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How big do index funds need to be to destabilise markets?

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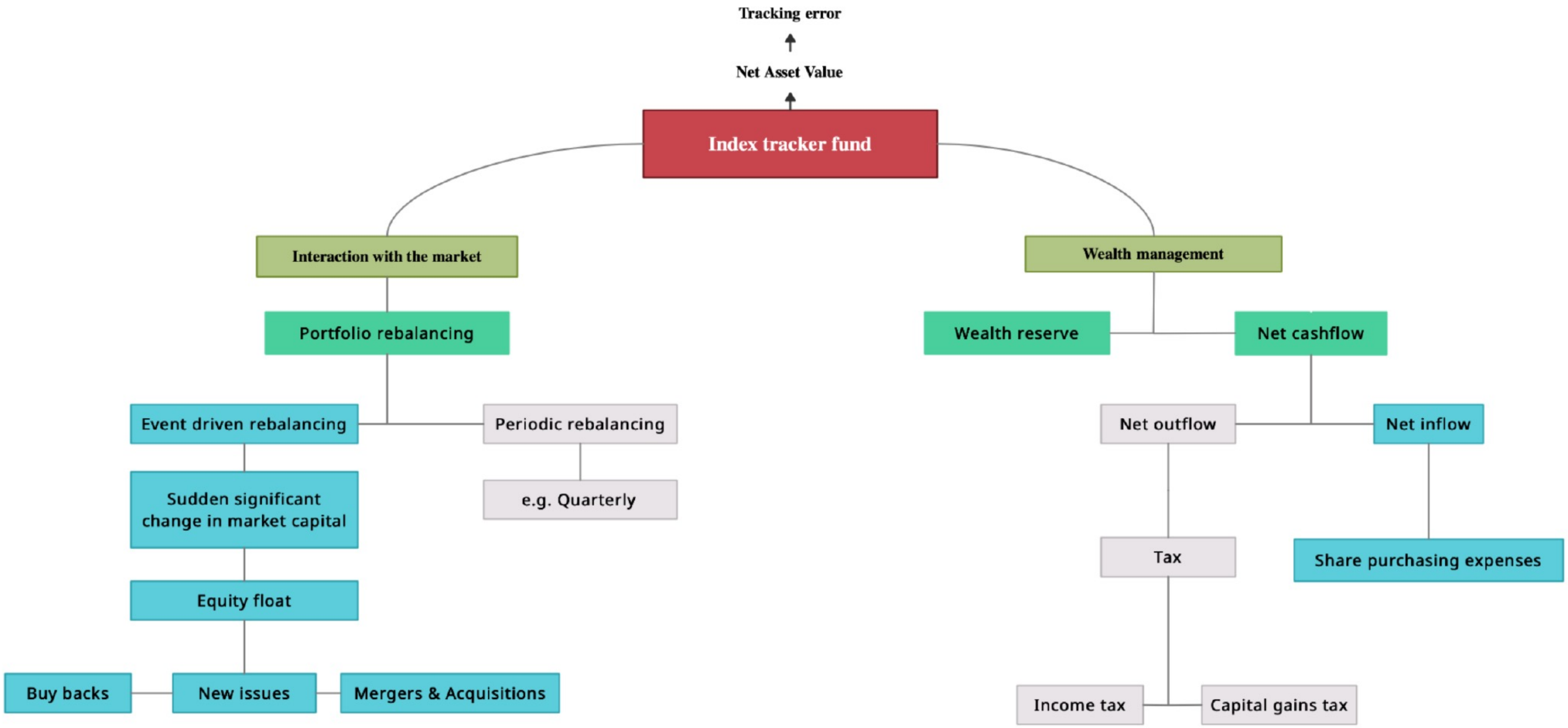
Index funds invest in a basket of stocks in order to match the performance of an index, such as the S&P 500.

The resulting trading can move prices for reasons that have nothing to do with the fundamentals of the underlying stocks.

