetilde{\boldsymbol{\Delta}}_{(1)}\boldsymbol{x}_{t}\|_{2}^{2} \gtrsim \kappa_{1} \|\widetilde{\boldsymbol{\Delta}}\|_{\mathrm{F}}^{2}\}$, where $\kappa_{1} = \lambda_{\min}(\boldsymbol{\Sigma}_{\varepsilon})\mu_{\min}(\boldsymbol{\Psi}_{*})\min\{1,c_{\bar{\rho}}^{2}\}$ and $c_{\bar{\rho}}$ is an absolute constant defined in Lemma S.2 in the Appendix. Moreover, we assume that the initialization error terms have a upper bound, and denote the event

$$\mathcal{I}_3 = \{ S_1(\widetilde{\Delta}) \lesssim \kappa_1 \|\widetilde{\Delta}\|_F^2 / 4, S_k(\widetilde{\Delta}) \lesssim \lambda \|\widetilde{\Delta}\|_F / 4, k = 2 \text{ or } 3 \}.$$

By Assumptions 3 and 4, let $\bar{s}_i = (s_i + \underline{u}^{-1})\mathcal{R}_i$, and then $\widetilde{\Delta}_{U_i}$ has an elementwise sparsity of at most \bar{s}_i for i = 1 or 2. Hence, by the perturbation bounds in Lemma S.17 and Lemma 1,

$$\|\widetilde{\Delta}_{U_i}\|_1 \leq \sqrt{\overline{s}_i} \|\widetilde{\Delta}_{U_i}\|_{\mathcal{F}} \leq c_{\Lambda}^{-1} \beta^{-1} C \eta_i \sqrt{\overline{s}_i} \|\widetilde{\Delta}\|_{\mathcal{F}}.$$

Suppose that $c_{\Delta}^{-2}\beta^{-2}C^2\eta_1\eta_2\sqrt{\bar{s}_1\bar{s}_2}\tau \lesssim \kappa_1/8$, on the events $\mathcal{I}_1,\mathcal{I}_2$ and \mathcal{I}_3 ,

$$\kappa_{1} \|\widetilde{\boldsymbol{\Delta}}\|_{F}^{2} \lesssim \lambda \|\widetilde{\boldsymbol{\Delta}}\|_{F} + \lambda (\|\widetilde{\boldsymbol{\Delta}}_{s}\|_{F} + \alpha \|\widetilde{\boldsymbol{\Delta}}_{\omega}\|_{2}) + 3\lambda \sum_{i=1}^{2} \|(\widetilde{\boldsymbol{\Delta}}_{U_{i}})_{\mathbb{S}_{i}}\|_{1}$$

$$\lesssim \lambda \|\widetilde{\boldsymbol{\Delta}}\|_{F} + \lambda (\|\widetilde{\boldsymbol{\Delta}}_{s}\|_{F} + \alpha \|\widetilde{\boldsymbol{\Delta}}_{\omega}\|_{2}) + 3\lambda \sum_{i=1}^{2} \sqrt{s_{i}} \|\widetilde{\boldsymbol{\Delta}}_{U_{i}}\|_{F},$$

where by the perturbation bounds in Lemma S.17 and Lemma 1, $\|\widetilde{\Delta}_{U_i}\|_{\text{F}} \leqslant c_{\Delta}^{-1}\beta^{-1}C\eta_i\|\widetilde{\Delta}\|_{\text{F}}$. Similarly, we can show

$$\|\widetilde{\boldsymbol{\Delta}}_{\mathbf{s}}\|_{F} + \alpha \|\widetilde{\boldsymbol{\Delta}}_{\omega}\|_{2} \overset{\text{(Lemma S.17)}}{\leqslant} \beta^{-1} C(\eta_{1} + \eta_{2}) \|\widetilde{\boldsymbol{G}} - \boldsymbol{G}^{*}\|_{F} + \alpha \|\widetilde{\boldsymbol{\omega}} - \boldsymbol{\omega}^{*}\|_{2}$$

$$\overset{\text{(Lemma 1)}}{\leqslant} c_{\Delta}^{-1} \max \left(\beta^{-1} C(\eta_{1} + \eta_{2}), 1\right) \|\widetilde{\boldsymbol{\Delta}}\|_{F},$$

Since $\sqrt{x} + \sqrt{y} \le 2\sqrt{x+y}$ for any $x, y \ge 0$, we have

$$\kappa_1 \|\widetilde{\Delta}\|_{\mathrm{F}}^2 \lesssim c_{\Delta}^{-1} \beta^{-1} (\eta_1 + \eta_2) \sqrt{s_1 + s_2} \lambda \|\widetilde{\Delta}\|_{\mathrm{F}}.$$

And the estimation error bound and in-sample prediction error bound are given by

$$\|\widetilde{\Delta}\|_{\mathrm{F}} \lesssim (c_{\Delta}\beta\kappa_{1})^{-1}(\eta_{1}+\eta_{2})\sqrt{s_{1}+s_{2}}\lambda \quad \text{and} \quad \frac{1}{T}\sum_{t=1}^{T}\|\widetilde{\Delta}_{(1)}\widetilde{x}_{t}\|_{2}^{2} \lesssim (c_{\Delta}^{2}\beta^{2}\kappa_{1})^{-1}(\eta_{1}+\eta_{2})^{2}(s_{1}+s_{2})\lambda^{2},$$

respectively.

In the second part, we show the conditions that events \mathcal{I}_1 , \mathcal{I}_2 and \mathcal{I}_3 occur with high probability. First, denote by $d_{\mathcal{S}} = \mathcal{R}_1 \mathcal{R}_2 d + \sum_{i=1}^2 s_i \mathcal{R}_i (1 + \log N \mathcal{R}_i)$ the sample complexity of the model. If $T \gtrsim (\kappa_2/\kappa_1)^2 d_{\mathcal{S}} \log(\kappa_2/\kappa_1)$, given that $\lambda \gtrsim \sqrt{\kappa_2 \lambda_{\max}(\Sigma_{\varepsilon}) d_{\mathcal{S}}/T}$ and $\tau = \sqrt{(d + \log N)/T}$, it follows from Lemma S.15 that the event \mathcal{I}_1 holds with probability at least $1 - 3e^{-cd_{\mathcal{S}}\log(\kappa_2/\kappa_1)} - 4e^{-cs_2\log N(\mathcal{R}_1 \wedge \mathcal{R}_2)}$. Moreover, if $T \gtrsim \beta^{-4}\eta_1^{-2}\eta_2^{-2}\bar{s}_1\bar{s}_2(d + \log N)$, the condition that $c_{\Delta}^{-2}\beta^{-2}C^2\eta_1\eta_2\sqrt{\bar{s}_1\bar{s}_2}\tau \lesssim \kappa_1/8$ holds. Secondly, if $T \gtrsim (\kappa_2/\kappa_1)^2\bar{d}_{\mathcal{S}}\log(\kappa_2/\kappa_1)$, it follows from Lemma S.14 that the event \mathcal{I}_2 holds with probability at least $1 - 2e^{-c\bar{d}_{\mathcal{S}}\log(\kappa_2/\kappa_1)} - 3e^{-c\bar{s}_2\log N}$, where $\bar{d}_{\mathcal{S}} = \mathcal{R}_1\mathcal{R}_2d + \sum_{i=1}^2 \bar{s}_i\mathcal{R}_i(1 + \log N\mathcal{R}_i)$. Finally, if $T \gtrsim \bar{s}_2$, then it follows from Lemma S.16 that with probability at least $1 - c\sqrt{\bar{s}_2/T}(1 + \sqrt{\bar{s}_1/d_{\mathcal{S}}})$, the event \mathcal{I}_3 holds.