Internal operations of a passively managed investment fund [3]



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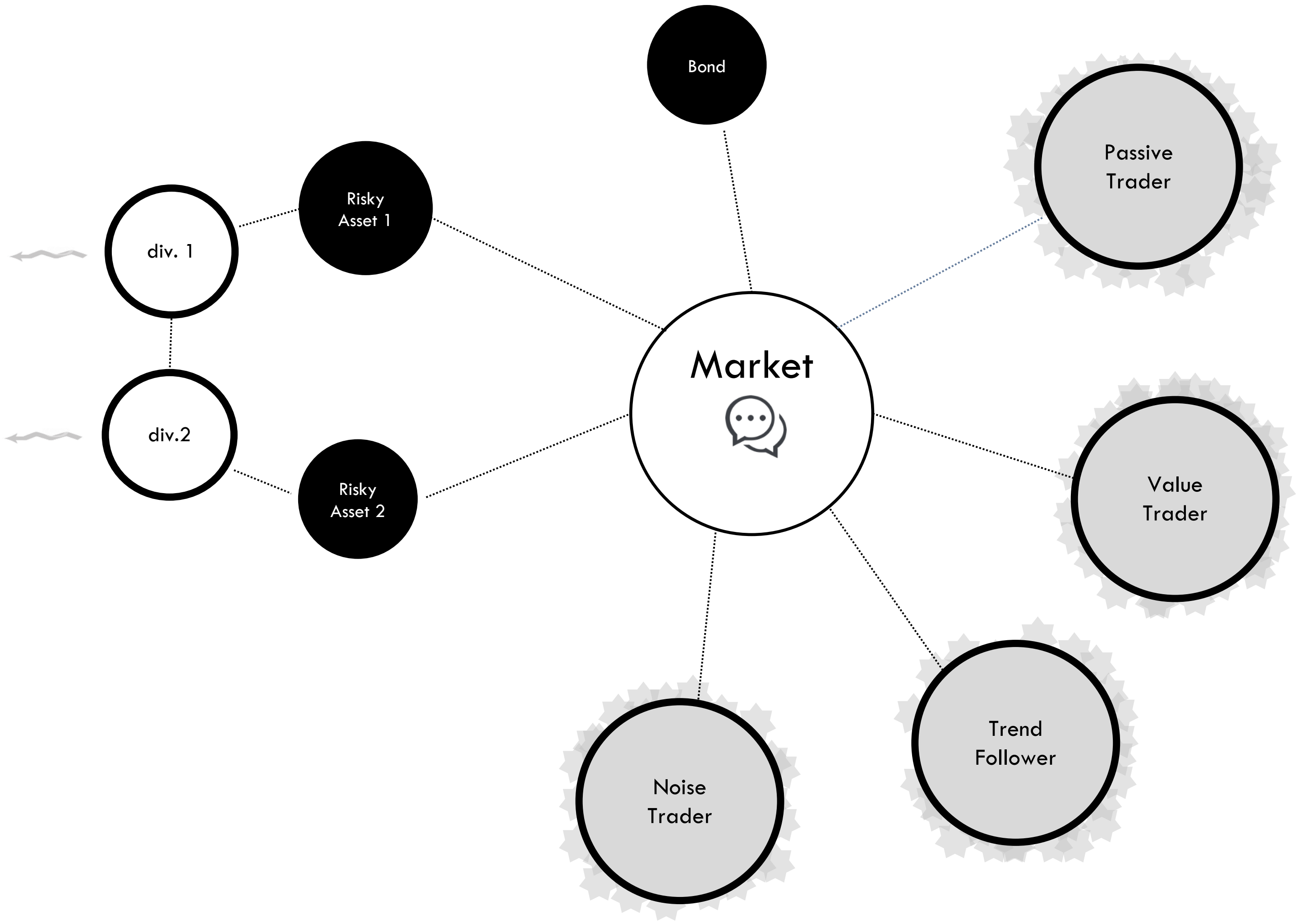
Multi-Asset Stock Market Ecology



Passive Trader

Excess demand by a passive investor n, on stock i, at time t

$$\mathrm{E}_{n,i,t} = S_{n,i,t} - S_{n,i,t-1}$$

This is the number of shares that the fund targets to end up holding minus what it currently holds

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$S_{n,i,t}$ - is the number of shares that a passive investor n, targets to hold on stock i at time t,

$S_{n,i,t-1}$ - is the number of shares of stock i held by investor n at time t-1

Passive Trader

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The targeted holdings, $S_{n,i,t}$, are a function of current wealth and the market capitalization ratio of the stock in the overall stock market [3]

$$E_{n,i,t} = W_{n,t}\left(\frac{m_{i,\tau}}{\sum_x m_{x,\tau}}\right).\frac{1}{P_{i,t}} - S_{n,i,t-1}$$

$E_{n,i,t}$ - excess demand by passive investor n on stock i at time t.

$W_{n,t}$ - Total wealth of the passive investor n at time t

$m_{i,\tau}$ - market capital of stock i at time τ

τ - the time when the fund was last rebalanced.

α - rebalancing interval measured in days

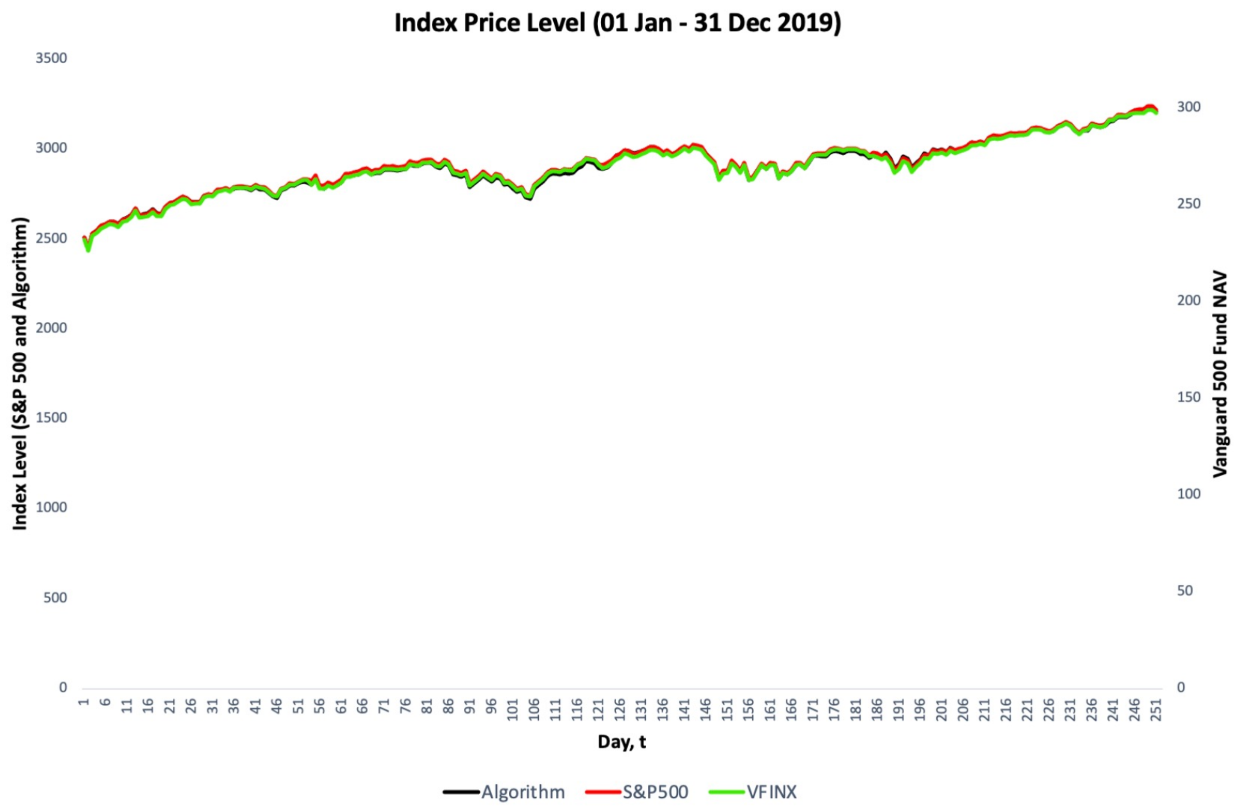
$P_{i,t}$ - market price of stock i at time t

$S_{n,i,t-1}$ - number of shares held by investor n, on stock i at time t-1.