S4.18 Proof of Lemma S.14

The proof of this lemma is largely based on some existing results in the proof of Lemma S.3 in Section S4.7. First of all, under Assumptions 3 & 4, there are at most $\underline{u}^{-1}\mathcal{R}_i$ nonzero rows in U_i and $s_i\mathcal{R}_i$ ones in U_i^* , for all $U_i \in \mathcal{U}_i$ with i = 1 or 2. Hence, for any $\Delta = \mathcal{A} - \mathcal{A}^* \in \Upsilon_{\mathbb{S}}$, we only need to consider those that satisfy $\mathfrak{G}_{\text{stack}} \in \Xi_{\mathbb{S}}(\bar{s}_1\mathcal{R}_1, \bar{s}_2\mathcal{R}_2, 2\mathcal{R}_1, 2\mathcal{R}_2)$ and $\mathbf{R}_j \in \Pi(\bar{s}_1(\mathcal{R}_1 \wedge \mathcal{R}_2), \bar{s}_2(\mathcal{R}_1 \wedge \mathcal{R}_2), 2(\mathcal{R}_1 \wedge \mathcal{R}_2))$ for all $j \geq 1$.

Then, it remains to show that the following two results hold for all $\Delta \in \Upsilon_{\mathbb{S}} \cap \mathcal{S}(\delta)$ that satisfy the above sparsity conditions.

(i) If
$$T \gtrsim (\kappa_2/\kappa_1)^2 \bar{d}_S \log(\kappa_2/\kappa_1)$$
, then

$$\mathbb{P}\left\{\forall \boldsymbol{\Delta} \in \boldsymbol{\Upsilon}_{\mathbb{S}} \cap \mathcal{S}(\delta) : \frac{c_{\mathbf{M}}\delta^{2}\kappa_{1}}{8} \lesssim \frac{1}{T} \sum_{t=1}^{T} \|(\boldsymbol{\mathcal{G}}_{\text{stack}})_{(1)}\boldsymbol{z}_{t}\|_{2}^{2} \lesssim 6C_{\mathbf{M}}\delta^{2}\kappa_{2}\right\} \geqslant 1 - 2e^{-c\bar{d}_{\mathcal{S}}\log(\kappa_{2}/\kappa_{1})}.$$

(ii) If $T \gtrsim \bar{s}_2 \log N$, then

$$\mathbb{P}\left\{\sup_{\boldsymbol{\Delta}\in\boldsymbol{\Upsilon}_{\mathbb{S}}\cap\mathcal{S}(\delta)}\frac{1}{T}\sum_{t=1}^{T}\|\boldsymbol{\mathcal{R}}_{(1)}\boldsymbol{x}_{t-p}\|_{2}^{2}\lesssim\delta^{2}\delta_{\boldsymbol{\omega}}^{2}\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})\mu_{\max}(\boldsymbol{\Psi}_{*})\right\}\geqslant1-3e^{-c\bar{s}_{2}\log N}.$$

The result in (i) can be jointly obtained from (S28), (S30) and Lemma S.20. While to obtain the result in (ii), we first denote by $\mathbb{K}(s) = \{ \boldsymbol{v} \in \mathbb{R}^N : \|\boldsymbol{v}\|_0 \leq s, \|\boldsymbol{v}\|_2 \leq 1 \}$ the set of s-sparse vectors. Note that for all $j \geq 1$,

$$\frac{1}{T} \sum_{t=1}^{T} \| \boldsymbol{R}_{j} \boldsymbol{y}_{t-p-j} \|_{2}^{2} = \operatorname{tr} \left\{ \boldsymbol{R}_{j} \left(\frac{1}{T} \sum_{t=1}^{T} \boldsymbol{y}_{t-p-j} \boldsymbol{y}'_{t-p-j} \right) \boldsymbol{R}'_{j} \right\} \\
\leq \| \boldsymbol{R}_{j} \|_{F}^{2} \left(\sup_{\boldsymbol{v} \in \mathbb{K}(\bar{s}_{2})} \boldsymbol{v}' \frac{1}{T} \sum_{t=1}^{T} \boldsymbol{y}_{t-p-j} \boldsymbol{y}'_{t-p-j} \boldsymbol{v} \right).$$

Combining this with (S32) & (S34) and (S83), if $T \gtrsim \bar{s}_2 \log N$, with probability at least $1 - 3e^{-\bar{s}_2 \log N \log 9}$, we have

$$\sup_{\boldsymbol{\Delta} \in \boldsymbol{\Upsilon}_{\mathbb{S}} \cap \mathcal{S}(\delta)} \frac{1}{T} \sum_{t=1}^{T} \|\boldsymbol{\mathcal{R}}_{(1)} \boldsymbol{x}_{t-p}\|_{2}^{2} \lesssim \delta^{2} \delta_{\boldsymbol{\omega}}^{2} \lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon}) \mu_{\max}(\boldsymbol{\Psi}_{*}) \left(\sum_{j=1}^{\infty} \bar{\rho}^{j} \sqrt{j\sigma^{2} + 1} \right)^{2} \lesssim \delta^{2} \delta_{\boldsymbol{\omega}}^{2} \lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon}) \mu_{\max}(\boldsymbol{\Psi}_{*}),$$

where the second inequality follows from the fact that $\sum_{j=1}^{\infty} \bar{\rho}^j \sqrt{j\sigma^2 + 1} \approx 1$. Thus (ii) is verified. The rest of the proof is same as the proof of Lemma S.3 and hence is omitted here.

S4.19 Proof of Lemma S.15

The proof of this lemma closely follows the proof of Lemma S.4. Essentially, we only need to show the following two intermediate results.

(i) If $T \gtrsim (\kappa_2/\kappa_1)^2 d_{\mathcal{S}} \log(\kappa_2/\kappa_1)$, then

$$\mathbb{P}\left\{\sup_{\boldsymbol{\Delta}\in\boldsymbol{\Upsilon}_{\mathbb{S}}}\frac{1}{T}\left|\sum_{t=1}^{T}\langle(\mathbf{9}_{\text{stack}})_{(1)}\boldsymbol{z}_{t},\boldsymbol{\varepsilon}_{t}\rangle\right| \leqslant \lambda\left(\delta_{8}+\alpha\delta_{\boldsymbol{\omega}}+\sum_{i=1}^{2}\|\boldsymbol{\Delta}_{\boldsymbol{U}_{i}}\|_{1}\right)/4+\tau\|\boldsymbol{\Delta}_{\boldsymbol{U}_{1}}\|_{1}\|\boldsymbol{\Delta}_{\boldsymbol{U}_{2}}\|_{1}\right\}$$

$$\geqslant 1-e^{-cd_{\mathcal{S}}}-2e^{-cd_{\mathcal{S}}\log(\kappa_{2}/\kappa_{1})}.$$

(ii) If $T \gtrsim (s_1 + s_2)(\mathcal{R}_1 \wedge \mathcal{R}_2 + \log N(\mathcal{R}_1 \wedge \mathcal{R}_2))$, then

$$\mathbb{P}\left\{\sum_{j=1}^{\infty} \sup_{\boldsymbol{\Delta} \in \boldsymbol{\Upsilon}_{\mathbb{S}}} \frac{1}{T} \left| \sum_{t=1}^{T} \langle \boldsymbol{R}_{j} \boldsymbol{y}_{t-p-j}, \boldsymbol{\varepsilon}_{t} \rangle \right| \leq \lambda \delta_{\boldsymbol{\omega}} \left(\delta_{\mathbf{S}} + \alpha \delta_{\boldsymbol{\omega}} + \sum_{i=1}^{2} \|\boldsymbol{\Delta}_{\boldsymbol{U}_{i}}\|_{1} \right) / 4 + \tau \delta_{\boldsymbol{\omega}} \|\boldsymbol{\Delta}_{\boldsymbol{U}_{1}}\|_{1} \|\boldsymbol{\Delta}_{\boldsymbol{U}_{2}}\|_{1} \right\}$$

$$\geqslant 1 - 4e^{-s_{2} \log N(\mathcal{R}_{1} \wedge \mathcal{R}_{2}) \log 9}.$$