Passive Trader

Assessment of how well the algorithm's portfolio tracks the S&P 500 index on Bloomberg data.

We use the Vanguard 500 Fund as a benchmark for the algorithm's performance



Passive Trader

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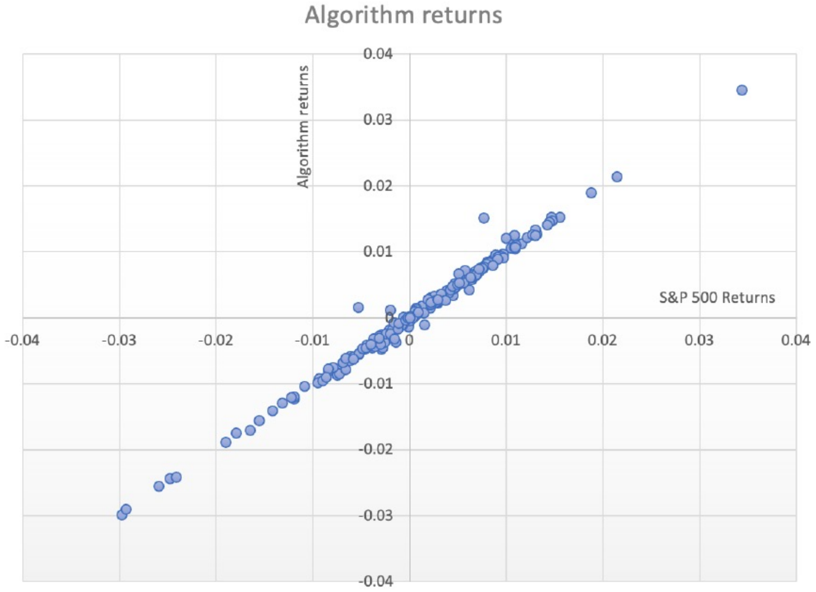
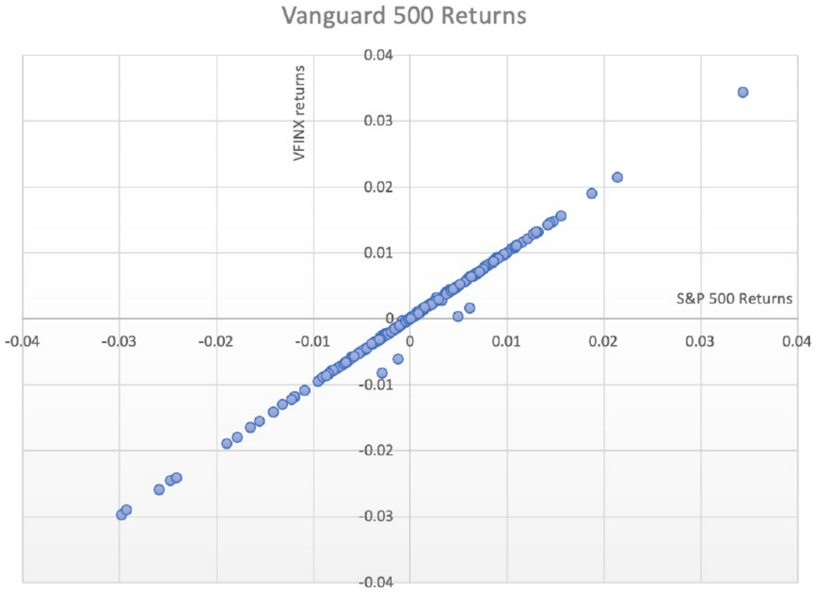
Assessment of the returns from the algorithm's portfolio compared to the S&P 500 index returns





Mean absolute error	0.00041461
Standard deviation of error	0.00086369

Mean absolute error	0.00015777
Standard deviation of error	0.00062014

Passive Trader

Tracking error

Absolute tracking error

$$\varepsilon_T = \frac{\sum_t^T |r_{index}^t - r_{tracker}^t|}{T - t}$$

Vanguard : $\varepsilon_T = 0.000158$
Algorithm : $\varepsilon_T = 0.000415$

Standard deviation [10]

$$\varepsilon_T = \sqrt{\frac{\sum_t^T (r_{index}^t - r_{tracker}^t)^2}{T - t}}$$

Vanguard : $\varepsilon_T = 0.000620$
Algorithm : $\varepsilon_T = 0.000863$

Risky Asset

The risky asset gives a dividend which, for simplicity we assume is payable continuously. We model the process as illustrated in Scholl *et al* (2020) [11]

$$dD_{i,t} = g_i D_{i,t-\delta t}.dt + \sigma_i D_{i,t-\delta t} dU_t^i$$

To replicate empirical observations, we make the dividend process temporally correlated with its historical payouts.