Proof of (i): Note that $\mathcal{G} - \mathcal{G}^* = \mathcal{S} \times_1 \Delta_{U_1} \times_2 U_2^* + \mathcal{S} \times_1 U_1^* \times_2 \Delta_{U_2} + \mathcal{S} \times_1 \Delta_{U_1} \times_2 \Delta_{U_2} + \Delta_{U_2} \times_2 \Delta_{U_2} + \times_2 \Delta_{U_2} \times_2 \Delta_{U_2} + \Delta_{U_2} \times_2 \Delta$

$$\mathbf{M}_{1} = \sum_{i=1}^{N} \sum_{m=1}^{\mathcal{R}_{1}} \mathbf{M}_{1,i,m}, \quad \mathbf{M}_{2} = \sum_{i=1}^{N} \sum_{m=1}^{\mathcal{R}_{2}} \mathbf{M}_{2,i,m}, \quad \mathbf{M}_{3} = \sum_{i,j=1}^{N} \sum_{m=1}^{\mathcal{R}_{1}} \sum_{h=1}^{\mathcal{R}_{2}} \mathbf{M}_{3,i,j,m,h}, \text{ and}$$

$$\mathbf{M}_{4} = \operatorname{stack}(\mathbf{\Delta}_{s} \times_{1} \mathbf{U}_{1}^{*} \times_{2} \mathbf{U}_{2}^{*}, \mathbf{\mathcal{D}}(\boldsymbol{\omega})) = \mathbf{\mathcal{M}}(\boldsymbol{\omega} - \boldsymbol{\omega}^{*}, \mathbf{\Delta}_{s} \times_{1} \mathbf{U}_{1}^{*} \times_{2} \mathbf{U}_{2}^{*}),$$

where for any $\mathbf{a} = (a_1, \dots, a_{r+2s})' \in \mathbb{R}^{r+2s}$ and $\mathbf{B} \in \mathbb{R}^{N \times N \times d}$, the bilinear functional $\mathbf{M}(\mathbf{a}, \mathbf{B})$ is defined as in (S3), and moreover, for $1 \leq i \leq N$, the $N \times N \times (d+r+2s)$ tensors $\mathbf{M}_{1,i}$, $\mathbf{M}_{2,i}$ and $\mathbf{M}_{3,i,m}$ are defined respectively by

$$\begin{split} \mathbf{\mathcal{M}}_{1,i,m} &= \mathbf{\mathcal{M}}(\mathbf{0},\mathbf{S}\times_{1}(\mathbf{\Delta}_{\boldsymbol{U}_{1}}^{(i,m)}\boldsymbol{e}_{i}\overline{\boldsymbol{e}}_{m}^{\prime})\times_{2}\boldsymbol{U}_{2}^{*\prime}), \quad \mathbf{\mathcal{M}}_{2,i,m} = \mathbf{\mathcal{M}}(\mathbf{0},\mathbf{S}\times_{1}\boldsymbol{U}_{1}^{*\prime}\times_{2}(\mathbf{\Delta}_{\boldsymbol{U}_{2}}^{(i,m)}\boldsymbol{e}_{i}\widetilde{\boldsymbol{e}}_{m}^{\prime})), \\ \text{and } \mathbf{\mathcal{M}}_{3,i,j,m,h} &= \mathbf{\mathcal{M}}(\mathbf{0},\mathbf{S}\times_{1}(\mathbf{\Delta}_{\boldsymbol{U}_{1}}^{(i,m)}\boldsymbol{e}_{i}\overline{\boldsymbol{e}}_{m}^{\prime})\times_{2}(\mathbf{\Delta}_{\boldsymbol{U}_{2}}^{(j,h)}\boldsymbol{e}_{j}\widetilde{\boldsymbol{e}}_{h}^{\prime})) \end{split}$$

where e_{ℓ} , \bar{e}_{ℓ} and \tilde{e}_{ℓ} are coordinate vectors whose ℓ -th element is 1 and the others are 0 of dimensional N, \mathcal{R}_1 and \mathcal{R}_2 respectively, and $\Delta_{U_i}^{(k,\ell)}$ is the (k,ℓ) -th element of Δ_{U_i} , i=1,2 with $1 \leq \ell \leq N$. Since $\|U_i^*\|_{\text{op}} = 1$ and $\|S_{(i)}\|_{\text{op}} = \|S_{(i)}\|_{\text{op}}$ for i=1 or 2, by Assumption 2, the norms of \mathcal{M}_k 's further satisfy

$$\|\mathbf{M}_{1,i,m}\|_{F} \leq \|\mathbf{S}_{(1)}\|_{\text{op}}|\mathbf{\Delta}_{U_{1}}^{(i,m)}| \leq C_{g}|\mathbf{\Delta}_{U_{1}}^{(i,m)}|, \quad \|\mathbf{M}_{2,i,m}\|_{F} \leq \|\mathbf{S}_{(2)}\|_{\text{op}}|\mathbf{\Delta}_{U_{2}}^{(i,m)}| \leq C_{g}|\mathbf{\Delta}_{U_{2}}^{(i,m)}|,$$

$$\|\mathbf{M}_{3,i,j,m,h}\|_{F} \leq \|\mathbf{S}_{(1)}\|_{\text{op}}|\mathbf{\Delta}_{U_{1}}^{(i,m)}||\mathbf{\Delta}_{U_{2}}^{(j,h)}| \leq C_{g}|\mathbf{\Delta}_{U_{1}}^{(i,m)}||\mathbf{\Delta}_{U_{2}}^{(j,h)}|, \text{ and } \|\mathbf{M}_{4}\|_{F} \leq C_{\Delta}\left(\delta_{s} + \alpha\delta_{\omega}\right),$$
(S73)

where the norm bound on \mathfrak{M}_4 follows from Lemma 1.

As a result,

$$\begin{split} \frac{1}{T} \sum_{t=1}^{T} \langle \boldsymbol{\varepsilon}_{t}, (\boldsymbol{\mathcal{G}}_{\text{stack}})_{(1)} \boldsymbol{z}_{t} \rangle &= \sum_{k=1}^{2} \sum_{i=1}^{N} \sum_{m=1}^{\mathcal{R}_{k}} \frac{1}{T} \sum_{t=1}^{T} \langle \boldsymbol{\varepsilon}_{t}, (\boldsymbol{\mathcal{M}}_{k,i,m})_{(1)} \boldsymbol{z}_{t} \rangle + \frac{1}{T} \sum_{t=1}^{T} \langle \boldsymbol{\varepsilon}_{t}, (\boldsymbol{\mathcal{M}}_{4})_{(1)} \boldsymbol{z}_{t} \rangle \\ &+ \sum_{i,j=1}^{N} \sum_{m=1}^{\mathcal{R}_{1}} \sum_{h=1}^{\mathcal{R}_{2}} \frac{1}{T} \sum_{t=1}^{T} \langle \boldsymbol{\varepsilon}_{t}, (\boldsymbol{\mathcal{M}}_{3,i,j,m,h})_{(1)} \boldsymbol{z}_{t} \rangle \\ &\leqslant \sum_{i=1}^{N} \sum_{m=1}^{\mathcal{R}_{1}} \| \boldsymbol{\mathcal{M}}_{1,i,m} \|_{F} \sup_{\boldsymbol{\mathcal{M}} \in \boldsymbol{\Xi}_{S,1}(1,s_{2}\mathcal{R}_{2},1,\mathcal{R}_{2})} \frac{1}{T} \sum_{t=1}^{T} \langle \boldsymbol{\varepsilon}_{t}, \boldsymbol{\mathcal{M}}_{(1)} \boldsymbol{z}_{t} \rangle \\ &+ \sum_{i=1}^{N} \sum_{m=1}^{\mathcal{R}_{2}} \| \boldsymbol{\mathcal{M}}_{2,i,m} \|_{F} \sup_{\boldsymbol{\mathcal{M}} \in \boldsymbol{\Xi}_{S,1}(s_{1}\mathcal{R}_{1},1,\mathcal{R}_{1},1)} \frac{1}{T} \sum_{t=1}^{T} \langle \boldsymbol{\varepsilon}_{t}, \boldsymbol{\mathcal{M}}_{(1)} \boldsymbol{z}_{t} \rangle \\ &+ \sum_{i,j=1}^{N} \sum_{m=1}^{\mathcal{R}_{1}} \sum_{h=1}^{\mathcal{R}_{2}} \| \boldsymbol{\mathcal{M}}_{3,i,j,m,h} \|_{F} \sup_{\boldsymbol{\mathcal{M}} \in \boldsymbol{\Xi}_{S,1}(1,1,1,1)} \frac{1}{T} \sum_{t=1}^{T} \langle \boldsymbol{\varepsilon}_{t}, \boldsymbol{\mathcal{M}}_{(1)} \boldsymbol{z}_{t} \rangle \\ &+ \| \boldsymbol{\mathcal{M}}_{4} \|_{F} \sup_{\boldsymbol{\mathcal{M}} \in \boldsymbol{\Xi}_{S,1}(s_{1}\mathcal{R}_{1},s_{2}\mathcal{R}_{2},\mathcal{R}_{1},\mathcal{R}_{2})} \frac{1}{T} \sum_{t=1}^{T} \langle \boldsymbol{\varepsilon}_{t}, \boldsymbol{\mathcal{M}}_{(1)} \boldsymbol{z}_{t} \rangle. \end{split}$$

Combine this with (S73) and (S82) in Lemma S.20, we can show (i).

Proof of (ii): Note that $\mathbf{R}_j = \mathbf{R}_{1j} + \mathbf{R}_{2j} + \mathbf{R}_{3j}$, where \mathbf{R}_{kj} , $1 \le k \le 3$ are defined in (S9). In \mathbf{R}_{1j} and \mathbf{R}_{2j} , we have $\mathbf{G}_k - \mathbf{G}_k^* = \Delta_{U_1} \mathbf{S}_k \mathbf{U}_2^{*'} + \mathbf{U}_1^* \mathbf{S}_k \Delta_{U_2}' + \Delta_{U_1} \mathbf{S}_k \Delta_{U_2}' + \mathbf{U}_1^* (\mathbf{S}_k - \mathbf{S}_k^*) \mathbf{U}_2^{*'}$ for all \mathbf{G}_k -matrices, while in \mathbf{R}_{3j} , we have $\mathbf{G}_k^* = \mathbf{U}_1^* \mathbf{S}_k^* \mathbf{U}_2^{*'}$ for all \mathbf{G}_k^* -matrices. It is then possible to further break down \mathbf{R}_{1j} into $\mathbf{M}_{1j} + \mathbf{M}_{3j} + \mathbf{M}_{5j} + \mathbf{M}_{7j}$ and \mathbf{R}_{2j} into $\mathbf{M}_{2j} + \mathbf{M}_{4j} + \mathbf{M}_{6j} + \mathbf{M}_{8j}$, respectively, where

$$\begin{split} \boldsymbol{M}_{kj} &= \sum_{i=1}^{N} \sum_{m=1}^{\mathcal{R}_{1}} \boldsymbol{M}_{kj,i,m}, \ 1 \leqslant k \leqslant 2, \ \boldsymbol{M}_{kj} = \sum_{i=1}^{N} \sum_{m=1}^{\mathcal{R}_{2}} \boldsymbol{M}_{kj,i,m}, \ 3 \leqslant k \leqslant 4, \\ \boldsymbol{M}_{kj} &= \sum_{i,\ell=1}^{N} \sum_{m=1}^{\mathcal{R}_{1}} \sum_{h=1}^{\mathcal{R}_{2}} \boldsymbol{M}_{kj,i,\ell,m,h}, \ 5 \leqslant k \leqslant 6, \\ \boldsymbol{M}_{7j} &= \sum_{k=1}^{r} \nabla \ell_{j}^{I}(\lambda_{k}^{*})(\lambda_{k} - \lambda_{k}^{*}) \boldsymbol{U}_{1}^{*}(\boldsymbol{S}_{k}^{I} - \boldsymbol{S}_{k}^{I*}) \boldsymbol{U}_{2}^{*\prime} \\ &+ \sum_{k=1}^{s} \sum_{h=1}^{2} (\boldsymbol{\eta}_{k} - \boldsymbol{\eta}_{k}^{*})' \nabla \ell_{j}^{II,h}(\boldsymbol{\eta}_{k}^{*}) \boldsymbol{U}_{1}^{*}(\boldsymbol{S}_{k}^{II,h} - \boldsymbol{S}_{k}^{II,h*}) \boldsymbol{U}_{2}^{*\prime}, \\ \boldsymbol{M}_{8j} &= \frac{1}{2} \sum_{k=1}^{r} \nabla^{2} \ell_{j}^{I}(\widetilde{\lambda}_{k})(\lambda_{k} - \lambda_{k}^{*})^{2} \boldsymbol{U}_{1}^{*}(\boldsymbol{S}_{k}^{I} - \boldsymbol{S}_{k}^{I*}) \boldsymbol{U}_{2}^{*\prime} \\ &+ \frac{1}{2} \sum_{k=1}^{s} \sum_{h=1}^{2} (\boldsymbol{\eta}_{k} - \boldsymbol{\eta}_{k}^{*})' \nabla^{2} \ell_{j}^{II,h}(\widetilde{\boldsymbol{\eta}}_{k})(\boldsymbol{\eta}_{k} - \boldsymbol{\eta}_{k}^{*}) \boldsymbol{U}_{1}^{*}(\boldsymbol{S}_{k}^{II,h} - \boldsymbol{S}_{k}^{II,h*}) \boldsymbol{U}_{2}^{*\prime}, \end{split}$$

where $\boldsymbol{M}_{kj,i,m} = \boldsymbol{\Delta}_{\boldsymbol{U}_1}^{(i,m)} \boldsymbol{e}_i \bar{\boldsymbol{e}}_m' \boldsymbol{T}_{kj} \boldsymbol{U}_2^*$ for $1 \leqslant k \leqslant 2$, $\boldsymbol{M}_{kj,i,m} = \boldsymbol{\Delta}_{\boldsymbol{U}_2}^{(i,m)} \boldsymbol{U}_1^* \boldsymbol{T}_{(k-2)j} \tilde{\boldsymbol{e}}_m \boldsymbol{e}_i'$ for $3 \leqslant k \leqslant 4$ and $\boldsymbol{M}_{kj,i,\ell,m,h} = \boldsymbol{\Delta}_{\boldsymbol{U}_1}^{(i,m)} \boldsymbol{\Delta}_{\boldsymbol{U}_2}^{(\ell,h)} \boldsymbol{e}_i \bar{\boldsymbol{e}}_m' \boldsymbol{T}_{(k-4)j} \tilde{\boldsymbol{e}}_h \boldsymbol{e}_\ell'$ for $5 \leqslant k \leqslant 6$ are all $N \times N$ matrices, with