We also correlate it cross-sectionally with the dividend processes of the other risky assets.

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dD_t is the change in dividend payout at time t

D_t is the dividend at time t

g is the average dividend growth rate

σ is the standard deviation of historical dividend growth rate

dU_t is a standard weiner process

Risky Asset

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We model temporal correlation by modifying the Weiner term by the function,

$$dU_t^i = (\sqrt{1-\alpha_i^2}).dZ_t^i + \alpha_i dU_{t-\delta t}^i$$ [12]

This process is consistent with both the standard Weiner process and Ito's lemma and one would find that,

$$E(dU_t^i) = 0 \qquad Var(dU_t^i) = \delta t \qquad Corr(dU_t^i, dU_{t-\delta t}^i) = \alpha_i$$

δt is a small change in time

ω_i is the auto-correlation of the process for stock i

dZ_t^i and $dU_{t-\delta t}^i$ are independant weiner processes$\sim N(0, \delta t)$

Risky Asset

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For cross sectional correlation, we do a Cholesky decomposition of the correlation matrix of the two assets [2]

$$\Sigma = L.L^T$$

We then multiply the Lower matrix, L by the uncorrelated Weiner processes, dU_t^i, and

end up with a cross sectionally correlated process

Risky Asset

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The overall model

$$\begin{pmatrix} dD_{1,t} \\ dD_{2,t} \\ dD_{3,t} \end{pmatrix} = dt. \begin{pmatrix} g_1.dD_{1,t-\delta t} \\ g_2.dD_{2,t-\delta t} \\ g_3.dD_{3,t-\delta t} \end{pmatrix} + \begin{pmatrix} \rho_1^{'} & 0 & 0 \\ \rho_2^{'} & \rho_3^{'} & 0 \\ \rho_4^{'} & \rho_5^{'} & \rho_6^{'} \end{pmatrix} \begin{pmatrix} \sigma_1 D_{1,t-\delta t}.dU_t^i \\ \sigma_2 D_{2,t-\delta t}.dU_t^i \\ \sigma_3 D_{3,t-\delta t}.dU_t^i \end{pmatrix}$$

$$dU_t^i = (\sqrt{1-\alpha_i^2}).dZ_t^i + \alpha_i dU_{t-\delta t}^i$$

$\rho_i^{'}$ is a lower triangle term of a cholesky decompostion of the cross-sectional correlation matrix

α_i is the auto-correlation of the process of stock i where dZ_t^i and dU_t^i are independent Weiner processes