$$\mathbf{T}_{1j} = \sum_{k=1}^{r} \nabla \ell_j^I(\lambda_k^*) (\lambda_k - \lambda_k^*) \mathbf{S}_k^I + \sum_{k=1}^{s} \sum_{h=1}^{2} (\boldsymbol{\eta}_k - \boldsymbol{\eta}_k^*)' \nabla \ell_j^{II,h} (\boldsymbol{\eta}_k^*) \mathbf{S}_k^{II,h} \text{ and}$$

$$\mathbf{T}_{2j} = \frac{1}{2} \sum_{k=1}^{r} \nabla^2 \ell_j^I(\widetilde{\lambda}_k) (\lambda_k - \lambda_k^*)^2 \mathbf{S}_k^I + \sum_{k=1}^{s} \sum_{h=1}^{2} (\boldsymbol{\eta}_k - \boldsymbol{\eta}_k^*)' \nabla^2 \ell_j^{II,h} (\widetilde{\boldsymbol{\eta}}_k) (\boldsymbol{\eta}_k - \boldsymbol{\eta}_k^*) \mathbf{S}_k^{II,h}.$$
(S74)

For the tidiness of notation, we further let $M_{9j} = R_{3j}$, and then $R_j = \sum_{k=1}^{9} M_{kj}$. Note that the norms of M_{kj} 's further satisfy

$$\|\boldsymbol{M}_{1j,i,m}\|_{F} \leq \sqrt{2}C_{L}C_{g}\bar{\rho}^{j}\delta_{\omega}|\boldsymbol{\Delta}_{\boldsymbol{U}_{1}}^{(i,m)}|, \qquad \|\boldsymbol{M}_{2j,i,m}\|_{F} \leq \frac{\sqrt{2}}{2}C_{L}C_{g}\bar{\rho}^{j}\delta_{\omega}^{2}|\boldsymbol{\Delta}_{\boldsymbol{U}_{1}}^{(i,m)}|,$$

$$\|\boldsymbol{M}_{3j,i,m}\|_{F} \leq \sqrt{2}C_{L}C_{g}\bar{\rho}^{j}\delta_{\omega}|\boldsymbol{\Delta}_{\boldsymbol{U}_{2}}^{(i,m)}|, \qquad \|\boldsymbol{M}_{4j,i,m}\|_{F} \leq \frac{\sqrt{2}}{2}C_{L}C_{g}\bar{\rho}^{j}\delta_{\omega}^{2}|\boldsymbol{\Delta}_{\boldsymbol{U}_{2}}^{(i,m)}|,$$

$$\|\boldsymbol{M}_{5j,i,\ell,m,h}\|_{F} \leq \sqrt{2}C_{L}\bar{\rho}^{j}C_{g}\delta_{\omega}|\boldsymbol{\Delta}_{\boldsymbol{U}_{1}}^{(i,m)}||\boldsymbol{\Delta}_{\boldsymbol{U}_{2}}^{(\ell,h)}|, \quad \|\boldsymbol{M}_{6j,i,\ell,m,h}\|_{F} \leq \frac{\sqrt{2}}{2}C_{L}\bar{\rho}^{j}C_{g}\delta_{\omega}^{2}|\boldsymbol{\Delta}_{\boldsymbol{U}_{1}}^{(i,m)}||\boldsymbol{\Delta}_{\boldsymbol{U}_{2}}^{(\ell,h)}|,$$

$$\|\boldsymbol{M}_{7j}\|_{F} \leq \sqrt{2}C_{L}\bar{\rho}^{j}\delta_{\omega}\delta_{s}, \qquad \|\boldsymbol{M}_{8j}\|_{F} \leq \frac{\sqrt{2}}{2}C_{L}\bar{\rho}^{j}\delta_{\omega}^{2}\delta_{s}, \quad \text{and} \quad \|\boldsymbol{M}_{9j}\|_{F} \leq C_{L}C_{g}\bar{\rho}^{j}\alpha\delta_{\omega}^{2},$$

$$(S75)$$

As a result,

$$\frac{1}{T} \sum_{t=1}^{T} \left| \left\langle \boldsymbol{\varepsilon}_{t}, \boldsymbol{R}_{j} \boldsymbol{y}_{t-p-j} \right\rangle \right| = \sum_{k=1}^{2} \sum_{t=1}^{N} \sum_{m=1}^{R_{1}} \frac{1}{T} \sum_{t=1}^{T} \left\langle \boldsymbol{\varepsilon}_{t}, \boldsymbol{M}_{kj,i,m} \boldsymbol{y}_{t-p-j} \right\rangle + \sum_{k=3}^{4} \sum_{i=1}^{N} \sum_{m=1}^{R_{2}} \frac{1}{T} \sum_{t=1}^{T} \left\langle \boldsymbol{\varepsilon}_{t}, \boldsymbol{M}_{kj,i,m} \boldsymbol{y}_{t-p-j} \right\rangle + \sum_{k=7}^{4} \sum_{m=1}^{N} \sum_{t=1}^{T} \left\langle \boldsymbol{\varepsilon}_{t}, \boldsymbol{M}_{kj,i,m} \boldsymbol{y}_{t-p-j} \right\rangle + \sum_{k=7}^{4} \sum_{t=1}^{T} \left\langle \boldsymbol{\varepsilon}_{t}, \boldsymbol{M}_{kj} \boldsymbol{y}_{t-p-j} \right\rangle \\
\leq \sum_{k=1}^{2} \sum_{i=1}^{N} \sum_{m=1}^{R_{1}} \left\| \boldsymbol{M}_{kj,i,m} \right\|_{F} \sup_{\boldsymbol{M} \in \boldsymbol{\Pi}_{S,1}(1,s_{2}R_{2},1)} \frac{1}{T} \sum_{t=1}^{T} \left\langle \boldsymbol{\varepsilon}_{t}, \boldsymbol{M} \boldsymbol{y}_{t-p-j} \right\rangle \\
+ \sum_{k=3}^{4} \sum_{i=1}^{N} \sum_{m=1}^{R_{2}} \left\| \boldsymbol{M}_{kj,i,m} \right\|_{F} \sup_{\boldsymbol{M} \in \boldsymbol{\Pi}_{S,1}(s_{1}R_{1},1,1)} \frac{1}{T} \sum_{t=1}^{T} \left\langle \boldsymbol{\varepsilon}_{t}, \boldsymbol{M} \boldsymbol{y}_{t-p-j} \right\rangle \\
+ \sum_{k=5}^{6} \sum_{i,\ell=1}^{N} \sum_{m=1}^{R_{2}} \sum_{h=1}^{T} \sum_{t=1}^{T} \left\| \boldsymbol{M}_{kj,i,\ell,m,h} \right\|_{F} \sup_{\boldsymbol{M} \in \boldsymbol{\Pi}_{S,1}(1,1,1)} \left\langle \boldsymbol{\varepsilon}_{t}, \boldsymbol{M} \boldsymbol{y}_{t-p-j} \right\rangle \\
+ \sum_{k=7}^{9} \left\| \boldsymbol{M}_{kj} \right\|_{F} \sup_{\boldsymbol{M} \in \boldsymbol{\Pi}_{S,1}(s_{1}R_{1},s_{2}R_{2},R_{1}\wedge R_{2})} \frac{1}{T} \sum_{t=1}^{T} \left\langle \boldsymbol{\varepsilon}_{t}, \boldsymbol{M} \boldsymbol{y}_{t-p-j} \right\rangle$$
(S76)

Then by (S75), (S76) and (S84) in Lemma S.22, (ii) is verified.

S4.20 Proof of Lemma S.16

The proof of this lemma follows closely from the proof of Lemma S.5. The main difference lies in the sparsity conditions. Specifically, denote the index sets of the nonzero rows and columns in \mathbf{A}_{j}^{*} by \mathbb{S}_{1} and \mathbb{S}_{2} , and those in $\mathbf{\Delta}_{j}$ by $\bar{\mathbb{S}}_{1}$ and $\bar{\mathbb{S}}_{2}$, respectively. By Assumptions 3 and 4, it holds that the cardinality of \mathbb{S}_{i} and $\bar{\mathbb{S}}_{i}$ satisfy $|\mathbb{S}_{i}| \leq s_{i}\mathcal{R}_{i}$ and $|\bar{\mathbb{S}}_{i}| \leq \bar{s}_{i}$ for i = 1 or 2. Then, note that $\mathbb{E}(\|(\mathbf{y}_{t})_{\mathbb{S}_{2}}\|_{2}^{2}) \leq s_{2}\mathcal{R}_{2}\lambda_{\max}(\mathbf{\Sigma}_{\varepsilon})\mu_{\max}(\mathbf{\Psi}_{*})$ holds by Lemma S.11, and thus for all $j \geq 1$,

$$\mathbb{E}(\|\boldsymbol{A}_{j}^{*}\boldsymbol{y}_{t-j}\|_{2}) \leqslant \left\{\mathbb{E}(\|\boldsymbol{A}_{j}^{*}\boldsymbol{y}_{t-j}\|_{2}^{2})\right\}^{1/2} \leqslant \|\boldsymbol{A}_{j}^{*}\|_{\text{op}}\mathbb{E}(\|(\boldsymbol{y}_{t-j})_{\mathbb{S}_{2}}\|_{2}^{2})^{1/2} \leqslant C_{*}\bar{\rho}^{j}\sqrt{\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})\mu_{\max}(\boldsymbol{\Psi}_{*})s_{2}\boldsymbol{\mathcal{R}}_{2}}$$
(S77)

and

$$\mathbb{E}\left(\sup_{\boldsymbol{\Delta}\in\boldsymbol{\Upsilon}_{\mathbb{S}}\cap\mathcal{S}(\delta)}\|\boldsymbol{\Delta}_{j}\boldsymbol{y}_{t-j}\|_{2}\right) \leqslant \left\{\mathbb{E}\left(\sup_{\boldsymbol{\Delta}\in\boldsymbol{\Upsilon}_{\mathbb{S}}\cap\mathcal{S}(\delta)}\|\boldsymbol{\Delta}_{j}\boldsymbol{y}_{t-j}\|_{2}^{2}\right)\right\}^{1/2}$$

$$\leqslant \delta C_{1}\bar{\rho}^{j}\sqrt{\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})\mu_{\max}(\boldsymbol{\Psi}_{*})\bar{s}_{2}}.$$
(S78)

By the Cauchy-Schwarz inequality and (S78),

$$\mathbb{E}\left\{\sup_{\boldsymbol{\Delta}\in\boldsymbol{\Upsilon}_{\mathbb{S}}\cap\mathcal{S}(\delta)}|S_{1}(\boldsymbol{\Delta})|\right\}\leqslant\frac{2}{T}\sum_{t=1}^{T}\sum_{j=1}^{\infty}\sum_{k=t}^{\infty}\delta^{2}C_{1}^{2}\bar{\rho}^{j+k}\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})\mu_{\max}(\boldsymbol{\Psi}_{*})\bar{s}_{2}\leqslant\frac{\delta^{2}C_{2}\kappa_{2}\bar{s}_{2}}{T},$$

where $C_2 = 2C_1^2 \bar{\rho}^2/(1-\bar{\rho})^3 \approx 1$. Similarly, by (S77) and (S78),

$$\mathbb{E}\left\{\sup_{\boldsymbol{\Delta}\in\boldsymbol{\Upsilon}_{\mathbb{S}}\cap\mathcal{S}(\delta)}|S_{2}(\boldsymbol{\Delta})|\right\}\leqslant\frac{2}{T}\sum_{t=1}^{T}\sum_{j=t}^{\infty}\sum_{k=1}^{t-1}\delta C_{*}C_{1}\bar{\rho}^{j+k}\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})\mu_{\max}(\boldsymbol{\Psi}_{*})\sqrt{s_{2}\bar{s}_{2}\mathcal{R}_{2}}\leqslant\frac{\delta C_{3}\kappa_{2}\sqrt{s_{2}\bar{s}_{2}\mathcal{R}_{2}}}{T},$$

where $C_3 = 2C_*C_1\bar{\rho}^2/(1-\bar{\rho})^3 \approx 1$. Moreover, note that $\mathbb{E}(\|(\boldsymbol{\varepsilon}_t)_{\bar{\mathbb{S}}_1}\|_2) \leqslant \sqrt{\mathbb{E}(\|(\boldsymbol{\varepsilon}_t)_{\bar{\mathbb{S}}_1}\|_2^2)} \leqslant \sqrt{\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})\bar{s}_1}$. Then by (S78) and a method similar to the above,

$$\mathbb{E}\left\{\sup_{\boldsymbol{\Delta}\in\boldsymbol{\Upsilon}_{\mathbb{S}}\cap\mathcal{S}(\delta)}|S_{3}(\boldsymbol{\Delta})|\right\} \leqslant \frac{2}{T}\sum_{t=1}^{T}\sum_{j=t}^{\infty}\delta C_{1}\bar{\rho}^{j}\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})\sqrt{\mu_{\max}(\boldsymbol{\Psi}_{*})}\sqrt{\bar{s}_{1}\bar{s}_{2}}$$
$$\leqslant \frac{\delta C_{4}\sqrt{\kappa_{2}\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})}\sqrt{\bar{s}_{1}\bar{s}_{2}}}{T},$$

where $C_4 = 2C_1\bar{\rho}/(1-\bar{\rho})^2 \approx 1$. By Markov's inequality, we can show that

$$\mathbb{P}\left\{\sup_{\boldsymbol{\Delta}\in\boldsymbol{\Upsilon}_{\mathbb{S}}\cap\mathcal{S}(\delta)}|S_1(\boldsymbol{\Delta})|\geqslant \delta^2C_2\kappa_1\right\}\leqslant \frac{\mathbb{E}\{\sup_{\boldsymbol{\Delta}\in\boldsymbol{\Upsilon}_{\mathbb{S}}\cap\mathcal{S}(\delta)}|S_1(\boldsymbol{\Delta})|\}}{\delta^2C_2\kappa_1}\leqslant \frac{\kappa_2\bar{s}_2}{\kappa_1T}\leqslant \sqrt{\frac{\bar{s}_2}{T}},$$

$$\mathbb{P}\left\{\sup_{\boldsymbol{\Delta}\in\boldsymbol{\Upsilon}_{\mathbb{S}}\cap\mathcal{S}(\delta)}|S_{2}(\boldsymbol{\Delta})|\geqslant\delta C_{3}\sqrt{\frac{\kappa_{2}\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})d_{\mathcal{S}}}{T}}\right\}\leqslant\sqrt{\frac{\kappa_{2}s_{2}\bar{s}_{2}\mathcal{R}_{2}}{\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})Td_{\mathcal{S}}}}\leqslant\sqrt{\frac{\kappa_{2}\bar{s}_{2}}{\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})T}},$$

and

$$\mathbb{P}\left\{\sup_{\boldsymbol{\Delta}\in\boldsymbol{\Upsilon}_{\mathbb{S}}\cap\mathcal{S}(\delta)}|S_{3}(\boldsymbol{\Delta})|\geqslant\delta C_{4}\sqrt{\frac{\kappa_{2}\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})d_{\mathcal{S}}}{T}}\right\}\leqslant\sqrt{\frac{\bar{s}_{1}\bar{s}_{2}}{Td_{\mathcal{S}}}},$$

where the last inequality in (S43) uses the condition that $T \gtrsim \bar{s}_2$. Then the sum of the above three tail probabilities is $(1 + \sqrt{\kappa_2/\lambda_{\max}(\Sigma_{\varepsilon})})\sqrt{\bar{s}_2/T}(1 + \sqrt{\bar{s}_1/d_{\mathcal{S}}})$.