Risky Asset

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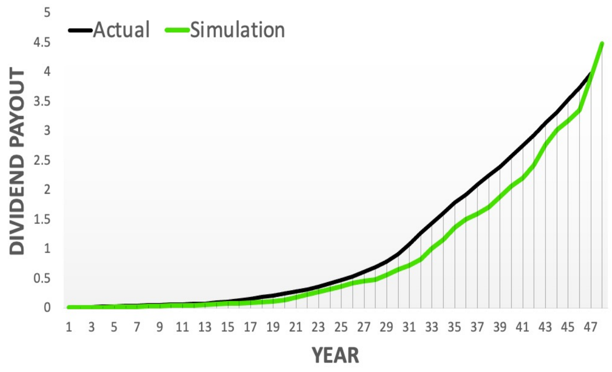
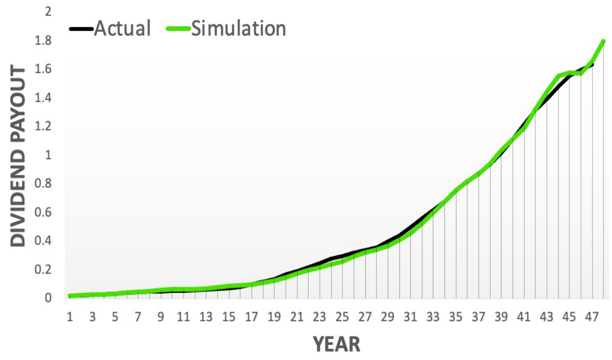
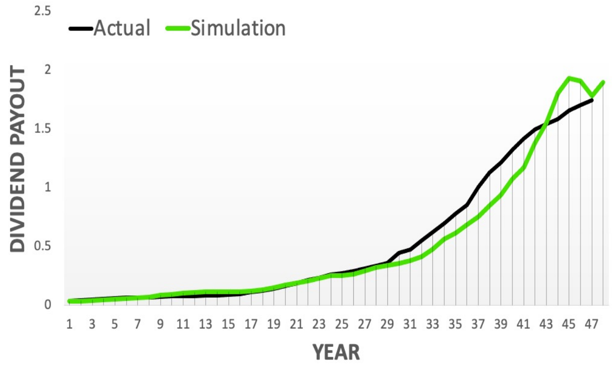
The model's simulation Vs. Actual data



JNJ



Coca-Cola



Colgate

Value Trader

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The Value Trader has a fixed rule of allocating wealth across asset classes:

Bonds	40%
Risky assets	60%

Value Trader

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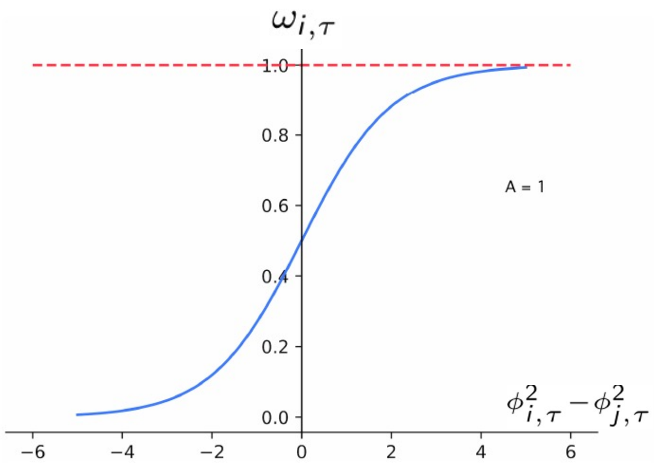
Within the Risky asset category, the 60% portion of wealth is further divided between the two risky assets depending on the relative strength of each asset's signal.

We model the investor's choice by a sigmoid function [5]

$$\omega_{i,\tau} = \frac{1}{1+e^{-A[\phi^2_{i,\tau} - \phi^2_{j,\tau}]}}$$

$$\omega_{j,\tau} = 1 - \omega_{i,\tau}$$



$\omega_{i,\tau}$ is the proportion of wealth invested in stock i at time τ

$\phi^2_{i,\tau}$ is the square of the signal of stock i at time τ

we raise the signal to the power of 2 so that the strength of a signal is independant of its sign since the trader is able to either short-sell or go long on an asset.