S4.21 Auxiliary lemmas for the proofs of Lemmas S.14 and S.15

The proofs of Lemmas S.14 and S.15 rely on the following auxiliary results.

Lemma S.17 (HOSVD perturbation bound). Suppose that $\mathfrak{G} = \mathfrak{S} \times U_1 \times U_2$ and $\widetilde{\mathfrak{G}} = \widetilde{\mathfrak{S}} \times \widetilde{U}_1 \times \widetilde{U}_2$ are two HOSVD for \mathfrak{G} and $\widetilde{\mathfrak{G}}$, with the same multilinear ranks $(\mathcal{R}_1, \mathcal{R}_2)$ along the first and second modes. Under Assumptions 4 and 5, we have

$$\|\widetilde{\mathbf{S}} - \mathbf{S}\|_{F} \leqslant \frac{C(\eta_{1} + \eta_{2})}{\beta} \|\widetilde{\mathbf{G}} - \mathbf{G}\|_{F} \quad and$$

$$\|\widetilde{\mathbf{U}}_{i} - \mathbf{U}_{i}\|_{F} \leqslant \frac{C\eta_{i}}{\beta} \|\widetilde{\mathbf{G}} - \mathbf{G}\|_{F},$$
(S79)

where $\eta_i = \sum_{j=1}^{\mathcal{R}_i} \sigma_1^2(\mathbf{G}_{(i)}) / \sigma_j^2(\mathbf{G}_{(i)})$ for i = 1 or 2. Moreover,

$$\|\widetilde{\mathbf{G}} - \mathbf{G}\|_{\mathbf{F}} \leqslant \|\widetilde{\mathbf{S}} - \mathbf{S}\|_{\mathbf{F}} + C_{\mathbf{S}} \sum_{i=1}^{2} \|\widetilde{\boldsymbol{U}}_{i} - \boldsymbol{U}_{i}\|_{\mathbf{F}}.$$
 (S80)

Proof of Lemma S.17. The proof of this lemma follows trivially from Lemma 1 in Wang et al. (2021b).

Lemma S.18 (Covering number for sparse-and-low-Tucker-rank tensors). Let

$$\boldsymbol{\Pi}_{\mathrm{S}}(s_1, s_2, \mathcal{R}_1, \mathcal{R}_2) = \{\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{S}} \times_1 \boldsymbol{U}_1 \times_2 \boldsymbol{U}_2 : \|\boldsymbol{\mathcal{M}}\|_{\mathrm{F}} \leqslant 1, \boldsymbol{\mathcal{S}} \in \mathbb{R}^{\mathcal{R}_1 \times \mathcal{R}_2 \times d}, \boldsymbol{U}_i \in \mathcal{U}_{\mathrm{S},i}, i = 1, 2\},$$

where $\mathcal{U}_{S,i} = \{ \boldsymbol{U} \in \mathbb{R}^{N \times \mathcal{R}_i} \mid \boldsymbol{U}'\boldsymbol{U} = \boldsymbol{I}_{\mathcal{R}_i}, \|\boldsymbol{U}\|_0 \leqslant s_i \}$. For any $\epsilon > 0$, let $\bar{\boldsymbol{\Pi}}_S(\epsilon; s_1, s_2, \mathcal{R}_1, \mathcal{R}_2)$ be a minimal ϵ -net for $\boldsymbol{\Pi}_S(s_1, s_2, \mathcal{R}_1, \mathcal{R}_2)$ in the Frobenius norm. Then $\bar{\boldsymbol{\Pi}}_S(\epsilon; s_1, s_2, \mathcal{R}_1, \mathcal{R}_2)$ has cardinality satisfying

$$|\bar{\mathbf{\Pi}}_{\mathrm{S}}(\epsilon; s_1, s_2, \mathcal{R}_1, \mathcal{R}_2)| \leq \binom{N\mathcal{R}_1}{s_1} \binom{N\mathcal{R}_2}{s_2} \binom{9}{\epsilon}^{\mathcal{R}_1\mathcal{R}_2d + s_1 + s_2}.$$

Proof of Lemma S.18. The proof of this lemma follows trivially from Lemma S.13.

Recall from (S71) that

$$\mathbf{\Xi}_{\mathrm{S}}(s_1, s_2, \mathcal{R}_1, \mathcal{R}_2) = \left\{ \mathbf{M}(\boldsymbol{a}, \mathbf{B}) \in \mathbb{R}^{N \times N \times (d+r+2s)} \mid \boldsymbol{a} \in \mathbb{R}^{r+2s}, \mathbf{B} \in \Gamma_{\mathrm{S}}(s_1, s_2, \mathcal{R}_1, \mathcal{R}_2) \right\},$$

and
$$\Xi_{S,1}(s_1, s_2, \mathcal{R}_1, \mathcal{R}_2) = \Xi_S(s_1, s_2, \mathcal{R}_1, \mathcal{R}_2) \cap \{ \mathbf{M} \in \mathbb{R}^{N \times N \times (d+r+2s)} \mid \| \mathbf{M} \|_F = 1 \}.$$

Lemma S.19 (Covering number and discretization for $\Xi_{S,1}$). For any $0 < \epsilon < 2/3$, let $\bar{\Xi}_{S}(\epsilon; s_1, s_2, \mathcal{R}_1, \mathcal{R}_2)$ be a minimal generalized ϵ -net of $\Xi_{S,1}(s_1, s_2, \mathcal{R}_1, \mathcal{R}_2)$. In (ii) – (iii), denote $\Xi_{S,1}(s_1, s_2, \mathcal{R}_1, \mathcal{R}_2)$ and $\bar{\Xi}_{S}(s_1, s_2, \mathcal{R}_1, \mathcal{R}_2)$ by Ξ_1 and $\bar{\Xi}$, respectively.

(i) The cardinality of $\bar{\Xi}_{S}(\epsilon; s_1, s_2, \mathcal{R}_1, \mathcal{R}_2)$ satisfies

$$\log |\bar{\Xi}_{\mathrm{S}}(\epsilon; s_1, s_2, \mathcal{R}_1, \mathcal{R}_2)| \lesssim (\mathcal{R}_1 \mathcal{R}_2 d + s_1 + s_2) \log(1/\epsilon) + \sum_{i=1}^2 s_i \log N \mathcal{R}_i,$$