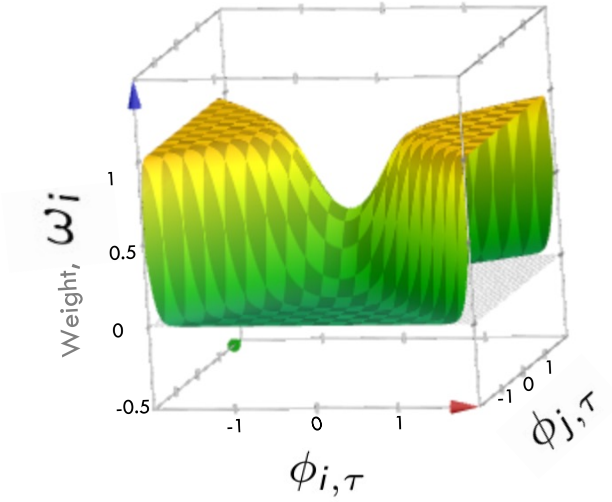
τ is the previous rebalancing date

Value Trader

The function is better represented on a 3D-space

$$\omega_{i,\tau} = \frac{1}{1+e^{-A[\phi^2_{i,\tau} - \phi^2_{j,\tau}]}}$$

$$\omega_{j,\tau} = 1 - \omega_{i,\tau}$$



$\omega_{i,\tau}$ is the proportion of wealth invested in stock i at time τ

$\phi^2_{i,\tau}$ is the square of the signal of stock i at time τ

τ is the previous rebalancing date

Value Trader

Re-engineering the sigmoid function gives us a more generic formula.

$$\omega_{i,\tau} = \frac{1}{1+e^{-\left[\phi^2_{i,\tau} - \phi^2_{j,\tau}\right]}}$$

$$= \frac{e^{\phi^2_{i,\tau}}}{e^{\phi^2_{i,\tau}} + e^{\phi^2_{j,\tau}}}$$

$$= \frac{e^{\phi^2_{i,\tau}}}{\sum_x e^{\phi^2_{x,\tau}}}$$

This result is identical to the choice function illustrated by McFadden, D. (1973) [9]

$\omega_{i,\tau}$ is the proportion of wealth invested in stock i at time τ

$\phi^2_{i,\tau}$ is the square of the signal of stock i at time τ

τ is the previous rebalancing date

Value Trader [11]

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The Value trader believes market prices revert to the asset's intrinsic value

$$\phi^{vt}_{i,t} = \log V^{vt}_{i,t} - \log P_{i,t}$$

$P_{i,t}$ is price of stock i at time t

$V^{vt}_{i,t}$ is the Value of the stock i at time t computed by the value trader as $V^{vi}_{i,t} = \mathbb{E}\sum_t^{\infty}[\frac{D_{i,t}}{(1+k)^t}]$

k is the required rate of return and we assume k $= 2\%$ [11]

the value trader calculates $E[D_{i,t}]$ from historical dividends as $D_{(i,o)}(1+g)^t$

Value Trader

Excess demand is then computed as

$$E_{n,i,t} = sign(\phi_{i,\tau}) \times \frac{0.6 W_{n,t} \lambda_n \omega_{i,\tau}}{P_{i,t}} - S_{n,i,t-1}$$

$sign(\phi_{i,\tau})$ is the sign of the trade signal

$W_{n,t}$ is the total wealth of investor n at time t

λ_n is the leverage of investor n

$\omega_{i,\tau}$ is the weight of wealth that the investor targets to invest in stock i at time t

$P_{i,t}$ is the price of the stock i at time t

$S_{n,i,t-1}$ is the number of shares held by investor n on stock i at time t

Noise Trader

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Everything is identical to the Value trader's rules of operation except for the function used to compute the signal of a risky asset

[11]

Signal, $\phi^{nt}_{i,t} = \log X_t V^{vt}_{i,t} - \log P_{i,t}$

$P_{i,t}$ is price of stock i at time t

$V^{vt}_{i,t}$ is the Value of the stock i at time t computed by the value trader as $V^{vi}_{i,t} = \mathbb{E}\sum_t^{\infty}[\frac{D_{i,t}}{(1+k)^t}]$

X_t is a discretised Ornstein Uhlenbeck process with the form $X_t = X_t + \rho(\mu - X_{t-1}) + \gamma\epsilon$

ρ is mean reversion rate, μ is long term mean of the process γ is a volatility parameter and ϵ is a standard normal random variable

Trend Follower

Everything is identical to the Value trader's rules of operation except for the function used to compute the signal of an asset

[11]

Signal, $\phi_{i,t}^{tf} = \log P_{i,t} - \log P_{i,(t-1)}$

$P_{i,t}$ is price of stock i at time t

The Market

The market regulates communication in the ecology.

It sends quotes (asset prices) to traders, receives orders and computes new asset prices depending on excess demand.

We use the Walrasian market clearing mechanism to determine new asset prices.

The algorithm searches for a price that minimizes the net excess demand.

Bond

We assume that the Bond gives a constant interest rate of 2% on all wealth invested as Bonds

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Thank You!