- (ii) There exist absolute constants $c_{\mathfrak{M}}, C_{\mathfrak{M}} > 0$ such that for any $\mathfrak{M} \in \bar{\Xi}_{S}(s_{1}, s_{2}, \mathcal{R}_{1}, \mathcal{R}_{2})$, it holds $c_{\mathfrak{M}} \leqslant \|\mathfrak{M}\|_{F} \leqslant C_{\mathfrak{M}}$.
- (iii) For any $\mathbf{X} \in \mathbb{R}^{N \times N(d+r+2s)}$ and $\mathbf{Z} \in \mathbb{R}^{N(d+r+2s) \times T}$, it holds

$$\sup_{\mathbf{M} \in \mathbf{\Xi}_{\mathrm{S},1}(s_{1},s_{2},\mathcal{R}_{1},\mathcal{R}_{2})} \langle \mathbf{M}_{(1)}, \boldsymbol{X} \rangle \leqslant (1 - 1.5\epsilon)^{-1} \max_{\mathbf{M} \in \bar{\mathbf{\Xi}}_{\mathrm{S}}(s_{1},s_{2},\mathcal{R}_{1},\mathcal{R}_{2})} \langle \mathbf{M}_{(1)}, \boldsymbol{X} \rangle,$$

$$\sup_{\mathbf{M} \in \mathbf{\Xi}_{\mathrm{S},1}(s_{1},s_{2},\mathcal{R}_{1},\mathcal{R}_{2})} \|\mathbf{M}_{(1)}\boldsymbol{Z}\|_{\mathrm{F}} \leqslant (1 - 1.5\epsilon)^{-1} \max_{\mathbf{M} \in \bar{\mathbf{\Xi}}_{\mathrm{S}}(s_{1},s_{2},\mathcal{R}_{1},\mathcal{R}_{2})} \|\mathbf{M}_{(1)}\boldsymbol{Z}\|_{\mathrm{F}}.$$

Proof of Lemma S.19. The proof of this lemma follows trivially from Lemma S.18 and the proof of Lemma S.8. \Box

Lemma S.20. Suppose that Assumptions 1 and 2 hold and $T \gtrsim (\kappa_2/\kappa_1)^2 d_1 \log(\kappa_2/\kappa_1)$. Let $\boldsymbol{z}_t = \{\boldsymbol{L}'_{\text{stack}}(\boldsymbol{\omega}^*) \otimes \boldsymbol{I}_N\} \boldsymbol{x}_t$ be defined as in (S25). Then

$$\mathbb{P}\left(\frac{c_{\mathbf{M}}\kappa_{1}}{8} \leqslant \inf_{\mathbf{M} \in \mathbf{\Xi}_{S,1}(s_{1},s_{2},\mathcal{R}_{1},\mathcal{R}_{2})} \frac{1}{T} \sum_{t=1}^{T} \|\mathbf{M}_{(1)}\boldsymbol{z}_{t}\|_{2}^{2} \leqslant \sup_{\mathbf{M} \in \mathbf{\Xi}_{S,1}(s_{1},s_{2},\mathcal{R}_{1},\mathcal{R}_{2})} \frac{1}{T} \sum_{t=1}^{T} \|\mathbf{M}_{(1)}\boldsymbol{z}_{t}\|_{2}^{2} \leqslant 6C_{\mathbf{M}}\kappa_{2}\right)
\geqslant 1 - 2e^{-cd_{1}\log(\kappa_{2}/\kappa_{1})}.$$
(S81)

and

$$\mathbb{P}\left\{\sup_{\mathbf{M}\in\mathbf{\Xi}_{S,1}(s_1,s_2,\mathcal{R}_1,\mathcal{R}_2)}\frac{1}{T}\sum_{t=1}^{T}\langle\mathbf{M}_{(1)}\boldsymbol{z}_t,\boldsymbol{\varepsilon}_t\rangle\lesssim\sqrt{\frac{\kappa_2\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})d_1}{T}}\right\}\geqslant 1-e^{-cd_1}-2e^{-cd_1\log(\kappa_2/\kappa_1)},\quad(S82)$$

where $d_1 = \mathcal{R}_1 \mathcal{R}_2 d + \sum_{i=1}^2 s_i (1 + \log N \mathcal{R}_i)$.

Proof of Lemma S.20. The proof of this lemma follows trivially from Lemma S.19 and the proof of Lemma S.6. \Box

Lemma S.21 (Covering number and discretization for sparse low-rank matrices). Let $\Pi_{S,1}(s_1, s_2, \mathcal{R}) = \{ \boldsymbol{M} = \boldsymbol{U}_1 \boldsymbol{S} \boldsymbol{U}_2' \mid \|\boldsymbol{M}\|_F = 1, \boldsymbol{S} \in \mathbb{R}^{\mathcal{R} \times \mathcal{R}}, \boldsymbol{U}_i \in \mathbb{R}^{N \times \mathcal{R}}, \|\boldsymbol{U}_i\|_0 \leqslant s_i, i = 1 \text{ or } 2 \}, \text{ and let } \bar{\Pi}_S(s_1, s_2, \mathcal{R}) \text{ be a minimal } 1/2\text{-net of } \Pi_{S,1}(s_1, s_2, \mathcal{R}) \text{ in the Frobenius norm. Then the cardinality of } \bar{\Pi}_S(s_1, s_2, \mathcal{R}) \text{ satisfies}$

$$\log |\bar{\Pi}_{S}(s_1, s_2, \mathcal{R})| \leq [\mathcal{R} + (s_1 + s_2)(1 + \log N\mathcal{R})] \log 18.$$

Moreover, for any $\mathbf{X} \in \mathbb{R}^{N \times N}$, it holds

$$\sup_{\boldsymbol{M}\in\boldsymbol{\Pi}_{\mathrm{S},1}(s_{1},s_{2},\mathcal{R})}\langle\boldsymbol{M},\boldsymbol{X}\rangle\leqslant 4\max_{\boldsymbol{M}\in\bar{\boldsymbol{\Pi}}_{\mathrm{S}}(s_{1},s_{2},\mathcal{R})}\langle\boldsymbol{M},\boldsymbol{X}\rangle.$$

Proof of Lemma S.21. The proof of this lemma follows trivially from Lemma S.12. \Box

Lemma S.22. Suppose that Assumptions 1 and 2 hold.

(i) Let $\mathbb{K}(s) = \{ \boldsymbol{v} \in \mathbb{R}^N : \|\boldsymbol{v}\|_0 \leqslant s, \|\boldsymbol{v}\|_2 \leqslant 1 \}$ the set of s-sparse vectors. If $T \gtrsim s \log N$, then

$$\mathbb{P}\left\{\forall j \geqslant 1 : \sup_{\boldsymbol{v} \in \mathbb{K}(s)} \boldsymbol{v}' \frac{1}{T} \sum_{t=1}^{T} \boldsymbol{y}_{t-p-j} \boldsymbol{y}'_{t-p-j} \boldsymbol{v} \leqslant 2\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon}) \mu_{\max}(\boldsymbol{\Psi}_{*}) (j\sigma^{2} + 1)\right\}$$

$$\geqslant 1 - 3e^{-s \log N \log 9}.$$
(S83)

(ii) If $T \gtrsim (s_1 + s_2)(\mathcal{R} + \log N\mathcal{R})$, then

$$\mathbb{P}\left\{\forall j \geqslant 1: \sup_{\boldsymbol{M} \in \boldsymbol{\Pi}_{S}(s_{1}, s_{2}, \mathcal{R})} \frac{1}{T} \sum_{t=1}^{T} \langle \boldsymbol{M} \boldsymbol{y}_{t-p-j}, \boldsymbol{\varepsilon}_{t} \rangle \leqslant 24\lambda_{\max}(\boldsymbol{\Sigma}_{\varepsilon})(2j\sigma^{2} + 1) \\
\cdot \sqrt{\frac{\mu_{\max}(\boldsymbol{\Psi}_{*})[\mathcal{R} + (s_{1} + s_{2})(1 + \log N\mathcal{R})]}{T}}\right\} \geqslant 1 - 4e^{-s_{2}\log N\mathcal{R}\log 9}.$$
(S84)

Proof of Lemma S.22. The proof of this lemma follows trivially from Lemma S.21, Lemma F.2 in Basu and Michailidis (2015) and the proof of Lemma S.7. □

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